LSZ-reduction, resonances and non-diagonal propagators: fermions and scalars

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Abstract

We analyze in details the effects associated with mixing of fermionic fields. In a system with an arbitrary number of Majorana or Dirac particles, a simple proof of factorizability of residues of non-diagonal propagators at the complex poles is given, together with a prescription for finding the "square-rooted" residues to all orders of perturbation theory, in an arbitrary renormalization scheme. Corresponding prescription for the scalar case is provided as well.

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1 Introduction

Finding the correct form of renormalization conditions in gauge theories of particle physics is a nontrivial task, as the improper conditions can spoil the gauge symmetry even in the presence of a symmetry preserving regularization. For this reason, renormalization schemes with implicit renormalization conditions are often employed. A famous representative of this class is the $\overline{\mathrm{MS}}$ scheme of dimensional regularization [1]. Zinn-Justin proved that non-Abelian gauge symmetries are preserved in minimal subtraction schemes based on a symmetry preserving regularization [2], what makes $\overline{\mathrm{MS}}$ the most convenient choice, at least in non-chiral theories. In fact, the violation of gauge symmetry induced by regularization can be handled according to the general rules of "algebraic renormalization" [3, 4, 5] (see also [6] for a discussion in the context of dimensional regularization with the consistent 't Hooft-Veltman-Breitenlohner-Maison prescription for γ^5). This approach does not preclude the use of schemes with implicit renormalization conditions. For instance, in [7] a renormalization scheme for gauge theories in the cutoff regularization was constructed by carefully selecting a set of vertices to which non-minimal (cutoff-independent) counterterms are added. These non-minimal counterterms are completely fixed by investigating the Slavnov-Taylor identities for the gauge symmetry, what leads to a mass independent renormalization scheme with implicit renormalization conditions. In such a framework, renormalization can be considered as the first stage in the study of correlation functions, a stage which is independent of the physical interpretation of the studied model.

In this paper we are interested *only* in the second stage, i.e. in the extraction of scattering amplitudes from *renormalized* correlation functions in an arbitrary renormalization scheme.[3] More specifically, we are interested

[3]Of course, it is assumed that the scheme preserves Slavnov-Taylor identities of the

in amplitudes corresponding to the particles described by the system of mixed fields with non-diagonal propagators. In this context, the crucial objects are residues of the (renormalized) propagators; for instance in the case of scalar fields one finds the following expression for the full propagator

$$\langle T(\phi^k(x)\phi^j(y))\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, e^{-i\,p(x-y)} \left\{ \sum_\ell \zeta^k_{S[\ell]} \frac{i}{p^2 - m^2_{S(\ell)}} \zeta^j_{S[\ell]} + \begin{bmatrix}\text{non-pole}\\ \text{part}\end{bmatrix} \right\}. \tag{1}$$

The factorization of the residues at the (complex) poles is a well-known property [8]. For the mixed Z-photon system of the Standard Model it was explicitly demonstrated in [9]. For fermionic systems the factorization was shown in [10, 11] (see also [12, 13]). [4]

The coefficients $\zeta^k_{S[\ell]}$ associated with real poles $m^2_{S(\ell)}$ are crucial for obtaining correctly normalized (i.e. consistent with unitarity) transition amplitudes between the states of stable particles. On account of Cutkosky-Veltman rules, it is well-known [15] that the S-matrix is unitary provided that (1.) unstable particles appear only as internal lines, and (2.) asymptotic (free) fields appearing in the LSZ-reduced formula for the S-operator (see e.g. [16] and Eq. (37) below) are normalized so as to reproduce the behavior of full propagators about the poles associated with stable particles. Thus, the asymptotic field $\boldsymbol{\phi}^j$ associated with ϕ^j can be written as $\boldsymbol{\phi}^j = \sum'_\ell \zeta^j_{S[\ell]} \Phi^\ell$, where Φ^ℓ is a canonically normalized free scalar field of mass $m_{S(\ell)}$, and the summation runs over indices labeling real poles.

For resonances the external lines, aka the "in" and "out" states, do not really exist. Nonetheless, coefficients $\zeta^k_{S[\ell]}$ associated with the complex poles are useful in studying properties of unstable particles [8, 9, 10, 11]. In this connection, the problem of finding a convenient prescription for coefficients $\zeta^k_{S[\ell]}$ parameterizing the residues in Eq. (1) have gained renewed interest in recent years. In Ref. [17] the case of 3-by-3 mixing of scalar particles was analyzed in details. The factorization property (1) was demonstrated and explicit formulae for the coefficients $\zeta^k_{S[\ell]}$ were given. These results were applied to the neutral Higgs sector of MSSM; it was shown that cross-sections obtained by neglecting the non-pole part in Eq. (1) agree to good accuracy with the cross-sections based on full propagators. Analysis of a generic n-by-n mixing in fermionic systems was given in [18, 19, 20].

The purpose of this paper is to generalize and to simplify the available in

gauge symmetry.

[4] We also mention in this context Ref. [14] where the factorization of the residues was shown for scalar, vector and fermionic systems, on the basis of the presupposed existence of the on-shell renormalization scheme. This approach makes the generalization to the complex poles rather difficult.

the literature procedure of calculating coefficients ζ for fermions and scalars. Our analysis is closest in spirit to the one given in [18, 19, 20]; there are, however, some differences. First, we follow the philosophy of keeping the renormalization scheme as general as possible. In particular, we do not impose any concrete renormalization conditions on the two-point functions. Second, we offer a technical improvement in comparison with the analyzes of [8, 18, 19, 20], where the cofactor matrix was used to get the formulae for ζ. By contrast, the coefficients ζ in our approach are expressed directly in terms of properly normalized eigenvectors of certain "mass-squared matrices", so that the case of degenerated eigenvalues is naturally covered by our prescription. Thus, the proposed prescription for finding ζ can be considered as a generalization of the standard procedure for finding tree-level mass eigenstates. The analysis of mixed vector fields along these lines will be given elsewhere [21].

The formulae presented below are valid in the generic case of mixed unstable Majorana or/and Dirac fermions (a scalar version is also given). For completeness, the expression for the Landau-gauge one-loop fermionic self-energy of a general renormalizable model in the $\overline{\mathrm{MS}}$ scheme is provided below. The paper is therefore intended to cultivate a long tradition by providing generic formulae that can be easily applied to (almost) any model at hand, especially in the present computer era, see e.g. [22], [23], [24], [25], [27], [28], [26], [7].

The remainder is organized as follows. In the next section the notation is specified, together with basic assumptions. In Sec. 3 the prescription for ζ matrices is given for massive Majorana particles (3.A), massive Dirac particles (3.B), generic spin-1/2 fermions (3.C) and scalars (3.D), together with the generic expression for fermionic one-loop self-energy (3.E). The correctness of the prescription is proved in Sec. 4 and the last section is reserved for conclusions.

2 Notation and assumptions

In most formulae given below indices are suppressed and matrix multiplication is understood. The summation convention is used only when an upper index is contracted with a lower one; whenever ambiguities may arise, sums are explicitly present. The Minkowski metric has the form

$$\eta = [\eta_{\mu\nu}] = \mathrm{diag}(+1, -1, -1, -1)\,. \tag{2}$$

Recall that a Majorana field [35] $\psi^{\tilde{a}}$ is a pair of a Weyl field χ^a_A, below referred to as the left-chiral Weyl field (LW), and its Weyl conjugate $\overline{\chi}^{a\dot{A}}$, alias the right-chiral Weyl field (RW)

$$\psi^{\widetilde{a}} = \left[\begin{array}{c} \chi^a_A \\ \overline{\chi}^{a\dot{A}} \end{array} \right], \tag{3}$$

here $a = 1, \ldots, n$ is a generalized-flavor index, A and $\dot{A}$ are $SL(2, \mathbb{C})$ indices, while $\widetilde{a} = (a, (A, \dot{A}))$.

Take, for instance, (a toy version of) the Standard Model [31] in which all Weyl fields except for these that describe the electron-positron pair of states have been forgotten. Let λ_A be a LW representing the charged component of the lepton (would-be) $SU(2)_L$-doublet and let ρ_A be a LW of the charged lepton $SU(2)_L$-singlet. In this case $n = 2$ and the fields with the definite generalized-flavor (henceforth called flavor) can be chosen as

$$\chi^1 = \lambda \,, \qquad\qquad \chi^2 = \rho, \tag{4}$$

though nothing (but common sense) prevents a more general choice

$$\chi^a = u^a{}_1\, \lambda + u^a{}_2\, \rho \,, \tag{5}$$

with an arbitrary unitary matrix u, which off-diagonalizes the charge generator.

With chiral projections $P_{L,R}$

$$P_L\, \psi \simeq \left[\begin{array}{c} \chi \\ 0 \end{array} \right], \qquad P_R\, \psi \simeq \left[\begin{array}{c} 0 \\ \overline{\chi} \end{array} \right],$$

and the charge conjugation matrix $\mathcal{C}$ that expresses the Dirac conjugate of ψ in terms of ψ itself

$$\bar{\psi} = \psi^{\top} \mathcal{C} \,,$$

the renormalized (in some renormalization scheme) one-particle-irreducible (1PI) two-point function of Majorana fileds can be written in the following form

$$\begin{aligned} \widetilde{\Gamma}_{\widetilde{a}\widetilde{b}}(-p,p) = \Big[\mathcal{C} \Big\{ \big(\not{p}\, \mathscr{Z}_L(p^2) - \mathscr{M}_L(p^2) \big)\, P_L + \\ + \big(\not{p}\, \mathscr{Z}_R(p^2) - \mathscr{M}_R(p^2) \big)\, P_R \Big\} \Big]_{\widetilde{a}\widetilde{b}} \,, \end{aligned} \tag{6}$$

where

$$\mathscr{Z}_{L,R}(p^2) = \mathbb{1} + \mathcal{O}(\hbar) \,. \tag{7}$$

Clearly, matrices $P_{R,L}$, $\mathcal{C}$ and $\not{p}$ carry only the $SL(2,\mathbb{C})$ indices, while $\mathscr{Z}_{L,R}$ and $\mathscr{M}_{L,R}$ carry only the flavor indices; the tensor products $\otimes$ are not explicitly shown in Eq. (6). ($\mathscr{Z}_{L,R}$ and $\mathscr{M}_{L,R}$ are, essentially, the 1PI functions of different pairs of Weyl fields; Majorana fields are used here and below only for bookkeeping reasons.)

The full propagator of Majorana fields is given by

$$\widetilde{G}^{\widetilde{a}\widetilde{b}}(p,-p) = i\Big[\widetilde{\Gamma}(-p,p)^{-1}\Big]^{\widetilde{a}\widetilde{b}} = i\Big[\hat{\mathscr{D}}_{\mathcal{F}}(p)\,\mathcal{C}^{-1}\Big]^{\widetilde{a}\widetilde{b}} . \tag{8}$$

Inverting the two-point function in Eq. (6) one finds

$$\hat{\mathscr{D}}_{\mathcal{F}}(p) = P_L\,\hat{\mathscr{D}}_L(p) + P_R\,\hat{\mathscr{D}}_R(p), \tag{9}$$

where ($s \equiv p^2$)

$$\begin{aligned}
\hat{\mathscr{D}}_L(p) &= \left[s\mathbb{1} - \mathbb{M}_L^2(s)\right]^{-1} \mathscr{Z}_L(s)^{-1}\left[\not{p} + \mathscr{M}_R(s)\mathscr{Z}_R(s)^{-1}\right] , \\
\hat{\mathscr{D}}_R(p) &= \left[s\mathbb{1} - \mathbb{M}_R^2(s)\right]^{-1} \mathscr{Z}_R(s)^{-1}\left[\not{p} + \mathscr{M}_L(s)\mathscr{Z}_L(s)^{-1}\right] ,
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
\mathbb{M}_L^2(s) &= \mathscr{Z}_L(s)^{-1}\,\mathscr{M}_R(s)\,\mathscr{Z}_R(s)^{-1}\,\mathscr{M}_L(s) , \\
\mathbb{M}_R^2(s) &= \mathscr{Z}_R(s)^{-1}\,\mathscr{M}_L(s)\,\mathscr{Z}_L(s)^{-1}\,\mathscr{M}_R(s) .
\end{aligned} \tag{11}$$

The antisymmetry of the fermionic two-point function, Eq. (6), yields

$$\begin{aligned}
\mathscr{M}_X(s) &= \mathscr{M}_X(s)^\top , \qquad X = L, R, \\
\mathscr{Z}_R(s) &= \mathscr{Z}_L(s)^\top ,
\end{aligned} \tag{12}$$

and thus

$$\mathbb{M}_R^2(s)^\top = \mathscr{Z}_L(s)\,\mathbb{M}_L^2(s)\,\mathscr{Z}_L(s)^{-1} , \tag{13}$$

what gives

$$\mathscr{X}(s) \equiv \det(s\mathbb{1} - \mathbb{M}_L^2(s)) = \det(s\mathbb{1} - \mathbb{M}_R^2(s)) , \tag{14}$$

hence the poles of both chiral parts $\hat{\mathscr{D}}_{L,R}$ of propagator in Eq. (9) appear at the same points $s = m_{(a)}^2$, obeying the condition

$$\mathscr{X}(m_{(a)}^2) = 0 . \tag{15}$$

In this paper three technical assumptions are made about the solutions to Eq. (15) and the matrices $\mathbb{M}_L^2(m_{(a)}^2)$. First, it is assumed that each generalized eigenvector (see e.g. [29]) of $\mathbb{M}_L^2(m_{(a)}^2)$ associated with the eigenvalue

$m^2_{(a)}$ is an (ordinary) eigenvector; in other words, it is assumed that in the Jordan basis for $\mathbb{M}^2_L(m^2_{(a)})$ the block corresponding to $m^2_{(a)}$ is diagonal. This excludes standard pathologies associated with non-diagonalizable matrices (e.g. second order poles of gauge-field propagators in covariant non-Feynman gauges caused by *pseudo*Hermiticity of the Hamiltonian [34]).

Second, it is assumed that each nonzero solution $m^2_{(a)}$ is nonzero at the tree level, as is usually the case in the common seesaw models.

Third, it is assumed that, roughly speaking, the quantum corrections do not change the total number of solutions to Eq. (15). More specifically, suppose that the a label distinguishes different solutions $m^2_{(a)}$. Let $\xi_{[a_1]}$, $\xi_{[a_2]}$, . . . , be a basis of the eigenspace of $\mathbb{M}^2_L(m^2_{(a)})$ associated with the eigenvalue $m^2_{(a)}$. It is assumed that each element in the sequence

$$\xi_{[1_1]}, \; \ldots \; , \xi_{[2_1]} \; \ldots \; ,$$

has the form $\xi_{[a_r]} = \xi^0_{[a_r]} + \mathcal{O}(\hbar)$, where vectors

$$\xi^0_{[1_1]}, \; \ldots \; , \xi^0_{[2_1]} \; \ldots \; ,$$

are of zeroth order in $\hbar$ and form a basis of $\mathbb{C}^n$, with n denoting the total number of LWs. [5]

The pole masses $m^2_{(a)}$ are formal power series in $\hbar$. Thus, if all of the tree-level masses of fermions are different, then $\mathbb{M}^2_L(s)$ is diagonalizable as a formal power series

$$\mathbb{M}^2_L(s) = W(s)^{-1} \, \mathrm{diag}\big(d_1(s), ..., d_n(s)\big) \, W(s) \, ,$$

and

$$\mathscr{X}(s) = \prod_{p=1}^{n} \Big(s - d_p(s) \Big) \, .$$

If $d_a(s) = (m^{\text{tree}}_{(a)})^2 + \mathcal{O}(\hbar)$, then Eq. (15) reads

$$d_a(m^2_{(a)}) = m^2_{(a)} \, ,$$

and has a unique solution $m^2_{(a)} = (m^{\text{tree}}_{(a)})^2 + \mathcal{O}(\hbar)$. In particular, the first and the third assumption are satisfied in this case. In general, assuming non-degeneracy of the tree-level masses is however not an option as physics

[5]The reader should be warned that the a label on pole masses is the same as the index on flavor eigenfields χ^a, even though χ^a are not assumed to be the eigenstates of the tree-level (nor the pole) masses. This little abuse of notation will not lead to any misunderstandings.

is about symmetries. Therefore it is convenient (and desirable from practical point of view) to distinguish two special situations called below the Majorana case and the Dirac case.

Let $\mathscr{G}$ be the group of exact, linearly realized, internal global symmetries of the tree-level action that are respected by the renormalization conditions and let $\mathcal{U}(\cdot)$ be the representation of $\mathscr{G}$ on the left-chiral flavor eigenfields χ^a. The two-point function (6) obeys the following conditions

$$\begin{aligned}\mathscr{M}_L(s) &= \mathcal{U}(g)^\top \mathscr{M}_L(s)\,\mathcal{U}(g)\,,\\ \mathscr{M}_R(s) &= \mathcal{U}(g)^\dagger \mathscr{M}_R(s)\,\mathcal{U}(g)^\star\,,\\ \mathscr{Z}_L(s) &= \mathcal{U}(g)^\dagger \mathscr{Z}_L(s)\,\mathcal{U}(g)\,,\\ \mathscr{Z}_R(s) &= \mathcal{U}(g)^\top \mathscr{Z}_R(s)\,\mathcal{U}(g)^\star\,, \qquad \forall_{g\in\mathscr{G}}\,.\end{aligned} \tag{16}$$

Consider first a toy model in which fermions form three families, each one consisting of gluinos of the Minimal Supersymmetric Standard Model [35]. In this case the flavor index is a pair of an adjoint color index and a family index, and the most general matrices $\mathscr{Z}_{L,R}$ and $\mathscr{M}_{L,R}$ consistent with the $SU(3)_C$ symmetry have the form

$$\begin{aligned}\mathscr{M}_{L,R}(s) &= \mathbb{1}_{k\times k}\otimes \mathscr{M}^{\rm fam}_{L,R}(s)\,,\\ \mathscr{Z}_{L,R}(s) &= \mathbb{1}_{k\times k}\otimes \mathscr{Z}^{\rm fam}_{L,R}(s)\,,\end{aligned} \tag{17}$$

and thus

$$\mathbb{M}^2_L(s) = \mathbb{1}_{k\times k}\otimes \mathbb{M}^{2\,\rm fam}_L(s)\,, \tag{18}$$

where $\mathbb{1}_{k\times k}$ (with $k=8$) is the identity matrix in the adjoint color space, while $\mathscr{M}^{\rm fam}_{L,R}$ and $\mathscr{Z}^{\rm fam}_{L,R}$ are 3×3 matrices in the family space. In particular, $\mathbb{M}^2_L(m^2_{(a)})$ are diagonalizable if e.g. the tree-level contribution to $\mathbb{M}^{2\,\rm fam}_L(0)$ has non-degenerate eigenvalues. A situation in which the two-point functions have the form (17) with an arbitrary number f of "families", an arbitrary k, and with f different and nonvanishing eigenvalues of the tree-level contribution to $\mathbb{M}^{2\,\rm fam}_L(0)$ is called below the Majorana case; the total number of flavors equals $n = f\times k$. As far as the propagator and mixing are concerned, one can in this case restrict attention to a single color. [6] It is worth emphasizing that the Majorana case (as well as the Dirac case below) is defined here by demanding $m^{\rm tree}_{(a)} \neq 0$ for all a, in order to make the corresponding

[6] A more physical representative of the Majorana case is the type I seesaw mechanism with $k=1$ and $f=3+3$ neutrinos.

prescription in Sec. 3.A (respectively, 3.B) as simple and practical as possible.[7] Vanishing masses require a separate treatment and they are dealt with in Sec. 3.C devoted to the generic case.

Consider next a more interesting example of three families of down-type quarks in the SM (clearly, the $SU(3)_C \times U(1)_Q$ symmetry of the SM prohibits down-type quarks from mixing with other SM fermions). Without loss of generality, it can be assumed that the flavor eigenfields χ^a have been chosen so that the anti-Hermitian generator of $U(1)_Q$ is diagonal

$$\begin{aligned} \mathscr{f}_Q &= \mathbb{1}_{\ell\times\ell} \otimes \left[\begin{array}{cc} -\frac{ie}{3}\mathbb{1}_{3\times3} & 0 \\ 0 & \frac{ie}{3}\mathbb{1}_{3\times3} \end{array} \right] , \\ \mathcal{U}(\mathscr{g}_t^Q) &= \exp(t\, \mathscr{f}_Q) , \end{aligned}$$

where $\mathbb{1}_{\ell\times\ell}$ (with $\ell = 3$) is the identity matrix in the color space. The most general matrices $\mathscr{Z}_{L,R}$ and $\mathscr{M}_{L,R}$ consistent with Eqs. (12) and the $SU(3)_C \times U(1)_Q$ symmetry read

$$\begin{aligned} \mathscr{Z}_L(s) = \mathscr{Z}_R(s)^\top &= \mathbb{1}_{\ell\times\ell} \otimes \left[\begin{array}{cc} \mathscr{I}_+(s)^{-1} & 0 \\ 0 & \mathscr{I}_-(s)^{-1} \end{array} \right] , \\ \mathscr{M}_X(s) &= \mathbb{1}_{\ell\times\ell} \otimes \left[\begin{array}{cc} 0 & \mu_X(s) \\ \mu_X(s)^\top & 0 \end{array} \right] , \quad X = L, R, \end{aligned} \tag{19}$$

where $\mu_{L,R}$ and $\mathscr{I}_\pm(s)$ are arbitrary 3×3 matrices and, in addition, $\mathscr{I}_\pm(s)$ are nonsingular. Thus

$$\mathbb{M}_L^2(s) = \mathbb{1}_{\ell\times\ell} \otimes \left[\begin{array}{cc} \mathbb{M}_+^2(s) & 0 \\ 0 & \mathbb{M}_-^2(s) \end{array} \right] , \tag{20}$$

with

$$\begin{aligned} \mathbb{M}_+^2(s) &= \mathscr{I}_+(s)\mu_R(s)\mathscr{I}_-(s)^\top\mu_L(s)^\top , \\ \mathbb{M}_-^2(s) &= \mathscr{I}_-(s)\mu_R(s)^\top\mathscr{I}_+(s)^\top\mu_L(s) . \end{aligned} \tag{21}$$

Using the relation (valid if, e.g. tree-level masses are non-vanishing, so that $\mu_L(s)$ is nonsingular)

$$\mathbb{M}_-^2(s)^\top = \mu_L(s)^\top\, \mathbb{M}_+^2(s)\, \{\mu_L(s)^\top\}^{-1} , \tag{22}$$

[7] In light of neutrino oscillations, theories with massless spin-1/2 fermions are no longer so appealing. In fact even in the *pure* SM, symmetries exclude not only neutrino masses but also any mixing between, say, the muon-neutrino and other fermions, and thus allow to restrict the attention to the block of massive fermions.

one gets

$$\mathscr{X}_+(s) \equiv \det(s\mathbb{1} - \mathbb{M}^2_+(s)) = \det(s\mathbb{1} - \mathbb{M}^2_-(s))\,, \tag{23}$$

hence the determinant in Eq. (14) reads

$$\mathscr{X}(s) = \mathscr{X}_+(s)^{2\ell}\,. \tag{24}$$

It follows from Eq. (23) that complex poles corresponding to the left-chiral flavor eigenfields with opposite charges are located at the same points $s = m^2_{(a)}$. A situation in which the two-point functions have the form (19) with an arbitrary number f of families, an arbitrary ℓ, and with f different and nonvanishing eigenvalues of the tree-level contribution to $\mathbb{M}^2_+(0)$ is called below the Dirac case (the total number of flavors equals $n = 2\times f\times\ell$). Once again, as far as the propagator and mixing are concerned, one can in this case restrict attention to a single color, i.e. one can effectively neglect color factors $\mathbb{1}_{\ell\times\ell}$ in Eqs. (19)-(20).

A simple prescription for the pole part of the propagator (9) is given in the next section for these two special cases. A generalization to arbitrary $\mathbb{M}^2_L(m^2_{(a)})$ matrices consistent with three assumptions stated above is provided as well.

It should be noted, however, that infrared problems (see e.g. [30]) are *not* discussed in this paper. In other words, it is assumed that an IR regulator was introduced (if necessary) so that the propagators do have simple poles at the points obeying Eq. (15).

3 Prescription

3.A. Majoran case. Consider first the Majorana case, Eqs. (17). In order not to obscure the notation it is assumed that $k = 1$ and $^{\rm fam}$ superscripts are omitted; thus the total number of LWs is $n = f$. On the assumptions stated in Sec. 2, it is clear that Eq. (15) has, in the sense of formal power series, n different and non-vanishing solutions

$$m_{(a)} = m^{\rm tree}_{(a)} + \mathcal{O}(\hbar)\,, \tag{25}$$

such that $\mathrm{Re}(m_{(a)}) > 0$. Define

$$m = \mathrm{diag}(m_{(1)}\,, \dots\,, m_{(n)})\,. \tag{26}$$

It will be shown (in Sec. 4) that the $\hat{\mathscr{D}}_{\mathcal{F}}(p)$ matrix in the full propagator of two Majorana fields, Eq. (8), has the following simple form

$$\hat{\mathscr{D}}_{\mathcal{F}}(p) = \hat{\zeta}\,[p^2 - m^2]^{-1}[\not{p} + m]\,\hat{\zeta}^\top + [\text{non-pole part}]\,, \tag{27}$$

where

$$\hat{\zeta} = \zeta_L \, P_L \, + \, \zeta_R \, P_R \, ,$$
$$\hat{\zeta}^\top = {\zeta_L}^\top P_L + {\zeta_R}^\top P_R \, , \tag{28}$$

matrices $\zeta_{L,R}$ (as well as m) carry only flavor indices, while columns of ζ_L and ζ_R are given, respectively, by vectors $\zeta_{L[a]}$ and $\zeta_{R[a]}$ in the flavor space

$$\zeta_X = \Big[\big[\zeta_{X[1]} \big] \; \cdots \; \big[\zeta_{X[n]} \big] \Big] \, , \qquad X = L, R,$$

obtained in the following way. Let $\xi_{[a]}$ be an eigenvector of $\mathbb{M}^2_L(m^2_{(a)})$, Eq. (11), corresponding to the eigenvalue $m^2_{(a)}$

$$\mathbb{M}^2_L(m^2_{(a)}) \, \xi_{[a]} = m^2_{(a)} \, \xi_{[a]} \, , \tag{29}$$

and obeying the following normalization condition

$$\xi_{[a]}^\top \, \mathscr{M}_L(m^2_{(a)}) \, \xi_{[a]} = m_{(a)} \, , \tag{30}$$

then

$$\zeta_{L[a]} = \mathcal{N}(a) \, \xi_{[a]} \, , \tag{31}$$

with a normalizing factor

$$\mathcal{N}(a) = \Big\{ 1 - \frac{1}{m_{(a)}} \xi_{[a]}^\top \, \mathscr{M}_L(m^2_{(a)}) \, \mathbb{M}^{2\,\prime}_L(m^2_{(a)}) \, \xi_{[a]} \Big\}^{-1/2} \, , \tag{32}$$

where $\mathbb{M}^{2\,\prime}_L(s) \equiv \mathrm{d}\mathbb{M}^2_L(s)/\mathrm{d}s$, and

$$\zeta_{R[a]} = \frac{1}{m_{(a)}} \mathscr{Z}_R(m^2_{(a)})^{-1} \mathscr{M}_L(m^2_{(a)}) \, \zeta_{L[a]} \, . \tag{33}$$

(Note that, on the assumptions stated above, Eqs. (29)-(30), determine $\xi_{[a]}$ uniquely up to a sign; one could worry that the condition (30) cannot be imposed since e.g. $[1, \;\; i] \, [1, \;\; i]^\top = 0$, however such a pathology is impossible at the tree-level, and thus it is impossible for formal power series.)

Moreover it will be shown that, if Feynman integrals contributing to $\mathscr{Z}_{L,R}(p^2)$ and $\mathscr{M}_{L,R}(p^2)$ do not acquire imaginary parts in a left neighborhood $\mathscr{U}_a \subset \mathbb{R}$ of $p^2 = (m^{\mathrm{tree}}_{(a)})^2$

$$\mathscr{U}_a \equiv \big\{ p^2 \in \mathbb{R} | \;\; (m^{\mathrm{tree}}_{(a)})^2 - \varepsilon < p^2 \leq (m^{\mathrm{tree}}_{(a)})^2 \big\} \, , \qquad \varepsilon > 0 \, ,$$

so that the following reality conditions are satisfied

$$\mathscr{Z}_R(s) = \mathscr{Z}_L(s)^\star\,, \quad \mathscr{M}_R(s) = \mathscr{M}_L(s)^\star\,, \quad \forall_{s\in\mathscr{U}_a}\,, \tag{34}$$

then all terms of a formal power series $m_{(a)}$, Eq. (25), are real, and conditions (29)-(33) imply that $\zeta_{R[a]}$ is the complex conjugation of $\zeta_{L[a]}$.

If, in particular, conditions (34) are satisfied for all $a = 1, \ldots, n$, then matrices appearing in Eq. (28) obey $\zeta_R = \zeta_L{}^\star$ and Eq. (27) has a simple interpretation: the Majorana field ψ in, e.g., the $\overline{\text{MS}}$ scheme of dimensional regularization can be expressed in terms of its on-shell scheme counterpart ψ_{OS} (see e.g. [14]) as follows

$$\psi = \hat{\zeta}\,\psi_{\text{OS}}. \tag{35}$$

What if only some of the particles are stable? If $\text{Im}(m_{(a_S)}) = 0$, then one can introduce a free (interaction picture) Majorana field $\Psi^{\widetilde{a}_S}$ of mass $m_{(a_S)}$ with canonically normalized propagator and define (recall that $\widetilde{b}$ is the "total" index, cf. Eq. (3))

$$\Psi^{\widetilde{b}}_{\text{red}} = \sum_{\widetilde{a}_S} \big[\,\hat{\zeta}\,\big]^{\widetilde{b}}{}_{\widetilde{a}_S}\,\Psi^{\widetilde{a}_S}\,, \tag{36}$$

where the summation runs over all "stable indices". Clearly, Ψ_{red} is a free quantum field and Eq. (27) implies that the chronological propagator of Ψ_{red} reproduces the behavior of propagator in Eq. (8) about all poles located on the real axis. Thus, Ψ_{red} is the field that appears in the LSZ-reduced formula for the S-operator describing the transitions between stable states [16]

$$S = \,:\!\exp(\Sigma)\!:\, \exp(i\,W[J])\Big|_{J=0}\,, \tag{37}$$

with

$$\Sigma = -\int \mathrm{d}^4x\,\Psi^{\widetilde{b}}_{\text{red}}(x)\int \mathrm{d}^4y\,\Gamma_{\widetilde{b}\widetilde{c}}(x,y)\,\frac{\delta}{\delta J_{\widetilde{c}}(y)}\,, \tag{38}$$

where $\Gamma_{\widetilde{b}\widetilde{c}}(x,y)$ is the Fourier transform of (6), normal ordering in Eq. (37) refers to free quantum fields Ψ_{red}, while the connected generating functional $W[J]$ is related through the Legendre transform to the (renormalized) 1PI effective action $\Gamma[\psi]$

$$\Gamma[\psi] = W[\mathcal{J}^\psi] - \mathcal{J}^\psi\!\cdot\psi\,, \qquad \frac{\delta W[J]}{\delta J_{\widetilde{b}}(x)}\bigg|_{J=\mathcal{J}^\psi} = \psi^{\widetilde{b}}(x)\,,$$

(in the last three equations, ψ and Ψ represent not only fermions but also scalars and vectors).

What about unstable particles? Consider, for instance, a theory in which heavy neutrinos described in terms of Majorana fields $\psi_N^{\bar{a}}$ carrying a family index $\bar{a}$, interact with a Hermitian scalar field h and massless SM (anti)neutrinos, described in terms of Majorana fields $\psi_\nu^{\check{b}}$ carrying a family index $\check{b}$, through the following Lagrangian density (spinor indices are suppressed)

$$\mathcal{L}_{\rm int} = -h\,\bar{\psi}_N^{\bar{a}}\left(\mathcal{Y}_{\bar{a}\check{b}}P_L + \mathcal{Y}^\star_{\bar{a}\check{b}}P_R\right)\psi_\nu^{\check{b}}\,. \tag{39}$$

The CP-asymmetry

$$\varepsilon_{\bar{a}\check{b}} = \frac{\Gamma(N_{\bar{a}}\to h\nu_{\check{b}}) - \Gamma(N_{\bar{a}}\to h\bar{\nu}_{\check{b}})}{\Gamma(N_{\bar{a}}\to h\nu_{\check{b}}) + \Gamma(N_{\bar{a}}\to h\bar{\nu}_{\check{b}})}\,, \tag{40}$$

was calculated in [12, 10, 11, 13] by looking at diagrams in which $N_{\bar{a}}$ is an internal (rather than an external) line, what leads to the following expression

$$\varepsilon_{\bar{a}\check{b}} = \frac{|\mathcal{Y}^R_{\bar{a}\check{b}}|^2 - |\mathcal{Y}^L_{\bar{a}\check{b}}|^2}{|\mathcal{Y}^R_{\bar{a}\check{b}}|^2 + |\mathcal{Y}^L_{\bar{a}\check{b}}|^2}\,, \tag{41}$$

with

$$\mathcal{Y}^L_{\bar{a}\check{b}} = \mathcal{Y}_{\bar{a}'\check{b}}(\zeta_L)^{\bar{a}'}{}_{\bar{a}} + \ldots\,, \tag{42}$$

$$\mathcal{Y}^R_{\bar{a}\check{b}} = \mathcal{Y}^\star_{\bar{a}'\check{b}}(\zeta_R)^{\bar{a}'}{}_{\bar{a}} + \ldots\,, \tag{43}$$

where $\zeta_{R,L}$ are ζ matrices for the $\psi_N^{\bar{a}}$ fields, while the ellipsis indicates contributions of corrections to external lines of h and $\psi_\nu^{\check{b}}$ fields, as well as loop corrections to the 1PI vertices (for simplicity it is assumed here that the mixing between light and heavy neutrinos is negligible, even though the present formalism is capable of describing quantum corrections to the mixing in the full 6×6 system).

3.B Dirac case. Consider now the Dirac case, Eqs. (19). For simplicity of the notation it is assumed that $\ell = 1$, thus the total number of LWs is $n = 2f$. On the assumptions stated in Sec. 2, it is clear that Eq. (15), cf. Eqs. (23)-(24), has f different and non-vanishing solutions

$$m_{(a)} = m^{\rm tree}_{(a)} + \mathcal{O}(\hbar)\,, \tag{44}$$

such that $\mathrm{Re}(m_{(a)}) > 0$. Define

$$m_D = \mathrm{diag}(m_{(1)}\,,\ldots\,,m_{(f)})\,, \tag{45}$$

and

$$\widetilde{m} = \begin{bmatrix} 0 & m_D \\ m_D & 0 \end{bmatrix} . \tag{46}$$

It will be shown that the $\hat{\mathscr{D}}_{\mathcal{F}}(p)$ matrix in the full propagator of two Majorana fields, Eq. (8), has the form

$$\hat{\mathscr{D}}_{\mathcal{F}}(p) = \hat{\zeta}\,[p^2 - \widetilde{m}^2]^{-1}[\not{p} + \widetilde{m}]\,\hat{\zeta}^{\top} + [\text{non-pole part}] , \tag{47}$$

where

$$\hat{\zeta} = \zeta_L\, P_L + \zeta_R\, P_R , \tag{48}$$

while the $\zeta_{L,R}$ matrices have a block-diagonal form

$$\zeta_X = \begin{bmatrix} \bar{\zeta}_{X+} & 0 \\ 0 & \bar{\zeta}_{X-} \end{bmatrix} , \qquad X = L, R, \tag{49}$$

with matrices $\bar{\zeta}_{X\pm}$ built out of vectors $\bar{\zeta}_{X[a\pm]}$

$$\bar{\zeta}_{X\pm} = \Big[\big[\bar{\zeta}_{X[1\pm]}\big] \cdots \big[\bar{\zeta}_{X[f\pm]}\big] \Big] , \qquad X = L, R,$$

obtained in the following way. Let $\bar{\xi}_{[a\pm]}$ be arbitrary *but fixed* eigenvectors of $\mathbb{M}^2_{\pm}(m^2_{(a)})$, Eqs. (21), corresponding to the eigenvalue $m^2_{(a)}$

$$\begin{aligned} \mathbb{M}^2_{+}(m^2_{(a)})\,\bar{\xi}_{[a+]} &= m^2_{(a)}\,\bar{\xi}_{[a+]} , \\ \mathbb{M}^2_{-}(m^2_{(a)})\,\bar{\xi}_{[a-]} &= m^2_{(a)}\,\bar{\xi}_{[a-]} , \end{aligned} \tag{50}$$

(eigenspaces of $\mathbb{M}^2_{\pm}(m^2_{(a)})$ are one-dimensional on the assumptions stated in Sec. 2), and obeying the following normalization condition

$$\bar{\xi}^{\top}_{[a+]}\,\mu_L(m^2_{(a)})\,\bar{\xi}_{[a-]} = m_{(a)} . \tag{51}$$

Then

$$\begin{aligned} \bar{\zeta}_{L[a+]} &= \mathscr{c}(a)\,\bar{\mathscr{N}}(a)\,\bar{\xi}_{[a+]} , \\ \bar{\zeta}_{L[a-]} &= \mathscr{c}(a)^{-1}\,\bar{\mathscr{N}}(a)\,\bar{\xi}_{[a-]} , \end{aligned} \tag{52}$$

and

$$\begin{aligned} \bar{\zeta}_{R[a+]} &= \frac{1}{m_{(a)}}\,\mathscr{I}_{+}(m^2_{(a)})^{\top}\mu_L(m^2_{(a)})\,\bar{\zeta}_{L[a-]} , \\ \bar{\zeta}_{R[a-]} &= \frac{1}{m_{(a)}}\,\mathscr{I}_{-}(m^2_{(a)})^{\top}\mu_L(m^2_{(a)})^{\top}\,\bar{\zeta}_{L[a+]} , \end{aligned} \tag{53}$$

with a normalizing factor

$$\mathcal{N}(a) = \Big\{1 - \frac{1}{m_{(a)}}\bar{\xi}_{[a-]}^{\top}\,\mu_L(m_{(a)}^2)^{\top}\,\mathbb{M}_{+}^{2}{}'(m_{(a)}^2)\,\bar{\xi}_{[a+]}\Big\}^{-1/2}, \tag{54}$$

where $\mathbb{M}_{+}^{2}{}'(s) \equiv \mathrm{d}\mathbb{M}_{+}^{2}(s)/\mathrm{d}s$, while $\mathfrak{c}(a) \in \mathbb{C}\setminus\{0\}$ is an arbitrary number which does not affect the pole part of the propagator.

Moreover it will be shown that, if the reality conditions (34) hold in a left neighborhood $\mathscr{U}_a \subset \mathbb{R}$ of $p^2 = (m_{(a)}^{\mathrm{tree}})^2$, then all terms of a formal power series $m_{(a)}$, Eq. (44), are real, and there exists $\mathfrak{c}(a) \in \mathbb{C}\setminus\{0\}$ such that

$$\bar{\zeta}_{R[a+]} = \bar{\zeta}_{L[a+]}^{\star}, \qquad \text{and} \qquad \bar{\zeta}_{R[a-]} = \bar{\zeta}_{L[a-]}^{\star}. \tag{55}$$

With fixed $\bar{\xi}_{[a\pm]}$, conditions (55) determine $\mathfrak{c}(a)$ uniquely up to a phase. Thus, if conditions (34) are satisfied for all $a = 1,\ldots,f$, then matrices appearing in Eq. (48) obey $\zeta_R = \zeta_L{}^{\star}$.

3.C. Generic fermionic case. The above prescriptions can be generalized to the case constrained only by the three conditions discussed below Eq. (15). Recall that these conditions imply that the number of poles of the full propagator is equal to the total number n of LWs. The a label is assumed to distinguish different solutions $m_{(a)}^2$ to Eq. (15); indices corresponding to this eigenvalue are labeled with a_1, a_2, etc.

Let $\widetilde{m} = \widetilde{m}^{\top}$ be an *arbitrarily* chosen symmetric $n\times n$ matrix such that

$$\widetilde{m}^2 = \mathrm{diag}(m_{(1)}^2, \ldots), \tag{56}$$

and

$$\widetilde{m}_{a_r b_q} = 0 \quad \forall_{a_r}, \qquad \text{if} \qquad m_{(b)}^2 = 0. \tag{57}$$

Clearly,

$$[\widetilde{m}^2]_{a_r a_q} = m_{(a)}^2\,\delta_{rq}.$$

It will be shown that the $\hat{\mathscr{D}}_{\mathcal{F}}(p)$ matrix in the full propagator of two Majorana fields, Eq. (8), has the following form

$$\hat{\mathscr{D}}_{\mathcal{F}}(p) = \hat{\zeta}\,[p^2 - \widetilde{m}^2]^{-1}[\not{p} + \widetilde{m}]\,\hat{\zeta}^{\top} + [\text{non-pole part}], \tag{58}$$

where

$$\hat{\zeta} = \zeta_L P_L + \zeta_R P_R,$$

matrices $\zeta_{L,R}$ (as well as $\widetilde{m}$) carry only flavor indices, while columns of $\zeta_{L,R}$ are given by vectors $\zeta_{L,R[a_r]}$

$$\zeta_X = \Bigg[\Big[\zeta_{X[1_1]}\Big] \;\; \cdots\Bigg], \qquad X = L, R, \tag{59}$$

(the order of columns reflects the order of eigenvalues in Eq. (56)) obtained in the following way.

1. Nonzero $m^2_{(a)}$.

Let $\xi_{[a_1]}\,,\ldots,$ be a basis of the eigenspace

$$\mathbb{M}_L^2(m^2_{(a)})\,\xi_{[a_r]} = m^2_{(a)}\,\xi_{[a_r]}\,, \tag{60}$$

obeying the following normalization conditions

$$\xi^{\top}_{[a_r]}\,\mathscr{M}_L(m^2_{(a)})\,\xi_{[a_q]} = \widetilde{m}_{a_r a_q}\,, \tag{61}$$

(recall that for each pair of nonsingular complex symmetric matrices $S_{1,2}$ there always exists a nonsingular matrix N such that $S_1 = N^{\top}S_2N$, thus starting with an accidentally chosen basis of eigenspace one can always find vectors obeying Eq. (61); the non-singularity of the left-hand side of Eq. (61) is ensured by the assumptions listed below Eq. (15)). Define the following matrix

$$\Xi(a)_{qr} = \xi^{\top}_{[a_q]}\,\mathscr{M}_L(m^2_{(a)})\,\mathbb{M}_L^{2\,\prime}(m^2_{(a)})\,\xi_{[a_r]}\,, \tag{62}$$

which is symmetric (see Sec. 4) and find a matrix $\mathscr{N}(a)$ such that

$$\frac{1}{m^2_{(a)}}\,\mathscr{N}(a)\,\overline{m}(a)\,\mathscr{N}(a)^{\top} = \big(\overline{m}(a) - \Xi(a)\big)^{-1}\,, \tag{63}$$

where

$$\overline{m}(a)_{rq} = \widetilde{m}_{a_r a_q}\,. \tag{64}$$

(Clearly, $\mathscr{N}(a)$ is determined only up to a complex orthogonal matrix.) Then

$$\zeta_{L[a_r]} = \sum_q \mathscr{N}(a)^q{}_r\,\xi_{[a_q]}\,, \tag{65}$$

and

$$\zeta_{R[a_r]} = \frac{1}{m^2_{(a)}}\,\mathscr{Z}_R(m^2_{(a)})^{-1}\mathscr{M}_L(m^2_{(a)})\,\sum_q \zeta_{L[a_q]}\overline{m}(a)_{qr}\,. \tag{66}$$

Moreover it will be shown that, if the $\widetilde{m}$ matrix is chosen to be diagonal

$$\widetilde{m} = \mathrm{diag}(m_{(1)}\,,\ldots)\,,$$

with $\mathrm{Re}(m_{(a)}) > 0$, and if reality conditions (34) are satisfied in a left neighborhood $\mathscr{U}_a \subset \mathbb{R}$ of $p^2 = (m^{\mathrm{tree}}_{(a)})^2$, then all terms of a formal power series $m_{(a)}$ are real and there exists a $\mathscr{N}(a)$ matrix obeying Eq. (63) and such that

$\zeta_{R[a_r]} = \zeta^\star_{L[a_r]}$ for all r. With fixed $\{\xi_{[a_r]}\}$ eigenvectors this matrix is unique up to a *real* orthogonal matrix $\mathcal{R}(a)$, i.e. $\mathcal{N}(a) = \mathcal{N}_0(a)\mathcal{R}(a)$.

2. Vanishing $m^2_{(a)}$.

Let $\xi_{[0_1]}, \ldots,$ be a basis of the null eigenspace

$$\mathbb{M}^2_L(0)\, \xi_{[0_r]} = 0\,, \tag{67}$$

obeying the following normalization conditions

$$\xi^\dagger_{[0_r]}\, \mathscr{Z}_L(0)\, \xi_{[0_q]} = \delta_{rq}\,, \tag{68}$$

(for $p^2 = 0$ reality conditions (34) cannot be violated and thus $\mathscr{Z}_L(0)$ is a Hermitian and positive matrix, cf. Eqs. (7) and (12)). Define the following matrix

$$\Xi(0)_{qr} = \xi^\dagger_{[0_q]}\, \mathscr{Z}_L(0)\, \mathbb{M}^{2\,\prime}_L(0)\, \xi_{[0_r]}\,, \tag{69}$$

which is Hermitian (see Sec. 4) and find a matrix $\mathcal{N}(0)$ such that

$$\mathcal{N}(0)\, \mathcal{N}(0)^\dagger = \big(\mathbb{1} - \Xi(0)\big)^{-1}\,. \tag{70}$$

Then

$$\zeta_{L[0_r]} = \sum_q \mathcal{N}(0)^q{}_r\, \xi_{[0_q]}\,, \tag{71}$$

and

$$\zeta_{R[0_r]} = \zeta^\star_{L[0_r]}\,. \tag{72}$$

It should be stressed that auxiliary normalization conditions (61) and (68) are, in fact, redundant, i.e. prescriptions (63) and (70) for normalizing factors can be easily generalized to the case when the basis $\{\xi_{[a_r]}\}$ of eigenspace is completely arbitrary. Nonetheless, Eqs. (61) and (68) are imposed here, since the resulting equations (63) and (70) show immediately that, if flavor eigenfields are chosen to be canonically normalized eigenstates of the tree-level mass matrix, as is usually the case, then the $\mathcal{N}(a)$ matrix can be chosen as an $\mathcal{O}(\hbar)$ perturbation of the identity matrix, while $\xi_{[a_r]}$ can be chosen as $\mathcal{O}(\hbar)$ perturbations of vectors belonging to the canonical basis of $\mathbb{R}^n \subset \mathbb{C}^n$.

3.D. Scalar case. Consider a set $\{\phi^\ell\}$ of n scalar fields. Without loss of generality it is assumed that ϕ^ℓ are Hermitian. The renormalized 1PI two-point function

$$
\begin{aligned}
\widetilde{\Gamma}_{\ell j}(-p,p) &= \Big[p^2\mathbb{1} - (M^{\text{tree}})^2 - \Sigma(p^2)\Big]_{\ell j} \\
&\equiv \Big[p^2\mathbb{1} - M^2(p^2)\Big]_{\ell j} ,
\end{aligned}
\tag{73}
$$

where $M^2(s) = M^2(s)^\top$ is a symmetric matrix, leads to the propagator

$$
\widetilde{G}^{\ell j}(p,-p) = i\Big[\big(p^2\mathbb{1} - M^2(p^2)\big)^{-1}\Big]^{\ell j} ,
\tag{74}
$$

and the gap equation

$$
\mathscr{X}_S(m^2_{(\ell)}) = 0 ,
\tag{75}
$$

with

$$
\mathscr{X}_S(s) \equiv \det(s\mathbb{1} - M^2(s)) .
\tag{76}
$$

It is assumed that assumptions listed below Eq. (15) for fermionic solutions $m^2_{(a)}$ and matrices $\mathbb{M}^2_L(m^2_{(a)})$, are satisfied also for their scalar counterparts, $m^2_{(\ell)}$ and $M^2(m^2_{(\ell)})$.

Let m^2 be a diagonal $n \times n$ matrix

$$
m^2 = \operatorname{diag}(m^2_{(1)}, \ldots) .
\tag{77}
$$

The ℓ label is assumed to distinguish different values $m^2_{(\ell)}$; indices corresponding to this value in Eq. (77) are labeled with ℓ_1, ℓ_2, etc.

It will be shown that the propagator (74) has the form

$$
\widetilde{G}(p,-p) = i\,\zeta\,[p^2 - m^2]^{-1}\,\zeta^\top + [\text{non-pole part}] ,
\tag{78}
$$

where columns of ζ are given by vectors $\zeta_{[\ell_r]}$

$$
\zeta = \Big[\Big[\zeta_{[1_1]}\Big] \;\; \cdots \;\Big] ,
\tag{79}
$$

(the order of columns reflects the order of eigenvalues in Eq. (77)) obtained in the following way. Let $\xi_{[\ell_1]}, \ldots,$ be a basis of the eigenspace

$$
M^2(m^2_{(\ell)})\,\xi_{[\ell_r]} = m^2_{(\ell)}\,\xi_{[\ell_r]} ,
\tag{80}
$$

obeying the following normalization conditions

$$
\xi^\top_{[\ell_r]}\,\xi_{[\ell_q]} = \delta_{rq} ,
\tag{81}
$$

(starting with an arbitrary basis of eigenspace one can always find vectors obeying Eq. (81), just as in the fermionic case). Define the following matrix

$$\Xi(\ell)_{qr} = \xi_{[\ell_q]}^{\top} M^{2\prime}(m_{(\ell)}^2)\, \xi_{[\ell_r]}\,, \tag{82}$$

which is manifestly symmetric, and find a matrix $\mathcal{N}(\ell)$ such that

$$\mathcal{N}(\ell)\,\mathcal{N}(\ell)^{\top} = \big(\mathbb{1} - \Xi(\ell)\big)^{-1}\,. \tag{83}$$

(Clearly, $\mathcal{N}(\ell)$ is determined only up to a complex orthogonal matrix.) Then

$$\zeta_{[\ell_r]} = \sum_q \mathcal{N}(\ell)^q{}_r\, \xi_{[\ell_q]}\,. \tag{84}$$

Moreover it will be shown that, if Feynman integrals contributing to $M^2(p^2)$ do not acquire imaginary parts in a left neighborhood $\mathscr{U}_\ell \subset \mathbb{R}$ of $p^2 = (m_{(\ell)}^{\text{tree}})^2$, so that the following reality conditions are satisfied

$$M^2(s) = M^2(s)^\star\,, \qquad \forall_{s\in\mathscr{U}_\ell}\,, \tag{85}$$

then all terms of a formal power series $m_{(\ell)}^2$ are real and there exists a $\mathcal{N}(\ell)$ matrix obeying Eq. (83) and such that $\zeta_{[\ell_r]} = \zeta_{[\ell_r]}^\star$ for all r. With fixed $\{\xi_{[\ell_r]}\}$ eigenvectors this matrix is unique up to a *real* orthogonal matrix $\mathcal{R}(\ell)$, i.e. $\mathcal{N}(\ell) = \mathcal{N}_0(\ell)\mathcal{R}(\ell)$.

3.E. Fermionic one-loop self-energy. It is convenient to supplement the prescription for fermionic $\zeta_{L,R}$ matrices by providing generic expressions for one-loop contributions in the $\overline{\text{MS}}$ scheme with anticommuting γ^5 to the two-point functions $\mathscr{Z}_{L,R}$ and $\mathscr{M}_{L,R}$ in Eq. (6). Consider an arbitrary renormalizable model, in which Majorana fields ψ^a (spinor indices are suppressed for simplicity) interact with Hermitian scalar fields ϕ^ℓ (already shifted if necessary, so that $\langle\phi\rangle = 0$) and Hermitian gauge fields A_μ^α via the following Lagrangian density

$$\begin{aligned}\mathcal{L}_{\text{int}}^{\text{tree}} = &+\frac{1}{2!}\, i\, A_\mu^\alpha\, \bar{\psi}^a\, \gamma^\mu\, (\mathfrak{f}_{\alpha ab}\, P_L + \mathfrak{f}_{\alpha ab}^\star\, P_R)\, \psi^b + \\ &-\frac{1}{2!}\, \phi^\ell\, \bar{\psi}^a\, (Y_{\ell ab}\, P_L + Y_{\ell ab}^\star\, P_R)\, \psi^b\,.\end{aligned} \tag{86}$$

Here $\mathfrak{f}_{\alpha ab} = -\mathfrak{f}_{\alpha ba}^\star$ are matrix elements of ordinary anti-Hermitian gauge-group generators (already containing the coupling constants), while $Y_{\ell ab} = Y_{\ell ba}$ are matrix elements of symmetric Yukawa matrices. It is assumed that all fields are chosen to be the eigenfields of the tree-level mass-squared matrices, so that

$$\mathcal{L}^{\text{tree}}_{\text{mass}} = +\frac{1}{2}\sum_{\beta} m^2_{V\beta}\,\eta^{\mu\nu} A^{\beta}_{\mu} A^{\beta}_{\nu} \; - \frac{1}{2}\sum_{\ell} m^2_{S\ell}\,\phi^{\ell}\phi^{\ell} + \\ -\frac{1}{2}\,\bar{\psi}^{a}\left(M_{Fab}\,P_L + M^{\star}_{Fab}\,P_R\right)\psi^{b}\,,$$

where

$$M_F M_F^{\star} = \operatorname{diag}(m^2_{F1}\,,\; m^2_{F2}\,,\; \ldots\,,\; m^2_{Fn})\,, \tag{87}$$

(clearly, without loss of generality one could assume that M_F itself is diagonal; such a choice is however completely impractical for Dirac particles, as it implies that, for instance, the u matrix, Eq. (5), in the SM is non-diagonal).

Functions $\mathscr{Z}_{L,R}$ and $\mathscr{M}_{L,R}$ can be parametrized in the following way

$$\begin{aligned} \mathscr{Z}_{L,R}(p^2) &= \mathbb{1} + \frac{\hbar}{(4\pi)^2}\,\mathscr{Z}^{(1)}_{L,R}(p^2) + \mathcal{O}(\hbar^2)\,, \\ \mathscr{M}_{L}(p^2) &= M_F + \hbar\,Y_{\ell}\,v^{\ell}_{(1)} + \frac{\hbar}{(4\pi)^2}\,\mathscr{M}^{(1)}_{L}(p^2) + \mathcal{O}(\hbar^2)\,, \\ \mathscr{M}_{R}(p^2) &= M_F^{\star} + \hbar\,Y^{\star}_{\ell}\,v^{\ell}_{(1)} + \frac{\hbar}{(4\pi)^2}\,\mathscr{M}^{(1)}_{R}(p^2) + \mathcal{O}(\hbar^2)\,, \end{aligned} \tag{88}$$

where $v^{\ell}_{(1)}$ represents the one-loop contribution to the scalar vacuum expectation value (VEV), while $\mathscr{Z}^{(1)}_{L,R}$ and $\mathscr{M}^{(1)}_{L,R}$ are produced by one-loop diagrams shown in Figure 1 (we work in the Landau gauge). Using the standard, minimally subtracted one-loop functions a^R and b_0^R in the dimensional regularization [32]

$$a^R(m) = m^2\left\{\ln\frac{m^2}{\bar{\mu}^2} - 1\right\}\,,$$

$$B_M\big(p^2, m_1, m_2\big) \equiv b_0^R\big(p^2, m_1, m_2\big) = \\ = \int_0^1 \mathrm{d}x\,\ln\frac{x(x-1)p^2 + (1-x)m_1^2 + x\,m_2^2 - i\,0}{\bar{\mu}^2}\,,$$

(here $\bar{\mu}$ is the renormalization scale of the $\overline{\text{MS}}$ scheme, related to the usual 't Hoot mass unit via $\bar{\mu} \equiv \mu_H\sqrt{4\pi}\,e^{-\gamma_E/2}$), together with their combinations $B_M(\equiv b_0^R)$, B_Z, A_M and A_Z

$$B_Z\big(p^2, m_S, m_F\big) = \frac{1}{2\,p^2}\Big\{a^R(m_F) - a^R(m_S) + \\ + \big(m_S^2 - m_F^2 - p^2\big)\,b_0^R\big(p^2, m_S, m_F\big)\Big\}\,,$$

$$A_M(p^2, m_V, m_F) = 3\, b_0^R\big(p^2, m_V, m_F\big) + 2\,,$$

$$\begin{aligned} A_Z(p^2, m_V, m_F) &= \frac{m_F^2 + 2m_V^2 - p^2}{2p^2}\,\frac{a^R(m_V)}{m_V^2} + \\ &+1 - \frac{a^R(m_F)}{p^2} + \frac{p^2 + m_F^2 - 2m_V^2}{2p^2}\; b_0^R\big(p^2, m_V, m_F\big) + \\ &+ \frac{(p^2 - m_F^2)^2}{2p^2}\,\frac{b_0^R(p^2, m_V, m_F) - b_0^R(p^2, 0, m_F)}{m_V^2}\,, \end{aligned}$$

one gets [8]

$$\begin{aligned} [\mathscr{Z}_L^{(1)}(s)]_{ac} = &\sum_{\beta,b} A_Z(s\,,\, m_{V\beta}\,,\, m_{Fb})\, \mathfrak{f}_{\beta ab}\, \mathfrak{f}_{\beta bc} + \\ &+ \sum_{\ell,b} B_Z(s\,,\, m_{S\ell}\,,\, m_{Fb})\, Y^\star_{\ell ab}\, Y_{\ell bc}\,, \end{aligned}$$

$$\begin{aligned} [\mathscr{Z}_R^{(1)}(s)]_{ac} = &\sum_{\beta,b} A_Z(s\,,\, m_{V\beta}\,,\, m_{Fb})\, \mathfrak{f}^\star_{\beta ab}\, \mathfrak{f}^\star_{\beta bc} + \\ &+ \sum_{\ell,b} B_Z(s\,,\, m_{S\ell}\,,\, m_{Fb})\, Y_{\ell ab}\, Y^\star_{\ell bc}\,, \end{aligned}$$

$$\begin{aligned} [\mathscr{M}_L^{(1)}(s)]_{ac} = &\sum_{\beta,b,d} A_M(s\,,\, m_{V\beta}\,,\, m_{Fb})\, \mathfrak{f}^\star_{\beta ab}\, M_{Fbd}\, \mathfrak{f}_{\beta dc} + \\ &+ \sum_{\ell,b,d} B_M(s\,,\, m_{S\ell}\,,\, m_{Fb})\, Y_{\ell ab}\, M^\star_{Fbd}\, Y_{\ell dc}\,, \end{aligned}$$

$$\begin{aligned} [\mathscr{M}_R^{(1)}(s)]_{ac} = &\sum_{\beta,b,d} A_M(s\,,\, m_{V\beta}\,,\, m_{Fb})\, \mathfrak{f}_{\beta ab}\, M^\star_{Fbd}\, \mathfrak{f}^\star_{\beta dc} + \\ &+ \sum_{\ell,b,d} B_M(s\,,\, m_{S\ell}\,,\, m_{Fb})\, Y^\star_{\ell ab}\, M_{Fbd}\, Y^\star_{\ell dc}\,. \end{aligned}$$

In particular, reality conditions (34) are violated whenever b_0^R has a non-vanishing imaginary part.

Clearly, in the expression for A_Z, the limits $m_{V\beta} \to 0$ are to be taken for contributions of massless gauge bosons. On the other hand, the last term

[8] Correctness of these results was *checked* with the aid of FeynCalc [33].

Figure 1: One-loop contributions to $\mathscr{M}_{L,R}$ and $\mathscr{Z}_{L,R}$ in the Landau gauge.

in A_Z, even for spontaneously broken gauge symmetries, contains contributions of unphysical massless modes; as far as corrections to the pole masses are concerned, they cancel with similar contributions of would-be Goldstone bosons, as the gauge symmetry leads to the following relation

$$s_\gamma\, m_{V\gamma}\, Y_g = M_F\, \mathfrak{f}_\gamma - \mathfrak{f}^\star_\gamma\, M_F\,, \tag{89}$$

where Y_g is a Yukawa matrix of the (massless) would-be Goldstone boson ϕ^g associated with a broken generator $\mathfrak{f}_\gamma$, while $s_\gamma = -1$ or $s_\gamma = +1$. By contrast, contributions of unphysical modes to the $\zeta_{L,R}$ matrices do not cancel completely, but Eq. (89) ensures that they do not contain resonant factors $(m^2_{Fa} - m^2_{Fb})^{-1}$ [12]. Thus, in the CP-asymmetry (41) these contributions cancel with similar "unphysical" corrections to the 1PI vertices (indicated by the ellipsis in Eqs. (42)-(43)).

Finally, the one-loop contribution $v_{(1)}$ to the VEV can be obtained from the tadpole cancellation condition in the Landau gauge

$$0 = -\mathcal{V}'_i(v_{(0)} + \hbar\, v_{(1)}) + \frac{\hbar}{(4\pi)^2}\Big\{\ 3\sum_{\alpha j}[\mathcal{T}^2_\alpha]_{ij}\, v^j_{(0)}\left[a^R(m_{V\alpha}) + \frac{2}{3}m^2_{V\alpha}\right] +$$

$$-\frac{1}{2}\sum_j \rho_{ijj}\, a^R(m_{Sj}) + \sum_{bc}(M_{Fbc}Y^\star_{icb} + M^\star_{Fbc}Y_{icb})a^R(m_{Fb})\Big\} + \mathcal{O}(\hbar^2)\,.$$

Here $\mathcal{V}$ is the gauge-invariant tree-level potential of scalar fields, $v_{(0)}$ represents the tree-level VEV (i.e. $\mathcal{V}'_i(v_{(0)}) = 0$), $\rho_{ijk} = \mathcal{V}'''_{ijk}(v_{(0)})$, while $\mathcal{T}_\alpha$ is the generator of the gauge group on scalar fields; $\mathcal{T}_\alpha$ is normalized in such a way that the covariant derivative reads

$$(D_\mu\phi)^j = \partial_\mu\phi^j + A^\alpha_\mu[\mathcal{T}_\alpha]^j{}_k(\phi^k + v^k_{(0)} + \hbar\, v^k_{(1)} + \ldots)\,.$$

4 Proof

4.A. Proof of generic fermionic prescription.
The proof is a simple exercise in linear algebra.

0. Generalities. First of all, one has to calculate the following limits (cf. Eqs. (10) and (15))

$$\Delta_{L,R}(a) = \lim_{s\to m_{(a)}^2} \left\{ (s - m_{(a)}^2) \left[s\mathbb{1} - \mathbb{M}_{L,R}^2(s) \right]^{-1} \right\} . \tag{90}$$

It is convenient to start with something simpler

$$\mathbb{P}(a) = \lim_{s\to m_{(a)}^2} \left\{ (s - m_{(a)}^2) \left[s\mathbb{1} - \mathbb{M}_{L}^2(m_{(a)}^2) \right]^{-1} \right\} . \tag{91}$$

On the assumptions stated in Sec. 2, this limit exists and gives a projection onto the eigenspace of $\mathbb{M}_L^2(m_{(a)}^2)$ associated with $m_{(a)}^2$ along the direct sum of remaining generalized eigenspaces of $\mathbb{M}_L^2(m_{(a)}^2)$. To verify this statement, it is enough to calculate the action of the right-hand side of Eq. (91) on generalized eigenvectors of $\mathbb{M}_L^2(m_{(a)}^2)$. Introducing the resolvent

$$R(s) = \left(s\mathbb{1} - \mathbb{M}_L^2(m_{(a)}^2) \right)^{-1} ,$$

one has

$$R(s)\xi_{[a_r]} = (s - m_{(a)}^2)^{-1}\xi_{[a_r]} ,$$

for all eigenvectors $\xi_{[a_r]}$ associated with $m_{(a)}^2$. Let $\lambda_\theta \neq m_{(a)}^2$ be another eigenvalue of $\mathbb{M}_L^2(m_{(a)}^2)$; the generalized eigenspace associated with it is spanned by (in general more than one) Jordan chain $\theta_1 , \ldots , \theta_p$ (a subsequence of the Jordan basis for $\mathbb{M}_L^2(m_{(a)}^2)$, see e.g. [29])

$$\theta_r = \left(\lambda_\theta \mathbb{1} - \mathbb{M}_L^2(m_{(a)}^2) \right) \theta_{r+1} , \qquad r = 0, \ldots , p-1 , \tag{92}$$

where $\theta_0 \equiv 0$, i.e. θ_1 is an eigenvector. Let $Q(s) = (s - \lambda_\theta)^{-1}$, the following identity can be easily checked by induction

$$R(s)\,\theta_r = \sum_{k=1}^{r} (-1)^{k+1}\, Q(s)^k \,\theta_{r+1-k} . \tag{93}$$

Thus

$$\mathbb{P}(a)\,\xi_{[a_k]} = \xi_{[a_k]} , \qquad \mathbb{P}(a)\,\theta_r = 0 , \tag{94}$$

as was to be shown. In particular

$$\mathbb{P}(a)^2 = \mathbb{P}(a) . \tag{95}$$

Expanding $\mathbb{M}_L^2(s)$ in Eq. (90) about $s = m_{(a)}^2$ one gets

$$\Delta_L(a) = \left\{ \mathbb{1} - \mathbb{P}(a)\, \mathbb{M}_L^{2\prime}(m_{(a)}^2) \right\}^{-1} \mathbb{P}(a) . \tag{96}$$

Eq. (13) now yields

$$\Delta_R(a) = \mathscr{Z}_R(m^2_{(a)})^{-1}\, \Delta_L(a)^\top \mathscr{Z}_R(m^2_{(a)})\,.$$

Introducing another family $\{\widetilde{P}(a)\}$ of projections

$$\widetilde{P}(a) = \lim_{s\to m^2_{(a)}} \left\{ (s - m^2_{(a)}) \left[s\mathbb{1} - \widetilde{m}^2 \right]^{-1} \right\},$$

one can decompose the $\widetilde{m}^2$ matrix in Eq. (56) as follows

$$\widetilde{m}^2 = \sum_a m^2_{(a)}\, \widetilde{P}(a)\,, \tag{97}$$

clearly

$$\sum_a \widetilde{P}(a) = \mathbb{1}\,, \qquad \text{and} \qquad \widetilde{P}(a)\widetilde{P}(b) = \delta_{ab}\widetilde{P}(a)\,.$$

Now one sees that the formula that needs to be proven, Eq. (58), is equivalent to the following four sets of conditions ($\bar{s}_a \equiv m^2_{(a)}$)

$$\zeta_L\, \widetilde{P}(a)\, \zeta_R^\top = \Delta_L(a) \mathscr{Z}_L(m^2_{(a)})^{-1}\,, \qquad \forall a, \tag{98}$$

$$\zeta_R\, \widetilde{P}(a)\, \zeta_L^\top = \mathscr{Z}_R(m^2_{(a)})^{-1} \Delta_L(a)^\top, \quad \forall a, \tag{99}$$

$$\zeta_L\, \widetilde{P}(a)\, \widetilde{m}\, \zeta_L^\top = \Delta_L(a) \mathscr{Z}_L(\bar{s}_a)^{-1} \mathscr{M}_R(\bar{s}_a) \mathscr{Z}_R(\bar{s}_a)^{-1}\,, \;\; \forall a,$$
$$\zeta_R\, \widetilde{P}(a)\, \widetilde{m}\, \zeta_R^\top = \mathscr{Z}_R(\bar{s}_a)^{-1} \Delta_L(a)^\top \mathscr{M}_L(\bar{s}_a) \mathscr{Z}_L(\bar{s}_a)^{-1}\,, \; \forall a.$$

Thus, one has to show that there exist matrices $\zeta_{L,R}$ obeying, in addition to Eqs. (98)-(99), the following conditions

$$\zeta_L\, \widetilde{P}(a)\, \widetilde{m}\, \zeta_L^\top = \zeta_L\, \widetilde{P}(a)\, \zeta_R^\top \mathscr{M}_R(m^2_{(a)}) \mathscr{Z}_R(m^2_{(a)})^{-1}\,,$$
$$\zeta_R\, \widetilde{P}(a)\, \widetilde{m}\, \zeta_R^\top = \zeta_R\, \widetilde{P}(a)\, \zeta_L^\top \mathscr{M}_L(m^2_{(a)}) \mathscr{Z}_L(m^2_{(a)})^{-1}\,.$$

It is enough to impose, instead of the last two equations, the following two (cf. Eq. (12))

$$\zeta_R\, \widetilde{m}\, \widetilde{P}(a) = \mathscr{Z}_R(m^2_{(a)})^{-1} \mathscr{M}_L(m^2_{(a)})\, \zeta_L\, \widetilde{P}(a)\,, \tag{100}$$

$$\zeta_L\, \widetilde{m}\, \widetilde{P}(a) = \mathscr{Z}_L(m^2_{(a)})^{-1} \mathscr{M}_R(m^2_{(a)})\, \zeta_R\, \widetilde{P}(a)\,. \tag{101}$$

Using the following relation (cf. Eq. (64))

$$\left[\widetilde{P}(a)\, \widetilde{m}\, \widetilde{P}(a) \right]_{a_r a_q} = \overline{m}(a)_{rq}\,,$$

together with the identity [9] $\widetilde{m}\,\widetilde{P}(a) = \widetilde{P}(a)\,\widetilde{m}$, which yields in turn

$$\widetilde{m}\,\widetilde{P}(a) = \widetilde{P}(a)\,[\widetilde{P}(a)\,\widetilde{m}\,\widetilde{P}(a)]\,,$$

one can rewrite Eqs. (100)-(101) in terms of respective columns of matrices $\zeta_{L,R}$ in Eq. (59)

$$\sum_r \zeta_{R[a_r]}\,\overline{m}(a)_{rq} = \mathscr{Z}_R(m^2_{(a)})^{-1}\mathscr{M}_L(m^2_{(a)})\,\zeta_{L[a_q]}\,, \tag{102}$$

$$\sum_r \zeta_{L[a_r]}\,\overline{m}(a)_{rq} = \mathscr{Z}_L(m^2_{(a)})^{-1}\mathscr{M}_R(m^2_{(a)})\,\zeta_{R[a_q]}\,. \tag{103}$$

1. Nonzero $m^2_{(a)}$. Consider first the case $m^2_{(a)} \neq 0$; then Eq. (102) is nothing more than the relation (66), since

$$\overline{m}(a)^2 = m^2_{(a)}\,\mathbb{1}\,.$$

In turn, Eq. (66) allows to rewrite Eq. (103) in an equivalent form

$$\begin{aligned} m^2_{(a)}\,\zeta_{L[a_r]} &= \\ &= \mathscr{Z}_L(m^2_{(a)})^{-1}\mathscr{M}_R(m^2_{(a)})\mathscr{Z}_R(m^2_{(a)})^{-1}\mathscr{M}_L(m^2_{(a)})\zeta_{L[a_r]}\,, \end{aligned} \tag{104}$$

hence columns $\zeta_{L[a_1]}, \dots$, of ζ_L are eigenvectors of $\mathbb{M}^2_L(m^2_{(a)})$ corresponding to the eigenvalue $m^2_{(a)}$, just as in Eqs. (60) and (65).

Without loss of generality one can thus assume that, for $m^2_{(a)} \neq 0$, $\zeta_{L[a_r]}$ are linear combinations of linearly independent eigenvectors $\xi_{[a_q]}$ obeying the normalization conditions (61), with (yet unspecified) coefficients $\mathcal{N}(a)^q{}_r$, as in Eq. (65). It remains to be shown that Eqs. (98)-(99) are equivalent to the condition (63) on the matrix $\mathcal{N}(a)$. In fact Eq. (99), being a transposition of (98), can be skipped. Clearly,

$$\zeta_L\,\widetilde{P}(a)\,\zeta_R^\top = \sum_q \zeta_{L[a_q]}\zeta^\top_{R[a_q]}\,.$$

Employing Eq. (66), and defining

$$\mathbb{Y}(a) \equiv \sum_{q,r} \tau(i)^{qr}\,\xi_{[a_q]}\,\xi^\top_{[a_r]}\,\mathscr{M}_L(m^2_{(a)})\,, \tag{105}$$

with

$$\tau(a) \equiv \frac{1}{m^2_{(a)}}\,\mathcal{N}(a)\,\overline{m}(a)\,\mathcal{N}(a)^\top\,, \tag{106}$$

[9] This identity follows from $(p^2 - \widetilde{m}^2)^{-1}\,\widetilde{m} = \widetilde{m}\,(p^2 - \widetilde{m}^2)^{-1}$, in the limit $p^2 \to m^2_{(a)}$.

one can rewrite Eq. (98) as

$$\mathbb{Y}(a) = \Delta_L(a)\,,$$

or, using Eq. (96), as

$$\mathbb{Y}(a) = \Big\{\mathbb{1} - \mathbb{P}(a)\,\mathbb{M}_L^{2\,\prime}(m_{(a)}^2)\Big\}^{-1}\mathbb{P}(a)\,. \tag{107}$$

Eq. (107) can be further rewritten as

$$\mathbb{P}(a) = \mathbb{S}(a)\,, \tag{108}$$

where

$$\mathbb{S}(a) \equiv \mathbb{Y}(a)\Big\{\mathbb{1} + \mathbb{M}_L^{2\,\prime}(m_{(a)}^2)\,\mathbb{Y}(a)\Big\}^{-1}\,. \tag{109}$$

Note that the left-hand side of Eq. (108) is a projection, cf. Eq. (95), thus it remains to be shown that $\mathcal{N}(a)$ can be chosen in such a way that $\mathbb{S}(a)$ is a projection operator with the same image and the same kernel as $\mathbb{P}(a)$, cf. Eq. (94). To that end, it is convenient to simplify first the explicit expression (109) for $\mathbb{S}(a)$. Expanding a geometric series and appropriately changing the order of infinite sum with the summation over q and r appearing in Eq. (105) one ends up with another geometric series, thus

$$\mathbb{S}(a) = \sum_{q,r}\Big[\Omega(a)^{-1}\tau(a)\Big]^{qr}\,\xi_{[a_q]}\,\xi_{[a_r]}^{\top}\,\mathscr{M}_L(m_{(a)}^2)\,, \tag{110}$$

where

$$\Omega(a) \equiv \mathbb{1} + \tau(a)\,\Xi(a)\,, \tag{111}$$

with the $\Xi(a)$ matrix defined in Eq. (62). The normalization condition for $\xi_{[a_r]}$ eigenvectors, Eq. (61), gives (cf. Eq. (64))

$$\mathbb{S}(a)\xi_{[a_r]} = \sum_q\big[\Omega(a)^{-1}\tau(a)\overline{m}(a)\big]^q{}_r\;\xi_{[a_q]}\,, \tag{112}$$

hence

$$\begin{aligned}\mathbb{S}(a)^2 = \sum_{q,r}\big[\Omega(a)^{-1}\tau(a)\overline{m}(a)\Omega(a)^{-1}\tau(a)\big]^{qr}\;\times\\ \times\,\xi_{[a_q]}\,\xi_{[a_r]}^{\top}\,\mathscr{M}_L(m_{(a)}^2)\,.\end{aligned} \tag{113}$$

Comparing this with Eq. (110) one sees that $\mathbb{S}(a)$ is a projection operator if, for instance, the following equation is satisfied

$$\tau(a)\overline{m}(a) = \Omega(a)\,, \tag{114}$$

this is nothing more than the condition (63). To prove that a matrix $\mathcal{N}(a)$ obeying Eq. (63) indeed exists, it is necessary to show that the $\Xi(a)$ matrix, defined in Eq. (62), is symmetric. The following identity

$$\begin{aligned}&\mathbb{M}_L^{2\prime}(s)^\top \mathscr{M}_L(s) - \mathscr{M}_L(s)\,\mathbb{M}_L^{2\prime}(s) = \\ &\qquad = \mathscr{M}_L'(s)\,\mathbb{M}_L^2(s) - \mathbb{M}_L^2(s)^\top \mathscr{M}_L'(s)\,,\end{aligned}$$

is easy to verify (cf. Eq. (11)); sandwiched between $\xi_{[a_q]}^\top$ and $\xi_{[a_r]}$ it gives

$$\Xi(a)_{rq} - \Xi(a)_{qr} = 0\,, \tag{115}$$

since $\xi_{[a_{q,r}]}$ are eigenvectors of $\mathbb{M}_L^2(m_{(a)}^2)$ corresponding to the same eigenvalue.

Moreover, Eqs. (112) and (114) show that

$$\mathbb{S}(a)\xi_{[a_r]} = \xi_{[a_r]}\,. \tag{116}$$

Hence, to complete the proof of the generalized prescription for $m_{(a)}^2 \neq 0$, one has to show that the $\mathbb{S}(a)$ operator annihilates these generalized eigenvectors of $\mathbb{M}_L^2(m_{(a)}^2)$ which correspond to eigenvalues different than $m_{(a)}^2$, so that $\mathbb{S}(a) = \mathbb{P}(a)$. Because of Eq. (110) it is enough to prove the following property: let η be an eigenvector of $\mathbb{M}_L^2(m_{(a)}^2)$ corresponding to the eigenvalue λ_η and let θ be a generalized eigenvector of $\mathbb{M}_L^2(m_{(a)}^2)$ associated with $\lambda_\theta \neq \lambda_\eta$; then η and θ are $\mathscr{M}_L(m_{(a)}^2)$-orthogonal

$$\eta^\top \mathscr{M}_L(m_{(a)}^2)\theta = 0\,. \tag{117}$$

This fact follows from the identity (cf. Eqs. (11) and (12))

$$\mathbb{M}_L^2(m_{(a)}^2)^\top \mathscr{M}_L(m_{(a)}^2) - \mathscr{M}_L(m_{(a)}^2)\,\mathbb{M}_L^2(m_{(a)}^2) = 0\,.$$

Sandwiched between $\eta^\top$ and θ_1, i.e. the first element of a Jordan chain (92), it gives

$$(\lambda_\eta - \lambda_\theta) \times \eta^\top \mathscr{M}_L(m_{(a)}^2)\theta_1 = 0\,,$$

while sandwiched between $\eta^\top$ and θ_{r+1} yields

$$(\lambda_\eta - \lambda_\theta) \times \eta^\top \mathscr{M}_L(m_{(a)}^2)\theta_{r+1} = -\eta^\top \mathscr{M}_L(m_{(a)}^2)\theta_r\,.$$

This proves Eq. (117) by induction.

$1\frac{1}{2}$. Reality conditions. Suppose now that conditions (34) are satisfied for $s \in \mathscr{U}_a \subset \mathbb{R}$. Then $\mathscr{Z}_L(s)$ is a Hermitian matrix, cf. Eq. (12), and thus one can parametrize it locally as

$$\mathscr{Z}_L(s) = U(s)^\dagger\, \Lambda(s)\, U(s)\,,$$

where $U(s)$ is unitary, while $\Lambda(s)$ is diagonal (and positive, cf. Eq. (7)). On the other hand, a symmetric matrix

$$\widetilde{\mathscr{M}_L}(s) \equiv \left[\left(\sqrt{\Lambda(s)}U(s)\right)^{-1}\right]^{\top} \mathscr{M}_L(s) \left(\sqrt{\Lambda(s)}U(s)\right)^{-1},$$

can be written in the following form

$$\widetilde{\mathscr{M}_L}(s) = V(s)^{\top}\mu(s)\,V(s)\,,$$

where $V(s)$ is unitary, while $\mu(s)$ is diagonal, real and nonnegative. Hence

$$\mathscr{Z}_L(s) = \omega(s)^{\dagger}\omega(s)\,, \tag{118}$$

and

$$\mathscr{M}_L(s) = \omega(s)^{\top}\mu(s)\omega(s)\,, \tag{119}$$

where

$$\omega(s) = V(s)\sqrt{\Lambda(s)}U(s)\,. \tag{120}$$

Eq. (11) now reads

$$\mathbb{M}_L^2(s) = \omega(s)^{-1}\mu^2(s)\omega(s)\,, \tag{121}$$

where $\mu^2(s) \equiv \mu(s)^2$, and thus (cf. Eq. (14))

$$\mathscr{X}(s) = \prod_{\bar{c}} \left(s - \mu_{\bar{c}\bar{c}}(s)^2\right). \tag{122}$$

Let $\{\bar{a}_r\}$ be a set of indices for which $\mu_{\bar{a}_r\bar{a}_r}(s) = m_{(a)}^{\text{tree}} + \mathcal{O}(\hbar)$. The gap equation (15) reduces to

$$\mu_{\bar{a}_r\bar{a}_r}(m_{(\bar{a}_r)}^2) = m_{(\bar{a}_r)}\,. \tag{123}$$

A formal-power-series solution $m_{(\bar{a}_r)} = m_{(a)}^{\text{tree}} + \mathcal{O}(\hbar)$ to this equation obviously exists and is real, since all the derivatives $\mu_{\bar{a}_r\bar{a}_r}^{(k)}(s)$ at $s = (m_{(a)}^{\text{tree}})^2$ are real. Let $\{a_r\} \subset \{\bar{a}_r\}$ be a set of indices for which $m_{(a_r)} = m_{(a)}$; in other words, a situation in which the degeneracy of the tree-level masses is lifted by quantum corrections is not excluded here. Let $[\omega(m_{(a)}^2)^{-1}]_{[a_1]}, \ldots,$ be the columns of the $\omega(m_{(a)}^2)^{-1}$ matrix such that

$$\mu_{a_ra_r}(m_{(a)}^2) = m_{(a)}\,, \tag{124}$$

clearly

$$[\omega(m_{(a)}^2)^{-1}]_{[a_r]} = \omega(m_{(a)}^2)^{-1}\,\mathbb{1}_{[a_r]}\,.$$

Eigenvectors $\{\xi_{[a_u]}\}$, cf. Eq. (60), have the form

$$\xi_{[a_u]} = \sum_q C(a)^q{}_u \, \omega(m^2_{(a)})^{-1} \mathbb{1}_{[a_q]} \,,$$

where $C(a)$ is a square matrix. The normalization condition (61) reduces to (recall that $\widetilde{m}$ is now assumed to be diagonal)

$$C(a)^\top C(a) = \mathbb{1} \,, \tag{125}$$

i.e. $C(a)$ is a complex orthogonal matrix. The $\Xi(a)$ matrix, Eq. (62), reads

$$\Xi(a) = C(a)^\top \Theta(a) \, C(a) \,, \tag{126}$$

where

$$\Theta(a)_{ur} = m_{(a)} \, \mathbb{1}^\top_{[a_u]} \, \mu^{2\prime}(m^2_{(a)}) \, \mathbb{1}_{[a_r]} = m_{(a)} \, \mu^{2\prime}_{a_u a_r}(m^2_{(a)}) \,,$$

since terms with derivatives of $\omega(s)$ cancel. This shows that $\Theta(a)$ is real. (Since an accidental degeneracy of masses is not excluded, it is in principle possible that $\Theta(a)$ is not proportional to the identity matrix.)

Eqs. (65) and (66) now read

$$\zeta_{L[a_r]} = \sum_q \left[C(a)\mathcal{N}(a)\right]^q{}_r \left[\omega(m^2_{(a)})^{-1}\right] \mathbb{1}_{[a_q]} \,, \tag{127}$$

$$\zeta_{R[a_r]} = \sum_q \left[C(a)\mathcal{N}(a)\right]^q{}_r \left[\omega(m^2_{(a)})^{-1}\right]^\star \mathbb{1}_{[a_q]} \,, \tag{128}$$

thus $\zeta_{R[a_r]} = \zeta^\star_{L[a_r]}$, if $C(a)\mathcal{N}(a)$ is a real matrix. Finally, with the aid of Eq. (125), the condition (63) for $\mathcal{N}(a)$ can be rewritten as

$$\left[C(a)\,\mathcal{N}(a)\right]\left[C(a)\,\mathcal{N}(a)\right]^\top = \left\{\mathbb{1} - \frac{1}{m_{(a)}}\Theta(a)\right\}^{-1} . \tag{129}$$

Since the right-hand side of Eq. (129) is a real diagonal and positive (in perturbation theory) matrix, there always exists a real matrix $C(a)\,\mathcal{N}(a)$ obeying this condition. Clearly, Eq. (129), together with the reality condition $\zeta_{R[a_r]} = \zeta^\star_{L[a_r]}$, determine $\mathcal{N}(a)$ up to a rotation, as was to be shown.

2. Vanishing $m^2_{(a)}$. Consider the case $m^2_{(a)} = 0$. Reality conditions (34) cannot be violated for $p^2 = 0$, and thus Eqs. (15) and (11) give

$$\left|\det\left(\mathscr{M}_L(0)\right)\right|^2 = 0 = \left|\det\left(\mathscr{M}_R(0)\right)\right|^2.$$

Eqs. (102)-(103) now show that columns $\zeta_{L,R[0_r]}$ have to belong to the kernel of $\mathscr{M}_{L,R}(0)$, cf. Eq. (57), and therefore one can assume that Eq. (72) holds. One needs also the relation

$$\ker \mathbb{M}_L^2(0) = \ker \mathscr{M}_L(0)\,,$$

which follows immediately from the parametrization employed for analysis of reality conditions in the massive case, see Eqs. (119) and (121). Hence, one can assume that $\zeta_{L[0_r]}$ are linear combinations of linearly independent vectors $\xi_{[0_q]}$ obeying Eq. (67) and the normalization conditions (68), with (yet unspecified) coefficients $\mathscr{N}(a)^q{}_r$, as in Eq. (71).

It remains to be shown that Eq. (98) reduces to the condition (70) on the matrix $\mathscr{N}(0)$. This can be done just as before, by rewriting (98) as $\mathbb{P}(0) = \mathbb{S}(0)$ with $\mathbb{S}(0)$ defined by (109) and appropriately adjusted matrix $\mathbb{Y}(0)$

$$\mathbb{Y}(0) \equiv \sum_{q,r} \left[\mathscr{N}(0)\,\mathscr{N}(0)^\dagger\right]^{qr} \xi_{[0_q]}\,\xi^\dagger_{[0_r]}\,\mathscr{Z}_L(0)\,. \tag{130}$$

If Eq. (70) is satisfied, one finds

$$\mathbb{S}(0) = \sum_{q,r} \delta^{qr}\,\xi_{[0_q]}\,\xi^\dagger_{[0_r]}\,\mathscr{Z}_L(0)\,, \tag{131}$$

and Eq. (68) gives

$$\operatorname{im}\mathbb{S}(0) \supset \ker \mathbb{M}_L^2(0) \equiv \operatorname{im}\mathbb{P}(0)\,.$$

The existence of matrices $\mathscr{N}(0)$ obeying Eq. (70) is ensured by the Hermiticity of $\Xi(0)$, which follows from the identity

$$\begin{aligned}\mathbb{M}_L^2{}'(0)^\dagger \mathscr{Z}_L(0)^\dagger - \mathscr{Z}_L(0)\,\mathbb{M}_L^2{}'(0) &= \\ = \mathscr{Z}_L'(0)\,\mathbb{M}_L^2(0) - \mathbb{M}_L^2(0)^\dagger \mathscr{Z}_L'(0)^\dagger\,,\end{aligned}$$

sandwiched between $\xi^\dagger_{[0_q]}$ and $\xi_{[0_r]}$.

To complete the proof of $\mathbb{P}(0) = \mathbb{S}(0)$, one has to show that the generalized eigenvectors θ of $\mathbb{M}_L^2(0)$ associated with non-vanishing eigenvalues satisfy

$$\xi^\dagger_{[0_r]}\,\mathscr{Z}_L(0)\,\theta = 0\,, \qquad \forall_r\,. \tag{132}$$

To that end one can employ once again the parametrization from Eqs. (118)-(121). In particular, $\mathbb{M}_L^2(0)$ is diagonalizable, and both $\xi_{[0_r]}$ and θ are linear combinations of (disjoint sets of) columns of $\omega(0)^{-1}$; hence Eq. (132) follows immediately from Eq. (118).

4.B. Proof of Majorana prescription. The prescription for Majorana case follows immediately from the generalized prescription, since one can take $\widetilde{m} = m$, with a diagonal matrix m, Eq. (26). In particular, the assumed non-degeneracy of the tree-level masses implies that $\mathcal{N}(a)$ is a 1×1 matrix and thus the freedom in Eq. (63) reduces to a choice of sign. Hence, regardless of which sign is chosen, $\zeta_{R[a]} = \zeta^{\star}_{L[a]}$, if reality conditions (34) are satisfied for $s \in \mathscr{U}_a \subset \mathbb{R}$.

4.C. Proof of Dirac prescription. Apart from Eq. (55) for stable particles, the prescription for the Dirac case can be easily obtained from the generalized prescription. Having eigenvectors $\bar{\xi}_{[a\pm]}$, Eq. (50), it is convenient to choose eigenvectors $\xi_{[a_r]}$ (with $[a_r] = [a+], [a-]$), Eq. (60), as

$$\xi_{[a+]} = \begin{bmatrix} \bar{\xi}_{[a+]} \\ 0 \end{bmatrix} , \qquad \xi_{[a-]} = \begin{bmatrix} 0 \\ \bar{\xi}_{[a-]} \end{bmatrix} , \tag{133}$$

and take $\mathcal{N}(a)$ in Eq. (65) to be the following 2×2 matrix

$$\mathcal{N}(a) = \begin{pmatrix} \bar{\mathcal{N}}(a)c(a) & 0 \\ 0 & \bar{\mathcal{N}}(a)c(a)^{-1} \end{pmatrix} . \tag{134}$$

Similarly, a convenient choice of the $\widetilde{m}$ matrix in Eq. (56) is given by Eqs. (46) and (45). With these choices, one-particle states corresponding to the columns of $\zeta_{L,R}$ matrices, Eqs. (49), carry the definite charge. The normalization condition (61) now reduces to Eq. (51), while Eq. (63) is solved by (54).

Suppose that the reality conditions (34) are satisfied for $s \in \mathscr{U}_a \subset \mathbb{R}$. Using the explicit form (19) of $\mathscr{Z}_{L,R}$ and $\mathscr{M}_{L,R}$ matrices as well as the fact that an arbitrary nonsingular complex matrix $\widetilde{\mu_L}(s)$ can be written as

$$\widetilde{\mu_L}(s) = V_+(s)^\top \mu(s)\, V_-(s) ,$$

where $V_\pm(s)$ are unitary, while $\mu(s)$ is diagonal and positive, one finds (similarly as in the generic case) the following local parametrization of μ_L and $\mathscr{I}_\pm^{-1}$ matrices

$$\begin{aligned} \mathscr{I}_\pm(s)^{-1} &= \omega_\pm(s)^\dagger \omega_\pm(s) , \\ \mu_L(s) &= \omega_+(s)^\top \mu(s) \omega_-(s) , \end{aligned} \tag{135}$$

where

$$\omega_\pm(s) = V_\pm(s)\Lambda_\pm(s)U_\pm(s) , \tag{136}$$

with unitary $U_\pm(s)$, $V_\pm(s)$ matrices and positive-diagonal $\mu(s)$, $\Lambda_\pm(s)$ matrices. Hence

$$\mathbb{M}^2_\pm(s) = \omega_\pm(s)^{-1}\mu^2(s)\omega_\pm(s) , \tag{137}$$

where $\mu^2(s) \equiv \mu(s)^2$, and thus (cf. Eq. (23))

$$\mathscr{X}_+(s) = \prod_c \Big(s - \mu_{cc}(s)^2\Big)\,. \tag{138}$$

Thus, just like before, one sees that the solution $m_{(a)} = m^{\rm tree}_{(a)} + \mathcal{O}(\hbar)$ to the gap equation (15) is real and that the eigenvectors $\bar{\xi}_{[a\pm]}$ can be chosen as

$$\bar{\xi}_{[a\pm]} = [\omega_\pm(m^2_{(a)})^{-1}]_{[a]}\,, \tag{139}$$

what ensures that the normalization condition (51) is obeyed. Eq. (54) now yields

$$\bar{\mathscr{N}}(a) = \frac{1}{\sqrt{1 - \mu^{2\prime}_{aa}(m^2_{(a)})}}\,,$$

showing that $\bar{\mathscr{N}}(a)$ is real. On the other hand, Eqs. (53) give

$$\bar{\zeta}_{R[a\pm]} = \bar{\mathscr{N}}(a)\, \mathfrak{c}(a)^{\mp 1}\, (\bar{\xi}_{[a\pm]})^\star\,.$$

Comparing this with Eqs. (52) one sees that Eqs. (55) hold provided that $\mathfrak{c}(a)$ is, for a *particular* choice (139), a phase factor, as was to be proved.

4.D. Proof of scalar prescription. Similarly to the fermionic case one sees that Eq. (78) is equivalent to the following conditions

$$\zeta\, \widetilde{P}(\ell)\, \zeta^\top = \Delta(\ell)\,, \qquad \forall \ell, \tag{140}$$

where

$$\widetilde{P}(\ell) \equiv \lim_{s\to m^2_{(\ell)}} \Big\{(s - m^2_{(\ell)})\, \big[s\mathbb{1} - m^2\big]^{-1}\Big\}\,,$$

is a diagonal projection, while

$$\begin{aligned}\Delta(\ell) &\equiv \lim_{s\to m^2_{(\ell)}} \Big\{(s - m^2_{(\ell)})\, \big[s\mathbb{1} - M^2(s)\big]^{-1}\Big\} \\ &= \Big\{\mathbb{1} - \mathbb{P}(\ell)\, M^{2\prime}(m^2_{(\ell)})\Big\}^{-1}\, \mathbb{P}(\ell)\,, \end{aligned} \tag{141}$$

with $\mathbb{P}(\ell)$ being the projection onto the eigenspace of $M^2(m^2_{(\ell)})$ corresponding to $m^2_{(\ell)}$ along the direct sum of remaining generalized eigenspaces of $M^2(m^2_{(\ell)})$.

Clearly,

$$\zeta\, \widetilde{P}(\ell)\, \zeta^\top = \sum_q \zeta_{[\ell_q]} \zeta^\top_{[\ell_q]}\,. \tag{142}$$

In the scalar case, there is no counterpart of Eq. (104); suppose then that $\zeta_{[\ell_r]}$ are linear combinations of (*yet unspecified*) linearly independent vectors $\xi_{[\ell_q]}$ obeying the normalization condition (81), with (yet unspecified) coefficients $\mathcal{N}(\ell)^q{}_r$, as in Eq. (84).

Defining

$$\mathbb{Y}(\ell) \equiv \sum_{q,r} \left[\mathcal{N}(\ell)\,\mathcal{N}(\ell)^\top\right]^{qr} \xi_{[\ell_q]}\,\xi_{[\ell_r]}^\top\,, \tag{143}$$

one can (similarly to the fermionic case) rewrite Eq. (140) as

$$\mathbb{P}(\ell) = \mathbb{S}(\ell)\,, \tag{144}$$

where

$$\mathbb{S}(\ell) \equiv \mathbb{Y}(\ell)\Big\{\mathbb{1} + M^{2\prime}(m^2_{(\ell)})\,\mathbb{Y}(\ell)\Big\}^{-1}\,,$$

what can be simplified to

$$\mathbb{S}(\ell) = \sum_{q,r} \left[\sigma(\ell)\right]^{qr} \xi_{[\ell_q]}\,\xi_{[\ell_r]}^\top\,, \tag{145}$$

where

$$\sigma(\ell) \equiv \Big\{\mathbb{1} + \mathcal{N}(\ell)\,\mathcal{N}(\ell)^\top\,\Xi(\ell)\Big\}^{-1} \mathcal{N}(\ell)\,\mathcal{N}(\ell)^\top\,, \tag{146}$$

with $\Xi(\ell)$ defined by Eq. (82).

The normalization condition for $\xi_{[\ell_r]}$ eigenvectors, Eq. (81), gives

$$\mathbb{S}(\ell)\xi_{[\ell_r]} = \sum_{q,s} \left[\sigma(\ell)\right]^{qs} \delta_{sr}\,\xi_{[\ell_q]}\,, \tag{147}$$

hence

$$\mathbb{S}(\ell)^2 = \sum_{q,s,t,r} \sigma(\ell)^{qs}\,\delta_{st}\,\sigma(\ell)^{tr}\,\xi_{[\ell_q]}\,\xi_{[\ell_r]}^\top\,. \tag{148}$$

Thus, the following condition (equivalent to Eq. (83))

$$\sigma(\ell)^{rs} = \delta^{rs}\,, \tag{149}$$

ensures that $\mathbb{S}(\ell)$ is a projection and that the image of $\mathbb{S}(\ell)$ contains the subspace spanned by $\{\xi_{[\ell_q]}\}$. Therefore Eq. (144) requires $\{\xi_{[\ell_q]}\}$ to be a basis of the eigenspace of $M^2(m^2_{(\ell)})$ associated with $m^2_{(\ell)}$. To complete the proof of Eq. (144), one still has to show that the kernel of $\mathbb{S}(\ell)$ is equal to the direct sum of generalized eigenspaces of $M^2(m^2_{(\ell)})$ associated with eigenvalues different from $m^2_{(\ell)}$. This follows from the fact that a generalized eigenvector θ of $M^2(m^2_{(\ell)})$ associated with an eigenvalue λ_θ is orthogonal to an eigenvector η associated with $\lambda_\eta \neq \lambda_\theta$

$$\eta^\top \theta = 0\,. \tag{150}$$

Eq. (150) can be proved in an analogous way to its fermionic counterpart, Eq. (117), with the aid of the relation $M^2(s)^\top \equiv M^2(s)$.

Suppose now that reality conditions (85) are satisfied in a left neighborhood of $p^2 = (m^{\mathrm{tree}}_{(\ell)})^2$. A real symmetric matrix $M^2(s)$ can be written as

$$M^2(s) = \omega(s)^{-1}\mu^2(s)\omega(s)\,, \tag{151}$$

where $\mu^2(s)$ is diagonal and real,while $\omega(s)$ is a real orthogonal matrix. A similar argument to the one given below Eq. (122) shows that the pole mass squares are real.

Let $[\omega(m^2_{(\ell)})^{-1}]_{[\ell_1]}\,,\ldots,$ be the columns of the $\omega(m^2_{(\ell)})^{-1}$ matrix such that

$$\mu^2_{\ell_r\ell_r}(m^2_{(\ell)}) = m^2_{(\ell)}\,. \tag{152}$$

Following the same reasoning as for the fermionic case in Sec. 4.A, one finds that columns (associated with $m^2_{(\ell)}$) of the ζ matrix, defined according to Eq. (84), have the form

$$\zeta_{[\ell_r]} = \sum_q \left[C(\ell)\mathcal{N}(\ell)\right]^q{}_r \left[\omega(m^2_{(\ell)})^{-1}\right] \mathbb{1}_{[\ell_q]}\,, \tag{153}$$

where the $C(\ell)\mathcal{N}(\ell)$ matrix obeys

$$\left[C(\ell)\,\mathcal{N}(\ell)\right]\left[C(\ell)\,\mathcal{N}(\ell)\right]^\top = \left\{\mathbb{1} - \Theta(\ell)\right\}^{-1}, \tag{154}$$

with a real symmetric matrix $\Theta(\ell)$

$$\Theta(\ell)_{qp} = \mathbb{1}^\top_{[\ell_q]}\,\omega(m^2_{(\ell)})\,M^{2\prime}(m^2_{(\ell)})\,\omega(m^2_{(\ell)})^\top\,\mathbb{1}_{[\ell_p]}\,.$$

In particular, there exists a matrix $C(\ell)\,\mathcal{N}(\ell)$ which obeys Eq. (154) and is real, what ensures the reality of $\zeta_{[\ell_r]}$.

5 Conclusions

We have analyzed in details the effects associated with mixing of scalar and fermionic fields. Presented results, together with their counterparts for vector fields [21], can be useful in the study of extensions of the Standard Model.

In particular, the prescription for “square-rooted residues” ζ is formulated entirely in terms of eigenvectors of certain matrices, and thus it can be efficiently employed in numerical calculations.

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