Conformal Bootstrap to Rényi Entropy in 2D Liouville and Super-Liouville CFTs

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Abstract

We mainly study the Rényi entanglement entropy (REE) of the states excited by local operators in two dimensional irrational conformal field theories (CFTs), especially in Liouville field theory (LFT) and $\mathcal{N} = 1$ super Liouville field theory (SLFT). In particular, we consider the excited states obtained by acting on the vacuum with primary operators. These states can be divided into three classes in LFT and SLFT. We show that the 2nd REE of such local excited states becomes divergent in early and late time limits. Choosing a target state and reference state in the same class, the variation of REE between target and reference states can be well defined. The difference of such variation of REE between in early and late time limit always coincides with the log of the ratio of fusion matrix element between target states and reference states. Furthermore, we also study the locally excited states by acting generic descendent operators on the vacuum and the difference of variation of REE will be sum of the log of the ratio of the fusion matrix element and some additional normalization factors. Because the identity operator does not live in Hilbert space of LFT and SLFT, we found that all these properties are quite different from those of excited states in 1+1 dimensional rational CFTs.

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1 Introduction

One can define some observables to detect the property of the vacuum or excited states in a local quantum field theory. For example, entanglement entropy (EE) and the Rényi entanglement entropy (REE) are very helpful quantities to study global or non-local structures in QFTs. For a subsystem A, both of them are defined as functions of the reduced density matrix ρ_A which can be obtained by tracing out the degree freedom of complementary of A in the original density matrix ρ.

One might wonder whether there is a kind of topological contribution to the entanglement entropy even for gapless theories, e.g. conformal field theories (CFTs). (For example, computing topological contributions in entanglement entropy called topological entanglement entropy [1] can quantify some topological properties.) In this paper, we focus on extracting such kind of topological quantity from both Rényi and von-Neumann entropies of locally excited primary and descendent states in two dimensional irrational CFTs. Earlier work [2] pointed out a connection between the topological entanglement entropy and boundary entropy. Furthermore, the connection between the boundary entropy and entanglement entropy has been explored in [3].

By analytical continuation, the n-th Rényi entanglement entropy is defined by $S_A^{(n)} = \log \mathrm{Tr}[\rho_A^n]/(1-n)$. In the limit $n \to 1$, $S_A^{(n)}$ coincides with the von-Neumann entropy. By so called standard replica trick, one can calculate the entanglement entropy in field theory. One can extend [3] from vacuum states to locally excited states in CFTs. The computations of entanglement entropies for locally excited states have been worked out in [4][5][6] in various dimensional field theories. The entanglement entropy for free scalar fields have been investigated in [6][7][8]. N-th REE for 2D rational conformal field theories has been obtained in [9]. In large N CFTs with holographic dual, the entanglement entropy for locally excited states has been investigated in [10][11]. In this paper, we mainly focus on the variation of $S_A^{(n)}$ between excited states and a reference states, where the excited states are obtained by acting primary or descendent fields on the vacuum in irrational CFTs. The variation of n-th REE is denoted by $\Delta S_A^{(n)}$.

In 2D rational CFTs, it was found [9] that for the locally primary excited states, the variation of n-th Rényi entropy is related to the quantum dimension [12][13] of the primary operator. The quantum dimensions are measures of effective degrees of freedom of local operator and it is a kind of topological quantity. In various dimensional CFTs, REE has been studied by [14] [15][16][17][18][19][20][21][22][23] [24][25][26][27][28][29][30][31][32] from various perspectives. These papers [16][17][19][20] mainly focus on entanglement entropy in higher dimensional field theory. The authors of [18] have found the REE of local excited states in large central charge $1+1$ dimensional CFTs from holography. Especially, [14][20] have provided a point of view to study Rényi entropy from string theory and it provides us one loop correction to the large black hole entropy. In [21][22], entanglement entropy of local excited state in some specific quantum Lifshitz models has been presented. More recently, the authors of [33] mainly focus on the local states of product form of local operators in rational CFTs and they find that the variation of REE is consistent with the scattering process during entanglement propagation in RCFTs.

In this paper, we generalize the previous study [9][15][34] on the Rényi entropy for the primary and descendent states to irrational CFTs, especially for Liouville field theory (LFT) and super Liouville field theory (SLFT). There are two main motivations to go in this direction. The first one is the representation of spectrum will be infinite dimensional in irrational CFTs, therefore extracting entanglement entropy for local excited states will be highly nontrivial. A priori, one can not expect the variation of REE will be still the logarithm of the quantum dimension. Furthermore, the quantum dimension of local primary operator in irrational CFTs will be quite different from that in the 1+1 dimensional rational CFTs. How to get variation of REE in irrational CFTs in a precise and solid way is our main goal. The second is that LFT can be reformulated as 3 dimensional Chern Simons theory [35] or 3d gravity theory. In large central charge limit Liouville field theory might have connections [36][37][38] with 3d gravity. Basically, the boundary conditions in Chern-Simons theory are related to the Virasoro conformal blocks. The Liouville primary fields can be regarded as monodromy defects, which is proposed by [39]. To understand whether these connections are AdS/CFT-like, we would like to work out large central charge properties of local excited states by primary fields in LFT or SLFT. Because EE and REE can be probed on field theory side and holographic side, both of them will be good objects to test the properties of these connections. In this sense, we can generate large c universal properties from these data to compare with the holographic expectation [10] about REE. In this paper, we mainly focus on $1+1$ dimensional LFT and SLFT to show how to extract the variation of entanglement entropy for locally excited states between the early time limit and the late time limit properly. Firstly, we will show 2nd REE of local primary excited states by using CFTs techniques in a precise way. And then we can extend these calculation to n-th REE of primary and descendent states following [9][34]. From these studies, we can exactly see how the fusion matrix appears in REE and variation behavior $\Delta S^{(n)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t\to\infty) - \Delta S^{(n)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t\to 0)$ of REE in irrational CFTs.

The layout of this paper is as follows. In section 2, we give the 1+1 dimensional setup and study the 2nd REE in a precise way in Liouville field theory (LFT) and $\mathcal{N}=1$ super Liouville field theory (SLFT). The variations of REE $\Delta S^{(2)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t\to\infty) - \Delta S^{(2)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t\to 0)$ have been obtained. In section 3, we extend our calculation to n-th REE in LFT to show the variations of REE are log of fusion matrix ratio as 2nd REE, which are quite different from that in the rational CFTs. In section 4, we further study variation of REE $\Delta S^{(n)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle]$ between states with generated by acting descendent operators on the vacuum state in this setup. Finally, we devote to the conclusions and discussions and also mention the future problems. In appendices, we would like to list some techniques and relevant notations which are very useful in our analysis.

2 Second Rényi Entropy

2.1 Setup in 2D CFT

Consider an excited state defined by acting an operator O_a on the vacuum $|0\rangle$ in a two dimensional CFT. The operator can be primary or descendent. We can make use of the Euclidean formulation and introduce the complex coordinate $(w,\bar{w}) = (x+i\tau, x-i\tau)$ on R^2 such that τ and x denote the Euclidean time and the space respectively. We insert the operator O_a at $x=-l<0$ and investigate its real time-evolution from time 0 to t under the Hamiltonian H. We present our setup in fig. [1]. The corresponding density matrix reads as following

$$\begin{aligned}\rho(t) &= C_a \cdot e^{-iHt} e^{-\epsilon H} O_a(-l)|0\rangle\langle 0|O_a^\dagger(-l) e^{-\epsilon H} e^{iHt} \\ &= C_a \cdot O_a(w_2, \bar{w_2})|0\rangle\langle 0|O_a^\dagger(w_1, \bar{w}_1),\end{aligned} \tag{2.1}$$

where C_a is determined by requiring $\mathrm{Tr}\,\rho(t) = 1$. Here we can define coordinates as

$$w_1 = i(\epsilon - it) - l, \quad w_2 = -i(\epsilon + it) - l, \tag{2.2}$$

$$\bar{w}_1 = -i(\epsilon - it) - l, \quad \bar{w}_2 = i(\epsilon + it) - l. \tag{2.3}$$

ϵ as an infinitesimal positive parameter is an ultraviolet regulator. Till the end of the calculations, we treat $\epsilon \pm it$ as purely imaginary numbers as in [6, 7, 9].

To calculate variation of n-th REE $\Delta S_A^{(n)}$, we can make use of the replica method in the path-integral formalism by generalizing the formulation for the ground states [3] to our excited states [6]. In this paper, we choose the subsystem A to be an interval $0 \le x \le L$ at $\tau = 0$. For simplification, we just only consider $L \to \infty$ in the whole paper. It leads to a n-sheeted Riemann surface Σ_n with $2n$ operators O_a inserted.

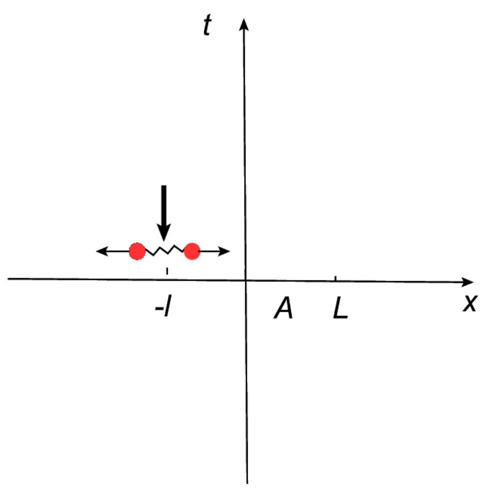

Figure 1: This figure is to show our basic setup in two dimensional plane $w = x + it$. Lower case we consider the subsystem A $0 < x < L$ with $L \to \infty$. We just put the local operators at $x = -l, t = 0$. The local operator will trigger left- and right-moving quasi-particles with time evolution.

Finally, the $\Delta S_A^{(n)}$ can be computed as

$$\begin{aligned}\Delta S_A^{(n)} &= \frac{1}{1-n}\Big[\log\left\langle O_a^{\dagger}(w_l,\bar{w}_1)O_a(w_2,\bar{w}_2)\cdots O_a(w_{2n},\bar{w}_{2n})\right\rangle_{\Sigma_n} \\ &\quad -n\log\left\langle O_a^{\dagger}(w_l,\bar{w}_1)O_a(w_2,\bar{w}_2)\right\rangle_{\Sigma_1}\Big],\end{aligned} \tag{2.4}$$

where (w_{2k+1}, w_{2k+2}) for $k = 1, 2, ..., n-1$ are $n-1$ replicas of (w_1, w_2) in the k-th sheet of Σ_n. The term in the first line in eq.(2.4) is given by a $2n$ points correlation function on Σ_n. Here Δ_a is the (chiral and anti-chiral) conformal dimension of the operator O_a.

2.2 Convention

Firstly, we study $n = 2$ i.e. the second Rényi entanglement entropy in detail as the calculations of $\Delta S_A^{(2)}$ is reduced to four point functions in CFTs.

For $n = 2$ case, one can connect the coordinate w_i with z_i by a conformal mapping $w_i = z_i^2$ which looks like

$$\begin{aligned} w_1 &= i\epsilon + t - l \equiv re^{i\theta_1} = (z_1)^2, \\ w_2 &= -i\epsilon + t - l \equiv se^{i\theta_2} = (z_2)^2, \\ w_3 &= (i\epsilon + t - l)e^{2\pi i} \equiv re^{i(2\pi+\theta_1)} = (z_3)^2, \\ w_4 &= (-i\epsilon + t - l)e^{2\pi i} \equiv se^{i(2\pi+\theta_2)} = (z_4)^2. \end{aligned} \tag{2.5}$$

Thus one can find

$$\begin{aligned} z_1 &= -z_3 = \sqrt{w_1} = \sqrt{r}e^{i\theta_1/2} = i\sqrt{l-t-i\epsilon}, \\ z_2 &= -z_4 = \sqrt{w_2} = \sqrt{s}e^{i\theta_2/2} = i\sqrt{l-t+i\epsilon}. \end{aligned} \tag{2.6}$$

If the readers are interested in finite size formula, ones can refer to [9].

We will follow a standard procedure of analytical continuation of Euclidean theory into its Lorentzian version. The most important and subtle point is that we should treat $\pm i\epsilon + t$ as a pure imaginary number in whole algebraic calculations. Finally, we take t to be real only in the final expression of the variation of entropy. Here we identify

$$(r\cos\theta_1, r\sin\theta_1) = (-l, \epsilon - it), \qquad (s\cos\theta_2, s\sin\theta_2) = (-l, -\epsilon - it), \tag{2.7}$$

which leads to

$$\begin{aligned} r &= \sqrt{l^2 + (\epsilon - it)^2}, \quad s = \sqrt{l^2 + (-\epsilon - it)^2}, \\ rs &= \sqrt{(l^2 + \epsilon^2 - t^2)^2 + 4\epsilon^2 t^2}, \quad r^2 + s^2 = 2(l^2 + \epsilon^2 - t^2), \\ \cos(\theta_1 - \theta_2) &= 2\cos^2\left(\frac{\theta_1 - \theta_2}{2}\right) - 1 = \frac{l^2 - \epsilon^2 - t^2}{\sqrt{(l^2 + \epsilon^2 - t^2)^2 + 4\epsilon^2 t^2}}. \end{aligned} \tag{2.8}$$

To REE, we just focus on the conformal cross ratio

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}} = \frac{-(l-t)+\sqrt{(l-t)^2+\epsilon^2}}{2\sqrt{(l-t)^2+\epsilon^2}},$$
$$\bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}} = \frac{-(l+t)+\sqrt{(l+t)^2+\epsilon^2}}{2\sqrt{(l+t)^2+\epsilon^2}}, \tag{2.9}$$

where $z_{ij} = z_i - z_j$.

It is useful to note the relation

$$1 - z = \frac{z_{14}z_{23}}{z_{13}z_{24}}. \tag{2.10}$$

We are interested in the two limits (i) $l >> t >> \epsilon$ (early time) and (ii) $t >> l >> \epsilon$ (late time) and from (2.9) we can show that separately correspond to

$$(i) \qquad z \simeq \bar{z} \simeq \frac{\epsilon^2}{4l^2}\ (\to 0),$$
$$(ii) \qquad z \simeq 1 - \frac{\epsilon^2}{4t^2}\ (\to 1), \quad \bar{z} \simeq \frac{\epsilon^2}{4t^2}\ (\to 0). \tag{2.11}$$

Note that the late time limit is quite non-trivial which originates from our analytical continuation of t.

2.3 2nd Rényi Entropy

The four point function on Σ_n is mapped into that on R^2 by the conformal map $w = z^n$. Thus we find

$$\begin{aligned}
&\langle O_a(w_1,\bar{w}_1)O_a(w_2,\bar{w}_2)O_a(w_3,\bar{w}_3)O_a(w_4,\bar{w}_4)\rangle_{\Sigma_2} \\
&= \prod_{i=1}^{4}\left|\frac{dw_i}{dz_i}\right|^{-2\Delta}\langle O_a(z_1,\bar{z}_1)O_a(z_2,\bar{z}_2)O_a(z_3,\bar{z}_3)O_a(z_4,\bar{z}_4)\rangle_{\Sigma_1} \\
&= 2^{-8\Delta}|z_1z_2z_3z_4|^{-2\Delta}\cdot\langle O_a(z_1,\bar{z}_1)O_a(z_2,\bar{z}_2)O_a(z_3,\bar{z}_3)O_a(z_4,\bar{z}_4)\rangle_{\Sigma_1} \\
&= 2^{-8\Delta}\cdot(rs)^{-2\Delta}\cdot\langle O_a(z_1,\bar{z}_1)O_a(z_2,\bar{z}_2)O_a(z_3,\bar{z}_3)O_a(z_4,\bar{z}_4)\rangle_{\Sigma_1},
\end{aligned} \tag{2.12}$$

where Δ_a is the chiral conformal dimension of the operator O_a.

The two point function looks like

$$\langle O_a(w_1,\bar{w}_1)O_a(w_2,\bar{w}_2)\rangle_{\Sigma_1} = \frac{C_a}{|w_{12}|^{4\Delta}} = \frac{C_a}{(2\epsilon)^{4\Delta}}, \tag{2.13}$$

where C_a is the normalization. Note that the four point function is proportional to C_a^2 and the $\Delta S_A^{(2)}$ is of course independent from C_a.

It is useful to note that owing to the conformal symmetry, the four point function on R^2 can be expressed as

$$\langle O_a(z_1,\bar{z}_1)O_a(z_2,\bar{z}_2)O_a(z_3,\bar{z}_3)O_a(z_4,\bar{z}_4)\rangle_{\Sigma_1} = |z_{13}z_{24}|^{-4\Delta}\cdot G(z,\bar{z}), \tag{2.14}$$

where $(z,\bar{z})$ are given by (2.9).

In the late time limit (ii), we finally find the ratio in (2.16) is expressed in terms of the four point function on R^2:

$$\begin{aligned}
\mathrm{Tr}\rho_A^2 &= \frac{\langle O_a(w_1,\bar{w}_1)O_a(w_2,\bar{w}_2)O_a(w_3,\bar{w}_3)O_a(w_4,\bar{w}_4)\rangle_{\Sigma_2}}{\left(\langle O_a(w_1,\bar{w}_1)O_a(w_2,\bar{w}_2)\rangle_{\Sigma_1}\right)^2}\\
&\simeq \frac{1}{C_a^2}\cdot\left(\frac{\epsilon^2}{t}\right)^{4\Delta}\cdot\langle O_a(z_1,\bar{z}_1)O_a(z_2,\bar{z}_2)O_a(z_3,\bar{z}_3)O_a(z_4,\bar{z}_4)\rangle_{\Sigma_1}\\
&\simeq \frac{1}{C_a^2}|z^{4\Delta}(1-z)^{4\Delta}|G(z,\bar{z}) \simeq \frac{1}{C_a^2}\cdot\left(\frac{\epsilon^2}{4t^2}\right)^{4\Delta}\cdot G(z,\bar{z}).
\end{aligned} \tag{2.15}$$

In rational CFTs, we can calculate $\Delta S_A^{(n)}$ between local excited states and vacuum state as follows

$$\Delta S_A^{(n)} = S_A^{(n)}(O_a|0\rangle) - S_A^{(n)}(1|0\rangle) = \frac{1}{1-n}\log\left[\frac{\langle O_a(w_1,\bar{w}_1)O_a(w_2,\bar{w}_2)\cdots O_a(w_{2n},\bar{w}_{2n})\rangle_{\Sigma_n}}{\left(\langle O_a(w_1,\bar{w}_1)O_a(w_2,\bar{w}_2)\rangle_{\Sigma_1}\right)^n}\right]. \tag{2.16}$$

Here Σ_n denotes the $n-$sheeted Euclidean surface given by the metric

$$ds^2 = d\rho^2 + \rho^2(d\theta)^2, \tag{2.17}$$

where θ has the $2\pi n$ periodicity $\theta \sim \theta + 2\pi n$.

Extra care should be taken when we generalize (2.16) to the case of LFT and SLFT. We note that normally the vacuum expectation value of n operators is defined as

$$\langle O_a(w_1,\bar{w}_1)O_a(w_2,\bar{w}_2)\cdots O_a(w_{2n},\bar{w}_{2n})\rangle_{\Sigma_n} \equiv \frac{Z_n}{Z_{0n}}, \tag{2.18}$$

where Z_n, Z_{0n} are the partition functions with or without operators inserted on Σ_n. Following the replica method in the Euclidean path-integral formalism we can express the reduced density as $\mathrm{Tr}\rho^n = Z_n/Z_1^n$. As a result $\Delta S_A^{(n)}$ can be written as

$$\Delta S_A^{(n)} = \frac{1}{1-n}\left(\log\mathrm{Tr}\rho^n - \log\mathrm{Tr}\rho_0^n\right) = \frac{1}{1-n}\left(\log\frac{Z_n}{Z_{0n}} - n\log\frac{Z_1}{Z_{01}}\right). \tag{2.19}$$

One can easily see that (2.16) follows. We note however that in Liouville theory, the n-point function is defined by the path integral and hence is unnormalized

$$\langle V_{\alpha_1}(w_1,\bar{w}_1)V_{\alpha_2}(w_2,\bar{w}_2)\cdots V_{\alpha_n}(w_{2n},\bar{w}_{2n})\rangle_{\Sigma_n} \equiv Z_n \tag{2.20}$$

It is not difficult to check that when we take all V's to be 1 *i.e.*, analytically continue $\alpha = \frac{Q}{2} + ip$ to $p = iQ/2$, the n-point function will not becomes 1 (as a normalized one (2.18) would).

In other words, (2.16) applied to the case of Liouville theory gives the Rényi entropy $S_A^{(n)}(|V_\alpha\rangle)$ instead of the difference and

$$S_A^{(n)}(V_\alpha|0\rangle) = \frac{1}{1-n}\log\frac{\langle V^\dagger_{\bar{\alpha}}(w_1,\bar{w}_1)V_\alpha(w_2,\bar{w}_2)\dots V^\dagger_{\bar{\alpha}}(w_{2n-1},\bar{w}_{2n-1})V_\alpha(w_{2n},\bar{w}_{2n})\rangle_{\Sigma_n}}{\left(\langle V^\dagger_{\bar{\alpha}}(w_1,\bar{w}_1)V_\alpha(w_2,\bar{w}_2)\rangle_{\Sigma_1}\right)^n}. \tag{2.21}$$

In LFT and SLFT considered in this paper, the identity operator does not belong to the Hilbert space and the vacuum state [40] can not be considered as a good reference state like in rational CFTs. Therefore, the $S_A^{(n)}(1|0\rangle)$ in the formula eq.(2.16) can not be applied in LFT and SLFT. We can define the difference $\Delta S_A^{(n)}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t)$ between of two excited states. In this paper, we call one the target state and the other the reference state. Alternatively in this paper, we can study $S_A^{(n)}(O_a|0\rangle)$ in early time and late time limit and define

$$\Delta S_A^{(n)}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t) = S_A^{(n)}[V_\alpha(t)]|0\rangle(t) - S_A^{(n)}[V_{\alpha_r}(t)|0\rangle](t) \tag{2.22}$$

for the study of time evolution.

2.4 2nd REE in Liouville Field theory

In our setup, we are mainly interested in $\langle V_{\bar{\alpha}}V_\alpha V_{\bar{\alpha}}V_\alpha\rangle_{\Sigma_1}$ given by eq.(6.11) to obtain 2nd REE. In the Liouville field theory, four point Green function of primary operator V_α in the s-channel can be expressed by

$$\begin{aligned}\langle V_{\alpha_1}V_{\alpha_2}V_{\alpha_3}V_{\alpha_4}\rangle_{\Sigma_1} &= |z_{13}|^{-4\Delta}|z_{24}|^{-4\Delta}G_{1234}(z,\bar{z})\\ &= \frac{1}{2}|z_{13}|^{-4\Delta}|z_{24}|^{-4\Delta}\int_{\mathbb{R}}\frac{dp}{2\pi}C(\alpha_1,\alpha_2,\frac{Q}{2}+ip)C(\alpha_3,\alpha_4,\frac{Q}{2}-ip)\\ &\quad F_s(\Delta_{i=1,2,3,4},\Delta_p,z)F_s(\Delta_{i=1,2,3,4},\Delta_p,\bar{z}).\end{aligned} \tag{2.23}$$

The $\Delta_i = \alpha_i(Q-\alpha_i)$ is the conformal dimension of external leg momentum $\alpha_i \in \frac{Q}{2} + i\mathbb{R}^+$. The integration over intermediate momentum p stands for contour integration over $p \in \mathbb{R}$. $F_s(\Delta_{i=1,2,3,4},\Delta_p,z)$ and $F_s(\Delta_{i=1,2,3,4},\Delta_p,\bar{z})$ are the holomorphic conformal block and anti-holomorphic conformal block respectively. The DOZZ formula $C(\bar{\alpha},\alpha,\frac{Q}{2}+ip)$ are given in appendix 6.1. More precisely, for $\alpha_i \in (0, Q)$ the $G_{1234}(z, \bar{z})$ in s-channel can be modified and expressed as follows

$$G_{1234}(z,\bar{z}) = \sum_{\alpha'_s \in D} D_{\alpha'_s}(z,\bar{z}) + \int \frac{dp}{2\pi} C(\alpha_1, \alpha_2, \frac{Q}{2}+ip) C(\alpha_3, \alpha_4, \frac{Q}{2}-ip) F_s(\Delta_{i=1,2,3,4}, \Delta_p, z) F_s(\Delta_{i=1,2,3,4}, \Delta_p, \bar{z}) \tag{2.24}$$

Where D denotes the discrete terms (6.68) [1] reviewed in the appendix 6.5 and appendix 6.6.

Firstly, let us study the REE in the early time limit and one can make use of s-channel expression eq.(6.11) for $\langle V_{\bar{\alpha}} V_{\alpha} V_{\bar{\alpha}} V_{\alpha} \rangle$

$$G_{\bar{\alpha}\alpha\bar{\alpha}\alpha}(z,\bar{z}) = \frac{1}{2} \int'_{\mathbb{R}} \frac{dp}{2\pi} C(\bar{\alpha}, \alpha, \frac{Q}{2}+ip) C(\bar{\alpha}, \alpha, \frac{Q}{2}-ip) F_s(\Delta_{i=\bar{\alpha},\alpha,\bar{\alpha},\alpha}, \Delta_p, z) F_s(\Delta_{i=\bar{\alpha},\alpha,\bar{\alpha},\alpha}, \Delta_p, \bar{z}) \tag{2.25}$$

$$= \frac{1}{2} \int_{\mathbb{R}} \frac{dp}{2\pi} \Big[C(\bar{\alpha}, \alpha, \frac{Q}{2}+ip) C(\bar{\alpha}, \alpha, \frac{Q}{2}-ip) \bar{z}^{\Delta_{Q/2+ip}-2\Delta_\alpha}(1+\ldots) z^{\Delta_{Q/2+ip}-2\Delta_\alpha}(1+\ldots) \Big] \tag{2.26}$$

Once we take early time limit of eq.(2.26),

$$\lim_{z\to 0, \bar{z}\to 0} \langle V_{\bar{\alpha}} V_{\alpha} V_{\bar{\alpha}} V_{\alpha} \rangle_{\Sigma_1} \simeq |z_{13}|^{-4\Delta} |z_{24}|^{-4\Delta} \frac{d^2 f_\alpha(p)}{dp^2}\Big|_{p\to 0} \int_{\mathbb{R}} |z|^{-4\Delta_\alpha + 2\Delta_{Q/2+ip}} p^2 \, dp$$
$$\simeq \frac{\sqrt{\pi}}{8 \times 2!} \frac{d^2 f_\alpha(p)}{dp^2}\Big|_{p\to 0} |z|^{-2(2\Delta_\alpha - \Delta_{Q/2})} \ln^{-\frac{3}{2}} |1/z| \tag{2.27}$$

Where we define

$$f_\alpha(p) = C(\bar{\alpha}, \alpha, \frac{Q}{2}+ip) C(\bar{\alpha}, \alpha, \frac{Q}{2}-ip) \,. \tag{2.28}$$

Further details are presented in appendix 6.6. The two-point green function for primary operator in LFT is following:

$$\langle V_\alpha(x_1) V_\alpha(x_2) \rangle_{\Sigma_1} = \frac{S(\alpha)\delta(0)}{(x_{12}\bar{x}_{12})^{2\Delta_{\alpha_1}}} \,. \tag{2.29}$$

The $\delta(0)$ is proportional to the volume of the dilaton group Vol(dilaton) $= \int_0^\infty \frac{d\lambda}{\lambda} = \infty$. Then using "reflection relation" [41] $V_\alpha = S(\alpha) V_{Q-\alpha}$, one can obtain

$$\langle V_{\bar{\alpha}}(x_1) V_\alpha(x_2) \rangle_{\Sigma_1} = \frac{\delta(0)}{(x_{12}\bar{x}_{12})^{2\Delta_{\alpha_1}}} \,. \tag{2.30}$$

[1] In this paper, the four point green function does not involve in discrete terms.

In terms of formula (12) in [9] and early time limit, the ratio becomes following

$$R^{(2)}_{EE} \underset{(z,\bar{z})\to(0,0)}{\simeq} \lim_{z\to 0,\bar{z}\to 0} \frac{\langle V_{\bar{\alpha}} V_{\alpha} V_{\bar{\alpha}} V_{\alpha}\rangle_{\Sigma_2}}{\langle V_{\bar{\alpha}} V_{\alpha}\rangle^2_{\Sigma_1}} \underset{(z,\bar{z})\to(0,0)}{\simeq} \frac{\sqrt{\pi}}{8\times 2!}\frac{1}{\delta^2(0)}\frac{d^2 f_{\alpha}(p)}{dp^2}\Big|_{p\to 0}|z|^{2\Delta_{Q/2}} \ln^{-\frac{3}{2}}|1/z|.$$

One can choose the proper normalization condition to remove the $\delta^2(0)$ dependence[2].

Then

$$S^{(2)}_{EE} \underset{(z,\bar{z})\to(0,0)}{\simeq} -\log(R^{(2)}_{EE}) \underset{(z,\bar{z})\to(0,0)}{\simeq} -\log(\frac{\sqrt{\pi}}{8\times 2!}\frac{1}{\delta^2(0)}\frac{d^2 f_{\alpha}(p)}{dp^2}\Big|_{p\to 0}|z|^{2\Delta_{Q/2}} \ln^{-\frac{3}{2}}|1/z|). \quad (2.31)$$

The 2nd REE will be divergent in the early time limit, which is quite different from the quasi-particle picture proposed by [9][6]. The main reason is that the identity operator does not belong to the spectrum of LFT. When the external legs of four point function are $\alpha_i \in \frac{Q}{2} + i\mathbb{R}^+$, the primary operators will not fuse into the identity operator. Therefore, the vacuum module will not contribute to four point function in this case. In order to restore the quasi-particle picture, we should choose a proper reference state $V_{\alpha_r}|0\rangle$ which is not vacuum state. In this paper, we choose a reference state which lives in the same class of target states and we can define the difference of 2nd REE in the early time limit as follows

$$\begin{aligned}\Delta S^{(2)}_{EE}[V_{\alpha}|0\rangle, V_{\alpha_r}|0\rangle](t\to 0) &= S^{(2)}_{EE}[V_{\alpha}|0\rangle](t\to 0) - S^{(2)}_{EE}[V_{\alpha_r}|0\rangle](t\to 0) = -\log\Big(\frac{f''_{\alpha}(p)}{f''_{\alpha_r}(p)}\Big)\Big|_{p\to 0},\\ &\alpha, \alpha_r \in \{Q/2 + ip | p \in \mathbb{R}\}.\end{aligned} \quad (2.32)$$

For the four point function with external legs α_i with $\mathrm{Re}(\alpha_i) \in (0, Q/2)$, we have to consider the discrete terms' contributions which have been reviewed in the appendix 6.5 and appendix 6.6. There are two cases, (a)-(b):

(a) $\{Q/2 > \mathrm{Re}(\alpha) > Q/4\} \bigcup \{Q/4 > \mathrm{Re}(\alpha) > 0\}$ (no pole crossing).

There are no discrete terms, and the minimal conformal dimension is given by $\alpha_s = Q/2$ in the continuous integral. These factors $C(\bar{\alpha}, \alpha, \alpha_s)$ and $C(\bar{\alpha}, \alpha, \bar{\alpha}_s)$ have a simple zero at $\alpha_s = Q/2$, so eq.(6.61) and eq.(6.60) imply

$$\begin{aligned}\langle V_{\bar{\alpha}}(0) V_{\alpha}(z) V_{\bar{\alpha}}(1) V_{\alpha}(\infty)\rangle_{\Sigma_1} &\underset{(z,\bar{z})\to(0,0)}{\simeq} \frac{1}{2!}\frac{d^2 f_{\alpha}(p)}{dp^2}\Big|_{p\to 0}\int_{\mathbb{R}} |z|^{-4\Delta_{\alpha}+2\Delta_{Q/2}-2p^2} p^2\, dp\\ &\underset{(z,\bar{z})\to(0,0)}{\simeq} \frac{\sqrt{\pi}}{8\times 2!}\frac{d^2 f_{\alpha}(p)}{dp^2}\Big|_{p\to 0}|z|^{-2(2\Delta_{\alpha}-\Delta_{Q/2})} \ln^{-\frac{3}{2}}|1/z|.\end{aligned} \quad (2.33)$$

[2] The normalization condition followed from eq.(5.13) in [42] to to remove the $\delta^2(0)$ dependence. In this paper, we just keep the factor $\delta^2(0)$ without defining normalization factor.

One can refer to the appendix 6.6 for details. For this class of excited states, the ratio for computing 2nd REE in the early time limit reads

$$R_{EE}^{(2)} \underset{(z,\bar{z})\to(0,0)}{\simeq} \frac{\sqrt{\pi}}{8\times 2!}\frac{1}{\delta(0)^2}\frac{d^2 f_\alpha(p)}{dp^2}\Big|_{p\to 0}|z|^{2\Delta_{Q/2}}\ln^{-\frac{3}{2}}|1/z|. \tag{2.34}$$

Then the 2nd REE in the early time becomes

$$S_{EE}^{(2)}[V_\alpha|0\rangle](t\to 0) \underset{(z,\bar{z})\to(0,0)}{\simeq} -\log\left(\frac{\sqrt{\pi}}{8\times 2!}\frac{1}{\delta^2(0)}\frac{d^2 f_\alpha(p)}{dp^2}\Big|_{p\to 0}|z|^{-2(2\Delta_\alpha-\Delta_{Q/2})}\ln^{-\frac{3}{2}}|1/z|\right). \tag{2.35}$$

Finally, the difference in the early time limit is

$$\begin{aligned}&\Delta S_{EE}^{(2)}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t\to 0) = S_{EE}^{(2)}[V_\alpha|0\rangle](t\to 0) - S_{EE}^{(2)}[V_{\alpha_r}|0\rangle](t\to 0)\\ =\ &-\log\Big(\frac{f''_\alpha(p)}{f''_{\alpha_r}(p)}\Big)\Big|_{p\to 0},\\ &\alpha,\alpha_r\in\{a\in\mathbb{C}|\{Q/2>\mathrm{Re}(a)>Q/4\}\bigcup\{Q/4>\mathrm{Re}(a)>0\}\}.\end{aligned} \tag{2.36}$$

(b) $\mathrm{Re}(\alpha) = Q/4, \mathrm{Im}(\alpha)\neq 0$ (marginal case).

This case is similar to the above one, except that $C(\bar{\alpha},\alpha,\alpha_s)C(\bar{\alpha},\alpha,Q-\alpha_s)$ does not vanish at $\alpha_s = Q/2$, so we have

$$\begin{aligned}&\langle V_\alpha(0)V_\alpha(z)V_\alpha(1)V_\alpha(\infty)\rangle_{\Sigma_1}\\ \underset{(z,\bar{z})\to(0,0)}{\simeq}\ & C(\bar{\alpha},\alpha,\frac{Q}{2})^2\int_{\mathbb{R}}|z|^{-2(2\Delta_\alpha-\Delta 2\alpha)-2p^2}dp\\ \underset{(z,\bar{z})\to(0,0)}{\simeq}\ & \frac{\sqrt{\pi}}{2}C(\bar{\alpha},\alpha,\frac{Q}{2})^2|z|^{-2(2\Delta_\alpha-\Delta_{Q/2})}\ln^{-\frac{1}{2}}|1/z|\,.\end{aligned} \tag{2.37}$$

For $\mathrm{Re}(\alpha) = Q/4$, the ratio for 2nd REE

$$R_{EE}^{(2)} \underset{(z,\bar{z})\to(0,0)}{\simeq} \frac{\sqrt{\pi}}{2}\frac{1}{\delta^2(0)}C(\bar{\alpha},\alpha,\frac{Q}{2})^2|z|^{2\Delta_{Q/2}}\ln^{-\frac{1}{2}}|1/z|$$

in the early time limit. Then

$$S_{EE}^{(2)}[V_\alpha|0\rangle](t\to 0) \underset{(z,\bar{z})\to(0,0)}{\simeq} -\log\left(\frac{\sqrt{\pi}}{2}\frac{1}{\delta^2(0)}C(\bar{\alpha},\alpha,\frac{Q}{2})^2|z|^{2\Delta_{Q/2}}\ln^{-\frac{1}{2}}|1/z|\right). \tag{2.38}$$

In this class, the difference of 2nd REE between target states and the reference state is

$$\Delta S^{(2)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t\to 0) = S^{(2)}_{EE}[V_\alpha|0\rangle](t\to 0) - S^{(2)}_{EE}[V_{\alpha_r}|0\rangle](t\to 0)$$
$$= -\log\Big(\frac{f_\alpha(p)}{f_{\alpha_r}(p)}\Big)\Big|_{p\to 0}, \quad \alpha,\alpha_r \in \{a\in\mathbb{C}|\mathrm{Re}(a) = Q/4, \mathrm{Im}(a)\neq 0\}. \tag{2.39}$$

The four point function of primary fields in Liouville theory in holomorphic t-channel can be written down as given in appendix 6.1

$$\langle V_{\bar\alpha}V_\alpha V_{\bar\alpha}V_\alpha\rangle_{\Sigma_1} = \frac{1}{2}|z_{13}|^{-4\Delta_\alpha}|z_{24}|^{-4\Delta_\alpha}\int'_{\mathbb{R}}\frac{dp}{2\pi}C(\bar\alpha,\alpha,\frac{Q}{2}+ip)C(\bar\alpha,\alpha,\frac{Q}{2}-ip)$$
$$F_s(\Delta_{i=\bar\alpha,\alpha,\bar\alpha,\alpha},\Delta_t,\bar z)\int_{\mathbb{S}} d\alpha_t F^L_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{smallmatrix}\right]F_t(\Delta_{i=\bar\alpha,\alpha,\bar\alpha,\alpha},\Delta_t,z) \tag{2.40}$$

The prime of integration over intermediate momentum p stands for contour integration over reals with some additional so-called discrete terms' contributions. The integral in α_t is over $\mathbb{S} = \frac{Q}{2} + i\mathbb{R}^+$. $F^L_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{smallmatrix}\right]$ is the fusion matrix from s-channel to t-channel and it has been revisited in 2.5. More precisely, the $G_{\bar\alpha,\alpha,\bar\alpha,\alpha}(z,\bar z)$ can be expressed by holomorphic t-channel conformal block in terms of conformal boostrap equation as follows

$$G_{\bar\alpha,\alpha,\bar\alpha,\alpha}(z,\bar z) = \sum_{\alpha'_s\in D}\tilde D_{\alpha'_s}(z,\bar z) + \int_{\mathbb{R}^+}\frac{dp}{2\pi}C(\bar\alpha,\alpha,\frac{Q}{2}+ip)C(\bar\alpha,\alpha,\frac{Q}{2}-ip)$$
$$F_s(\Delta_{i=\bar\alpha,\alpha,\bar\alpha,\alpha},\Delta_t,\bar z)\int_{\mathbb{S}} d\alpha_t F^L_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{smallmatrix}\right]F_t(\Delta_{i=\bar\alpha,\alpha,\bar\alpha,\alpha},\Delta_t,z) \tag{2.41}$$

where $\tilde D$ is the finite set of discrete terms which have been reviewed in appendix 6.5 and appendix 6.6. The D is the set of double poles induced by the factors $C(\bar\alpha,\alpha,\alpha_s)C(\bar\alpha,\alpha,\bar\alpha_s)$ and $\tilde D_{\alpha'_s}(z,\bar z)$[3] is given by last line in eq.(6.60).

Once we consider the late time limit to obtain the REE, the leading contributions to the

[3]For external Liouville momentum $\alpha = \frac{Q}{2} + ip$, $p\in\mathbb{R}$ and $p\neq 0$, there are no discrete terms.

REE will be the following

$$\begin{aligned}\langle V_{\bar{\alpha}}V_{\alpha}V_{\bar{\alpha}}V_{\alpha}\rangle_{\Sigma_1} \underset{(z,\bar{z})\to(1,0)}{\simeq} & \frac{1}{2}|z_{13}|^{-4\Delta}|z_{24}|^{-4\Delta}\int_{\mathbb{R}}' \frac{dp}{2\pi}C(\bar{\alpha},\alpha,\frac{Q}{2}+ip)C(\bar{\alpha},\alpha,\frac{Q}{2}-ip) \\ & \bar{z}^{\Delta_p-2\Delta}\int_{\mathbb{S}} d\alpha_t F^L_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right](1-z)^{\Delta_t-2\Delta} \\ \underset{(z,\bar{z})\to(1,0)}{\simeq} & \frac{1}{2}|z_{13}|^{-4\Delta}|z_{24}|^{-4\Delta}\int_{\mathbb{R}} \frac{dp}{2\pi}C(\bar{\alpha},\alpha,\frac{Q}{2}+ip)C(\bar{\alpha},\alpha,\frac{Q}{2}-ip) \\ & \bar{z}^{\Delta_p-2\Delta}\int_{\mathbb{S}} d\alpha_t F^L_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right](1-z)^{\Delta_t-2\Delta} + |z_{13}|^{-4\Delta}|z_{24}|^{-4\Delta}\sum_{\alpha'_s\in D}\tilde{D}_{\alpha'_s}(z,\bar{z})\end{aligned} \tag{2.42}$$

For external Liouville momentum $\alpha = \frac{Q}{2} + ip, p \in \mathbb{R}$ and $p \neq 0$, we take late time limit of eq.(2.42) and the eq.(2.15) will be

$$\begin{aligned}\lim_{(z,\bar{z})\to(1,0)}\langle V_{\bar{\alpha}}V_{\alpha}V_{\bar{\alpha}}V_{\alpha}\rangle_{\Sigma_1} \underset{(z,\bar{z})\to(1,0)}{\simeq} & \frac{\pi}{64\times 2!\sqrt{\pi}}\frac{d^2 f_\alpha(p_s)}{dp_s^2}\Big|_{p_s\to 0}2(s_b'(Q))^2\frac{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]}{|s_b(Q)|^2} \\ & (1-z)^{\Delta_{Q/2}-2\Delta_\alpha}\bar{z}^{\Delta_{Q/2}-2\Delta_\alpha}\ln^{-3/2}\left(\frac{1}{(1-z)}\right)\ln^{-3/2}\left(\frac{1}{\bar{z}}\right).\end{aligned} \tag{2.43}$$

We have used the late time limit to extract the leading contribution from relevant term $\alpha_s = Q/2, \alpha_t = Q/2$.

In terms of formula (12) in [9] and late time limit, the ratio becomes following

$$\begin{aligned}R^{(2)}_{EE} \underset{(z,\bar{z})\to(1,0)}{\simeq} & \lim_{(z,\bar{z})\to(1,0)}\frac{\langle V_{\bar{\alpha}}V_{\alpha}V_{\bar{\alpha}}V_{\alpha}\rangle_{\Sigma_2}}{\langle V_{\bar{\alpha}}V_{\alpha}\rangle^2_{\Sigma_1}} \\ \underset{(z,\bar{z})\to(1,0)}{\simeq} & \frac{\pi}{64\times 2!\sqrt{\pi}}\frac{1}{\delta^2(0)}\frac{d^2 f_\alpha(p_s)}{dp_s^2}\Big|_{p_s\to 0}2(s_b'(Q))^2\frac{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]}{|s_b(Q)|^2} \\ & (1-z)^{\Delta_{Q/2}}\bar{z}^{\Delta_{Q/2}}\ln^{-3/2}\left(\frac{1}{(1-z)}\right)\ln^{-3/2}\left(\frac{1}{\bar{z}}\right).\end{aligned} \tag{2.44}$$

We use same normalization for two point Green function given in eq.(2.35).

Then the 2nd REE in late time limit reads

$$\begin{aligned}S^{(2)}_{EE}[V_\alpha|0\rangle](t\to\infty) \underset{(z,\bar{z})\to(1,0)}{\simeq} & -\log(R^{(2)}_{EE}) \\ \underset{(z,\bar{z})\to(1,0)}{\simeq} & -\log\Big(\frac{\pi}{64\times 2!\sqrt{\pi}}\frac{1}{\delta^2(0)}\frac{d^2 f_\alpha(p)}{dp^2}\Big|_{p\to 0}2(s_b'(Q))^2\frac{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]}{|s_b(Q)|^2} \\ & (1-z)^{\Delta_{Q/2}}\bar{z}^{\Delta_{Q/2}}\ln^{-3/2}\left(\frac{1}{(1-z)}\right)\ln^{-3/2}\left(\frac{1}{\bar{z}}\right)\Big)\end{aligned} \tag{2.45}$$

and the difference is

$$
\begin{aligned}
&\Delta S^{(2)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t\to\infty) = S^{(2)}_{EE}[V_\alpha|0\rangle](t\to\infty) - S^{(2)}_{EE}[V_{\alpha_r}|0\rangle](t\to\infty)\\
= &\ -\log\Big(\frac{f''_\alpha(p)F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{smallmatrix}\right]}{f''_{\alpha_r}(p)F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha_r & \alpha_r\\ \alpha_r & \bar\alpha_r\end{smallmatrix}\right]}\Big)\Big|_{p\to 0},\quad \alpha,\alpha_r\in\{Q/2+ip|p\in\mathbb{R}\}. \qquad (2.46)
\end{aligned}
$$

For the four point function with external legs α_i with $\mathrm{Re}(\alpha_i)\in(0,\frac{Q}{2})$, we have to check whether the discrete terms make contributions or not, which have been reviewed in the appendix 6.5 and appendix 6.6. We have to show two cases, (a)-(b):

(a) $\{Q/2>\mathrm{Re}(\alpha)>Q/4\}\bigcup\{Q/4>\mathrm{Re}(\alpha)>0\}$ (no pole crossing).

There are no discrete terms in eq.(2.47), and the dominant contribution in late time will be the minimal conformal dimension is given by $\alpha_s = Q/2$ in the continuous integral . These factors $C(\bar\alpha,\alpha,\alpha_s)$ and $C(\bar\alpha,\alpha,\bar\alpha_s)$ have a simple zero at $\alpha_s = Q/2$, so eq.(6.61) and eq.(6.60) imply

$$
\begin{aligned}
&\langle V_{\bar\alpha}(0)V_\alpha(z)V_{\bar\alpha}(1)V_\alpha(\infty)\rangle_{\Sigma_1}\\
&\underset{(z,\bar z)\to(1,0)}{\simeq} \frac{1}{2!}\frac{d^2 f_\alpha(p_s)}{dp_s^2}\Big|_{p_s\to 0}\int_{\mathbb{R}} F^L_{\alpha_s=Q/2,\alpha_t=Q/2+ip_t}\left[\begin{smallmatrix}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{smallmatrix}\right](1-z)^{\Delta_{Q/2+ip_t}-2\Delta_\alpha}(\bar z)^{\Delta_{Q/2+ip_s}-2\Delta_\alpha}p_s^2\,dp_s dp_t\\
&\underset{(z,\bar z)\to(1,0)}{\simeq} \frac{\pi}{64\times 2!}\frac{d^2 f_\alpha(p_s)}{dp_s^2}\Big|_{p_s\to 0}2(s_b'(Q))^2\frac{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{smallmatrix}\right]}{|s_b(Q)|^2}\\
&(1-z)^{\Delta_{Q/2}-2\Delta_\alpha}\bar z^{\Delta_{Q/2}-2\Delta_\alpha}\ln^{-3/2}\left(\frac{1}{(1-z)}\right)\ln^{-3/2}\left(\frac{1}{\bar z}\right)\Big]. \qquad (2.47)
\end{aligned}
$$

In the late time limit, one can get the ratio for 2nd REE

$$
\begin{aligned}
R^{(2)}_{EE} \underset{(z,\bar z)\to(1,0)}{\simeq} &\frac{\pi}{64\times 2!\sqrt{\pi}}\frac{1}{\delta^2(0)}\frac{d^2 f_\alpha(p_s)}{dp_s^2}\Big|_{p_s\to 0}2(s_b'(Q))^2\frac{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{smallmatrix}\right]}{|s_b(Q)|^2}\\
&(1-z)^{\Delta_{Q/2}-2\Delta_\alpha}\bar z^{\Delta_{Q/2}-2\Delta_\alpha}\ln^{-3/2}\left(\frac{1}{(1-z)}\right)\ln^{-3/2}\left(\frac{1}{\bar z}\right)\Big]. \qquad (2.48)
\end{aligned}
$$

The 2nd REE is then given by

$$
\begin{aligned}
S^{(2)}_{EE}[V_\alpha|0\rangle](t\to\infty) \underset{(z,\bar z)\to(1,0)}{\simeq} &-\log\Big(\frac{\pi}{64\times 2!\sqrt{\pi}}\frac{1}{\delta^2(0)}\frac{d^2 f_\alpha(p)}{dp^2}\Big|_{p\to 0}2(s_b'(Q))^2\frac{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{smallmatrix}\right]}{|s_b(Q)|^2}\\
&(1-z)^{\Delta_{Q/2}}\bar z^{\Delta_{Q/2}}\ln^{-3/2}\left(\frac{1}{(1-z)}\right)\ln^{-3/2}\left(\frac{1}{\bar z}\right), \qquad (2.49)
\end{aligned}
$$

and the difference with that of the reference state in late time limit becomes

$$\begin{aligned}&\Delta S^{(2)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t\to\infty) = S^{(2)}_{EE}[V_\alpha|0\rangle](t\to\infty) - S^{(2)}_{EE}[V_{\alpha_r}|0\rangle](t\to\infty)\\ =\ & -\log\Big(\frac{f''_\alpha(p)F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha\alpha\\ \alpha\bar\alpha\end{smallmatrix}\right]}{f''_{\alpha_r}(p)F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha_r\alpha_r\\ \alpha_r\bar\alpha_r\end{smallmatrix}\right]}\Big)\Big|_{p\to 0},\\ &\alpha,\alpha_r\in\{a\in\mathbb{C}|\{Q/2>\mathrm{Re}(a)>Q/4\}\bigcup\{Q/4>\mathrm{Re}(a)>0\}\}.\end{aligned}$$

(b) $\mathrm{Re}(\alpha)=Q/4, \mathrm{Im}(\alpha)\neq 0$ (marginal case).

This case is similar to the above one, except that $C(\bar\alpha,\alpha,\alpha_s)C(\bar\alpha,\alpha,Q-\alpha_s)$ does not vanish at $\alpha_s=Q/2$, so we have

$$\begin{aligned}&\langle V_{\bar\alpha}(0)V_\alpha(z)V_{\bar\alpha}(1)V_\alpha(\infty)\rangle_{\Sigma_1}\\ &\underset{(z,\bar z)\to(1,0)}{\simeq} C(\bar\alpha,\alpha,\frac{Q}{2})^2\int_{\mathbb{R}} F^L_{\alpha_s=Q/2+ip_s,\alpha_t=Q/2+ip_t}\left[\begin{smallmatrix}\alpha\alpha\\ \alpha\alpha\end{smallmatrix}\right](1-z)^{\Delta_{2\alpha}-2\Delta_\alpha+p_s^2}(\bar z)^{\Delta_{2\alpha}-2\Delta_\alpha+p_t^2}dp_s dp_t\\ &\underset{(z,\bar z)\to(1,0)}{\simeq} \frac{\pi}{32}C(\bar\alpha,\alpha,\frac{Q}{2})^2 2(s'_b(Q/2))^2\frac{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha\alpha\\ \alpha\bar\alpha\end{smallmatrix}\right]}{|s_b(Q/2)|^2}(1-z)^{\Delta_{Q/2}-2\Delta_\alpha}(\bar z)^{\Delta_{Q/2}-2\Delta_\alpha}\ln^{-\frac12}\left(\frac{1}{(1-z)}\right)\ln^{-\frac32}\left(\frac{1}{\bar z}\right).\end{aligned} \tag{2.50}$$

In the late time limit, one can get the ratio for 2nd REE as

$$\begin{aligned}R^{(2)}_{EE}\underset{(z,\bar z)\to(1,0)}{\simeq}\ & \frac{\pi}{32}\frac{1}{\delta^2(0)}C(\bar\alpha,\alpha,\frac{Q}{2})^2 2(s'_b(Q/2))^2\frac{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha\alpha\\ \alpha\bar\alpha\end{smallmatrix}\right]}{|s_b(Q/2)|^2}(1-z)^{\Delta_{Q/2}}(\bar z)^{\Delta_{Q/2}}\\ &\ln^{-\frac12}\left(\frac{1}{(1-z)}\right)\ln^{-\frac32}\left(\frac{1}{\bar z}\right).\end{aligned} \tag{2.51}$$

Then

$$\begin{aligned}S^{(2)}_{EE}[V_\alpha|0\rangle](t\to\infty)\underset{(z,\bar z)\to(1,0)}{\simeq}\ & -\log\Big[\frac{\pi}{32}\frac{1}{\delta^2(0)}C(\bar\alpha,\alpha,\frac{Q}{2})^2 2(s'_b(Q/2))^2\frac{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha\alpha\\ \alpha\bar\alpha\end{smallmatrix}\right]}{|s_b(Q/2)|^2}\\ &(1-z)^{\Delta_{Q/2}}\bar z^{\Delta_{Q/2}}\ln^{-\frac12}\left(\frac{1}{(1-z)}\right)\ln^{-\frac32}\left(\frac{1}{\bar z}\right)\Big],\end{aligned} \tag{2.52}$$

and the difference of 2nd REE between target states and the reference state in late time is

$$\begin{aligned}&\Delta S^{(2)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t\to\infty) = S^{(2)}_{EE}[V_\alpha|0\rangle](t\to\infty) - S^{(2)}_{EE}[V_{\alpha_r}|0\rangle](t\to\infty)\\ =\ & -\log\Big(\frac{f_\alpha(p)F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha\alpha\\ \alpha\bar\alpha\end{smallmatrix}\right]}{f_{\alpha_r}(p)F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha_r\alpha_r\\ \alpha_r\bar\alpha_r\end{smallmatrix}\right]}\Big)\Big|_{p\to 0}\quad \alpha,\alpha_r\in\{a\in\mathbb{C}|\mathrm{Re}(a)=Q/4,\mathrm{Im}(a)\neq 0\}.\end{aligned} \tag{2.53}$$

2.5 Comments on Fusion matrix in Liouville field theory

For late time limit of 2nd REE, the fusion matrix element $F^{L}_{Q/2,Q/2}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]$ will be presented. This matrix element can not be identify the quantum dimension in LFT. With following the definition of quantum dimension [37] LFT, we will show the $F^{L}_{0,0}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]$ will be the quantum dimension.

Just to follow the convention in [43][44][45], we introduce $F^{\rm L}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ as followings

$$F^{\rm L}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = \frac{N(\alpha_s,\alpha_2,\alpha_1)N(\alpha_4,\alpha_3,\alpha_s)}{N(\alpha_t,\alpha_3,\alpha_2)N(\alpha_4,\alpha_t,\alpha_1)} F^{\rm PT}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right], \tag{2.54}$$

where

$$\begin{aligned} N(\alpha_3,\alpha_2,\alpha_1) &= \\ &= \frac{\Gamma_b(2Q-2\alpha_3)\Gamma_b(2\alpha_2)\Gamma_b(2\alpha_1)\Gamma_b(Q)}{\Gamma_b(2Q-\alpha_1-\alpha_2-\alpha_3)\Gamma_b(Q-\alpha_1-\alpha_2+\alpha_3)\Gamma_b(\alpha_1+\alpha_3-\alpha_2)\Gamma_b(\alpha_2+\alpha_3-\alpha_1)}. \end{aligned} \tag{2.55}$$

The b-$6j$ symbol has the following explicit form

$$\begin{aligned} F^{\rm PT}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right] &= \frac{S_b(\alpha_2+\alpha_s-\alpha_1)S_b(\alpha_t+\alpha_1-\alpha_4)}{S_b(\alpha_2+\alpha_t-\alpha_3)S_b(\alpha_s+\alpha_3-\alpha_4)}|S_b(2\alpha_t)|^2 \\ &\times \int_C du\, S_b(-\alpha_2 \pm (\alpha_1 - Q/2)+u) S_b(-\alpha_4 \pm (\alpha_3 - Q/2)+u) \\ &\qquad \times S_b(\alpha_2+\alpha_4 \pm (\alpha_t - Q/2) - u) S_b(Q \pm (\alpha_s - Q/2) - u), \end{aligned} \tag{2.56}$$

where the following notation has been used $S_b(\alpha \pm u) := S_b(\alpha+u)S_b(\alpha-u)$. The function $S_b(x)$ is defined by eq.(6.35) in appendix 6.4. The integral can be performed using the identity

$$\int_{i\mathbb{R}} dz \prod_{i=1}^{3} S_b(\mu_i - z) S_b(\nu_i + z) = \prod_{i,j=1}^{3} S_b(\mu_i+\nu_j), \tag{2.57}$$

where the balancing condition is $\sum_{i=1}^{3}\mu_i+\nu_i = Q$. We are interested in the $F^{\rm PT}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]$ which has been shown in section 2.4. In terms of notation of [46], the $6j$ symbols correspond to $F^{\rm PT}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = \left\{\begin{smallmatrix}\alpha_1 & \alpha_2 & | & \alpha_s\\ \alpha_3 & \alpha_4 & | & \alpha_t\end{smallmatrix}\right\}^{\rm an}_b$ and canonical $6j$ symbols are defined as

$$\left\{\begin{smallmatrix}\alpha_1 & \alpha_2 & | & \alpha_s\\ \alpha_3 & \alpha_4 & | & \alpha_t\end{smallmatrix}\right\}_b = \frac{M(\alpha_s,\alpha_2,\alpha_1)M(\alpha_4,\alpha_3,\alpha_s)}{M(\alpha_t,\alpha_3,\alpha_2)M(\alpha_4,\alpha_t,\alpha_1)} \left\{\begin{smallmatrix}\alpha_1 & \alpha_2 & | & \alpha_s\\ \alpha_3 & \alpha_4 & | & \alpha_t\end{smallmatrix}\right\}^{\rm an}_b. \tag{2.58}$$

with

$$\begin{aligned} M(\alpha_3,\alpha_2,\alpha_1) &= \Big(S_b(2Q-\alpha_1-\alpha_2-\alpha_3)S_b(Q-\alpha_1-\alpha_2+\alpha_3) \\ &\qquad S_b(\alpha_1+\alpha_3-\alpha_2)S_b(\alpha_2+\alpha_3-\alpha_1)\Big)^{-\frac{1}{2}}. \end{aligned} \tag{2.59}$$

With following relation given in [46],

$$\left\{\begin{smallmatrix}\alpha_1 & \alpha_2 & \alpha_s\\ \alpha_3 & \alpha_4 & \alpha_t\end{smallmatrix}\right\}_b = \left\{\begin{smallmatrix}\alpha_1 & \alpha_2 & | & \alpha_s\\ \alpha_3 & \bar{\alpha}_4 & | & \alpha_t\end{smallmatrix}\right\}_b, \qquad \bar{\alpha}_4 := Q - \alpha_4. \tag{2.60}$$

one can obtain that

$$F^{\mathrm{PT}}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right] = \frac{M(\alpha_4,\alpha_t,\alpha_1)M(\bar{\alpha}_4,\alpha_3,\alpha_s)}{M(\bar{\alpha}_4,\alpha_t,\alpha_1)M(\alpha_4,\alpha_3,\alpha_s)} F^{\mathrm{PT}}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \bar{\alpha}_4 & \alpha_1\end{smallmatrix}\right]. \tag{2.61}$$

Then

$$\begin{aligned} F^{\mathrm{L}}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right] \;=\; & \frac{\Gamma_b(2\alpha_s)}{\Gamma_b(\alpha_s)} \frac{\Gamma_b(\alpha_t)}{\Gamma_b(2\alpha_t)} \frac{\Gamma_b(2Q-2\alpha_s)\Gamma_b(Q-\alpha_t)^3}{\Gamma_b(2\alpha-\alpha_s)\Gamma_b(Q-\alpha_s)^3} \\ & \frac{\Gamma_b(2\alpha-\alpha_t)\Gamma_b(-2\alpha+Q+\alpha_t)\Gamma_b(2\alpha-Q+\alpha_t)^2}{\Gamma_b(-2\alpha+Q+\alpha_s)\Gamma_b(2\alpha-Q+\alpha_s)^2\Gamma_b(2Q-2\alpha_t)} \\ & \frac{\Gamma_b(\alpha_t)}{\Gamma_b(\alpha_s)} \frac{\Gamma_b(Q-\alpha_s)\Gamma_b(2\alpha-Q+\alpha_s)\Gamma_b(-2\alpha+2Q-\alpha_t)}{\Gamma_b(-2\alpha+2Q-\alpha_s)\Gamma_b(Q-\alpha_t)\Gamma_b(2\alpha-Q+\alpha_t)} \\ & \frac{\Gamma_b(\alpha_s)\Gamma_b(-2\alpha+Q+\alpha_s)\Gamma_b(2\alpha-Q+\alpha_s)}{\Gamma_b(2\alpha-\alpha_s)\Gamma_b(Q-\alpha_s)\Gamma_b(-2\alpha+2Q-\alpha_s)} |S_b(2\alpha_t)|^2 \end{aligned} \tag{2.62}$$

The factor in the second line of (2.62) comes from 4 normalization factors $N(\alpha_3,\alpha_2,\alpha_1)$ in eq.(2.54), the factor in the third line of (2.62) is from 4 factors associated with $M(\alpha_3,\alpha_2,\alpha_1)$ in eq.(2.61) and the factor in the last line of (2.62) is mainly from $F^{\mathrm{PT}}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right]$ eq.(2.56). We have already made use of eq.(2.57) to do the u integration to obtain the simple expression (2.62). From (2.62) and meromorphic property of $\Gamma_b(\alpha)$ shown in appendix, one can see that there is no pole structure in $F^{\mathrm{L}}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]$ for $\alpha_s \to 0$. We then obtain the following expansion near $\alpha_s \to 0$,

$$\begin{aligned} F^{\mathrm{L}}_{0\alpha_t}\left[\begin{smallmatrix}\alpha_2 & \alpha_1\\ \alpha_2 & \alpha_1\end{smallmatrix}\right] \;=\; & \frac{1}{2}|S_b(2\alpha_t)|^2 \frac{\Gamma_b(\alpha_t)^2}{\Gamma_b(2\alpha_t)} \frac{\Gamma_b(2Q)\Gamma_b(Q-\alpha_t)^2\Gamma_b(-2\alpha+2Q-\alpha_t)}{\Gamma_b(Q)^3\Gamma_b(2Q-2\alpha)^2} \\ & \frac{\Gamma_b(2\alpha-\alpha_t)\Gamma_b(-2\alpha+Q+\alpha_t)\Gamma_b(2\alpha-Q+\alpha_t)}{\Gamma_b(2\alpha)^2\Gamma_b(2Q-2\alpha_t)} \end{aligned} \tag{2.63}$$

The factor $|S_b(2\alpha_t)|^2$ can be taken care of using

$$|S_b(\alpha)|^2 = -4\sin\pi b(2\alpha-Q)\sin\pi b^{-1}(2\alpha-Q),$$

but we will temporarily keep it. The only divergence is from the simple pole of $\Gamma_b(\alpha_t)$ eq.(6.34),

$$\Gamma_b(x) \sim \frac{\Gamma_b(Q)}{2\pi x}, \tag{2.64}$$

and the residue is given by simple pole of $F^{L}_{0\alpha_t\to 0}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]$

$$F^{L}_{0\alpha_t\to 0}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right] = \frac{1}{2\pi}|S_b(\alpha)|^2.\frac{\Gamma_b(Q-2\alpha)\Gamma_b(2\alpha-Q)}{\Gamma_b(2\alpha)\Gamma_b(2Q-2\alpha)} = \frac{1}{2\pi}|S_b(\alpha)|^2.\frac{S_b(2\alpha-Q)}{S_b(2\alpha)} \tag{2.65}$$

With the help of the following identity eq.(6.37)

$$S_b(x+b^{\pm1}) = 2\,\sin(\pi b^{\pm1}x)\,S_b(x)\,, \tag{2.66}$$

we can express the residue as (where we also use $S_b(x)=S_{b^{-1}}(x)$)

$$\begin{aligned} F_{0\alpha_t\to0}\left[\begin{smallmatrix}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{smallmatrix}\right] &= \frac{1}{2\pi\alpha_t}\frac{|S_b(2\alpha_t)|^2 S_b(2\alpha-Q)}{4\sin[\pi b(2\alpha-Q+\frac{1}{b})]\sin[\pi b^{-1}(2\alpha-Q)]S_b(2\alpha-Q)} \\ &= \frac{1}{2\pi\alpha_t}\frac{\sin\pi bQ\sin\pi b^{-1}Q}{\sin[\pi b(2\alpha-Q)]\sin[\pi b^{-1}(2\alpha-Q)]}\,. \end{aligned} \tag{2.67}$$

With comparing the definition of quantum dimension [37] in LFT, we will show the $F^L_{0,0}\left[\begin{smallmatrix}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{smallmatrix}\right]$ will be the quantum dimension.

2.6 2nd REE in Super Liouville field theory

In this section, we would like to consider the states excited by local operators in super Liouville field theory which is briefly reviewed in appendix 6.2. We will mainly focus on the four point function to derive the 2nd REE in SLFT.

The four point Green function for NS-NS operator in s-channel eq.(6.28) is the following,

$$\begin{aligned} \langle V_{\bar\alpha}V_\alpha V_{\bar\alpha}V_\alpha\rangle_{\Sigma_1} &= |z_{13}|^{-4\Delta}|z_{24}|^{-4\Delta}G_{\bar12\bar34}(z,\bar z) \\ &= |z_{13}|^{-4\Delta}|z_{24}|^{-4\Delta}\Big[\int_S' d\alpha_s C_{NS}(\bar\alpha,\alpha,\alpha_s)C_{NS}(\bar\alpha,\alpha,\bar\alpha_s)F^e_s(\Delta_{i=1,2,3,4},\Delta_p,\bar z)F^e_s(\Delta_{i=1,2,3,4},\Delta_p,z) \\ &+ \int_S' d\alpha_s\tilde C_{NS}(\bar\alpha,\alpha,\alpha_s)\tilde C_{NS}(\bar\alpha,\alpha,\alpha_s)F^o_s(\Delta_{i=1,2,3,4},\Delta_p,\bar z)F^o_s(\Delta_{i=1,2,3,4},\Delta_p,z)\Big] \end{aligned} \tag{2.68}$$

The four point green function for R-R operator reads similarly,

$$\begin{aligned} \langle R_{\bar\alpha}R_\alpha R_{\bar\alpha}R_\alpha\rangle_{\Sigma_1} &= |z_{13}|^{-4\Delta}|z_{24}|^{-4\Delta}G_{\bar12\bar34}(z,\bar z) \\ &= |z_{13}|^{-4\Delta}|z_{24}|^{-4\Delta}\Big[\int_S' d\alpha_s C_R(\bar\alpha,\alpha,\alpha_s)C_R(\bar\alpha,\alpha,\bar\alpha_s)F^e_s(\Delta_{i=\bar\alpha,\alpha,\bar\alpha,\alpha},\Delta_p,\bar z)F^e_s(\Delta_{i=\bar\alpha,\alpha,\bar\alpha,\alpha},\Delta_p,z) \\ &+ \int_S' d\alpha_s\tilde C_R(\bar\alpha,\alpha,\alpha_s)\tilde C_R(\bar\alpha,\alpha,\alpha_s)F^o_s(\Delta_{i=\bar\alpha,\alpha,\bar\alpha,\alpha},\Delta_p,\bar z)F^o_s(\Delta_{i=\bar\alpha,\alpha,\bar\alpha,\alpha},\Delta_p,z)\Big] \end{aligned} \tag{2.69}$$

and the intermediate states are descendants of NS-NS primaries as in (2.68). All the conclusions for the NS-NS states can be generalized trivially to states excited by the R-R operators and hence we will carry out the analysis in the former case.

We start with external super Liouville momentum $\alpha_i = \frac{Q}{2} + ip_i$, $p_i \in \mathbb{R}$ and $p_i \neq 0$. Once we take the early time limit of eq.(2.68),

$$\begin{aligned}\lim_{(z,\bar{z})\to(0,0)} \langle V_{\bar{\alpha}} V_{\alpha} V_{\bar{\alpha}} V_{\alpha} \rangle_{\Sigma_1} &\underset{(z,\bar{z})\to(0,0)}{\simeq} \frac{1}{2!} \frac{d^2 f_{\alpha NS}(p)}{dp^2}\Big|_{p\to 0} \int_{\mathbb{R}} |z|^{-2\delta_1 + 2p^2} p^2\, dp \\ &\underset{(z,\bar{z})\to(0,0)}{\simeq} \frac{\sqrt{\pi}}{4\times 2!} \frac{d^2 f_{\alpha NS}(p)}{dp^2}\Big|_{p\to 0} |z|^{-2(2\Delta_\alpha - \Delta_{Q/2})} \ln^{-\frac{3}{2}} |1/z|\,, \quad (2.70)\end{aligned}$$

where $f_{\alpha NS}(p_s) = C_{NS}(\bar{\alpha}, \alpha, \alpha_s) C_{NS}(\bar{\alpha}, \alpha, \bar{\alpha}_s)$ and $\tilde{f}_{\alpha NS} = \tilde{C}_{NS}(\bar{\alpha}, \alpha, \alpha_s)\tilde{C}_{NS}(\bar{\alpha}, \alpha, \alpha_s)$. We note that the intermediate state with smallest conformal dimension in the odd conformal block $F^o_s(\Delta_{i=\bar{\alpha},\alpha,\bar{\alpha},\alpha}, \Delta_p, \bar{z}) F^o_s(\Delta_{i=\bar{\alpha},\alpha,\bar{\alpha},\alpha}, \Delta_p, z)$ is $G_{-1/2}\tilde{G}_{-1/2} V_{\alpha_s}$ and hence its contribution in the early time limit ($z \to 0$) will be smaller by a factor of $z^{1/2}$ compared to the even conformal block (see [47] for more details). As a result we can drop the contribution from parity odd conformal block [48]. For R-R sector, we replace $C_{NS}(\alpha_1, \alpha_2, \alpha_3)$ by $C_R(\alpha_1, \alpha_2, \alpha_3)$. The structure constant $C_R(\alpha_1, \alpha_2, \alpha_s)$ has the simple pole at $\alpha_s = Q/2$) as in the NS-NS case because both follow from $\Upsilon_{\rm NS}(\alpha_3)$ in the numerator of (6.23) and (6.29).

The two-point green function for primary operator in NS sector is following:

$$\langle V_\alpha(x_1) V_\alpha(x_2) \rangle_{\Sigma_1} = \frac{D_{NS}(\alpha)\delta(0)}{(x_{12}\bar{x}_{12})^{2\Delta_\alpha}} \quad (2.71)$$

with

$$D_{NS}(\alpha) = (\pi\mu\gamma(\frac{bQ}{2}))^{\frac{(Q-2\alpha)}{b}} \frac{b^2 \gamma\left(b\alpha - \frac{1}{2} - \frac{b^2}{2}\right)}{\gamma\left(\frac{1}{2} + \frac{b^{-2}}{2} - \alpha b^{-1}\right)}. \quad (2.72)$$

Then using "reflection relation" $V_\alpha = D_{NS}(\alpha) V_{Q-\alpha}$, one can obtain

$$\langle V_{\bar{\alpha}}(x_1) V_\alpha(x_2) \rangle_{\Sigma_1} = \frac{\delta(0)}{(x_{12}\bar{x}_{12})^{2\Delta_\alpha}}\,. \quad (2.73)$$

The associated ratio in early time limit is

$$R^{(2)}_{EE} \underset{(z,\bar{z})\to(0,0)}{\simeq} \frac{\langle V_{\bar{\alpha}} V_\alpha V_{\bar{\alpha}} V_\alpha \rangle_{\Sigma_2}}{\langle V_{\bar{\alpha}} V_\alpha \rangle^2_{\Sigma_1}} \underset{(z,\bar{z})\to(0,0)}{\simeq} \frac{\sqrt{\pi}}{8\times 2!} \frac{1}{\delta^2(0)} \frac{d^2 f_{\alpha NS}(p)}{dp^2}\Big|_{p\to 0} |z|^{2\Delta_{Q/2}} \ln^{-\frac{3}{2}} |1/z|$$

As we have done in LFT, we also keep the normalization factor delta function.

Then

$$S^{(2)}_{EE}(t\to 0) = -\log(R^{(2)}_{EE}) \underset{(z,\bar{z})\to(0,0)}{\simeq} -\log\Big(\frac{\sqrt{\pi}}{8\times 2!} \frac{1}{\delta^2(0)} \frac{d^2 f_{\alpha NS}(p)}{dp^2}\Big|_{p\to 0} |z|^{2\Delta_{Q/2}} \ln^{-\frac{3}{2}} |1/z|\Big).$$

Finally, the early time of difference of 2nd REE between $V_\alpha|0\rangle$ and $V_{\alpha_r}|0\rangle$ is

$$\begin{aligned} \Delta S^{(2)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t\to 0) &= S^{(2)}_{EE}[V_\alpha|0\rangle](t\to 0) - S^{(2)}_{EE}[V_{\alpha_r}|0\rangle](t\to 0) \\ = \quad -\log\Big(\frac{f''_{\alpha NS}(p)}{f''_{\alpha_r NS}(p)}\Big)\Big|_{p\to 0}, &\quad \alpha,\alpha_r \in \{Q/2+ip|p\in\mathbb{R}\}. \end{aligned} \tag{2.74}$$

For the four point function with external legs α_i with $\mathrm{Re}(\alpha_i) \in (0, Q/2)$, we have to check whether the discrete terms'(6.68) contributions which have been reviewed in the appendix 6.5 and appendix 6.6. We have to show two cases, (a)-(b):

(a) $\{Q/2 > \mathrm{Re}(\alpha) > Q/4\} \bigcup \{Q/4 > \mathrm{Re}(\alpha) > 0\}$ (no pole crossing).

There are no discrete terms in eq.(2.75), and the leading contribution comes from the s-channel with minimal conformal dimension $\alpha_s = Q/2$ in the continuous integral. These factors $C_{NS}(\bar{\alpha}, \alpha, \alpha_s)$ and $C_{NS}(\bar{\alpha}, \alpha, \bar{\alpha}_s)$ have a simple zero at $\alpha_s = Q/2$, so eq.(6.61) and eq.(6.60) imply

$$\begin{aligned} &\langle V_{\bar{\alpha}}(0)V_\alpha(z)V_{\bar{\alpha}}(1)V_\alpha(\infty)\rangle_{\Sigma_1} \\ &\underset{(z,\bar{z})\to(0,0)}{\simeq} \frac{1}{2!}\frac{d^2 f_{\alpha NS}(p)}{dp^2}\Big|_{p\to 0}\int_{\mathbb{R}} |z|^{-2\delta_1+2p^2} p^2\, dp \\ &\underset{(z,\bar{z})\to(0,0)}{\simeq} \frac{\sqrt{\pi}}{8\times 2!}\frac{d^2 f_{\alpha NS}(p)}{dp^2}\Big|_{p\to 0}|z|^{-2(2\Delta_\alpha-\Delta_{Q/2})}\ln^{-\frac{3}{2}}|1/z|. \end{aligned} \tag{2.75}$$

The ratio for 2nd REE in the early time limit is given by

$$R^{(2)}_{EE} \underset{(z,\bar{z})\to(0,0)}{\simeq} \frac{\sqrt{\pi}}{8\times 2!}\frac{1}{\delta^2(0)}\frac{d^2 f_{\alpha NS}(p)}{dp^2}\Big|_{p\to 0}|z|^{2\Delta_{Q/2}}\ln^{-\frac{3}{2}}|1/z|. \tag{2.76}$$

Then

$$S^{(2)}_{EE}[V_\alpha|0\rangle](t\to 0) \underset{(z,\bar{z})\to(0,0)}{\simeq} -\log\left(\frac{\sqrt{\pi}}{8\times 2!}\frac{1}{\delta^2(0)}\frac{d^2 f_{\alpha NS}(p)}{dp^2}\Big|_{p\to 0}|z|^{2\Delta_{Q/2}}\ln^{-\frac{3}{2}}|1/z|\right).$$

Finally, the difference in the early time limit between $V_\alpha|0\rangle$ and $V_{\alpha_r}|0\rangle$ is

$$\begin{aligned} \Delta S^{(2)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t\to 0) &= S^{(2)}_{EE}[V_\alpha|0\rangle](t\to 0) - S^{(2)}_{EE}[V_{\alpha_r}|0\rangle](t\to 0) \\ = \quad -\log\Big(\frac{f''_{\alpha NS}(p)}{f''_{\alpha_r NS}(p)}\Big)\Big|_{p\to 0}&, \\ \alpha,\alpha_r \in \{a\in\mathbb{C}|\{Q/2 > \mathrm{Re}(a) > Q/4\} &\bigcup \{Q/4 > \mathrm{Re}(a) > 0\}\}. \end{aligned} \tag{2.77}$$

(b) $\mathrm{Re}(\alpha) = \frac{Q}{4}, \mathrm{Im}(\alpha) \neq 0$ (marginal case).

This case is similar to the above one, except that $C_{NS}(\bar{\alpha},\alpha,\alpha_s,)C_{NS}(\bar{\alpha},\alpha,Q-a_s)$ does not vanish at $\alpha_s = Q/2$, so we have

$$\begin{aligned}
&\langle V_{\bar{\alpha}}(0)V_\alpha(z)V_{\bar{\alpha}}(1)V_\alpha(\infty)\rangle_{\Sigma_1}\\
\underset{(z,\bar{z})\to(0,0)}{\simeq}\ & C_{NS}(\bar{\alpha},\alpha,\frac{Q}{2})^2\int_R |z|^{-2(2\Delta_{2\alpha}-\Delta_\alpha)+2p^2}dp\\
\underset{(z,\bar{z})\to(0,0)}{\simeq}\ & \frac{\sqrt{\pi}}{2}C_{NS}(\bar{\alpha},\alpha,\frac{Q}{2})^2|z|^{-2(\Delta_\alpha-\Delta_{Q/2})}\ln^{-\frac{1}{2}}|1/z|\,.
\end{aligned}\tag{2.78}$$

The ratio for 2nd REE in the early time limit is

$$R^{(2)}_{EE}\underset{(z,\bar{z})\to(0,0)}{\simeq}\frac{\sqrt{\pi}}{2}\frac{1}{\delta^2(0)}C_{NS}(\bar{\alpha},\alpha,\frac{Q}{2})^2|z|^{2\Delta_{Q/2}}\ln^{-\frac{1}{2}}|1/z|. \tag{2.79}$$

Then

$$S^{(2)}_{EE}[V_\alpha|0\rangle](t\to 0)\underset{(z,\bar{z})\to(0,0)}{\simeq} -\log\left(\frac{\sqrt{\pi}}{2}\frac{1}{\delta^2(0)}C_{NS}(\bar{\alpha},\alpha,\frac{Q}{2})^2|z|^{2\Delta_{Q/2}}\ln^{-\frac{1}{2}}|1/z|\right). \tag{2.80}$$

In this case, the difference of 2nd REE between $V_\alpha|0\rangle$ and $V_{\alpha_r}|0\rangle$ is

$$\begin{aligned}
&\Delta S^{(2)}_{EE}[V_\alpha|0\rangle,V_{\alpha_r}|0\rangle](t\to 0) = S^{(2)}_{EE}[V_\alpha|0\rangle](t\to 0) - S^{(2)}_{EE}[V_{\alpha_r}|0\rangle](t\to 0)\\
=\ & -\log\Big(\frac{C_{NS}(\bar{\alpha},\alpha,\frac{Q}{2})^2}{C_{NS}(\bar{\alpha}_r,\alpha_r,\frac{Q}{2})^2}\Big)\\
&\alpha,\alpha_r\in\{a\in\mathbb{C}|\mathrm{Re}(a)=Q/4,\mathrm{Im}(a)\in\mathbb{R},\mathrm{Im}(\alpha)\neq 0\}.
\end{aligned}\tag{2.81}$$

Secondly, we consider the 2nd REE in SLFT in the late time limit. For convenience in late time limit, we have to use conformal boostrap equation to express the four point function $G_{1234}(z,\bar{z})$, which is similar to the procedures shown in LFT. The four point function with fusing holomorphic conformal block from s-channel to t-channel is following

$$\begin{aligned}
&\langle V_{\alpha_1}(0,0)V_{\alpha_2}(z,\bar{z})V_{\alpha_3}(1,1)V_{\alpha_4}(\infty,\infty)\rangle_{\Sigma_1} = G_{1234}(z,\bar{z})\\
=\ & \frac{1}{2}\Big(\int_S' d\alpha_s C_{NS}(\alpha_1,\alpha_2,\alpha_s)C_{NS}(\alpha_3,\alpha_4,\bar{\alpha}_s)\\
& F^e_s(\Delta_{i=1,2,3,4},\Delta_{\alpha_s},\bar{z})\int d\alpha_t\sum_{\rho=e,o}F^{SL}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\bar{\alpha}&\alpha\\ \alpha&\bar{\alpha}\end{smallmatrix}\right]^e_\rho F^\rho_t(\Delta_{i=1,2,3,4},\Delta_{\alpha_t},z)\\
+\ & \int_S' d\alpha_s\tilde{C}_{NS}(\alpha_1,\alpha_2,\alpha_s)\tilde{C}_{NS}(\alpha_3,\alpha_4,\bar{\alpha}_s)F^o_s(\Delta_{i=1,2,3,4},\Delta_{\alpha_s},\bar{z})\\
& \int d\alpha_t\sum_{\rho=e,o}F^{SL}_{\alpha_s\alpha_t}\left[\begin{smallmatrix}\alpha_3&\alpha_2\\ \alpha_4&\alpha_1\end{smallmatrix}\right]^o_\rho F^\rho_t(\Delta_{i=1,2,3,4},\Delta_{\alpha_t},z)\Big).
\end{aligned}\tag{2.82}$$

Once we take the late time limit of eq.(2.82), the four point function becomes

$$\begin{aligned}
\lim_{z\to 1,\bar z\to 0}\langle V_{\bar\alpha}V_\alpha V_{\bar\alpha}V_\alpha\rangle_{\Sigma_1} \quad &\underset{(z,\bar z)\to(1,0)}{\simeq} \quad \frac{1}{2!}\frac{d^2 f_{\alpha NS}(p_s)}{dp_s^2}\Big|_{p_s\to 0} \\
&\int_{\mathbb{R}} F^{SL}_{\alpha_s=Q/2+ip_s,\alpha_t=Q/2+ip_t}\left[\begin{smallmatrix}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{smallmatrix}\right]^e_e (1-z)^{\Delta_{Q/2}-2\Delta_\alpha+p^2}\,(\bar z)^{\Delta_{Q/2}-2\Delta_\alpha+p_s^2}\,p_s^2\,dp_s dp_t. \\
&\underset{(z,\bar z)\to(1,0)}{\simeq} \quad \frac{\pi}{64\times 2!}\frac{d^2 f_{\alpha NS}(p_s)}{dp_s^2}\Big|_{p_s\to 0} 2(s'_{NS}(Q))^2\frac{F^{SL}_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{smallmatrix}\right]^e_e}{|s_{NS}(Q)|^2} \\
&(1-z)^{\Delta_{Q/2}-2\Delta_\alpha}\bar z^{\Delta_{Q/2}-2\Delta_\alpha}\ln^{-3/2}\left(\frac{1}{(1-z)}\right)\ln^{-3/2}\left(\frac{1}{\bar z}\right). \qquad (2.83)
\end{aligned}$$

where the exact expression for $F^{SL}_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{smallmatrix}\right]$ is revisited in 2.7. We have used the late time limit to extract the leading contribution. One can show that the contribution from odd parity part of conformal block $F^o_t(\Delta_{i=1,2,3,4},\Delta_p,z)$ in late time limit will be subleading [48] and we drop the subleading contributions in eq.(2.83).

Then the ratio associated with 2nd REE in super Liouville field theory can be defined as following

$$\begin{aligned}
R^{(2)}_{EE} \quad &= \quad \lim_{z\to 1,\bar z\to 0}\frac{\langle V_{\bar\alpha}V_\alpha V_{\bar\alpha}V_\alpha\rangle_{\Sigma_2}}{\langle V_{\bar\alpha}V_\alpha\rangle^2_{\Sigma_1}} \\
&\underset{(z,\bar z)\to(1,0)}{\simeq} \quad \frac{\pi}{64\times 2!}\frac{d^2 f_{\alpha NS}(p_s)}{dp_s^2}\Big|_{p_s\to 0}\frac{1}{\delta^2(0)}2(s'_{NS}(Q))^2\frac{F^{SL}_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{smallmatrix}\right]^e_e}{|s_{NS}(Q)|^2} \\
&(1-z)^{\Delta_{Q/2}}\bar z^{\Delta_{Q/2}}\ln^{-3/2}\left(\frac{1}{(1-z)}\right)\ln^{-3/2}\left(\frac{1}{\bar z}\right). \qquad (2.84)
\end{aligned}$$

Then the corresponding REE in late time is

$$\begin{aligned}
S^{(2)}_{EE}[V_\alpha|0\rangle](t\to\infty) \quad &\underset{(z,\bar z)\to(1,0)}{\simeq} \quad -\log\Big(\frac{\pi}{64\times 2!}\frac{1}{\delta^2(0)}\frac{d^2 f_{\alpha NS}(p_s)}{dp_s^2}\Big|_{p_s\to 0}2(s'_{NS}(Q))^2\frac{F^{SL}_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{smallmatrix}\right]^e_e}{|s_{NS}(Q)|^2} \\
&(1-z)^{\Delta_{Q/2}}\bar z^{\Delta_{Q/2}}\ln^{-3/2}\left(\frac{1}{(1-z)}\right)\ln^{-3/2}\left(\frac{1}{\bar z}\right)\Big), \qquad (2.85)
\end{aligned}$$

and the difference is

$$\begin{aligned}
&\Delta S^{(2)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t\to\infty) = S^{(2)}_{EE}[V_\alpha|0\rangle](t\to\infty) - S^{(2)}_{EE}[V_{\alpha_r}|0\rangle](t\to\infty), \\
= \quad &-\log\Big(\frac{f''_{\alpha NS}(p)F^{SL}_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{smallmatrix}\right]}{f''_{\alpha_r NS}(p)F^{SL}_{Q/2,Q/2}\left[\begin{smallmatrix}\bar\alpha_r & \alpha_r\\ \alpha_r & \bar\alpha_r\end{smallmatrix}\right]}\Big)\Big|_{p\to 0} \quad \alpha,\alpha_r\in\{Q/2+ip|p\in\mathbb{R}\}. \qquad (2.86)
\end{aligned}$$

For the four point function with external legs α_i with $\mathrm{Re}(\alpha_i)\in(0,Q/2)$, we have to check whether the discrete terms make contribution or not, which have been reviewed in appendix 6.5. We have to consider two cases, (a)-(b):

(a) $\{Q/2 > \mathrm{Re}(\alpha) > Q/4\} \bigcup \{Q/4 > \mathrm{Re}(\alpha) > 0\}$ (no pole crossing).

There are no discrete terms in eq.(2.87), and the leading contribution in late time comes from the s-channel anti-holomorphic (t-channel holomorphic) conformal block with minimal conformal dimension $\alpha_s = Q/2$ ($\alpha_t = Q/2$) in the continuous integral. These factors $C_{NS}(\bar{\alpha}, \alpha, \alpha_s)$ and $C_{NS}(\bar{\alpha}, \alpha, \bar{\alpha}_s)$ have simple zeros at $\alpha_s = Q/2$, and hence eq.(6.61) and eq.(6.60) imply

$$\begin{aligned}\langle V_{\bar{\alpha}}(0) V_\alpha(z) V_{\bar{\alpha}}(1) V_\alpha(\infty)\rangle_{\Sigma_1} \underset{(z,\bar{z})\to(1,0)}{\simeq} & \frac{1}{2!} \frac{d^2 f_{\alpha NS}(p_s)}{dp_s^2}\Big|_{p_s\to 0} \\ & \int_{\mathbb{R}} F^{SL}_{\alpha_s=Q/2+ip_s, \alpha_t=Q/2+ip_t}\left[\begin{smallmatrix}\bar{\alpha} & \alpha \\ \alpha & \bar{\alpha}\end{smallmatrix}\right]^e_e (1-z)^{\Delta_{Q/2}-2\Delta_\alpha+p_s^2} (\bar{z})^{\Delta_{Q/2}-2\Delta_\alpha+p_t^2} p_s^2\, dp_s dp_t. \\ \underset{(z,\bar{z})\to(1,0)}{\simeq} & \frac{\pi}{64\times 2!} \frac{d^2 f_{\alpha NS}(p_s)}{dp_s^2}\Big|_{p_s\to 0} 2(s'_{NS}(Q))^2 \frac{F^{SL}_{Q/2,Q/2}\left[\begin{smallmatrix}\bar{\alpha} & \alpha \\ \alpha & \bar{\alpha}\end{smallmatrix}\right]^e_e}{s_{NS}(Q)} \\ & (1-z)^{2\Delta_{Q/2}-2\Delta_\alpha} \bar{z}^{2\Delta_{Q/2}-2\Delta_\alpha} \ln^{-3/2}\left(\frac{1}{(1-z)}\right) \ln^{-3/2}\left(\frac{1}{\bar{z}}\right).\end{aligned} \tag{2.87}$$

In the late time limit, one can get the ratio for 2nd REE

$$\begin{aligned}R^{(2)}_{EE} \underset{(z,\bar{z})\to(1,0)}{\simeq} & \frac{\pi}{64\times 2!} \frac{1}{\delta^2(0)} \frac{d^2 f_{\alpha NS}(p_s)}{dp_s^2}\Big|_{p_s\to 0} 2(s'_{NS}(Q))^2 \frac{F^{SL}_{Q/2,Q/2}\left[\begin{smallmatrix}\bar{\alpha} & \alpha \\ \alpha & \bar{\alpha}\end{smallmatrix}\right]^e_e}{s_{NS}(Q)} \\ & (1-z)^{2\Delta_{Q/2}-2\Delta_\alpha} \bar{z}^{2\Delta_{Q/2}-2\Delta_\alpha} \ln^{-3/2}\left(\frac{1}{(1-z)}\right) \ln^{-3/2}\left(\frac{1}{\bar{z}}\right).\end{aligned} \tag{2.88}$$

Then

$$\begin{aligned}S^{(2)}_{EE}(t\to\infty) \underset{(z,\bar{z})\to(1,0)}{\simeq} & -\log\Big(\frac{\pi}{64\times 2!} \frac{1}{\delta^2(0)} \frac{d^2 f_{\alpha NS}(p_s)}{dp_s^2}\Big|_{p_s\to 0} 2(s'_{NS}(Q))^2 \frac{F^{SL}_{Q/2,Q/2}\left[\begin{smallmatrix}\bar{\alpha} & \alpha \\ \alpha & \bar{\alpha}\end{smallmatrix}\right]^e_e}{s_{NS}(Q)} \\ & (1-z)^{2\Delta_{Q/2}} \bar{z}^{2\Delta_{Q/2}} \ln^{-3/2}\left(\frac{1}{(1-z)}\right) \ln^{-3/2}\left(\frac{1}{\bar{z}}\right)\Big),\end{aligned} \tag{2.89}$$

and the difference of 2nd REE in the late time limit between $V_\alpha|0\rangle$ and $V_{\alpha_r}|0\rangle$ is

$$\begin{aligned} & \Delta S^{(2)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t\to\infty) = S^{(2)}_{EE}[V_\alpha|0\rangle](t\to\infty) - S^{(2)}_{EE}[V_{\alpha_r}|0\rangle](t\to\infty) \\ = & -\log\Big(\frac{f''_{\alpha NS}(p) F^{SL}_{Q/2,Q/2}\left[\begin{smallmatrix}\bar{\alpha} & \alpha \\ \alpha & \bar{\alpha}\end{smallmatrix}\right]}{f''_{\alpha NS}(p) F^{SL}_{Q/2,Q/2}\left[\begin{smallmatrix}\bar{\alpha}_r & \alpha_r \\ \alpha_r & \bar{\alpha}_r\end{smallmatrix}\right]}\Big)\Big|_{p\to 0}, \\ & \alpha, \alpha_r \in \{a \in \mathbb{C} | \{Q/2 > \mathrm{Re}(a) > Q/4\} \bigcup \{Q/4 > \mathrm{Re}(a) > 0\}\}.\end{aligned} \tag{2.90}$$

(b) $\mathrm{Re}(\alpha) = \frac{Q}{4}, \mathrm{Im}(\alpha) \neq 0$ (marginal case).

This case is similar to the one above, except that $C_{NS}(\bar{\alpha}, \alpha, \frac{Q}{2})^2$ does not vanish at $\alpha_s = Q/2$, so we have

$$\begin{aligned}
&\langle V_{\bar{\alpha}}(0)V_{\alpha}(z)V_{\bar{\alpha}}(1)V_{\alpha}(\infty)\rangle_{\Sigma_1} \\
&\underset{(z,\bar{z})\to(1,0)}{\simeq} C_{NS}(\bar{\alpha}, \alpha, \frac{Q}{2})^2 \int_R F^{SL}_{\alpha_s=Q/2+ip_s,\alpha_t=Q/2+ip_t}\big[{}^{\alpha\alpha}_{\alpha\alpha}\big]^e_e (1-z)^{\Delta_{2\alpha}-2\Delta_\alpha+p_s^2}(\bar{z})^{\Delta_{2\alpha}-2\Delta_\alpha+p_t^2} dp_s dp_t \\
&\underset{(z,\bar{z})\to(1,0)}{\simeq} \frac{\pi}{16\times 2} C_{NS}(\bar{\alpha}, \alpha, \frac{Q}{2})^2 2(s'_{NS}(Q))^2 \frac{F^{SL}_{Q/2,Q/2}\big[{}^{\alpha\alpha}_{\alpha\alpha}\big]^e_e}{s_{NS}(Q)^2} (1-z)^{\Delta_{2\alpha}-2\Delta_\alpha}(\bar{z})^{\Delta_{2\alpha}-2\Delta_\alpha} \ln^{-\frac{1}{2}}\left(\frac{1}{(1-z)}\right)\ln^{-\frac{3}{2}}\left(\frac{1}{\bar{z}}\right).
\end{aligned} \tag{2.91}$$

For $\mathrm{Re}(\alpha) = \frac{Q}{4}$, in the late time limit, one can guess the ratio for 2nd REE

$$\begin{aligned}
R^{(2)}_{EE} \underset{(z,\bar{z})\to(1,0)}{\simeq} & \frac{\pi}{16\times 2}\frac{1}{\delta^2(0)} C_{NS}(\bar{\alpha}, \alpha, \frac{Q}{2})^2 2(s'_{NS}(Q))^2 \frac{F^{SL}_{Q/2,Q/2}\big[{}^{\alpha\alpha}_{\alpha\alpha}\big]^e_e}{s_{NS}(Q)^2} \\
& (1-z)^{\Delta_{Q/2}}\bar{z}^{\Delta_{Q/2}} \ln^{-\frac{1}{2}}\left(\frac{1}{(1-z)}\right)\ln^{-\frac{3}{2}}\left(\frac{1}{\bar{z}}\right).
\end{aligned} \tag{2.92}$$

Then

$$\begin{aligned}
S^{(2)}_{EE}[V_\alpha|0\rangle](t\to\infty) \underset{(z,\bar{z})\to(1,0)}{\simeq} & -\log\Big[\frac{\pi}{16\times 2}\frac{1}{\delta^2(0)} C_{NS}(\bar{\alpha}, \alpha, \frac{Q}{2})^2 2(s'_{NS}(Q))^2 \frac{F^{SL}_{Q/2,Q/2}\big[{}^{\alpha\alpha}_{\alpha\alpha}\big]^e_e}{s_{NS}(Q)^2} \\
& (1-z)^{\Delta_{Q/2}}\bar{z}^{\Delta_{Q/2}} \ln^{-\frac{1}{2}}\left(\frac{1}{(1-z)}\right)\ln^{-\frac{3}{2}}\left(\frac{1}{\bar{z}}\right)\Big],
\end{aligned} \tag{2.93}$$

and

$$\begin{aligned}
& \Delta S^{(2)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t\to\infty) = S^{(2)}_{EE}[V_\alpha|0\rangle](t\to\infty) - S^{(2)}_{EE}[V_{\alpha_r}|0\rangle](t\to\infty) \\
= & -\log\Big(\frac{f_{\alpha NS}(p) F^{SL}_{Q/2,Q/2}\big[{}^{\bar{\alpha}\alpha}_{\alpha\bar{\alpha}}\big]}{f_{\alpha_r NS}(p) F^{SL}_{Q/2,Q/2}\big[{}^{\bar{\alpha}_r\alpha_r}_{\alpha_r\bar{\alpha}_r}\big]}\Big)\Big|_{p\to 0} \\
& \alpha, \alpha_r \in \{a \in \mathbb{C}|\mathrm{Re}(a) = Q/4, \mathrm{Im}(a)\in\mathbb{R}, \mathrm{Im}(\alpha)\neq 0\},.
\end{aligned} \tag{2.94}$$

2.7 Comments on Fusion matrix in super Liouville field theory

Similar to the subsection, we would like to comment on quantum dimension and fusion matrix presented in 2nd REE in SLFT. Generically, the fusion matrices take the following form $i, j = 1, 2$ which correspond to parity of e and o respectively [48]

$$F_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]_j^i = \tag{2.95}$$
$$\frac{\Gamma_i(2Q-\alpha_t-\alpha_2-\alpha_3)\Gamma_i(Q-\alpha_t+\alpha_3-\alpha_2)\Gamma_i(Q+\alpha_t-\alpha_2-\alpha_3)\Gamma_i(\alpha_3+\alpha_t-\alpha_2)}{\Gamma_j(2Q-\alpha_1-\alpha_s-\alpha_2)\Gamma_j(Q-\alpha_s-\alpha_2+\alpha_1)\Gamma_j(Q-\alpha_1-\alpha_2+\alpha_s)\Gamma_j(\alpha_s+\alpha_1-\alpha_2)}$$
$$\times\ \frac{\Gamma_i(Q-\alpha_t-\alpha_1+\alpha_4)\Gamma_i(\alpha_1+\alpha_4-\alpha_t)\Gamma_i(\alpha_t+\alpha_4-\alpha_1)\Gamma_i(\alpha_t+\alpha_1+\alpha_4-Q)}{\Gamma_j(Q-\alpha_s-\alpha_3+\alpha_4)\Gamma_j(\alpha_3+\alpha_4-\alpha_s)\Gamma_j(\alpha_s+\alpha_4-\alpha_3)\Gamma_j(\alpha_s+\alpha_3+\alpha_4-Q)}$$
$$\times\frac{\Gamma_{\rm NS}(2Q-2\alpha_s)\Gamma_{\rm NS}(2\alpha_s)}{\Gamma_{\rm NS}(Q-2\alpha_t)\Gamma_{\rm NS}(2\alpha_t-Q)}\frac{1}{i}\int_{-i\infty}^{i\infty}d\tau J_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]_j^i,$$

We will consider $i = j = 1$, which gives the fusion matrix for NS sector. In this case, we have

$$J_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]_1^1 = \tag{2.96}$$
$$\frac{S_{\rm NS}(Q+\tau-\alpha_1)S_{\rm NS}(\tau+\alpha_4+\alpha_2-\alpha_3)S_{\rm NS}(\tau+\alpha_1)S_{\rm NS}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{\rm NS}(Q+\tau+\alpha_4-\alpha_t)S_{\rm NS}(\tau+\alpha_4+\alpha_t)S_{\rm NS}(Q+\tau+\alpha_2-\alpha_s)S_{\rm NS}(\tau+\alpha_2+\alpha_s)}$$
$$+\ \frac{S_R(Q+\tau-\alpha_1)S_R(\tau+\alpha_4+\alpha_2-\alpha_3)S_R(\tau+\alpha_1)S_R(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_R(Q+\tau+\alpha_4-\alpha_t)S_R(\tau+\alpha_4+\alpha_t)S_R(Q+\tau+\alpha_2-\alpha_s)S_R(\tau+\alpha_2+\alpha_s)},$$

where $S_{NS,R}$ are defined by eq.(6.41). In terms of explicit form of $F_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]$ in eq.(2.95), we can show that

$$F_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\bar\alpha & \alpha\\ \alpha & \bar\alpha\end{array}\right]_1^1 = F_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha & \alpha\\ \alpha & \alpha\end{array}\right]_1^1. \tag{2.97}$$

Alternatively, it can be shown that for $\alpha_s = 0$, the fusion matrix becomes

$$F_{0,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_1\\ \alpha_3 & \alpha_1\end{array}\right]_1^1 = C_{\rm NS}(\alpha_t,\alpha_1,\alpha_3)\frac{W_{\rm NS}(Q)W_{\rm NS}(\alpha_t)}{\pi W_{\rm NS}(Q-\alpha_1)W_{\rm NS}(Q-\alpha_3)}. \tag{2.98}$$

Where $W_{\rm NS}$ is defined in (6.45) and (6.46). It is not difficult to see that near $\alpha_t \sim 0$ (and $\alpha_1 = \alpha_3 = \alpha$), the DOZZ function has a single pole

$$\begin{aligned} C_{\rm NS}(\alpha_t,\alpha,\alpha) &\sim \lambda^{(Q-2\alpha)/b}\frac{\Upsilon'_{\rm NS}(0)\Upsilon_{\rm NS}(2\alpha_t)\Upsilon_{\rm NS}(2\alpha)}{\Upsilon_{\rm NS}(2\alpha-Q)\Upsilon_{\rm NS}(\alpha_t)^2},\\ &\sim \frac{\Upsilon'_{\rm NS}(0)\Upsilon_{\rm NS}(2\alpha_t)W_{\rm NS}(Q-\alpha)}{W_{\rm NS}(\alpha)\Upsilon^2_{\rm NS}(\alpha_t)} \sim \frac{2W_{\rm NS}(Q-\alpha)}{W_{\rm NS}(\alpha)\pi\alpha_t}, \end{aligned} \tag{2.99}$$

where in the second line we use

$$\Gamma_{\rm NS}(x) \sim \frac{\Gamma_{\rm NS}(Q)}{\pi x} \Rightarrow \Upsilon_{\rm NS}(x) \sim \frac{\pi x}{\Gamma^2_{\rm NS}(Q)}\,. \tag{2.100}$$

In the second line of (2.99) we use

$$\frac{\Upsilon_{\rm NS}(2x)}{\Upsilon_{\rm NS}(2x-Q)} = \mathcal{G}_{\rm NS}(x)\lambda^{-\frac{Q-2x}{b}} = \frac{W_{\rm NS}(Q-x)}{W_{\rm NS}(x)}\lambda^{-\frac{Q-2x}{b}}\,. \tag{2.101}$$

We also need the values of the derivative $\Upsilon'_{\rm NS}(0)$

$$\Upsilon'_{\rm NS}(0) = \frac{\pi}{\Gamma^2_{\rm NS}(Q)}\,. \tag{2.102}$$

We have made use of definition of $\Gamma_{\rm NS}, \Gamma_{\rm R}$ given in eq.(6.41). Substituting (2.99) back into (2.98) and with the help of the following identity

$$W_{\rm NS}(x)W_{\rm NS}(Q-x) = -4\sin\pi b(x-Q/2)\sin\pi\frac{1}{b}(x-Q/2)\,, \tag{2.103}$$

we obtain the pole structure of the fusion matrix

$$F_{0,\alpha_t}\begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix}^1_1 = \frac{2}{\pi^2\alpha_t}\frac{\sin\frac{\pi}{2}bQ\sin\frac{\pi}{2}b^{-1}Q}{\sin\pi b(\alpha-Q/2)\sin\pi b^{-1}(\alpha-Q/2)}\,. \tag{2.104}$$

So we can again relate the entanglement entropy due to the the local operator to its quantum dimension

$$\mathrm{Res}_{\alpha_t=0}F_{00} \sim \sin\pi b(\alpha-Q/2)\sin\pi b^{-1}(\alpha-Q/2)\,. \tag{2.105}$$

Similar to the situation in LFT, fusion matrix element for $F^{SL}_{Q/2,Q/2}[^{\alpha\alpha}_{\alpha\alpha}]^e_e$ presented in 2nd REE in SLFT can not be identify as quantum dimension.

3 n-th Rényi Entropy in Irrational CFTs

Here we give a sketch of the proof of (2.45) following [9] which is similar to procedure in rational CFTs with slight modification. First we define the following matrix elements F_{nm} (similar to the F matrix in eq.(2.10) in [49]) by

$$F(\alpha_s|1-z) = \int_{\alpha_t} F_{\alpha_s\alpha_t}\cdot F(\alpha_t|z), \tag{3.1}$$

where $F(\alpha|z)$ is the conformal block for the four point function $\langle V_{\bar{\alpha}}(z_1,\bar{z}_1)\cdots V_\alpha(z_4,\bar{z}_4)\rangle$ with the intermediate operator V_n. One should note that the fusion matrix is infinite dimensional which is different from the matrix in rational CFTs. One can extract the dominant contribution to difference of REE in $F^{L,SL}_{\alpha_s=Q/2,\alpha_t=Q/2}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]$ in the late time limit and the details have been given by section 2 in LFT and SLFT. In the late time limit, this transformation $z \to 1-z$ is equivalent to the exchange of z_2 and z_4.

We can define F_α as the F matrix with α_s and α_t taken to be those that give the dominant contribution in the late time limit:

$$
\begin{aligned}
F_\alpha \quad &\sim \quad F^{L,SL}_{\alpha_s=Q/2,\alpha_t=Q/2}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right] \\
&\times \begin{cases} (1-z)^{\Delta_{Q/2}}\bar{z}^{\Delta_{Q/2}} \ln^{-3/2}\left(\frac{1}{(1-z)}\right)\ln^{-3/2}\left(\frac{1}{\bar{z}}\right) & \alpha \in \{Q/2+ip|p\in\mathbb{R}\}, \\ (1-z)^{2\Delta_{Q/2}}\bar{z}^{2\Delta_{Q/2}} \ln^{-3/2}\left(\frac{1}{(1-z)}\right)\ln^{-3/2}\left(\frac{1}{\bar{z}}\right) & \{Q/2>\mathrm{Re}(\alpha)>Q/4\}\bigcup\{Q/4>\mathrm{Re}(\alpha)>0\}, \\ (1-z)^{\Delta_{Q/2}}\bar{z}^{\Delta_{Q/2}} \ln^{-\frac{1}{2}}\left(\frac{1}{(1-z)}\right)\ln^{-\frac{3}{2}}\left(\frac{1}{\bar{z}}\right) & \mathrm{Re}(\alpha)=Q/4, \mathrm{Im}(\alpha)\neq 0. \end{cases}
\end{aligned} \tag{3.2}
$$

where $\sim$ denotes the normalization factors of two point functions and the factors associated with the structure constants $C(\alpha_1,\alpha_2,\alpha_3)$ and $C_{NS}(\alpha_1,\alpha_2,\alpha_3)$ in LFT and SLFT respectively[4].

The n-th Rényi entanglement entropy can be obtained from the formula (2.16). We find

$$
\begin{aligned}
&\langle V_{\bar{\alpha}}(w_1,\bar{w}_1)V_\alpha(w_2,\bar{w}_2)\cdots V_\alpha(w_{2n},\bar{w}_{2n})\rangle_{\Sigma_n} \\
&= n^{-4n\Delta}\cdot(rs)^{-2(n-1)\Delta}\cdot\langle V_{\bar{\alpha}}(z_1,\bar{z}_1)V_\alpha(z_2,\bar{z}_2)\cdots V_\alpha(z_{2n},\bar{z}_{2n})\rangle_{\Sigma_1},
\end{aligned} \tag{3.3}
$$

where we defined

$$
|z_{2k+1}|^n = r, \quad |z_{2k+2}|^n = s. \tag{3.4}
$$

We normalize the two point function such that $\mathcal{N}=1$ or equally

$$
\langle V_{\bar{\alpha}}(w_1,\bar{w}_1)V_\alpha(w_2,\bar{w}_2)\rangle_{\Sigma_1} = \frac{1}{|w_{12}|^{4\Delta}} = \frac{1}{(2\epsilon)^{4\Delta}}. \tag{3.5}
$$

Then we get

$$
\begin{aligned}
&\frac{\langle V_{\bar{\alpha}}(w_1,\bar{w}_1)V_\alpha(w_2,\bar{w}_2)\cdots V_\alpha(w_{2n},\bar{w}_{2n})\rangle_{\Sigma_n}}{\left(\langle V_{\bar{\alpha}}(w_1,\bar{w}_1)V_\alpha(w_2,\bar{w}_2)\rangle_{\Sigma_1}\right)^n} \\
&= \left(\frac{2\epsilon}{n}\right)^{4\Delta n}\cdot(rs)^{-2(n-1)\Delta}\cdot\langle V_{\bar{\alpha}}(z_1,\bar{z}_1)V_\alpha(z_2,\bar{z}_2)\cdots V_\alpha(z_{2n},\bar{z}_{2n})\rangle_{\Sigma_1} \\
&\to \left(\frac{2\epsilon}{nt^{\frac{n-1}{n}}}\right)^{4\Delta n}\cdot\langle V_\alpha(z_1,\bar{z}_1)V_\alpha(z_2,\bar{z}_2)\cdots V_\alpha(z_{2n},\bar{z}_{2n})\rangle_{\Sigma_1},
\end{aligned} \tag{3.6}
$$

where we took the late time limit in the final expression.

[4]Refer to eq.(2.43)eq.(2.47)eq.(2.50) in LFT and eq.(2.83)eq.(2.87)eq.(2.91) in SLFT. All these factors are divergent in late time limit, namely $(z,\bar{z})\to(1,0)$.

The $2n$ points $z_1, z_2, \cdots, z_n$ in the z coordinate are given by

$$\begin{aligned}
z_{2k+1} &= e^{2\pi i\frac{k}{n}}(i\epsilon + t - l)^{\frac{1}{n}} = e^{2\pi i\frac{k+1/2}{n}}(l - t - i\epsilon)^{\frac{1}{n}} \\
z_{2k+2} &= e^{2\pi i\frac{k}{n}}(-i\epsilon + t - l)^{\frac{1}{n}} = e^{2\pi i\frac{k+1/2}{n}}(l - t + i\epsilon)^{\frac{1}{n}}, \\
\bar{z}_{2k+1} &= e^{-2\pi i\frac{k}{n}}(-i\epsilon - t - l)^{\frac{1}{n}} = e^{-2\pi i\frac{k+1/2}{n}}(l + t + i\epsilon)^{\frac{1}{n}} \\
\bar{z}_{2k+2} &= e^{-2\pi i\frac{k}{n}}(i\epsilon - t - l)^{\frac{1}{n}} = e^{-2\pi i\frac{k+1/2}{n}}(l + t - i\epsilon)^{\frac{1}{n}}.
\end{aligned} \tag{3.7}$$

In the early time limit $t \ll l$ we find

$$z_{2k+1} \to z_{2k+2}, \quad \bar{z}_{2k+1} \to \bar{z}_{2k+2}, \tag{3.8}$$

for all k. On the other hand, if we take the late time limit $t \gg l$, we find the asymmetric limit:

$$z_{2k+1} \to z_{2k+4}, \quad \bar{z}_{2k+1} \to \bar{z}_{2k+2}, \tag{3.9}$$

for all k.

If we regard the $2n$ point functions as n products of two point functions (3.5) in the late time limit, we have

$$\begin{aligned}
&\langle V_{\bar{\alpha}}(z_1, \bar{z}_1) \cdots V_{\alpha}(z_n, \bar{z}_n)\rangle_{\Sigma_1} \\
&\to \langle V_{\bar{\alpha}}(z_1, \bar{z}_1)V_{\alpha}(z_2, \bar{z}_2)\rangle_{\Sigma_1} \otimes \langle V_{\bar{\alpha}}(z_3, \bar{z}_3)V_{\alpha}(z_4, \bar{z}_4)\rangle_{\Sigma_1} \otimes \cdots \otimes \langle V_{\bar{\alpha}}(z_{2n-1}, \bar{z}_{2n-1})V_{\alpha}(z_{2n}, \bar{z}_{2n})\rangle_{\Sigma_1},
\end{aligned} \tag{3.10}$$

which respects the limit in the anti holomorphic sectors and $\otimes$ denotes that we just consider the dominant contribution fusion matrix with divergent factor defined by eq.(3.2). In the holomorphic sector we would like to take the late time limit. For this aim, we need to exchange some of z_is with z_js as

$$(z_1, z_2)(z_3, z_4) \cdots (z_{2n-1}, z_{2n}) \to (z_1, z_4)(z_3, z_6) \cdots (z_{2n-1}, z_2). \tag{3.11}$$

This transformation is realized by acting the F-transformation $n-1$ times.

We can estimate the difference in the late time limit as follows

$$\begin{aligned}
z_{2k+1} - z_{2k+4} &\simeq -e^{2\pi i\frac{k+1/2}{n}} \cdot \frac{2i\epsilon t^{(1/n-1)}}{n}. \\
\bar{z}_{2k+1} - \bar{z}_{2k+2} &\simeq e^{-2\pi i\frac{k+1/2}{n}} \cdot \frac{2i\epsilon t^{(1/n-1)}}{n}.
\end{aligned} \tag{3.12}$$

Their absolute values are all the same and this is given by

$$\delta \equiv \frac{2\epsilon}{nt^{\frac{n-1}{n}}}. \tag{3.13}$$

Thus we can estimate the $2n$ point function in the late time limit as follows

$$\begin{aligned}&\langle V_{\tilde{\alpha}}(z_1,\bar{z}_1)V_{\alpha}(z_2,\bar{z}_2)\cdots V_{\alpha}(z_{2n},\bar{z}_{2n})\rangle_{\Sigma_1}\\&\simeq (F_{\alpha})^{n-1}\cdot\left[\lim_{|z_{2k+1}-z_{2k+2}|=\delta\to 0}\langle V_{\tilde{\alpha}}(z_1,\bar{z}_1)V_{\alpha}(z_2,\bar{z}_2)\cdots V_{\alpha}(z_{2n},\bar{z}_{2n})\rangle_{\Sigma_1}\right]\\&\simeq (F_{\alpha})^{n-1}\cdot\delta^{-4\Delta n}.\end{aligned}\tag{3.14}$$

$$\frac{\langle V_{\tilde{\alpha}}(w_1,\bar{w}_1)V_{\alpha}(w_2,\bar{w}_2)\cdots V_{\alpha}(w_{2n},\bar{w}_{2n})\rangle_{\Sigma_n}}{\left(\langle V_{\tilde{\alpha}}(w_1,\bar{w}_1)V_{\alpha}(w_2,\bar{w}_2)\rangle_{\Sigma_1}\right)^n}=(F_{\alpha})^{n-1}.\tag{3.15}$$

The late time of 2nd REE is

$$\begin{aligned}&\Delta S_A^{(n)}[V_{\alpha}|0\rangle,V_{\alpha_r}|0\rangle](t\to\infty)\\&=\begin{cases}-\log\left(\dfrac{f_{\alpha}''(p)F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\tilde{\alpha}&\alpha\\\alpha&\tilde{\alpha}\end{smallmatrix}\right]}{f_{\alpha_r}''(p)F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\tilde{\alpha}_r&\alpha_r\\\alpha_r&\tilde{\alpha}_r\end{smallmatrix}\right]}\right)\Bigg|_{p\to 0} & \alpha\in\{Q/2+ip|p\in\mathbb{R}\},\\-\log\left(\dfrac{f_{\alpha}''(p)F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\tilde{\alpha}&\alpha\\\alpha&\tilde{\alpha}\end{smallmatrix}\right]}{f_{\alpha_r}''(p)F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\tilde{\alpha}_r&\alpha_r\\\alpha_r&\tilde{\alpha}_r\end{smallmatrix}\right]}\right)\Bigg|_{p\to 0} & \{Q/2>\mathrm{Re}(\alpha)>Q/4\}\bigcup\{Q/4>\mathrm{Re}(\alpha)>0\},\\-\log\left(\dfrac{f_{\alpha}(p)F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\tilde{\alpha}&\alpha\\\alpha&\tilde{\alpha}\end{smallmatrix}\right]}{f_{\alpha_r}(p)F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\tilde{\alpha}_r&\alpha_r\\\alpha_r&\tilde{\alpha}_r\end{smallmatrix}\right]}\right)\Bigg|_{p\to 0} & \mathrm{Re}(\alpha)=Q/4,\mathrm{Im}(\alpha)\neq 0.\end{cases}\end{aligned}\tag{3.16}$$

Then the difference of n-th REE between early time and late time limit is

$$\begin{aligned}&\Delta S_A^{(n)}[V_{\alpha}|0\rangle,V_{\alpha_r}|0\rangle](t\to\infty)-\Delta S_A^{(n)}[V_{\alpha}|0\rangle,V_{\alpha_r}|0\rangle](t\to 0)\\&=\begin{cases}-\log\left(\dfrac{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\tilde{\alpha}&\alpha\\\alpha&\tilde{\alpha}\end{smallmatrix}\right]}{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\tilde{\alpha}_r&\alpha_r\\\alpha_r&\tilde{\alpha}_r\end{smallmatrix}\right]}\right)\Bigg|_{p\to 0} & \alpha\in\{Q/2+ip|p\in\mathbb{R}\},\\-\log\left(\dfrac{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\tilde{\alpha}&\alpha\\\alpha&\tilde{\alpha}\end{smallmatrix}\right]}{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\tilde{\alpha}_r&\alpha_r\\\alpha_r&\tilde{\alpha}_r\end{smallmatrix}\right]}\right)\Bigg|_{p\to 0} & \{Q/2>\mathrm{Re}(\alpha)>Q/4\}\bigcup\{Q/4>\mathrm{Re}(\alpha)>0\},\\-\log\left(\dfrac{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\tilde{\alpha}&\alpha\\\alpha&\tilde{\alpha}\end{smallmatrix}\right]}{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\tilde{\alpha}_r&\alpha_r\\\alpha_r&\tilde{\alpha}_r\end{smallmatrix}\right]}\right)\Bigg|_{p\to 0} & \mathrm{Re}(\alpha)=Q/4,\mathrm{Im}(\alpha)\neq 0.\end{cases}\end{aligned}\tag{3.17}$$

Here we have restored the structure constants in F_{α} defined by eq.(3.2).

4 n-th REE for Generic Descendent States

In this section, we would like to extend to generic descendent states in LFT following [34]. In the late time limit, the authors of [34] show that the difference of the n-th REE only depends on the most singular term in the two-point function and the $2n$-point function and nothing will change from including the operators with smaller conformal dimensions. They define the following descendent operator

$$\tilde{V}(w,\bar{w})=L^{(-)}V_{\alpha}(w,\bar{w}),\tag{4.1}$$

where $L^{(-)}$ is a complicated product form of holomorphic generators, $V_\alpha(w,\bar{w})$ is a primary operator of conformal dimension h and $\tilde{V}(w,\bar{w})$ is a quasi-primary operator. The operator $L^{(-)}$ has a fixed conformal dimension m

$$[L_0, L^{(-)}] = mL^{(-)}. \tag{4.2}$$

The conformal transformation for the descendant operators can be derived from energy momentum tensor and the conformal transformation from z plane to w plane is

$$L_{-m}\mid_{w_1} \;=\; L_{-m}\mid_{z_1}\left(\frac{\partial z_1}{\partial w_1}\right)^n + ... \tag{4.3}$$

where the ellipsis denotes the terms of lower conformal dimensions leading to less divergent terms in the limit $\epsilon \to 0$. By the conformal transformation, the four-point function transforms as follows

$$\begin{aligned}
&\langle \tilde{V}^+(w_1,\bar{w}_1)\tilde{V}(w_1^{'},\bar{w}_1^{'})\tilde{V}^+(w_2,\bar{w}_2)\tilde{V}(w_2^{'},\bar{w}_2^{'})\rangle \\
= &\left(\frac{\partial w_1}{\partial z_1}\right)^{h+m}\left(\frac{\partial \bar{w}_1}{\partial \bar{z}_1}\right)^{h+\bar{m}}\left(\frac{\partial w_2}{\partial z_2}\right)^{h+m}\left(\frac{\partial \bar{w}_2}{\partial \bar{z}_2}\right)^{h+\bar{m}}\left(\frac{\partial w_1^{'}}{\partial z_1^{'}}\right)^{h+m}\left(\frac{\partial \bar{w}_1^{'}}{\partial \bar{z}_1^{'}}\right)^{h+\bar{m}}\left(\frac{\partial w_2^{'}}{\partial z_2^{'}}\right)^{h+m}\left(\frac{\partial \bar{w}_2^{'}}{\partial \bar{z}_2^{'}}\right)^{h+\bar{m}} \\
&\langle \tilde{V}^+(z_1,\bar{z}_1)\tilde{V}(z_1^{'},\bar{z}_1^{'})\tilde{V}^+(z_2,\bar{z}_2)\tilde{V}(z_2^{'},\bar{z}_2^{'})\rangle + \text{less divergent terms.}
\end{aligned} \tag{4.4}$$

The coefficient for the leading term is the same as the one for the primary operator. The terms with lower conformal dimensions do not change the final result.

The two-point function for V can be expressed as follows

$$\langle (L^{(-)}V_\alpha)^+(z), L^{(-)}V_\alpha(z^{'})\rangle = \frac{(-1)^m\langle h \mid L^{(-)\dagger}L^{(-)} \mid h\rangle}{(z-z^{'})^{2(h+m)}}. \tag{4.5}$$

In the late time limit, $(z_1, z_2^{'})$ $(z_2, z_1^{'})$ $(\bar{z}_1, \bar{z}_1^{'})$ $(\bar{z}_2, \bar{z}_2^{'})$ approach each other. The four-point correlation function of V can be transformed into

$$\begin{aligned}
&\langle \tilde{V}^+(z_1,\bar{z}_1)\tilde{V}(z_1^{'},\bar{z}_1^{'})\tilde{V}^+(z_2,\bar{z}_2)\tilde{V}(z_2^{'},\bar{z}_2^{'})\rangle \\
= \;& \mathcal{D}\langle V_{\bar{\alpha}}(z_1,\bar{z}_1)V_\alpha(z_1^{'},\bar{z}_1^{'})V_{\bar{\alpha}}(z_2,\bar{z}_2)V_\alpha(z_2^{'},\bar{z}_2^{'})\rangle \\
= \;& \mathcal{D}\int_m^{'} c_m\langle V_{\bar{\alpha}}(z_1)V_\alpha(z_1^{'})\mid_m V_{\bar{\alpha}}(z_2)V_\alpha(z_2^{'})\rangle\langle V_{\bar{\alpha}}(\bar{z}_1)V_\alpha(\bar{z}_1^{'})\mid_m V_{\bar{\alpha}}(\bar{z}_2)V_\alpha(\bar{z}_2^{'})\rangle \\
= \;& \mathcal{D}\int_{m,n}^{'} c_{m,n}\langle V_{\bar{\alpha}}(z_1)V_\alpha(z_2^{'})\mid_m V_{\bar{\alpha}}(z_2)V_\alpha(z_1^{'})\rangle\langle V_{\bar{\alpha}}(\bar{z}_1)V_\alpha(\bar{z}_1^{'})\mid_n V_{\bar{\alpha}}(\bar{z}_2)V_\alpha(\bar{z}_2^{'})\rangle \\
= \;& \int_{m,n}^{'} c_{m,n}\langle L^{(-)}V_\alpha^+(z_1)L^{(-)}V_\alpha(z_2^{'})\mid_m L^{(-)}V_\alpha^+(z_2)L^{(-)}V_\alpha(z_1^{'})\rangle\langle V_{\bar{\alpha}}(\bar{z}_1)V_\alpha(\bar{z}_1^{'})\mid_n V_{\bar{\alpha}}(\bar{z}_2)V_\alpha(\bar{z}_2^{'})\rangle.
\end{aligned} \tag{4.6}$$

Here $\langle V_{\bar{\alpha}}(z_1)V_\alpha(z_1^{'}) \mid_m V_{\bar{\alpha}}(z_2)V_\alpha(z_2^{'})\rangle$ denotes the conformal block expansion with the Virasoro module $[m]$ which is of continuous spectrum in Liouville field theory and $[m]$ contains the dominant contribution satisfying the fusion rule[5]. This is the main difference from the rational CFTs. The correlation function of four descendants has been transformed into the differential operator $\mathcal{D}$ acting on the correlation function of corresponding primaries in the first equality. In the second equality, the partition function is expanded by the conformal blocks and here c_m denote the OPE coefficients. In the third equality, the holomorphic part can be expressed by t-channel like [9] in the late time limit. In the fourth equality, we pull the differential operator back into the Virasoro operators acting on the primaries in the correlation function. In LFT and SLFT, we have shown that the dominant contribution to REE comes from the discrete terms in primed contour integration over m.

In section 2 and section 3, we have already found that the most divergent term only comes from the one with $m = n =$ lower bound in fusion channel which depends on the external legs and $c_{\mathrm{Min}(\alpha_s),\mathrm{Min}(\alpha_t)} = \left(F^{L}_{\mathrm{Min}(\alpha_s),\mathrm{Min}(\alpha_t)}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]\right)^{-1} = \frac{1}{F^{L,SL}_{\alpha}}$ [6] which is much similar to the one in rational CFTs [9]. We use upper index L, SL to distinguish the quantities in LFT and SLFT. Actually even in the vacuum block, only the identity operator gives the most divergent term,

$$\begin{aligned}&\langle \tilde{V}^{+}(z_1,\bar{z}_1)\tilde{V}(z_1^{'},\bar{z}_1^{'})\tilde{V}^{+}(z_2,\bar{z}_2)\tilde{V}(z_2^{'},\bar{z}_2^{'})\rangle\\ =\;&\frac{1}{F^{L,SL}_{\mathrm{Min}(\alpha_s),\mathrm{Min}(\alpha_t)}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]}\langle L^{(-)}V_\alpha^{+}(z_1)L^{(-)}V_\alpha(z_2^{'})\rangle\otimes\langle L^{(-)}V_\alpha^{+}(z_2)L^{(-)}V_\alpha(z_1^{'})\rangle\otimes\langle V_{\bar{\alpha}}(\bar{z}_1)V_\alpha(\bar{z}_1^{'})\rangle\\ &\otimes\langle V_{\bar{\alpha}}(\bar{z}_2)V_\alpha(\bar{z}_2^{'})\rangle + \text{less divergent terms.}\end{aligned}\tag{4.7}$$

So the four-point function in w-coordinate keeping the most divergent term is

$$\begin{aligned}&\langle \tilde{V}^{+}(w_1,\bar{w}_1)\tilde{V}(w_1^{'},\bar{w}_1^{'})\tilde{V}^{+}(w_2,\bar{w}_2)\tilde{V}(w_2^{'},\bar{w}_2^{'})\rangle\\ =\;&\frac{1}{F^{L,SL}_{\mathrm{Min}(\alpha_s),\mathrm{Min}(\alpha_t)}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]}\langle L^{(-)}V_\alpha^{+}(w_1)L^{(-)}V_\alpha(w_2^{'})\rangle\otimes\langle L^{(-)}V_\alpha^{+}(w_2)L^{(-)}V_\alpha(w_1^{'})\rangle\otimes\langle V_{\bar{\alpha}}(\bar{w}_1)V_\alpha(\bar{w}_1^{'})\rangle\\ &\otimes\langle V_{\bar{\alpha}}(\bar{w}_2)V_\alpha(\bar{w}_2^{'})\rangle + \text{less divergent terms.}\end{aligned}\tag{4.8}$$

Therefore, for a quasi-primary operator we still have $S^{(2)}[\tilde{V}|0\rangle] = -\log F^{L,SL}_{\mathrm{Min}(\alpha_s),\mathrm{Min}(\alpha_t)}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]$ in LFT.

Finally, we just present the main results with the more generic descendent operators like

$$\tilde{V} = \sum_{m,j,r,k} d_{m,j,r,k}\partial^m L^{(-,j)}\;\bar{\partial}^r\bar{L}^{(-,k)}V_\alpha(z,\bar{z})\tag{4.9}$$

[5] Due the fusion rule in LFT or SLFT, $[m]$ is not vacuum module which does not belong to the spectrum of lFT or SLFT.

[6] Here the $F^{L,SL}_{\alpha}$ contains the divergent piece shown in eq.(3.2).

Here j and k denote the quasi-primary operators and V_α is primary operator.

We do not repeat the calculation in details here which has been studied carefully in [34]. The differences have been already presented in eq.(4.6). Here we just give the final answer as follows

$$
\begin{aligned}
S_n[\tilde{V}|0\rangle](t\to\infty) &= \log F^L_{\mathrm{Min}(\alpha_s),\mathrm{Min}(\alpha_t)}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right] - \frac{1}{n-1}\log \mathrm{Tr}\rho_0^n \\
&= S_n^{\text{primary}} - \frac{1}{n-1}\log \mathrm{Tr}\rho_0^n,
\end{aligned} \tag{4.10}
$$

with the normalized density matrix $\rho_0 = \frac{\rho}{\mathrm{Tr}\rho}$, where

$$
S_n^{\text{primary}}[V_\alpha|0\rangle](t\to\infty) = -\log F^L_{\mathrm{Min}(\alpha_s),\mathrm{Min}(\alpha_t)}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right] \tag{4.11}
$$

is the quantum entanglement of V_α and it is divergent as shown in previous sections. The density matrix is defined following

$$
\rho = BMB^\dagger M^\dagger, \tag{4.12}
$$

and these matrices are associated with coefficients in eq.(4.9)

$$
\begin{aligned}
B_{\{m,j\},\{r,k\}} &= d^*_{m,j,r,k}, \\
M_{\{m,j\},\{r,k\}} &= \langle h \mid L^{(-,j)\dagger} L^{(-,j)} \mid h\rangle \delta_{j,k} i^{r-m},
\end{aligned} \tag{4.13}
$$

As we see from eq.(4.10), ΔS_n has similar structure in LFT and SLFT with the one in rational CFTs. ΔS_n has two main contributions. The first one contains the universal part depending on the fusion matrix element of the corresponding primary operator and it is divergent due to the divergent piece in the first term. The other comes from the normalization scheme of local descendent operator. We also see that

$$
\begin{aligned}
&\Delta S^{(n)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t\to\infty) - \Delta S^{(n)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t\to 0) = \\
&\begin{cases}
-\log\left(\frac{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]}{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar{\alpha}_r & \alpha_r\\ \alpha_r & \bar{\alpha}_r\end{smallmatrix}\right]}\right)\Big|_{p\to 0} - \frac{1}{n-1}\log\frac{tr\rho_0^n}{tr\tilde{\rho}_0^n} & \alpha\in\{Q/2+ip|p\in\mathbb{R}\}, \\
-\log\left(\frac{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]}{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar{\alpha}_r & \alpha_r\\ \alpha_r & \bar{\alpha}_r\end{smallmatrix}\right]}\right)\Big|_{p\to 0} - \frac{1}{n-1}\log\frac{tr\rho_0^n}{tr\tilde{\rho}_0^n} & \{Q/2>\mathrm{Re}(\alpha)>Q/4\}\bigcup\{Q/4>\mathrm{Re}(\alpha)>0\}, \\
-\log\left(\frac{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]}{F^L_{Q/2,Q/2}\left[\begin{smallmatrix}\bar{\alpha}_r & \alpha_r\\ \alpha_r & \bar{\alpha}_r\end{smallmatrix}\right]}\right)\Big|_{p\to 0} - \frac{1}{n-1}\log\frac{tr\rho_0^n}{tr\tilde{\rho}_0^n} & \mathrm{Re}(\alpha)=Q/4, \mathrm{Im}(\alpha)\neq 0.
\end{cases}
\end{aligned} \tag{4.14}
$$

where we choose $\tilde{\rho}_0$ defined by the reference state (4.9) associated with primary operator V_{α_r}.

5 Conclusions and Discussions

In this paper, we would like to study the time evolution of the difference in Rényi entropy $\Delta S^{(n)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t)$ between locally excited states $V_\alpha|0\rangle$ and the reference states $V_{\alpha_r}|0\rangle$ in $1+1$ dimensional irrational CFT, especially in Liouville field theory and super Liouville field theory. In rational CFTs, there are a finite number of primary operators and one needs to do the finite summation to extract the difference of REE shown in [9]. However, in irrational CFTs, there are infinite dimensional spectrums which are also highly degenerate. One might doubt the differences of REE in irrational CFTs have very different structures compared with that [9] in rational CFTs.

To answer this question, we especially choose LFT and SLFT as playgrounds to calculate the difference of REE $\Delta S^{(n)}_{EE}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t)$ between two excited states, e.g. $V_{\alpha_r}|0\rangle$ and $V_{\alpha_r}|0\rangle$. Furthermore, there is no well studied holographic dual of local primary operator in integrable rational CFTs [50] and we can not make use AdS/CFT to calculate REE in this case. The conformal field theory with holographic dual can be probed in the large central charge limit. Although the holographic dual of LFT or SLFT is not clear. We can study time evolution of REE $S^{(n)}_{EE}[V_\alpha|0\rangle]$ by replica trick in LFT and SLFT and we might get some properties of large c CFTs.

To understand these properties, we calculate the 2nd REE $S^{(2)}_{EE}[V_\alpha|0\rangle]$ of local excited states in LFT and SLFT. For a state excited by a local primary operator V_α, we can show the REE is divergent both in the early time and the late time limit. The divergent behaviors of REE $S^{(2)}_{EE}[V_\alpha|0\rangle](t)$ in the early time and late time are different, which contradicts with the quasi-particle picture proposed in rational CFTs [6][9]. That also means that $S^{(2)}_{EE}[V_\alpha|0\rangle](t\to\infty) - S^{(2)}_{EE}[V_\alpha|0\rangle](t\to 0)$ is divergent [7]. Because the identity operator does not live in the Hilbert space of LFT and SLFT, the vacuum block does not make contribution to REE in LFT or SLFT. That is the main reason leading to the different divergent behavior of REE in the early and late time limit. To define finite quantities, e.g. $\Delta S^{(n)}_{A}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t)$, we have to classify all excited states in LFT and SLFT. In this paper, we analyze the zero point of structure constant (DOZZ formula) presented in 2nd-REE to classify the primary operators in LFT and SLFT. In LFT and SLFT, the primary operators have been divided into three classes in terms of real part of Liouville momentum α, e.g. $\alpha \in \{Q/2 + ip|p \in \mathbb{R}\}$, $\{Q/2 > \mathrm{Re}(\alpha) > Q/4\}\bigcup\{Q/4 > \mathrm{Re}(\alpha) > 0\}$ and $\mathrm{Re}(\alpha) = Q/4, \mathrm{Im}(\alpha) \neq 0$. Due to the fact that the 2nd REE of excited states are divergent, we have to choose a proper reference state $V_{\alpha_r}|0\rangle$ which lives in the same class with target states $V_\alpha|0\rangle$. We can show that the difference $\Delta S^{(2)}_{A}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t)$ of 2nd REE between target states $V_\alpha|0\rangle$ and reference states $V_{\alpha_r}|0\rangle$ will be finite in early time and late limit. We can study the time evolution behavior of $\Delta S^{(2)}_{A}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t)$. The difference of the REE $\Delta S^{(2)}_{A}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle]$ between in early time limit and in late time limit always coin-

[7] Equivalently, $\Delta S^{(n)}_{A}[V_\alpha|0\rangle, 1|0\rangle](t\to\infty) - \Delta S^{(n)}_{A}[V_\alpha|0\rangle, 1|0\rangle](t\to 0)$ is divergent.

cides with the log of ratio $\frac{F_{\alpha_s=Q/2,\alpha_t=Q/2}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]}{F_{\alpha_s=Q/2,\alpha_t=Q/2}\left[\begin{smallmatrix}\bar{\alpha}_r & \alpha_r\\ \alpha_r & \bar{\alpha}_r\end{smallmatrix}\right]}$ of fusion matrix element between two excited states e.g. $V_\alpha|0\rangle, V_{\alpha_r}|0\rangle$. The precise difference has been listed in the eq.(4.14). Following these logic, we can extend this analysis to generic n-th REE following [9]. The difference $\Delta S_A^{(n)}[V_\alpha|0\rangle, V_{\alpha_r}|0\rangle](t)$ between the early and late time is independent on n and it still contains log of the ratio of fusion matrix element $\frac{F_{\alpha_s=Q/2,\alpha_t=Q/2}\left[\begin{smallmatrix}\bar{\alpha} & \alpha\\ \alpha & \bar{\alpha}\end{smallmatrix}\right]}{F_{\alpha_s=Q/2,\alpha_t=Q/2}\left[\begin{smallmatrix}\bar{\alpha}_r & \alpha_r\\ \alpha_r & \bar{\alpha}_r\end{smallmatrix}\right]}$ between two excited states. Furthermore, we also study the generic descendent states following [15]. Comparing with the case of primary states, the difference $\Delta S_A^{(n)}[\tilde{V}_\alpha|0\rangle, \tilde{V}_{\alpha_r}|0\rangle](t)$ about descendants states $\tilde{V}_\alpha|0\rangle, \tilde{V}_{\alpha_r}|0\rangle$ in LFT or SLFT now contains one more additional term which is associated with normalization factor of descendent operator. The difference $\Delta S_A^{(n)}[\tilde{V}_\alpha|0\rangle, \tilde{V}_{\alpha_r}|0\rangle](t)$ between early and late time limit contains log of the ratio of fusion matrix element of local operators and an additional normalization factor.

One can use our techniques to calculate the out of time ordered correlation function to check whether the Liouville field theory has chaotic behavior or not [51]. Recently, the authors of [52] proposed there is correspondence between Liouville field theory and one dimensional conformal quantum mechanics SYK model. One can compare the OTOC in Liouville with the late time behavior of two point function of bi-local operators in SYK and then see check the correspondence [52] between LFT and SYK. Recently, authors of [53][54][55] proposed that Liouville field theory action as optimization of complexity of static states in conformal field theory. Also the associated measurements of complexity in generic field theory and holographic aspects have also been proposed in [56][57] and [58][59] [60] [61][62] respectively. We would like to extend to study complexity of the local excited state in conformal field theory to define the optimization procedure. Hopefully, we can report some progresses in these directions in the near future.

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6 Appendix

6.1 Brief review of LFT

The full Liouville action (See review [42]) is

$$S_L = \frac{1}{4\pi}\int d^2\xi\,\sqrt{g}\left[\partial_a\phi\partial_b\phi g^{ab} + QR\phi + 4\pi\mu e^{2b\phi}\right], \tag{6.1}$$

where $Q = b + \frac{1}{b}$. The conformal dimension of corresponding primary operator $e^{2\alpha\phi}$ is

$$\Delta(e^{2\alpha\phi}) = \bar{\Delta}(e^{2\alpha\phi}) = \alpha(Q-\alpha), \tag{6.2}$$

The stress tensor is

$$T(z) = -(\partial\phi)^2 + Q\partial^2\phi, \tag{6.3}$$

and the central charge of the conformal algebra is

$$c_L = 1 + 6Q^2 = 1 + 6(b + b^{-1})^2. \tag{6.4}$$

The three point function of primary operator in LFT is

$$\langle V_{\alpha_1}(z_1,\bar{z}_1)V_{\alpha_2}(z_2,\bar{z}_2)V_{\alpha_3}(z_3,\bar{z}_3)\rangle = \frac{C(\alpha_1,\alpha_2,\alpha_3)}{|z_{12}|^{2(\Delta_1+\Delta_2-\Delta_3)}|z_{13}|^{2(\Delta_1+\Delta_3-\Delta_2)}|z_{23}|^{2(\Delta_2+\Delta_3-\Delta_1)}}, \tag{6.5}$$

with $z_{ij} = z_i - z_j$. The function $C(\alpha_1,\alpha_2,\alpha_3)$ is called structure constants associated with dynamical data of any CFT. The DOZZ formula is an analytic expression for C in LFT by [63, 64]. The DOZZ formula gives the three-point function

$$C(\alpha_1,\alpha_2,\alpha_3) = \lambda^{(Q-\sum_{i=1}^{3}\alpha_i)/b}\times \tag{6.6}$$
$$\frac{\Upsilon_b'(0)\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)},$$

where

$$\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)}, \tag{6.7}$$

and

$$\lambda = \pi\mu\gamma\left(b^2\right)b^{2-2b^2}. \tag{6.8}$$

The $\Gamma_b(x)$ is given by eq.(6.31). In this paper, we need four point function of primary operator which reads

$$\left\langle V_{\alpha_1}(z_1,\bar{z}_1)V_{\alpha_2}(z_2,\bar{z}_2)V_{\alpha_3}(z_3,\bar{z}_3)V_{\alpha_4}(z_4,\bar{z}_4)\right\rangle$$
$$= |z_{13}|^{2(\Delta_4-\Delta_1-\Delta_2-\Delta_3)}|z_{14}|^{2(\Delta_2+\Delta_3-\Delta_1-\Delta_4)}|z_{24}|^{-4\Delta_2}|z_{34}|^{2(\Delta_1+\Delta_2-\Delta_3-\Delta_4)}G_{1234}(z,\bar{z}), \tag{6.9}$$

with the harmonic cross ratio z defined as:

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}}. \tag{6.10}$$

and $G_{1234}(z, \bar{z})$

$$G_{1234}(z,\bar{z}) = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dp}{2\pi}C(\alpha_1,\alpha_2,Q/2+ip)R_L(\alpha_s)C(\alpha_3,\alpha_4,Q/2-ip) \times F_{1234}(\Delta_i,\Delta_p,z)F_{1234}(\Delta_i,\Delta_p,\bar{z}). \tag{6.11}$$

Here $\Delta_p = p^2 + Q^2/4$, and the function $F_{1234}(z)$ and $F_{1234}(\bar{z})$ are the Virasoro conformal block. In this paper, we follow the notation for four point Green function given in [41] with respect to normalization of two point function.

6.2 Brief review of SLFT

The $\mathcal{N} = 1$ supersymmetric Liouville field theory may be defined by the action (See review [42])

$$S_{SL} = \frac{1}{4\pi}\int d^2\xi\,\sqrt{g}\frac{1}{2\pi}g_{ab}\partial_a\varphi\partial_b\varphi + \frac{1}{2\pi}(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}) + 2i\mu b^2\bar{\psi}\psi e^{b\varphi} + 2\pi\mu^2 b^2 e^{2b\varphi}\,, \tag{6.12}$$

where φ is a bosonic and ψ a fermionic field, μ denotes a two-dimensional cosmological constant and b is a Liouville coupling constant.

The theory has $\mathcal{N} = 1$ superconformal symmetry. The energy-momentum tensor and the superconformal current are

$$\begin{aligned} T &= -\frac{1}{2}(\partial\varphi\partial\varphi - Q\partial^2\varphi + \psi\partial\psi)\,, \\ G &= i(\psi\partial\varphi - Q\partial\psi). \end{aligned}$$

and the superconformal algebra is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n}\,, \tag{6.13}$$

$$[L_m, G_k] = \frac{m-2k}{2}G_{m+k}\,, \tag{6.14}$$

$$\{G_k, G_l\} = 2L_{l+k} + \frac{c}{3}\left(k^2 - \frac{1}{4}\right)\delta_{k+l}\,. \tag{6.15}$$

The central charge in SLFT is given by

$$c = \frac{3}{2} + 3Q^2. \tag{6.16}$$

The NS-NS primary fields $e^{\alpha\varphi(z,\bar{z})}$ in the $\mathcal{N}=1$ SLFT have conformal dimensions

$$\Delta_{\alpha}^{NS} = \frac{1}{2}\alpha(Q-\alpha). \tag{6.17}$$

As before, physical states have $\alpha = \frac{Q}{2} + ip$ with $Q = b + \frac{1}{b}$. The R-R primary field is defined as

$$R_{\alpha}^{\epsilon}(z,\bar{z}) = \sigma^{\epsilon}(z,\bar{z})e^{\alpha\varphi(z,\bar{z})}\,, \tag{6.18}$$

where σ is the spin field and $\epsilon = \pm$ is the fermion parity. For simplicity we can take all $\epsilon = +$ and drop this index.

The dimension of the R-R operator is

$$\Delta_{\alpha}^{R} = \frac{1}{16} + \frac{1}{2}\alpha(Q-\alpha)\,. \tag{6.19}$$

To consider both NS and R sectors, we will need various functions defined differently for each sector. Here we use the notations $C_i, \Upsilon_i, \Gamma_i$, where $i = 1 \bmod 2$ for $C_{\rm NS}, \Upsilon_{\rm NS}, \Gamma_{\rm NS}$ and $i = 0 \bmod 2$ for R. One can refer to appendix 6.4 to find out the exact definition of these special functions.

The four point function for NS-NS operator is

$$G_4(z,\bar{z}) = \left\langle V_{\alpha_4}(\infty,\infty)V_{\alpha_3}(1,1)V_{\alpha_2}(z,\bar{z})V_{\alpha_1}(0,0)\right\rangle, \tag{6.20}$$

which can be written in the "$s-$channel" representation:

$$\begin{aligned} G_4(z,\bar{z}) &= \int_{\frac{Q}{2}+i\mathbb{R}^+} \frac{d\alpha_s}{i}\left[C(\alpha_4,\alpha_3,\alpha_s)C(\bar{\alpha}_s,\alpha_2,\alpha_1)\left|F^{\rm e}_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z)\right|^2\right. & (6.21)\\ &\qquad \left. - \tilde{C}(\alpha_4,\alpha_3,\alpha_s)\tilde{C}(\bar{\alpha}_s,\alpha_2,\alpha_1)\left|F^{\rm o}_{\alpha_s}\left[\begin{smallmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{smallmatrix}\right](z)\right|^2\right]. & (6.22)\end{aligned}$$

The e,o in $F^{\rm e}_{\alpha_s}$ and $F^{\rm o}_{\alpha_s}$ denote $\mathcal{N}=1$ Neveu-Schwarz blocks with even and odd fermion parity as in [65][66].

Following [67, 68] the structure constants have the following explicit form (α stands here for $\alpha_1+\alpha_2+\alpha_3$)

$$\begin{aligned} C(\alpha_1,\alpha_2,\alpha_3) &= \left(\pi\mu\gamma\left(\frac{Qb}{2}\right)b^{1-b^2}\right)^{(Q-\alpha)/b}\frac{\Upsilon'_{\rm NS}(0)\Upsilon_{\rm NS}(2\alpha_1)\Upsilon_{\rm NS}(2\alpha_2)\Upsilon_{\rm NS}(2\alpha_3)}{\Upsilon_{\rm NS}(\alpha-Q)\Upsilon_{\rm NS}(\alpha_{1+2-3})\Upsilon_{\rm NS}(\alpha_{2+3-1})\Upsilon_{\rm NS}(\alpha_{3+1-2})}\\ \tilde{C}(\alpha_1,\alpha_2,\alpha_3) &= \left(\pi\mu\gamma\left(\frac{Qb}{2}\right)b^{1-b^2}\right)^{(Q-\alpha)/b}\frac{2i\Upsilon'_{\rm NS}(0)\Upsilon_{\rm NS}(2\alpha_1)\Upsilon_{\rm NS}(2\alpha_2)\Upsilon_{\rm NS}(2\alpha_3)}{\Upsilon_{\rm R}(\alpha-Q)\Upsilon_{\rm R}(\alpha_{1+2-3})\Upsilon_{\rm R}(\alpha_{2+3-1})\Upsilon_{\rm R}(\alpha_{3+1-2})}\end{aligned} \tag{6.23}$$

Where we define $\alpha_{i+j-k} = \alpha_i + \alpha_j - \alpha_k$ for short and

$$\Upsilon_1(x) \equiv \Upsilon_{\rm NS}(x) = \Upsilon_b\left(\frac{x}{2}\right)\Upsilon_b\left(\frac{x+Q}{2}\right) = \frac{1}{\Gamma_{\rm NS}(x)\Gamma_{\rm NS}(Q-x)}\,, \tag{6.24}$$

$$\Gamma_1(x) \equiv \Gamma_{\rm NS}(x) = \Gamma_b\left(\frac{x}{2}\right)\Gamma_b\left(\frac{x+Q}{2}\right). \tag{6.25}$$

Functions for R sector are defined differently. For example, we have

$$\Gamma_0(x) \equiv \Gamma_{\rm R}(x) = \Gamma_b\left(\frac{x+b}{2}\right)\Gamma_b\left(\frac{x+b^{-1}}{2}\right). \tag{6.26}$$

The four point function for R-R operators

$$G_4(z,\bar{z}) = \Big\langle R_{\alpha_4}(\infty,\infty)R_{\alpha_3}(1,1)R_{\alpha_2}(z,\bar{z})R_{\alpha_1}(0,0)\Big\rangle, \tag{6.27}$$

can be also written in a similar form

$$\begin{aligned} G_4(z,\bar{z}) \;&=\; \int_{\frac{Q}{2}+i\mathbb{R}^+}\frac{d\alpha_s}{i}\left[C_R^+(\alpha_4,\alpha_3|\alpha_s)C_R^+(\alpha_2,\alpha_1|\bar{\alpha}_s)\left|F^{\rm e}_{\alpha_s}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}(z)\right|^2\right. \\ &\qquad \left. -\,\tilde{C}_R^+(\alpha_4,\alpha_3|\alpha_s)\tilde{C}_R^+(\alpha_2,\alpha_1|\bar{\alpha}_s)\left|F^{\rm o}_{\alpha_s}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}(z)\right|^2\right]. \end{aligned} \tag{6.28}$$

The corresponding structure constants become

$$\begin{aligned} C_R^\epsilon(\alpha_1,\alpha_2|\alpha_3) &= \left(\pi\mu\gamma\left(\frac{Qb}{2}\right)b^{1-b^2}\right)^{(Q-\alpha)/b} \times\left[\frac{\Upsilon'_{NS}(0)\Upsilon_R(2\alpha_1)\Upsilon_R(2\alpha_2)\Upsilon_{NS}(2\alpha_3)}{\Upsilon_R(\alpha-Q)\Upsilon_R(\alpha_{1+2-3})\Upsilon_{NS}(\alpha_{2+3-1})\Upsilon_{NS}(\alpha_{3+1-2})}\right. \\ &+\left.\frac{\epsilon\Upsilon'_{NS}(0)\Upsilon_R(2\alpha_1)\Upsilon_R(2\alpha_2)\Upsilon_{NS}(2\alpha_3)}{\Upsilon_{NS}(\alpha-Q)\Upsilon_{NS}(\alpha_{1+2-3})\Upsilon_R(\alpha_{2+3-1})\Upsilon_R(\alpha_{3+1-2})}\right] \end{aligned} \tag{6.29}$$

$$\tilde{C}_R^\epsilon(\alpha_1,\alpha_2|\alpha_3) = -\frac{i\epsilon}{2}[(p_1^2+p_2^2)C_R^\epsilon(\alpha_1,\alpha_2|\alpha_3) - 2p_1p_2C_R^{-\epsilon}(\alpha_1,\alpha_2|\alpha_3)]\,. \tag{6.30}$$

6.3 The function $\Gamma_b(x)$

The function $\Gamma_b(x)$ is a close relative of the double Gamma function studied in [69]. It can be expressed by means of the integral representation

$$\log\Gamma_b(x) = \int_0^\infty\frac{dt}{t}\left(\frac{e^{-xt}-e^{-Qt/2}}{(1-e^{-bt})(1-e^{-t/b})} - \frac{(Q-2x)^2}{8e^t} - \frac{Q-2x}{t}\right). \tag{6.31}$$

Important properties of $\Gamma_b(x)$ are listed as follows

functional relation $\qquad \Gamma_b(x+b) = \sqrt{2\pi}b^{bx-\frac{1}{2}}\Gamma^{-1}(bx)\Gamma_b(x).$ (6.32)

analyticity $\qquad \Gamma_b(x)$ is meromorphic function and it has poles only at $x = -nb - mb^{-1}, n, m \in \mathbb{Z}^{\geq 0}$. (6.33)

The further properties is listed in [70].

In terms of functional relation eq.(6.32), one can find the residues near by various pole as following

$$\Gamma_b(x) = \frac{\Gamma_b(Q)}{2\pi x} + O(x), \quad x \to 0 \tag{6.34}$$

6.4 Double Sine-function

In terms of $\Gamma_b(x)$, the double Sine-function is given as follows

$$S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}. \tag{6.35}$$

We will use the properties

$$\text{Self duality} \quad S_b(x) = S_{b^{-1}}(x)\,, \tag{6.36}$$

$$\text{Functional relation} \quad S_b(x+b^{\pm 1}) = 2\,\sin(\pi b^{\pm 1}x)\,S_b(x)\,, \tag{6.37}$$

$$\text{Reflection relation} \quad S_b(x)\,S_b(Q-x) = 1\,. \tag{6.38}$$

The asymptotics behavior of $S_b(x)$ is,

$$S_b(x = x_0 + \mathrm{i}x_1) \sim \begin{cases} e^{\pi x_1\left(\frac{b}{2}+\frac{1}{2b}-x_0\right)} e^{i\pi\left(\frac{b^2}{12}+\frac{1}{12b^2}-\frac{bx_0}{2}-\frac{x_0}{2b}+\frac{x_0^2}{2}-\frac{x_1^2}{2}+\frac{1}{4}\right)} & \text{for } |x| \to \infty, \ x_1 < 0\,, \\ e^{-\pi x_1\left(\frac{b}{2}+\frac{1}{2b}-x_0\right)} e^{-i\pi\left(\frac{b^2}{12}+\frac{1}{12b^2}-\frac{bx_0}{2}-\frac{x_0}{2b}+\frac{x_0^2}{2}-\frac{x_1^2}{2}+\frac{1}{4}\right)} & \text{for } |x| \to \infty, \ x_1 > 0\,. \end{cases} \tag{6.39}$$

We define following functions,

$$\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)}, S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}, G_b(x) = \mathrm{e}^{-\frac{i\pi}{2}x(Q-x)}S_b(x), \tag{6.40}$$

and, in SFLT, we follow notations from [71] to denote:

$$\Gamma_{\mathrm{NS}}(x) = \Gamma_b\left(\frac{x}{2}\right)\Gamma_b\left(\frac{x+Q}{2}\right), \qquad \Gamma_{\mathrm{R}}(x) = \Gamma_b\left(\frac{x+b}{2}\right)\Gamma_b\left(\frac{x+b^{-1}}{2}\right), \tag{6.41}$$

$$\Upsilon_{\mathrm{NS}}(x) = \Upsilon_b\left(\frac{x}{2}\right)\Upsilon_b\left(\frac{x+Q}{2}\right), \qquad \Upsilon_{\mathrm{R}}(x) = \Upsilon_b\left(\frac{x+b}{2}\right)\Upsilon_b\left(\frac{x+b^{-1}}{2}\right), \tag{6.42}$$

etc. Using relations, basic properties of these functions can be established easily.

$$\frac{\Gamma_{\mathrm{NS}}(2\alpha)}{\Gamma_{\mathrm{NS}}(2\alpha-Q)} = W_{\mathrm{NS}}(\alpha)\lambda^{\frac{Q-2\alpha}{2b}}\,, \tag{6.43}$$

$$\frac{\Gamma_{\mathrm{R}}(2\alpha)}{\Gamma_{\mathrm{R}}(2\alpha-Q)} = W_{\mathrm{R}}(\alpha)\lambda^{\frac{Q-2\alpha}{2b}}\,, \tag{6.44}$$

where $W_{NS}(\alpha)$, $W_{\rm R}(\alpha)$ are defined in (6.45) and (6.46), and $\lambda = \pi\mu\gamma\left(\frac{bQ}{2}\right)b^{1-b^2}$. The functions W_i are defined as

$$W_{\rm NS}(\alpha) = \frac{2(\pi\mu\gamma(bQ/2))^{-\frac{Q-2\alpha}{2b}}\pi(\alpha - Q/2)}{\Gamma(1 + b(\alpha - Q/2))\Gamma(1 + \frac{1}{b}(\alpha - Q/2))}\,, \tag{6.45}$$

$$W_R(\alpha) = \frac{2\pi(\pi\mu\gamma(bQ/2))^{-\frac{Q-2\alpha}{2b}}}{\Gamma(1/2 + b(\alpha - Q/2))\Gamma(1/2 + \frac{1}{b}(\alpha - Q/2))} \tag{6.46}$$

In the literature, one can define following equations for convenience,

$$S_1(x) \equiv S_{\rm NS}(x) = \frac{\Gamma_{\rm NS}(x)}{\Gamma_{\rm NS}(Q - x)}\,, \tag{6.47}$$

$$S_0(x) \equiv S_{\rm R}(x) = \frac{\Gamma_{\rm R}(x)}{\Gamma_{\rm R}(Q - x)}\,. \tag{6.48}$$

They have the following relations with $S_{NS,R}$ functions:

$$\frac{S_{\rm NS}(2x)}{S_{\rm NS}(2x - Q)} = W_{NS}(x)W_{NS}(Q - x)\,, \tag{6.49}$$

$$\frac{S_{\rm R}(2x)}{S_{\rm R}(2x - Q)} = W_R(x)W_R(Q - x)\,. \tag{6.50}$$

In the paper we used following

- Reflection properties:

$$S_{\rm NS}(x)S_{\rm NS}(Q - x) = S_{\rm R}(x)S_{\rm R}(Q - x) = 1 \tag{6.51}$$

- Locations of zeroes and poles can be obtained from eq.(6.41):

$$S_{\rm NS}(x) = 0 \quad\Leftrightarrow\quad x = Q + mb + nb^{-1}, \qquad m, n \in \mathbb{Z}_{\geq 0},\ m + n \in 2\mathbb{Z}, \tag{6.52}$$

$$S_{\rm R}(x) = 0 \quad\Leftrightarrow\quad x = Q + mb + nb^{-1}, \qquad m, n \in \mathbb{Z}_{\geq 0},\ m + n \in 2\mathbb{Z} + 1, \tag{6.53}$$

$$S_{\rm NS}(x)^{-1} = 0 \quad\Leftrightarrow\quad x = -mb - nb^{-1}, \qquad m, n \in \mathbb{Z}_{\geq 0},\ m + n \in 2\mathbb{Z}, \tag{6.54}$$

$$S_{\rm R}(x)^{-1} = 0 \quad\Leftrightarrow\quad x = -mb - nb^{-1}, \qquad m, n \in \mathbb{Z}_{\geq 0},\ m + n \in 2\mathbb{Z} + 1. \tag{6.55}$$

- Basic residue:

$$\lim_{x\to 0} x S_{\rm NS}(x) = \frac{1}{\pi}. \tag{6.56}$$

6.5 Poles Structure and discrete terms

With following the appendix about the LFT in [72], the 4-point functions of primary operators is:

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle = \int_{i\mathbb{R}+\frac{Q}{2}} C(\alpha_1,\alpha_2,\alpha_s)C(\alpha_3,\alpha_4,\bar{\alpha}_s)|F_s(\Delta_{\alpha_{i=1,2,3,4}},\Delta_{\alpha_s},z)|^2 d\alpha_s\,, \tag{6.57}$$

where $\mathrm{Re}(\alpha_1),\ldots,\mathrm{Re}(\alpha_4) = Q/2$. For $\alpha_i \in (0, Q/2)$ cases, The eq. (6.57) needs to be extended. The proper way to integration should preserve crossing symmetry with assumptions given by [73][74][75].

The integrand of (6.57), as a function of $\alpha \in \mathbb{C}$, has many poles. The 2 structure constants $C(\alpha_1,\alpha_2,\alpha_s)$ and $C(\alpha_3,\alpha_4,\bar{\alpha}_s)$ in (6.57) have poles and these poles comes from the zeros of the Υ's in the denominator of the DOZZ formula in LFT[8]. These poles are as follows,

1	2	3	4	5	6	7	8
$Q-\alpha_s-L$	α_s+L	$2Q-\alpha_s-L$	$-Q+\alpha_s+L$	α_d-L	$-\alpha_d+Q+L$	$-\alpha_d-L$	α_d+Q+L

$$\begin{aligned} &\text{where } \alpha_s \equiv \alpha_1+\alpha_2\,,\ \alpha_d \equiv \alpha_1-\alpha_2\,,\ \text{for LFT}:\quad L=\left\{bm+b^{-1}n : m,n\in\mathbb{Z}^{\geq 0}\right\},\\ &\text{for SLFT NS sector}:\quad L=\left\{bm+b^{-1}n : m+n\in 2\mathbb{Z}^{\geq 0}\right\},\\ &\text{for SLFT R sector}:\quad L=\left\{bm+b^{-1}n : m+n\in 2\mathbb{Z}^{\geq 0}+1\right\} \end{aligned} \tag{6.58}$$

Note that the row 1 and 2 are related by $\alpha \mapsto Q-\alpha$ symmetry and so do 3 and 4, *etc*. The poles coming from $C(\alpha_2,\alpha_3,\bar{\alpha})$ are got by replacing $1,2 \to 3,4$ in the above equations.

If $\mathrm{Re}(\alpha_i) = \frac{Q}{2}$, the real part of the poles belongs to the intervals $(-\infty,0]\cup[Q,+\infty)$ and the intervals do not intersect with the integration contour $Q/2+iR$ (6.57). When $\mathrm{Re}(\alpha_s) = \mathrm{Re}(\alpha_1+\alpha_2)$ starts to decrease from Q into the interval $(0,Q)$, the poles start to move on the plane α_s. One can show that only the rows 1 and 2 may cross the line $Q/2+iR$. When $\mathrm{Re}(\alpha_s)$ decreases to $Q/2$, row 1 crosses the line from its left, and row 2 cross the line from it right. As $\mathrm{Re}(\alpha_s)$ further decreases, several poles from those rows in the table will have crossed the line. These poles are:

$$P_+ \equiv \{x\in Q-(\alpha_1+\alpha_2)-L : \mathrm{Re}(x)\in(Q/2,Q)\}\,,\ P_- \equiv \{x\in \alpha_1+\alpha_2+L : \mathrm{Re}(x)\in(0,Q/2)\}\,. \tag{6.59}$$

To extend analytically the integral (6.57), the integration contour has to be deformed to avoid the poles from crossing it. Using Cauchy formula, this amounts to adding $\pm 2\pi i$ times the residues of the integrand of (6.57) at points in $P_\pm$, respectively. These terms are the so called discrete terms [74] [73][75] known in the literature.

[8] The similar structure constant in SLFT is given in above appendix 6.2 and we just list the relevant results.

Using the $\alpha_s \mapsto Q - \alpha_s$ symmetry of Υ, the contribution of P_+ equals that of P_-. Then, the poles from $\alpha_{2,3}$ can be similarly treated. For LFT 4-point function for values of $\text{Re}(\alpha_i) \in (0, Q/2)$, the resulting form is as follows [72]:

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_3}(1)V_{\alpha_4}(\infty)\rangle = \int_{\frac{Q}{2}+i\mathbb{R}} C(\alpha_1,\alpha_2,\alpha_s)C(\alpha_3,\alpha_4,\bar{\alpha}_s)|F_s(\Delta_{\alpha_i},\Delta_{\alpha_s},z)|^2 d\alpha_s + \\ -2\sum_{p\in P_-} \text{Res}_{\alpha_s\to p}\left[C(\alpha_1,\alpha_2,\alpha_s)C(\alpha_3,\alpha_4,\bar{\alpha}_s)\right]|F_s(\Delta_{\alpha_i},\Delta_{\alpha_s},z)|^2 + [(1,2)\leftrightarrow(3,4)] \tag{6.60}$$

In this paper, we will use asymptotic form of conformal block as $z \to 0$, or $z \to 1$. The $z \to 0$ series expansion of conformal block is:

$$F_s(\Delta_{\alpha_i},\Delta_{\alpha_s},z) = z^{-\Delta_{\alpha_1}-\Delta_{\alpha_2}+\Delta_{\alpha_s}}\left(1 + \frac{(\Delta_{\alpha_2}-\Delta_{\alpha_1}+\Delta_{\alpha_s})(\Delta_{\alpha_3}-\Delta_{\alpha_4}+\Delta_{\alpha_s})}{2\Delta_{\alpha_s}}z + O(z^2)\right). \tag{6.61}$$

Once s-channel blocks are known, t-channel blocks can be obtained by a permutation of the arguments, with taking global conformal symmetry into account and this becomes [76]

$$F_t(\Delta_{\alpha_i},\Delta_{\alpha_t},z) = (1-z)^{\Delta_2+\Delta_3-\Delta_1-\Delta_4}F_s(\Delta_{\alpha_i},\Delta_{\alpha_s},1-z)\,. \tag{6.62}$$

In this paper, to compute the dominant asymptotic behavior of (6.59) as $z \to 0$ or $z \to 1$ is very important. We need to consider the internal charges $\alpha_s \in P_+ \cup (Q/2 + i\mathbb{R})$ involved, and find the smallest scaling dimension $\Delta_\alpha = \alpha(Q-\alpha)$. In this paper, we will make use of these details to obtain REE. Some further calculations have been given in subsection 6.6.

6.6 To calculate the dominant contribution in early time

In this section, we would like to show the some details about how to do the early time integral appeared in four point function in LFT. The early time limit $(z,\bar{z}) \to (0,0)$ is a short distance limit, or equivalently $z_1 \to z_2$ or $z_3 \to z_4$. Hence one can insert a corresponding OPE [74][75] in eq.(6.57). If the OPE would be given by a sum as in rational CFTs, the dominant contribution trivially would be realized by the contribution of the r.h.s. operator with lowest conformal dimension. In LFT, we have an integral as given in formula (1.10) of [74].

For $\text{Re}(\alpha_1+\alpha_2) > Q/2$ [74], the integral is

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_1}(1)V_{\alpha_2}(\infty)\rangle \underset{(z,\bar{z})\to(0,0)}{\simeq} \int_0^\infty (z\bar{z})^{P^2} f(P)dP \tag{6.63}$$

whose asymptotics for $z \to 0$ is

$$\sqrt{\frac{\pi}{-\log z}}\frac{f(0)}{2} + \frac{1}{2\log z}f'(0) + \frac{\sqrt{\pi}}{8(-\log z)^{\frac{3}{2}}}f''(0) + \ldots \tag{6.64}$$

One can apply to following OPE [74]

$$V_{\alpha_1}(0)V_{\alpha_2}(z) = \int' \frac{dP}{4\pi} C(\alpha_1, \alpha_2, \alpha_s)(z\bar{z})^{\Delta_{Q/2+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2}} [V_{Q/2+iP}(0)] \tag{6.65}$$

The integration contour here is the real axis if α_1 and α_2 are in the basic domain$|Q/2 - \mathrm{Re}(\alpha_1)| + |Q/2 - \mathrm{Re}(\alpha_2)| < Q/2$. In this case, one can find that

$$\frac{1}{4\pi}\sqrt{\frac{\pi}{-\log(z\bar{z})}}(z\bar{z})^{\frac{Q^2}{4} - \Delta_1 - \Delta_2} C(\alpha_1, \alpha_2, \frac{Q}{2}) V_{Q/2}(0) + ... \tag{6.66}$$

But we have to take into account that $C(\alpha_1, \alpha_2, \frac{Q}{2}) = 0$. The function $f(P)$ is $C(\alpha_1, \alpha_2, \frac{Q}{2} + iP)C(\alpha_1, \alpha_2, \frac{Q}{2} - iP)$. Due to $f(0) = 0$ and the first nonvanishing term is the one with $f''(0)$ which is then

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_1}(1)V_{\alpha_2}(\infty)\rangle \underset{(z,\bar{z})\to(0,0)}{\simeq} \frac{\sqrt{\pi}}{8} f''(0)(z\bar{z})^{\frac{Q^2}{4} - \Delta_1 - \Delta_2}(-\log(z\bar{z}))^{-\frac{3}{2}} \tag{6.67}$$

This conclusion corresponds to a statement by Seiberg in [77], see comment after his eq. (4.15).[9]

For α_1 and α_2 stay outside of basic domain[74], we can use following OPE

$$\begin{aligned} V_{\alpha_1}(0)V_{\alpha_2}(z) &= \frac{1}{2}(z\bar{z})^{-2\alpha_1\alpha_2}[V_{\alpha_1+\alpha_2}(0)] + \frac{1}{2}(z\bar{z})^{-2\alpha_1\alpha_2} S(\alpha_1 + \alpha_2)[V_{Q-\alpha_1-\alpha_2}(0)] \\ &+ \frac{1}{2}\int \frac{dP}{4\pi} C(\alpha_1, \alpha_2, \alpha_s)(z\bar{z})^{\Delta_{Q/2+iP} - \Delta_{\alpha_1} - \Delta_{\alpha_2}}[V_{Q/2+iP}(0)] \end{aligned} \tag{6.68}$$

The integral has again an asymptotics as above eq.(6.64). The first two terms in eq.(6.68) are so called discrete terms. But now the more dominant term is given by the contribution of the discrete term, i.e.

$$\begin{aligned} &\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_1}(1)V_{\alpha_2}(\infty)\rangle \\ \underset{(z,\bar{z})\to(0,0)}{\simeq} \; & C(\alpha_1, \alpha_2, \alpha_1 + \alpha_2)C(\alpha_1, \alpha_2, Q - \alpha_1 - \alpha_2)(z\bar{z})^{\Delta_{\alpha_1+\alpha_2} - \Delta_1 - \Delta_2} V_{\alpha_1+\alpha_2}(0) \end{aligned} \tag{6.69}$$

Depending on the value of α_1 and α_2 other discrete terms can take over, as discussed in [74].

To calculate the late time limit of $\langle V_{\alpha_1}(0)V_{\alpha_2}(z)V_{\alpha_1}(1)V_{\alpha_2}(\infty)\rangle$, we have to use conformal boostrap equations firstly and following the above procedures in this subsection. We have present the main results and will not repeat all the details here.

[9] The fact that author writes $Q^2/8$ instead of $Q^2/4$ due to different normalisations.

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