Ryu-Takayanagi Formula for Symmetric Random Tensor Networks

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We consider the special case of Random Tensor Networks (RTN) endowed with gauge symmetry constraints on each tensor. We compute the Rènyi entropy for such states and recover the Ryu-Takayanagi (RT) formula in the large bond regime. The result provides first of all an interesting new extension of the existing derivations of the RT formula for RTNs. Moreover, this extension of the RTN formalism brings it in direct relation with (tensorial) group field theories (and spin networks), and thus provides new tools for realizing the tensor network/geometry duality in the context of background independent quantum gravity, and for importing quantum gravity tools in tensor network research.

I. INTRODUCTION

Tensor networks [1] have developed into a powerful and ubiquitous formalism in quantum information and in the analysis of quantum many-body systems. It is first of all a very efficient way to capture the entanglement properties of such many-body systems, as well as quantum field theories (including lattice gauge theories), and it provides a general framework for describing (and identifying) quantum states characterized by area-laws, which indeed include the ground states of several interesting quantum many-body systems [2]. Somewhat surprisingly, it also provides a natural framework for investigating the nature of spacetime at the Planck scale. This comes about from two main directions. First, different theoretical frameworks, from background independent quantum gravity to string theory, hint at a scenario, where continuum spacetime geometry is replaced, at a more fundamental level, by pre-geometric quantum degrees of freedom, often of purely combinatorial and algebraic nature. In (Tensorial) Group Field Theories (GFT) [3] and random tensor models [4], as well as in Loop Quantum Gravity (LQG) and Spin Foam Models [5], pre-geometric quantum degrees of freedom are encoded in *random combinatorial network structures*, labelled by algebraic data. In particular, they are encoded in spin networks, graphs labeled by irreps of $SU(2)$ and endowed with a gauge symmetry at each node. These type of quantum states, in fact, are very close to tensor networks [6], and tensor network techniques have found already a number of quantum gravity applications [7]. A discrete spacetime and geometry is naturally associated, at a semi-classical level, to such structures and their quantum dynamics is directly related to (non-commutative) discrete gravity path integrals [8]. The outstanding issue is then to show the emergence of *continuum* spacetime and geometry from the full quantum dynamics of the same pre-geometric degrees of freedom, which in fact describe a quantum spacetime as a peculiar sort of quantum many-body system [9]. It is natural to expect that the entanglement between the fundamental entities constituting spacetime is crucial for its emergence and for understanding the quantum nature of geometry at the Planck scale, and thus that tensor networks techniques will provide relevant tools in this context. Second, a different, but probably related relation between entanglement and geometry has been unraveled n the context of holographic gauge/gravity duality, and also here tensor network states recently acquired a prominent role. One key example is the Ryu-Takayanagi (RT) formula, which relates the entanglement entropy of a boundary region to the area of the minimal surface in the bulk homologous to the same region [10]. The main intuition [11] consists in interpreting the geometry of the auxiliary tensor network representation of a quantum many-body state as a representation of the dual spatial geometry.

More recently, a lot of interest focussed on random tensor network (RTN) states, which are shown also to satisfy the Ryu-Takayanagi formula, as well as quantum error correction properties, in the large bond dimension limit [12]. The random character plays a central role in the study of the entanglement area laws in tensor networks. Indeed, random pure states are nearly maximally entangled states [13], hence can be used as a toy model of a thermal state [14, 15]. This in particular implies that the computation of typical entropies and other quantities of interest for these states can be mapped to the evaluation of partition functions of classical statistical models [12]. On the other hand, the interpretation of GFT fields as tensors provides an actual generalisation of the tensor network decomposition techniques, and any given GFT model would then provide tensor network states with a specific probability measure, defined through its partition function Z and momenta (correlations), and tools for its evaluation

For the above reasons, we think it is very important not only to develop further all the above directions, but also to strengthen the links between these different uses of tensor networks. The present work goes in this direction, by extending the formalism of RTN to incorporate one key feature of the random networks appearing in quantum gravity: the *gauge symmetry constraint* [16], and deriving the RT formula in this extension.

The paper is organized as follows. In the next section, we recall the definition of tensor network states, and of their random version. Then, we introduce the symmetric random tensor networks that we use in the rest of the

paper. Having done so, we compute the 2nd Rényi entropy for random tensor networks endowed with a local gauge symmetry constraint and derive the RT formula for them. We end up with some concluding remarks.

II. TENSOR NETWORK STATES AND HOLOGRAPHIC BEHAVIOUR

A tensor network is a collection of tensors, associated to nodes of a network, connected by contractions operations, associated to links of the same network.

Generically, a rank-v (or v-valent) tensor T is a multidimensional array of complex numbers with v indices λ, each taking values from a set of dimension ('size') $D_{|\lambda|} \in \mathbb{N}^+$ [16]. For simplicity, all indices are assumed to have the same size $D_{|\lambda|} = D$.

At the quantum level, to each leg of the tensor one associates a Hilbert space $\mathbb{H}_D$, with dimension D, so that a covariant tensor of rank v is a multilinear form on the Hilbert space of the vertex $T\colon \mathbb{H}_n \equiv \mathbb{H}_D^{\otimes v}$. Given an orthonormal basis $|\lambda_n\rangle$, $n = 1, \ldots, D$ in $\mathbb{H}_D$, we can denote its components by

$$\widehat{T}_{\lambda_1 \cdots \lambda_d} \equiv T(\lambda_1, \ldots, \lambda_v), \qquad (1)$$

hence generally define a *tensor state* as

$$|T\rangle = \sum_{\lambda_1, \ldots \lambda_d} \widehat{T}_{\lambda_1 \cdots \lambda_v} |\lambda_1\rangle \otimes \cdots \otimes |\lambda_v\rangle \qquad . \qquad (2)$$

Graphically, we can represent any such tensor state as a vertex with v open links emanating from it.

A state corresponding to a set of unconnected vertices is given by a tensor product of individual tensor states

$$|\mathcal{T}_{\mathcal{N}}\rangle \equiv \bigotimes_n |T_n\rangle \qquad . \qquad (3)$$

In a connected network, individual tensor states are *glued* by links, to each end of which we associate a Hilbert space $\mathbb{H}_D$. The Hilbert space of the link ℓ is then $\mathbb{H}_\ell = \mathbb{H}_D^{\otimes 2}$ while a link state can be written as

$$|M\rangle = M_{\lambda_1 \lambda_2} |\lambda_1\rangle \otimes |\lambda_2\rangle \qquad (4)$$

where the coefficients $M_{\lambda_1 \lambda_2}$ indicate generic quantum correlations between the links ends.[1] One could picture

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[1] One can observe it by defining a density matrix $\rho_M \equiv |M\rangle \langle M|$ and tracing out one of the Hilbert space, without losing generality, tracing out $\mathbb{H}_D$ of $|\lambda_2\rangle$, then computing the von Neumann entropy of the reduced density matrix $\rho_1 \equiv \mathrm{Tr}_2 \rho_M = M^\dagger M$. The entropy $S = \mathrm{Tr} \rho_1 \ln \rho_1$ is non-zero unless $M_{\lambda_1 \lambda_2}$ can split as $M_{\lambda_1 \lambda_2} = A_{\lambda_1} B_{\lambda_2}$. For simplicity, in the next sections we will often assume that the link state is maximally entangled, i.e.

$$|M\rangle = \frac{1}{\sqrt{D}} \delta_{\lambda_1 \lambda_2} |\lambda_1\rangle \otimes |\lambda_2\rangle. \qquad (5)$$

The latter corresponds to the simplest type of gluing, imposing gauge invariance, in the spin networks states used in quantum gravity, and forming indeed a special type of tensor networks [6, 16].

this gluing as the joining of two of the open links emanating from the original vertices (along their open ends), to form a link of the resulting network.

The entanglement of links encodes the information on the connectivity of the graph: two nodes are connected if their corresponding states contract with an entangled link state,

$$|\mathcal{T}_{12}\rangle \equiv \langle M| |T_1\rangle |T_2\rangle \qquad (6)$$
$$= T^{(1)}_{\lambda_1 \cdots \lambda_a \cdots \lambda_v} \overline{M}_{\lambda_a \lambda_b} T^{(2)}_{\lambda'_1 \cdots \lambda_b \cdots \lambda'_u} \bigotimes_{i \neq a}^{v} |\lambda_i\rangle \otimes \bigotimes_{j \neq b}^{u} |\lambda'_i\rangle$$

Accordingly, given a network $\mathcal{N}$ with N nodes and L links, the corresponding *tensor network state* is given by the contraction

$$|\Psi_{\mathcal{N}}\rangle \equiv \bigotimes_\ell^L \langle M_\ell| \bigotimes_n^N |T_n\rangle \qquad . \qquad (7)$$

As all but the open links of the network are contracted with nodes, $|\Psi_{\mathcal{N}}\rangle$ can be thought as an element of the Hilbert space $\mathbb{H}_{\partial \mathcal{N}}$ associated to the boundary (formed by the remaining open ends) of the network graph.

In lattice gauge theory, formulated in terms of tensor networks, the open links carry the physical degrees of freedom of the theory, while the fully contracted internal graph is interpreted as a virtual structure, whose correlation structure can be tuned to match the desired properties of the boundary lattice state $|\Psi_{\mathcal{N}}\rangle$ in $\mathbb{H}_{\partial \mathcal{N}}$. In quantum gravity formalisms based on spin networks (thus on tensor networks), like GFT and LQG, the internal graphs carry the degrees of freedom a spatial manifold, while the open links are associated to its boundary (corners of the spacetime manifold) and carry indeed additional degrees of freedom, related to the breaking of diffeomorphism symmetry on the same boundary [?].

A. Random Tensor Network States

Recently, a lot of interest has focussed on the study of networks of large-dimensional *random* tensors (RTN).

A convenient way to deal with RTNs is to consider the tensors $\widehat{T}_{\lambda_1 \cdots \lambda_v}$ on each node are unit complex vectors T_μ chosen independently at random in their respective Hilbert spaces $\mathbb{H}_T \simeq \mathbb{H}_D^{\otimes v}$ (indeed, a D^v dimensional vector space), with inner product $\overline{T}_\mu T'_\mu$. One can represent these vectors by choosing a gauge such that $T_\mu \equiv T_{\lambda_1 \cdots \lambda_v}$, with

$$\mu = \sum_{a=1}^{v} \lambda_a D^{v-a} = 0, 1, \ldots, D^4 - 1. \qquad (8)$$

Moreover, $|T\rangle \in \mathbb{H}_n$ being normalized, one has as well

$$\langle T|T\rangle = \overline{T}_{\lambda_1 \cdots \lambda_v} T_{\lambda_1 \cdots \lambda_v} = \overline{T}_\mu T_\mu = 1 \qquad (9)$$

Notice also that the Hilbert space $\mathbb{H}_T$ corresponds to the fundamental representation of the group $\mathrm{U}(D^v)$.

Given an arbitrary reference state T_μ^0, a group element $U \in \mathrm{U}(D^v)$ will transform T_μ^0 to a new vector $T_\mu^U \equiv (UT^0)_\mu$. A random tensor $T_{\lambda_1 \cdots \lambda_v}$ corresponds then to a random choice of the group element $U \in \mathrm{U}(D^v)$ defining T_μ^U,[2] where the group element U_n is independently chosen for each node of the network.

Idealised versions of RTNs, so-called pluperfect tensors, have been used to define bidirectional holographic codes, which simultaneously satisfies the Ryu-Takayanagi (RT) formula for a subset of boundary states, error correction properties of bulk local operators, a kind of bulk gauge invariance, and the possibility of sub-AdS locality.

More recently, in particular, building on the statistical properties of large dimensional random tensor network, the technique of *random state averaging* was used by [12] to compute Rényi entropies and other quantities of interest in the corresponding tensor network states, by means of a mapping to the evaluation of partition functions of classical statistical models, like generalized Ising models with boundary pinning fields.

In what follows, we shall reconsider the random state averaging technique for a specific class of large dimensional random tensor networks endowed with extra symmetry constraints.

III. RANDOM TENSOR NETWORK STATES WITH GAUGE SYMMETRY CONSTRAINTS

Now let us consider a tensor $T_{\lambda_1 \cdots \lambda_v}$ which satisfies the following symmetry

$$T_{\lambda_1 \cdots \lambda_v} = T_{[\lambda_1 + \ell]_D \cdots [\lambda_v + \ell]_D}, \quad \forall \ell \in \mathbb{Z} \qquad (10)$$

The square bracket $[\cdots]_D$ denotes the modular arithmetic: for all $k \in \mathbb{Z}$ and $D \in \mathbb{Z}^+$

$$[k]_D \equiv k \bmod D, \quad [k]_D \in \mathbb{Z}/D \qquad (11)$$

where $\mathbb{Z}/D$ is the the set of integers modulo D. Such a tensor state can be considered as a particular case of a GFT tensor field, with arguments/indices taking values on a finite group.

For all $k_1, k_2 \in \mathbb{Z}$ and $D \in \mathbb{Z}^+$, it satisfies:

$$\begin{aligned} [k_1]_D + [k_2]_D &= [k_1 + k_2]_D, \\ [[k_1]_D + [k_2]_D]_D &= [k_1 + k_2]_D, \quad [D]_D = 0 \end{aligned} \qquad (12)$$

In the presence of the symmetry (10), only D^{v-1} components are independent. We choose a new gauge

$$\mu = \lambda_1 D^{v-1} + \sum_{a=2}^v [\lambda_a - \lambda_1]_D D^{v-a} \qquad (13)$$

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[2] The random average of an arbitrary function $f(|Vx\rangle)$ of the state $|V_x\rangle$ is equivalent to an integration over U according to the Haar probability measure $\int \mathrm{d}U f(U |0_x\rangle)$, with normalization $\int \mathrm{d}U = 1$.

such that

$$T_\mu = \bigoplus_{i=0}^{D-1} T_{\mu_i} \equiv T_{\lambda_1 \cdots \lambda_v} \qquad (14)$$

and numerically

$$T_{\mu_i} = T_{\mu_j}, \quad \forall i, j = 0, 1, \ldots, D-1 \qquad (15)$$

In a different representation, for a given component T_μ one has

$$T_\mu = T_{[\mu + \ell D^{v-1}]_{D^v}}, \quad \forall \ell \in \mathbb{Z} \qquad (16)$$

From now on, for simplicity, we denote the tensor with the symmetry (10) with two indices: μ and $i = \lambda_1 \in \mathbb{Z}/D$ as $T_{\mu_i} = T_{\lambda_1 \cdots \lambda_v}$, with

$$\mu_i = \sum_{a=2}^v [\lambda_a - \lambda_1]_D D^{v-a} \in \mathbb{Z}/D^{v-1} \qquad (17)$$

For a given i, the vector T_{μ_i} is lying on a D^{v-1} dimensional space, which is a fundamental representation space of the group $\mathrm{U}(D^{v-1})$. Because $\overline{T}_\mu T_\mu = 1$ and (10), T_{μ_i} is also normalized

$$\overline{T}_{\mu_i} T_{\mu_i} = D^{-1}, \quad \forall i = 0, 1, \cdots, D-1 \qquad (18)$$

Then the $r$andom nature of the tensor $T_{\lambda_1 \cdots \lambda_v}$ implies that, with respect to the same T_0^μ, the group element $U_n \in \mathrm{U}(D^{v-1})$ is randomly chosen for each node.

IV. ENTANGLEMENT ENTROPY OF A RTN SUBREGION

For the specific class of RTN states defined in Section III, we shall then investigate the effect of the symmetry constraint on the holographic behaviour of the entanglement entropy. We follow a similar procedure as in [12].

Given our tensor network state,

$$|\Psi_\mathcal{N}\rangle \equiv \bigotimes_\ell^L \langle M_\ell| \bigotimes_n^N |T_n\rangle \in \mathbb{H}_{\partial \mathcal{N}}, \qquad (19)$$

with tensor states satisfying the relation given in (10), we consider a bipartition of the boundary Hilbert space

$$\mathbb{H}_{\partial \mathcal{N}} = \mathbb{H}_A \otimes \mathbb{H}_B \qquad (20)$$

associated to the definition of two – a priori not adjacent – boundary subregions A and B (see Figure 1).

A measure of the entanglement between the two subsystems is given by the von Neumann entropy of the reduced density matrix of the subsystem, either A or B, defined by partial tracing over the full system Hilbert space. Focussing on subsystem A, for $\rho \equiv |\Psi_\mathcal{N}\rangle \langle \Psi_\mathcal{N}|$, we have
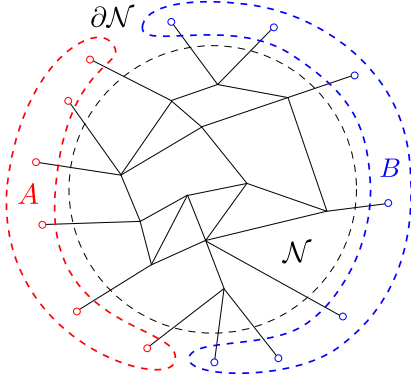
$$\rho_A = \mathrm{Tr}_B(\rho), \qquad (21)$$

Figure 1. Boundary $\partial\mathcal{N}$ of network $\mathcal{N}$ divided into two parts A and B.

and the entanglement entropy between A and B is given by the von Neumann entropy

$$S_{\text{EE}}(A) = -\text{Tr}[\,\widehat{\rho}_A \ln \widehat{\rho}_A\,], \qquad (22)$$

where now

$$\widehat{\rho}_A \equiv \frac{\rho_A}{\text{Tr}\rho} \qquad (23)$$

is the normalized reduced density matrix.

Because of the difficulties in computing directly the von Neumann entropy, one uses the so-called replica trick to approach the problem. Contracting N copies of the reduced density matrix ρ_A and taking the logarithm of the trace of ρ_A^N, one obtains the Nth-order Rényi entropy

$$S_N(A) = -\frac{1}{N-1} \ln \text{Tr}\widehat{\rho}_A^N \qquad . \qquad (24)$$

We can also define

$$Z_A^{(N)} \equiv \text{Tr}\rho_A^N = \text{Tr}[\rho^{\otimes N}\mathbb{P}(\pi_A^0; N, d)], \qquad (25)$$
$$Z_0^N \equiv (\text{Tr}\rho)^N,$$

where $\mathbb{P}(\pi_A^0; N, d)$ is the 1-cycle permutation operator acting on the states in region A,

$$\mathbb{P}(\pi_A^0; N, d) = \prod_{s=1}^{N} \delta_{\mu_A^{([s+1]_N)}\mu_A^{(s)}} \qquad (26)$$

and d is the dimension of the Hilbert space in the same region A.

The replica trick is useful because the Rényi entropy S_N, which is easier to compute, coincides with the von Neumann entropy of region A, and thus with the entanglement entropy between regions A and B, in the limit $N \to 1$

$$S_{\text{EE}}(A) = \lim_{N \to 1} S_N(A) \qquad (27)$$

A. Derivation of the Second Rényi Entropy

As a first step, let us write down the second order Rényi entropy, S_2, for a generic tensor network state $|\Psi_\mathcal{N}\rangle$ in

$\mathbb{H}_{\partial\mathcal{N}}$. We rewrite states in the following synthetic index notation,

$$|\Psi_\mathcal{N}\rangle \Longleftrightarrow \Psi_{\{\lambda_A\}\{\lambda_B\}} \equiv \Psi_{AB} \qquad (28)$$

$$\bigotimes_n |T_n\rangle \Longleftrightarrow \left(\bigotimes_n T_n\right)_{\{\lambda_A\}\{\lambda_B\}\{\lambda_C\}} \equiv \mathcal{T}_{ABC} \qquad (29)$$

$$\bigotimes_\ell \langle M_\ell| \Longleftrightarrow \left(\bigotimes_\ell \overline{M}_\ell\right)_{\{\lambda_C\}} \equiv \overline{\mathcal{M}}_C \qquad (30)$$

Based on the definition (7), the tensor network state is rewritten as

$$\Psi_{AB} = \overline{\mathcal{M}}_C \mathcal{T}_{ABC} \qquad (31)$$

where A and B label the two regions of the boundary $\partial\mathcal{N}$.

All internal links are contracted with nodes. The density matrix corresponding to Ψ_{AB} is

$$\rho_{\overline{A}A\overline{B}B} = \overline{\Psi}_{\overline{AB}}\Psi_{AB} \qquad (32)$$

Then S_2 is defined as

$$\text{e}^{-S_2} = \frac{Z_A^{(2)}}{Z_0^{(2)}} \qquad (33)$$

with $Z_0^{(2)} = \text{Tr}[\rho^{\otimes 2}]$ and $Z_A^{(2)} = \text{Tr}[\rho_A^{\otimes 2}]$.

In particular, for $N = 2$, the cyclic group $\mathcal{S}_n$ only has two elements: the identity $\mathbb{1}$ and swap operator $\mathbb{F}$, so that

$$\mathbb{P}(\pi_A^0; 2, d) \equiv \mathbb{F}^{(A)} \qquad (34)$$

Then

$$Z_A^{(2)} = \text{Tr}[\rho_A^{\otimes 2}] = \text{Tr}[\rho^{\otimes 2}\mathbb{F}^{(A)}] \qquad (35)$$
$$= \rho_{\overline{A}_1 A_1 \overline{B}_1 B_1}\rho_{\overline{A}_2 A_2 \overline{B}_2 B_2}\mathbb{F}^{(A)}_{\overline{A}_1 A_1 \overline{A}_2 A_2}\mathbb{1}^{(B)}_{\overline{B}_1 B_1 \overline{B}_2 B_2}$$

$$Z_0^{(2)} = \text{Tr}[\rho^{\otimes 2}] = \qquad (36)$$
$$= \rho_{\overline{A}_1 A_1 \overline{B}_1 B_1}\rho_{\overline{A}_2 A_2 \overline{B}_2 B_2}\mathbb{1}^{(A)}_{\overline{A}_1 A_1 \overline{A}_2 A_2}\mathbb{1}^{(B)}_{\overline{B}_1 B_1 \overline{B}_2 B_2}$$

Following [12], given the random nature of the tensor network, one can consider the average of the relevant quantities over all states in the single-vertex Hilbert space and look for the typical value of the entropy.

The entropy average can be expanded in powers of the fluctuations $\delta Z_A^{(2)} = Z_A^{(2)} - \mathbb{E}(Z_A^{(2)})$ and $\delta Z_0^{(2)} = Z_0^{(2)} - \mathbb{E}(Z_0^{(2)})$, so that [12]

$$\mathbb{E}(S_N(A)) = -\mathbb{E}\left(\log\frac{\mathbb{E}(Z_A^{(2)}) + \delta Z_A^{(2)}}{\mathbb{E}(Z_0^{(2)}) + \delta Z_0}\right) \qquad (37)$$

$$= -\log\frac{\mathbb{E}(Z_A^{(2)})}{\mathbb{E}(Z_0^{(2)})} + \text{fluctuations}$$

In large enough bond dimensions D, as a direct consequence of the *concentration of measure phenomenon* [17],

the statistical fluctuations around the average value are exponentially suppressed. Therefore, it is possible to approximate the entropy with high probability by the averages of $Z_A^{(2)}$ and $Z_0^{(2)}$,

$$\mathbb{E}(S_N(A)) \simeq -\log \frac{\mathbb{E}(Z_A^{(2)})}{\mathbb{E}(Z_0^{(2)})} \qquad (38)$$

In order to get the typical Rényi entropy one needs then to compute $\mathbb{E}(Z_A^{(2)})$ and $\mathbb{E}(Z_0^{(2)})$ separately.

B. Random State Averaging

Let us first consider the case without the gauge symmetry (10) for a given graph with only one node. The corresponding density matrix is

$$\rho_{\mu\overline{\mu}} \equiv T_\mu \overline{T}_{\overline{\mu}} \qquad (39)$$

Consider N copies of the density matrix $\rho^{\otimes N}$. If T is uniformly distributed, then the average $\rho^{\otimes N}$ over T is given by

$$\mathbb{E}_T(\rho^{\otimes N}) \equiv \int_{\mathrm{U}(D^v)} \mathrm{d}U \prod_{s=1}^N (UT^0)_{\mu^{(s)}} \overline{(UT^0)}_{\overline{\mu}^{(s)}} \qquad (40)$$

Because of the Schur's lemma, since $\mathbb{H}_{D^v}$ is the irrep of $\mathrm{U}(D^v)$, the result of the average is the identity matrix on the symmetric subspace of $\mathbb{H}_{D^v}^{\otimes n}$. When $\mathrm{Tr}\rho = 1$, the result is then

$$\mathbb{E}_T(\rho^{\otimes N}) = \frac{1}{D^v[N]} \sum_{\pi \in \mathcal{S}_N} \prod_{s=1}^N \delta_{\mu^{(s)}\overline{\mu}^{(\pi(s))}} \qquad (41)$$

$$\equiv \frac{1}{D^v[N]} \sum_{\pi \in \mathcal{S}_N} \mathbb{P}(\pi; N, D^v)$$

where

$$D^v[N] \equiv D^v(D^v + 1) \cdots (D^v + N - 1) \quad . \qquad (42)$$

$\mathcal{S}_N$ is the symmetric group on N objects and

$$\mathbb{P}(\pi; N, D^v) \equiv \prod_{s=1}^N \delta_{\mu^{(s)}\overline{\mu}^{(\pi(s))}} \qquad (43)$$

which is the representation matrix of $\pi \in \mathcal{S}_N$ on $\mathbb{H}_{D^v}^{\otimes N}$. Using the gauge (8) or (13), we can get the relation between representations on $\mathbb{H}_{D^v}$ and $\mathbb{H}_D^{\otimes v}$

$$\delta_{\mu^{(s)}\overline{\mu}^{(\pi(s))}} \equiv \prod_{a=1}^v \delta_{\lambda_a^{(s)}\overline{\lambda}_a^{(\pi(s))}} \qquad (44)$$

Then we have

$$\mathbb{P}(\pi; N, D^v) = \prod_{a=1}^v \mathbb{P}(\pi_a; N, D) \qquad (45)$$

If T is an random Gaussian vector, then the average is

$$\mathbb{E}_T(\rho^{\otimes N}) \equiv \int \mathcal{D}T \; \mathrm{e}^{-\beta|T|^2} \prod_{s=1}^N T_{\mu^{(s)}} \overline{T}_{\overline{\mu}^{(s)}} \qquad (46)$$

If we ask $\mathbb{E}_T(\rho) = \mathbb{1}/D^v$ and $T = x\hat{T}$, where $|\hat{T}| = 1$, then $\beta = D^v$ and the average $\mathbb{E}_T(\rho^{\otimes N})$ becomes

$$\int \mathrm{d}x \; |x|^{2N} \mathrm{e}^{-D^v|x|^2} \int \mathrm{d}U \; \prod_{s=1}^N (U\hat{T}^0)_{\mu^{(s)}} \overline{(U\hat{T}^0)}_{\overline{\mu}^{(s)}}$$

$$= (D^v)^{-N} \sum_{\pi \in \mathcal{S}_N} \prod_{s=1}^N \delta_{\mu^{(s)}\overline{\mu}^{(\pi(s))}} \qquad (47)$$

Now let us consider the case with symmetry (10). The corresponding density matrix is

$$\rho_{\mu\mu'} \equiv T_\mu \overline{T}_{\overline{\mu}} = T_{\mu_i} \overline{T}_{\overline{\mu}_{\overline{i}}} \equiv \widetilde{\rho}_{\mu_i \overline{\mu}_{\overline{i}}} \qquad (48)$$

The expression for the N copies of the density matrix reads

$$\rho^{\otimes N} = \prod_{s=1}^N \widetilde{\rho}_{\mu_{i(s)}\overline{\mu}_{\overline{i}(s)}} = \prod_{s=1}^N T_{\mu_{i(s)}} \overline{T}_{\overline{\mu}_{\overline{i}(s)}} \qquad (49)$$

If T is uniformly distributed, then the average of $\rho^{\otimes N}$ over T is

$$\mathbb{E}_T(\rho^{\otimes N}) = \int_{\mathrm{U}(D^{v-1})} \mathrm{d}U \prod_{s=1}^N (UT^0)_{\mu_{i(s)}} \overline{(UT^0)}_{\overline{\mu}_{\overline{i}(s)}} \qquad (50)$$

As shown in the first case, the result of the integral is the identity matrix in the symmetric subspace of $\mathbb{H}_{D^{v-1}}^{\otimes N}$.

$$\mathbb{E}_T(\rho^{\otimes N}) \propto \sum_{\pi \in \mathcal{S}_N} \prod_{s=1}^N \delta_{\mu_{i(s)}\overline{\mu}_{\overline{i}(\pi(s))}} \qquad (51)$$

$$\equiv \sum_{\pi \in \mathcal{S}_N} \mathbb{P}_{\{i(s)\}\{\overline{i}(s)\}}(\pi; N, D^{v-1})$$

where $\mathbb{P}_{\{i(s)\}\{\overline{i}(s)\}}(\pi; N, D^{v-1})$ is the representation matrix of $\pi \in \mathcal{S}_N$ on $\mathbb{H}_{D^{v-1}}^{\otimes N}$ with a set of $\{i(s)\}\{\overline{i}(s)\}$. Similarly, when T_μ is a Gaussian vector, the result of the average is the same as the above equation up to a normalization. The details of the matrix in (51) are given in Appendix A.

By using the gauge (13), one can show the relation between the representations μ_i and λ_a. Because of (13), $\delta_{\mu_{i(s)}\overline{\mu}_{\overline{i}(s')}}$ is not zero only when

$$[\lambda_a(s) - i(s)]_D = [\overline{\lambda}_a(s') - \overline{i}(s')]_D \qquad (52)$$

because of the modular rules (12), the above equation can be rewritten as

$$[\lambda_a(s) - \overline{\lambda}_a(s')]_D = [i(s) - \overline{i}(s')]_D \qquad (53)$$

If $[i(s) - \bar{i}(s')]_D = \ell \in \mathbb{Z}/D$, then

$$\delta_{\mu_{i(s)}\overline{\mu}_{\bar{i}(s')}} = \prod_{a=1}^{v} \delta_{[\lambda_a(s)-\bar{\lambda}_a(s')]_D,\ell} = \prod_{a=1}^{v} \delta_{[\lambda_a(s)-\ell]_D,\bar{\lambda}_a(s')}$$

Notice that ℓ is a uniform shift for all legs of each node as long as the step $[i(s) - \bar{i}(s')]_D = \ell$ is fixed. Finally, we define for later use the trace on a tensor $\mathbb{T}_{\{\mu^{(s)}\}\{\overline{\mu}^{(s)}\}}$, as

$$\text{Tr}\mathbb{T} = \mathbb{T}_{\{\mu^{(s)}\}\{\overline{\mu}^{(s)}\}} \prod_{s=1}^{N} \delta_{\mu^{(s)}\overline{\mu}^{(s)}} \quad , \tag{54}$$

which becomes, with the symmetry (10),

$$\text{Tr}\mathbb{T} = \sum_{\substack{\{i(s)\} \\ \{\bar{i}(s)\}}} \mathbb{T}_{\{\mu_{i(s)}\}\{\overline{\mu}_{\bar{i}(s)}\}} \prod_{s=1}^{N} \delta_{\mu_{i(s)}\overline{\mu}_{\bar{i}(s)}} \delta_{i(s)\bar{i}(s)} \quad . \tag{55}$$

C. S_2 Rényi with symmetry constraint

Coming back to the case of $N = 2$, we can now explicitly write down the expression for the average of the single symmetric tensor defined in (10),

$$\mathbb{E}_T(\rho_{T_n}^{\otimes 2}) = \frac{1}{D^2 D^{v-1}[2]} \sum_{\substack{m(1) \\ m(2)}} \left(\mathbb{1}_{m(1)m(2)} + \mathbb{F}_{m(1)m(2)} \right) \tag{56}$$

Given (42), $D^{v-1}[2] = D^{v-1}(D^{v-1} + 1)$, we can define the normalization as $\mathbb{D}_2 = D^2 D^{v-1}[2]$. Therefore, for the density matrix of a tensor network with N nodes we have

$$\mathbb{E}_T(\rho^{\otimes 2}) = \text{Tr}_C \left[\bigotimes_n \mathbb{E}_T(\rho_{T_n}^{\otimes 2}) \, \rho_M^{\otimes 2} \right] = \frac{1}{\mathbb{D}_2^N} \text{Tr}_C \left[\bigotimes_n^N \sum_{m_n(1)m_n(2)} \left(\mathbb{1}_{m_n(1)m_n(2)} + \mathbb{F}_{m_n(1)m_n(2)} \right) \, \rho_M^{\otimes 2} \right]$$

$$= \frac{1}{\mathbb{D}_2^N} \sum_{\{m_n(1)\}\{m_n(2)\}} \text{Tr}_C \left[\bigotimes_n \mathbb{1}_{m_n(1)m_n(2)} \bigotimes_{n'} \mathbb{F}_{m_{n'}(1)m_{n'}(2)} \, \rho_M^{\otimes 2} \right] \tag{57}$$

In the case of a network of N-nodes, the above sum is naively given by a sum of $(2D^2)^N$ terms, $2D^2$ choices for each node, but with several terms vanishing.

In order to calculate (57) explicitly, it is convenient to separately consider the case of internal and boundary links. For an internal links connecting two nodes, we have the following three cases:

- $\mathbb{1}_{m(1)m(2)}$ and $\mathbb{1}_{m'(1)m'(2)}$

$$\text{Tr}\left[\mathbb{1}_{m(1)m(2)} \, \rho_{M_\ell}^{\otimes 2} \, \mathbb{1}_{m'(1)m'(2)} \right]$$
$$= D^2 \, \delta_{m(1)m'(1)}\delta_{m(2)m'(2)} \tag{58}$$

- $\mathbb{F}_{m(1)m(2)}$ and $\mathbb{F}_{m'(1)m'(2)}$

$$\text{Tr}\left[\mathbb{F}_{m(1)m(2)} \, \rho_{M_\ell}^{\otimes 2} \, \mathbb{F}_{m'(1)m'(2)} \right]$$
$$= D^2 \, \delta_{m(1)m'(1)}\delta_{m(2)m'(2)} \tag{59}$$

- $\mathbb{1}_{m(1)m(2)}$ and $\mathbb{F}_{m'(1)m'(2)}$

$$\text{Tr}\left[\mathbb{1}_{m(1)m(2)} \, \rho_{M_\ell}^{\otimes 2} \, \mathbb{F}_{m'(1)m'(2)} \right]$$
$$= D \, \delta_{[m(1)+m(2)]_D,[m'(1)+m'(2)]_D} \tag{60}$$

On the boundary of region A, for $\mathbb{F}^{(A)} = \mathbb{F}_{00}$ at one end of the boundary link, we have instead only two cases:

- $\mathbb{F}^{(A)}$ and $\mathbb{F}_{m'(1)m'(2)}$

$$\text{Tr}\left[\mathbb{F}^{(A)} \, \rho_{M_\ell}^{\otimes 2} \, \mathbb{F}_{m'(1)m'(2)} \right] = \text{Tr}\left[\mathbb{F}_{00} \, \rho_{M_\ell}^{\otimes 2} \, \mathbb{F}_{m'(1)m'(2)} \right]$$
$$= D^2 \, \delta_{0m'(1)}\delta_{0m'(2)} \tag{61}$$

- $\mathbb{F}^{(A)}$ and $\mathbb{1}_{m'(1)m'(2)}$

$$\text{Tr}\left[\mathbb{F}^{(A)} \, \rho_{M_\ell}^{\otimes 2} \, \mathbb{1}_{m'(1)m'(2)} \right] = \text{Tr}\left[\mathbb{F}_{00} \, \rho_{M_\ell}^{\otimes 2} \, \mathbb{1}_{m'(1)m'(2)} \right]$$
$$= D \, \delta_{0,[m'(1)+m'(2)]_D} \tag{62}$$

Finally, on the boundary of region B, where $\mathbb{1}^{(B)} = \mathbb{1}_{00}$ at one end of the boundary link, there exist two cases,

- $\mathbb{1}^{(B)}$ and $\mathbb{F}_{m'(1)m'(2)}$

$$\text{Tr}\left[\mathbb{1}^{(B)} \, \rho_{M_\ell}^{\otimes 2} \, \mathbb{F}_{m'(1)m'(2)} \right] = \text{Tr}\left[\mathbb{1}_{00} \, \rho_{M_\ell}^{\otimes 2} \, \mathbb{F}_{m'(1)m'(2)} \right]$$
$$= D \, \delta_{0,[m'(1)+m'(2)]_D} \tag{63}$$

- $\mathbb{1}^{(B)}$ and $\mathbb{1}_{m'(1)m'(2)}$

$$\text{Tr}\left[\mathbb{1}^{(B)} \, \rho_{M_\ell}^{\otimes 2} \, \mathbb{1}_{m'(1)m'(2)} \right] = \text{Tr}\left[\mathbb{1}_{00} \, \rho_{M_\ell}^{\otimes 2} \, \mathbb{1}_{m'(1)m'(2)} \right]$$
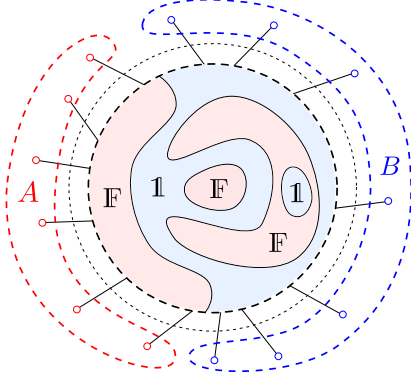$$= D^2 \, \delta_{0m'(1)}\delta_{0m'(2)} \tag{64}$$

Figure 2. Network with boundary condition: $A = \mathbb{F}^{(A)}$ and $B = \mathbb{1}^{(B)}$ is divided into regions. The nodes in red regions are assigned with $\mathbb{F}$ and the ones in blue are with $\mathbb{1}$.

D. Remarks on the calculation

Instead of giving all details about the calculation of the terms identified above, it is enough to sketch the main features of the same.

We see from the results of the previous section that the averaging $Z_A^{(2)}$ and $Z_0^{(2)}$ over T is equivalent to defining a class of networks where each node is assigned a matrix $\mathbb{1}_{m(1)m(2)}$ or a matrix $\mathbb{F}_{m(1)m(2)}$, and the boundary is assigned $\mathbb{F}^{(A)}$, and $\mathbb{1}^{(B)}$ for $Z_A^{(2)}$, and $\mathbb{1}^{(A)}$ and $\mathbb{1}^{(B)}$ for $Z_0^{(2)}$.

For all $\mathbb{1}_{m(1)m(2)}$ and $\mathbb{F}_{m(1)m(2)}$, the ones with $[m(1) + m(2)]_D \neq 0$ will never contribute to $Z_A^{(2)}$ and $Z_0^{(2)}$. In fact, if there is one node has $[m(1) + m(2)]_D \neq 0$, it will make its neighbouring node satisfying $[m(1) + m(2)]_D \neq 0$, and these nodes will make their neighbouring nodes satisfying $[m(1) + m(2)]_D \neq 0$. Because all nodes connect to the boundary through a certain number of links, the consequence is that the boundary should be $[m(1) + m(2)]_D \neq 0$, but we have assumed that the boundary is assigned by $\mathbb{1}_{00}$ or $\mathbb{F}_{00}$, i.e. $[m(1) + m(2)]_D = 0$. Therefore, none of the matrices at each node can satisfy $[m(1) + m(2)]_D \neq 0$. So in the following discussion we only consider the matrices of $\mathbb{F}_{m(1)m(2)}$ and $\mathbb{1}_{m(1)m(2)}$ with $[m(1) + m(2)]_D = 0$

If a node is $\mathbb{F}_{m(1)m(2)}$, then its neighbouring nodes can only be $\mathbb{F}_{m'(1)m'(2)}$ or $\mathbb{1}_{m'(1)m'(2)}$ with $[m'(1) + m'(2)]_D = 0$. So the network ends up being divided into several regions, where all nodes are associated with the same matrix. If a region is associated with $\mathbb{F}_{m(1)m(2)}$, its neighbouring region can only be with $\mathbb{1}_{m'(1)m'(2)}$, and vice versa. An example is shown in Figure (IV C). Each regions a labeled with $\mathbb{F}$ or $\mathbb{1}$. The boundaries of these regions are called domain walls []. The domain walls are also in correspondence with links, such that one end of each is assigned with $\mathbb{F}$ and the other end with $\mathbb{1}$.

As shown in figure IV C, such domain walls form different *patterns* $\mathcal{P}$ for the network. For a given pattern, changing a region's label from $\mathbb{F}_{m(1)m(2)}$ to $\mathbb{F}_{m'(1)m'(2)}$

will not change the value of its corresponding term in $Z_A^{(2)}$ or $Z_0^{(2)}$. This implies a degeneracy. The degeneracy of the region that does not connect to the boundary is D, which is the number of the possible choice of the pair $(m(1), m(2))$. Therefore we have

$$\mathbb{E}_T(Z_A^{(2)}) = \frac{1}{\mathbb{D}_2^N} \sum_{\mathcal{P}_A} d_{\mathcal{P}_A} Z_{\mathcal{P}_A}^{(2)}, \qquad (65)$$

$$\mathbb{E}_T(Z_0^{(2)}) = \frac{1}{\mathbb{D}_2^N} \sum_{\mathcal{P}_0} d_{\mathcal{P}_0} Z_{\mathcal{P}_0}^{(2)} \qquad (66)$$

where $d_{\mathcal{P}}$ is the degeneracy of the pattern, which is the product of the degeneracies of each region in this pattern. $Z_{\mathcal{P}}^{(2)}$ is given as

$$Z_{\mathcal{P}}^{(2)} = D^{2(L-L_{\mathcal{P}})} D^{L_{\mathcal{P}}} = D^{2L - L_{\mathcal{P}}} \qquad (67)$$

where L is the total link number in a given network $\mathcal{N}$, including links across $\partial \mathcal{N}$; $L_{\mathcal{P}}$ is the links across the domain walls in $\mathcal{P}$.

E. Holographic behaviour and Ryu-Takanayagi formula

The main contribution of $\mathbb{E}_T(Z_A^{(2)})$ is the pattern with the least number of links through the domain walls. We call this domain wall with the least number of links the *minimal surface*. One can show that this is true even after the degeneracy $d_{\mathcal{P}}$ is taken into account. In fact, all patterns can be generated from the one only with minimal surface by deforming the minimal surface or adding new regions. Starting from the pattern corresponding to a minimal surface, any deformation will lead to a surface which is not minimal. When adding a region, on the other hand, this will contribute to $d_{\mathcal{P}}$ with D but to the number of domain wall links at least $v > 1$, i.e. the valence of a single node, thus in total one has to consider the product $D^{1-v} < 1$ to the original $Z_{\mathcal{P}}^{(2)}$. This gives a contribution that is smaller than the original one. So the main contribution to $\mathbb{E}_T(Z_A^{(2)})$ comes from the pattern with only the minimal surface. And in this pattern there are only two regions, which are labeled by $\mathbb{F}^{(A)}$ and $\mathbb{1}^{(B)}$, respectively:

$$\mathbb{E}_T(Z_A^{(2)}) = \frac{1}{\mathbb{D}_2^N} D^{2L - L_{\min}} \left(1 + O(D^{-1})\right) \qquad (68)$$

In this sense, in agreement with the results in [?], we find that the main contribution of $\mathbb{E}_T(Z_0^{(2)})$ is given by the pattern without any domain wall. This is because its boundary condition is $\mathbb{1}^{(A)} = \mathbb{1}^{(B)} = \mathbb{1}_{00}$. Such pattern exists and all its nodes are assigned $\mathbb{1}_{00}$. Then

$$\mathbb{E}_T(Z_0^{(2)}) = \frac{1}{\mathbb{D}_2^N} D^{2L} \left(1 + O(D^{-1})\right) \qquad (69)$$
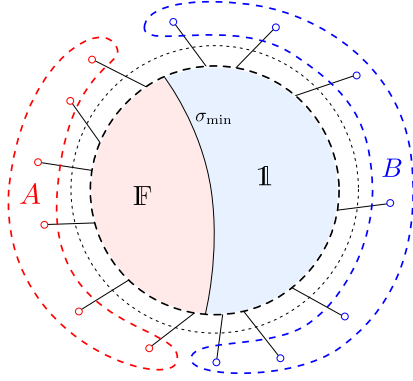
Figure 3. The pattern with only the minimal surface $\sigma_{\min}$. Boundary condition: $A = \mathbb{F}^{(A)}$ and $B = \mathbb{1}^{(B)}$.
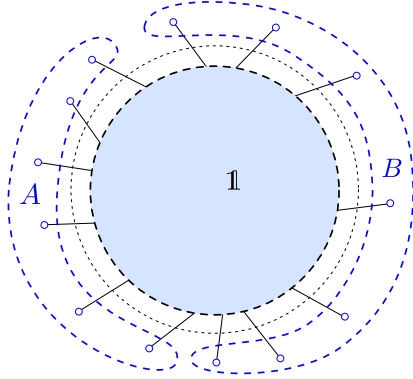

Figure 4. The pattern without the minimal surface. Boundary condition: $A = \mathbb{1}^{(A)}$ and $B = \mathbb{1}^{(B)}$.

Finally, collecting the above results, we find that the leading contribution to S_2 is given as

$$\mathbb{E}_T(\mathrm{e}^{-S_2}) \sim \frac{\mathbb{E}_T(Z_A^{(2)})}{\mathbb{E}_T(Z_0^{(2)})} = D^{-L_{\min}}\left(1 + O(D^{-1})\right) \quad (70)$$

which gives, when $D \gg 1$,

$$S_2 = L_{\min}\ln D + O(D^{-1}) \quad . \quad (71)$$

This is the Ryu-Takayanaki formula for S_2.

The Nth Rényi entropy S_N can be calculated in a similar way, and with analogous result.

V. CONCLUSIONS

We have computed the Renyi entropy and derived the Ryu-Takanyagi entropy formula, for random tensor networks with an additional gauge invariance property. This is an interesting extension of existing derivations. On the one hand it shows the generality of the holographic behaviour of (random) tensor network states, thus their role in the entenglement/geometry correspondence, and confirming their interest also for applications in a quantum gravity context. On the other hand, the type of

gauge symmetry we imposed is suggested by the correspondence with GFT and LQG states, and it is indeed required for the exact matching with the spin network states used in these quantum gravity approaches. Thus our results will facilitate the application of techniques from quantum gravity to quantum many-body systems (beyond the AdS/CFT framework), and the exploration of the same entanglement/geometry correspondence in new quantum gravity contexts.

Acknowledgements

Appendix A: Structure of the matrix of $\mathbb{E}_T(\rho^{\otimes n})$

In this appendix we analyse the structure of the matrix in (51)

$$\mathbb{M} \equiv \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^{n} \delta_{\mu_{i(s)}\overline{\mu}_{\bar{i}(\pi(s))}} = \sum_{\pi \in \mathcal{S}_n} \mathbb{P}_{\{i(s)\}\{\bar{i}(s)\}}(\pi; n, D^{v-1}) \quad (A1)$$

The sum over $\mathbb{P}(\pi; n, D^{v-1})$ is proportional to the projector operator $\mathbb{P}_{\text{sym}}^{n,D^{v-1}}$ which projects vectors in $\mathbb{H}_{D^{v-1}}^{\otimes n}$ into its symmetric subspace.

$$\mathbb{P}_{\text{sym}}^{n,D^{v-1}} = \frac{1}{n!}\sum_{\pi \in \mathcal{S}_n} \mathbb{P}(\pi; n, D^{v-1}) \quad (A2)$$

Given a set of $\{i(s)\} = \{i(1), i(2), \cdots, i(n)\}$ and $\{\bar{i}(s)\} = \{\bar{i}(1), \bar{i}(2), \cdots, \bar{i}(n)\}$ where $i(s)$ and $\bar{i}(s)$ are from 0 to $D-1$, there is a projector $\mathbb{P}_{\text{sym}}^{n,D^{v-1}}$, which is a $D^{(v-1)n} \times D^{(v-1)n}$ matrix. Write (51) as a matrix, with $\{i(s)\}$ labeling its rows and $\{\bar{i}(s)\}$ labels its columns:

$$\mathbb{M} = n!\begin{pmatrix} \mathbb{P}_{\text{sym}}^{n,D^{v-1}} & \cdots & \mathbb{P}_{\text{sym}}^{n,D^{v-1}} \\ \vdots & \ddots & \vdots \\ \mathbb{P}_{\text{sym}}^{n,D^{v-1}} & \cdots & \mathbb{P}_{\text{sym}}^{n,D^{v-1}} \end{pmatrix} \quad (A3)$$

This matrix is a $D^n \times D^n$ block matrix.

The trace of $\mathbb{M}$ is

$$\text{Tr}\mathbb{M} = \sum_{\substack{\{i(s)\} \\ \{\bar{i}(s)\}}} \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^{n} \delta_{\mu_{i(s)}\overline{\mu}_{\bar{i}(\pi(s))}}\delta_{\mu_{i(s)}\overline{\mu}_{\bar{i}(s)}}\delta_{i(s)\bar{i}(s)}$$

$$= \sum_{\{i(s)\}} \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^{n} \delta_{\mu_{i(s)}\overline{\mu}_{i(\pi(s))}}\delta_{\mu_{i(s)}\overline{\mu}_{i(s)}}$$

$$= \sum_{\{i(s)\}} \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^{n} \delta_{\mu_{i(s)}\mu_{i(\pi(s))}}$$

$$= \sum_{\{i(s)\}} n!\, \text{Tr}\mathbb{P}_{\text{sym}}^{n,D^{v-1}}$$

$$= D^n D^{v-1}[n] \quad (A4)$$

where we use the trace of the projector $\mathbb{P}^n_{\text{sym}}$

$$\text{Tr}\mathbb{P}^{n,D^{v-1}}_{\text{sym}} = \binom{D^{v-1}+n-1}{n} \tag{A5}$$

The matrix $\mathbb{M}$ can be written as a sum of matrices

$$\mathbb{M} = \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^{n} \delta_{\mu_{i(s)}\overline{\mu}_{\overline{i}(\pi(s))}} \tag{A6}$$

$$= \sum_{\{m(s)\}} \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^{n} \delta_{\mu_{[i(s)-m(s)]_D}\overline{\mu}_{\overline{i}(\pi(s))}} \delta_{i(s)\overline{i}(\pi(s))}$$

$$= \sum_{\{m(s)\}} \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^{n} \delta_{[\mu^{(s)}-m(s)D^{v-1}]_{D^v},\overline{\mu}^{(\pi(s))}} \tag{A7}$$

$$= \sum_{\{m(s)\}} \sum_{\pi \in \mathcal{S}_n} \prod_{s=1}^{n} \prod_{a=1}^{v} \delta_{[\lambda_a(s)-m(s)]_D,\overline{\lambda}_a(\pi(s))} \tag{A8}$$

Let us define a class of new matrices $\mathbb{P}(\pi)_{\{m(s)\}}$ as

$$\mathbb{P}(\pi)_{\{m(s)\}} \equiv \prod_{s=1}^{n}\prod_{a=1}^{v} \delta_{[\lambda_a(s)-m(s)]_D,\overline{\lambda}_a(\pi(s))} \tag{A9}$$

$$= \prod_{s=1}^{n} \delta_{[\mu^{(s)}-m(s)D^{v-1}]_{D^v},\overline{\mu}^{(\pi(s))}}$$

Then matrix $\mathbb{M}$ becomes

$$\mathbb{M} = \sum_{\{m(s)\}} \sum_{\pi \in \mathcal{S}_n} \mathbb{P}(\pi)_{\{m(s)\}} \qquad . \tag{A10}$$

When $m(s) = 0$ for all s, $\mathbb{P}(\pi)_{\{0\}} = \mathbb{P}(\pi; n, D^v)$, the representation matrix of $\pi \in \mathcal{S}_n$ on $\mathbb{H}^{\otimes n}_{D^v}$.

[1] R. Orus, Annals Phys. **349**, 117 (2014), arXiv:1306.2164 [cond-mat.str-el]
J. C. Bridgeman and C. T. Chubb, (2016), arXiv:1603.03039 [quant-ph].
[2] J. I. Cirac and F. Verstraete, J. Phys. **A42**, 504004 (2009)
F. Verstraete, V. Murg, and J. I. Cirac, Advances in Physics **57**, 143 (2008)
F. Pastawski, B. Yoshida, D. Harlow, and J. Preskill, JHEP **06**, 149 (2015), arXiv:1503.06237 [hep-th]
G. Vidal, Physical review letters **101**, 110501 (2008)
S. Singh, R. N. C. Pfeifer, and G. Vidal, Phys. Rev. **A82**, 050301 (2010), arXiv:0907.2994 [cond-mat.str-el]
G. Evenbly and G. Vidal, Journal of Statistical Physics **145**, 891 (2011).
[3] D. Oriti, in Proceedings, Foundations of Space and Time: Reflections on Quantum Gravity: Cape Town, South Africa (2011) pp. 257–320, arXiv:1110.5606 [hep-th]
in to appear in "Loop Quantum Gravity - 100 Years of General Relativity Series", edited by A. Ashtekar and J. Pullin (2014) arXiv:1408.7112 [gr-qc]
A. Baratin and D. Oriti, Proceedings, International Conference on Non-perturbative / background independent quantum gravity (Loops 11): Madrid, Spain, May 23-28, 2011, J. Phys. Conf. Ser. **360**, 012002 (2012), arXiv:1112.3270 [gr-qc].
[4] R. Gurau and J. P. Ryan, SIGMA **8**, 020 (2012), arXiv:1109.4812 [hep-th]
R. Gurau, SIGMA **12**, 094 (2016), arXiv:1609.06439 [hep-th]
V. Rivasseau, SIGMA **12**, 069 (2016), arXiv:1603.07278 [math-ph].
[5] A. Ashtekar and J. Lewandowski, Class. Quant. Grav. **21**, R53 (2004), arXiv:gr-qc/0404018
C. Rovelli, Quantum Gravity (Cambridge University Press, London, 2004)
T. Thiemann, Modern canonical quantum general relativity, Cambridge Monographs on Mathematical Physics (Cambridge University Press, London, 2007)
A. Perez, Living Rev.Rel. **16**, 3 (2012), arXiv:1205.2019 [gr-qc]
.
[6] G. Chirco, D. Oriti, and M. Zhang arXiv:1701.01383 [gr-qc]].
[7] B. Dittrich, S. Mizera, and S. Steinhaus, New J. Phys. **18**, 053009 (2016), arXiv:1409.2407 [gr-qc]C. Delcamp and B. Dittrich, (2016), arXiv:1612.04506 [gr-qc]
B. Dittrich, F. C. Eckert, and M. Martin-Benito, New J. Phys. **14**, 035008 (2012), arXiv:1109.4927 [gr-qc]
B. Dittrich, E. Schnetter, C. J. Seth, and S. Steinhaus, Phys. Rev. **D94**, 124050 (2016), arXiv:1609.02429 [gr-qc].
[8] A. Baratin and D. Oriti, Phys. Rev. Lett. **105**, 221302 (2010), arXiv:1002.4723 [hep-th]
Phys. Rev. **D85**, 044003 (2012), arXiv:1111.5842 [hep-th]
M. Han, and T. Krajewski , Class.Quant.Grav. **31**, 015009 (2014), arXiv:1304.5626 [gr-qc]
M. Han, and M. Zhang , Class.Quant.Grav. **30**, 165012 (2013), arXiv:1109.0499 [gr-qc]
M. Han and L.-Y. Hung, (2016), arXiv:1610.02134 [hep-th]
(2016), arXiv:1705.01964 [hep-th].
[9] D. Oriti, Stud. Hist. Phil. Sci. **B46**, 186 (2014), arXiv:1302.2849 [physics.hist-ph]
C. Cao, S. M. Carroll, and S. Michalakis, (2016), arXiv:1606.08444 [hep-th]
N. Seiberg, in The Quantum Structure of Space and Time: Proceedings of the 23rd Solvay Conference on Physics. Brussels, Belgium. 1 - 3 December 2005 (2006) pp. 163–178, arXiv:hep-th/0601234 [hep-th]
D. Oriti, PoS **QG-PH**, 030 (2007), arXiv:0710.3276 [gr-qc]
V. Rivasseau, Proceedings, 8th International Conference on Progress in theoretical physics (ICPTP 2011): Constantine, Algeria, October 23-25, 2011, AIP Conf. Proc. **1444**, 18 (2011), arXiv:1112.5104 [hep-th]
D. Oriti, arXiv:1710.02807 [gr-qc] .
[10] S. Ryu and T. Takayanagi, Phys.Rev.Lett. **96**, 181602 (2006), arXiv:hep-th/0603001 [hep-th].
[11] B. Swingle, Phys. Rev. **D86**, 065007 (2012), arXiv:0905.1317 [cond-mat.str-el].
[12] P. Hayden, S. Nezami, X.-L. Qi, N. Thomas, M. Walter, and Z. Yang, JHEP **11**, 009 (2016), arXiv:1601.01694

[hep-th]

.

[13] D. N. Page, Phys. Rev. Lett. **71**, 1291 (1993), arXiv:gr-qc/9305007 [gr-qc].

[14] J. Deutsch, Physical Review A **43**, 2046 (1991).

[15] M. Srednicki, Physical Review E **50**, 888 (1994).

[16] S. Singh, R. N. Pfeifer, and G. Vidal, Physical Review B **83**, 115125 (2011)Y. Li, M. Han, D. Ruan, B. Zeng, arXiv:1709.08370 [quant-ph].

[17] P. Hayden, D. Leung, and A. Winter, Commun. Math. Phys. (2006) 265: 95. https://doi.org10.1007s00220-006-1535-6