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4-particle Amplituhedron at 3-loop and its Mondrian Diagrammatic Implication

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Abstract: This article provides a direct calculation of the 4-particle amplituhedron at 3-loop level, by introducing a set of practical tricks. After delicately rearranging each piece by this calculation, we find a suggestive connection between positivity conditions and Mondrian diagrams, which will be quantitatively defined. Such a pattern can be generalized for all Mondrian diagrams among all those contribute to the 4-particle integrand of planar $\mathcal{N}=4$ SYM to all loop orders, as a subsequent work will show.

Keywords: Amplitudes, Loop integrands.

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1. Introduction

The amplituhedron proposal for 4-particle integrand of planar $\mathcal{N}=4$ SYM to all loop orders [1, 2, 3, 4] is a novel reformulation which only uses positivity conditions for all physical poles to construct the loop integrand. At 2-loop level, as the first nontrivial case, we have just one (mutual) positivity condition

$$D_{12} \equiv (x_2 - x_1)(z_1 - z_2) + (y_2 - y_1)(w_1 - w_2) > 0, \tag{1.1}$$

where $x_i = \langle A_i B_i\, 14\rangle$, $y_i = \langle A_i B_i\, 34\rangle$, $z_i = \langle A_i B_i\, 23\rangle$, $w_i = \langle A_i B_i\, 12\rangle$ and $D_{ij} = \langle A_i B_i\, A_j B_j\rangle$ are all possible physical poles in terms of momentum twistor contractions, and x_i, y_i, z_i, w_i are trivially set to be positive for the i-th loop. The resulting integrand is the double-box topology of two possible orientations, and it is symmetrized for two sets of loop variables [2]. As the loop level increases, its calculational complexity grows explosively due to the highly nontrivial intertwining of all $L(L-1)/2$ positivity conditions of D_{ij}'s. As far as the 3-loop case, it is done under significant simplification brought by double cuts [2], still there is considerable complexity that obscures its somehow simple mathematical structure, as we will reveal in this article and the subsequent work.

As an illuminating appetizer, we reformulate the 2-loop case in the following. As usual, let's preserve z_1, z_2 for imposing $D_{12} > 0$, and triangulate the space spanned by $x_1, x_2, y_1, y_2, w_1, w_2$. We introduce the *ordered subspaces* characterized by, for instance:

$$X(12)Y(12)W(12) \equiv \frac{1}{x_1(x_2-x_1)}\frac{1}{y_1(y_2-y_1)}\frac{1}{w_1(w_2-w_1)}, \tag{1.2}$$

which is a $d\log$ form (omitting the measure factor) of the orderings $x_1 < x_2$, $y_1 < y_2$ and $w_1 < w_2$. In this particular subspace, positivity condition (1.1) unambiguously demands

$$z_1 - z_2 > \frac{y_{21}w_{21}}{x_{21}}, \tag{1.3}$$

where $x_{21} \equiv x_2 - x_1$ and so forth. Here, x_{21}, y_{21}, w_{21} can be treated as genuinely positive variables which replace the original x_2, y_2, w_2. Then the relevant $d\log$ form of z_1, z_2 is simply

$$\frac{1}{z_2(z_1 - z_2 - y_{21}w_{21}/x_{21})} = \frac{x_{21}z_1}{z_1 z_2\, D_{12}}, \tag{1.4}$$

analogously, for $X(12)Y(12)W(21)$ we have

$$\frac{1}{z_1}\left(\frac{1}{z_2} - \frac{1}{z_2 - z_1 - y_{21}w_{12}/x_{21}}\right) = \frac{x_{21}z_1 + y_{21}w_{12}}{z_1 z_2\, D_{12}}. \tag{1.5}$$

A seemingly farfetched observation is, after we flip $W(12)$ to $W(21)$, the additional term $y_{21}w_{12}$ appears in the numerator above due to the orderings of y_1, y_2 and w_1, w_2 are now opposite, allowing one to orient the double box "vertically", as explained in a diagrammatic way below.

In figure 1, we have chosen two perpendicular directions for x and y, while the z and w directions are opposite to those of x and y respectively. Then we assign each loop with a number as usual, but now these numbers have a meaning of orderings of positive variables. Since loop number 2 is below 1, we naturally interpret this as $y_2 > y_1$, and similarly $w_1 > w_2$. In this way, it is straightforward to conclude that, if we flip $w_1 > w_2$ back to $w_2 > w_1$, there is no consistent way to place loop numbers $1, 2$ vertically so the double box can be only oriented horizontally!

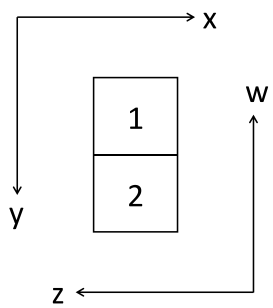

Figure 1: Giving the orderings of positive variables x_i, y_i, z_i, w_i a directional meaning.

If we sum the numerators above over $W(12)$ and $W(21)$ respectively as

$$\frac{1}{w_1(w_2-w_1)}(x_{21}z_1)+\frac{1}{w_2(w_1-w_2)}(x_{21}z_1+y_{21}w_{12})=\frac{x_{21}z_1+y_{21}w_1}{w_1w_2}, \tag{1.6}$$

the ordering of w_1, w_2 is nicely wiped off as expected. Similarly, after summing all such $d\log$ forms over y- and x-space, we get the symmetrized numerator $(x_1z_2+x_2z_1+y_1w_2+y_2w_1)$ [2], and the corresponding denominator is $x_1x_2\,y_1y_2\,z_1z_2\,w_1w_2\,D_{12}$, free of any ordering of x_i, y_i, z_i, w_i.

Actually, positivity condition (1.1) already has a diagrammatic implication. If both $(x_2-x_1)(z_1-z_2)$ and $(y_2-y_1)(w_1-w_2)$ are positive, which means both the orderings of x_1, x_2 and z_1, z_2, and those of y_1, y_2 and w_1, w_2, are opposite, D_{12} is trivially positive. In this case both horizontal and vertical orientations of the double box are allowed and we call them *seed diagrams* with respect to a particular ordered subspace, while those final diagrams are obtained after we sum seed diagrams over all ordered subspaces that admit them for each distinct topology with an orientation. If $(y_2-y_1)(w_1-w_2)$ is negative, for instance, after we switch to ordered subspace $Y(12)W(12)$ or $Y(21)W(21)$, D_{12} is then conditionally positive, since $x_{21}z_{12}$ must be greater than $y_{21}w_{21}$ or $y_{12}w_{12}$, and the double box can be only oriented horizontally. But if both $(x_2-x_1)(z_1-z_2)$ and $(y_2-y_1)(w_1-w_2)$ are negative, D_{12} is by no means positive, and diagrammatically it means there is no legal seed diagram in this ordered subspace.

One may already note that, such a diagrammatic setting only makes sense when all internal lines can be oriented either horizontally or vertically (which are also borders of adjacent loops), and consequently, only 3- and 4-vertex are admitted. This category is called the *Mondrian diagrams* [5], as we will exactly define them based on the amplituhedron setting in the subsequent work. Inside a Mondrian diagram, the relation between any two loops is a horizontal contact, or a vertical contact, or no contact. To characterize a Mondrian diagram in this way turns out to be sufficient for completely identifying its topology!

However, it is known beyond 3-loop, non-Mondrian diagrams also contribute to the planar 4-particle integrand [6, 7]. Therefore, so far, we have to limit ourselves to the Mondrian types until further generalization is available. Nevertheless, at 3-loop such a setting can perfectly connect the amplituhedron and the actual loop integrand. To understand its motivation, we will first present the direct calculation in the context of amplituhedron (or positivity conditions more precisely).

This article is organized as follows. Section 2 presents the fundamentals of positive $d\log$ forms which are necessary for posterior formulations. Section 3 introduces the trick of intermediate variables to handle the 4-particle amplituhedron at 3-loop, as a bridge towards the precise description of imposing 3 positivity conditions simultaneously. Section 4 continues to sum the former results over all ordered subspaces, as we find it "almost" reaches the correct answer. Section 5 further refines co-positive products for each ordered subspace of y and w, based on the discussions in terms of intermediate variables, as a precise description. Section 6 sums the refined co-positive products by delicately separating the contributing and the spurious parts, where the former manifest the Mondrian diagrammatic interpretation, while the latter sum to zero at the end as their name implies.

2. Fundamentals of Positive $d\log$ Forms

First, we will extend the fundamentals of positive $d\log$ forms in [2], as the minimal techniques necessary for the posterior sections. It is known that, for a general positive variable ranging from a to b $(a\!<\!b)$, its form is given by

$$\frac{b-a}{(x-a)(b-x)}, \tag{2.1}$$

in particular, for $a\!=\!0$ it becomes

$$\frac{b}{x(b-x)}, \tag{2.2}$$

as well as for $b\!=\!\infty$ it becomes

$$\frac{1}{x-a}, \tag{2.3}$$

and finally if $b\!=\!a$ for the two special cases above, we have

$$\frac{1}{x-a}+\frac{a}{x(a-x)}=\frac{1}{x}, \tag{2.4}$$

which will be named as the *completeness relation*. It has a natural interpretation as the sum of projective lengths of two complementary positive intervals. Note that we have treated a as a constant above, while it could also be a positive variable. In that case, we only need an additional form $1/a$, so the completeness relation is now

$$\frac{1}{a(x-a)}+\frac{1}{x(a-x)}=\frac{1}{a\,x}, \tag{2.5}$$

where the LHS characterize nothing but two ordered subspaces in which $x\!>\!a$ and $x\!<\!a$ respectively. A trivial generalization of (2.4) for n x_i's satisfying $x_1\ldots x_n\!>\!a$ and $x_1\ldots x_n\!<\!a$ is then

$$\frac{1}{x_1\ldots x_n-a}+\frac{a}{x_1\ldots x_n(a-x_1\ldots x_n)}=\frac{1}{x_1\ldots x_n}, \tag{2.6}$$

here, for example, $x_1\ldots x_n\!>\!a$ is characterized by

$$\frac{1}{x_2\ldots x_n}\frac{1}{x_1-a/(x_2\ldots x_n)}=\frac{1}{x_1\ldots x_n-a}. \tag{2.7}$$

Another less direct generalization of (2.4) for $x_1\!+\!\ldots\!+\!x_n\!>\!a$ and $x_1\!+\!\ldots\!+\!x_n\!<\!a$ is

$$\frac{x_1+\ldots+x_n}{x_1\ldots x_n(x_1+\ldots+x_n-a)}+\frac{a}{x_1\ldots x_n(a-x_1-\ldots-x_n)}=\frac{1}{x_1\ldots x_n}, \tag{2.8}$$

where both terms of the LHS can be proved recursively. If we assume they hold for $x_1\!+\!\ldots\!+\!x_{n-1}\!>\!a$ and $x_1\!+\!\ldots\!+\!x_{n-1}\!<\!a$, to obtain the form of $x_1\!+\!\ldots\!+\!x_n\!>\!a$ we must separate it into two parts as

$$\begin{aligned}&\frac{x_1+\ldots+x_{n-1}}{x_1\ldots x_{n-1}(x_1+\ldots+x_{n-1}-a)}\frac{1}{x_n}+\frac{a}{x_1\ldots x_{n-1}(a-x_1-\ldots-x_{n-1})}\frac{1}{x_1+\ldots+x_n-a}\\&=\frac{x_1+\ldots+x_n}{x_1\ldots x_n(x_1+\ldots+x_n-a)},\end{aligned} \tag{2.9}$$

where in the first line x_n is simply positive in the first term, while it is greater than $(a-x_1-\ldots-x_{n-1})$ in the second, and their sum nicely returns to the form for n x_i's. To obtain the form of $x_1+\ldots+x_n<a$, we can simply insert the form

$$\frac{(a-x_1-\ldots-x_{n-1})}{x_n((a-x_1-\ldots-x_{n-1})-x_n)} \tag{2.10}$$

into that for $(n-1)$ x_i's, and note that $(a-x_1-\ldots-x_{n-1})$ is treated as one positive variable above. These two forms, as well as (2.8) itself, are often useful in the subsequent derivation.

It is also convenient to introduce the *co-positive* product of forms. For example, for $y>x_1,\ldots,x_n$, to obtain its form we can divide it into $n!$ parts with respect to $n!$ ordered subspaces in which $x_{\sigma_1}<\ldots<x_{\sigma_n}$ as $\{\sigma_1,\ldots,\sigma_n\}$ is a permutation of $\{1,\ldots,n\}$. Then we need to simplify

$$I_n(X_n,y) \equiv \sum_{\sigma_n} \frac{1}{x_{\sigma_1}} \frac{1}{x_{\sigma_2}-x_{\sigma_1}} \cdots \frac{1}{x_{\sigma_n}-x_{\sigma_{n-1}}} \frac{1}{y-x_{\sigma_n}} \tag{2.11}$$

with $X_n=\{x_1,\ldots,x_n\}$, by induction. Now let's focus on x_n's location in each permutation while omitting those of $x_1,\ldots,x_{n-1}$, it is straightforward to regroup the sum in order to reach

$$\begin{aligned}\frac{I_n}{I_{n-1}} &= \frac{y-x_{\sigma_{n-1}}}{(x_n-x_{\sigma_{n-1}})(y-x_n)} + \frac{x_{\sigma_{n-1}}-x_{\sigma_{n-2}}}{(x_n-x_{\sigma_{n-2}})(x_{\sigma_{n-1}}-x_n)} + \ldots + \frac{x_{\sigma_2}-x_{\sigma_1}}{(x_n-x_{\sigma_1})(x_{\sigma_2}-x_n)} + \frac{x_{\sigma_1}}{x_n(x_{\sigma_1}-x_n)} \\ &= \frac{y-x_{\sigma_{n-1}}}{(x_n-x_{\sigma_{n-1}})(y-x_n)} + \frac{x_{\sigma_{n-1}}}{x_n(x_{\sigma_{n-1}}-x_n)} = \frac{y}{x_n(y-x_n)},\end{aligned} \tag{2.12}$$

therefore

$$I_n = \frac{y^{n-1}}{x_1\ldots x_n(y-x_1)\ldots(y-x_n)} \equiv \frac{1}{x_1(y-x_1)} \cap \cdots \cap \frac{1}{x_n(y-x_n)}, \tag{2.13}$$

here the symbol $\cap$ is the co-positive product operation. This product denotes the intersected subspace of a number of different subspaces as one form. If we evaluate the residue of I_n at $y=\infty$, it returns to

$$I_n(X_n,\infty) = \int_\infty \frac{dy}{y} \times \frac{y^n}{x_1\ldots x_n(y-x_1)\ldots(y-x_n)} = \frac{1}{x_1\ldots x_n}, \tag{2.14}$$

which is the completeness relation of n positive variables as all x_i's are trivially less than infinity.

Analogously, for $y<x_1,\ldots,x_n$, we need to simplify

$$J_n(y,X_n) = \sum_{\sigma_n} \frac{1}{y} \frac{1}{x_{\sigma_1}-y} \frac{1}{x_{\sigma_2}-x_{\sigma_1}} \cdots \frac{1}{x_{\sigma_n}-x_{\sigma_{n-1}}} \tag{2.15}$$

with the aid of

$$\begin{aligned}\frac{J_n}{J_{n-1}} &= \frac{1}{x_n-x_{\sigma_{n-1}}} + \frac{x_{\sigma_{n-1}}-x_{\sigma_{n-2}}}{(x_n-x_{\sigma_{n-2}})(x_{\sigma_{n-1}}-x_n)} + \ldots + \frac{x_{\sigma_2}-x_{\sigma_1}}{(x_n-x_{\sigma_1})(x_{\sigma_2}-x_n)} + \frac{x_{\sigma_1}-y}{(x_n-y)(x_{\sigma_1}-x_n)} \\ &= \frac{1}{x_n-x_{\sigma_1}} + \frac{x_{\sigma_1}-y}{(x_n-y)(x_{\sigma_1}-x_n)} = \frac{1}{x_n-y},\end{aligned} \tag{2.16}$$

therefore

$$J_n = \frac{1}{y(x_1-y)\ldots(x_n-y)} \equiv \frac{1}{y(x_1-y)} \cap \cdots \cap \frac{1}{y(x_n-y)}. \tag{2.17}$$

If we evaluate the residue of J_n at $y=0$, it returns to

$$J_n(0,X_n) = \int_0 \frac{dy}{y} \times \frac{1}{(x_1-y)\ldots(x_n-y)} = \frac{1}{x_1\ldots x_n}, \tag{2.18}$$

which is also the completeness relation as all x_i's are trivially greater than zero. In fact, (2.13) and (2.17) can be trivially obtained, if we switch to the perspectives which consider $x_1,\ldots,x_n<y$ and $x_1,\ldots,x_n>y$ respectively. Such an equivalent but much simpler approach can be further generalized to

$$\frac{1}{c_1x_1(y-x_1-c_1)} \cap \cdots \cap \frac{1}{c_nx_n(y-x_n-c_n)} = \frac{y^{n-1}}{c_1\ldots c_n\, x_1\ldots x_n(y-x_1-c_1)\ldots(y-x_n-c_n)}, \tag{2.19}$$

as well as

$$\frac{x_1+c_1}{c_1x_1(x_1+c_1-y)\,y} \cap \cdots \cap \frac{x_n+c_n}{c_nx_n(x_n+c_n-y)\,y} = \frac{(x_1+c_1)\ldots(x_n+c_n)}{c_1\ldots c_n\, x_1\ldots x_n(x_1+c_1-y)\ldots(x_n+c_n-y)\,y}, \tag{2.20}$$

where we have used the expressions in (2.8). A mixed product of these two types is, for example,

$$\frac{1}{c_1x_1(y-x_1-c_1)} \cap \frac{x_2+c_2}{c_2x_2(x_2+c_2-y)\,y} = \frac{1}{c_1x_1(y-x_1-c_1)} \times \frac{x_2+c_2}{c_2x_2(x_2+c_2-y)}. \tag{2.21}$$

From these formulas of co-positive products, it is easy to observe that: for n forms that impose positivity conditions on a number of variables, and there is only one common variable among all conditions of these forms, denoted by y for instance, we have

$$I_1 \cap \cdots \cap I_n = I_1 \times \cdots \times I_n \times y^{n-1}, \tag{2.22}$$

which is trivial to prove if we adopt the perspective above. When there are two or more common variables, this simplification is no longer valid, in general. Two such examples are given below:

$$\begin{aligned} &\frac{y_1}{x_1x_2(y_1-x_1)(y_1-x_2)} \cap \frac{y_2}{x_1x_2(y_2-x_1)(y_2-x_2)} \\ &= \frac{y_1y_2-x_1x_2}{x_1x_2(y_1-x_1)(y_1-x_2)(y_2-x_1)(y_2-x_2)}, \end{aligned} \tag{2.23}$$

as well as

$$\begin{aligned} &\frac{y_1^2}{x_1x_2x_3(y_1-x_1)(y_1-x_2)(y_1-x_3)} \cap \frac{y_2^2}{x_1x_2x_3(y_2-x_1)(y_2-x_2)(y_2-x_3)} \\ &= \frac{y_1^2y_2^2 - y_1y_2(x_1x_2+x_1x_3+x_2x_3) + (y_1+y_2)x_1x_2x_3}{x_1x_2x_3(y_1-x_1)(y_1-x_2)(y_1-x_3)(y_2-x_1)(y_2-x_2)(y_2-x_3)}. \end{aligned} \tag{2.24}$$

3. The Trick of Intermediate Variables at 3-loop

Now, we are ready to introduce the trick of intermediate variables to handle the 3 intertwining positivity conditions of the 4-particle amplituhedron at 3-loop. This is not the final answer that we pursuit, but it divides a difficult problem into two parts in a pedagogical way, and it is a nice mathematical warmup for the more precise description. These three positivity conditions are

$$\begin{aligned} D_{12} &= (x_2 - x_1)(z_1 - z_2) + (y_2 - y_1)(w_1 - w_2) > 0, \\ D_{13} &= (x_3 - x_1)(z_1 - z_3) + (y_3 - y_1)(w_1 - w_3) > 0, \\ D_{23} &= (x_3 - x_2)(z_2 - z_3) + (y_3 - y_2)(w_2 - w_3) > 0. \end{aligned} \tag{3.1}$$

Without loss of generality, let's work in the ordered subspace $X(123)$ so that $x_1 < x_2 < x_3$. Then $D_{12} > 0$ unambiguously demands

$$z_1 - z_2 + \frac{(y_2 - y_1)(w_1 - w_2)}{x_{21}} > 0, \tag{3.2}$$

for instance. Depending on the sign of $(y_2 - y_1)(w_1 - w_2)$, we have

$$\begin{aligned} (y_2 - y_1)(w_1 - w_2) > 0: &\quad z_2 < z_1 + c_{12}, \quad c_{12} \equiv \frac{(y_2 - y_1)(w_1 - w_2)}{x_{21}}, \\ (y_2 - y_1)(w_1 - w_2) < 0: &\quad z_1 > z_2 + c_{21}, \quad c_{21} \equiv \frac{-(y_2 - y_1)(w_1 - w_2)}{x_{21}}, \end{aligned} \tag{3.3}$$

where c_{12} and c_{21} are defined as the positive *intermediate variables*. The corresponding forms are then

$$\begin{aligned} Z^-_{12} &\equiv \frac{1}{z_1}\left(\frac{1}{z_2} - \frac{1}{z_2 - z_1 - c_{12}}\right) = \frac{1}{z_1}\frac{z_1 + c_{12}}{z_2(z_1 + c_{12} - z_2)}, \\ Z^+_{21} &\equiv \frac{1}{z_2}\frac{1}{z_1 - z_2 - c_{21}}, \end{aligned} \tag{3.4}$$

in ordered subspaces $(Y(12)W(21)+Y(21)W(12))$ and $(Y(12)W(12)+Y(21)W(21))$ respectively, and the symbols Z^+ and Z^- are related by the completeness relation

$$Z^+_{21} + Z^-_{21} = \frac{1}{z_1 z_2} \equiv I_{12}, \tag{3.5}$$

here the identity I_{12} denotes no positivity condition is imposed on z_1 and z_2.

Therefore, in subspace $X(123)$, for $D_{12}, D_{13}, D_{23} > 0$ we need to figure out the product

$$\begin{aligned} &\left[Z^+_{31}(Y(13)W(13)+Y(31)W(31)) + Z^-_{13}(Y(13)W(31)+Y(31)W(13))\right] \\ \cap &\left[Z^+_{32}(Y(23)W(23)+Y(32)W(32)) + Z^-_{23}(Y(23)W(32)+Y(32)W(23))\right] \\ \cap &\left[Z^+_{21}(Y(12)W(12)+Y(21)W(21)) + Z^-_{12}(Y(12)W(21)+Y(21)W(12))\right], \end{aligned} \tag{3.6}$$

where the products involving y- and w-space are easy, so we mainly focus on the products of $Z^{\pm}$'s. There are $2^3\!=\!8$ such triple co-positive products, as listed below:

$$\begin{aligned} T_1 &\equiv Z^+_{31} \cap Z^+_{32} \cap Z^+_{21}, \quad T_2 \equiv Z^+_{31} \cap Z^-_{23} \cap Z^+_{21}, \\ T_3 &\equiv Z^-_{13} \cap Z^+_{32} \cap Z^+_{21}, \quad T_4 \equiv Z^-_{13} \cap Z^-_{23} \cap Z^+_{21}, \\ T_5 &\equiv Z^+_{31} \cap Z^+_{32} \cap Z^-_{12}, \quad T_6 \equiv Z^+_{31} \cap Z^-_{23} \cap Z^-_{12}, \\ T_7 &\equiv Z^-_{13} \cap Z^+_{32} \cap Z^-_{12}, \quad T_8 \equiv Z^-_{13} \cap Z^-_{23} \cap Z^-_{12}, \end{aligned} \tag{3.7}$$

and now we will determine them one by one.

For T_1, it demands that z_1 is greater than both $(z_2\!+\!c_{21})$ and $(z_3\!+\!c_{31})$, so we need to separately discuss the situations of $z_2\!+\!c_{21}\!>\!z_3\!+\!c_{31}$ and $z_2\!+\!c_{21}\!<\!z_3\!+\!c_{31}$. The extra complexity is, z_2 and z_3 are restricted to the subspace of $z_2\!>\!z_3\!+\!c_{32}$. If $c_{31}\!<\!c_{32}\!+\!c_{21}$, we find $z_2\!+\!c_{21}\!>\!z_3\!+\!c_{32}\!+\!c_{21}\!>\!z_3\!+\!c_{31}$, so $z_1\!>\!z_2\!+\!c_{21}$ already implies $z_1\!>\!z_3\!+\!c_{31}$. If $c_{31}\!>\!c_{32}\!+\!c_{21}$, and $z_2\!>\!z_3\!+\!c_{31}\!-\!c_{21}$ which already implies $z_2\!>\!z_3\!+\!c_{32}$, we again have $z_1\!>\!z_2\!+\!c_{21}$. Finally if $z_3\!+\!c_{32}\!<\!z_2\!<\!z_3\!+\!c_{31}\!-\!c_{21}$, we switch to $z_1\!>\!z_3\!+\!c_{31}$.

As c_{31} is treated as a positive variable, instead of a rational function of other positive variables as it actually should be, the discussion above leads to the sum

$$\begin{aligned} T_1 =& \left(\frac{1}{c_{31}} - \frac{1}{c_{31}-c_{32}-c_{21}}\right) \frac{1}{z_3} \frac{1}{z_2-z_3-c_{32}} \frac{1}{z_1-z_2-c_{21}} \\ &+ \frac{1}{c_{31}-c_{32}-c_{21}} \frac{1}{z_3} \Bigg(\frac{1}{z_2-z_3-c_{31}+c_{21}} \frac{1}{z_1-z_2-c_{21}} \\ &\qquad + \left(\frac{1}{z_2-z_3-c_{32}} - \frac{1}{z_2-z_3-c_{31}+c_{21}}\right) \frac{1}{z_1-z_3-c_{31}} \Bigg) \\ =& \frac{1}{c_{31}} \frac{z_1-z_3}{z_3(z_1-z_2-c_{21})(z_1-z_3-c_{31})(z_2-z_3-c_{32})}, \end{aligned} \tag{3.8}$$

and the $1/c_{31}$ part will be dropped for later convenience. We see this sum wipes off the subspace division of c_{31}, physically it means there is no "spurious pole". And if $c_{31}\!=\!0$, $(z_1\!-\!z_3\!-\!c_{31})$ is cancelled, since this positivity condition becomes redundant as $z_1\!>\!z_2\!+\!c_{21}\!>\!z_3\!+\!c_{32}\!+\!c_{21}$.

Then for T_2, z_2 and z_3 are restricted to the subspace of $z_3\!<\!z_2\!+\!c_{23}$ while other two conditions remain the same as T_1's. Analogously, we have the following discussion:

$$\begin{aligned} &c_{21} < c_{31}, && \begin{cases} z_2 > z_3 + c_{31} - c_{21}, & z_1 > z_2 + c_{21}, \\ z_2 < z_3 + c_{31} - c_{21},\ z_3 < z_2 + c_{23}, & z_1 > z_3 + c_{31}, \end{cases} \\ &c_{31} < c_{21} < c_{23} + c_{31}, && \begin{cases} z_3 < z_2 + c_{21} - c_{31}, & z_1 > z_2 + c_{21}, \\ z_2 + c_{21} - c_{31} < z_3 < z_2 + c_{23}, & z_1 > z_3 + c_{31}, \end{cases} \\ &c_{21} > c_{23} + c_{31}, && \quad z_3 < z_2 + c_{23}, \qquad z_1 > z_2 + c_{21}, \end{aligned} \tag{3.9}$$

note that we have divided the c_{21}-space. This leads to

$$
\begin{aligned}
T_2 =& \left(\frac{1}{c_{21}} - \frac{1}{c_{21}-c_{31}}\right)\left(\frac{1}{z_3}\frac{1}{z_2-z_3-c_{31}+c_{21}}\frac{1}{z_1-z_2-c_{21}}\right. \\
&\qquad\qquad\left. + \left(\frac{1}{z_2}\left(\frac{1}{z_3} - \frac{1}{z_3-z_2-c_{23}}\right) - \frac{1}{z_3}\frac{1}{z_2-z_3-c_{31}+c_{21}}\right)\frac{1}{z_1-z_3-c_{31}}\right) \\
&+ \left(\frac{1}{c_{21}-c_{31}} - \frac{1}{c_{21}-c_{23}-c_{31}}\right)\frac{1}{z_2}\left[\left(\frac{1}{z_3} - \frac{1}{z_3-z_2-c_{21}+c_{31}}\right)\frac{1}{z_1-z_2-c_{21}}\right. \\
&\qquad\qquad\left. + \left(\frac{1}{z_3-z_2-c_{21}+c_{31}} - \frac{1}{z_3-z_2-c_{23}}\right)\frac{1}{z_1-z_3-c_{31}}\right] \\
&+ \frac{1}{c_{21}-c_{23}-c_{31}}\frac{1}{z_2}\left(\frac{1}{z_3} - \frac{1}{z_3-z_2-c_{23}}\right)\frac{1}{z_1-z_2-c_{21}} \\
=& \frac{1}{c_{21}}\frac{z_1(z_2+c_{23})-z_2z_3}{z_2z_3(z_1-z_2-c_{21})(z_1-z_3-c_{31})(z_2+c_{23}-z_3)}.
\end{aligned} \tag{3.10}
$$

In general, we find that for c_{ij}, c_{jk}, c_{ik} with respect to $T_1, T_2, T_4, T_5, T_7, T_8$, it is most convenient to divide the c_{ik}-space. While for c_{ij}, c_{jk}, c_{ki} with respect to T_3, T_6, there is no need to divide any of them.

Then for T_3, since $z_1 > z_2+c_{21} > z_3+c_{32}+c_{21}$ already implies $z_3 < z_1+c_{13}$, Z_{13}^- becomes redundant, which leads to

$$
T_3 = Z_{13}^- \cap Z_{32}^+ \cap Z_{21}^+ = Z_{32}^+ \cap Z_{21}^+ = \frac{1}{z_3(z_1-z_2-c_{21})(z_2-z_3-c_{32})}, \tag{3.11}
$$

now we see there is indeed no need to consider any c_{ij}.

Then for T_4, it demands that z_3 is less than both (z_1+c_{13}) and (z_2+c_{23}) while z_1 and z_2 are restricted to the subspace of $z_1 > z_2+c_{21}$. Analogously, we have the following discussion:

$$
\begin{aligned}
&c_{23} < c_{21}+c_{13}, && z_1 > z_2+c_{21}, \quad z_3 < z_2+c_{23}, \\
&c_{23} > c_{21}+c_{13}, && \begin{cases} z_1 > z_2+c_{23}-c_{13}, & z_3 < z_2+c_{23}, \\ z_2+c_{21} < z_1 < z_2+c_{23}-c_{13}, & z_3 < z_1+c_{13}, \end{cases}
\end{aligned} \tag{3.12}
$$

which leads to

$$
\begin{aligned}
T_4 =& \left(\frac{1}{c_{23}} - \frac{1}{c_{23}-c_{21}-c_{13}}\right)\frac{1}{z_2}\frac{1}{z_1-z_2-c_{21}}\left(\frac{1}{z_3} - \frac{1}{z_3-z_2-c_{23}}\right) \\
&+ \frac{1}{c_{23}-c_{21}-c_{13}}\frac{1}{z_2}\left[\frac{1}{z_1-z_2-c_{23}+c_{13}}\left(\frac{1}{z_3} - \frac{1}{z_3-z_2-c_{23}}\right)\right. \\
&\qquad\qquad\left. + \left(\frac{1}{z_1-z_2-c_{21}} - \frac{1}{z_1-z_2-c_{23}+c_{13}}\right)\left(\frac{1}{z_3} - \frac{1}{z_3-z_1-c_{13}}\right)\right] \\
=& \frac{1}{c_{23}}\frac{(z_1+c_{13})(z_2+c_{23})-z_2z_3}{z_2z_3(z_1-z_2-c_{21})(z_1+c_{13}-z_3)(z_2+c_{23}-z_3)}.
\end{aligned} \tag{3.13}
$$

Then for T_5, $z_3{+}c_{32}{<}z_2{<}z_1{+}c_{12}$ implies that $(z_1{+}c_{12})$ must be greater than $(z_3{+}c_{32})$ while z_1 and z_3 are restricted to the subspace of $z_1{>}z_3{+}c_{31}$. Analogously, we have the following discussion:

$$\begin{aligned} &c_{32} < c_{31} + c_{12},\ z_1 > z_3 + c_{31}, \\ &c_{32} > c_{31} + c_{12},\ z_1 > z_3 + c_{32} - c_{12}, \end{aligned} \tag{3.14}$$

which leads to

$$\begin{aligned} T_5 =& \left[\left(\frac{1}{c_{32}} - \frac{1}{c_{32} - c_{31} - c_{12}}\right)\frac{1}{z_1 - z_3 - c_{31}} + \frac{1}{c_{32} - c_{31} - c_{12}}\frac{1}{z_1 - z_3 - c_{32} + c_{12}}\right] \\ &\times \frac{1}{z_3}\left(\frac{1}{z_2 - z_3 - c_{32}} - \frac{1}{z_2 - z_1 - c_{12}}\right) \\ =& \frac{1}{c_{32}}\frac{(z_1 + c_{12}) - z_3}{z_3(z_1 + c_{12} - z_2)(z_1 - z_3 - c_{31})(z_2 - z_3 - c_{32})}. \end{aligned} \tag{3.15}$$

Then for T_6, similar to T_3, there is no need to consider any c_{ij}, the sum is simply

$$\begin{aligned} T_6 =& \frac{1}{z_1 - z_3 - c_{31}}\left(\frac{1}{z_2}\left(\frac{1}{z_3} - \frac{1}{z_3 - z_2 - c_{23}}\right) - \frac{1}{z_3}\frac{1}{z_2 - z_1 - c_{12}}\right) \\ =& \frac{(z_1 + c_{12})(z_2 + c_{23}) - z_2 z_3}{z_2 z_3(z_1 + c_{12} - z_2)(z_2 + c_{23} - z_3)(z_1 - z_3 - c_{31})}. \end{aligned} \tag{3.16}$$

Then for T_7, similar to T_5, $(z_1{+}c_{12})$ must be greater than $(z_3{+}c_{32})$ while z_1 and z_3 are restricted to the subspace of $z_3{<}z_1{+}c_{13}$. Analogously, we have the following discussion:

$$\begin{aligned} &c_{12} < c_{32}, && z_1 > z_3 + c_{32} - c_{12}, \\ &c_{32} < c_{12} < c_{13} + c_{32}, && z_3 < z_1 + c_{12} - c_{32}, \\ &c_{12} > c_{13} + c_{32}, && z_3 < z_1 + c_{13}, \end{aligned} \tag{3.17}$$

which leads to

$$\begin{aligned} T_7 =& \left[\left(\frac{1}{c_{12}} - \frac{1}{c_{12} - c_{32}}\right)\frac{1}{z_3}\frac{1}{z_1 - z_3 - c_{32} + c_{12}}\right. \\ &+ \left(\frac{1}{c_{12} - c_{32}} - \frac{1}{c_{12} - c_{13} - c_{32}}\right)\frac{1}{z_1}\left(\frac{1}{z_3} - \frac{1}{z_3 - z_1 - c_{12} + c_{32}}\right) \\ &\left. + \frac{1}{c_{12} - c_{13} - c_{32}}\frac{1}{z_1}\left(\frac{1}{z_3} - \frac{1}{z_3 - z_1 - c_{13}}\right)\right]\left(\frac{1}{z_2 - z_3 - c_{32}} - \frac{1}{z_2 - z_1 - c_{12}}\right) \\ =& \frac{1}{c_{12}}\frac{(z_1 + c_{12})(z_1 + c_{13}) - z_1 z_3}{z_1 z_3(z_1 + c_{12} - z_2)(z_1 + c_{13} - z_3)(z_2 - z_3 - c_{32})}. \end{aligned} \tag{3.18}$$

Finally for T_8, similar to T_4, z_3 is less than both $(z_1{+}c_{13})$ and $(z_2{+}c_{23})$ while z_1 and z_2 are restricted to the subspace of $z_2{<}z_1{+}c_{12}$. Analogously, we have the following discussion:

$$
\begin{aligned}
&c_{13}<c_{23}, && \begin{cases} z_2<z_1+c_{12}, & z_1<z_2+c_{23}-c_{13},\ z_3<z_1+c_{13}, \\ & z_1>z_2+c_{23}-c_{13},\ z_3<z_2+c_{23}, \end{cases} \\
&c_{23}<c_{13}<c_{12}+c_{23}, && \begin{cases} z_1+c_{13}-c_{23}<z_2<z_1+c_{12}, & z_3<z_1+c_{13}, \\ & z_2<z_1+c_{13}-c_{23},\ z_3<z_2+c_{23}, \end{cases} \\
&c_{12}+c_{23}<c_{13}, && \qquad\qquad z_2<z_1+c_{12}, \quad z_3<z_2+c_{23},
\end{aligned}
\tag{3.19}
$$

which leads to

$$
\begin{aligned}
T_8 = &\left(\frac{1}{c_{13}}-\frac{1}{c_{13}-c_{23}}\right)\left[\left(\frac{1}{z_1}\left(\frac{1}{z_2}-\frac{1}{z_2-z_1-c_{12}}\right)-\frac{1}{z_2}\frac{1}{z_1-z_2-c_{23}+c_{13}}\right)\left(\frac{1}{z_3}-\frac{1}{z_3-z_1-c_{13}}\right)\right. \\
&\left.+\frac{1}{z_2}\frac{1}{z_1-z_2-c_{23}+c_{13}}\left(\frac{1}{z_3}-\frac{1}{z_3-z_2-c_{23}}\right)\right] \\
&+\left(\frac{1}{c_{13}-c_{23}}-\frac{1}{c_{13}-c_{12}-c_{23}}\right)\frac{1}{z_1}\left[\left(\frac{1}{z_2-z_1-c_{13}+c_{23}}-\frac{1}{z_2-z_1-c_{12}}\right)\left(\frac{1}{z_3}-\frac{1}{z_3-z_1-c_{13}}\right)\right. \\
&\left.+\left(\frac{1}{z_2}-\frac{1}{z_2-z_1-c_{13}+c_{23}}\right)\left(\frac{1}{z_3}-\frac{1}{z_3-z_2-c_{23}}\right)\right] \\
&+\frac{1}{c_{13}-c_{12}-c_{23}}\frac{1}{z_1}\left(\frac{1}{z_2}-\frac{1}{z_2-z_1-c_{12}}\right)\left(\frac{1}{z_3}-\frac{1}{z_3-z_2-c_{23}}\right) \\
=&\frac{1}{c_{13}}\frac{(z_1+c_{12})(z_1+c_{13})(z_2+c_{23})-z_1z_2z_3}{z_1z_2z_3(z_1+c_{12}-z_2)(z_1+c_{13}-z_3)(z_2+c_{23}-z_3)}.
\end{aligned}
\tag{3.20}
$$

Now we have known all eight T_i's. A consistency check via the completeness relations gives

$$
T_2(2{\leftrightarrow}3)+T_1 = Z^+_{31}\cap Z^+_{21} = Z^+_{31}\times Z^+_{21}\times z_1 = \frac{z_1}{z_2z_3(z_1-z_2-c_{21})(z_1-z_3-c_{31})},
\tag{3.21}
$$

where we have used (2.22), similarly we also have (dropping all $1/c_{ij}$ prefactors)

$$
\begin{aligned}
T_2(1{\leftrightarrow}2)+T_7 &= Z^-_{13}\cap Z^+_{32}, \\
T_7(2{\leftrightarrow}3)+T_8 &= Z^-_{13}\cap Z^-_{12}, \\
T_5(1{\leftrightarrow}2)+T_1 &= Z^+_{31}\cap Z^+_{32}, \\
T_5(2{\leftrightarrow}3)+T_4 &= Z^-_{13}\cap Z^+_{21}, \\
T_4(1{\leftrightarrow}2)+T_8 &= Z^-_{13}\cap Z^-_{23}, \\
T_6(1{\leftrightarrow}2)+T_3 &= Z^-_{13}\cap Z^+_{32}.
\end{aligned}
\tag{3.22}
$$

These relations, in fact, serve as an equivalent approach to obtain all other T_i's one by one, once we know T_1 and T_3, following the sequence below:

$$
T_1\to T_2\to T_7\to T_8,\ T_1\to T_5\to T_4\to T_8,\ T_3\to T_6.
\tag{3.23}
$$

In addition, we have also observed that from

$$T_8 = \frac{(z_1+c_{12})(z_1+c_{13})(z_2+c_{23}) - z_1z_2z_3}{z_1z_2z_3(z_1+c_{12}-z_2)(z_1+c_{13}-z_3)(z_2+c_{23}-z_3)}, \tag{3.24}$$

all other T_i's can be obtained via flipping c_{ij} to $-c_{ji}$ in the denominator with respect to flipping each Z^-_{ij} to Z^+_{ji}, as well as setting c_{ij} to zero in the numerator. Therefore, T_8 is named as the *master form.*

There is still another equivalent approach to get the master form which divides the z-space instead of the c-space. Defining

$$\eta_{12} \equiv z_1 - z_2 + c_{12} > 0, \quad \eta_{13} \equiv z_1 - z_3 + c_{13} > 0, \quad \eta_{23} \equiv z_2 - z_3 + c_{23} > 0, \tag{3.25}$$

we find the sum is then

$$\begin{aligned} T_8 = &\ \frac{1}{c_{12}}\left(\frac{1}{c_{13}c_{23}} \times Z(321) + \frac{1}{c_{13}\eta_{23}} \times Z(231) + \frac{1}{\eta_{13}\eta_{23}} \times Z(213)\right) \\ &+ \frac{1}{\eta_{12}}\left(\frac{1}{c_{13}c_{23}} \times Z(312) + \frac{1}{c_{13}\eta_{23}} \times Z(132) + \frac{1}{\eta_{13}\eta_{23}} \times Z(123)\right) \\ = &\ \frac{1}{c_{12}c_{13}c_{23}} \frac{(z_1+c_{12})(z_1+c_{13})(z_2+c_{23}) - z_1z_2z_3}{z_1z_2z_3\,\eta_{12}\eta_{13}\eta_{23}}, \end{aligned} \tag{3.26}$$

as expected. Both ways to get the master form using the completeness relations and dividing the z-space can be generalized beyond 3-loop. Once it is known, we can apply the observation above to get all $2^{\frac{L(L-1)}{2}}$ co-positive products of arbitrary $Z^{\pm}$'s. This observation has not been proved, but it turns out to be valid at 4-loop. In appendix A, we use the latter way to get the master form at 4-loop and after that, we check this observation explicitly via two examples, as a mathematical exercise of curiosity.

4. A Naive Sum

Next, we continue to sum the former results over all ordered subspaces, and we find this naive sum which takes the advantage of intermediate variables "almost" reaches the correct answer, as it can reproduce 96 out of the total 120 monomials in the latter.

To figure out the co-positive products involving y- and w-space, we define, for instance:

$$S(12) \equiv Y(12)W(12) + Y(21)W(21), \;\; A(12) \equiv Y(12)W(21) + Y(21)W(12), \tag{4.1}$$

in which the orderings of y_1, y_2 and w_1, w_2 are the same or opposite respectively. According to (3.6), each Z^+_{ij} is associated with an $S(ij)$, as well as Z^-_{ij} with an $A(ij)$. Then we explicitly calculate the products of S's and A's with respect to all T_i's as

$$\begin{aligned}
T_1:\;\; S_1 \equiv S(13)\cap S(23)\cap S(12) = \;& Y(123)W(123) + Y(132)W(132) + Y(213)W(213) \\
&+ Y(231)W(231) + Y(312)W(312) + Y(321)W(321), \\
T_2:\;\; S_2 \equiv S(13)\cap A(23)\cap S(12) = \;& Y(123)W(132) + Y(132)W(123) \\
&+ Y(231)W(321) + Y(321)W(231), \\
T_3:\;\; S_3 \equiv A(13)\cap S(23)\cap S(12) = \;& Y(132)W(312) + Y(312)W(132) \\
&+ Y(213)W(231) + Y(231)W(213), \\
T_4:\;\; S_4 \equiv A(13)\cap A(23)\cap S(12) = \;& Y(123)W(312) + Y(312)W(123) \\
&+ Y(213)W(321) + Y(321)W(213), \\
T_5:\;\; S_5 \equiv S(13)\cap S(23)\cap A(12) = \;& Y(123)W(213) + Y(213)W(123) \\
&+ Y(312)W(321) + Y(321)W(312), \\
T_6:\;\; S_6 \equiv S(13)\cap A(23)\cap A(12) = \;& Y(132)W(213) + Y(213)W(132) \\
&+ Y(312)W(231) + Y(231)W(312), \\
T_7:\;\; S_7 \equiv A(13)\cap S(23)\cap A(12) = \;& Y(123)W(231) + Y(231)W(123) \\
&+ Y(132)W(321) + Y(321)W(132), \\
T_8:\;\; S_8 \equiv A(13)\cap A(23)\cap A(12) = \;& Y(123)W(321) + Y(132)W(231) + Y(213)W(312) \\
&+ Y(231)W(132) + Y(312)W(213) + Y(321)W(123),
\end{aligned} \tag{4.2}$$

note that in particular, above we have used $Y(13)\cap Y(32)\cap Y(21)=0$ and so forth. These results are for subspace $X(123)$ only, and we need to consider all other ordered subspaces of x, such as

$$\begin{aligned}
X(123):&\;\; S_1T_1 + S_2T_2 + S_3T_3 + S_4T_4 + S_5T_5 + S_6T_6 + S_7T_7 + S_8T_8, \\
\to X(132):&\;\; S_1T_1 + S_2T_2 + S_3T_5 + S_4T_6 + S_5T_3 + S_6T_4 + S_7T_7 + S_8T_8,
\end{aligned} \tag{4.3}$$

where switching $2 \leftrightarrow 3$ for x, y, z, w leads to switching $T_3 \leftrightarrow T_5$ and $T_4 \leftrightarrow T_6$, as can be easily verified, and the rest pieces are similarly given by

$$\begin{aligned} X(213):\ & S_1T_1 + S_2T_3 + S_3T_2 + S_4T_4 + S_5T_5 + S_6T_7 + S_7T_6 + S_8T_8, \\ X(231):\ & S_1T_1 + S_2T_5 + S_3T_2 + S_4T_6 + S_5T_3 + S_6T_7 + S_7T_4 + S_8T_8, \\ X(312):\ & S_1T_1 + S_2T_3 + S_3T_5 + S_4T_7 + S_5T_2 + S_6T_4 + S_7T_6 + S_8T_8, \\ X(321):\ & S_1T_1 + S_2T_5 + S_3T_3 + S_4T_7 + S_5T_2 + S_6T_6 + S_7T_4 + S_8T_8. \end{aligned} \tag{4.4}$$

Therefore, summing them over all ordered subspaces of x, y, w, we obtain

$$\text{Sum} = (\text{Correct answer}) - \text{Difference}, \tag{4.5}$$

where

$$\begin{aligned} &(\text{Correct answer}) \times \text{Denominator} \\ &= (x_2x_3z_1z_2 + y_2y_3w_1w_2)D_{13} + \left(x_3^2z_1z_2y_2w_1 + x_2x_3z_1^2y_3w_2 + x_2z_1y_3^2w_1w_2 + x_3z_2y_2y_3w_1^2\right) \\ &\quad + (5 \text{ permutations of } 1,2,3), \end{aligned} \tag{4.6}$$

as well as

$$\begin{aligned} &\text{Difference} \times \text{Denominator} \\ &= x_2x_3z_1z_2(-y_1w_1 - y_3w_3) + y_2y_3w_1w_2(-x_1z_1 - x_3z_3) + (5 \text{ permutations of } 1,2,3), \end{aligned} \tag{4.7}$$

and we have defined the product of all physical poles as

$$\text{Denominator} \equiv x_1x_2x_3\, y_1y_2y_3\, z_1z_2z_3\, w_1w_2w_3\, D_{12}D_{13}D_{23}. \tag{4.8}$$

Since D_{13} contains 8 monomials, the correct answer has $(2\times8+4)\times6=120$ monomials, and the sum has 96 so their difference has $4\times6=24$ monomials. It is important to notice that terms such as $x_2x_3z_1z_2(-y_1w_1)$ are not dual conformally invariant by themselves, but grouped as $x_2x_3z_1z_2D_{13}$ they are.

This tentative answer simplified by the trick of intermediate variables captures more than we expect. Even though it oversimplifies the complexity of c_{ij}'s which are functions of x, y, w, it still gives most parts of the correct answer. If we manually heal the dual conformal invariance, it is then correct. Remarkably, even if it does not give the full numerator, it can wipe off the subspace division of all positive variables, which frees it from "spurious poles". After we refine the calculation and reach the correct answer, we will return to discuss the diagrammatic interpretation of (4.6).

5. Refined Co-positive Products

To precisely describe the 4-particle amplituhedron at 3-loop, we need to further refine co-positive products for each ordered subspace of y and w, based on the former discussions using intermediate variables. These seemingly lengthy results can be nicely rearranged in order to manifest its simple mathematical structure, namely the Mondrian diagrammatic interpretation.

From the previous setting we know that, for each ordered subspace of x, there are eight T_i's, namely the co-positive products in terms of intermediate variables. From (4.2), each of T_1, T_8 corresponds to six ordered subspaces of y and w, while each of $T_2, T_3, T_4, T_5, T_7, T_8$ corresponds to four, so that in total their number is $6\times6=36$ as expected. If we abandon intermediate variables, in principle we have to figure out 36 co-positive products instead of 8, as elaborated in the following.

For T_1, the six different ordered subspaces lead to six different sets of c_{ij}'s. First for $Y(123)W(123)$, the condition $c_{31}>c_{32}+c_{21}$ is now replaced by

$$\frac{(y_{32}+y_{21})(w_{32}+w_{21})}{x_{32}+x_{21}} > \frac{y_{32}w_{32}}{x_{32}} + \frac{y_{21}w_{21}}{x_{21}} \Longrightarrow \left(\frac{y_{32}}{y_{21}} - \frac{x_{32}}{x_{21}}\right)\left(\frac{w_{32}}{w_{21}} - \frac{x_{32}}{x_{21}}\right) < 0, \tag{5.1}$$

where $x_{32}, x_{21}, y_{32}, y_{21}, w_{32}, w_{21}$ are positive variables in this subspace (as usual, we first work in $X(123)$). The corresponding form is

$$\begin{aligned} &\frac{1}{y_{21}}\frac{1}{y_{32}-y_{21}x_{32}/x_{21}}\frac{1}{w_{32}}\frac{1}{w_{21}-w_{32}x_{21}/x_{32}} + (y \leftrightarrow w) \\ &= \frac{x_{21}x_{32}(y_{21}w_{32}+y_{32}w_{21})}{y_{21}y_{32}\,w_{21}w_{32}(x_{32}y_{21}-x_{21}y_{32})(-x_{32}w_{21}+x_{21}w_{32})}, \end{aligned} \tag{5.2}$$

then using the completeness relation, the form of $c_{31}<c_{32}+c_{21}$ is

$$\begin{aligned} &\frac{1}{y_{21}y_{32}\,w_{21}w_{32}} - \frac{x_{21}x_{32}(y_{21}w_{32}+y_{32}w_{21})}{y_{21}y_{32}\,w_{21}w_{32}(x_{32}y_{21}-x_{21}y_{32})(-x_{32}w_{21}+x_{21}w_{32})} \\ &= \frac{x_{32}^2y_{21}w_{21}+x_{21}^2y_{32}w_{32}}{y_{21}y_{32}\,w_{21}w_{32}(x_{32}y_{21}-x_{21}y_{32})(x_{32}w_{21}-x_{21}w_{32})}. \end{aligned} \tag{5.3}$$

Making relevant replacements in (3.8) gives the full form

$$T_{1,\,Y(123)W(123)} = \frac{1}{D_{12}D_{13}D_{23}}\,Y(123)W(123)\,\frac{D_{13}+y_{21}w_{32}+y_{32}w_{21}}{x_1z_3}, \tag{5.4}$$

analogously, for $Y(321)W(321)$ it becomes

$$T_{1,\,Y(321)W(321)} = \frac{1}{D_{12}D_{13}D_{23}}\,Y(321)W(321)\,\frac{D_{13}+y_{12}w_{23}+y_{23}w_{12}}{x_1z_3}. \tag{5.5}$$

For convenience, we will drop the prefactors $Y(\ldots)W(\ldots)/D_{12}D_{13}D_{23}$ below.

Next for $Y(132)W(132)$, we trivially have $c_{31}<c_{21}$ since

$$\frac{y_{31}w_{31}}{x_{32}+x_{21}} < \frac{(y_{23}+y_{31})(w_{23}+w_{31})}{x_{21}} \tag{5.6}$$

always holds, which belongs to the situation of $c_{31}<c_{32}+c_{21}$. Therefore the sum in (3.8) trivially has one term only, which gives

$$T_{1,\,Y(132)W(132)}=\frac{D_{13}}{x_1z_3}, \tag{5.7}$$

analogously, for $Y(231)W(231)$ it becomes

$$T_{1,\,Y(231)W(231)}=\frac{D_{13}}{x_1z_3}. \tag{5.8}$$

Last for $Y(213)W(213)$, we trivially have $c_{31}<c_{32}$ since

$$\frac{y_{31}w_{31}}{x_{32}+x_{21}}<\frac{(y_{31}+y_{12})(w_{31}+w_{12})}{x_{32}} \tag{5.9}$$

always holds, which belongs to the situation of $c_{31}<c_{32}+c_{21}$. Therefore the sum in (3.8) trivially has one term only, which gives

$$T_{1,\,Y(213)W(213)}=\frac{D_{13}}{x_1z_3}, \tag{5.10}$$

as well as for $Y(312)W(312)$,

$$T_{1,\,Y(312)W(312)}=\frac{D_{13}}{x_1z_3}. \tag{5.11}$$

Then for T_2 in $Y(123)W(132)$, the condition $c_{21}<c_{31}$ is replaced by

$$\frac{y_{21}(w_{23}+w_{31})}{x_{21}}<\frac{(y_{32}+y_{21})w_{31}}{x_{32}+x_{21}}\Longrightarrow\frac{y_{32}}{y_{21}}>\frac{x_{32}}{x_{21}}+\frac{w_{23}}{w_{31}}+\frac{x_{32}}{x_{21}}\frac{w_{23}}{w_{31}}\equiv\alpha, \tag{5.12}$$

and the corresponding form is

$$\frac{1}{y_{21}}\frac{1}{y_{32}-y_{21}\alpha}, \tag{5.13}$$

similarly, $c_{21}>c_{23}+c_{31}$ is replaced by

$$\frac{y_{21}(w_{23}+w_{31})}{x_{21}}>\frac{y_{32}w_{23}}{x_{32}}+\frac{(y_{32}+y_{21})w_{31}}{x_{32}+x_{21}}\Longrightarrow\frac{y_{32}}{y_{21}}<\frac{x_{32}}{x_{21}}, \tag{5.14}$$

and the corresponding form is

$$\frac{1}{y_{32}}\frac{1}{y_{21}-y_{32}x_{21}/x_{32}}, \tag{5.15}$$

hence the form of $c_{31}<c_{21}<c_{23}+c_{31}$ is

$$\frac{1}{y_{21}y_{32}}-\frac{1}{y_{21}}\frac{1}{y_{32}-y_{21}\alpha}-\frac{1}{y_{32}}\frac{1}{y_{21}-y_{32}x_{21}/x_{32}}. \tag{5.16}$$

Making relevant replacements in (3.10) gives

$$T_{2,\,Y(123)W(132)}=\frac{x_{32}+x_{21}}{x_1z_2z_3}\left[\left(z_1-\frac{y_{21}w_{31}}{x_{32}+x_{21}}\right)\left(z_2+\frac{y_{32}w_{23}}{x_{32}}\right)-z_2z_3\right], \tag{5.17}$$

analogously, for $Y(321)W(231)$ it becomes

$$T_{2,\,Y(321)W(231)} = \frac{x_{32}+x_{21}}{x_1 z_2 z_3}\left[\left(z_1 - \frac{y_{12}w_{13}}{x_{32}+x_{21}}\right)\left(z_2 + \frac{y_{23}w_{32}}{x_{32}}\right) - z_2 z_3\right]. \tag{5.18}$$

Switching $y \leftrightarrow w$, for $Y(132)W(123)$ and $Y(231)W(321)$ we obtain

$$\begin{aligned} T_{2,\,Y(132)W(123)} &= \frac{x_{32}+x_{21}}{x_1 z_2 z_3}\left[\left(z_1 - \frac{y_{31}w_{21}}{x_{32}+x_{21}}\right)\left(z_2 + \frac{y_{23}w_{32}}{x_{32}}\right) - z_2 z_3\right], \\ T_{2,\,Y(231)W(321)} &= \frac{x_{32}+x_{21}}{x_1 z_2 z_3}\left[\left(z_1 - \frac{y_{13}w_{12}}{x_{32}+x_{21}}\right)\left(z_2 + \frac{y_{32}w_{23}}{x_{32}}\right) - z_2 z_3\right]. \end{aligned} \tag{5.19}$$

Then for T_3 in $Y(132)W(312)$, $Y(312)W(132)$, $Y(213)W(231)$ and $Y(231)W(213)$, there is no difference in the discussions as we do not divide any c_{ij}. Immediately, (3.11) gives

$$\begin{aligned} T_{3,\,Y(132)W(312)} &= \frac{D_{13}}{x_1 z_3}, \quad T_{3,\,Y(231)W(213)} = \frac{D_{13}}{x_1 z_3}, \\ T_{3,\,Y(312)W(132)} &= \frac{D_{13}}{x_1 z_3}, \quad T_{3,\,Y(213)W(231)} = \frac{D_{13}}{x_1 z_3}, \end{aligned} \tag{5.20}$$

note that we must separately write these identical results, since they have different hidden prefactors.

Then for T_4 in $Y(123)W(312)$, the condition $c_{23}<c_{21}+c_{13}$ is replaced by

$$\frac{y_{32}(w_{21}+w_{13})}{x_{32}} < \frac{y_{21}w_{21}}{x_{21}} + \frac{(y_{32}+y_{21})w_{13}}{x_{32}+x_{21}} \Longrightarrow \frac{y_{32}}{y_{21}} < \frac{x_{32}}{x_{21}}, \tag{5.21}$$

and the corresponding form is

$$\frac{1}{y_{32}}\frac{1}{y_{21}-y_{32}x_{21}/x_{32}}, \tag{5.22}$$

similarly, the form of $c_{23}>c_{21}+c_{13}$ is

$$\frac{1}{y_{21}}\frac{1}{y_{32}-y_{21}x_{32}/x_{21}}. \tag{5.23}$$

Making relevant replacements in (3.13) gives

$$T_{4,\,Y(123)W(312)} = \frac{x_{32}+x_{21}}{x_1 z_2 z_3}\left[\left(z_1 + \frac{(y_{32}+y_{21})w_{13}}{x_{32}+x_{21}}\right)\left(z_2 + \frac{y_{32}(w_{21}+w_{13})}{x_{32}}\right) - \left(z_2 + \frac{y_{32}w_{13}}{x_{32}+x_{21}}\right) z_3\right], \tag{5.24}$$

analogously, for $Y(321)W(213)$ it becomes

$$T_{4,\,Y(321)W(213)} = \frac{x_{32}+x_{21}}{x_1 z_2 z_3}\left[\left(z_1 + \frac{(y_{12}+y_{23})w_{31}}{x_{32}+x_{21}}\right)\left(z_2 + \frac{y_{23}(w_{31}+w_{12})}{x_{32}}\right) - \left(z_2 + \frac{y_{23}w_{31}}{x_{32}+x_{21}}\right) z_3\right]. \tag{5.25}$$

Switching $y \leftrightarrow w$, for $Y(312)W(123)$ and $Y(213)W(321)$ we obtain

$$\begin{aligned} T_{4,\,Y(312)W(123)} &= \frac{x_{32}+x_{21}}{x_1 z_2 z_3}\left[\left(z_1 + \frac{y_{13}(w_{32}+w_{21})}{x_{32}+x_{21}}\right)\left(z_2 + \frac{(y_{21}+y_{13})w_{32}}{x_{32}}\right) - \left(z_2 + \frac{y_{13}w_{32}}{x_{32}+x_{21}}\right) z_3\right], \\ T_{4,\,Y(213)W(321)} &= \frac{x_{32}+x_{21}}{x_1 z_2 z_3}\left[\left(z_1 + \frac{y_{31}(w_{12}+w_{23})}{x_{32}+x_{21}}\right)\left(z_2 + \frac{(y_{31}+y_{12})w_{23}}{x_{32}}\right) - \left(z_2 + \frac{y_{31}w_{23}}{x_{32}+x_{21}}\right) z_3\right]. \end{aligned} \tag{5.26}$$

Then for T_5 in $Y(123)W(213)$, the condition $c_{32}<c_{31}+c_{12}$ is replaced by

$$\frac{y_{32}(w_{31}+w_{12})}{x_{32}} < \frac{(y_{32}+y_{21})w_{31}}{x_{32}+x_{21}} + \frac{y_{21}w_{12}}{x_{21}} \Longrightarrow \frac{y_{32}}{y_{21}} < \frac{x_{32}}{x_{21}}, \tag{5.27}$$

and the corresponding form is

$$\frac{1}{y_{32}}\frac{1}{y_{21}-y_{32}x_{21}/x_{32}}. \tag{5.28}$$

Making relevant replacements in (3.15) gives

$$T_{5,\,Y(123)W(213)} = \frac{x_{32}+x_{21}}{x_1z_3}\left(z_1-z_3-\frac{y_{32}w_{31}}{x_{32}+x_{21}}+\frac{y_{21}w_{12}}{x_{21}}\right), \tag{5.29}$$

analogously, for $Y(321)W(312)$ it becomes

$$T_{5,\,Y(321)W(312)} = \frac{x_{32}+x_{21}}{x_1z_3}\left(z_1-z_3-\frac{y_{23}w_{13}}{x_{32}+x_{21}}+\frac{y_{12}w_{21}}{x_{21}}\right). \tag{5.30}$$

Switching $y\leftrightarrow w$, for $Y(213)W(123)$ and $Y(312)W(321)$ we obtain

$$\begin{aligned} T_{5,\,Y(213)W(123)} &= \frac{x_{32}+x_{21}}{x_1z_3}\left(z_1-z_3-\frac{y_{31}w_{32}}{x_{32}+x_{21}}+\frac{y_{12}w_{21}}{x_{21}}\right), \\ T_{5,\,Y(312)W(321)} &= \frac{x_{32}+x_{21}}{x_1z_3}\left(z_1-z_3-\frac{y_{13}w_{23}}{x_{32}+x_{21}}+\frac{y_{21}w_{12}}{x_{21}}\right). \end{aligned} \tag{5.31}$$

Then for T_6 in $Y(132)W(213)$, $Y(213)W(132)$, $Y(312)W(231)$ and $Y(231)W(312)$, there is no difference in the discussions, similar to the case of T_3. But the difference among c_{ij}'s now matters, as (3.16) gives

$$\begin{aligned} T_{6,\,Y(132)W(213)} &= \frac{x_{32}+x_{21}}{x_1z_2z_3}\left[\left(z_1+\frac{(y_{23}+y_{31})w_{12}}{x_{21}}\right)\left(z_2+\frac{y_{23}(w_{31}+w_{12})}{x_{32}}\right)-z_2z_3\right], \\ T_{6,\,Y(231)W(312)} &= \frac{x_{32}+x_{21}}{x_1z_2z_3}\left[\left(z_1+\frac{(y_{13}+y_{32})w_{21}}{x_{21}}\right)\left(z_2+\frac{y_{32}(w_{21}+w_{13})}{x_{32}}\right)-z_2z_3\right], \end{aligned} \tag{5.32}$$

switching $y\leftrightarrow w$, we obtain

$$\begin{aligned} T_{6,\,Y(213)W(132)} &= \frac{x_{32}+x_{21}}{x_1z_2z_3}\left[\left(z_1+\frac{y_{12}(w_{23}+w_{31})}{x_{21}}\right)\left(z_2+\frac{(y_{31}+y_{12})w_{23}}{x_{32}}\right)-z_2z_3\right], \\ T_{6,\,Y(312)W(231)} &= \frac{x_{32}+x_{21}}{x_1z_2z_3}\left[\left(z_1+\frac{y_{21}(w_{13}+w_{32})}{x_{21}}\right)\left(z_2+\frac{(y_{21}+y_{13})w_{32}}{x_{32}}\right)-z_2z_3\right]. \end{aligned} \tag{5.33}$$

Then for T_7 in $Y(123)W(231)$, the condition $c_{12}<c_{32}$ is replaced by

$$\frac{y_{21}(w_{13}+w_{32})}{x_{21}} < \frac{y_{32}w_{32}}{x_{32}} \Longrightarrow \frac{y_{32}}{y_{21}} > \frac{x_{32}}{x_{21}}\left(1+\frac{w_{13}}{w_{32}}\right) \equiv \beta, \tag{5.34}$$

and the corresponding form is

$$\frac{1}{y_{21}}\frac{1}{y_{32}-y_{21}\beta}, \tag{5.35}$$

similarly, $c_{12}>c_{13}+c_{32}$ is replaced by

$$\frac{y_{21}(w_{13}+w_{32})}{x_{21}} > \frac{(y_{32}+y_{21})w_{13}}{x_{32}+x_{21}} + \frac{y_{32}w_{32}}{x_{32}} \Longrightarrow \frac{y_{32}}{y_{21}} < \frac{x_{32}}{x_{21}}, \tag{5.36}$$

and the corresponding form is

$$\frac{1}{y_{32}}\frac{1}{y_{21}-y_{32}x_{21}/x_{32}}. \tag{5.37}$$

Making relevant replacements in (3.18) gives

$$T_{7,Y(123)W(231)} = \frac{x_{32}+x_{21}}{x_1z_2z_3}\left[\left(z_1+\frac{y_{21}(w_{13}+w_{32})}{x_{21}}\right)\left(z_1+\frac{(y_{32}+y_{21})w_{13}}{x_{32}+x_{21}}\right)-\left(z_1+\frac{y_{21}w_{13}}{x_{32}+x_{21}}\right)z_3\right], \tag{5.38}$$

analogously, for $Y(321)W(132)$ it becomes

$$T_{7,Y(321)W(132)} = \frac{x_{32}+x_{21}}{x_1z_2z_3}\left[\left(z_1+\frac{y_{12}(w_{23}+w_{31})}{x_{21}}\right)\left(z_1+\frac{(y_{12}+y_{23})w_{31}}{x_{32}+x_{21}}\right)-\left(z_1+\frac{y_{12}w_{31}}{x_{32}+x_{21}}\right)z_3\right]. \tag{5.39}$$

Switching $y\leftrightarrow w$, for $Y(231)W(123)$ and $Y(132)W(321)$ we obtain

$$\begin{aligned} T_{7,Y(231)W(123)} &= \frac{x_{32}+x_{21}}{x_1z_2z_3}\left[\left(z_1+\frac{(y_{13}+y_{32})w_{21}}{x_{21}}\right)\left(z_1+\frac{y_{13}(w_{32}+w_{21})}{x_{32}+x_{21}}\right)-\left(z_1+\frac{y_{13}w_{21}}{x_{32}+x_{21}}\right)z_3\right], \\ T_{7,Y(132)W(321)} &= \frac{x_{32}+x_{21}}{x_1z_2z_3}\left[\left(z_1+\frac{(y_{23}+y_{31})w_{12}}{x_{21}}\right)\left(z_1+\frac{y_{31}(w_{12}+w_{23})}{x_{32}+x_{21}}\right)-\left(z_1+\frac{y_{31}w_{12}}{x_{32}+x_{21}}\right)z_3\right]. \end{aligned} \tag{5.40}$$

Finally for T_8 in $Y(123)W(321)$, the condition $c_{13}<c_{23}$ is replaced by

$$\frac{(y_{32}+y_{21})(w_{12}+w_{23})}{x_{32}+x_{21}} < \frac{y_{32}w_{23}}{x_{32}} \Longrightarrow \frac{x_{21}}{x_{32}} > \frac{y_{21}}{y_{32}} + \frac{w_{12}}{w_{23}} + \frac{y_{21}}{y_{32}}\frac{w_{12}}{w_{23}} \equiv \gamma, \tag{5.41}$$

and the corresponding form is

$$\frac{1}{x_{32}}\frac{1}{x_{21}-x_{32}\gamma}, \tag{5.42}$$

similarly, $c_{13}>c_{12}+c_{23}$ is replaced by

$$\frac{(y_{32}+y_{21})(w_{12}+w_{23})}{x_{32}+x_{21}} > \frac{y_{21}w_{12}}{x_{21}} + \frac{y_{32}w_{23}}{x_{32}} \Longrightarrow \left(\frac{y_{32}}{y_{21}}-\frac{x_{32}}{x_{21}}\right)\left(\frac{w_{23}}{w_{12}}-\frac{x_{32}}{x_{21}}\right) < 0, \tag{5.43}$$

by trivially adjusting the prefactors of (5.2), we get the corresponding form

$$\frac{(y_{21}w_{32}+y_{32}w_{21})}{(x_{32}y_{21}-x_{21}y_{32})(-x_{32}w_{21}+x_{21}w_{32})}. \tag{5.44}$$

Making relevant replacements in (3.20) gives

$$T_{8,\,Y(123)W(321)} = \frac{x_{32}+x_{21}}{x_1 z_1 z_2 z_3}\left[\left(z_2+\frac{y_{32}w_{23}}{x_{32}}\right)\left(z_1+\frac{y_{21}w_{12}}{x_{21}}\right)\left(z_1+\frac{(y_{32}+y_{21})(w_{12}+w_{23})}{x_{32}+x_{21}}\right)\right.$$
$$\left. -\left(\frac{y_{21}w_{12}}{x_{32}+x_{21}}\left(z_2+\frac{y_{32}w_{23}}{x_{21}}\right)+\left(z_2+\frac{y_{32}w_{23}}{x_{32}+x_{21}}\right)z_1\right)z_3\right], \tag{5.45}$$

analogously, for $Y(321)W(123)$ it becomes

$$T_{8,\,Y(321)W(123)} = \frac{x_{32}+x_{21}}{x_1 z_1 z_2 z_3}\left[\left(z_2+\frac{y_{23}w_{32}}{x_{32}}\right)\left(z_1+\frac{y_{12}w_{21}}{x_{21}}\right)\left(z_1+\frac{(y_{12}+y_{23})(w_{32}+w_{21})}{x_{32}+x_{21}}\right)\right.$$
$$\left. -\left(\frac{y_{12}w_{21}}{x_{32}+x_{21}}\left(z_2+\frac{y_{23}w_{32}}{x_{21}}\right)+\left(z_2+\frac{y_{23}w_{32}}{x_{32}+x_{21}}\right)z_1\right)z_3\right]. \tag{5.46}$$

Next for $Y(132)W(231)$, we trivially have $c_{13}<c_{12}$ since

$$\frac{y_{31}w_{13}}{x_{32}+x_{21}} < \frac{(y_{23}+y_{31})(w_{13}+w_{32})}{x_{21}} \tag{5.47}$$

always holds, which forbids the situation of $c_{13}>c_{12}+c_{23}$ so we only need to discuss whether c_{13} is greater than c_{23}. The condition $c_{13}<c_{23}$ is replaced by

$$\frac{y_{31}w_{13}}{x_{32}+x_{21}} < \frac{y_{23}w_{32}}{x_{32}} \Longrightarrow \frac{y_{31}}{y_{23}} < \left(1+\frac{x_{21}}{x_{32}}\right)\frac{w_{32}}{w_{13}} \equiv \delta, \tag{5.48}$$

and the corresponding form is

$$\frac{1}{y_{31}}\frac{1}{y_{23}-y_{31}/\delta}. \tag{5.49}$$

Making relevant replacements in (3.20) gives

$$T_{8,\,Y(132)W(231)} = \frac{x_{32}+x_{21}}{x_1 z_1 z_2 z_3}\left(z_1+\frac{y_{31}w_{13}}{x_{32}+x_{21}}\right)\left[\left(z_1+\frac{(y_{23}+y_{31})(w_{13}+w_{32})}{x_{21}}\right)\left(z_2+\frac{y_{23}w_{32}}{x_{32}}\right)-z_2 z_3\right], \tag{5.50}$$

analogously, for $Y(231)W(132)$ it becomes

$$T_{8,\,Y(231)W(132)} = \frac{x_{32}+x_{21}}{x_1 z_1 z_2 z_3}\left(z_1+\frac{y_{13}w_{31}}{x_{32}+x_{21}}\right)\left[\left(z_1+\frac{(y_{13}+y_{32})(w_{23}+w_{31})}{x_{21}}\right)\left(z_2+\frac{y_{32}w_{23}}{x_{32}}\right)-z_2 z_3\right]. \tag{5.51}$$

Last for $Y(213)W(312)$, we trivially have $c_{13}<c_{23}$ since

$$\frac{y_{31}w_{13}}{x_{32}+x_{21}} < \frac{(y_{31}+y_{12})(w_{21}+w_{13})}{x_{32}} \tag{5.52}$$

always holds. Therefore the sum in (3.20) trivially has one term only, which gives

$$T_{8,\,Y(213)W(312)} = \frac{x_{32}+x_{21}}{x_1 z_1 z_2 z_3}\left[\left(z_1+\frac{y_{12}w_{21}}{x_{21}}\right)\left(z_1+\frac{y_{31}w_{13}}{x_{32}+x_{21}}\right)\left(z_2+\frac{(y_{31}+y_{12})(w_{21}+w_{13})}{x_{32}}\right)\right.$$
$$\left. -z_3\left(\frac{y_{31}w_{13}}{x_{32}+x_{21}}\left(z_1+\frac{y_{12}w_{21}}{x_{21}}\right)+z_1 z_2\right)\right], \tag{5.53}$$

analogously, for $Y(312)W(213)$ it becomes

$$
\begin{aligned}
T_{8,\,Y(312)W(213)} = \frac{x_{32}+x_{21}}{x_1 z_1 z_2 z_3} \Bigg[&\left(z_1 + \frac{y_{21}w_{12}}{x_{21}}\right)\left(z_1 + \frac{y_{13}w_{31}}{x_{32}+x_{21}}\right)\left(z_2 + \frac{(y_{21}+y_{13})(w_{31}+w_{12})}{x_{32}}\right) \\
&- z_3 \left(\frac{y_{13}w_{31}}{x_{32}+x_{21}}\left(z_1 + \frac{y_{21}w_{12}}{x_{21}}\right) + z_1 z_2\right)\Bigg] .
\end{aligned}
\tag{5.54}
$$

6. The Correct Sum and its Mondrian Diagrammatic Interpretation

Collecting the 36 co-positive products for all ordered subspaces of y and w, we can continue to sum these results. And this time it indeed reaches the correct answer. Instead of a brute-force summation, for each piece we delicately separate the contributing and the spurious parts. The former manifest the Mondrian diagrammatic interpretation with which they nicely sum to (4.6), while the latter sum to zero at the end.

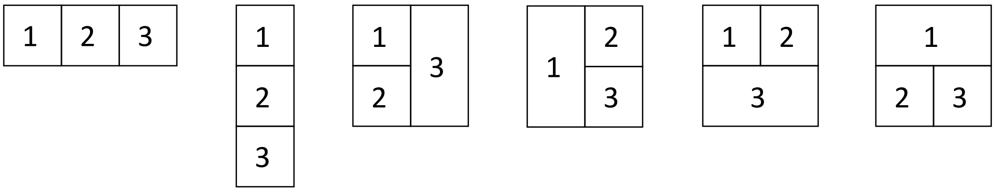

Figure 2: Legal seed diagrams of $X(123)Y(123)$: ladders and tennis courts.

The first example is $Y(123)W(123)$, for which the form (5.4) can be rewritten as

$$\begin{aligned} T_{1,\,Y(123)W(123)} &= \frac{1}{D_{12}D_{13}D_{23}}\,Y(123)W(123)\,\frac{D_{13}+y_{21}w_{32}+y_{32}w_{21}}{x_1z_3} \\ &= \frac{1}{D_{12}D_{13}D_{23}\,z_1z_2z_3}\,X(123)Y(123)W(123)\times x_{21}x_{32}z_1z_2(D_{13}+y_{21}w_{32}+y_{32}w_{21}) \\ &= (\text{prefactors})\times(x_{21}x_{32}z_1z_2D_{13}+x_{21}x_{32}z_1z_2(y_{21}w_{32}+y_{32}w_{21})), \end{aligned} \tag{6.1}$$

where again we will drop the prefactors that simply encode its information of ordered subspaces as well as physical poles. The first term above denotes the seed diagram which pictorially is a horizontal ladder, the first diagram given in figure 2. According to the contact rules conceived in the introduction, since boxes $1,2$ have a horizontal contact and so do boxes $2,3$, we can trivially read off the factor $x_{21}x_{32}z_1z_2D_{13}$ from that ladder diagram. In fact, this factor originates from $x_{21}x_{32}z_{12}z_{23}D_{13}$ in the ordered subspace $Z(321)$, before we sum over all subspaces of z that admit it. As we have stated, $Y(123)W(123)$ forbids any vertical contact of boxes (or loops), so we only have a horizontal ladder for this subspace, while the rest terms are spurious. For later convenience, we can define

$$\begin{aligned} T_{1,\,Y(123)W(123)} &= (100000) + S_{1,\,Y(123)W(123)}, \\ S_{1,\,Y(123)W(123)} &\equiv x_{21}x_{32}z_1z_2(y_{21}w_{32}+y_{32}w_{21}), \end{aligned} \tag{6.2}$$

where the symbol (100000) denotes which ones are present for $W(123)$ among the six legal seed diagrams for $X(123)Y(123)$ in figure 2, as the latter cover two distinct topologies of various orientations at 3-loop. The numbers filled in the boxes above are uniquely determined by the orderings $X(123)Y(123)$, and if the ordering of w conflicts with that of y, the relevant diagram is excluded.

Analogously, for $W(132)$ we have

$$\begin{aligned} T_{2,\,Y(123)W(132)} &= x_{21}x_{32}z_1z_2D_{13} + x_{21}(x_{32}+x_{21})z_1^2y_{32}w_{23} + S_{2,\,Y(123)W(132)}, \\ &= (100100) + S_{2,\,Y(123)W(132)}, \\ S_{2,\,Y(123)W(132)} &= x_{21}z_1(x_{32}z_2 - y_{21}w_{23})y_{32}w_{31}, \end{aligned} \tag{6.3}$$

where (100100) denotes the first and fourth diagrams in figure 2 are present. Note that, $(x_{32}+x_{21})$ in the first line is nothing but (x_3-x_1), which means a horizontal contact between boxes $1,3$.

The results for the rest four orderings of w for $Y(123)$ are given by

$$\begin{aligned} T_{5,\,Y(123)W(213)} &= x_{21}x_{32}z_1z_2D_{13} + (x_{32}+x_{21})x_{32}z_1z_2y_{21}w_{12} + S_{5,\,Y(123)W(213)}, \\ &= (101000) + S_{5,\,Y(123)W(213)}, \\ S_{5,\,Y(123)W(213)} &= x_{21}x_{32}z_1z_2y_{21}w_{31} \end{aligned} \tag{6.4}$$

for $W(213)$, and

$$\begin{aligned} T_{7,\,Y(123)W(231)} &= x_{21}x_{32}z_1z_2D_{13} + (x_{32}+x_{21})x_{32}z_1z_2y_{21}(w_{13}+w_{32}) + x_{32}z_2y_{21}(y_{32}+y_{21})(w_{13}+w_{32})w_{13} \\ &\quad + S_{7,\,Y(123)W(231)}, \\ &= (101001) + S_{7,\,Y(123)W(231)}, \\ S_{7,\,Y(123)W(231)} &= -x_{21}x_{32}z_2z_3y_{21}w_{13} \end{aligned} \tag{6.5}$$

for $W(231)$, and

$$\begin{aligned} T_{4,\,Y(123)W(312)} &= x_{21}x_{32}z_1z_2D_{13} + x_{21}(x_{32}+x_{21})z_1^2y_{32}(w_{21}+w_{13}) + x_{21}z_1(y_{32}+y_{21})y_{32}w_{13}(w_{21}+w_{13}) \\ &\quad + S_{4,\,Y(123)W(312)}, \\ &= (100110) + S_{4,\,Y(123)W(312)}, \\ S_{4,\,Y(123)W(312)} &= -x_{21}x_{32}z_1z_3y_{32}w_{13} \end{aligned} \tag{6.6}$$

for $W(312)$, and

$$\begin{aligned} T_{8,\,Y(123)W(321)} &= x_{21}x_{32}z_1z_2D_{13} + y_{21}y_{32}w_{12}w_{23}D_{13} \\ &\quad + (x_{32}+x_{21})x_{32}z_1z_2y_{21}w_{12} + x_{21}(x_{32}+x_{21})z_1^2y_{32}w_{23} \\ &\quad + x_{21}z_1(y_{32}+y_{21})y_{32}(w_{12}+w_{23})w_{23} + x_{32}z_2y_{21}(y_{32}+y_{21})w_{12}(w_{12}+w_{23}) \\ &\quad + S_{8,\,Y(123)W(321)}, \\ &= (111111) + S_{8,\,Y(123)W(321)}, \\ S_{8,\,Y(123)W(321)} &= -x_{21}x_{32}z_3(z_2y_{21}w_{12} + z_1y_{32}w_{23}) + x_{21}z_3y_{21}y_{32}w_{12}w_{23} \end{aligned} \tag{6.7}$$

for $W(321)$. It is clear that for different orderings of w, although their positive variables are different, the factors corresponding to any contact between boxes are the same. For example, both $W(213)$ and $W(231)$ admit the third diagram in figure 2, so the relevant w factors are w_{12} and $(w_{13}+w_{32})$ respectively, both of which equal to (w_1-w_2). We also see that $Y(123)W(321)$ admits all six diagrams, since the orderings of y and w are completely opposite.

Let's sum the six spurious parts over subspaces of w for $Y(123)$, which gives

$$\begin{aligned} S_{Y(123)} &= \frac{1}{D_{12}D_{13}D_{23}\, z_1z_2z_3\, w_1w_2w_3} X(123)Y(123) \times x_{21}y_{21}(x_{32}z_2 - y_{32}w_2)(z_1w_3 - z_3w_1) \\ &= (\text{prefactors}) \times x_{21}y_{21}(x_{32}z_2 - y_{32}w_2)(z_1w_3 - z_3w_1), \end{aligned} \tag{6.8}$$

and as usual the prefactors are dropped. For the sum of each seed diagram over all subspaces that admit it, we will present examples of two distinct topologies below.

First, for the first diagram in figure 2, $x_{21}x_{32}z_1z_2D_{13}$ trivially remains the same after we sum it over subspaces of y and w, since the completeness relation gives

$$\sum_Y Y(\ldots)\sum_W W(\ldots) = \frac{1}{y_1y_2y_3\, w_1w_2w_3}, \tag{6.9}$$

then it becomes $x_2x_3z_1z_2D_{13}$ after we sum it over subspaces of x that admit it, since

$$\sum_{\text{admitting } X} x_{21}x_{32} = X(123)\, x_{21}x_{32} = \frac{1}{x_1} = \frac{1}{x_1x_2x_3}\, x_2x_3, \tag{6.10}$$

and this is the correct answer, as one of those in (4.6).

Then, for the third diagram in figure 2, $x_{31}x_{32}z_1z_2y_{21}w_{12}$ becomes $x_3^2z_1z_2y_2w_1$ since

$$\sum_{\text{admitting } Y}\ \sum_{\text{admitting } W} y_{21}w_{12} = \frac{1}{y_3w_3} Y(12)W(21)\, y_{21}w_{12} = \frac{1}{y_1y_2y_3\, w_1w_2w_3}\, y_2w_1, \tag{6.11}$$

as well as

$$\sum_{\text{admitting } X} x_{31}x_{32} = X(\sigma(12)\,3)\, x_{31}x_{32} = \frac{1}{x_1x_2x_3}\, x_3^2, \tag{6.12}$$

where

$$X(\sigma(12)\,3) = X(123) + X(213) = \frac{x_3}{x_1x_2(x_3-x_1)(x_3-x_2)}, \tag{6.13}$$

and this is another one in (4.6). The rest four diagrams of different orientations are similar.

We can continue the separation for the rest five orderings of y, each of which contains six orderings of w. Since we still work in $X(123)$, the general seed diagrams for different orderings of y are given in figure 3, where some boxes are kept blank as the ordering of x alone can only fix part of numbers filled in these boxes. Straightforwardly, for $Y(132)$ we have

$$\begin{aligned} T_{2,\,Y(132)W(123)} &= (100100) + S_{2,\,Y(132)W(123)}, \\ S_{2,\,Y(132)W(123)} &= x_{21}z_1(x_{32}z_2 - y_{23}w_{21})y_{31}w_{32} \end{aligned} \tag{6.14}$$

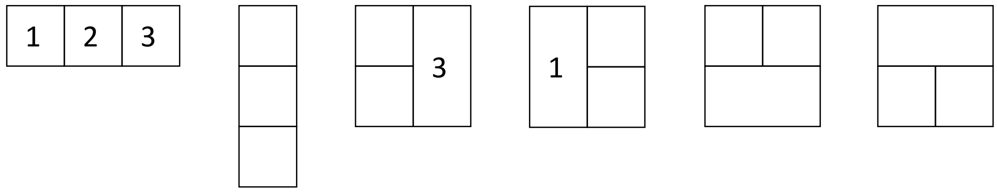

Figure 3: Legal seed diagrams of $X(123)$ of which some boxes are kept blank.

for $W(123)$, and

$$\begin{aligned} T_{1,\,Y(132)W(132)} &= (100000) + S_{1,\,Y(132)W(132)}, \\ S_{1,\,Y(132)W(132)} &= 0 \end{aligned} \tag{6.15}$$

for $W(132)$, and

$$\begin{aligned} T_{6,\,Y(132)W(213)} &= (101110) + S_{6,\,Y(132)W(213)}, \\ S_{6,\,Y(132)W(213)} &= x_{21}x_{32}z_1z_2y_{31}w_{31} \end{aligned} \tag{6.16}$$

for $W(213)$, and

$$\begin{aligned} T_{8,\,Y(132)W(231)} &= (111111) + S_{8,\,Y(132)W(231)}, \\ S_{8,\,Y(132)W(231)} &= -x_{21}z_2(x_{32}z_3 - y_{23}w_{32})y_{31}w_{13} \end{aligned} \tag{6.17}$$

for $W(231)$, and

$$\begin{aligned} T_{3,\,Y(132)W(312)} &= (100000) + S_{3,\,Y(132)W(312)}, \\ S_{3,\,Y(132)W(312)} &= 0 \end{aligned} \tag{6.18}$$

for $W(312)$, and

$$\begin{aligned} T_{7,\,Y(132)W(321)} &= (101001) + S_{7,\,Y(132)W(321)}, \\ S_{7,\,Y(132)W(321)} &= -x_{21}x_{32}z_2z_3y_{31}w_{12} \end{aligned} \tag{6.19}$$

for $W(321)$. The sum of six spurious parts over subspaces of w for $Y(132)$ is

$$S_{Y(132)} = x_{21}y_{31}(x_{32}z_2(z_1w_3 - z_3w_1) - y_{23}w_3(z_1w_2 - z_2w_1)). \tag{6.20}$$

Then for $Y(213)$, we have

$$\begin{aligned} T_{5,\,Y(213)W(123)} &= (101000) + S_{5,\,Y(213)W(123)}, \\ S_{5,\,Y(213)W(123)} &= x_{21}x_{32}z_1z_2y_{31}w_{21} \end{aligned} \tag{6.21}$$

for $W(123)$, and

$$\begin{aligned} T_{6,\,Y(213)W(132)} &= (101101) + S_{6,\,Y(213)W(132)}, \\ S_{6,\,Y(213)W(132)} &= x_{21}x_{32}z_1z_2y_{31}w_{31} \end{aligned} \tag{6.22}$$

for $W(132)$, and

$$\begin{aligned} T_{1,\,Y(213)W(213)} &= (100000) + S_{1,\,Y(213)W(213)}, \\ S_{1,\,Y(213)W(213)} &= 0 \end{aligned} \tag{6.23}$$

for $W(213)$, and

$$\begin{aligned} T_{3,\,Y(213)W(231)} &= (100000) + S_{3,\,Y(213)W(231)}, \\ S_{3,\,Y(213)W(231)} &= 0 \end{aligned} \tag{6.24}$$

for $W(231)$, and

$$\begin{aligned} T_{8,\,Y(213)W(312)} &= (111111) + S_{8,\,Y(213)W(312)}, \\ S_{8,\,Y(213)W(312)} &= -x_{21}x_{32}z_1z_3y_{31}w_{13} \end{aligned} \tag{6.25}$$

for $W(312)$, and

$$\begin{aligned} T_{4,\,Y(213)W(321)} &= (100110) + S_{4,\,Y(213)W(321)}, \\ S_{4,\,Y(213)W(321)} &= -x_{21}x_{32}z_1z_3y_{31}w_{23} \end{aligned} \tag{6.26}$$

for $W(321)$. The sum of six spurious parts over subspaces of w for $Y(213)$ is

$$S_{Y(213)} = 0. \tag{6.27}$$

Then for $Y(231)$, we have

$$\begin{aligned} T_{7,\,Y(231)W(123)} &= (101010) + S_{7,\,Y(231)W(123)}, \\ S_{7,\,Y(231)W(123)} &= -x_{21}x_{32}z_2z_3y_{13}w_{21} \end{aligned} \tag{6.28}$$

for $W(123)$, and

$$\begin{aligned} T_{8,\,Y(231)W(132)} &= (111111) + S_{8,\,Y(231)W(132)}, \\ S_{8,\,Y(231)W(132)} &= -x_{21}z_2(x_{32}z_3 - y_{32}w_{23})y_{13}w_{31} \end{aligned} \tag{6.29}$$

for $W(132)$, and

$$\begin{aligned} T_{3,\,Y(231)W(213)} &= (100000) + S_{3,\,Y(231)W(213)}, \\ S_{3,\,Y(231)W(213)} &= 0 \end{aligned} \tag{6.30}$$

for $W(213)$, and

$$\begin{aligned} T_{1,\,Y(231)W(231)} &= (100000) + S_{1,\,Y(231)W(231)}, \\ S_{1,\,Y(231)W(231)} &= 0 \end{aligned} \tag{6.31}$$

for $W(231)$, and

$$\begin{aligned} T_{6,\,Y(231)W(312)} &= (101101) + S_{6,\,Y(231)W(312)}, \\ S_{6,\,Y(231)W(312)} &= x_{21}x_{32}z_1z_2y_{13}w_{13} \end{aligned} \tag{6.32}$$

for $W(312)$, and

$$\begin{aligned} T_{2,\,Y(231)W(321)} &= (100100) + S_{2,\,Y(231)W(321)}, \\ S_{2,\,Y(231)W(321)} &= x_{21}z_1(x_{32}z_2 - y_{32}w_{12})y_{13}w_{23} \end{aligned} \tag{6.33}$$

for $W(321)$. The sum of six spurious parts over subspaces of w for $Y(231)$ is

$$S_{Y(231)} = -x_{21}w_2y_{13}y_{32}(z_1w_1 - z_2w_3). \tag{6.34}$$

Then for $Y(312)$, we have

$$\begin{aligned} T_{4,\,Y(312)W(123)} &= (100101) + S_{4,\,Y(312)W(123)}, \\ S_{4,\,Y(312)W(123)} &= -x_{21}x_{32}z_1z_3y_{13}w_{32} \end{aligned} \tag{6.35}$$

for $W(123)$, and

$$\begin{aligned} T_{3,\,Y(312)W(132)} &= (100000) + S_{3,\,Y(312)W(132)}, \\ S_{3,\,Y(312)W(132)} &= 0 \end{aligned} \tag{6.36}$$

for $W(132)$, and

$$\begin{aligned} T_{8,\,Y(312)W(213)} &= (111111) + S_{8,\,Y(312)W(213)}, \\ S_{8,\,Y(312)W(213)} &= -x_{21}x_{32}z_1z_3y_{13}w_{31} \end{aligned} \tag{6.37}$$

for $W(213)$, and

$$\begin{aligned} T_{6,\,Y(312)W(231)} &= (101110) + S_{6,\,Y(312)W(231)}, \\ S_{6,\,Y(312)W(231)} &= x_{21}x_{32}z_1z_2y_{13}w_{13} \end{aligned} \tag{6.38}$$

for $W(231)$, and

$$\begin{aligned} T_{1,\,Y(312)W(312)} &= (100000) + S_{1,\,Y(312)W(312)}, \\ S_{1,\,Y(312)W(312)} &= 0 \end{aligned} \tag{6.39}$$

for $W(312)$, and

$$\begin{aligned} T_{5,\,Y(312)W(321)} &= (101000) + S_{5,\,Y(312)W(321)}, \\ S_{5,\,Y(312)W(321)} &= x_{21}x_{32}z_1z_2y_{13}w_{12} \end{aligned} \tag{6.40}$$

for $W(321)$. The sum of six spurious parts over subspaces of w for $Y(312)$ is

$$S_{Y(312)} = x_{21}x_{32}z_1y_{13}(z_2w_1 - z_3w_3). \tag{6.41}$$

Finally for $Y(321)$, we have

$$\begin{aligned} T_{8,\,Y(321)W(123)} &= (111111) + S_{8,\,Y(321)W(123)}, \\ S_{8,\,Y(321)W(123)} &= -x_{21}x_{32}z_3(z_2y_{12}w_{21} + z_1y_{23}w_{32}) + x_{21}z_3y_{12}y_{23}w_{21}w_{32} \end{aligned} \tag{6.42}$$

for $W(123)$, and

$$\begin{aligned} T_{7,\,Y(321)W(132)} &= (101010) + S_{7,\,Y(321)W(132)}, \\ S_{7,\,Y(321)W(132)} &= -x_{21}x_{32}z_2z_3y_{12}w_{31} \end{aligned} \tag{6.43}$$

for $W(132)$, and

$$\begin{aligned} T_{4,\,Y(321)W(213)} &= (100101) + S_{4,\,Y(321)W(213)}, \\ S_{4,\,Y(321)W(213)} &= -x_{21}x_{32}z_1z_3y_{23}w_{31} \end{aligned} \tag{6.44}$$

for $W(213)$, and

$$\begin{aligned} T_{2,\,Y(321)W(231)} &= (100100) + S_{2,\,Y(321)W(231)}, \\ S_{2,\,Y(321)W(231)} &= x_{21}z_1(x_{32}z_2 - y_{12}w_{32})y_{23}w_{13} \end{aligned} \tag{6.45}$$

for $W(231)$, and

$$\begin{aligned} T_{5,\,Y(321)W(312)} &= (101000) + S_{5,\,Y(321)W(312)}, \\ S_{5,\,Y(321)W(312)} &= x_{21}x_{32}z_1z_2y_{12}w_{13} \end{aligned} \tag{6.46}$$

for $W(312)$, and

$$\begin{aligned} T_{1,\,Y(321)W(321)} &= (100000) + S_{1,\,Y(321)W(321)}, \\ S_{1,\,Y(321)W(321)} &= x_{21}x_{32}z_1z_2(y_{12}w_{23} + y_{23}w_{12}) \end{aligned} \tag{6.47}$$

for $W(321)$. The sum of six spurious parts over subspaces of w for $Y(321)$ is

$$S_{Y(321)} = x_{21}y_{23}(x_{32}z_1(z_2w_1 - z_3w_3) - y_{12}w_3(z_1w_1 - z_3w_2)). \tag{6.48}$$

Collecting the six spurious sums for all ordered subspaces of y, namely (6.8), (6.20), (6.27), (6.34), (6.41) and (6.48), we can further sum them over y-space as

$$S_{123} = x_{21}(-2z_1y_2y_3w_2w_3 - z_1y_1w_1(y_2w_3 + y_3w_2) + z_2y_3w_3(y_1w_2 + y_2w_1) + z_3y_2w_2(y_1w_3 + y_3w_1)), \tag{6.49}$$

where subscript 123 denotes this sum belongs to the sector of $X(123)$. In order to obtain the full result, which is permutation invariant of loop numbers, we calculate the final sum:

$$S_{123}X(123) + (5 \text{ permutations of } 1,2,3) = 0, \tag{6.50}$$

which nicely vanishes as expected. Therefore, the contributing parts indeed form the correct answer (4.6) which includes the six Mondrian diagrams in figure 2 and their permutations. Note the hidden prefactors are nothing but the reciprocal of 'Denominator' in (4.8).

7. Summary: a Mondrian Preamble

By separating the contributing and the spurious parts of each form in all ordered subspaces and assigning the former with corresponding Mondrian factors, which follow simple rules given by

$$\begin{aligned} \text{horizontal contact: } & (x_j - x_i)(z_i - z_j) \\ \text{vertical contact: } & (y_j - y_i)(w_i - w_j) \\ \text{no contact: } & D_{ij} = (x_j - x_i)(z_i - z_j) + (y_j - y_i)(w_i - w_j) \end{aligned} \tag{7.1}$$

between any two loops labelled by i, j, we obtain the seed diagrams. If we assume the spurious terms will always sum to zero at the end, there is no need to sum the seed diagrams over all ordered subspaces since they are already topologically valuable. There is a simple way to find seed diagrams: let's work in simply one ordered subspace $X(12)Z(21)Y(12)W(21)$ at 2-loop, as the first nontrivial example. Then, it is clear that D_{12} is trivially positive, so there is no positivity condition to be imposed. But as a physical pole D_{12} must appear in the denominator, which identically turns the form into

$$\frac{1}{x_1 x_2\, z_1 z_2\, y_1 y_2\, w_1 w_2} \frac{D_{12}}{D_{12}} = \frac{1}{x_1 x_2\, z_1 z_2\, y_1 y_2\, w_1 w_2} \frac{(x_2 - x_1)(z_1 - z_2) + (y_2 - y_1)(w_1 - w_2)}{D_{12}}. \tag{7.2}$$

As usual, dropping the prefactors which contain all physical poles, we precisely obtain two 2-loop ladders of horizontal and vertical orientations (the vertical one is shown in figure 1).

The 3-loop example is more interesting. Similarly in ordered subspace $X(123)Z(321)Y(123)W(321)$, we can separate the triple product as

$$\begin{aligned} D_{12}D_{13}D_{23} = & x_{21}z_{12} \cdot x_{32}z_{23} \cdot D_{13} + y_{21}w_{12} \cdot y_{32}w_{23} \cdot D_{13} \\ & + x_{31}z_{13} \cdot x_{32}z_{23} \cdot y_{21}w_{12} + x_{21}z_{12} \cdot x_{31}z_{13} \cdot y_{32}w_{23} \\ & + x_{21}z_{12} \cdot y_{31}w_{13} \cdot y_{32}w_{23} + y_{21}w_{12} \cdot y_{31}w_{13} \cdot x_{32}z_{23}, \end{aligned} \tag{7.3}$$

which precisely correspond to the six diagrams in figure 2 (including two ladders and four tennis courts). Here, for notational compactness we have defined $x_{31} \equiv x_{32} + x_{21}$ for instance, as x_{32} and x_{21} are primitive positive variables in this subspace while x_{31} is not.

In general, Mondrian diagrams of higher loop levels satisfy this neat pattern: the product of all D_{ij}'s can be expanded as a sum of all topologies of all orientations, in an ordered subspace where the orderings of x, z are opposite, and so are those of y, w. However, there are more subtle issues to be clarified, and we will discuss them more systematically in the subsequent work.

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A. The Master Form at 4-loop

To obtain the master form T_{64} (its subscript is due to $2^6=64$) at 4-loop by dividing the z-space, we define

$$\begin{aligned} &\eta_{12} \equiv z_1 - z_2 + c_{12} > 0, \quad \eta_{13} \equiv z_1 - z_3 + c_{13} > 0, \quad \eta_{23} \equiv z_2 - z_3 + c_{23} > 0, \\ &\eta_{14} \equiv z_1 - z_4 + c_{14} > 0, \quad \eta_{24} \equiv z_2 - z_4 + c_{24} > 0, \quad \eta_{34} \equiv z_3 - z_4 + c_{34} > 0, \end{aligned} \tag{A.1}$$

the sum is then

$$\begin{aligned} T_{64} \equiv\ & Z^-_{14} \cap Z^-_{24} \cap Z^-_{34} \cap Z^-_{13} \cap Z^-_{23} \cap Z^-_{12} \\ =\ & \frac{1}{c_{12}} \Bigg[\frac{1}{c_{13}c_{23}} \left(\frac{1}{c_{14}c_{24}c_{34}} Z(4321) + \frac{1}{c_{14}c_{24}\eta_{34}} Z(3421) + \frac{1}{c_{14}\eta_{24}\eta_{34}} Z(3241) + \frac{1}{\eta_{14}\eta_{24}\eta_{34}} Z(3214) \right) \\ & + \frac{1}{c_{13}\eta_{23}} \left(\frac{1}{c_{14}c_{24}c_{34}} Z(4231) + \frac{1}{c_{14}\eta_{24}c_{34}} Z(2431) + \frac{1}{c_{14}\eta_{24}\eta_{34}} Z(2341) + \frac{1}{\eta_{14}\eta_{24}\eta_{34}} Z(2314) \right) \\ & + \frac{1}{\eta_{13}\eta_{23}} \left(\frac{1}{c_{14}c_{24}c_{34}} Z(4213) + \frac{1}{c_{14}\eta_{24}c_{34}} Z(2413) + \frac{1}{\eta_{14}\eta_{24}c_{34}} Z(2143) + \frac{1}{\eta_{14}\eta_{24}\eta_{34}} Z(2134) \right) \Bigg] \\ & + \frac{1}{\eta_{12}} \Bigg[\frac{1}{c_{13}c_{23}} \left(\frac{1}{c_{14}c_{24}c_{34}} Z(4312) + \frac{1}{c_{14}c_{24}\eta_{34}} Z(3412) + \frac{1}{\eta_{14}c_{24}\eta_{34}} Z(3142) + \frac{1}{\eta_{14}\eta_{24}\eta_{34}} Z(3124) \right) \\ & + \frac{1}{\eta_{13}c_{23}} \left(\frac{1}{c_{14}c_{24}c_{34}} Z(4132) + \frac{1}{\eta_{14}c_{24}c_{34}} Z(1432) + \frac{1}{\eta_{14}c_{24}\eta_{34}} Z(1342) + \frac{1}{\eta_{14}\eta_{24}\eta_{34}} Z(1324) \right) \\ & + \frac{1}{\eta_{13}\eta_{23}} \left(\frac{1}{c_{14}c_{24}c_{34}} Z(4123) + \frac{1}{\eta_{14}c_{24}c_{34}} Z(1423) + \frac{1}{\eta_{14}\eta_{24}c_{34}} Z(1243) + \frac{1}{\eta_{14}\eta_{24}\eta_{34}} Z(1234) \right) \Bigg] \\ =\ & \frac{1}{c_{12}c_{13}c_{14}c_{23}c_{24}c_{34}} \frac{N}{z_1z_2z_3z_4\, \eta_{12}\eta_{13}\eta_{14}\eta_{23}\eta_{24}\eta_{34}}, \end{aligned} \tag{A.2}$$

where

$$\begin{aligned} N = &(z_1+c_{12})(z_1+c_{13})(z_1+c_{14})(z_2+c_{23})(z_2+c_{24})(z_3+c_{34}) \\ & - z_1z_2z_3(z_1+c_{14})(z_2+c_{24})(z_3+c_{34}) - z_1z_2z_4(z_1+c_{13})(z_2+c_{23})(z_3+c_{34}) \\ & - z_1z_3z_4(z_1+c_{12})(z_2+c_{23})(z_2+c_{24}) - z_2z_3z_4(z_1+c_{12})(z_1+c_{13})(z_1+c_{14}) \\ & + z_1z_2z_3z_4 \big(-(z_1+c_{12})(z_3+c_{34}) - (z_1+c_{13})(z_2+c_{24}) - (z_1+c_{14})(z_2+c_{23}) \\ & \qquad + (z_1+c_{12})(z_3+z_4) + (z_1+c_{13})(z_2+z_4) + (z_1+c_{14})(z_2+z_3) \\ & \qquad + (z_2+c_{23})(z_1+z_4) + (z_2+c_{24})(z_1+z_3) + (z_3+c_{34})(z_1+z_2) \\ & \qquad - z_1z_2 - z_1z_3 - z_1z_4 - z_2z_3 - z_2z_4 - z_3z_4 \big). \end{aligned} \tag{A.3}$$

Now let's determine $T_1 \equiv Z^+_{41} \cap Z^+_{42} \cap Z^+_{43} \cap Z^+_{31} \cap Z^+_{32} \cap Z^+_{21}$, for instance, by flipping all c_{ij}'s to $-c_{ji}$'s in the denominator and setting all c_{ij}'s to zero in the numerator, which gives

$$T_1 = \frac{1}{c_{12}c_{13}c_{14}c_{23}c_{24}c_{34}} \frac{(z_1-z_3)(z_1-z_4)(z_2-z_4)}{z_4\, \zeta_{12}\zeta_{13}\zeta_{14}\zeta_{23}\zeta_{24}\zeta_{34}}, \tag{A.4}$$

where we have similarly defined

$$\begin{aligned}&\zeta_{12}\equiv z_1-z_2-c_{21}>0,\quad \zeta_{13}\equiv z_1-z_3-c_{31}>0,\quad \zeta_{23}\equiv z_2-z_3-c_{32}>0,\\&\zeta_{14}\equiv z_1-z_4-c_{41}>0,\quad \zeta_{24}\equiv z_2-z_4-c_{42}>0,\quad \zeta_{34}\equiv z_3-z_4-c_{43}>0.\end{aligned} \tag{A.5}$$

To confirm this is indeed the correct answer, we can separate it into two parts as

$$T_1=\left(Z^+_{41}\cap Z^+_{42}\cap Z^+_{31}\right)\cap\left(Z^+_{43}\cap Z^+_{32}\cap Z^+_{21}\right)=\frac{(z_1-z_4)(z_2-z_4)(z_1-z_3)}{c_{14}c_{24}c_{13}\,\zeta_{14}\zeta_{24}\zeta_{13}}\times\frac{1}{c_{34}c_{23}c_{12}\,z_4\,\zeta_{34}\zeta_{23}\zeta_{12}}, \tag{A.6}$$

where the second part can be trivially obtained if we treat $\zeta_{34},\zeta_{23},\zeta_{12}$ as genuinely positive variables. For the first part, in terms of $\zeta_{34},\zeta_{23},\zeta_{12}$ we have

$$\zeta_{13}=\zeta_{12}+\zeta_{23}+c_{21}+c_{32}-c_{31}>0\Longrightarrow c_{31}<\zeta_{12}+\zeta_{23}+c_{21}+c_{32}, \tag{A.7}$$

and the corresponding form is

$$\frac{1}{c_{31}}-\frac{1}{c_{31}-(\zeta_{12}+\zeta_{23}+c_{21}+c_{32})}=\frac{z_1-z_3}{c_{13}\,\zeta_{13}}. \tag{A.8}$$

Analogously we have

$$\zeta_{24}=\zeta_{23}+\zeta_{34}+c_{32}+c_{43}-c_{42}>0\Longrightarrow\frac{z_2-z_4}{c_{24}\,\zeta_{24}}, \tag{A.9}$$

as well as

$$\zeta_{14}=\zeta_{12}+\zeta_{23}+\zeta_{34}+c_{21}+c_{32}+c_{43}-c_{41}>0\Longrightarrow\frac{z_1-z_4}{c_{14}\,\zeta_{14}}, \tag{A.10}$$

therefore, we have neatly confirmed expression (A.4) of T_1.

Another check of the master form is the completeness relation

$$Z^-_{14}\cap Z^-_{24}\cap Z^-_{34}\cap Z^-_{13}\cap Z^-_{23}\cap Z^-_{12}+Z^-_{14}\cap Z^-_{24}\cap Z^+_{34}\cap Z^-_{13}\cap Z^-_{23}\cap Z^-_{12}=Z^-_{14}\cap Z^-_{24}\cap Z^-_{13}\cap Z^-_{23}\cap Z^-_{12}, \tag{A.11}$$

for instance, of which the essential part is $Z^-_{34}+Z^+_{34}=I_{34}$. To prove this, we can focus on the numerators while omitting their common denominator, so that the relation becomes

$$N-N(c_{34}=0,3\leftrightarrow4)=\left(\frac{N}{\eta_{34}}\right)_{c_{34}\to\infty}\eta_{34}, \tag{A.12}$$

where N is the numerator in (A.3). Let's immediately give some explanation of the quantities above. For the RHS, unlike the 3-loop case (3.21), it is much more nontrivial to fix this quintuple co-positive product. Hence we use another way to circumvent it, which is extremely simple: we set $c_{34}\to\infty$ and then evaluate its residue, since this trivializes $z_3-z_4+c_{34}>0$, after that we need a compensating factor η_{34} as terms of the LHS are purely numerators. For the second term of the LHS, the operation $(c_{34}=0,3\leftrightarrow4)$ is easy to understand, while the minus sign comes from the operation $(c_{34}\to-c_{43},3\leftrightarrow4)$ of the denominator

$$\frac{1}{z_3+c_{34}-z_4}\to\frac{1}{z_3-c_{43}-z_4}\to\frac{1}{z_4-c_{34}-z_3}=-\frac{1}{z_3+c_{34}-z_4}, \tag{A.13}$$

as this term demands $z_4-z_3-c_{34}>0$, but to have a common denominator produces an minus sign.

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