The double-copy structure of one-loop open-string amplitudes

Carlos R. Mafra[1,*] and Oliver Schlotterer[2,3,†]

[1] *STAG Research Centre and Mathematical Sciences, University of Southampton, Highfield, Southampton SO17 1BJ, UK*
[2] *Max–Planck–Institut für Gravitationsphysik, Albert–Einstein–Institut, 14476 Potsdam, Germany*
[3] *Perimeter Institute for Theoretical Physics, Waterloo, ON N2L 2Y5, Canada*

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In this letter, we provide evidence for a new double-copy structure in one-loop amplitudes of the open superstring. Their integrands w.r.t. the moduli space of genus-one surfaces are cast into a form where gauge invariant kinematic factors and elliptic functions of the punctures enter on completely symmetric footing. In particular, replacing the elliptic functions by a second copy of kinematic factors maps one-loop open-string correlators to gravitational matrix elements of the higher-curvature operator R^4.

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INTRODUCTION

Recent investigations of scattering amplitudes revealed a variety of hidden relations between field theories of seemingly unrelated particle content. The oldest and possibly most prominent example of such connections is the double-copy structure of gravity [1–3] whose scattering amplitudes can be reduced to squares of gauge-theory building blocks. This kind of double copy is geometrically intuitive from the realization of gravitons and gauge bosons as vibration modes of closed and open strings, respectively. Its first explicit realization at the level of scattering amplitudes in string theory was pinpointed by Kawai, Lewellen and Tye (KLT) in 1985 [1].

The first loop-level generalization of the gravitational double copy was found by Bern, Carrasco and Johansson (BCJ) [3]: Gauge-theory ingredients in a suitable gauge can be conjecturally squared to gravitational loop integrands at the level of cubic diagrams. The gauge dependence of the BCJ construction has been recently bypassed through a generalized double copy [4] – see [5] for an impressive five-loop application – and a one-loop KLT formula in field theory [6].

It has been recently discovered that tree-level amplitudes of the *open* superstring admit a double-copy representation [7] which mimics the field-theory version of the KLT formula [8]: Gauge-theory trees are double-copied with moduli-space integrals whose expansion in the inverse string tension α' suggests an interpretation as scattering amplitudes in effective scalar field theories [9].

One-loop open-string amplitudes exhibit two sorts of invariances that are intertwined through a similar double-copy structure: While gauge invariance is also required for field-theory amplitudes, string-theory correlators defined over a Riemann surface must be additionally invariant under monodromy variations, i.e. transporting their punctures around the homology cycles.

In this letter, we introduce a one-to-one map between gauge-invariant kinematic factors of the external states and elliptic functions on genus-one Riemann surfaces. The examples given up to six points provide evidence for a double-copy structure in one-loop open-string amplitudes. In particular, when the elliptic functions present in one-loop integrands are double-copied to their gauge-invariant kinematic counterparts, we obtain gravitational tree-level matrix elements: Those with a single insertion of the higher-curvature operator R^4 from an effective Lagrangian $\sim R + R^4$ along with its supersymmetrization. We have also checked this correspondence at seven points and will report on higher-multiplicity results in [10].

GENERALIZED ELLIPTIC FUNCTIONS

At tree level, the double-copy structure of the open superstring arises from a relation between kinematic factors and worldsheet functions defined on a disk [8]. The same Kleiss–Kuijf and BCJ relations among gauge theory amplitudes $A^{\text{tree}}_{\text{YM}}(1, 2, \ldots, n)$ [2, 11] are satisfied by the disk integrals of the so-called Parke–Taylor factors $(z_{12}z_{23}\ldots z_{n-1,n}z_{n,1})^{-1}$, where z_j represent the locations of the punctures on the disk boundary and $z_{ij} \equiv z_i - z_j$.

We will now describe *generalized elliptic functions* on genus-one Riemann surfaces that will play a similar role in one-loop open-string amplitudes as the Parke–Taylor factors at tree level.

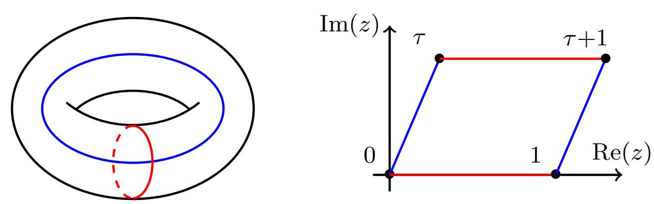

FIG. 1: Parameterization of a torus as a lattice $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with discrete identifications $z \cong z+1 \cong z+\tau$ of the punctures and modular parameter τ in the upper half plane.

Chiral splitting: The worldsheet topologies for one-loop open-string scattering – the cylinder and the Möbius strip – can be derived from a torus via suitable involutions [12]. Therefore, one-loop integrands must be doubly-periodic functions as $z \to z+1$ and $z \to z+\tau$, where z and τ correspond to the usual parameterization of the torus as depicted in fig. 1.

A convenient realization of double-periodicity that preserves meromorphic open-string constituents is provided by the *chiral-splitting* techniques of [13]. In this setup, a universal contribution to genus-one integrands is furnished by the Koba–Nielsen factor $|\mathcal{I}_n|^2$ with

$$\mathcal{I}_n \equiv \exp\Big(\sum_{i<j}^n s_{ij} \ln \theta(z_{ij},\tau) + \sum_{j=1}^n z_j (\ell\cdot k_j) + \frac{\tau \ell^2}{4\pi i}\Big) . \quad (1)$$

Here, $s_{ij} \equiv k_i\cdot k_j$ are the Mandelstam invariants in units $2\alpha'=1$ built from lightlike external momenta k_j, ℓ is the loop momentum, and θ is the odd Jacobi theta function

$$\theta(z,\tau) \equiv \sin(\pi z) \prod_{n=1}^\infty \big(1 - e^{2\pi i (n\tau+z)}\big)\big(1-e^{2\pi i (n\tau - z)}\big) . \quad (2)$$

By $\theta(z{+}\tau,\tau) = -e^{-i\pi\tau - 2\pi i z}\theta(z,\tau)$, the Koba–Nielsen factor (1) is not a doubly periodic function of the punctures, but its monodromies as $z_j \rightarrow z_j{+}\tau$ can be compensated by a shift in the loop momentum $\ell \rightarrow \ell{-}2\pi i k_j$,

$$\mathcal{I}_n \big|^{\ell\rightarrow \ell - 2\pi i k_j}_{z_j \rightarrow z_j + \tau} = \mathcal{I}_n . \quad (3)$$

We refer to meromorphic functions of z_j,ℓ,τ invariant under $(z_j,\ell) {\rightarrow} (z_j{+}\tau, \ell{-}2\pi i k_j)$ and $(z_j,\ell){\rightarrow}(z_j{+}1,\ell)$ as *generalized elliptic functions* (GEFs). After integrating the loop momentum in string amplitudes, GEFs give rise to doubly-periodic but non-meromorphic functions.

Scalar GEFs: A variety of GEFs can be generated from the Kronecker–Eisenstein series [14]

$$F(z,\alpha,\tau) \equiv \frac{\theta'(0,\tau)\theta(z{+}\alpha,\tau)}{\theta(\alpha,\tau)\theta(z,\tau)} \equiv \sum_{n=0}^\infty \alpha^{n-1} g^{(n)}(z,\tau) \quad (4)$$

whose expansion in α defines meromorphic functions such as $g^{(0)}(z,\tau)=1$ and $g^{(1)}(z,\tau) = \partial_z \ln\theta(z,\tau)$ as well as

$$2 g^{(2)}(z,\tau) {=} (\partial_z \ln \theta(z,\tau))^2 {+} \partial_z^2 \ln\theta(z,\tau) {-} \frac{\theta'''(0,\tau)}{3\theta'(0,\tau)} . \quad (5)$$

The importance of the Kronecker–Eisenstein series to the description of one-loop open-string integrands has been recently emphasized in [15], where it was shown to reproduce the spin-sum identities of [16].

The quasi-periodicity $F(z{+}\tau,\alpha,\tau) = e^{-2\pi i \alpha} F(z,\alpha,\tau)$ implies that the functions $g^{(n)}(z,\tau)$ are not elliptic,

$$g^{(n)}(z+\tau,\tau) = \sum_{k=0}^n \frac{(-2\pi i)^k}{k!} g^{(n-k)}(z,\tau) , \quad (6)$$

for example, $g^{(1)}(z{+}\tau,\tau) = g^{(1)}(z,\tau) - 2\pi i$. However, these monodromies cancel in cyclic products

$$F(z_{12},\alpha,\tau)F(z_{23},\alpha,\tau)\ldots F(z_{k-1,k},\alpha,\tau)F(z_{k,1},\alpha,\tau)$$

$$\equiv \alpha^{-k} \sum_{w=0}^\infty \alpha^w V_w(1,2,\ldots,k) \quad (7)$$

which define elliptic functions V_w in k variables with w simultaneous poles as $z_j \rightarrow z_{j+1}$ such as $V_0(1,2,\ldots,k){=}1$ and $V_1(1,2,\ldots,k) = \sum_{j=1}^k g^{(1)}_{j,j+1}$ as well as

$$V_2(1,2,\ldots,k) = \sum_{j=1}^k g^{(2)}_{j,j+1} {+} \sum_{i<j}^k g^{(1)}_{i,i+1} g^{(1)}_{j,j+1} \quad (8)$$

with $g^{(n)}_{ij} \equiv g^{(n)}(z_i{-}z_j,\tau)$ and $z_{k+1}\equiv z_1$. Therefore, the following functions are elliptic:

$$E_{1|23,4,5} \equiv V_1(1,2,3) , \qquad E_{1|234,5,6} \equiv V_2(1,2,3,4)$$
$$E_{1|23,45,6} \equiv V_1(1,2,3)V_1(1,4,5) . \quad (9)$$

Here and in the following, groups of external-state labels in a subscript that are separated through a comma (rather than a vertical bar) can be freely interchanged, e.g. $E_{1|23,45,6} = E_{1|45,23,6}$. In addition to the above scalar elliptic functions, we also introduce

$$E_{1|2|3,4,5,6} \equiv \partial_{z_1} g^{(1)}_{12} + s_{12} (g^{(1)}_{12})^2 - 2 s_{12} g^{(2)}_{12} . \quad (10)$$

Tensorial GEFs: Open-string integrands at ($n\geq 5$) points also involve loop momenta from the zero modes of certain worldsheet fields. Appearances of ℓ will be combined with the coefficients $g^{(n)}$ of the Kronecker–Eisenstein series (4) to form GEFs such as

$$E^m_{1|2,3,4,5} \equiv \ell^m + k_2^m g^{(1)}_{12} + k_3^m g^{(1)}_{13} + k_4^m g^{(1)}_{14} + k_5^m g^{(1)}_{15}$$
$$E^m_{1|23,4,5,6} \equiv \big(\ell^m {+} k_4^m g^{(1)}_{14} {+} k_5^m g^{(1)}_{15} {+} k_6^m g^{(1)}_{16}\big) V_1(1,2,3)$$
$$+ \big[k_2^m (g^{(1)}_{13} g^{(1)}_{23} {+} g^{(2)}_{12} {-} g^{(2)}_{13} {-} g^{(2)}_{23}) - (2\leftrightarrow 3)\big] \quad (11)$$
$$E^{mn}_{1|2,3,4,5,6} \equiv \ell^m \ell^n + \big[k_2^{(m} k_3^{n)} g^{(1)}_{12} g^{(1)}_{13} + (2,3|2,\ldots,6)\big]$$
$$+ \big[\ell^{(m} k_2^{n)} g^{(1)}_{12} + 2 k_2^m k_2^n g^{(2)}_{12} + (2\leftrightarrow 3,4,5,6)\big] .$$

Vector indices $m,n,\ldots = 0,1,\ldots,D{-}1$ in D spacetime dimensions are symmetrized according to $\ell^{(m}k_2^{n)} = \ell^m k_2^n {+} \ell^n k_2^m$, and $(i_1,\ldots,i_p|i_1,\ldots,i_q)$ denotes a sum over the $\binom{q}{p}$ choices of p indices $i_1,\ldots,i_p$ out of $i_1,\ldots,i_q$.

One can explicitly check that the above $E^{\cdots}_{1|\ldots}$ constitute GEFs after using (6) and momentum conservation. As detailed in the next section, these GEFs suffice to describe open-string correlators up to six points, and higher multiplicities or tensor ranks will be addressed in [10].

Shuffle symmetries: As a common property with kinematic factors of tree-level subdiagrams, the above GEFs obey shuffle symmetries within the individual groups of labels, e.g.

$$E^{\cdots}_{1|23,\ldots} = -E^{\cdots}_{1|32,\ldots} , \quad E^{\cdots}_{1|234,\ldots} + \mathrm{cyc}(2,3,4) = 0 . \quad (12)$$

These shuffle symmetries can be traced back to components of the Fay relations [17]

$$F(z_1,\alpha_1)F(z_2,\alpha_2) = F(z_1,\alpha_1{+}\alpha_2)F(z_2{-}z_1,\alpha_2) {+} (1\leftrightarrow 2) \quad (13)$$

such as [15] $g^{(1)}_{12} g^{(1)}_{23} + g^{(2)}_{13} + \mathrm{cyc}(1,2,3) = 0$.

OPEN-STRING CORRELATORS

Color-stripped one-loop amplitudes of the open string are described by the moduli-space integral [13]

$$A^{1-\text{loop}}_{\text{open}}(\lambda) = \int \mathrm{d}^D\ell \int_{D(\lambda)} \mathrm{d}\tau \prod_{j=2}^{n} \mathrm{d}z_j \, |\mathcal{I}_n|^2 \, \mathcal{K}_n \ , \quad (14)$$

see (1) for the Koba–Nielsen factor $\mathcal{I}_n$. The integration domain $D(\lambda)$ for the moduli τ and z_j depends on the topology of the genus-one worldsheet represented by λ. Finally, the correlators $\mathcal{K}_n$ comprise kinematic factors for the external states written in pure-spinor superspace as well as worldsheet functions expressed in terms of GEFs.

Pure spinors: In the pure-spinor formulation of the superstring [18] the gauge invariance and supersymmetry of the amplitudes are unified to an invariance under the BRST operator Q. A classification of BRST invariant kinematic factors of various tensor ranks that can arise from the one-loop amplitude prescription has been given in [19]. The simplest scalar BRST invariants can be expressed in terms of gauge-theory trees [20], e.g.

$$\begin{aligned} C_{1|2,3,4} &= s_{12}s_{23}A^{\text{tree}}_{\text{YM}}(1,2,3,4) \\ C_{1|23,4,5} &= s_{45}\big[s_{34}A^{\text{tree}}_{\text{YM}}(1,2,3,4,5) - (2\leftrightarrow 3)\big] \ , \end{aligned} \quad (15)$$

and further examples of various tensor ranks[1] are available for download under [21]. Their construction from Berends–Giele currents gives rise to the shuffle symmetries as seen for GEFs in (12), e.g. $C_{1|234,5,6} + \text{cyc}(2,3,4) = 0$ for scalar invariants at six points.

In addition to BRST-invariant kinematic factors, the six-point correlator [22] gives rise to *pseudo*-invariants with non-vanishing BRST variations

$$\begin{aligned} QC^{mn}_{1|2,3,4,5,6} &= -\eta^{mn}V_1 Y_{2,3,4,5,6} \\ QP_{1|2|3,4,5,6} &= -V_1 Y_{2,3,4,5,6} \ , \end{aligned} \quad (16)$$

where V_1 denotes an unintegrated vertex operator and $Y_{2,3,4,5,6}$ is related to the anomaly kinematic factor $\sim \varepsilon_{10}F^5$ of the gluon field-strength [23]. The BRST variation of the correlator localizes on the boundary of moduli space and the cancellation of the hexagon anomaly [24] thus follows as usual in the integrated amplitude (14).

With the formal definition $E_{1|2,3,4} \equiv 1$ in $\mathcal{K}_4$, the correlators up to multiplicity six are given by [10]

$$\begin{aligned} \mathcal{K}_4 &= C_{1|2,3,4}E_{1|2,3,4} \\ \mathcal{K}_5 &= C^m_{1|2,3,4,5}E^m_{1|2,3,4,5} + \big[s_{23}C_{1|23,4,5}E_{1|23,4,5} + (2,3|2,3,4,5)\big] \\ \mathcal{K}_6 &= \frac{1}{2}C^{mn}_{1|2,\ldots,6}E^{mn}_{1|2,\ldots,6} - \big[P_{1|2|3,4,5,6}E_{1|2|3,4,5,6} + (2\leftrightarrow 3,\ldots,6)\big] \\ &\quad + \big[s_{23}C^m_{1|23,4,5,6}E^m_{1|23,4,5,6} + (2,3|2,3,\ldots,6)\big] \\ &\quad + \Big(\big[s_{23}s_{45}C_{1|23,45,6}E_{1|23,45,6} + \text{cyc}(3,4,5)\big] + (6\leftrightarrow 5,4,3,2)\Big) \\ &\quad + \Big(\big[s_{23}s_{34}C_{1|234,5,6}E_{1|234,5,6} + \text{cyc}(2,3,4)\big] + (2,3,4|2,\ldots,6)\Big) \ . \end{aligned} \quad (17)$$

The notation for the permutations is explained below (11). Remarkably, kinematic factors and GEFs enter on completely symmetric footing and can be freely interchanged. This symmetry is at the heart of the double-copy structure of one-loop open-string amplitudes.

Note that chiral splitting allows to obtain closed-string one-loop amplitudes from a suitable moduli-space integral over the holomorphic square $|\mathcal{K}_n\mathcal{I}_n|^2$ [13].

THE DOUBLE-COPY STRUCTURE

In this section we will show surprising relations between the BRST-invariant kinematic factors and GEFs that underpin the double-copy structure of the open superstring at one loop. When trading the GEFs in the correlators (17) for another copy of kinematic factors, gravitational matrix elements of R^4 will be seen to emerge.

BRST-invariant kinematic factors versus GEFs: Given the GEFs defined above, one can show that

$$k_2^m E^m_{1|2,3,4,5} + \big[s_{23}E_{1|23,4,5} + (3\leftrightarrow 4,5)\big] = 0 \quad (18)$$

up to a total worldsheet derivative $\frac{\partial \ln \mathcal{I}_5}{\partial z_2}$ that vanishes under the integrals of (14). Rather surprisingly, in 2014 the following kinematic identity of identical structure was proven in the cohomology of the BRST operator [19],

$$k_2^m C^m_{1|2,3,4,5} + \big[s_{23}C_{1|23,4,5} + (3\leftrightarrow 4,5)\big] = 0 \ , \quad (19)$$

as can be explicitly verified with the data provided on the website [21]. Note that (19) enters the field-theory amplitudes of [25] as a kinematic Jacobi identity [3].

The striking resemblance between the identities (18) on a genus-one Riemann surface and (19) in the cohomology of the kinematic BRST operator motivates to search for further instances. Indeed, the symmetry properties [19]

$$\begin{aligned} C_{2|34,1,5} &= C_{1|34,2,5} + C_{1|23,4,5} - C_{1|24,3,5} \\ C_{2|13,4,5} &= -C_{1|23,4,5} \\ C^m_{2|1,3,4,5} &= C^m_{1|2,3,4,5} + \big[k_3^m C_{1|23,4,5} + (3\leftrightarrow 4,5)\big] \end{aligned} \quad (20)$$

hold for their dual GEFs in identical form

$$\begin{aligned} E_{2|34,1,5} &= E_{1|34,2,5} + E_{1|23,4,5} - E_{1|24,3,5} \\ E_{2|13,4,5} &= -E_{1|23,4,5} \\ E^m_{2|1,3,4,5} &= E^m_{1|2,3,4,5} + \big[k_3^m E_{1|23,4,5} + (3\leftrightarrow 4,5)\big] \ , \end{aligned} \quad (21)$$

as can be verified from their explicit expressions above. Similarly, the kinematic identities at six points [19]

$$\begin{aligned} k_{23}^m C^m_{1|23,4,5,6} &= P_{1|2|3,4,5,6} - P_{1|3|2,4,5,6} \\ &\quad + \big[s_{24}C_{1|324,5,6} - s_{34}C_{1|234,5,6} + (3\leftrightarrow 4,5,6)\big] \end{aligned} \quad (22)$$

[1] For instance, the bosonic components of $C^m_{1|2,3,4,5}$ involve tensor structures such as $e_1^m t_8(2,3,4,5)$ and $k_2^m t_8(12,3,4,5)/s_{12}$, where e_i denotes the polarization vector of the i^{th} gluon, and the t_8-tensor with multiparticle insertions is defined in [6].

$$k_1^m C^{mn}_{1|2,3,4,5,6} = -\big[k_2^n P_{1|2|3,4,5,6} + (2 \leftrightarrow 3,4,5,6)\big] \quad (23)$$

$$\eta_{mn} C^{mn}_{1|2,3,4,5,6} = 2\big[P_{1|2|3,4,5,6} + (2 \leftrightarrow 3,4,5,6)\big] \quad (24)$$

all have a direct counterpart in terms of GEFs up to boundary terms in moduli space. Under $C^{\cdots}_{1|\ldots} \to E^{\cdots}_{1|\ldots}$, (22) and (23) translate into total derivatives w.r.t. the punctures z_j that can be immediately discarded. The GEF-analogue of (24) additionally involves a τ-derivative

$$\eta_{mn} E^{mn}_{1|2,\ldots,6} = 2\big[E_{1|2|3,4,5,6} + (2 \leftrightarrow 3,\ldots,6)\big] + 4\pi i \frac{\partial \ln \mathcal{I}_6}{\partial \tau}\,, \quad (25)$$

resulting in the expected BRST anomaly $Q(\mathcal{K}_6 \mathcal{I}_6) \sim \frac{\partial \mathcal{I}_6}{\partial \tau}$.

The above identities among GEFs can be derived from the antisymmetry $g^{(1)}_{ij} = -g^{(1)}_{ji}$, momentum conservation, and the Fay identity (13). Together with the shuffle symmetries (12), these relations signal a fascinating *duality between BRST invariants and GEFs* which will be shown to persist at higher points [10].

The duality even extends to anomalies: The BRST variations (16) can be mapped to a modular anomaly in the ℓ-integral over $E^{mn}_{1|2,\ldots,6}$ and $E_{1|2|3,4,5,6}$ [10] which cancels by the kinematic identity (24) dual to (25).

Comparison with R^4: In the low-energy limit, one-loop amplitudes of the closed string are known to yield matrix elements of higher-curvature operators R^4 [26]. Up to and including six points, they have been expressed in terms of the above BRST (pseudo-)invariants [22],

$$\begin{aligned}
\mathcal{M}^{R^4}_4 &= C_{1|2,3,4}\tilde{C}_{1|2,3,4} \\
\mathcal{M}^{R^4}_5 &= C^m_{1|2,3,4,5}\tilde{C}^m_{1|2,3,4,5} + \big[s_{23} C_{1|23,4,5}\tilde{C}_{1|23,4,5} + (2,3|2,3,4,5)\big] \\
\mathcal{M}^{R^4}_6 &= \frac{1}{2} C^{mn}_{1|2,\ldots,6}\tilde{C}^{mn}_{1|2,\ldots,6} - \big[P_{1|2|3,4,5,6}\tilde{P}_{1|2|3,4,5,6} + (2\leftrightarrow 3,\ldots,6)\big] \\
&\quad + \big[s_{23} C^m_{1|23,4,5,6}\tilde{C}^m_{1|23,4,5,6} + (2,3|2,3,\ldots,6)\big] \qquad (26) \\
&\quad + \Big(\big[s_{23}s_{45} C_{1|23,45,6}\tilde{C}_{1|23,45,6} + \mathrm{cyc}(3,4,5)\big] + (6 \leftrightarrow 5,4,3,2)\Big) \\
&\quad + \Big(\big[s_{23}s_{34} C_{1|234,5,6}\tilde{C}_{1|234,5,6} + \mathrm{cyc}(2,3,4)\big] + (2,3,4|2,\ldots,6)\Big)\,.
\end{aligned}$$

The tilde refers to a second copy of the superspace kinematic factors, where the gravitational polarizations can be reconstructed from the tensor product of the gauge-theory polarizations. The double-copy structure of the above $\mathcal{M}^{R^4}_n$ is shared by the open-string correlators (17) which are converted to (26) by trading the GEFs for another copy of their kinematical correspondents: $E \leftrightarrow \{\tilde{C}, \tilde{P}\}$. This motivates to conjecture that

$$\mathcal{K}_n = \mathcal{M}^{R^4}_n \big|_{\tilde{C},\tilde{P}\to E} \quad (27)$$

for arbitrary multiplicities n, where all the vector indices and external-particle labels in the subscripts are understood to be inert under the replacements. We have checked that (27) also holds at $n = 7$ [10].

CONCLUSIONS AND OUTLOOK

In this letter, we have presented evidence for a duality between GEFs and BRST-invariant kinematic factors: identities among GEFs that vanish up to boundary terms in moduli space are mapped to identities among kinematic factors that vanish up to BRST-exact terms. This duality has been exploited to reveal a double-copy structure in one-loop amplitude of the open superstring. Trading GEFs in *open*-string correlators by another copy of BRST-invariant kinematic factors leads to *gravitational* matrix elements of supersymmetrized R^4 operators.

The duality between elliptic functions and BRST invariants presented here turns out to be even richer. Alternative double-copy representations of the above open-string correlators will be given in [10] which manifest their locality instead of gauge-invariance. These representations will illustrate further aspects of the duality between kinematic factors and worldsheet functions, in closer contact with conformal-field-theory techniques.

It will be interesting to playfully explore further double-copy formulae with the structure of R^4 matrix elements. Instead of (27), one could replace the kinematic factors of a gauge-theory side by color factors which opens up connections with heterotic-string amplitudes and F^4 matrix elements in the representation of [27].

There is a fascinating possibility that the duality between kinematic invariants and (generalized) elliptic functions is a generic feature of string-theory correlators. It would be rewarding to explore higher-loop analogues of the structures presented in this letter. At genus $g = 2,3$, the low-energy limits of closed-string amplitudes have been recently computed with the pure-spinor formalism, resulting in matrix elements of $D^{2g}R^4$ [28]. It appears conceivable that their double-copy structure applies to open-string correlators at the respective loop order.

The new double-copy structures unravelled in this letter should lead to great simplification of higher-order calculations in string theory, deducing the structure of the integrands from effective-field-theory quantities. Moreover, the study of GEFs is expected to trigger conceptual advances in the mathematics of string theory related to the interplay of higher-genus geometry and algebra. Finally, the $\alpha' \to 0$ limit our string-theory results yields new representations of field-theory amplitudes and will shed further light on the BCJ double copy at loop level.

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* Electronic address: c.r.mafra@soton.ac.uk
† Electronic address: olivers@aei.mpg.de

[1] H. Kawai, D. C. Lewellen and S. H. H. Tye, Nucl. Phys. B **269** (1986) 1.
[2] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. D **78** (2008) 085011 [arXiv:0805.3993 [hep-ph]].
[3] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. Lett. **105** (2010) 061602 [arXiv:1004.0476 [hep-th]].
[4] Z. Bern, J. J. Carrasco, W. M. Chen, H. Johansson and R. Roiban, Phys. Rev. Lett. **118** (2017) no.18, 181602 [arXiv:1701.02519 [hep-th]].
[5] Z. Bern, J. J. M. Carrasco, W. M. Chen, H. Johansson, R. Roiban and M. Zeng, arXiv:1708.06807 [hep-th].
[6] S. He and O. Schlotterer, Phys. Rev. Lett. **118** (2017) no.16, 161601 [arXiv:1612.00417 [hep-th]].
[7] C. R. Mafra, O. Schlotterer and S. Stieberger, Nucl. Phys. B **873** (2013) 419 [arXiv:1106.2645 [hep-th]].
[8] J. Broedel, O. Schlotterer and S. Stieberger, Fortsch. Phys. **61** (2013) 812 [arXiv:1304.7267 [hep-th]].
[9] J. J. M. Carrasco, C. R. Mafra and O. Schlotterer, JHEP **1706** (2017) 093 [arXiv:1608.02569 [hep-th]]; C. R. Mafra and O. Schlotterer, JHEP **1701** (2017) 031 [arXiv:1609.07078 [hep-th]]; J. J. M. Carrasco, C. R. Mafra and O. Schlotterer, JHEP **1708** (2017) 135 [arXiv:1612.06446 [hep-th]].
[10] C.R. Mafra and O.Schlotterer, work in progress
[11] R. Kleiss and H. Kuijf, Nucl. Phys. B **312** (1989) 616.
[12] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string,"
[13] E. D'Hoker and D. H. Phong, Rev. Mod. Phys. **60** (1988) 917; E. D'Hoker and D. H. Phong, Commun. Math. Phys. **125**, 469 (1989).
[14] L. Kronecker, Math. Werke IV (1881) 313; F. Brown and A. Levin, [arXiv:1110.6917].
[15] J. Broedel, C. R. Mafra, N. Matthes and O. Schlotterer, JHEP **1507** (2015) 112 [arXiv:1412.5535 [hep-th]].
[16] A. Tsuchiya, Phys. Rev. D **39** (1989) 1626.
[17] D. Mumford, M. Nori and P. Norman, Birkhäuser (1983, 1984).
[18] N. Berkovits, JHEP **0004** (2000) 018 [hep-th/0001035]; N. Berkovits, JHEP **0409** (2004) 047 [hep-th/0406055]; N. Berkovits, JHEP **0510** (2005) 089 [hep-th/0509120].
[19] C. R. Mafra and O. Schlotterer, arXiv:1408.3605 [hep-th].
[20] C. R. Mafra and O. Schlotterer, JHEP **1407** (2014) 153 [arXiv:1404.4986 [hep-th]].
[21] `https://tools.aei.mpg.de/purespinor/pss.html`
[22] C. R. Mafra and O. Schlotterer, JHEP **1604** (2016) 148 [arXiv:1603.04790 [hep-th]].
[23] N. Berkovits and C. R. Mafra, JHEP **0611** (2006) 079 [hep-th/0607187].
[24] M. B. Green and J. H. Schwarz, Nucl. Phys. B **255** (1985) 93; M. B. Green and J. H. Schwarz, Phys. Lett. **149B** (1984) 117.
[25] C. R. Mafra and O. Schlotterer, Fortsch. Phys. **63** (2015) no.2, 105 [arXiv:1410.0668 [hep-th]].
[26] M. B. Green, J. H. Schwarz and L. Brink, Nucl. Phys. B **198** (1982) 474.
[27] C. R. Mafra and O. Schlotterer, JHEP **1408** (2014) 099 [arXiv:1203.6215 [hep-th]].
[28] N. Berkovits, JHEP **0601**, 005 (2006) [hep-th/0503197]; H. Gomez, C. R. Mafra and O. Schlotterer, Phys. Rev. D **93** (2016) no.4, 045030 [arXiv:1504.02759 [hep-th]]; H. Gomez and C. R. Mafra, JHEP **1310** (2013) 217 [arXiv:1308.6567 [hep-th]].