Can a pure vector gravitational wave mimic a pure tensor one?

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In the general theory of relativity, gravitational waves have two possible polarizations, which are transverse and traceless with helicity ± 2. Some alternative theories contain additional helicity 0 and helicity ± 1 polarization modes. Here, we consider a hypothetical "pure vector" theory in which gravitational waves have only two possible polarizations, with helicity ± 1. We show that if these polarizations are allowed to rotate as the wave propagates, then for certain source locations on the sky, the strain outputs of three ideal interferometric gravitational wave detectors can exactly reproduce the strain outputs predicted by general relativity.

I. INTRODUCTION

In Einstein's general theory of relativity (GR), gravitational waves (GW) are radiative perturbations in the geometry of space-time that propagate at the speed of light [1,2]. These so-called "tensor" modes are spin-2 transverse traceless metric perturbations, which have only helicity ± 2 degrees of freedom [3]. In alternatives to GR, GWs can contain additional helicity ± 1 and helicity 0 degrees of freedom, which are often called "vector" and "scalar" modes [4–6].

Interferometric GW detectors such as the Laser Interferometer Gravitational-wave Observatory (LIGO) [7] and Virgo [8] would respond differently to vector and scalar modes than they respond to tensor modes. When more than two GW detectors [9] observe the same source, this makes it possible, at least in principle, to measure the wave polarization content and determine which helicity components are present [10–16].

On 14 August 2017, the two LIGO detectors in the USA and the Virgo detector in Italy made the first simultaneous 3-detector direct observation of GWs [17]; the signal was present in the band of the detectors for a fraction of a second. The waves were emitted by the merging binary black hole system GW170814, and excited the three detectors in a pattern that was consistent with the "pure tensor mode" predictions of GR. The discovery paper [17] also claimed that the observation strongly disfavored a pure vector model, which would not match the excitation pattern observed in the detectors.

The presentation in [17] suggests that this is a theory-independent observational result. Here we show that this is

not the case, by explicitly constructing a pure vector gravitational wave which for certain source locations on the sky can precisely mimic or match the predictions of GR. We assume that the signal has short duration compared to the 24h rotation period of the Earth, so that during the observation the detector orientations relative to the line of sight to the source do not change significantly.

To construct this counterexample, we allow the "direction" of the wave polarization to change as the GW propagates, in the same way that the Faraday effect can rotate the polarization of an electromagnetic wave [18–21]. With an appropriate amount of rotation, the GW remains pure vector but exactly mimics the detector response expected for tensor modes in GR.

The paper is organized as follows. In Sec. II we discuss the "pure tensor" weak-field plane wave solution of GR. In Sec. III we show how an array of three interferometric detectors would respond to these waves. In Sec. IV we then consider the same questions for a conventional pure vector theory, and discuss how null streams [11] can distinguish these from conventional GR. In Sec. V we construct an orthonormal basis for polarization tensors which rotates about the propagation axis. In Sec. VI we use the rotating basis to construct a pure vector gravitational wave which changes its polarization as it propagates (behavior that does not happen in GR). By correctly tuning the amount of rotation, we show that for some sky positions, this pure vector theory would produce exactly the same detector outputs as GR. This is followed by a short conclusion, and an Appendix giving quantities needed to reproduce the results, such as the antenna response tensors for the LIGO and Virgo detectors.

II. PLANE WAVES IN GENERAL RELATIVITY

In GR, a GW propagating in the positive z-direction may be written in the weak field limit as

$$h_{ab} = w_+(t - z/c)e_{ab}^+ + w_\times(t - z/c)e_{ab}^\times. \qquad (1)$$

Here c is the speed of light and w_+ and $w_\times$ are arbitrary real functions of one variable, which are the waveforms of the two different polarization modes. The function $t - z/c$ is called "retarded time" [22]. The transverse traceless polarization tensors are

$$
\begin{aligned}
e_{ab}^+ &= x_a x_b - y_a y_b \\
e_{ab}^\times &= x_a y_b + y_a x_b
\end{aligned}
\qquad (2)
$$

where x^a and y^a are unit vectors [23] in the x and y directions. The complex combinations $e_{ab}^+ \pm i e_{ab}^\times$ have helicity ± 2, because under rotation of the orthogonal pair x^a, y^a through angle ϕ in the plane orthogonal to the propagation direction z, the complex polarizations acquire phase factors $\exp(\pm 2i\phi)$ [24].

The strain response $h(t)$ of an ideal interferometric gravitational wave detector, with perpendicular arms of equal length is

$$h(t) = d^{ab} h_{ab}(t, z_0), \qquad (3)$$

where z_0 is the z coordinate of the detector's location, and

$$d^{ab} = L_1^a L_1^b - L_2^a L_2^b \qquad (4)$$

is the detector response tensor. The quantities L_1^a and L_2^a are unit vectors along the two detector arms, and the detector is assumed to be ideal. This means that it is free of instrumental noise, with a flat frequency response that is broad enough to reproduce the spectral content of the waveforms w^+ and $w^\times$.

III. THE CASE OF THREE DETECTORS

Now suppose that we have three ideal detectors, at locations with z coordinate z_i and with detector response tensors d_i^{ab}, for $i = 1, ..., 3$. We assume that the GW source is at a distance from the detectors which is much larger than their size, than the separation between them, and than the characteristic wavelength of the radiation. We assume that the source's sky location is known precisely, and (without loss of generality) that it lies on the negative z axis. We also assume that the signal is visible in the detectors for a time that is brief compared to the Earth's spin period of 24h, so that the instrument orientations may be treated as fixed.

In this case, the strain at the three detectors is given by combining (1) and (3):

$$h_i(t) = F_i^+ w_+(t - z_i/c) + F_i^\times w_\times(t - z_i/c). \qquad (5)$$

The strain is seen to be a linear combination of the two different waveforms, with weights F_i^+ and $F_i^\times$ that are determined by the geometrical overlap between the detector

response tensors and the polarization tensors. These weights,

$$F_i^+ = e_{ab}^+ d_i^{ab} \quad \text{and} \quad F_i^\times = e_{ab}^\times d_i^{ab}, \qquad (6)$$

are called "antenna functions" and are easily computed. Here, they should be thought of as dimensionless numbers, whose fixed values are determined *a priori* by the orientations of the detector arms [25] relative to the GW source. (For completeness, the Appendix gives expressions for d_i^{ab} for the LIGO and Virgo detectors.)

For later convenience, we introduce the non-negative lengths F_i of these three two-vectors

$$F_i = \sqrt{(F_i^+)^2 + (F_i^\times)^2} \qquad (7)$$

and the angles defined by the ratios of the two components,

$$\phi_i = \arctan \frac{F_i^\times}{F_i^+}. \qquad (8)$$

Here, the arc-tangent is defined as the principal value of $\arg(F^+ + iF^\times)$. In terms of these lengths and angles, we can write the two-vectors $(F_i^+, F_i^\times) = F_i(\cos\phi_i, \sin\phi_i)$. As before, F_i and ϕ_i should be thought of as constants, which are defined by the detector arm orientations relative to the direction to the source.

To compare the strain outputs of the different detectors, it is helpful to first shift the observed strains to a common fiducial arrival time, defining new time series by $\bar{h}_i(t) = h_i(t + z_i/c)$. In GR these time-shifted strains may be written as linear combinations of the two waveforms, with weights determined by the antenna patterns of the three instruments,

$$
\begin{bmatrix} \bar{h}_1(t) \\ \bar{h}_2(t) \\ \bar{h}_3(t) \end{bmatrix}
=
\begin{matrix} F_1 \\ F_2 \\ F_3 \end{matrix}
\begin{bmatrix} \cos\phi_1 & \sin\phi_1 \\ \cos\phi_2 & \sin\phi_2 \\ \cos\phi_3 & \sin\phi_3 \end{bmatrix}
\begin{bmatrix} w_+(t) \\ w_\times(t) \end{bmatrix},
\qquad (9)
$$

where the respective values of F_i are understood to multiply the corresponding row on the right-hand side (r.h.s.).

For generic detector orientations, the three two-vectors $(F_i^+, F_i^\times)$ are nonvanishing. Hence they must be linearly dependent, since a two-dimensional vector space can contain at most two linearly independent vectors. Thus, the three time-shifted strains must also be linearly dependent, with time-independent coefficients, since they are linear combinations of two independent waveform functions $w_+(t)$ and $w_\times(t)$.

This is the basis of "null tests", which construct a linear combination of these three time-shifted strains $\bar{h}_1(t)$, $\bar{h}_2(t), \bar{h}_3(t)$ which vanishes if GR is a correct description of GW [11]. This vanishing linear combination is called a "null stream." By the same reasoning, if there are N

generically-oriented detectors, then $N - 2$ "different" null stream combinations may be constructed from them [11].

IV. THE THREE-DETECTOR CASE IN "PURE VECTOR GRAVITY"

Now consider a fictitious pure vector theory of gravity, where the transverse traceless polarization tensors of GR are replaced by a pair which for convenience we denote with L and R:

$$e_{ab}^L = x_a z_b + z_a x_b$$
$$e_{ab}^R = y_a z_b + z_a y_b \qquad (10)$$

where z^a is a unit vector in the positive z direction. In this pure vector case, the GW is described by

$$h_{ab} = w_L(t - z/c)e_{ab}^L + w_R(t - z/c)e_{ab}^R, \qquad (11)$$

where, as before, the waveforms w_L and w_R are arbitrary real functions of one variable.

We assume that in this fictitious theory of gravity, the detectors respond as before (3) so that the strain in the ith detector is

$$h_i(t) = F_i^L w_L(t - z_i/c) + F_i^R w_R(t - z_i/c), \qquad (12)$$

where the weights F that appear are again the overlaps between the detector response tensors and the polarization tensors:

$$F_i^L = e_{ab}^L d_i^{ab} \quad \text{and} \quad F_i^R = e_{ab}^R d_i^{ab}. \qquad (13)$$

As before, after shifting to a common fiducial time, the three time series $\bar{h}_i(t) = h_i(t + z_i/c)$ must be linearly dependent, because they are linear combinations of the two functions $w_L(t)$ and $w_R(t)$.

As in the previous section, we introduce the non-negative lengths f_i of these three two-vectors

$$f_i = \sqrt{(F_i^L)^2 + (F_i^R)^2} \qquad (14)$$

and the angles defined by the ratios of the two components,

$$\psi_i = \arctan \frac{F_i^R}{F_i^L}, \qquad (15)$$

where the arc-tangent is defined as the principal value of $\arg(F^L + iF^R)$. In terms of these quantities, we can express the two-vectors as $(F_i^L, F_i^R) = f_i(\cos\psi_i, \sin\psi_i)$, where again f_i and ψ_i are constants defined by the detector arm orientations relative to the direction to the source.

The time-shifted strains resulting from this pure vector theory are then

$$\begin{bmatrix} \bar{h}_1(t) \\ \bar{h}_2(t) \\ \bar{h}_3(t) \end{bmatrix} = \begin{matrix} f_1 \\ f_2 \\ f_3 \end{matrix} \begin{bmatrix} \cos\psi_1 & \sin\psi_1 \\ \cos\psi_2 & \sin\psi_2 \\ \cos\psi_3 & \sin\psi_3 \end{bmatrix} \begin{bmatrix} w_L(t) \\ w_R(t) \end{bmatrix}, \qquad (16)$$

where the respective values of f_i are understood to multiply the ith row on the r.h.s. This should be compared with the corresponding Eq. (9) that arises in GR.

For generic detector orientations, the first two rows of this system are invertible, so the data at the first two detectors completely determines $w_L(t)$ and $w_R(t)$. With that choice, $\bar{h}_1(t)$ and $\bar{h}_2(t)$ will match the data perfectly. But if GR is correct, then $\bar{h}_3(t)$ in (16) will be incorrect: it will not match the data from the detector.

V. A ROTATING BASIS FOR POLARIZATION TENSORS

At this juncture, it is helpful to introduce a "rotating" vector polarization basis [26] analogous to (10). For this, we first define a basis that rotates with space and time as

$$x'^a = \cos\phi(t, z)x^a + \sin\phi(t, z)y^a$$
$$y'^a = -\sin\phi(t, z)x^a + \cos\phi(t, z)y^a$$
$$z'^a = z^a \qquad (17)$$

where the phase ϕ is an arbitrary real function of time t and spatial position z. The new basis vectors (x'^a, y'^a) are obtained by rotating the old basis vectors (x^a, y^a) counterclockwise through angle ϕ.

Using this basis, we define new polarization tensors by

$$e_{ab}^l = x'_a z'_b + z'_a x'_b = \cos\phi e_{ab}^L + \sin\phi e_{ab}^R$$
$$e_{ab}^r = y'_a z'_b + z'_a y'_b = -\sin\phi e_{ab}^L + \cos\phi e_{ab}^R \qquad (18)$$

and express the gravitational wave amplitude as a sum over these two polarizations:

$$h_{ab}(t, z) = w_l(t - z/c)e_{ab}^l + w_r(t - z/c)e_{ab}^r. \qquad (19)$$

As before, the waveform functions w_l and w_r are arbitrary real functions of one variable. Note however that the left-hand side is a function of retarded time $t - z/c$ if and only if the phase ϕ is *also* a function of retarded time.

We stress that (17) is *not* defining a new basis for the tensor space over our space-time manifold. If it were, then scalar quantities such as the strain h at a particular detector would not be changed. This is because the components of the field h_{ab} would transform covariantly, but the components of the detector tensors d^{ab} would transform contravariantly, and the scalar product $h_{ab}d^{ab}$ would be invariant. Here, we are explicitly *changing* the nature and physical content of the GW in ways that we will discuss shortly.

By using (11), (13), (18), and (19), we can express the time-shifted detector strains as the 2-dimensional vector inner products

$$\bar{h}_i(t) = [F_i^L F_i^R] \begin{bmatrix} \cos\phi(t+z_i/c,z_i) & -\sin\phi(t+z_i/c,z_i) \\ \sin\phi(t+z_i/c,z_i) & \cos\phi(t+z_i/c,z_i) \end{bmatrix}$$
$$\times \begin{bmatrix} w_l(t) \\ w_r(t) \end{bmatrix}. \tag{20}$$

If the phase ϕ only depends upon (a single variable, which is) the retarded time $t - z/c$, then this is completely equivalent to the case considered in the previous section with waveform functions $w_L(t)$ and $w_R(t)$ defined by

$$\begin{bmatrix} w_L(t) \\ w_R(t) \end{bmatrix} = \begin{bmatrix} \cos\phi(t) & -\sin\phi(t) \\ \sin\phi(t) & \cos\phi(t) \end{bmatrix} \begin{bmatrix} w_l(t) \\ w_r(t) \end{bmatrix}. \tag{21}$$

As before, with three strain functions observed at the detectors, but only two functional degrees of freedom available, the system is overconstrained.

VI. CONSTRUCTING A COUNTEREXAMPLE

In electrodynamics, when a wave is propagating in vacuum, then the polarization state is "carried" by the wavefronts. For an observer at a fixed point in space, the direction of the electric field polarization vector may change with time. But if we consider the polarization vector along the null path followed by the wave, then the polarization is constant along that path. In this sense, the wavefront carries the polarization state with it [21,27]. However this is only true in a vacuum.

If instead, the wave is propagating through a material, then the polarization vector can shift direction along the path followed by the wavefront. One example is Faraday rotation, when the wave is propagating through a medium containing free charges in the presence of a magnetic field [18–20].

GR is similar to vacuum electrodynamics, where by "polarization" we mean the relative "mixture" of e_{ab}^+ and $e_{ab}^\times$ that is present in the wave, for a fixed Cartesian basis. For a given fixed observer, this mixture will vary with time as the GW passes by. But if we look along the null path traced by a point on the wavefront, then it follows from (1) that the mixture does not change. However there is no reason to assume that the conjectured pure vector theory of gravity obeys this way.

To construct a pure vector gravitational wave which gives rise to the same strains as would be found in pure tensor GR, we assume that in the pure vector theory (for unspecified reasons) the polarization state can *change* as the wave propagates. For example, this might be due to the presence of the Earth, which hypothetically could introduce a dependence on the variable z, so that

$$\phi(t,z) = \Delta\psi(z). \tag{22}$$

As can be seen from Eqs. (16) and (21), the effect is equivalent to rotating the antenna pattern function through an additional angle $\Delta\psi_i = \Delta\psi(z_i)$ at the ith detector. Defining $\tilde{\psi}_i = \psi_i + \Delta\psi_i$, the response of the detectors will now be described by

$$\begin{bmatrix} \bar{h}_1(t) \\ \bar{h}_2(t) \\ \bar{h}_3(t) \end{bmatrix} = \begin{matrix} f_1 \\ f_2 \\ f_3 \end{matrix} \begin{bmatrix} \cos\tilde{\psi}_1 & \sin\tilde{\psi}_1 \\ \cos\tilde{\psi}_2 & \sin\tilde{\psi}_2 \\ \cos\tilde{\psi}_3 & \sin\tilde{\psi}_3 \end{bmatrix} \begin{bmatrix} w_L(t) \\ w_R(t) \end{bmatrix}, \tag{23}$$

where the factors f_i multiply the corresponding lines on the r.h.s.

Now that we have introduced additional degrees of freedom in the $\tilde{\psi}_i$, it will be possible *in some cases* for this hypothetical pure vector theory to exactly match the strains predicted or expected for GR. However this is not always possible. Roughly speaking, by changing the polarization angle shifts $\Delta\psi_i$, we can vary the proportions of the w_R and w_L waveforms that enter the detected strain at each site. However if the vector response f_i at a particular detector is too small compared to the GR response at that detector, then it won't be possible to match the GR prediction, because varying only the phase $\Delta\psi_i$ can not make the amplitude arbitrarily large. As an extreme example, suppose that the source is located at a position in the sky for which the third detector has no response to *either* of the pure vector modes, which means that f_3 vanishes. In this case there is no way for the pure vector theory to give the same observed strain as GR at the third detector [28].

Can we pick $\Delta\psi_i$ so that the pure vector theory masquerades as ordinary GR? It turns out that this is possible if and only if the antenna patterns obey a triangle inequality. This will only be satisfied for some incoming wave directions.

To understand what directions, invert the first two rows of Eq. (9) to obtain w_+ and $w_\times$ as linear functions of $\bar{h}_1$ and $\bar{h}_2$. Then use the first two rows of the pure vector theory Eq. (23) to replace $\bar{h}_1$ and $\bar{h}_2$, obtaining expressions for w_+ and $w_\times$ as linear functions of w_L and w_R. With these choices, we have ensured that the pure vector model will produce strains at detectors 1 and 2 which agree with those of GR.

Can we pick the $\Delta\psi_i$ (or equivalently, ψ_i) so that the strain at the third detector would agree for GR and the pure vector model? This implies that the third rows of (9) and (23) agree for all choices of w_R and w_L, which is possible if and only if

$$f_3(\cos\tilde{\psi}_3 w_L + \sin\tilde{\psi}_3 w_R) = F_3(\cos\phi_3 w_+ + \sin\phi_3 w_\times). \tag{24}$$

Replacing w_+ and $w_\times$ on the r.h.s. with the linear combinations of w_L and w_R obtained above, multiplying by a factor of $F_1 F_2 \cos(\phi_2 - \phi_1)$, and collecting terms, we obtain
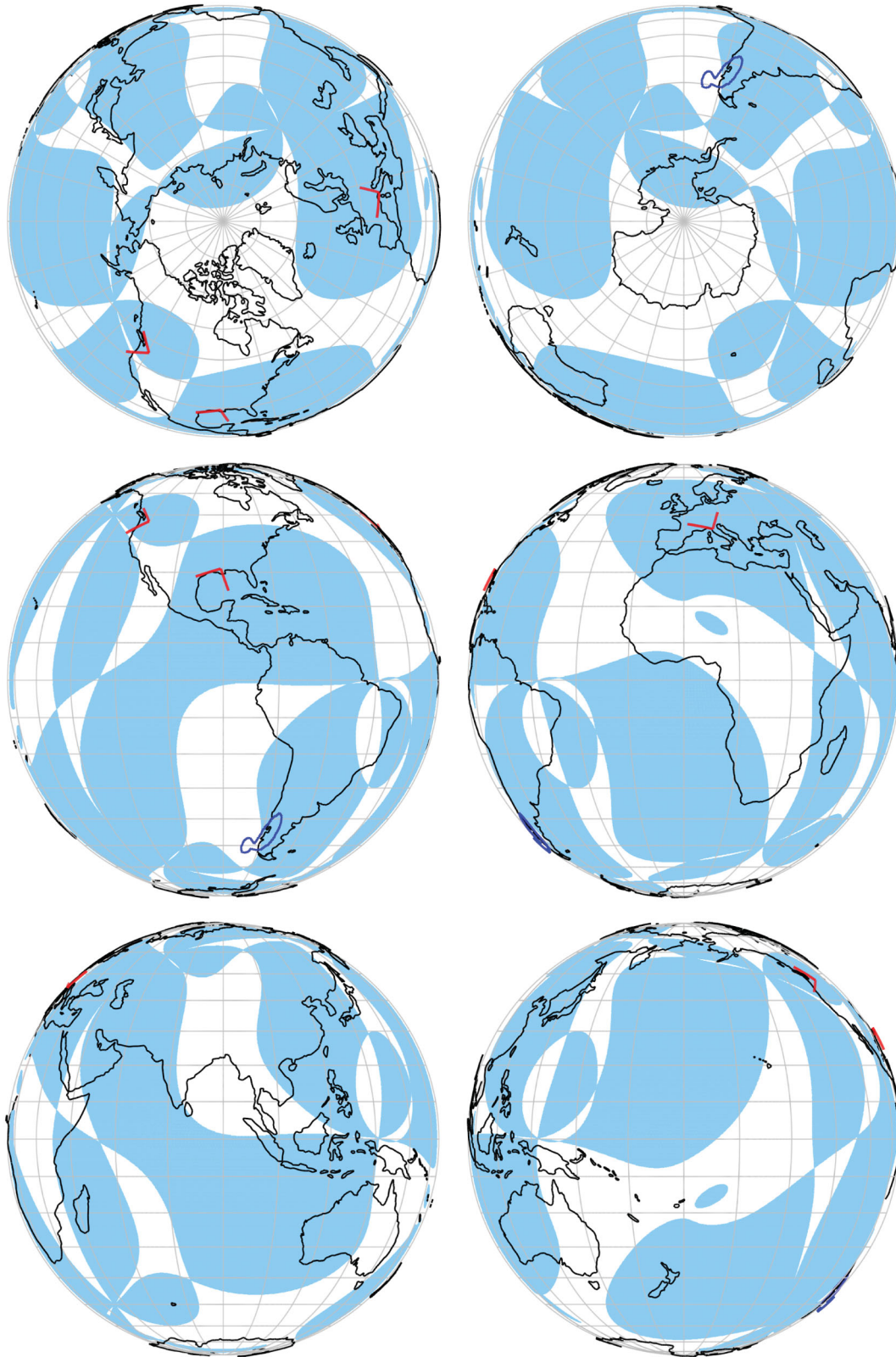
FIG. 1.   These plots show directions on the sky, relative to the currently-operating Earth-based gravitational wave detectors. The LIGO Hanford, LIGO Livingston, and Virgo detector arms are shown in red (not to scale). Sky directions in which the triangle inequality is *not* satisfied are shown in light blue; these are invariant under antipodal reflection. The remaining sky directions (white) are those in which the pure vector model can mimic GR. The dark blue path outlines the most likely region (at 90% Bayesian confidence) from which the GW170814 event originated. In part of that region the pure vector model presented here can give detector signals in the three instruments which are the same as those from GR.

$$w_L(t) \left[f_1 F_2 F_3 \sin(\phi_3 - \phi_2) \cos \tilde{\psi}_1 + F_1 f_2 F_3 \sin(\phi_1 - \phi_3) \cos \tilde{\psi}_2 + F_1 F_2 f_3 \sin(\phi_2 - \phi_1) \cos \tilde{\psi}_3 \right]$$

$$+ w_R(t) \left[f_1 F_2 F_3 \sin(\phi_3 - \phi_2) \sin \tilde{\psi}_1 + F_1 f_2 F_3 \sin(\phi_1 - \phi_3) \sin \tilde{\psi}_2 + F_1 F_2 f_3 \sin(\phi_2 - \phi_1) \sin \tilde{\psi}_3 \right] = 0. \quad (25)$$

If it is possible to pick $\Delta \psi_i$ (or equivalently $\tilde{\psi}_i$) so that both of the quantities in square brackets vanish, then the pure vector theory can masquerade as GR.

The problem may be visualized as follows. Each of the three $\tilde{\psi}_i$ define a unit-length vector $\hat{n}_i = (\cos \tilde{\psi}_i, \sin \tilde{\psi}_i)$ in the (L, R) plane. The quantities in square brackets will vanish if and only if the directions of these three unit vectors can be chosen so that the sum

$$f_1 F_2 F_3 \sin(\phi_3 - \phi_2)\hat{n}_1 + F_1 f_2 F_3 \sin(\phi_1 - \phi_3)\hat{n}_2$$
$$+ F_1 F_2 f_3 \sin(\phi_2 - \phi_1)\hat{n}_3 \quad (26)$$

vanishes. This in turn is possible if and only if the three lengths that multiply these unit vectors obey the triangle inequality: each of the three must not be larger than the sum of the other two.

If we use the definitions of the ϕ_i, then this can be re-expressed in terms of the original antenna pattern weights. This postulated pure vector theory can masquerade as GR provided that the three cyclically-related quantities

$$f_1(F_2^+ F_3^\times - F_2^\times F_3^+),$$
$$f_2(F_3^+ F_1^\times - F_3^\times F_1^+),$$
$$\text{and} \quad f_3(F_1^+ F_2^\times - F_1^\times F_2^+) \quad (27)$$

obey the triangle inequality: the sum of any two must be greater than or equal to the third.

Note that if a particular incoming wave direction $\hat{z}$ satisfies this inequality, then so does the antipodal direction $-\hat{z}$. This is because, under the antipodal transformation $\vec{x} \to -\vec{x}$, the antenna weights F_i^+ and $F_i^\times$ are invariant, as are the f_i. So a sky map of the regions where the triangle inequality is satisfied is symmetric under antipodal transformations.

At the moment, there are three sensitive interferometric gravitational wave detectors in operation: LIGO Hanford, LIGO Livingston, and Virgo. In Fig. 1 we show the regions of the sky where this triangle inequality is satisfied. Pure vector theory sources in those directions could mimic the spin-2 transverse traceless perturbations of GR. In particular, the area of the sky in which GW170814 was likely located, overlaps these directions.

VII. CONCLUSION

What has been presented here is a cautionary note which demonstrates that ruling out a pure vector polarization model, in the absence of a theory which defines its behavior, is difficult. We have constructed a pure vector model which, for certain source locations, can exactly mimic the signal predicted by GR in the LIGO and Virgo detectors. Of course the recent LIGO [29–31] and LIGO-Virgo [17,32] observations and many other experimental tests indicate that GR is an excellent description of nature, so the author disavows any belief in the contrived pure vector model presented here. It is only intended as a counterexample.

The pure vector model presented here is able to mimic GR for a three-detector network because the polarization mixture "rotates" as the gravitational wave propagates, so it demonstrates that such polarization rotation is *sufficient* to mimic GR. While we have not proved that such rotation is *necessary* to mimic GR, that may well be true.

Counting free parameters suggests that it should be possible to construct similar contrived pure vector models which would match the prediction of GR for short-duration signals for networks containing more than three detectors. However the set of allowed sky directions where this is possible would be further constrained (smaller total sky area).

At each detector location (labeled by i) this pure vector model introduces an additional angle $\Delta \psi_i$. So one might well ask: does it have any predictive ability at all? Can it be falsified? The answer to both questions is yes. Recall that the helicity of a field is defined by the properties of the complex phase under rotations [24]. Imagine that we have a set of co-located gravitational wave detectors, whose arms are all in the same plane tangent to the Earth, but which are rotated with respect to one another. Then the helicity can be directly measured by comparing the phase of the strain signals between the different instruments. For example suppose that we had 360 co-located interferometric detectors, each of which had its arms offset by 1° from the previous one, so that they spanned the entire 360° circle in 1° increments. In GR, the phase of the signal will go through two complete cycles (at any instant in time) as you move around the "circle of detectors." In the pure vector theory there would be only one cycle of phase change. This thought experiment demonstrates that the pure vector model presented here does make falsifiable predictions.

Similar reasoning indicates that this pure vector model would have difficulty in mimicking GR for long-duration signals. This is because the Earth's rotation changes the orientation of the source relative to the detectors. For example a detector located at the North Pole with its arms in the plane tangent to the Earth would undergo a complete 360° rotation each day. If a gravitational wave signal is

observed over that entire time span, then models like the one presented here could then be constrained or falsified [33].

The pure vector model presented here has one undetermined phase $\Delta\psi_i$ for each detector. We have shown that for specific values of these phases, the model can exactly reproduce the observed instrumental strains expected from general relativity. It follows that for a real instrument, in which detector noise makes it impossible to identify the signal exactly, some range of these values would be compatible with data. So there is a temptation to use Bayesian methods to compare the odds that this model (with uniform uninformed priors on $\Delta\psi_i \in [0, 2\pi]$) describes Nature, versus the predictions of general relativity. But this would miss the point: the model presented here is merely intended to demonstrate how difficult it is to rule something out, in the absence of a specific predictive model. One could compute odds ratios for the model presented here, but the number of alternative models is only limited by the imagination. Without definite predictions, their odds ratios are not computable.

APPENDIX: DETECTOR RESPONSE TENSORS FOR LIGO AND VIRGO

Here we list the quantities needed to easily reproduce Fig. 1. We work in a right-handed Cartesian coordinate system $[u, v, w]$ which is fixed to the Earth (so rotates with it) and whose origin is at the center of the Earth. The positive u-axis passes through the line of Greenwich at zero longitude where it crosses the equator, and the positive w-axis passes through the North Pole. The Earth is assumed to be spherical.

The first quantities of interest are unit vectors n pointing from the center of the Earth to the detectors, which may be obtained from the geographical information cataloged in [25]. These are

$$n_{\mathrm{LLO}} = -0.01157\hat{u} - 0.86102\hat{v} + 0.50844\hat{w}$$
$$n_{\mathrm{LHO}} = -0.33833\hat{u} - 0.60020\hat{v} + 0.72477\hat{w}$$
$$n_{\mathrm{Virgo}} = 0.71169\hat{u} + 0.13190\hat{v} + 0.69000\hat{w}.$$

The next useful quantities are unit vectors L_1 and L_2 along the two detector arms, derived from information in [25]:

$$\mathrm{LLO}: L_1 = -0.95308\hat{u} - 0.14432\hat{v} - 0.26609\hat{w}$$
$$L_2 = 0.30249\hat{u} - 0.48767\hat{v} - 0.81895\hat{w}$$
$$\mathrm{LHO}: L_1 = -0.23684\hat{u} + 0.79971\hat{v} + 0.55169\hat{w}$$
$$L_2 = -0.91073\hat{u} + 0.01500\hat{v} - 0.41272\hat{w}$$
$$\mathrm{Virgo}: L_1 = -0.70121\hat{u} + 0.19275\hat{v} + 0.68641\hat{w}$$
$$L_2 = -0.04246\hat{u} - 0.97234\hat{v} + 0.22967\hat{w}.$$

Working in the same u, v, w basis, the components of the detector response tensors (4) are given by:

$$d^{ab}_{\mathrm{LLO}} = \begin{bmatrix} 0.81686 & 0.28507 & 0.50134 \\ 0.28507 & -0.21699 & -0.36097 \\ 0.50134 & -0.36097 & -0.59988 \end{bmatrix}$$

$$d^{ab}_{\mathrm{LHO}} = \begin{bmatrix} -0.77334 & -0.17575 & -0.50654 \\ -0.17575 & 0.63931 & 0.44739 \\ -0.50654 & 0.44739 & 0.13403 \end{bmatrix}$$

$$d^{ab}_{\mathrm{Virgo}} = \begin{bmatrix} 0.48989 & -0.17644 & -0.47156 \\ -0.17644 & -0.90830 & 0.35562 \\ -0.47156 & 0.35562 & 0.41841 \end{bmatrix}.$$

Please note that since the Earth is not spherical, these quantities differ from the actual values by fractional amounts of order 10^{-3}. For our purposes, this may be neglected.

For a given sky location, the pure tensor polarization tensors $e^{+,\times}_{ab}$ in Eq. (2) and the pure vector polarization tensors $e^{L,R}_{ab}$ in Eq. (10) may be easily constructed from the right-handed basis

$$x = -\sin L\,\hat{u} + \cos L\,\hat{v}$$
$$y = \cos\ell \cos L\,\hat{u} + \cos\ell \sin L\,\hat{v} - \sin\ell\,\hat{w}$$
$$z = -\sin\ell \cos L\,\hat{u} - \sin\ell \sin L\,\hat{v} - \cos\ell\,\hat{w},$$

where (positive) $\ell \in [-90°, 90°]$ is latitude North of the equator, and (positive) $L \in [-180°, 180°]$ is longitude East of Greenwich.

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[22] This retarded time is not referred to the source, but rather to the time at which the corresponding wavefront passes the plane defined by $z = 0$.

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[26] This basis rotates in phase at frequency "one cycle per rotation" (ϕ) as opposed to (say) frequency "two cycles per rotation" (2ϕ) or "no cycles per rotation" (0ϕ). Since the polarization basis transforms as a rank-(0,2) tensor, this implies that the vector polarizations are helicity ± 1 components of a spin-2 field. The name "vector" perturbations is therefore somewhat misleading, since this implies a spin-1 object, which is a rank-(0,1) or (1,0) tensor. However it has historically been used in this context, so we perpetuate the misnomer.

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