

# Boosted Schwarzschild Metrics from a Kerr-Schild Perspective

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## Abstract.

The Kerr-Schild version of the Schwarzschild metric contains a Minkowski background which provides a definition of a boosted black hole. There are two Kerr-Schild versions corresponding to ingoing or outgoing principle null directions. We show that the two corresponding Minkowski backgrounds and their associated boosts have an unexpected difference. We analyze this difference and discuss the implications in the nonlinear regime for the gravitational memory effect resulting from the ejection of massive particles from an isolated system. We show that the nonlinear effect agrees with the linearized result based upon the retarded Green function only if the velocity of the ejected particle corresponds to a boost symmetry of the ingoing Minkowski background. A boost with respect to the outgoing Minkowski background is inconsistent with the absence of ingoing radiation from past null infinity.

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## 1. Introduction

By considering retarded solutions of the linearized Einstein equation on a Minkowski background, Zeldovich and Polnarev [1] pointed out the existence of a memory effect in the gravitational waves produced by the ejection of massive particles to infinity. Our previous work [2] has shown that this effect could also be obtained in linearized theory by considering the transition from an initial state whose exterior was described by a Schwarzschild metric at rest to a final state whose exterior was a boosted Schwarzschild metric. The results were based upon a Kerr-Schild version of the Schwarzschild metric to describe the far field exterior to what we referred to as a Schwarzschild body. For such a body in linearized theory which is initially at rest, then goes through a radiative stage

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and finally emerges in a boosted state, we showed that the proper retarded solution for the resulting memory effect is described in terms of the ingoing version of the Kerr-Schild metric for both the initial and final states. An outgoing Kerr-Schild or time symmetric Schwarzschild metric would give the wrong result. The result was independent of the details of the intervening radiative period. Because the Kerr-Schild metrics are solutions both in the linearized and nonlinear sense, we extrapolated this result to the nonlinear case.

Here, we investigate this problem from the purely nonlinear perspective. There are two major differences from the linearized view.

The first major difference is that the linearized result in [2] was obtained using the boost associated with the Lorentz symmetry of the unperturbed Minkowski background. The Kerr-Schild metrics [3, 4] have the form

$$g_{\mu\nu} = \eta_{\mu\nu} + H\ell_\mu\ell_\nu \quad (1)$$

where  $\eta_{\mu\nu}$  is a Minkowski metric,  $\ell_\mu$  is a principle null vector field (with respect to both  $\eta_{\mu\nu}$  and  $g_{\mu\nu}$ ) and  $H$  is a scalar function. In the nonlinear case, there are two natural choices of “Minkowski background”  $\eta_{\mu\nu}$  depending on whether the null vector  $\ell^\mu$  in the Kerr-Schild metric (1) is chosen to be in the ingoing or outgoing direction. §.

The second major difference in the nonlinear case is that there is no analogue of the Green function to construct a retarded solution. Instead, the retarded solution due to the emission of radiation from an accelerated particle is characterized by the absence of ingoing radiation from  $\mathcal{I}^-$ . A necessary condition that there be no ingoing radiation is that the analogue of the radiation memory at past null infinity  $\mathcal{I}^-$  vanishes. In that case the ingoing radiation strain, which forms the free characteristic initial data on  $\mathcal{I}^-$ , may be set to zero. Otherwise, as explained in Sec. 3, non-vanishing radiation memory at  $\mathcal{I}^-$  requires that there must ingoing radiation from  $\mathcal{I}^-$ .

Consequently, since an initial unboosted Kerr-Schild-Schwarzschild metric has vanishing radiation strain at  $\mathcal{I}^-$ , the final boosted metric must also have vanishing radiation strain at  $\mathcal{I}^-$  if there is no intervening ingoing radiation. This is the case if the boost belongs to the Lorentz subgroup of the BMS group at  $\mathcal{I}^-$ . This corresponds to the boost symmetry of the Minkowski metric associated with the ingoing version of the Kerr-Schild metric. On the contrary, a boost with respect to the Minkowski metric associated with the outgoing version of the Kerr-Schild metric leads to non-vanishing radiation memory at  $\mathcal{I}^-$  so that it is inconsistent with the requirement of vanishing ingoing radiation.

This leads to our main result: the calculation in the nonlinear regime of the memory effect due to the ejection of a massive particle is correctly described by the boost associated with Minkowski background of the ingoing Kerr-Schild metric. The key ingredient is that this boost is a BMS symmetry at  $\mathcal{I}^-$  but not at  $\mathcal{I}^+$ . This leads to vanishing radiation memory at  $\mathcal{I}^-$  but to non-zero radiation memory at  $\mathcal{I}^+$ , which is

§ Note that a time symmetric version of the Schwarzschild metric, see (2), does not single out such a preferred Minkowski background in the nonlinear case.

in precise agreement with the extrapolation from the linearized result based upon the retarded Green function.

In Sec. 2, we discuss unexpected features which result in defining the boost in terms of the Lorentz symmetries of the Minkowski backgrounds of either the ingoing or outgoing versions of the Kerr-Schild metric. This requires considerable notational care, which warrants the formalism presented in Sec. 2. We show that the boost symmetry of the Minkowski metric associated with the ingoing version of the Kerr-Schild metric corresponds to the boost symmetry of the Bondi-Metzner-Sachs (BMS) [5] asymptotic symmetry group at  $\mathcal{I}^-$  but is a singular transformation at future null infinity  $\mathcal{I}^+$ . Conversely, the boost symmetry of the Minkowski metric associated with the outgoing version of the Kerr-Schild metric corresponds to the boost symmetry of the BMS group at  $\mathcal{I}^+$  but is a singular transformation at  $\mathcal{I}^-$ .

The Kerr-Schild metrics have played an important role in the construction of exact solutions; see [6]. The most important examples are the Schwarzschild and Kerr black hole metrics. Because their metric form (1) is invariant under the Lorentz symmetry of the Minkowski background  $\eta_{\mu\nu}$ , the boosted Kerr-Schild versions of the Schwarzschild and Kerr metrics have also played an important role in numerical relativity in prescribing initial data for superimposed black holes in a binary orbit [7, 8]. The initial data for the numerical simulations are prescribed in terms of the ingoing version of the Kerr-Schild metric, whose coordinatization in terms of advanced time covers the interior of the future event horizon. The initial velocities of the black holes are generated by the boost symmetry of the Minkowski background for the ingoing version of the Kerr-Schild metric. Surprisingly, this boost symmetry, which is a well-behaved BMS transformation at  $\mathcal{I}^-$ , has singular behavior at  $\mathcal{I}^+$ . This overlooked asymptotic property of the boost symmetry could possibly introduce spurious asymptotic behavior in the Kerr-Schild construction of binary black hole initial data.

We concentrate here on the boosted Schwarzschild metric. The Kerr case is more complicated because the twist of the principle null direction  $\ell_\mu$  does not allow a straightforward construction of well-behaved advanced or retarded null coordinate systems. Although an exact Schwarzschild exterior is unrealistic in a dynamic spacetime it is reasonable to expect that our results are valid if it is a good far field approximation in the neighborhood of null infinity in the limit of both infinite future and infinite past retarded and advanced times. In this respect, our results might also apply to the Kerr case since the metric terms involving the angular momentum parameter fall off faster with  $r$  than the terms involving the mass.

## 2. Boosts and the Kerr-Schild-Schwarzschild metrics

In time symmetric coordinates  $x^\mu = (t, r, x^A)$ , with  $x^A = (\theta, \phi)$  being standard spherical coordinates, the Schwarzschild metric is

$$g_{\mu\nu} = - \left(1 - \frac{2M}{r}\right) t_{,\mu} t_{,\nu} + \left(1 - \frac{2M}{r}\right)^{-1} r_{,\mu} r_{,\nu} + r^2 q_{\mu\nu}(x^A). \quad (2)$$

Here we use standard comma notation to denote partial derivatives, e.g.  $f_{,\mu} = \partial_\mu f$ , and  $q_{\mu\nu}(x^A)$  is the round unit sphere metric defined with respect to the Cartesian coordinates  $x^i = rr^i$ ,  $r^i(x^A) = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$  so that

$$q_{\mu\nu}dx^\mu dx^\nu = \delta_{ij}r^i_{,A}r^j_{,B}dx^A dx^B = d\theta^2 + \sin^2\theta d\phi^2.$$

Introduction of the ‘‘tortoise’’ coordinate  $r^* = r + 2M \ln(\frac{r}{2M} - 1)$ , with  $r^*_{,\mu} = (rr_{,\mu})(r - 2M)^{-1}$ , gives

$$g_{\mu\nu} = -\left(1 - \frac{2M}{r}\right)(t_{,\mu}t_{,\nu} - r^*_{,\mu}r^*_{,\nu}) + r^2q_{\mu\nu}(x^A). \quad (3)$$

In terms of the retarded time  $u = t - r^*$ ,

$$g_{\mu\nu} = -\left(1 - \frac{2M}{r}\right)u_{,\mu}u_{,\nu} - u_{,\mu}r_{,\nu} - u_{,\nu}r_{,\mu} + r^2q_{\mu\nu} \quad (4a)$$

and in terms of the advanced time  $v = t + r^*$ ,

$$g_{\mu\nu} = -\left(1 - \frac{2M}{r}\right)v_{,\mu}v_{,\nu} + v_{,\mu}r_{,\nu} + v_{,\nu}r_{,\mu} + r^2q_{\mu\nu}. \quad (4b)$$

The retarded time version of the Schwarzschild metric (4a) has the Kerr-Schild form with Minkowski metric  $\eta_{\mu\nu}^{(-)}$ ,

$$g_{\mu\nu} = \eta_{\mu\nu}^{(-)} + \frac{2M}{r}k_\mu k_\nu, \quad k_\mu = -u_{,\mu}. \quad (5)$$

Here, in the associated inertial coordinates  $x^{(-)\mu} = (t^{(-)}, x^{(-)i}) = (t^{(-)}, x^{(-)}, y^{(-)}, z^{(-)})$ , with  $t^{(-)} = u + r$  and  $x^{(-)i} = rr^i(x^A)$ ,

$$\eta_{\mu\nu}^{(-)}dx^{(-)\mu}dx^{(-)\nu} = -dt^{(-)2} + \delta_{ij}dx^{(-)i}dx^{(-)j}. \quad (6)$$

Similarly, the advanced time version (4b) has the Kerr-Schild form with the background Minkowski metric  $\eta_{\mu\nu}^{(+)}$ ,

$$g_{\mu\nu} = \eta_{\mu\nu}^{(+)} + \frac{2M}{r}n_\mu n_\nu, \quad n_\mu = -v_{,\mu} \quad (7)$$

where in the associated inertial coordinates  $x^{(+)\mu} = (t^{(+)}, x^{(+)i}) = (t^{(+)}, x^{(+)}, y^{(+)}, z^{(+)})$ , with  $t^{(+)} = v - r^{(-)}$  and  $x^{(+)i} = rr^i(x^A)$ ,

$$\eta_{\mu\nu}^{(+)}dx^{(+)\mu}dx^{(+)\nu} = -dt^{(+)\,2} + \delta_{ij}dx^{(+)i}dx^{(+)j}. \quad (8)$$

Note that the inertial time coordinates are related by

$$t^{(+)} = t^{(-)} + 4M \ln\left(\frac{r}{2M} - 1\right) \quad (9)$$

whereas the inertial spatial coordinates are related by  $x^{(+)i} = x^{(-)i}$ . As a result, it is unambiguous to write  $x^{(+)i} = x^{(-)i} = x^i$  and  $r^{(+)} = r^{(-)} = r$ , where  $r^{(+)\,2} := \delta_{ij}x^{(+)i}x^{(+)j}$  and  $r^{(-)\,2} := \delta_{ij}x^{(-)i}x^{(-)j}$ . However, the corresponding directional derivatives are related by

$$\partial_{t^{(+)}} = \partial_{t^{(-)}}, \quad \partial_{x^{(+)i}} = \partial_{x^{(-)i}} - \frac{4M}{r - 2M} \frac{x^i}{r} \partial_{t^{(-)}} \quad (10)$$

and

$$\partial_{r^{(+)}} = \partial_{r^{(-)}} - \frac{4M}{r - 2M} \partial_{t^{(-)}}. \quad (11)$$

As will be seen, these transformations have important bearing on the relation between the generators of the BMS group at past and future null infinity.

In [2], we showed that the linearized memory effect could be based upon the boosted version of the advanced time Kerr-Schild metric (7). In that linearized treatment, it was assumed that the boost was a Lorentz symmetry of the Minkowski background. However, this cannot be extended unambiguously to the nonlinear case, where there are two distinct Minkowski backgrounds  $\eta_{\alpha\beta}^{(-)}$  and  $\eta_{\alpha\beta}^{(+)}$  defined, respectively, by the retarded and advanced time Kerr-Schild metrics (5) and (7). Since the metrics (5) and (7) are algebraically identical, it cannot be the choice of retarded or advanced metric but the corresponding choice of boost that gives the essential result.

In spherical null coordinates, the Minkowski background metric (6) has the standard retarded time Bondi-Sachs form at  $\mathcal{I}^+$ ,

$$\eta_{\mu\nu}^{(-)} dx^\mu dx^\nu = -du^2 - 2dudr + r^2 q_{AB} dx^A dx^B \quad (12)$$

and (8) has the standard advanced time Bondi-Sachs form at  $\mathcal{I}^-$ ,

$$\eta_{\mu\nu}^{(+)} dx^\mu dx^\nu = -dv^2 + 2dvdr + r^2 q_{AB} dx^A dx^B. \quad (13)$$

(See [9] for a review of the Bondi-Sachs formalism.) These Minkowski line elements transform into each other under the Minkowski space relation  $u = v - 2r$  but not under the Schwarzschild relation between retarded and advanced time  $u = v - 2r^*$ . The retarded and advanced Minkowski metrics (6) and (8) are related by

$$\begin{aligned} \eta_{\mu\nu}^{(+)} &= \eta_{\mu\nu}^{(-)} + \frac{2M}{r} (k_\mu k_\nu - n_\mu n_\nu) \\ &= \eta_{\mu\nu}^{(-)} - \frac{4M}{r - 2M} (u_{,\mu} r_{,\nu} + u_{,\nu} r_{,\mu}) - \frac{8Mr}{(r - 2M)^2} r_{,\mu} r_{,\nu}. \end{aligned} \quad (14)$$

Because of the non-vanishing  $r_{,\mu} r_{,\nu}$  term in (14), although  $\eta_{\mu\nu}^{(+)}$  has the advanced time Bondi-Sachs form (13) near  $\mathcal{I}^-$  it does not have the retarded time Bondi-Sachs form near  $\mathcal{I}^+$ . The reverse is true of  $\eta_{\alpha\beta}^{(-)}$ , which has the retarded time Bondi-Sachs form (12) near  $\mathcal{I}^+$  but not the advanced time form near  $\mathcal{I}^-$ . This leads to a non-trivial difference between the boosts  $B^{(-)}$  and  $B^{(+)}$ , with generators  $B_{x^{(-)i}}$  and  $B_{x^{(+)i}}$ , which are symmetries of  $\eta_{\alpha\beta}^{(-)}$  and  $\eta_{\alpha\beta}^{(+)}$ , respectively.

To be specific, consider a boost in the  $z^{(-)}$ -direction intrinsic to  $\eta_{\mu\nu}^{(-)}$  with generator  $B_{z^{(-)}} = z^{(-)} \partial_{t^{(-)}} + t^{(-)} \partial_{z^{(-)}}$ . In retarded spherical null coordinates

$$\partial_{t^{(-)}} = \partial_u, \quad \partial_{z^{(-)}} = -\cos\theta(\partial_u - \partial_{r^{(-)}}) - \frac{\sin\theta}{r} \partial_\theta. \quad (15)$$

This leads to the retarded time dependence

$$B_{z^{(-)}} = -u \cos\theta \partial_u + (u + r) \cos\theta \partial_{r^{(-)}} - \left(\frac{u}{r} + 1\right) \sin\theta \partial_\theta, \quad (16)$$

which has the proper asymptotic behavior to be the generator of a BMS boost symmetry at  $\mathcal{I}^+$ .

However, expressed in terms of advanced null coordinates, using  $u = v - 2r^*$ ,

$$\begin{aligned}
 B_{z^{(-)}} &= -(v - 2r^*) \cos \theta \partial_v + (2\partial_r r^*)(v - 2r^* + r) \cos \theta \partial_v \\
 &+ (v - 2r^* + r) \cos \theta \partial_{r^{(+)}} - \left( \frac{v - 2r^*}{r} + 1 \right) \sin \theta \partial_\theta \\
 &= \left\{ \frac{(r + 2M)[v - 4M \ln(\frac{r}{2M} - 1)] - 4Mr}{r - 2M} \right\} \cos \theta \partial_v \\
 &+ \left[ v - r - 4M \ln \left( \frac{r}{2M} - 1 \right) \right] \left( \cos \theta \partial_{r^{(+)}} - \frac{1}{r} \sin \theta \partial_\theta \right). \tag{17}
 \end{aligned}$$

Here, because the  $\partial_v$  coefficient goes to infinity as  $\ln(\frac{r}{2M} - 1)$  for large  $r$ ,  $B_{z^{(-)}}$  generates a singular transformation at  $\mathcal{I}^-$ .

Now consider the corresponding boost in the  $z^{(+)}$ -direction intrinsic to  $\eta_{\mu\nu}^{(+)}$  with generator  $B_{z^{(+)}} = z^{(+)}\partial_{t^{(+)}} + t^{(+)}\partial_{z^{(+)}}$ . In advanced null coordinates,

$$\partial_{t^{(+)}} = \partial_v, \quad \partial_{z^{(+)}} = \cos \theta (\partial_v + \partial_{r^{(+)}}) - \frac{\sin \theta}{r} \partial_\theta. \tag{18}$$

This leads to the advanced time dependence

$$B_{z^{(+)}} = v \cos \theta \partial_v + (v - r) \cos \theta \partial_{r^{(+)}} - \left( \frac{v}{r} - 1 \right) \sin \theta \partial_\theta, \tag{19}$$

which has the proper asymptotic behavior to be the generator of a BMS boost symmetry at  $\mathcal{I}^-$ . However, expressed in terms of retarded null coordinates

$$\begin{aligned}
 B_{z^{(+)}} &= - \left\{ \frac{(r + 2M)[u + 4M \ln(\frac{r}{2M} - 1)] + 4Mr}{r - 2M} \right\} \cos \theta \partial_u \\
 &+ \left[ u + r + 4M \ln \left( \frac{r}{2M} - 1 \right) \right] \left( \cos \theta \partial_{r^{(-)}} - \frac{1}{r} \sin \theta \partial_\theta \right), \tag{20}
 \end{aligned}$$

which generates a singular transformation at  $\mathcal{I}^+$ . As will be seen in the next section, these unexpected gauge singularities of  $B_{z^{(-)}}$  at  $\mathcal{I}^-$  and  $B_{z^{(+)}}$  at  $\mathcal{I}^+$  do not affect calculation of the radiation memory.

### 3. Boosts and radiation memory

In retarded null coordinates  $x^\mu = (u, r, x^A)$ , where the angular coordinates  $x^A = (\theta, \phi)$ , are constant along the outgoing null rays and  $r$  is an areal coordinate which varies along the null rays, the metric takes the Bondi-Sachs form

$$g_{\mu\nu} dx^\mu dx^\nu = -\frac{V}{r} e^{2\beta} du^2 - 2e^{2\beta} du dr + r^2 h_{AB} \left( dx^A - U^A du \right) \left( dx^B - U^B du \right). \tag{21}$$

The choice of areal coordinate  $r$  and the choice  $x^A = (\theta, \phi)$  as angular coordinates requires

$$\det[h_{AB}] = \det[q_{AB}] = \sin^2 \theta. \tag{22}$$

As a result, the conformal 2-metric  $h_{AB}$  has only two degrees of freedom, which encode the two polarization modes of a gravitational wave.

In the neighborhood of  $\mathcal{I}^+$ , asymptotic flatness allows the construction of inertial coordinates such that the metric approaches the Minkowski metric,

$$\frac{V}{r} = 1 + O(1/r), \quad \beta = O(1/r^2), \quad U^A = -\frac{1}{2r^2} \bar{\delta}_E c^{EA} + O(1/r^3) \quad (23)$$

and

$$h_{AB} = q_{AB} + c_{AB}/r + O(1/r^2). \quad (24)$$

Here  $\bar{\delta}_A$  is the covariant derivative with respect to  $q_{AB}$  and the determinant condition (22) implies that  $c_{AB}(u, x^C)$  is traceless,  $q^{AB}c_{AB} = 0$ . Evaluation of the geodesic deviation equation in the linearised limit of the Bondi-Sachs metric shows that  $\sigma_{AB} = \frac{1}{2}c_{AB}$  is the  $O(1/r)$  strain tensor of the gravitational radiation.

The gravitational wave memory effect is determined by the change in the radiation strain between infinite future and past retarded time,

$$\Delta\sigma_{AB}(x^C) := \sigma_{AB}(u = \infty, \theta, \phi) - \sigma_{AB}(u = -\infty, \theta, \phi). \quad (25)$$

This produces a net displacement in the relative angular position of distant test particles,||

$$\Delta(x_2^A - x_1^A) = \frac{1}{r}(x_2^C - x_1^C)q^{AB}\Delta\sigma_{BC}. \quad (26)$$

A compact way to describe the radiation is in terms of a complex polarization dyad  $q_A$  satisfying

$$q_{AB} = \frac{1}{2}(q_A\bar{q}_B + \bar{q}_Aq_B), \quad q^A\bar{q}_A = 2, \quad q^Aq_A = 0. \quad (27)$$

For the standard form of the unit sphere metric in spherical coordinates  $x^A = (\theta, \phi)$ , we set  $q^A\partial_A = \partial_\theta + (i/\sin\theta)\partial_\phi$ . In the associated inertial Cartesian coordinates, the dyad  $q^A$  has components  $q^\mu = rQ^\mu = (0, rQ^i)$ , where

$$Q^i = r^i_{,A}q^A = (\cos\theta\cos\phi - i\sin\phi, \cos\theta\sin\phi + i\cos\phi, -\sin\theta)$$

and  $\delta_{ij}Q^i\bar{Q}^j = 2$ . The dyad decomposition

$$\sigma_{AB} = \frac{1}{4} \left[ (q^E q^F \sigma_{EF}) \bar{q}_A \bar{q}_B + (\bar{q}^E \bar{q}^F \sigma_{EF}) q_A q_B \right], \quad (28)$$

leads to the spin-weight-2 representation of the strain,

$$\sigma := \frac{1}{2} q^A q^B \sigma_{AB}. \quad (29)$$

Note that  $\sigma$  also corresponds to the leading ( $r^{-2}$ ) coefficient of the shear of the null hypersurfaces  $u = \text{const}$ . Its retarded time derivative  $N(u, x^A) := \partial_u \sigma(u, x^A)$  is the Bondi news function.

The shear-free property of the Schwarzschild metric in its rest frame implies that  $\sigma = 0$ . For a transition from an initially static Schwarzschild frame to a final boosted state, the resulting spin-weighted radiation memory is then

$$\Delta\sigma(x^C) = \sigma(u = \infty, x^C) - \sigma(u = -\infty, x^C), \quad (30)$$

|| Note (26) corrects a missing  $1/r$  factor in the corresponding equation in [11].

where  $\sigma(u = \infty, x^C)$  is the radiation strain of the final boosted state and initially  $\sigma(u = -\infty, x^C) = 0$ .

Under the retarded time transformation  $u \rightarrow u + \alpha(x^A)$ , which corresponds to the supertranslation freedom in the BMS group [5], the asymptotic strain has the gauge freedom

$$\sigma(u, x^A) \rightarrow \sigma(u, x^A) + \eth^2 \alpha(x^A), \quad (31)$$

where  $\eth$  is the Newman-Penrose spin-weight raising operator [10]. Since the finiteness of the radiative mass loss requires that the news function  $N = \partial_u \sigma$  vanish as  $u \rightarrow \pm\infty$ , the strain  $\sigma$  can be gauged to zero either as  $u \rightarrow \infty$  or  $u \rightarrow -\infty$ . The memory effect  $\Delta\sigma$  (25) is gauge invariant but determines a supertranslation  $\alpha(x^A)$  according to

$$\eth^2 \alpha(x^A) = \Delta\sigma(x^A),$$

which relates the strains at  $u = \pm\infty$ . The energy flux of the radiation is given by the absolute square,  $N\bar{N}$ , of the Bondi news function  $N$ , which is also gauge invariant. If the memory effect (30) is non-zero then there must be intervening radiation.

These attributes of  $\mathcal{I}^+$  have corresponding attributes at  $\mathcal{I}^-$ . In particular, the outgoing radiation strain  $\sigma(u, x^A)$  has as its analogue an ingoing radiation strain  $\Sigma(v, x^A)$ . In analogy with (30), the gravitational wave memory at  $\mathcal{I}^-$  due to ingoing radiation is

$$\Delta\Sigma(x^C) = \Sigma(v = \infty, x^C) - \Sigma(v = -\infty, x^C). \quad (32)$$

If there is no ingoing radiation, as required in the linearized case by a retarded solution, then  $\Delta\Sigma(x^C) = 0$ .

Of the BMS transformations, only the supertranslations (31) affect the radiation strain. As shown in Sec. 2, a  $B^{(-)}$  boost is a BMS boost symmetry at  $\mathcal{I}^+$  so that it does not introduce outgoing radiation memory  $\Delta\sigma$ . Conversely, a  $B^{(+)}$  boost is a BMS boost symmetry at  $\mathcal{I}^-$  so that it does not introduce ingoing radiation memory  $\Delta\Sigma$ . These results are explicitly demonstrated below.

### 3.1. Effect of a $B^{(-)}$ boost

Consider first the transition from a static Kerr-Schild-Schwarzschild metric to the  $B^{(-)}$  boosted version with 4-velocity  $v^\mu = \Gamma(1, V^i)$ , where  $\Gamma = (1 - \delta_{ij} V^i V^j)^{-1/2}$ . For a  $B^{(-)}$  boost,  $\eta_{\mu\nu}^{(-)} \rightarrow \eta_{\mu\nu}^{(-)}$ . The boosted version of the static retarded time Kerr-Schild-Schwarzschild metric (5), can be obtained by the further substitutions

$$\partial_\mu t^{(-)} \rightarrow -v_\mu, \quad r^2 \rightarrow R^{(-)2} = x_\mu^{(-)} x^{(-)\mu} + (x_\mu^{(-)} v^\mu)^2, \quad k_\mu \rightarrow K_\mu = v_\mu + R_\mu^{(-)}, \quad (33)$$

where

$$\partial_\mu r \rightarrow R_\mu^{(-)} = \frac{1}{R^{(-)}} (x_\mu^{(-)} + v_\mu x_\nu^{(-)} v^\nu). \quad (34)$$

The boosted metric is

$$g_{\mu\nu}^{(B^-)} = \eta_{\mu\nu}^{(-)} + \frac{2M}{R^{(-)}} K_\mu K_\nu. \quad (35)$$



This Lorentz covariant transformation reduces to the rest frame expression when  $V^i = 0$ .

In order to calculate the resulting radiation strain, we note that  $q^\mu q^\nu \eta_{\mu\nu}^{(-)} = 0$  and  $q^\mu x_\mu = 0$  so that

$$\sigma^{(B^-)} = \frac{1}{4} r q^\mu q^\nu g_{\mu\nu}^{(B^-)}|_{\mathcal{I}^+} = \frac{Mr}{2R^-} (q^\mu K_\mu)^2|_{\mathcal{I}^+},$$

where  $q^\mu K_\mu = q^\mu v_\mu (1 + R^{-1} x_\nu^{(-)} v^\nu)$ .

For the limit at  $\mathcal{I}^+$ , in retarded null coordinates

$$R^{(-)2} = r^2 \left[ -\frac{u^2}{r^2} - \frac{2u}{r} + \Gamma^2 \left( 1 - \frac{x_i V^i}{r} + \frac{u}{r} \right)^2 \right], \quad (36a)$$

$$x_\nu^{(-)} v^\nu = \Gamma(-u - r + x_i V^i), \quad (36b)$$

so that

$$\lim_{\substack{r \rightarrow \infty \\ u = \text{const}}} \frac{R^{(-)}}{r} = \Gamma \left( 1 - \frac{V^i x_i}{r} \right), \quad (37a)$$

$$\lim_{\substack{r \rightarrow \infty \\ u = \text{const}}} \frac{x_\nu^{(-)} v^\nu}{r} = -\Gamma \left( 1 - \frac{V^i x_i}{r} \right). \quad (37b)$$

Consequently,

$$\lim_{\substack{r \rightarrow \infty \\ u = \text{const}}} \frac{x_\nu^{(-)} v^\nu}{R^{(-)}} = -1 \quad (38)$$

and

$$\lim_{\substack{r \rightarrow \infty \\ u = \text{const}}} q^\mu K_\mu = 0. \quad (39)$$

Therefore  $\sigma^{(B^-)}(u, x^C) = 0$  and in particular  $\sigma^{(B^-)}(u = \infty, x^C) = 0$ . So, as expected from the BMS property of the  $B^{(-)}$  boost at  $\mathcal{I}^+$ , it produces no radiation memory at  $\mathcal{I}^+$ . Now consider the boosted strain on  $\mathcal{I}^-$ ,

$$\Sigma^{(B^-)} = \frac{r}{4} q^\mu q^\nu g_{\mu\nu}^{(B^-)}|_{\mathcal{I}^-} = \frac{Mr}{2R^{(-)}} (q^\mu v_\mu)^2 \left( 1 + \frac{x_\nu^{(-)} v^\nu}{R^{(-)}} \right)^2|_{\mathcal{I}^-}. \quad (40)$$

In order to calculate the limit at  $\mathcal{I}^-$ , for which  $r \rightarrow \infty$  holding  $v = t^{(+)} + r$  constant, we must express  $\Sigma^{(B^-)}$  as a function of the unboosted advanced coordinates  $(v, r, x^A)$ . Using (9), a straightforward calculation gives

$$x_\mu^{(-)} v^\mu = r \Gamma \left[ 1 - \frac{v}{r} + \frac{4M}{r} \ln \left( \frac{r}{2M} - 1 \right) + r_i V^i \right], \quad (41)$$

$$x_\mu^{(-)} x^{(-)\mu} = r \left[ v - 4M \ln \left( \frac{r}{2M} - 1 \right) \right] \left[ 2 - \frac{v}{r} + \frac{4M}{r} \ln \left( \frac{r}{2M} - 1 \right) \right], \quad (42)$$

which leads to the limits

$$\lim_{\substack{r \rightarrow \infty \\ v = \text{const}}} \frac{R^{(-)}}{r} = \Gamma(1 + V^i r_i), \quad (43)$$

$$\lim_{\substack{r \rightarrow \infty \\ v = \text{const}}} \frac{x_\mu^{(-)} v^\mu}{R^{(-)}} = \lim_{\substack{r \rightarrow \infty \\ v = \text{const}}} \left[ \frac{r}{R^{(-)}} \right] \left[ \frac{x_\mu^{(-)} v^\mu}{r} \right] = 1. \quad (44)$$

We then obtain

$$\Sigma^{(B^-)} = \frac{2Mr}{R^{(-)}}(q^\mu v_\mu)^2|_{\mathcal{I}^-} = \frac{2M\Gamma(q^i V_i)^2}{1 + V^i r_i} . \quad (45)$$

Consequently, for a non zero boost, the resulting radiation memory on  $\mathcal{I}^-$  does not vanish, which requires the existence of ingoing radiation.

Thus the  $B^{(-)}$  boost is inconsistent with vanishing ingoing radiation and produces zero radiation memory on  $\mathcal{I}^+$ . Both of these results contradict the linearized result based upon the retarded Green function so that  $B^{(-)}$  is not the appropriate boost to model the memory effect.

### 3.2. Effect of a $B^{(+)}$ boost

Consider now the transition from a static to a  $B^{(+)}$  boosted version of the Kerr-Schild-Schwarzschild metric with 4-velocity  $v^\mu = \Gamma(1, V^i)$ . For the  $B^{(+)}$  boost,  $\eta_{ab}^{(+)} \rightarrow \eta_{ab}^{(+)}$ . With respect to the advanced time version of the static Kerr-Schild-Schwarzschild metric (7), the boosted version can be obtained by the further substitutions

$$\partial_\mu t^{(+)} \rightarrow -v_\mu, \quad r^2 \rightarrow R^{(+2)} = x_\mu^{(+)} x^{(+)\mu} + (x_\mu^{(+)} v^\mu)^2, \quad n_\mu \rightarrow N_\mu = v_\mu - R_\mu^{(+)}, \quad (46)$$

where

$$\partial_\mu r \rightarrow R_\mu^{(+)} = \frac{1}{R^{(+)}}(x_\mu^{(+)} + v_\mu x_\nu^{(+)} v^\nu). \quad (47)$$

The boosted metric is

$$g_{\mu\nu}^{(B^+)} = \eta_{\mu\nu}^{(+)} + \frac{2M}{R^{(+)}} N_\mu N_\nu \quad (48)$$

with the corresponding boosted strain on  $\mathcal{I}^-$  given by

$$\Sigma^{(B^+)} = \frac{r}{4} q^\mu q^\nu g_{\mu\nu}^{(B^+)}|_{\mathcal{I}^-} = \frac{Mr}{2R^{(+)}}(q^\mu N_\mu)^2|_{\mathcal{I}^-} = \frac{Mr}{2R^{(+)}}(q^\mu v_\mu)^2 \left(1 - \frac{x_\nu^{(+)} v^\nu}{R^{(+)}}\right)^2|_{\mathcal{I}^-}. \quad (49)$$

The calculation of the limit proceeds in a time reversed sense as in Sec. 3.1.

In advanced null coordinates

$$R^{(+2)} = r^2 \left[ -\frac{v^2}{r^2} + \frac{2v}{r} + \Gamma^2 \left(1 + \frac{x_i V^i}{r} - \frac{v}{r}\right)^2 \right], \quad (50a)$$

$$x_\nu^{(+)} v^\nu = \Gamma(-v + r + x_i V^i), \quad (50b)$$

so that

$$\lim_{\substack{r \rightarrow \infty \\ v = \text{const}}} \frac{R^{(+)}}{r} = \Gamma \left(1 + \frac{V^i x_i}{r}\right), \quad (51a)$$

$$\lim_{\substack{r \rightarrow \infty \\ v = \text{const}}} \frac{x_\nu^{(+)} v^\nu}{r} = \Gamma \left(1 + \frac{V^i x_i}{r}\right). \quad (51b)$$

Consequently,

$$\lim_{\substack{r \rightarrow \infty \\ v = \text{const}}} \frac{x_\nu^{(+)} v^\nu}{R^{(+)}} = \lim_{\substack{r \rightarrow \infty \\ v = \text{const}}} \left[ \frac{r}{R^{(+)}} \right] \left[ \frac{x_\nu^{(+)} v^\nu}{r} \right] = 1 \quad (52)$$

and therefore  $\Sigma^{(B^+)}(v, x^C) = 0$ . So, as expected from the BMS property of the  $B^{(+)}$  boost at  $\mathcal{I}^-$ , there is no radiation memory at  $\mathcal{I}^-$ . It is thus consistent to set the free characteristic initial data  $\Sigma$  to zero on  $\mathcal{I}^-$  so that there is no ingoing radiation.

Now consider the boosted strain on  $\mathcal{I}^+$ ,

$$\sigma^{(B^+)} = \frac{r}{4} q^\mu q^\nu g_{\mu\nu}^{(B^+)}|_{\mathcal{I}^+} = \frac{Mr}{2R^{(+)}} (q^\mu v_\mu)^2 \left(1 - \frac{x_\nu^{(+)} v^\nu}{R^{(+)}}\right)^2 \Big|_{\mathcal{I}^+}. \quad (53)$$

In order to calculate the limit at  $\mathcal{I}^+$ , for which  $r \rightarrow \infty$  holding  $u = t^{(-)} - r$  constant, we must express  $\sigma^{(B^+)}$  as a function of the unboosted retarded coordinates  $(u, r, x^A)$ . A straightforward calculation gives

$$x_\mu^{(+)} v^\mu = -r\Gamma \left[1 + \frac{u}{r} + \frac{4M}{r} \ln\left(\frac{r}{2M} - 1\right) - r_i V^i\right], \quad (54)$$

$$x_\mu^{(+)} x^{(+)\mu} = -r \left[u + 4M \ln\left(\frac{r}{2M} - 1\right)\right] \left[2 + \frac{u}{r} + \frac{4M}{r} \ln\left(\frac{r}{2M} - 1\right)\right], \quad (55)$$

which leads to the limits

$$\lim_{\substack{r \rightarrow \infty \\ u = \text{const}}} \frac{R^{(+)}}{r} = \Gamma(1 - V^i r_i), \quad (56)$$

$$\lim_{\substack{r \rightarrow \infty \\ u = \text{const}}} \frac{x_\nu v^\nu}{R^{(+)}} = \lim_{\substack{r \rightarrow \infty \\ u = \text{const}}} \frac{-\Gamma r \left[1 - V^i r_i + \frac{u}{r} + \frac{4M}{r} \ln\left(\frac{r}{2M} - 1\right)\right]}{R^{(+)}} = -1. \quad (57)$$

We then obtain

$$\sigma^{(B^+)} = \frac{2Mr}{R^{(+)}} (q^\mu v_\mu)^2|_{\mathcal{I}^+} = \frac{2M\Gamma}{(1 - r_i V^i)} (q^i V_i)^2. \quad (58)$$

The resulting radiation memory due to the ejection of a Schwarzschild body is

$$\Delta\sigma^{(B^+)} = \frac{2M\Gamma}{1 - r_i V^i} (q^i V_i)^2. \quad (59)$$

This is in exact agreement with the linearized result.

#### 4. Discussion

We have shown that the boost symmetry  $B^{(+)}$  of the Minkowski background  $\eta_{\mu\nu}^{(+)}$  of the ingoing Kerr-Schild version of the Schwarzschild metric leads to a nonlinear model for determining the memory effect due to the ejection of a massive particle. An initially stationary Kerr-Schild-Schwarzschild metric followed by an accelerating interval which produces radiation and leads to a final  $B^{(+)}$  boosted state is consistent with the absence of ingoing radiation and produces outgoing radiation in agreement with the linearized memory effect obtained from a retarded solution. The corresponding results for a  $B^{(-)}$  boost of the Minkowski background  $\eta_{\mu\nu}^{(-)}$  produces results expected in the linearized limit from the use of an advanced Green function.

In [2], we have given an analysis of how radiation memory affects angular momentum conservation. In a non-radiative regime, a preferred Poincaré subgroup can be picked out from the BMS group. This difference  $\Delta\sigma$  between initial and final

radiation strains induces the supertranslation shift (31) between the preferred Poincaré groups at  $u = \pm\infty$ . The rotation subgroups associated with the initial and final Poincaré groups differ by a supertranslation. As a result, the corresponding components of angular momentum intrinsic to the initial and final states differ by supermomenta. This complicates the interpretation of angular momentum flux conservation laws. There might be a distinctly general relativistic mechanism for angular momentum loss. This is a ripe area for numerical investigation.

In prescribing initial data for the numerical simulation of binary black holes using superimposed Kerr-Schild metrics [3, 4],  $B^{(+)}$  is used to induce the orbital motion. Although  $B^{(+)}$  has a logarithmic singularity (20) at  $\mathcal{I}^+$ , this is a pure gauge effect which does not show up in the memory effect measured by the change in asymptotic strain  $\Delta\sigma$  but it could introduce spurious effects in the prescription of binary black hole initial data. Whether this adversely affects the asymptotic gauge behavior of the data deserves further study.

The model presented here provides a scheme for studying these issues. Although our example of a transition from a asymptotically stationary to boosted state is highly idealized, the chief criterion for the model is that, to an asymptotic approximation, the far field behavior of the initial and final states consist of the Kerr-Schild superposition of distant Schwarzschild bodies. The model is also applicable to an initial state whose far field is a superposition of boosted Schwarzschild bodies which, after some dynamic, radiative process, coalesce to form a boosted Kerr black hole. Of course, the intermediate radiative epoch, which determines the final mass and velocity, must be treated by numerical methods. The Kerr-Schild model offers a framework for interpreting such results.

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