The debate between Peletier and Clavius on superposition

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Abstract

This article considers the sixteenth-century debate between Jacques Peletier du Mans and Christoph Clavius over the admissi-bility of superposition as a means to demonstrate the equality of figures in Euclidean geometry. It notably aims to determine, in the first part, which understanding of superposition motivated its rejection by Peletier, especially whether and to which extent his critical position towards this method was related to its kinematic implications. In the second part, the article presents the critical response Clavius addressed to Peletier in order to defend the legitimacy of superposition in Euclid's Elements.

Résumé

Cet article traite du débat qui a pris place au seizième siècle entre Jacques Peletier du Mans et Christoph Clavius concernant la légitimité de la superposition comme mode de démontration de l'égalité des figures dans la géométrie euclidienne. Il s'agit notamment de considérer de plus près l'interprétation de la superposition de Peletier et les motifs qui l'ont poussé à rejeter cette méthode, cherchant à déterminer si et dans quelle mesure ce rejet est lié à ses implications cinématiques. L'article considère, dans un second temps, la défense de la superposition par Clavius en réponse aux critiques de Peletier.

MSC: primary 01A40; secondary 51-03

Keywords: Renaissance epistemology of geometry; Sixteenth-century Euclidean tradition; Geometrical procedures; Superposition; Peletier; Clavius

1. Introduction

The aim of this article is to bring forth new elements regarding the dispute that took place in the six-teenth century between Jacques Peletier du Mans and Christoph Clavius over the validity of superposition (ἐφάρμοσις) as a method to demonstrate the congruence and equality of geometrical figures, as it was used by Euclid in Prop. I.4, I.8 and III.24 of the Elements. This analysis intends to determine in particular which

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understanding of superposition might have motivated the critical position which was held by Peletier on this method for proving the equality of figures and the manner in which his interpretation of superposition was read by Clavius.In the *Elements*, superposition is first introduced in Prop. I.4 in order to demonstrate that in two triangles which have two sides equal to two sides respectively and the angles contained by these two sides mutually equal, the bases, the remaining angles and the whole triangles will also be equal.[1] Taking the case of two triangles ABC and DEF, which have the two sides AB and AC, respectively equal to the two sides DE and DF and the angle BAC equal to the angle EDF, Euclid intends to prove this in the following manner:

If the triangle ABC be applied to the triangle DEF, and if the point A be placed on the point D and the straight line AB on DE, then the point B will also coincide with E, because AB is equal to DE. Again, AB coinciding with DE, the straight line AC will also coincide with DF, because the angle BAC is equal to the angle EDF; hence the point C will also coincide with the point F, because AC is again equal to DF. But B also coincided with E; hence the base BC will coincide with the base EF. [For if, when B coincides with E and C with F, the base BC does not coincide with the base EF, two straight lines will enclose a space: which is impossible. Therefore the base BC will coincide with EF] and will be equal to it. Thus the whole triangle ABC will coincide with the whole triangle DEF, and will be equal to it. And the remaining angles will also coincide with the remaining angles and will be equal to them, the angle ABC to the angle DEF, and the angle ACB to the angle DFE.[2]In this proof, the equality of the various part of the triangles, and of the triangles themselves, is deduced from their coincidence, since, as stated by Euclid's fourth Common Notion: "Things which coincide with one another are equal to one another".[3] In Prop. I.8, the same argumentative method is introduced again to prove the equality of triangles[4] and, in Prop. III.24, to prove the equality of segment of circles.[5]In his commentary on the first six books of the *Elements*, Peletier described the mode of demonstration appealed to in these propositions as mechanical rather than as properly mathematical, and thus as deprived of the dignity of geometrical argumentative procedures, for which reason it should be excluded from the *Elements* and from geometry in general.[6] In order to prove the irrelevance of superposition to geometry, Peletier argued, as we will see later, that Euclid himself did not truly regard this procedure as a properly geometrical mode of demonstration, as he would have otherwise used it in many other occasions, notably in Prop. I.2 and I.3,[7] which require "to place at a given point (as an extremity) a straight line equal to a given straight line"[8] and, "given two unequal straight lines, to cut off from a greater straight line a line equal to the less".[9] To him, Euclid should rather have placed Prop. I.4 among the principles, as a definition of the equality of angles, which was missing in the argumentative structure of the *Elements*.[10]

1 [Euclid, 1956, 1, 247]: "If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend".

2 [Euclid, 1956, 1, 247–248]. T.L. Heath [Euclid, 1956, 1, 249] held that the passage in brackets was a later interpolation, as well as the related common notion "Two straight lines do not enclose a space" ([Euclid, 1956, 1, 232]).

3 [Euclid, 1956, 1, 224].

4 [Euclid, 1956, 1, prop. I.8, 261]: "If two triangles have the two sides equal to two sides respectively, and have also the base equal to the base, they will also have the angles equal which are contained by the equal straight lines".

5 [Euclid, 1956, 2, prop. III. 24, vol. II, 53]: "Similar segments of circles on equal straight lines are equal to one another".

6 [Peletier, 1557, Prop. I.4, 16]. For the full quotation, see *infra*, n. 109.

7 [Peletier, 1557, Prop. I.4, 15]. See *infra*, n. 120.

8 [Euclid, 1956, 1, Prop. I.2, 244].

9 [Euclid, 1956, 1, Prop. I.3, 246].

10 [Peletier, 1557, Prop. I.4, 15–16]. See *infra*, n. 131.

Peletier was directly challenged by Christoph Clavius on this issue in 1586, in his commentary on Theodosius's Spherics,[11] in other words, twenty-nine years after the publication of Peletier's commentary on the Elements and four years after Peletier's death. Clavius republished his arguments against Peletier's discus-sion of superposition in his commentary on Euclid's Elements, from the second edition published in 1589.[12] Clavius then defended the legitimacy of superposition in Euclid's Elements through a thorough rebuttal of Peletier's arguments against this mode of demonstration, aiming to display his opponent's misconception of the purpose of this theorem (and of geometrical theorems in general), as of the mode according to which the objects and procedures represented in this framework should be conceived.Modern commentators generally present Peletier as the earliest known mathematician to have openly re-jected the legitimacy of superposition in geometry.[13] Indeed, the procedure, which seems to have been quite common in the practice of geometers since Antiquity, is not known to have aroused objections before the sixteenth century. T.L. Heath wrote, in his Thirteen books of the Elements, that, although Euclid appealed to superposition in a very little number of propositions (however reluctantly, according to him),[14] the presence of this mode of demonstration in the Elements, where no other means of demonstration was possible would point to its traditional character at the time.[15] Beyond Euclid, this procedure was notably used in Antiquity by Archimedes in On the Equilibrium of Planes and On Conoids and Spheroids,[16] by Apollonius in the Conics[17] and by Pappus in the Collections.[18] The Neoplatonic philosopher Proclus, whose commentary on the first book of Euclid's Element became very influential in the sixteenth century,[19] did not object to Euclid's use of superposition in any manner[20] and even applied superposition to certain theorems which were demonstrated by Euclid through other means.[21] In the ninth century, Thabit ibn-Qurra, for whom geometry's first aim is to establish the measure of figures, and therefore their equality and inequality, even defined superposition as one of the most crucial procedures of geometry, along with the construction of circles.[22] J. Murdoch has shown that this mode of demonstration was also perfectly admitted by medieval commentators of Euclid's Elements.[23]After Peletier, the legitimacy of superposition was also contested by François de Foix-Candale, in his 1566 commentary on the Elements,[24] who also described superposition as a mechanical procedure, ex-plicitly intending the term mechanical as relating to the use of instruments, an interpretation of the term mechanical which is not apparent in Peletier's commentary. Although a very short lapse of time separates

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11 [Clavius, 1586, 343–344].

12 [Clavius, 1589, 368–370].

13 [Euclid, 1956, 1, 249], [Murdoch, 1964, p. 421], [Knorr, 1981, 160], [Mancosu, 1996, 29] and [Vitrac, 2005, 50].

14 Heath, in [Euclid, 1956, 1, 225]: "it is clear that Euclid disliked the method and avoided it wherever he could […]". See [Euclid, 1956, 1, 249]. This was asserted before him by [Killing, 1898, 3] and [Russell, 1938, 405] (quoted in n. 42) and was also confirmed by [Knorr, 1981, 161] and [Mancosu, 1996, 29], although these authors did not all agree on the reasons why Euclid avoided appealing to superposition where he could have.

15 [Euclid, 1956, 1, 225].

16 [Archimedes, 1953c, 1, Post. 4 and Prop. 9–10, 189 and 194–195] and [Archimedes, 1953a, Prop. 18, 128]. On this specific case, see also [Palmieri, 2009, 476–478] and [Knorr, 1981, 160].

17 [Apollonius, 2009, 5–6 and 9–12].

18 [Pappus, 1876–1878, III, 51, § 83; IV, 26, § 39 and 45–46; VI, 27, § 44, 1876–1878, 138, 244, 252 and 524]. For other cases, see [Mugler, 1959, 208].

19 This commentary, which was rediscovered at the end of the fifteenth century, was crucial to the transmission of Platonic and Neoplatonic philosophy of mathematics in the Renaissance. On its reception and diffusion in the sixteenth century, see [De Pace, 1993, 121–185], [Maierù, 1999], [Rommevaux, 2004] and [Higashi, 2007].

20 [Proclus, 1873, 233–241]. See also by [Euclid, 1990, 1, 202–203 and 295], [Vitrac, 2005, 50] and [Palmieri, 2009, 473–474].

21 [Proclus, 1873, 157–158 and 188–192]. See also [Knorr, 1981, 161].

22 [Jaouiche, 1986, 151–153] and [Vitrac, 2005, 5 and 51–52].

23 [Murdoch, 1964, 417–421].

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24 [Candale, 1566, Prop. I.4, I.8 and III.24, f. 5v, 6v and 27r]. See also [Mancosu, 1996, 30].

these two texts on the scale of the history of classical geometry, Peletier and Candale's critical opinions on superposition have remained, to our knowledge, relatively marginal. Their polemic discourse on the status of Euclid's congruence proofs did, however, have an impact on the following Euclidean tradition in the sense that, after the publication of their respective commentaries on the Elements, several sixteenth- and seventeenth-century mathematicians and philosophers discussed the issue, often in reaction to Peletier and Candale's claim that superposition is more mechanical than geometrical and in order to defend the foundational role of Prop. I.4 in the argumentative structure of Euclid's Elements.As early as 1569, Petrus Ramus took up Peletier and Candale's description of superposition as mechan-
ical in his Scholae arithmeticae et geometriae, only to assert the legitimacy of this procedure in geometry, as well as that of all the other procedures which may be considered as mechanical in geometry, such as the operations involved in the resolution of problems and those which allow one to measure magnitudes – to which superposition is then related.[25]As shown by P. Mancosu, in the seventeenth century, Giuseppe Biancani,[26] Henry Savile[27] and Isaac
Barrow[28] all positioned themselves in favour of the legitimacy of superposition, rejecting its representation as a mechanical procedure. Barrow, in particular, insisted on the foundational importance of superposition in the Elements as in geometry in general, given that it allows one to apply the axiom of congruence[29] to the demonstration of the equality of magnitudes.[30] The legitimacy of superposition was also defended by Bonaventura Cavalieri, who appealed to superposition for his new geometry of indivisibles,[31] against the criticisms raised by the Jesuit Paul Guldin regarding his use of superposition to compare and demonstrate the congruence of unequally shaped figures.[32]Certain difficulties regarding this mode of proof were raised in more modern times, some of which point to the solution envisaged by Peletier, which consists in formulating Prop. I.4 as a principle rather than as a
theorem. In the nineteenth and twentieth centuries, the issue of the legitimacy of superposition was mostly raised in relation to the assessment of the function and status of the axiom of congruence, which played a crucial role in theorems appealing to superposition. T.L. Heath, who offered in this regard a comprehen-sive historical analysis of the conceptions regarding this question at the beginning of the twentieth century, considered that the axiom of congruence was integrated by Euclid among the Common Notions in order to assert the legitimacy of the method of superposition as a means to demonstrate the congruence and thereby

[25] [Ramus, 1569, 2, 6–7]: "Neque vero nobis ἐφαρμόσεως axioma quisquam repudiet tanquam mechanicum, neque ideo geometricum. Nam Archimedem cum Euclide copulare libuit, et artis utilitatem cum veritate artis conjungere. Neque isto principio quicquam in geometria luculentius est, primaque omnino geometrica mensura, quae granis, digitis, palmis, pedibus, cubitis, passibus, decempedis, et similibus efficitur, ἐφαρμόσεως utique judicio efficitur, et falsum est, mechanicum quod sit, geometricum non esse: postulata enim et problemata omnia (quod geometricorum et principiorum et propositionum genus alterum est) mechanica sunt: et cum theorematis differentiam vulgo habent, quod haec contemplantur, illa fabricantur et machinantur. Symmetria et ratio totæ mechanicae sunt, dum eadem mensura diversas magnitudines metiuntur, et comparando numerant. Itaque ἐφαρμόσεως tota geometrica est, ut tot locis jam patuit, nec regulae et circini (quibus omnia geometræ problemata fabricas et machinas suas fabricantur et machinantur) vis alia atque alia facultas est, quam ἐφαρμόσεως geometricae: ut qui ἐφαρμόσεως à geometrica schola expellit, expellat ex eadem Euclidem et Archimedem, imo geometriam ipsam". The passage in which Ramus discusses this issue is quoted by Bernardino Baldi, in the part dedicated to Peletier in his Vite de' matematici, [Baldi, 1998, 474–475]. See also the notes Elio Nenci dedicated to this in [Baldi, 1998, 481–482].

[26] [Biancani, 1615, 24] and [Mancosu, 1996, 200–201].

[27] [Savile, 1621, 196] and [Mancosu, 1996, 31].

[28] [Barrow, 1683/1976, 167] and [Barrow, 1734/1970, 187], quoted by [Mancosu, 1996, 29].

[29] [Euclid, 1956, 1, 224]: "Things which coincide with one another are equal to one another".

[30] [Mancosu, 1996, 29 and 56].

[31] [Cavalieri, 1635] and [Cavalieri, 1647].

[32] [Mancosu, 1996, 39–56] and [Palmieri, 2009, 488–493]. [Cavalieri, 1647, p. 215] mentioned the names of Peletier and Clavius in reference to the status of superposition. See [Mancosu, 1996, 55] and [Palmieri, 2009, 476].

the equality of figures.[33] Correlatively, this axiom was considered as tacitly dependent on other fundamen-tal notions, such as the equality of angles and the motion of rigid figures, which were not made explicit at the foundation of Euclid's Elements and which were, however, regarded as playing an important role in Euclidean geometry.[34] In this respect, the connection between superposition (or superposability) and the coincidence of figures, as often noted in the historical analyses of the Euclidean notion of coincidence,[35] would be suggested by the twofold meaning of the verb ἐφαρμόζειν in ancient Greek, which in the active and intransitive form (apparent in C.N.4 through the active participle ἐφαρμόζοντα) implies the idea of coincidence and in the passive and transitive form ἐφαρμόζεσθαι (used in Prop. I.4 through the middle-passive participle ἐφαρμοζομένου) would mean to be applied to or superposed on, without necessarily implying coincidence.[36] Both forms are used in Prop. I.4, first in the passive (through the middle-passive participle ἐφαρμοζομένου), when the parts of the two triangles which are known to be equal are said to be superposed, then in the active (ἐφαρμόσει), when is demonstrated the coincidence of the parts previously asserted as equal and, from there, the coincidence of the remaining parts of the two triangles and of the triangles themselves.

As noted by T.L. Heath in reference to P. Tannery, this axiom sets forth a notion of equality which is specific to extended and continuous quantities,[37] which may be confirmed by the fact that David Hilbert, in his Grundlagen der Geometrie, attributed a key-role, among the principles of geometry, to the notion of congruence in his revision of the axiomatic of Euclidean geometry.[38]

Now, while the relation between the axiom of congruence and the method of superposition would seem to justify per se the admission of this mode of proof in Euclidean geometry, the legitimacy of superposition, as well as the very notion of equality as coincidence or congruence, were however questioned in view of the fact that they implied the mobility of figures, which in turn would point to their empirical character. Indeed, since superposition, such as introduced by Euclid in the Elements, requires one to suppose that one of the figures, if placed on the other, coincides with the latter with respect to its dimension and configura-tion, it was commonly interpreted as a kinematic process, implying in other words the local transport of a geometrical figure from one place to the other, even if only taking place in the imagination.[39] The notion of congruence itself, as shown by various modern analyses of the Euclidean axiomatic, was presented as indissociable from the motion of figures, namely, the rigid motion of figures.[40]

[33] [Euclid, 1956, 1, 225]: "It seems clear that the Common Notion, as here formulated, is intended to assert that superposition is a legitimate way of proving the equality of two figures which have the necessary parts respectively equal, or, in other words, to serve as an axiom of congruence".

[34] [Goldstein, 1972], [Euclid, 1990, 1, 202–203] and [De Risi, 2016, 615].

[35] [Euclid, 1956, 1, 224–225], [Murdoch, 1964, 417], [Euclid, 1990, 181, n. 13] and [Mancosu, 1996, 29]. See also [Goldstein, 1972, 331–332].

[36] This is notably suggested by Heath, in [Euclid, 1956, 1, 224–225] and [Murdoch, 1964, 417]. However, the idea that the middle-passive form does not necessarily imply the notion of coincidence may be nuanced by the fact that, in this context and in other similar Greek geometrical treatises in which this form is used, superposition is most often intended to demonstrate the coincidence of two points, lines or figures. Vitrac, in [Euclid, 1990, 181, n. 13], also suggests this, indicating that the distinction between application and coincidence is found in the natural, rather than mathematical, language. Moreover, in [Mugler, 1959, 208], all the examples given to illustrate the passive and transitive form of the verb ἐφαρμόζειν imply the notion of coincidence. We kindly thank N. Sidoli for his helpful remarks on this point.

[37] [Euclid, 1956, 1, 225] and [Tannery, 1884, 167].

[38] [Hilbert, 1903, 7–15]. See also [Euclid, 1956, 1, 228–231] and [Goldstein, 1972, 339].

[39] [Killing, 1898, II, 2–3], [Euclid, 1956, 1, 225], [Goldstein, 1972], [Mueller, 2006, 23], [Euclid, 1990, 1, 293–298] and [Vitrac, 2005, 5 and 49–50]. This kinematic implication of superposition was actually made explicit and integrated within the Euclidean axiomatic in the seventeenth century, as was recently put in evidence by [De Risi, 2016, 593, 632 and 661].

[40] [Hilbert, 1903, 7]: "Die Axiom dieser Gruppe definieren den Begriff der Kongruenz und damit auch den der Bewegung". See also [Goldstein, 1972].

Arthur Schopenhauer challenged the validity of the axiom of congruence, stating that the fact of defining coincidence as equality was either purely tautological or appealed to sensual experience, given that it would, as such, presuppose the mobility of figures.[41] Thus, if the axiom of congruence should be taken as implying the mobility of figures, the demonstrations which depend on it would invite one to deduce the equality of figures in a purely empirical manner.

Bertrand Russell, pointing to the lack of logical validity and to the empirical character of Prop. I.4, stated that Euclid should rather have formulated this proposition as an axiom, as was done to a certain extent by Hilbert.[42]

Now, as shown by B. Vitrac's analysis of the treatment of motion in geometry from Plato and Aristotle to Omar Khayyam,[43] the introduction of motion in geometrical definitions and demonstrations was regarded as problematic by Ancient Greek philosophers given that this practice seemed to contradict the ontological status they conferred to geometrical objects, these being defined, in this context, as intelligible entities, at least as objects essentially separate from matter and motion.[44]

The fact that geometrical objects were considered by philosophers as essentially deprived of movement certainly did not prevent ancient geometers from appealing to motion in the definition and study of geometrical objects,[45] as shown, for example, by Archimedes's definition of the spiral[46] or by ancient constructions of complex curves, such as the conchoid[47] or the quadratrix,[48] but also by Euclid's definitions of the sphere, the cone and the cylinder,[49] to which may be added the use of superposition in the *Elements*.

41 [Schopenhauer, 1988, 143]. Schopenhauer's position on this issue is mentioned by T.L. Heath, in [Euclid, 1956, 1, 227], [Goldstein, 1972, 336] and E. Nenci, in [Baldi, 1998, 482].

42 [Russell, 1938, 405]: "With regards to the fourth, there is a great deal to be said; indeed Euclid's proof is so bad that he would have done better to assume this proposition as an axiom [. . .]. The fourth proposition is the first in which Euclid employs the method of superposition – a method which, since he will make any *détour* to avoid it, he evidently dislikes, and rightly, since it has no logical validity, and strikes every intelligent child as a juggle". Russell pointed, in the footnote, to Hilbert's treatment of the equality of the remaining angles.

43 [Euclid, 1990, 1, 293–299] and [Vitrac, 2005, 37 and 49–52].

44 See in particular Plato, *Resp*. VII, 527a–b and Aristotle, *Phys*. II.2, 193b31–194a7, *Metaph*. II.8, 989 b32–33 and VI.1 1026a7–10. On this issue, see [Vitrac, 2005, 9–18].

45 A good analysis of these uses is provided by [Molland, 1976, 21–49], as well as in [Vitrac, 2005].

46 [Archimedes, 1953b, 165]: "If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and returns to the position from which it started, and if, at the same time as the line revolves, a point move at a uniform rate along the straight line beginning from the extremity which remains fixed, the point will describe a spiral in the plane".

47 [Pappus, 2009, Prop. 26, 126] ([Pappus, 1876–1878, I, 26, § 40–41, 244–246]): "Set out a straight line AB, and a straight line CDZ at right angles to it, and take a certain point E on CDZ as given. And assume that, while the point E remains in the place where it is, the straight line CDEZ *travels* along the straight line ADB, dragged via the point E in such a way that D *travels* on the straight line AB throughout and does not fall outside while CDEZ is dragged via E. Now, when such a *motion takes place* on both sides, it is obvious that the point C will describe a line such as LCM is, and its symptoma is of such a sort that, whenever some straight line starting from the point E toward the line meets it, the straight line cut off between the straight line AB and the line LCM is equal to the straight line CD".

48 [Pappus, 2009, Prop. 30, 131–132] ([Pappus, 1876–1878, I, 30, § 46, 254]): "Set out a square ABCD and describe the arc BED of a circle with centre A, and assume that AB *moves* in such a way that while the point A remains in place, the point B travels along the arc BED, whereas BC follows along with the *travelling* point B down the straight line BA, remaining parallel to AD throughout, and that in the same time both AB, *moving uniformly*, completes the angle BAD, i.e.: the point B completes the arc BED, and BC passes through the straight line BA, i.e.: the point B *travels* down BA. Clearly it will come to pass that both AB and BC reach the straight line AD at the same time. Now, while a *motion* of this kind *is taking place*, the straight lines BC and BA will intersect each other during their *travelling* in some point that is always changing its position together with them. By this point a certain line such as BZH is described in the space between the straight lines BA and AD and the arc BED, concave in the same direction as BED, which appears to be useful, among other things, for finding a square equal to a given circle".

49 [Euclid, 1956, 3, Df. XI.14, 261]: "When, the diameter of a semicircle remaining fixed, the semicircle is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a sphere."; [Euclid, 1956, 3, Df. XI.18, 261]: "When, one side of those about the right angle in a right-angled triangle remaining fixed, the triangle is

With regards to this, it is notable that T.L. Heath, who rejected the thesis that Aristotle condemned the use of motion in geometry (in response to W. Killing),[50] considered that Euclid disliked superposition and avoided it in view of its empirical and kinematic implications,[51] which is perhaps related to the fact that Heath described Euclid as a disciple of the Platonic school of Athens.[52] Nevertheless, although there is no doubt that Plato reproved the use of motion in geometry, the claim that Euclid avoided appealing to superposition, and to motion in general in the framework of his geometry, is disputable.[53]

Nevertheless, it is clear that the tension between the practice of geometers and the discourse of philosophers over the nature of geometrical objects did raise some concern from Antiquity, as different arguments were explicitly or implicitly presented by mathematicians or philosophers up to the early modern period (at least) in favour of or against the use of motion in geometry, or simply in order to state or restrict the conditions under which it should be admitted.[54] One of the most common attacks against the use of motion in geometry, particularly in the context of definitions, was aimed at the claim that it attributed sensible qualities to intelligible beings. This argument was notably made by Omar Khayyam in reference to the definition of the line as the movement of a point and to the definition of the sphere as resulting from the rotation of a semicircle.[55] Mathematicians sometimes anticipated such criticisms by pointing to the intelligible, metaphorical or imaginary nature of this motion.[56]

B. Vitrac showed that interpreting superposition, in the *Elements*, as a kinematic process was not systematic, pointing to the fact that Khayyam, who explicitly attacked the introduction of motion in geometrical definitions, seemed to accept superposition without reservation, as he made use of it in his own geometrical demonstrations, as well as in his justification of the principles of geometry.[57] It would, in any case, not be necessary to consider superposition as a motion in a straightforward and physical sense.[58] This was clearly expressed by B. Russell in order to address the problems raised by Prop. I.4 with regards to the motion it attributes to figures,[59] notably the fact that this motion would suppose the dissociation of geometrical points from their position in space.[60]

carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a cone." and [Euclid, 1956, 3, Df. XI.21, 262]: "When, one side of those about the right angle in a rectangular parallelogram remaining fixed, the parallelogram is carried round and restored again to the same position from which it began to be moved, the figure so comprehended is a cylinder".

50 [Heath, 1921, 226].

51 [Heath, 1921, 376]. On Euclid's avoidance of superposition, see *supra*, n. 14 and 42. This interpretation was also held by [Murdoch, 1964, 417].

52 [Euclid, 1956, 1, 2 and 168].

53 This thesis was notably refuted by [Mueller, 2006, 22–26]. See also [Euclid, 1990, 299] and [Vitrac, 2005]. [Knorr, 1981, 160–162] admitted that Euclid avoided appealing to superposition as much as he could, but rejected the hypothesis that he would have done so in order to comply with the Platonic doctrine.

54 On this issue, see [Jaouiche, 1986, 68–72], [Vitrac, 2005], [Dye and Vitrac, 2009], [Vinel, 2010] and [Rashed, 2013].

55 [Djebbar, 2002, 86–87] and [Vitrac, 2005, 5 and 57]. See also [Rashed, 2013, 60].

56 [Vitrac, 2005, 21 and 51–54].

57 [Vitrac, 2005, 5 and 57]. Vitrac, in [Euclid, 1990, 1, 295], also mentioned the fact that Proclus did not speak of motion when commenting on the Euclidean propositions involving superposition.

58 [Euclid, 1990, 1, 293–299] and [Vitrac, 2005, 37 and 49–52].

59 [Russell, 1938, p. 406]: "The fact is that motion, as the word is used by geometers, has a meaning entirely different from that which it has in daily life, just as a variable, in mathematics, is not something which changes, but is usually, on the contrary, something incapable of change. So it is with motion. [. . .] What is clear is, that a motion presupposes the existence, in different parts of space, of figures having the same metrical properties, and cannot be used to define those properties. And it is this sense of the word *motion*, not the usual material sense, which is relevant to Euclid's use of superposition". On Russell's interpretation of motion in geometry, see also [Euclid, 1956, 1, 227] and [Goldstein, 1972, 335–336].

60 [Russell, 1938, 405]: "to speak of motion implies that our triangles are not spatial, but material. For a point of space is a position, and can no more change its position than the leopard can change its spots".

But as shown again by Vitrac,[61] as well as by Heath's own interpretation of Euclid's intention in Prop. I.4 and I.8,[62] even when this procedure is conceived as an imaginary variation of position, or when it is conceived as resulting from a step-by-step transfer and superposition of points and lines onto other points and lines enabled by the first three propositions (instead of a direct rigid motion of figures),[63] it still conveys the notion of local motion,[64] maintaining some of the difficulties raised by its kinematic understanding, such as the empirical character of proofs by superposition.

In spite of these considerations, it was regularly claimed throughout the history of mathematics that Prop. I.4, which is the very first theorem of the Elements,[65] holds an essential place in the argumentative structure of this work of Euclid, given that various ulterior propositions are logically dependent on it.[66] This was, for that matter, one of the main reasons why Clavius strongly defended the legitimacy of the method of superposition in the Elements of Euclid.

It thus appears that, although the introduction of superposition in Euclid's Elements does not seem to have been challenged before sixteenth century and although Peletier and Candale's critical positions on superposition were widely refuted in the early modern period, this geometrical mode of demonstration, as well as the related axiom of congruence, remained a subject of discussion in the early modern and modern times. These discussions show that superposition did not only raise questions with regards to its logical validity and to its gnoseological value in the Euclidean framework, but also with regards to more general issues, such as the conditions and limits of the admissibility of motion in geometry and the requirements for geometrical proofs. They show, moreover, that Peletier's arguments against superposition resonated, to a certain degree, in the modern considerations on the nature of congruence.

In considering here what may be regarded as the starting point of these discussions, namely, the polemic exchange of arguments between Peletier and Clavius over superposition, we do not intend to present this debate in all its aspects, but rather to complete existing historical accounts[67] by focusing more closely

[61] [Vitrac, 2005, 5 and 50–52] and [Euclid, 1956, 1, 295 and 298].

[62] [Euclid, 1956, 1, 225]: "The phraseology of the propositions, e.g. I.4 and I.8, in which Euclid employs the method indicated, leaves no room for doubt that he regarded one figure as actually moved and placed upon the other".

[63] [Levi, 2003, 103–109] developed this interpretation to explain why the first theorem of the Elements (Prop. I.4) is preceded by three problems (Prop. I.1, I.2 and I.3). Levi thus hinted at the possibility that Euclid's demonstration of I.4 would have been modified in the version of the fourth proposition that was transmitted by Theon and Proclus into one that leaves aside the procedures of transfer of points, lines and angles enabled by the first three propositions, and this in order to offer a demonstration based on the rigid motion of one of the triangles onto the other. [Saito, 2009, 807–809] presents this reading as more plausible than the one given by [Proclus, 1873, 233–234] to explain the sequence of Euclid's first four propositions, according to which Euclid would have considered necessary to prove the existence of the triangle and of its parts before demonstrating its properties and relations. This interpretation is the one adopted and applied to the analysis of ancient geometrical practices by [Zeuthen, 1896] and which was refuted by [Knorr, 1983] (see also [Harari, 2003]). One should also compare this interpretation to the one proposed by [Wagner, 1983, 68–89], who supposes that, in Prop. I.4, superposition would have been intended by Euclid as the construction of a triangle identical to a given triangle (a copy of this triangle) on this given triangle through the tacit implementation of a construction algorithm, partially based on the first three propositions. B. Vitrac, in [Euclid, 1990, 1, 295–297], however dismisses this interpretation in view of its logical weakness, as it would require certain construction procedures which depend, in Euclid's Elements, on ulterior propositions. Levi's solution does not bear the same logical weakness insofar as it assumes that the angle EDF (equal to BAC) is given and not constructed.

[64] [Vitrac, 2005, 5].

[65] The first three propositions correspond to problems, which require to construct or find an object rather than to demonstrate a property or relation: Euclid, Elements, Prop. I.1, trans. T.L. Heath, p. 241: "On a given finite straight line to construct an equilateral triangle"; Prop. I.2, p. 244: "To place at a given point (as an extremity) a straight line equal to a given straight line"; Prop. I.3, p. 246: "Given two unequal straight lines, to cut off from the greater a straight line equal to the less".

[66] [Killing, 1898, 3] and [Euclid, 1956, 1, 225]. See also [Murdoch, 1964, 421]. [Mancosu, 1996, 29–32 and 56] shows that this assertion was often invoked in early modern arguments in defence of superposition, from Clavius to Barrow.

[67] That is to say, those we have found the most relevant, among those that have come to our knowledge. In addition to the previously mentioned works by Heath, Vitrac and Mancosu, ([Euclid, 1956, 1, 249], [Euclid, 1990, 1, 293–294], [Vitrac, 2005, 50]

on two key-elements of Peletier's argumentation, which are the assertion of the incompatibility between superposition and the constructive procedures presented in propositions such as I.2 and I.3, as well as his characterisation of superposition as a mechanical procedure, considering, afterwards, the manner in which Clavius received and replied to these arguments. We would like to determine, in particular, whether and to which extent the kinematic implications of superposition were a motive for Peletier's designation of this procedure as mechanical, and to see how Clavius, in responding to Peletier, distinguished his opponent's conceptions on the motion of figures in Prop. I.4, I.8 and III.24 from his own conceptions of this motion. Indeed, although Peletier, as we will see, did not explicitly mention the kinematic nature of superpo-sition as the reason for his rejection of this mode of proof, certain historians, such as P. Mancosu, have straightforwardly interpreted it as such.[68] This hypothesis seems in principle reasonable given that Peletier displayed, in his commentary on the Elements, as well as in other texts, Platonic and Neoplatonic ideas regarding the status of mathematical objects.[69] He presented indeed geometrical objects, and mathemati-cal objects in general, as entities of divine origin, which express the divine order that is deployed in and by Nature and that is reproduced in the human mind through the study of mathematics.[70] Thus, it would, in principle, be possible that Peletier considered superposition as mechanical and as unworthy of figuring among geometrical procedures because it would attribute a sensible quality to things which are by essence deprived of matter and which are prior to sensible realities in the order of causes. This would be further-more coherent with the fact that Peletier rarely used the term motion (motus), or terms derived from it, in his commentary on the Elements and, as he mentions the definition of the line as caused by the flow of a point in his commentary of Df. I.2, he never described the flow of the point as a motion, contrary to all the other sixteenth-century commentators of the Elements who offer a commentary on the definitions. He described it rather, notably in his poetic texts, as the image of the divine and instantaneous emanation of the multiple from the principial One and through which was caused not only all numbers and magnitudes, but also the universe, a conception that distinctly evokes the Platonic and Neoplatonic cosmogonical doctrines.[71] Such a process would indeed be very different from the more mechanical motion dealt with in a local form of displacement of figures, as in a straightforward understanding of superposition.

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and [Mancosu, **1996**, **29–31**]), we will refer to [Palmieri, **2009**, **474–476**], who presented the debate between Peletier and Clavius as a preamble to his analysis of superposition in Cavalieri's practice of mathematics. We will also draw on [Loget, **2000**, **171–177**], who presented Peletier and Clavius's respective discussions of superposition within his analysis of the early modern controversy on the angle of contact. While some of these works simply offer a brief summary of Peletier's arguments (Heath and Vitrac), some offer a more contextualised and detailed presentation (Mancosu and Palmieri), or even offer extensive translations of the relevant texts, as well as a systematic outline of the various arguments brought forth by both parts (Loget).

68 [Mancosu, **1996**, **29**].

69 See, for instance, [Pantin, **2002**] and [Axworthy, **2013**].

70 [Peletier, **1557**, Praefatio, sig. A4r–v]: "Ubique latet vis quaedam Geometriae: quae utrum plus naturae habeat an artificij, non satis perspici potest: nisi quatenus explicat ipsa exercitatio. In qua meditatione quanto maiores progressus fecerimus, tanto propiùs ad Deum accedere videmur. Ac quemadmodum Mens illa aeterna, praeteritorum meminit, praesentia cernit, futura perspicit, simul verò omnia amplectitur & moderatur: ita praeclarus Geometriae artifex suas cogitationes in unum collatas, ad rem suam convertit, et suum quendam Mundum universa speculatione intuetur. [...] Geometricae positiones, quae operas auxiliarias inter se praestant, omnia in rerum natura mutuis alternisque subsidijs niti & consistere declarant. Quinetiam amicitiae ipsius jura, in Figurarum similitudine, quarum colligationem Diameter efficit, conspicua sunt. Ad summam, haec imago & facies Geometrica ejusmodi est, ut in ea Mundi quandam θεωρίαν possis agnoscere. De cuius initijs huc nihil afferre constitui. Non ab Aegyptijs, non à Chaldaeis, non à Phoenicibus, illius originem requiram. Scientias quippè aeternas esse semper existimavi: atque ut in Mente divina, ab aeterno infixam fuisse Mundi constitutionem: sic disciplinas, caelestia quaedam semina esse: quae in nobis insita, & pro rata cuiusque portione exculta, fructum edunt". On this text, see [Axworthy, **2013**].

71 [Peletier, **1581**, f. **57**r]: "Tout cet Univers, à pris sa Forme ansamble: / Tous Nombres ont etè, e sont, aussi tót qu'Un: / Matiere, e Forme, e Tout, n'uret principe aucun: / A coup, e an l'instant les poins, qui s'estandiret, / Lignes, Eres, e Cors an l'Infini randiret". See also infra n. **145**. On the origin and signification of the definition of the line as flow of the point, see [Vinel, **2010**]. Concerning the interpretation of this notion by Peletier and by other Renaissance commentators of Euclid, see [Axworthy, **2017**].

It should be noted, with regard to this, that Clavius differed from Peletier in his ontological conception of geometrical objects in the sense that he assumed that mathematical objects do not exist separately from sensible things, although they are considered without matter by the mathematician.[72] In his commentary on the Elements, he actually asserted the structural relation between the level of the sensible and the level of the mathematical in the geometer's apprehension, saying, in the commentary on the definitions, that the mathematician places his objects before his eyes, in particular by the means of a flow or motion[73] of a point, a line or a surface by which he mentally brings his objects about.[74] In this context, this generative motion (or flow) of the point, which is explicitly related to the action of describing or prolonging a straight line in the two first postulates and which has thus a properly constructive meaning, is explicitly described as a local form motion. Nevertheless, as he explicitly and repeatedly specifies that the type of motion the geometer deals with then is imaginary,[75] there would be in this sense no problem, for Clavius, in attributing motion to geometrical figures. It would actually have a crucial function in the teaching of the properties of geometrical objects.[76]

As we will attempt to show by considering both Peletier and Clavius's texts, it would seem that Peletier's rejection of superposition was not motivated by its kinematic implication strictly speaking, but rather by the type of motion it would appeal to. This type of motion, contrary to other types of motion to which Euclid appeals more frequently, as in Prop. I.2 and I.3, would not express the equality of figures in a properly rational and demonstrative manner.

Moreover, Clavius' admission and defence of superposition would not be related to the important place he attributed to motion in his teaching of geometrical notions and geometrical construction processes, but, on the contrary, to the fact that, in propositions appealing to superposition, no motion may, to him, be properly said to take place.

Before considering, according to this perspective, Peletier and Clavius's arguments against and in favour of the legitimacy of superposition in Euclid's Elements, we will give a brief outline of Peletier and Clavius's works and careers as mathematicians, presenting also the place of their commentaries on Euclid in the history of mathematics and in the early modern Euclidean tradition, as well as a chronology of the debate. We will then consider Peletier's assumption of the incompatibility of superposition with the procedures presented in Prop. I.2 and I.3, and the distinction he thereby proposed between the processes that are authorised in geometry and those that are not. We will attempt to determine, thereby and also by considering other parts of his commentary on the Elements, the true grounds for this distinction and hence for his rejection of superposition as a means to demonstrate the equality of figures.In the last part, we will set forth the distinction Clavius made between his and Peletier's respective understandings of superposition in Euclid's Elements, focusing on the arguments he raised against Peletier's

[72] [Clavius, 1589, Prolegomena, 14]: "Quoniam disciplinae Mathematicae de rebus agunt, quae absque ulla materia sensibili considerantur, quamvis re ipsa materiae sint immersae; perspicuum est, eas medium inter Metaphysicam, & naturalem scientiam obtinere locum, si subjectum earum consideremus, ut recte a Proclo probatur. Metaphysices etenim subjectum ab omni est materia sejunctum & re, & ratione: Physices vero subjectum & re, & ratione materia sensibili est conjunctum: Unde cum subjectum Mathematicarum disciplinarum extra omnem materiam consideretur, quamvis re ipsa in ea reperiatur, liquido constat, hoc medium esse inter alia duo". See also [Claessens, 2009].

[73] The two terms are then used synonymously.

[74] [Clavius, 1589, Df. I.5, 33]: "Mathematici vero, ut nobis eam ob oculos ponant, monent, ut intelligamus lineam aliquam in transversum moveri" and [Clavius, 1589, Df. XI.1, 522]: "ut nobis ob oculos ponant corpus, seu solidum, hoc est, quantitatem trina dimensione praeditum, consulunt, ut concipiamus superficiem aliquam aequaliter elevari, sive in transversum moveri".

[75] [Clavius, 1589, Df. I.2, 29]: "ex isto motu imaginario"; [Clavius, 1589, Post. 1, 57]: "fluxus quidam puncti imaginarius"; [Clavius, 1589, Df. I.4, 30]: "fluxum puncti imaginarium".

[76] [Clavius, 1589, Df. I.2, 29]: "Mathematici quoque, ut nobis inculcent veram lineae intelligentiam, imaginantur punctum iam descriptum superiore definitione, e loco in locum moveri."; [Clavius, 1589, Df XI.1, 522]: "Mathematici, ut rectè intelligamus lineam, praecipiunt, ut imaginemur punctum aliquod è loco in locum moveri".

comparison between Prop. I.2–I.3 and Prop. I.4, notably by appealing to the distinction between theorems and problems in Euclidean geometry.

2. The lives and works of Peletier and Clavius

As a mathematician, Jacques Peletier du Mans (Le Mans, 1517 – Paris, 1582) is mostly known for his contributions to the sixteenth-century development of algebra[77] and for his polemic considerations on demonstrations by superposition and on the Euclidean definition of the angle, in his commentary on Euclid's *Elements*[78] and in various treatises relating to the controversy on the angle of contact.[79] Peletier also published treatises on arithmetic[80] and practical geometry.[81] As a member of the poetic circle *La Pléiade*, he contributed to promoting the mathematical disciplines in poetic and humanistic circles, notably through poetic writings in which he displayed the nature and perfection of mathematical knowledge.[82] He worked to promote the use of French as a scientific language,[83] notably through the publication of mathematical treatises in the vernacular, some of which were published in the reformed spelling of the French language he proposed in the *Dialogue de l'ortografe e prononciation françoese*.[84]

Among the numerous commentaries on the *Elements* published in France or by French mathematicians during the sixteenth century,[85] Peletier's commentary on the first six books of the *Elements*, published in Lyon in 1557, stands out by its relatively extensive methodological and philosophical digressions. In this context, Euclid's authority is treated with a greater freedom than in most sixteenth-century commentaries, Peletier not hesitating to challenge his definitions and proofs. He was criticised for this on several occasions by Jean Borrel (besides Clavius),[86] also with regards to his rejection of superposition.[87] Although Peletier's commentary circulated and was read within mathematical circles, as shown by the reactions of his contemporaries to his interpretations of Euclid's teaching, it was not reprinted before 1610.[88] It was however published in French a year later, in 1611, and again in 1628.[89] Hence, it does not seem to have benefited, in the sixteenth century, from the same popularity as other commentaries on the *Elements* published at the time, such as the one by Oronce Fine, published in Paris in 1536, 1544 and 1551,[90] or even Clavius's commentary, which was published seven times from 1574 to 1612.The Jesuit mathematician and professor of mathematics Christoph Clavius was born in Bamberg in Fran-conia in 1538 and taught at the Collegio Romano from 1564 until his death in Rome in 1612. He is mostly

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77 Peletier's works relating to algebra are [Peletier, 1554]. See also [Peletier, 1560]. See [Bosmans, 1907, 117–173]. More recently, [Cifoletti, 1992], [Cifoletti, 1995] and [Cifoletti, 2004] has notably put in evidence the interplay between algebra and rhetoric in Peletier's algebraic works. See also [Loget, 2012].

78 [Peletier, 1557].

79 [Peletier, 1563], [Peletier, 1567] and [Peletier, 1579a].

80 [Peletier, 1545] and [Peletier, 1549]. On Peletier's arithmetic, see [Davis, 1960].

81 [Peletier, 1572] and [Peletier, 1573].

82 See in particular *A ceuls qui blament les Mathematiques*, in [Peletier, 1547, ff. 77v–78v] and *Suite de la Sciance*, in [Peletier, 1581, ff. 51r–62r]. On this aspect of Peletier's poetic work, see notably [Bamforth, 1989, 202–211] and [Schmidt, 1938]. On Peletier's promotion of mathematics, see also his discourse for the inauguration of his chair of mathematics at the university of Poitiers, [Peletier, 1579b].

83 [Cifoletti, 1994] and [Davis, 1960].

84 [Peletier, 1550].

85 Considering altogether editions, commentaries and translations, more than twenty different editions of Euclid's *Elements* by French mathematicians and humanists were published between 1516 and 1602.

86 [Borrel, 1559] and [Borrel, 1562]. On Borrel and Peletier's polemic exchanges, see [Loget, 2000, 131–147].

87 [Borrel, 1559, 219–222] and [Borrel, 1562, 9 and 20].

88 [Peletier, 1610].

89 [Peletier, 1611] and [Peletier, 1628].

90 [Fine, 1536], [Fine, 1544] and [Fine, 1551].

known for his contribution to the reform of the Julian calendar and instauration of the Gregorian calendar[91] and, to a lesser degree, for his defence of the Ptolemaic cosmological model in the postcopernican era.[92] Through the numerous treatises and extensive commentaries he published on the different aspects of mathematics, theoretical and practical,[93] and through his contribution to the elaboration of the Jesuit *ratio studiorum*,[94] he played a crucial role in the promotion and development of mathematical teaching in the pedagogical programs of the Jesuit colleges. His edition and commentary of the fifteen books of the *Elements* (that is, the thirteen books properly attributed to Euclid and the two spurious books XIV and XV), to which he added a sixteenth book based on the treatise on regular polyhedra appended by François de Foix-Candale to his own commentary on the *Elements*,[95] was first published in Rome in 1574.[96] Although it did not display the philological precision of Federico Commandino's translation and commentary, published in Pesaro two years before,[97] this edition was crucial to the early modern tradition of the *Elements* in view of its pedagogical clarity and, more so, of its comprehensiveness, as Clavius not only commented on Euclid's principles and propositions, but compiled also many of the various interpretations set forth by his predecessors.[98] This edition, which was revised and reprinted six times until Clavius's death in 1612,[99] benefited from a great popularity within and beyond Jesuit institutions and remained the reference edition of the *Elements* up to the nineteenth century.[100] Clavius was engaged at different stages of his career in debates against Peletier over the legitimacy of Euclidean definitions and demonstrative procedures, in which he seems to have always positioned himself as a harsh defender of Euclid.

In the framework of the polemic exchanges between Peletier and Clavius, the debate over superposition is ancillary to the more widely known debate over the status of the angle of contact,[101] which dealt with the legitimacy of Euclid's quantitative treatment of the horn angle, that is, the angle contained by the circumference of a circle and one of its tangent lines, given its non-comparability with the rectilinear angle.[102] On this issue, Peletier raised, in his commentary on the *Elements*, objections against Euclid's notion of angle of contact, and more fundamentally, against his definition of angle as the mutual inclination of two lines, by stating that only angles resulting from the intersection of two lines and which are mutually compara-ble in quantity can be appropriately regarded as angles, which is not the case of the angle of contact such as defined by Euclid.[103] Peletier reasserted these objections in 1579, in the *Apologia In Christophorum*

91 [Clavius, 1588] and [Clavius, 1603b]. On Clavius and the reform of the calendar, see [Baldini, 1983], [Dutka, 1988], [Oñate Guillen, 2000] and [Bien, 2007].

92 [Lattis, 1994]. See also [Jardine, 1979].

93 Among Clavius's mathematical works, all published together in [Clavius, 1611–1612], see [Clavius, 1570], [Clavius, 1574], [Clavius, 1586], [Clavius, 1604], [Clavius, 1607c] and [Clavius, 1608].

94 *Ordo servandus in addiscendis disciplinis mathematicis, modus quo disciplinæ mathematicæ in scholis Societatis possent promoveri, De re mathematica instructio* and *Oratio de modo promovendi in Societate studia linguarum politioresque litteras ac mathematicas*, written between 1580–1593 [Lukacs, 1965–1992]. On Clavius's contribution to the *ratio studiorum*, see [Romano, 1999, 94–132 and 614–617], [Smolarski, 2002], [Gatto, 2006] and [Rommevaux, 2012].

95 [Candale, 1566, ff. 192r–201v].

96 [Clavius, 1574].

97 [Commandino, 1572].

98 [De Risi, 2016, 598].

99 [Clavius, 1589], [Clavius, 1591], [Clavius, 1603a], [Clavius, 1607a], [Clavius, 1607b] and [Clavius, 1611–1612].

100 [De Risi, 2016, 598].

101 On this debate, and its ulterior developments, see [Maierù, 1991], [Loget, 2000, 165–280], [Loget, 2002], [Rommevaux, 2006]. See also [Feldhay, 1998, 124–125].

102 [Euclid, 1956, 2, Prop. III.16, 37]: "The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed; further the angle of the semicircle is greater, and *the remaining angle less*, *than any acute rectilinear angle*" (our emphasis).

103 [Peletier, 1557, Prop. III.16, 76–78].

Clavium De contactu linearum,[104] where he condemned Clavius for having appealed to superposition in order to defend Euclid's notion of angle of contact in his 1574 commentary on the *Elements*,[105] explicitly referring to the arguments he presented against superposition in his own commentary on Euclid.[106] Clavius reacted to the *Apologia* in 1586, within his commentary on the *Spherics* of Theodosius.[107] The discourse he presented then, which aimed to defend not only Euclid's notion of angle of contact, but also his use of superposition, was later taken up in the 1589 edition[108] and in later editions of his commentary on Euclid's *Elements*.

3. Peletier's arguments against superposition

With regards to his allegation of the mechanical character of superposition, Peletier does not specify in which sense he then takes the term "mechanical", but he opposes *superponere* to *intelligere*, stating that: "To superpose figures on figures is something mechanical, but to perceive by the intellect only is mathematical".[109] In saying this, Peletier seems to point here to the empirical character of the demonstration, and to the fact that it appeals to the senses rather than to the intellect.François de Foix-Candale, in his 1566 commentary on the *Elements*,[110] also describes superposition as mechanical, as he says that, such as demonstrated by Campanus and Theon, the demonstration of the

104 [Peletier, 1579a].

105 [Clavius, 1574, f. 114r]: "Quod autem anguli contactus sint inaequales inter se, et non omnes aequales, ut vult Peletarius, similiter et anguli semicirculorum, ex eo manifestum est, quod angulus quilibet consistit in unico puncto, et linearum inclinatione, quae non in directum jacent, ut constat ex anguli plani definitione. Hinc enim fit, ut aequalitas angulorum ejusdem generis requirat eandem inclinationem linearum, ita ut lineae unius conveniant omnino lineis alterius si unus alteri superponatur. Ea enim æqualia sunt, quæ sibi mutuo congruunt, iuxta 8 pronunciatum. Cum igitur in angulis contactus, nec non in angulis semicirculorum, nequaquam reperiatur semper eadem inclinatio, quod (uno superposito alteri) lineæ eorum non sibi respondeant, sed prorsus inter se dissideant, ceu ex figuris superioribus perspicuum est [. . .]".

106 [Peletier, 1579a, ff. 6r–v]: "Hic etiam se torquet Clavius, ut probet angulos Contactus alios aliis esse inaequaleis, sic scribens, *Hinc fit*, inquit, *ut aequalitas angulorum eiusdem generis requirat eandem inclinationem linearum, ita ut lineae unius conveniant omnino lineis alterius, si unus alteri superponatur. Ea enim aequalia sunt, quae sibi mutuò congruunt, iuxta 8 Pronuntiatum.* [. . .] De Superpositione verò, quàm hic Clavius sibi adhibet, nihil ampliùs huc nobis reponendum est, quàm quod in quartam Propositionem libri Primi [Euclidis elementorum] disputavimus".

107 [Clavius, 1586, 343–344].

108 [Clavius, 1589, 368–370].

109 [Peletier, 1557, Prop. I.4, 16]: "Figuras Figuris superponere, Mechanicum quippiam esse: intelligere verò, id demùm esse Mathematicum". It may be noted that the use of the verb *intelligere* to describe the mathematical mode of study of figures and to distinguish it from the operation supposedly involved in the process of superposition (*superponere*) are significant, since for Clavius, as we will see, this verb actually describes the manner in which mathematicians make use of superposition (see *infra*, n. 172).

110 [Candale, 1566, Prop. I.4, f. 5v]: "Alteram demonstrationem huic quartae exhibere cogimur, ne praebeatur aditus, quo ulla mechanicorum usuum instrumenta in demonstrationes incidant. Nam Campanus ac Theon hanc demonstrantes, triangulum triangulo superponunt, angulumque angulo, sive latus lateri, demonstrationem potius instrumento palpantes, quàm ratione firmantes: quod tanquam prorsus alienum à vero disciplinarum cultu reijcientes, aliam demonstrationem absque figurae, anguli seu lineae transpositione, protulimus ratione elucidatam". See also [Mancosu, 1996, 30]. Cf. [Candale, 1566, Prop. I.8, f. 6v]: "Huius al-teram demonstrationis partem resecavimus eò quòd trianguli transpositione uteretur, quod quidem moechanicum spectat negotium à vera mathesi alienum, posita anguli qui ad z hypothesi ex quarta huius sumpta." and [Candale, 1566, Prop. III.24, f. 27r]: "Quo-niam Theon & Campanus hanc demonstrare conati sunt, aut hi à quibus demonstrationes sumpserunt, instrumento ferè mechanico, nempe coaptata figura supra figuram, quod indignum traditione mathematica supramodum existimatur. [. . .] Campanus verò unam sectionem per puram alterius superpositionem, tanquam instrumento mechanico metitur ut aequalem probet, quod esse argumentum verè mechanicum, patet. [. . .] Quare non intelligit figuras superponendas figuris ut aequales aut inaequales percipiantur, sed figurarum aut aliorum quorumvis subiectorum quantitates, ratiocinante argumento convenire cognitae, adinvicem sibimetipsis illae quantitates aequales dicentur, non autem quae experimento congruere palpantur, illae aequales dici debeant. Mathesis enim ex praeassumptis certis necessariò concludit, non autem ex sensibus externis praxim operantibus saepius fallacem".

equality of the two triangles in Prop. I.4 is carried out by moving and superposing lines and figures by the means of instruments rather than by rational means, that is, by deducing the conclusion from necessary principles. Looking at the editions of Euclid by Campanus and Bartolomeo Zamberti (who attributed the proofs of Euclid's propositions to Theon of Alexandria),[111] it is not clear how Candale conceived this process as carried out instrumentally, whether it is in the sense proposed by G. Ingrami,[112] who, as noted by Heath, proposed to establish the congruence of two triangles by the folding and cutting of sheets of paper, or by another means, such as the use of geometrical tools to measure lines and angles.

Peletier does not mention the use of instruments or the involvement of the external senses and, as we will see, it is not necessary to conceive this process as physically accomplished through a concrete manipulation of material objects, as Candale's description suggests. As we will see, the reliance of certain Euclidean demonstrations on the constructions of figures, which may be represented by constructions produced by the means of the compass and straightedge, are actually considered by Peletier as properly geometrical procedures, unlike superposition.[113]

In Peletier's text, "mechanical" may refer to the empirical process of deducing the coincidence of the two figures by looking at the diagram[114] and/or by simply representing in the imagination the superposition of the two triangles, while leaving aside all logical means of deduction. In relation to this interpretation, it is conceivable that "mechanical" then refers to the kinematic implications of this procedure, as was considered by P. Mancosu.[115] In any case, *intelligere*, which is then opposed to the process of superposing figures, clearly seems to refer, as will be expressed later by Candale, to the act of rationally deducing the conclusion of the demonstration from previously admitted principles, which represented at the time the necessary path to reach certainty in any science and which origin Peletier clearly situated in geometry.[116]

Now, in order to prove that superposition is not a properly mathematical mode of demonstration, Peletier argued that, although Euclid introduced it in three of his propositions (Prop. I.4, I.8 and III.24), he would not have himself considered it as a fully admissible procedure in geometry, as he would have otherwise appealed to it in many other occasions, starting with Prop. I.2 and I.3. In these two propositions, which require one to construct a line equal to a given line at a given point[117] and to cut off a line equal to a given line from one that is greater,[118] it would have indeed sufficed, if superposition were truly admitted

111 [Euclid, 1516, ff. 6v–7r].

112 [Ingrami, 1904, 66], quoted in [Euclid, 1956, 1, 227–228].

113 Indeed, the fact of transposing a figure onto another is not authorised by any postulate, contrary to the fact of drawing and prolonging a straight line and drawing a circle of any centre and of any size.

114 This is how [Palmieri, 2009, 476] and [Vitrac, 2005, 50] interpreted Peletier's understanding of proofs by superposition.

115 [Mancosu, 1996, 29].

116 [Peletier, 1557, 12]: "Demonstrationem verò appellant Dialectici, Syllogismum qui faciat scire: nempè qui ex probatissimis concludat. Atque hæc à Geometria ortum habet. Immò omnis quæ ad verùm perducit probatio, Geometrica est".

117 [Euclid, 1956, 1, Prop. I.2, 244]: "*To place at a given point* (*as an extremity*) *a straight line equal to a given straight line*. Let A be the given point, and BC the given straight line. Thus it is required to place at the point A (as an extremity) a straight line equal to the given straight line BC. From the point A to the point B let the straight line AB be joined; [Post. 1] and on it let the equilateral triangle DAB be constructed. [I. 1] Let the straight lines AE, BF be produced in a straight line with DA, DB; [Post. 2] with centre B and distance BC let the circle CGH be described; [Post. 3] and again, with centre D and distance DG let the circle GKL be described. [Post. 3] Then, since the point B is the centre of the circle CGH, BC is equal to BG. Again, since the point D is the centre of the circle GKL, DL is equal to DG. And in these DA is equal to DB; therefore the remainder AL is equal to the remainder BG. [C.N. 3] But BC was also proved equal to BG; therefore each of the straight lines AL, BC is equal to BG. And things which are equal to the same thing are also equal to one another; [C.N. 1] therefore AL is also equal to BC. Therefore at the given point A the straight line AL is placed equal to the given straight line BC. (Being) what it was required to do".

118 [Euclid, 1956, 1, Prop. I.3, 246]: "*Given two unequal straight lines, to cut off from the greater a straight line equal to the less*. Let AB, C be the two given unequal straight lines, and let AB be the greater of them. Thus it is required to cut off from AB the greater a straight line equal to C the less. At the point A let AD be placed equal to the straight line C; [I. 2] and with centre A and distance AD let the circle DEF be described. [Post. 3] Now, since the point A is the centre of the circle DEF, AE is equal

in geometry, to move the given line onto the given point (Prop. I.2) and to transport the shorter line onto the longer line in order to cut off the part that exceeds its length (Prop. I.3). In saying this, he did not only want to prove the incompatibility of Prop. I.4 with Prop. I.2 and I.3, but he also wanted to illustrate the mechanical, and thus non-geometrical, character of this procedure. Indeed, as Peletier says, just after presenting Euclid's demonstration of Prop. I.4:

> This demonstration is the common demonstration among all the commentators, if only it can be called a demonstration. For if we admit in the proof the superposition of lines and figures, nearly all of geometry will be full of this kind of application, and there would barely be a proposition that could not be proved by means of this method. For already the second and the third Propositions, which we have demonstrated, could have been proved this way. Because if a line equal to a given line should be drawn at a given point, our duty will have been completed as soon as the line will have been transported to the said point. Although application would be somewhat more tolerable than superposition, it is however refused in geometry.[119] Indeed, it is surely not allowed to transport a line, in order to describe a circle according to its magnitude [Prop. I.3], if a line equal to it has not first been drawn [Prop. I.2]. Otherwise, the second proposition would be completely useless. For then if a shorter line was to be cut off from a longer line, what else would we be doing than superposing the shorter on the longer, so as to cut off what is in excess? But I leave to those who understand the strength and power of the demonstration to judge how far this is from the dignity of geometry.[120]

As suggested here by Peletier, if superposition were authorised in geometry, instead of drawing a new line equal to the given line at a given point in Prop. I.2 through the construction of circles (Figure 1), the line should be carried out through a direct transfer of the line to the given point (Figure 2).
Also, in Prop. I.3, instead of drawing a new line equal to the given shorter line at one extremity of the longer line and to cut out, by means of a circle from the common extremity of the two lines, a segment equal to the shorter line on the longer line (Figure 3), the shorter line should be directly transported and placed onto the longer line (Figure 4).

This confirms that the demonstration of the congruence of the two triangles, in Prop. I.4, is conceived by Peletier as carried out through the local transport and superposition of one of the two compared figures on the other (Figure 5).
It should be noted that the appeal to superposition in Prop. I.2, which Peletier here envisages hypothetically, does not however seem to be regarded as equivalent to the operation of superimposing a figure onto another, but rather to what he calls here an *application*, although the term *applicare* and *superponere* were

to AD. [Def. 15] But C is also equal to AD. Therefore each of the straight lines AE, C is equal to AD; so that AE is also equal to C. [C.N. 1] Therefore, given the two straight lines AB, C, from AB the greater AE has been cut off equal to C the less. (Being) what it was required to do".

119 We distinguish here our translation of the term *applicatio* in this text from its paraphrase in [Mancosu, 1996, 29], who indirectly quotes the sentence "Applicatio verò quanvis superpositione sit tolerabilior" in this manner: "although this can be allowed in applications", suggesting that superposition would be allowed in mechanical arts. However, the term *applicatio*, as it is in the nominative, would rather designate an operation compared with superposition, which is here in the ablative.

120 [Peletier, 1557, Prop. I.4, 15]: "Haec est vulgata omnium Interpretum Demonstratio, si modò haec Demonstratio dici debeat. Nam si linearum figurarumque superpositiones in probationem recipiamus, tota ferè Geometria hujusmodi applicationibus erit referta: vixque ulla occurret Propositio, quae hac ratione non possit probari. Secunda enim iam indè ac tertia, quas modò demon-stravimus, sic probari poterant. Nam si ad datum punctum, linea datae lineae aequalis ducenda sit: illicò translata linea ad ipsum punctum, absolutum erit negotium: Applicatio verò quanvis superpositione sit tolerabilior, tamen in Geometria repudiatur: immò ne lineam quidem transportare licet, ut secundùm ipsius magnitudinem, Circulum describamus: quin prius aequalis linea ducta sit. Alioqui secunda prorsus vacaret. Tum si à maiori linea, minor sit abscindenda: quid aliud quàm maiori minorem superponemus, ut quod superat resecemus? Sed hoc quàm sit à Geometriae dignitate alienum, eorum judicio relinquo qui Demonstrationis vim & energiam animo concipiunt".

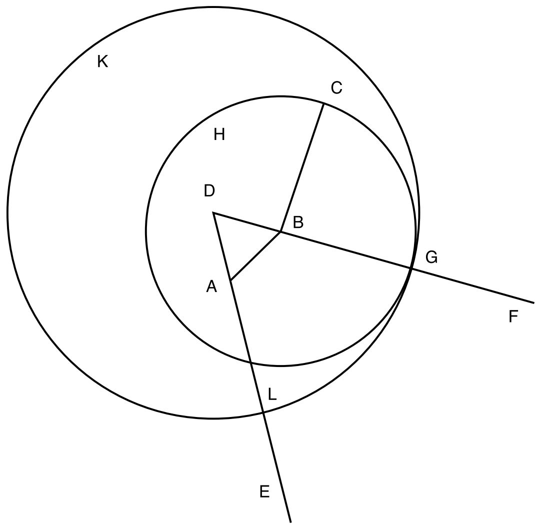

Figure 1. Prop. I.2: "To place at a given point (as an extremity) a straight line equal to a given straight line".

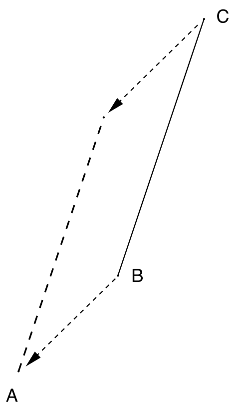

Figure 2. The construction in Prop. I.2 interpreted as produced through the direct transfer and application of the segment BC to the point A.

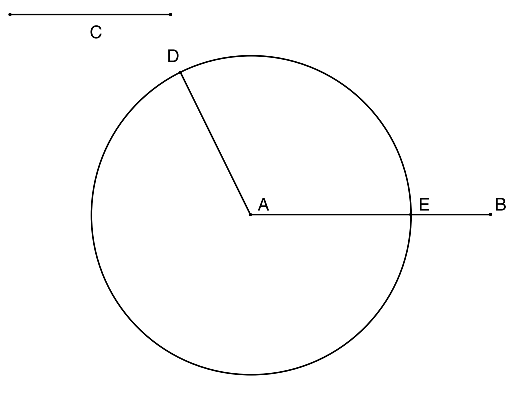

Figure 3. Prop. I.3: "Given two unequal straight lines, to cut off from a greater straight line a line equal to the less."

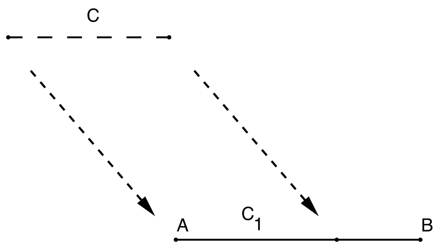

Figure 4. The construction in Prop. I.3 interpreted as produced through the direct transfer and superposition of the shorter segment C onto the longer segment AB.

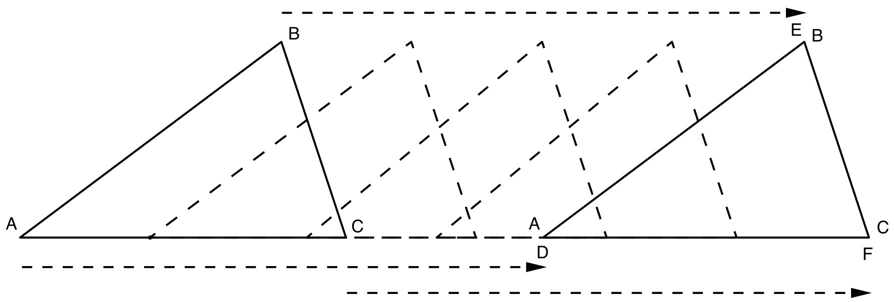

Figure 5. The direct motion and superposition of triangle ABC onto the triangle DEF in Prop. I.4.

presented as equivalent in certain contexts, as in Medieval commentaries on the *Elements*.[121] In Prop. I.2 therefore, there would be no superposition properly speaking, apart from that of extremities.[122] *Applicare* could also correspond to the fact of placing a point, a line or an angle, as opposed to a complete figure, onto a given point, line or angle and could be meant as a translation of τίθεσθαι in Euclid's Prop. I.4. Such a use of the term *applicare* would in any case not correspond to the use that is made of this term in the theory of application of areas, as illustrated by Euclid's Prop. I.44, as well as by certain propositions of Book VI (e.g. Prop. VI.28 and VI.29), and which was later applied by Apollonius to his theory of conic sections.[123]

[121] [Murdoch, 1964, 417–420].

[122] Peletier's comparison of *applicatio* and *superpositio* might be a reference to the medieval translations of the *Elements*, and notably to the so-called Adelard III version, in which application and superposition are presented as two procedures involved in the establishment of the congruence of two figures [Murdoch, 1964, 420–421]. However, while the medieval translations tended to assimilate the two procedures, Peletier clearly distinguished them, at least in so far as application would not imply the overlapping of the whole figure onto the other, but simply the superposition of one of its extremities onto a given point.

[123] In the context of the *Elements*, as illustrated by Prop. I.44, the notion of application designates the fact of placing or constructing a parallelogram equal to the area of a given figure onto a given length (and with an angle equal to a given angle), the side of the parallelogram perfectly coinciding with the given length. As explained by [Proclus, 1873, 419–421], this situation is to be distinguished from the situation when the side of the parallelogram equal to a given figure exceeds the given segment in length or on the contrary is surpassed in length by this given segment. In Apollonius's *Conics*, the distinction between these various cases is used to name and set out certain properties of the three main conic sections, the parabola (παραβολή, derived from παραβάλλειν, which means "to apply"), the hyperbola (ὑπερβολή, from ὑπερβάλλειν: "to exceed") and the ellipse (ἔλλειψις, from ἐλλείπω: "to fall short"). Using the term "ordinate" to designate the segment that joins perpendicularly any point of the curve to its diameter and the term "abscissa" to designate the interval between the ordinate and the tangent to the diameter (anticipating the notions of ordinate and abscissa used in the Cartesian coordinate system), the three curves are distinguished by the way the rectangle on the abscissa, equal in area to the square on the ordinate, relates, as for its base, to a determinate length on the tangent to the diameter called the *latus rectum* or the parameter. The parabola is the curve for which the rectangle on the abscissa and equal to the square on the ordinate has its base equal to the parameter; the hyperbola, the curve for which the base of the rectangle equal to the square exceeds the parameter; and the ellipse, the curve for which the base of the rectangle equal to the square is inferior to the parameter. On this topic, see Heath, in [Euclid, 1956, 1, 343–345].

Now, although Peletier concedes that application would be more tolerable than superposition, he says that it would not either be accepted in geometry. Indeed, as he explains it, if we admit, in Prop. I.3, that line C (the shorter line) can be simply moved onto point A (which corresponds to one of the two extremities of the greater line AB) in order to produce a circle around A that cuts the greater line AB according to its length (Figure 4) and, this, in the same manner as, in Prop. I.4, the first triangle would be moved towards and onto the other (Figure 5), then Prop. I.2, which demonstrates how to construct a line equal to a given line from a given point, would be completely superfluous.

On the contrary, Prop. I.2, in which the construction of a straight line equal to a given line from a given point fundamentally depends on the construction of circles (each circle allowing one to construct a line equal to another given line at a different place from a common point) (Figure 6), is precisely here to show, as it appears here from Peletier's discourse, that a line, or any magnitude, cannot in geometry be freely moved or transported from one place to another. It shows that the change of position of the line-segment needs to be caused and determined by a motion that allows one to guarantee or rather to demonstrate that this segment remains identical to itself in spite of this change of position, which amounts to ascertaining that we are then dealing with a motion without deformation or motion of rigid figures. This motion would thus allow one to demonstrate in a rational manner the relation of equality between the line which is given at the start and the line equal to it and which is constructed at a different place, since then the construction of a line equal to another at a different place would be understood as resulting from the transfer of a single line from one position to another by the intermediary of circles, as is suggested by Euclid's constructions.

Now since, in Prop. I.2 and I.3, the transfer of the line (or construction of a line equal to another at a different place) depends on the motion implied by the generation of the circle (as suggested by Figures 6 and 7), it can be considered as rationally determinable and therefore as able to demonstrate the equality of the two lines (the one given and the one, identical to it, constructed at a given point) in a properly geometrical manner, that is, through a procedure which appropriately displays (intellectually or rationally, rather than empirically) that the construction has been carried out in conformity with the principles of geometry and that the constructed figure possesses the required properties and relations.

As Peletier stated it in his commentary on Df. I.16, the motion of the line-segment which generates the circle would indeed perfectly express the essential properties of the circle, that is, the fact of having all the lines drawn from its centre to its circumference equal to each other, since all these could then be interpreted

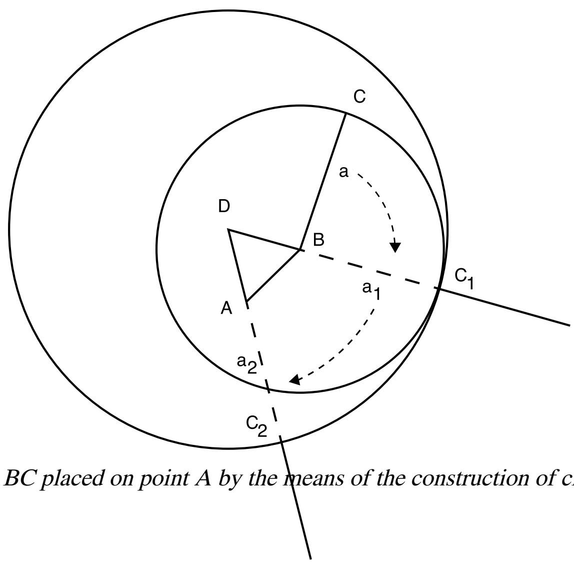

Figure 6. The transfer of the segment BC placed on point A by the means of the construction of circles in Prop. I.2.

This text was published on page 18 of the article "The Debate between Peletier and Clavius on Superposition.".

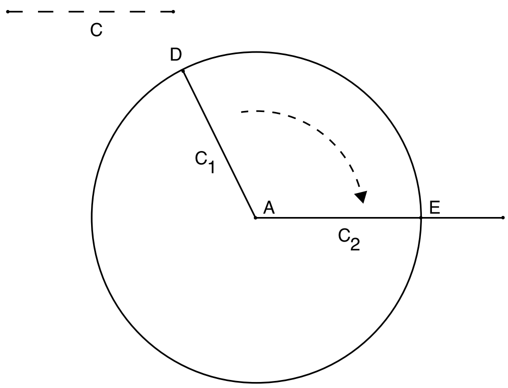

Figure 7. The section of the greater line AB by the length of the shorter line C through its rotation around the extremity A of the line AB in Prop. I.3.

as different traces (*vestigia*) or marks left by one single line at different points of its rotation around one of its extremities remained fixed.

The circle is a plane figure contained by one line called the periphery, which is such that all the lines extended to it from a single point existing within it are equal. Now, this point is called the centre of the circle. This is the most common definition of the circle, which explains its disposition or, as they say, its accident. However, if someone wished to be taught the mode of production or generation of the circle, as does the definition of the sphere that will be given in Euclid's Book XI, it will be as such: "The circle is the trace of the straight line carried around in the plane, one of its extremities remaining fixed, until it reaches the point where it started to be drawn". Just as, if we started to carry the line AB around the point A by the point B, through the points C, D and E, until it is made the same line AB again, the circle BCDE will be described. *And from this description, is perfectly expressed the entire property of the circle*. Indeed, the fixed point A will be called the centre, while the trace circumscribed by the mobile point B will be called the periphery. Finally, the entire line EB, which is carried around, describes the surface that is called circle. *Wherefrom, it clearly appears that all the lines that proceed from the centre of the circle are equal, since they are made by the trace of one single line*.[124]

The generation of the circle through the motion of a line-segment around one of its fixed extremities would therefore allow one to display in a rational and properly geometrical manner the relation of equality between two lines, since both are thereby demonstrated to be radii of the same circle. As opposed to this, the motion involved in the superposition of figures would, for Peletier, not take place in a determined and measurable spatial extension, such as that which is occupied by the resulting circle, or rather by the interval between the circle's centre and its periphery, which allows one to demonstrate that the line remains identical to itself by the very fact that its motion results in a circle. Indeed, although it is the motion of the line that produces the circle, and therefore delimits the extension and spatial properties of the circle, the

[124] [Peletier, 1557, Df. I. 15–16, 6]: "Circulus, est Figura plana, una linea contenta quae Peripheria appellatur: ad quam ab uno puncto introrsùm existente omnes porrectae lineae sunt aequales. Punctum autem illud, Centrum Circuli vocatur. Hæc Circuli definitio notissima est: quae ipsius affectionem, seu, ut dicunt, passionem explicat. Siquis verò factionem seu creationem Circuli sibi exponi petat, instar Definitionis Sphaerae quam Euclides libro undecimo daturus est: ea erit hujusmodi. Circulus, est vestigium lineae rectae in plano circunductae, altero extremorum manente fixo, donec ipsa unde duci coepit, redierit. Ut, si linea *ab* super *a* puncto duci incipiat in orbem à puncto *b*, per *c*, *d*, & *e* puncta, donec ipsa rursus *ab* facta sit: descriptus erit Circulus *bcde*. *Atque ex hac descriptione, graphicè exprimitur tota Circuli proprietas*. Punctum enim illud fixum *a*, Centrum dicetur: vestigium verò à puncto *b* mobili circunscriptum, Peripheria. Tota demùm linea *eb* circunducta, Superficiem describit quae Circulus dicitur. *Unde manifestum est omnes lineas à centro Circuli exeuntes, aequales esse: quum sint ex unius lineae vestigio*" (our emphasis).

This text was published on page 19 of the article "The Debate between Peletier and Clavius on Superposition.".

very fact that the produced figure can be defined as a circle guarantees that its parts are all equal, since a circle is essentially defined, as it is the case in Euclid's *Elements*, as a "plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure equal one another".[125] As opposed to the tracing of circles, as in the cases of Prop. I.2 and I.3, superposition would be unable to rationally demonstrate the equality, and in a sense the identity, of the parts of the two compared figures and hence of the two compared figures themselves. It may moreover be added, although it is not a point made here by Peletier, that, unlike the production of circles in geometry, which is legitimated by the third postulate,[126] superposition and the free transfer of figures it implies would not be authorised by any postulate.[127]

Although Peletier acknowledges that Prop. I.4 does not have the same status in Euclid's geometry as Prop. I.2 and I.3 – as he does say that Prop. I.4 is the very first theorem of the *Elements* –,[128] it appears that, through the comparison between the motion supposedly appealed to in Prop. I.4 and the local transfer of lines implied by the kinematic understanding of Prop. I.2 and I.3, Peletier considered himself entitled to condemn and reject the use of superposition in the *Elements* because he assumed that Euclid would have used it in Prop. I.2 and I.3 in the same manner as he used it in Prop. I.4, had he conceived this process as truly admissible in geometry.[129] In other words, for Peletier, the fact that Euclid did not appeal to this procedure in Prop. I.2 and I.3 would prove that Euclid himself did not consider it as a properly admissible geometrical procedure, contrary to the constructions depending on the production of circles in Prop. I.2 and I.3.

To explain why Euclid nevertheless made use of this mode of demonstration in Prop. I.4, I.8 and III.24, Peletier set forth several hypotheses. One of them is that this proposition would have been too complex to be placed among the principles,[130] although, to Peletier, it should really have been formulated as a definition of equal angles.[131] Indeed, as he says it then:

Nobody would explain the equality of angles more meaningfully than if they said that two angles are equal when the two sides that contain one angle are equal to the two sides that contain another, and when the bases that connect the two sides are equal. For it is certain that the angle is as wide as is the opening or

[125] [Euclid, 1956, 1, Df. I.15, 183].

[126] [Euclid, 1956, 1, Post. 1, 199]: "To describe a circle with any center and radius".

[127] As shown by [De Risi, 2016, 632–633], the free movement of rigid figures will be integrated in Euclidean axiomatics in the seventeenth century.

[128] [Peletier, 1557, Prop. I.4, 16]: "Huc accedit, quòd primum Theorema facile, perspicuum, ac sensui obvium esse debebat".

[129] This evokes the thesis, held by Heath among others, that Euclid disliked the method and tried to avoid it as much as he could. On this thesis, see *supra*, n. 51.

[130] [Peletier, 1557, Prop. I.4, 16]: "Quur ergo Euclides hoc inter Theoremata reposuit, non inter Principia praemisit? Nimirùm, quum speciem quodammodo mixtam Principii & Theorematis prae se ferret: Principij, quòd in communi animi judicio consisteret: Theorematis, quòd speciatim Triangula Triangulis comparanda proponeret: maluit Euclides inter Theoremata referre: praesertim quum multa haberet capita, Principium verò simplex ac velut nudum esse debeat. Ex hoc praetereà Axiomate tanquam ex locu-pletissimo Demonstrationum themate, multae Propositiones consequi debebant, eiusdem propè facilitatis & judicii: quas, quia erant notissimae, inter Principia annumerari non conveniebat. Paucis enim Principiis Geometriam contentam esse oportebat: immò multa Principia consultò supprimuntur, ne sit onerosa multitudo: ut etiam quae exprimuntur, tantùm ad exemplum exprimi videantur".

[131] [Peletier, 1557, Prop. I.4, 15–16]: "Quid ergo huc afferemus, ut Euclidem à reprehensione vindicemus? Neque enim ex tam paucis quas hactenus praemisit, Propositionibus, hoc Theorema confirmari posse videtur. Huic objectationi, meo judicio, sic occurri poterit: ut dicamus, hoc Theorema per se clarum esse, neque probatione egere: sed Definitionis cujusdam loco habendum esse. Nam quum de re aliqua sermonem instituimus: ea nobis tacitè per definitionem subit in animum: Non enim duos angulos aequales esse cogitabo, nisi quid sit aequales esse angulos concipiam. Quod respiciens Euclides, angulorum aequalitatem proponere, atque eadem opera definire voluit: ut hoc Theorema pro Definitione haberemus". *Cf.* [Peletier, 1557, Prop. I.8, 19]: "Hanc demonstrandi rationem in quarta hujus abundè refutavimus. Quare haec Propositio tanquam per se nota habenda est. Quis enim negaverit duas Superficies esse aequales, quarum latera & quantitate & numero sunt aequalia? vel ea demonstrabimus ratione quam illic tradidimus". See [Loget, 2000, 177].

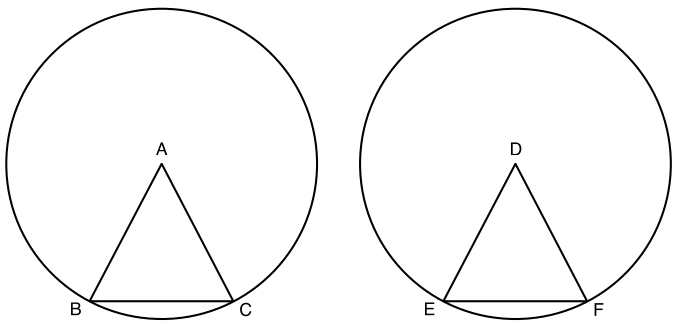

Figure 8. Peletier's tentative definition of the equality of angles.

expansion of the lines which contain it and that this opening is as wide as is the base or the line which connects them.[132]

This tentative definition of the equality of angles, which involves the consideration of the base joining the two legs of the triangle (or opposite side of the known angle) (Figure 8) and which, as such, seems only applicable to the comparison of rectilinear angles less than 180°, does not appear to be sufficiently general to correspond to a properly geometrical definition of the angle. Peletier in any case did not propose to add this definition to the Euclidean definitions in his edition of the *Elements*.

Another reason why Euclid would have appealed to superposition in the *Elements* in spite of the fact that he did not, according to Peletier, himself consider it as an acceptable mode of proof, is that Euclid would have wanted to start his geometrical teaching with a first theorem that depended on a more intuitive form of judgement, namely, one that appeals to the senses, and from which the geometer would be able thereafter to progressively reach more abstract demonstrations.[133]

Although, to Peletier, Prop. I.4 should have been placed among the definitions, he nevertheless attempted to give an alternative demonstration of it, which would, to him, display a "much more tolerable" use of superposition and which would depend on the construction of circles and on the comparison of the sides of the given figures to radii of circles.

If for any reason this superposition should be admitted, it will be much more tolerable in the following manner. Remaining with the constitution of the two triangles ABC and DEF, I will continue ED up to point G, by the first Postulate, and I will place DG equal to AB, by the second Proposition. And having also continued FD, I will place DH equal to AC. Then, on point D, I will describe two circles, one of radius DG, the other of radius DH. The first of these will evidently transit through point E, since DE and DG are equal, and the other, through point F, for the same reason. Now, from point D, I draw a straight line DL to point E, which will undoubtedly pass over DE. For if it transited beyond it, as in the case of DML or DNL, two straight lines would enclose a surface, against the last Common Notion. In a similar manner, from the same point D, I will draw the line DK, which will likewise be made one with line DF. But finally, line LK having been drawn, it will be made one with line EF. But now it clearly appears that line DL is equal to line DG and, on that account, to AB, by the construction and the Common Notion, and also that DK is equal to line

[132] [Peletier, 1557, Prop. I.4, 16]: "Nemo enim significantius explicabit angulorum aequalitatem, quàm si dixerit duos angulos aequales fieri, quum duo latera unum angulum continentia, duobus alterum angulum continentibus fiunt aequalia, & bases quae latera connectunt, æquales. Constat enim angulum tantum esse, quanta est duarum linearum ipsum continentium apertio, seu diductio, hanc verò tantam esse, quanta est basis, hoc est, linea ipsas connectens". See [Loget, 2000, 172] and [Palmieri, 2009, 475].

[133] [Peletier, 1557, Prop. I.4, p. 16]: "Hûc accedit, quòd primum Theorema facile, perspicuum, ac sensui obvium esse debebat, pro geometriae lege, quae ex parvis humilibusque initiis, in progressus mirabiles sese extollit".

This text was published on page 21 of the article "The Debate between Peletier and Clavius on Superposition.".

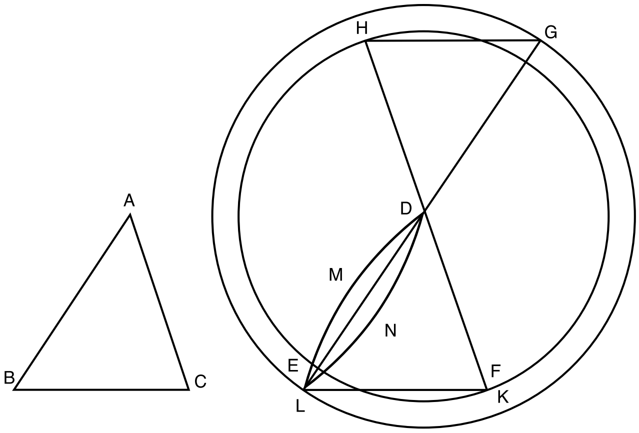

Figure 9. Peletier's alternative demonstration of Prop. I.4.

DH, or to AC, and that angle LDK is equal to angle DEF, or rather is the same angle, and therefore is equal to angle BAC; and that the space contained by the lines DL and DK is perfectly equal to the space contained by the lines AB and AC. But the space LDK is enclosed by a line equal to line EF. Therefore, EF will be equal to BC, which is what we were required to demonstrate. From this, the other parts of the theorem are made evident, which is the equality of the remaining angles and of the two triangles. And we should not believe that, in both methods, there is always the same application of triangles, as it is different to transpose triangles and to demonstrate by the similar and the equal. For this last proof depends on the capacity of the circle.[134]

Peletier's alternative demonstration of Prop. I.4 (Figure 9) thus invites one to construct lines DG and DH respectively equal to the sides AB and AC of the first triangle in the continuation of the sides DE and DF of the second triangle from point D and then to draw circles of radii DG and DH. The following step is then to draw the segments DL and DK from D to E and D to F respectively, establishing by the last Common Notion (last in Peletier's edition), "Two straight lines cannot enclose a space"[135] (as was done in Prop. I.4 through the interpolated passage referring to this Common Notion),[136] that they coincide

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[134] [Peletier, 1557, Prop. I.4, 16–17]: "At si haec superpositio aliqua ratione admittenda sit: tolerabilior sanè fuerit hoc qui sequitur modo. Manente duorum Triangularum ABC & DEF conditione, continuabo ED usque ad G punctum, per primam Petitionem: & ponam DG aequalem AB, per secundum Propositionem. Atque itidem continuata FD, ponam DH aequalem AC. Tum super puncto D, ducam duos Circulos: alterum spatio DG, alterum spatio DH. Quorum prior manifestò transit per punctum E, quum sint DE & DG aequales: alter verò per punctum F, ob eandem rationem. Iam à puncto D duco lineam rectam DL ad E punctum: quae omninò transibit super DE. Nam si extrà transeat ut DML aut DNL: duae rectae lineae concludent superficiem, contra ultimam animi Notionem. Itidem ab eodem D puncto, ducam lineam DK: quae etiam efficietur eadem cum linea DF. Ac demùm Linea LK ducta efficietur eadem cum linea EF. Iam verò manifestum est lineam DL esse aequalem lineae DG, ac proptereà ipsis AB, ex constructione & animi Notione: lineam quoque DK esse aequalem DH, seu AC: atque angulum LDK esse aequalem angulo DEF, immò eundem: ac proptereà aequalem angulo BAC: spatiumque comprehensum à lineis DL & DK, esse omninò aequale spatio comprehenso à lineis AB & AC. At spatium LDK claudetur linea aequali ipsi EF lineae. Quare aequalis EF ipsi BC, Quod erat demonstrandum. Hinc patent reliqua Theorematis capita: nempè reliquorum angulorum inter se, & duorum Triangulorum aequa-litas. Neque est quòd contendat quis, eandem esse utrobique rationem applicationis Triangulorum. Aliud nanque est, Triangula transponere, quàm per simile & aequale demonstrare. Probatio enim haec ultima è Circuli pendet officio".

[135] In Peletier's edition [Peletier, 1557, 11]: "Duae rectae lineae Superficiem non concludunt".

[136] [Euclid, 1956, 1, 248]: "[For if, when B coincides with E and C with F, the base BC does not coincide with the base EF, two straight lines will enclose a space: which is impossible. Therefore the base BC will coincide with EF]". As said in n. 2, this passage would correspond indeed to a later interpolation, so would the related common notion.

with DE and DF respectively and are thus equal to them. Taking as a principle the fact that, in the same circle, all the lines drawn from the centre to the circumference are mutually equal, Peletier shows that the segments DL and DK are equal to DG and DH (and thus to AB and AC respectively), that the angle LDK is equal to the angle DEF and hence to BAC. Deducing from this that the space contained by the segments DL and DK (or their opening, as he puts it in his envisaged definition of equal angles)[137] is equal to the space contained by the segments AB and AC, he concludes that LK = EF = BC and from there that DLK = DEF = ABC and DKL = DFE = ACB and that triangles ABC and DEF are congruent and thus equal. As he says at the end of the demonstration, Peletier therefore attempts here to demonstrate the congruence of the two triangles by appealing to the relation of equality between all the radii of any given circle. Indeed, the procedure he presents here as "more tolerable" than the operation he attributes to Euclid in Prop. I.4 implies the construction circles of which the radii are demonstrated to be equal to the sides of the compared triangles. We will not discuss here the validity of this demonstration, since Peletier himself only presented it as a compromise solution or as a concession to the admission of superposition in geometry, given that he rather would have placed Prop. I.4 among the definitions, as a definition of the equality of angles, since it was missing in Euclid's *Elements*. With regards to this, it should be noted that this alternative demonstration evokes his tentative definition of the equality of angles, which he illustrated by considering the case of two isosceles triangles which two equal sides are taken as two radii of the same circles and which bases are taken as the chords joining the two extremities of these lines, these extremities being situated on the circumferences of their respective circles and the chords joining them being of equal length (Figure 8).[138]

To Peletier, this demonstration would not possess the mechanical character of Euclid's demonstration of Prop. I.4 insofar as it would not imply the transposition, and therefore the free or undetermined mo-tion and direct overlapping of figures, but would ground the demonstration of the congruence of the two triangles ABC and DEF on the relation of equality between the radii of circles, circles which are in this case produced from the lines DG and DH, which are equal to the sides AB and AC of the tri-angle ABC and drawn in the continuation of DE and EF (by Prop. I.2). In other words, instead of a direct transfer of the triangle ABC onto the triangle DEF, this new version of Prop. I.4 would imply the successive displacement of the two sides AB and AC of the triangle ABC, through rationally deter-mined circular motions, onto the corresponding sides DE and DF of DEF through the construction of DG and DH in the continuation of DE and DF. Hence, as, in circles, all the lines drawn from the centre to the circumference may be conceived as different images or traces (*vestigia*) of the same line moved about one of its fixed extremity, DG and DL (coinciding with DE), and DH and DK (coinciding with DF), would be considered as different traces of the same lines turned about point D, and ultimately, by Prop. I.2, as different traces of the lines AB and AC. This conception would notably be allowed here by the fact that the two circles DLG and DKH of radii DG and DH are said to "transit" through point E and F.

[137] [Peletier, 1557, Prop. I.4, 15–16]. See *supra*, n. 132.

[138] [Peletier, 1557, Prop. I.4, 16]: "Atque ut clarè dicam, tantus est angulus BAC, quanta est remotio lineae AC ab ipsa AB: tanta ~~verò efficitur~~ remotio, quantam exhibet linea BC. Hoc autem in Isoscelibus est evidentius. Sint enim duo Isoscelia ABC & DEF: quorum unius duo latera AB & AC duobus DE & DF alterius sint aequalia: angulusque A angulo D. Ac positis centris in A & D punctis, ducantur duo Circuli: prior secundum AB, alter secundum DE spatium. Horum prior manifestò transibit per B & C: alter verò per E & F puncta: quum AB & AC, itemque DE & EF sint aequalia, & à centro utrinque exeuntia. Atque, ex definitione aequalium angulorum, erunt arcus BC & EF aequales. Angulorum enim magnitudo designatur ex arcubus Circulorum qui per extremas lineas quae angulos continent, transeunt. Ac converso modo, aequales anguli atque aequalibus lineis comprehensi, aequales subtendunt peripherias. Quum enim aequalia sint spatia BC & EF, ea aequalibus rectis lineis claudi oportet: propterea quòd recta linea, est à puncto ad punctum via brevissima. Atque haud dissimili judicio, ex laterum ratione & basium, quanta sit angulorum magnitudo aestimabimus." See *supra*, n. 132 and [Loget, 2000, 172–175].

This text was published on page 23 of the article "The Debate between Peletier and Clavius on Superposition.".

This demonstration confirms that, from an argumentative point of view, Peletier would here conceive a distinction between the motion involved in superposition and the motion involved in the generation of circles. This would also confirm that Peletier understood the proof in Prop. I.4 as depending on a constructive process.

It seems nevertheless that Peletier did not consider this alternative demonstration as absolutely satisfac-tory, as it is only said to be "more tolerable" (albeit "much more tolerable"), which implies that it is not entirely acceptable. In this sense, the comparative *tolerabilior* clearly evokes the discourse Peletier pre-sented at the beginning of his commentary of Prop. I.4, as he attempted to restrict the use of application in geometry by stating that: "although application would be somewhat *more tolerable* than superposition, it is however refused in geometry".[139]

In the above-quoted passage, the term "application" is certainly used to describe both procedures, that is, Euclid's method of superposition and the procedure by which Peletier intends to replace it here, as he then says: "we should not believe that, in both methods, there is always the same *application* of triangles". But as he makes it clear, these procedures would differ from each other, since the procedure he adopts in his alternative demonstration consists more in the placing of lines onto given points and lines rather than in the operation of superposing a figure onto a figure directly, which corresponds more properly to what he called an "application" at the beginning of the commentary and which he distinguished from superposition.[140] If this is so, although this procedure is said to be "much more tolerable" than superposition, it would nevertheless not belong to the procedures which are, to Peletier, fully authorised in geometry.

It seems in any case that, rather than resorting to any form of superposition or application, Peletier would prefer to appeal to the solution he proposed in the first place, which consists in admitting Prop. I.4 among the definitions (although he does not actually do), since, as he presents it in support of this solution, "the truth of this proposition should not be drawn from elsewhere than from the common judgement".[141] In Peletier's commentary on the *Elements*, the "common judgement" (*communis judicium*), also called the "common intelligence" (*communis intelligentia*), is described as the faculty which is explicitly associated with the Common Notions and which reaches its concepts through an immediate and non-discursive mode of appre-hension.[142] Therefore, as a definition of the equality of angles, the truth of Prop. I.4 should be drawn, for Peletier, from the common notions that govern the relation of equality between magnitudes, by being ap-prehended by the properly intellective and non-discursive faculty of the "common judgement" rather than by the faculties that are associated with discursive reasoning or empirical judgement. This confirms that Peletier did not found the intelligibility of this mode of demonstration on the axiom of congruence and that he did not consider it a sufficient foundation, contrary to how it was interpreted by other commentators.

Peletier's discussion of superposition therefore confirms that he did not plainly reject Euclid's use of superposition, and regard it as mechanical, because of its appeal to motion, but because it depended, according to him, on a type of motion which is not rationally determinable, and which in a sense would not rationally display the identity and therefore the rigidity of the moved figure or magnitude, contrary to the rotation of the line segment within the generation of the circle. In his commentary on the *Elements*, the circle defined by Euclid is indeed described as generated through the local motion or transit of a line segment around one of its fixed extremity, just as the line or the surface is regarded as resulting from the

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139 [Peletier, 1557, Prop. I.4, 15]: "Applicatio verò quanvis superpositione sit tolerabilior, tamen in Geometria repudiatur".

140 See *supra*, n. 122.

141 [Peletier, 1557, Prop. I.4, 16]: "Hujus itaque Propositionis veritatem non aliunde quàm *à communi judicio* petemus: cogitabimusque Figuras Figuris superponere, Mechanicum quippiam esse: intelligere verò, id demùm esse Mathematicum". See also, in the same text, "quòd in *communi animi judicio consisteret"* (our emphasis). See *supra*, n. 130.

142 On Peletier's conceptions on the status and function of the common judgement in geometry, see [Axworthy, 2013].

flow of a point or description of a line,[143] or just as the sphere is said to be generated through the rotation of the semicircle according to Euclid's definition XI.14, definition to which Peletier explicitly related the kinematic definition of the circle.[144]

This process would certainly not be regarded as offering the true genesis of the line, the circle or the sphere, as these, as previously mentioned, would, for Peletier, all have sprung instantaneously from God's infinite mind and would therefore be ungraspable as such by the human mind.[145] But unlike the motion entailed by the superposition of geometrical figures, the motion through which the geometer conceives magnitudes to be generated would allow to express the essential properties of the defined objects and to reach therefore a rational knowledge of the properties and relations of lines and figures. It would therefore enable us to reach a type of knowledge that can be discursively demonstrated on the basis of universal and necessary principles, and for which geometrical demonstrations would represent a source and a model according to Peletier.[146]

Therefore, if Peletier's comparison of Prop. I.2, I.3 and I.4 of the first book of the *Elements* shows that he did not clearly distinguish generation and superposition of geometrical figures with respect to their function and finality in geometry, as well as to their modes of apprehension and of execution, he however clearly distinguished them with respect to their ability to be used as a means to move geometrical figures in a manner suitable to the rational nature of mathematical knowledge and as means to demonstrate their relation of equality. If he rejected superposition, it is therefore not in view of its kinematic implications, but rather because he considered that the type of motion it involved is unfit to rationally demonstrate the identity and rigidity of the moved figure and thus to deduce any relation of congruence and equality between geometrical objects.

4. Clavius's response to Peletier

In responding to Peletier's arguments against superposition, Clavius's objective was first to add a new argument against his main opponent in the debate on the angle of contact, given that, to him, superposition was a properly relevant means to assess the equality of angles and therefore to compare angles with respect to quantity.[147] His aim was more fundamentally to defend the integrity and legitimacy of Euclid's geometry, as Prop. I.4, I.8 and III.24 would play, to him, a fundamental role in the argumentative structure

143 [Peletier, 1557, Df. I.1 and Df. I.4, 2–3]: "Ex Puncti fluxu perpetuo in longum, gigni intelligitur Linea. [...] Ac quemadmodum ex Puncti fluxu in continuum, exit Linea: ita ex Lineae in tranversum ductu, oritur Superficies". On the difference between *fluxus* and *ductus* in Peletier's commentary on Euclid, see [Axworthy, 2017].

144 [Peletier, 1557, Df. I.16–17, 6–7]: "Siquis verò factionem seu creationem Circuli sibi exponi petat, instar Definitionis Sphaerae quam Euclides libro undecimo daturus est, ea erit huiusmodi. *Circulus, est vestigium lineae rectae in plano circunductae, altero extremorum manente fixo, donec ipsa unde duci coepit, redierit.* [...] Caeterùm Diameter, et Dimetiens Circuli et Quadrati dicitur, immò et Diameter Quadrilaterorum, nulla vocum curiositate, quanvis horum proprius sit Diagonius. Axis autem, Sphaerae et solidorum est, ut Coni, Cylindri, & Piramidis".

145 [Peletier, 1557, Df. I.16, 6]: "Menti quippè nihil prius neque posterius. Immò puncta ante lineas: aut lineas ante Superficies: aut denique superficies ante corpora fuisse, vix cogitatio ipsa complecti potest. [...] Quid enim nos efficere posse putamus arte, in ijs quae Natura tam affabrè fecit? aut quid ingenio consequi, quum de his quae divinitùs emanarunt, humanitùs judicamus". *Cf.* [Peletier, 1581, ff. 56v–57r]: "Car tout cet Univers, à pris sa Forme ansamble: / Tous Nombres ont etè, e sont, aussi tòt qu'Un: / Matiere, e Forme, e Tout, n'urent principe aucun: / A coup, e an l'instant les Poins, qui s'estandiret, / Lignes, Eres, e Cors an l'Infini randiret. [...] Mes chercher dans le Cors, les Eres, plus ou moins, / E les Lignes an l'Ere, an la Ligne les Poins, / Ni pourquoe il an vient tele, ou tele facture, / C'et vouloer defonser l'armoere de Nature, / Pour comter le trezor de ce grand Immortel, / An soe seul infini, e seul se sachant tel". See *supra*, n. 71.

146 See *supra*, n. 120.

147 [Loget, 2000, 217 and 264].

of the *Elements*, standing as a necessary condition of the demonstration of many other propositions.[148] In this endeavour, Clavius actually intended to defend not only Euclid, but also Archimedes and all the ancient geometers who made use of superposition for their demonstrations.[149] Yet, beyond these polemic exchanges, Clavius seems to have respected Peletier as a mathematician, as he referred to his commentary on the *Elements* in more positive terms in other parts of his commentary on Euclid.[150]

In order to show the unfoundedness of Peletier's attack on superposition, Clavius argued that Peletier's solution to place Prop. I.4 among the principles as a definition of the equality of angles, an equality which Peletier described in his commentary on Prop. I.8 as dependent on the equal number and equal lengths of the sides of the compared figures,[151] would be properly irrelevant, since certain figures, such as the square and the rhombus, may have the same number of sides and have their sides equal without having equal angles and therefore without being congruent with each other.[152] Clavius, who thus referred to the formulation given by Peletier in his commentary on Prop. I.8, misunderstood his opponent, since, as Peletier expressed it in the commentary on Prop. I.4, "two angles are equal when the two sides which contain an angle are equal to the two sides which contain another, and when the bases which connect the two sides are equal".[153]

He also asserted that Peletier's alternative version of Prop. I.4 is deprived of any demonstrative power insofar as it consists in a *petitio principii*. As it stands, Peletier's demonstration would aim to prove the equality of the parts of the triangles and of the triangles themselves by constructing a third triangle DGH, which he assumes equal to ABC without any demonstration, for which reason the constructed circles are useless to his demonstration.[154] To this may be added that Peletier did not prove that the base EF of the

148 [Clavius, 1586, 343–344] and [Clavius, 1589, Prop. III.16, 369–370]: "Viderat Peletarius, (neque enim rem adeò manifestam videre non poterat) si hunc modum argumentandi è medio tollat, universam se Geometriam funditùs evertere, cùm plurimae, eaeque praecipuae propositiones in Geometria demonstrentur ex propos. 4 & 8. lib. I & ex 24. lib. 3. quae quidem alio modo demonstrari nequeunt, quam per illam figurarum superpositionem, non quidem re ipsa existentem, sed cogitatione duntaxat, ut dixi, comprehensam. [...] Quare neque propositio 4. Definitio, neque propos. 8. principium erit; ac proinde omnes propositiones, quae illis nituntur, quae innumerabiles propemodum sunt, corruant necesse est, nisi demonstrationes Euclidis recipiantur in illis propositionibus, cùm alio modo demonstrari non possint".

149 [Clavius, 1586, 343] and [Clavius, 1589, Prop. III.16, 369]: "Et certè hac in re non solùm Euclidem in crimen vocat Peletarius, verùm etiam Archimedem, quo omnium judicio, acutior in demonstrando, & subtilior fuit nemo, ejusque commentatorem gravissimum, eumque doctissimum Eutocium Ascalonitam, qui eodem argumentandi genere utuntur in aequeponderantibus, immò verò & omnes Geometras redarguat necesse est, qui non rarò hoc argumenti genus adhibent".

150 See, for instance, [Clavius, 1589, 184], about Prop. I.34: "Demonstrat quoque hic Peletarius problema non injucundum". 151 [Peletier, 1557, Prop. I.4, 19]: "Quis enim negaverit duas superficies esse aequales, quarum latera & quantitate & numero sunt aequalia?" See *supra* n. 131.

152 [Clavius, 1586, 343–344] and [Clavius, 1589, Prop. III.16, 369–370]: "Excogitavit sanè rem magis à Geometrìa alienam, quam est superpositio illa figurarum. Coactus enim est asserere, propos. 4. lib. 1 esse definitionem angulorum aequalium, (& quis unquam talem audivit definitionem?) atque adeò concedendam eam esse sine demonstratione: propositionem verò 8. ejusdem lib. principium esse per se quoque notum. Quod ut credibile magis efficiat, ita scribit in propos. 4. lib. 1. (Etenim nulla evidentiori specie aequalitas figurarum dignoscitur, quàm ex laterum aequalitate.) Idemque quasi confirmat, & repetit in propos. 8 ejusdem libri, dum ita loquitur, (Quis enim negaverit, duas superficies esse aequales, quarum latera & quantitate, & numero sunt aequalia?) [...] Assumpserat enim Peletarius propos. 4. & 8 lib. 1. pro principijs: quod quidem falsum est, atque absurdum. Unde ad eas absurditates necessariò devenit, quas etiam illi, qui vix adhuc principia Geometriae attigerunt, vel facilè vitare potuissent. Nam quis non videt, Rhombum & Quadratum, etiamsi latera habeant & quantitate, & numero aequalia, posse tamen inter se valde esse inaequalia? Id quod in Pentagonis quoque aequilateris, & in alijs figuris plurium laterum aequalium cerni potest: quod non est hujus loci pluribus verbis explicare. Cùm ergo in omnibus figuris multilateris inaequalitas reperiatur, licèt latera habeant & quantitate, & numero aequalia, demonstrandum fuit necessariò Euclidi, aequalitatem triangulorum colligi ex laterum aequalitate, quandoquidem in alijs figuris ea non colligitur".

153 [Peletier, 1557, Prop. I.4, 15–16]: "duos angulos aequales fieri, quum duo latera unum angulum continentia, duobus alterum angulum continentibus fiunt aequalia, & bases quae latera connectunt, aequales". See also *supra* n. 132.

154 [Clavius, 1586, 344] and [Clavius, 1589, Prop. III.16, 370]: "Demonstratio enim nova propos. 4. quam Peletarius confinxit, nihil aliud est, quam (ut cum Logicis loquamur) petitio principij. Id quod perspicuum erit cuilibet, qui eam diligentiùs considerare voluerit. Nam in ea solùm construitur unum triangulum posteriori ex duobus datis aequale, immò idem, atque hoc ipsum quidem

second triangle is equal to the base of the third triangle HG and that this base HG is equal to the base BC of the first triangle, according to the deductive path he followed in the rest of the demonstration to prove the equality of the other parts of the triangles.[155] Therefore, for Clavius, such a demonstration would not be able to replace Euclid's Prop. I.4 in the role it fulfils in the *Elements* and in geometry in general as a means to establish the equality of figures.

The main argument brought forth by Clavius against Peletier on this issue was to show that his opponent did not understand the manner in which geometers considered and appealed to superposition in geometry, which is not to construct figures, but rather to compare figures. He thus criticised Peletier for having com-pared the procedures employed in Prop. I.2 and I.3, which are problems, and in Prop. I.4., I.8 and III.24, which are theorems, and hence for having confused the level of concreteness of the motions introduced in each type of propositions.

> [In these propositions I.4, I.8 and III.24, Peletier] rejects as non-geometrical the very ancient demonstrations of Euclid, that is, those in which one must conceive in thought a figure as superposed on another, which he thinks is alien to the very dignity of geometry, only because he thinks that this superposition is something mechanical and that nearly all propositions could have been, as he says, demonstrated in this manner, even problems in which we are offered to construct something, and he brings forward, as an example of this, Prop. 2 and 3 of Book I, which are problems. And I could certainly discredit Peletier legitimately here, if this were my intention, as he falsely accuses me of doing. And this mainly because he thinks that superposition is used in the same manner in the demonstration of problems and in the demonstration of theorems. But he seems not to have sufficiently understood how geometers appeal to superposition. As they do not want superposition to be done in fact (this would certainly be mechanical), but only to be done in thought and in the mind, which is the duty of the reason and of the intellect.[156]

To Clavius, mathematicians would not conceive superposition as effectively carried-out on given constructed, dimensioned and in a certain sense manipulable figures, but would only be rationally conceded and therefore merely embraced by the intellect. As noted by B. Vitrac,[157] the superposition of figures which is intended in this proposition is only hypothetically assumed, which is notably indicated in the *Elements* by the use of the conditional form when speaking of the considered triangles and of their superposition in the text of the proposition.[158] The triangles themselves do not need to be regarded as given or constructed particular figures, since what is then dealt with is a general class of objects, which is why Clavius added,

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ineptissime, cùm ad id praestandum circulos describat Peletarius, quibus tamen in demonstratione non utitur, quod vitiosum omnino est in Geometria: Deinde infert, triangulum hoc constructum, quod à posteriori ex duobus propositis non differt, priori esse aequale, sine ulla demonstratione; certum autem est hoc ab initio propositum fuisse, ut demonstraretur. Quocirca manifestè principium petit, cùm eadem facilitate statim in principio concludere potuisset, etiamsi nullam adhibuisset constructionem, triangula proposita esse aequalia; quippe cùm constructio illa ad rem non faciat".

155 We thank V. De Risi for his insightful observations on this passage.

156 [Clavius, 1586, 342–343] and [Clavius, 1589, Prop. III.16, 368]: "In his enim omnibus [in propos. 4. & 8. lib. I atque in propos. 24 lib. 3.] [Peletarius] rejicit demonstrationes antiquissimas Euclidis, tanquam non Geometricas; quippe in quibus figuram unam alteri superponi concipere animo oporteat: quod ipse à Geometriae dignitate putat esse alienum, hac solùm inductus ratione, quòd superpositionem illam mechanicum quid esse arbitretur, & quod omnes ferè propositiones hoc modo, ut ait, possint demonstrari, etiam problemata in quibus aliquid proponitur construendum: atque in hujus rei exemplum adducit propos. 2. & 3. lib. I quae problemata sunt. Hic certè Peletarium jure carpere potuissem, si id mihi fuisset propositum, ut falsò criminatur; maximè in eo, quòd eadem ratione usui fore existimavit superpositionem in demonstrandis problematibus, ac theorematibus. Nam non satis intellexisse videtur, quo pacto Geometrae superpositionem illam usurpent. Neque enim volunt, re ipsa faciendam esse figurarum superpositionem, (hoc enim mechanicum quid esset) sed cogitatione tantùm, ac mente, quod opus est rationis atque intellectus".

157 [Euclide, 1990, 295] and [Vitrac, 2005, 50].

158 When the process of superposition is described in the proposition, the use of the conditional is implied by the use of the genitive absolute: "Ἐφαρμοζομένου γὰρ τοῦ ΑΒΓ τριγώνου ἐπὶ τὸ ΔΕΖ τρίγωνον καὶ τιθεμένου τοῦ μὲν Α σημείου ἐπὶ τὸ Δ σημεῖον... " (For, if the triangle ABC be applied to the triangle DEF and if the point A be placed on the point D...).

to conclude, that superposition cannot be said to be done or performed in fact (*re ipsa faciendam esse figurarum superpositionem*), but is only accomplished or rather assumed as accomplished by the intellect (*sed cogitatione tantùm, ac mente, quod opus est rationis atque intellectus*).[159] And this, for him, would suffice to justify its non-mechanical nature.

As Clavius clearly pointed it out then, Peletier's rejection of superposition fundamentally came from his confusion between problems and theorems. As Proclus expressed it in his *Commentary on the first book of the Elements*[160] (followed more or less faithfully on this point by most sixteenth-century commentators, including Peletier),[161] problems and theorems are essentially distinct from each other as for their finality and as for the nature of the procedures they require and allow. While theorems would aim to demonstrate a property or relation of a given class of figures in a universal manner, problems would require one to construct or produce a particular figure. Proclus certainly acknowledged that the purpose and status of problems and theorems could be defined differently according to the perspective one holds on the nature and aim of mathematics,[162] but this general definition remains the one transcribed and followed (more or less faithfully) by most sixteenth-century commentators of Euclid.[163]

Thus, in view of its hypothetical and non-effective nature, superposition would be, to Clavius, perfectly admissible in the context of a theorem. Consequently, superposition would have no relevance in problems such as intended and used in theorems, as it would not be able to serve the effective construction or production of a particular figure, but only to compare two given figures and to establish their congruence and hence their equality.

This is why this kind of argumentation will certainly have its place within theorems, but not within problems. For in theorems, on account of the equality and inequality of magnitudes, which we assume as known, the intellect easily understands without any hesitation that one does not exceed or exceeds the other, if one is conceived by the soul as superposed on the other, although this superposition does not occur in fact, as it is done in Prop. I.4. But in the problems in which we are invited to construct a magnitude equal to another,[164] although we think in our mind that the proposed magnitude is transported to another place, this does not however have any effect, since no translation has been done in fact. So much that it is surprising that Peletier

[159] *Cf.* n. 148: "non quidem re ipsa existentem, sed cogitatione duntaxat, ut dixi, comprehensam".

[160] [Proclus, 1992, 63] ([Proclus, 1873, 77]): "The propositions that follow from the first principles he divides into problems and theorems, the former including the construction of figures, the division of them into sections, subtractions from and additions to them, and in general the characters that result from such procedures, and the latter concerned with demonstrating inherent properties belonging to each figure".

[161] In spite of the confusion Clavius points out on Peletier's part between Prop. I.2, I.3 and I.4 as for the respective status and function of these propositions in the *Elements*, it remains that Peletier also followed, in its most important features, Proclus's distinction between problems and theorems ([Peletier, 1557, 12]: "Demonstrationum autem conclusiones, sunt Problemata & Theoremata. Problemata, ortus Figurarum comprehendunt, sectiones, additamenta: eaque omnia in arte, quae facienda proponuntur. [...] Theoremata: nempè quae factas Figuras comitantur, proprietates & affectiones: quaeque scientiae ipsi inhaerent & ipsam efficiunt".).

[162] Proclus dedicated a long note on the discussions in Antiquity concerning the distinction between theorems and problems [Proclus, 1873, 77–81]. Moreover, one cannot rule out the fact that many theorems also require constructions. On this aspect, see [Knorr, 1983], [Harari, 2003] and [Sidoli and Saito, 2009].

[163] [Clavius, 1589, 23–24]: "Demonstratio omnis Mathematicorum dividitur ab antiquis scriptoribus in Problema et Theorema. Problema vocant eam demonstrationem, quae iubet, ac docet aliquid constituere. Ut si quis conetur demonstrare, supra lineam rec-tam finitam posse triangulum aequilaterum constitui, appellabitur huiuscemodi demonstratio problema, quoniam docet, qua ratione triangulum aequilaterum constitui debeat supra rectam lineam finitam. [...] Theorema autem appellant eam demonstrationem, quae solum passionem aliquam, proprietatemus unius, vel plurium simul quantitatum perscrutatur. Ut si quis optet demonstrare, in omni triangulo tres angulos esse aequales duobus rectis, vocabunt talem demonstrationem Theorema, quia non jubet, aut docet tri-angulum, aut quippiam aliud construere, sed contemplatur tantummodo trianguli cujuslibet constituti passionem hanc, quod anguli illius duobus sint rectis aequales. Unde a contemplatione ipsa, haec demonstratio theorema dicitur".

[164] That will be, in Peletier's text, the case of Prop. I.2.

could have convinced himself that Prop. 2 and 3 of Book I and nearly all the other propositions could have been demonstrated by the means of superposition, or through the transport of lines or figures, if such a mode of argumentation were authorised in geometry.[165]

As he conceded here to Peletier, if superposition were considered and used, in geometrical proofs, as an effective procedure performed on particular figures, and if it were introduced in problems in a constructive aim, it would surely be regarded as mechanical and be excluded as such from geometry, given that it would invite one to deduce the relation or property of the figures through an empirical means of assessment rather than through a firm demonstrative reasoning.[166]
This concession is all the more significant in Clavius's epistemological discourse as, in the same 1589 edition of the *Elements*, he followed Pappus of Alexandria in questioning the admissibility of the classical construction of the quadratrix (curve used by Dinostratus to solve the quadrature of the circle), which Pappus (in reference to Sporus) acknowledged as mechanical in the Book IV of his *Mathematical collection*.[167] As formulated in this context, the combination of motions by which Dinostratus would have established the construction of the quadratrix would not be determined by a given ratio and its construction by this means would therefore not be admissible as a properly geometrical construction. While Clavius tried to solve this problem through a point-by-point construction of the quadratrix,[168] it led Descartes (among other reasons) to exclude the quadratrix from the category of geometrical curves and to place it among what he called mechanical curves.[169] However, it remains that, for Clavius, the hypothetical nature of superposition would

165 [Clavius, 1586, 343] and [Clavius, 1589, Prop. III.16, 368–369]: "Itaque in theorematibus quidem locum habebit genus hoc argumentandi, in problematibus vero non. Namque in theorematibus, propter magnitudinum aequalitatem, inaequalitatemque, quae, ut nota, ponitur, facile intellectus cujusvis sine ulla haesitatione comprehendit, unam vel non excedere alteram, vel excedere, si animo concipiatur una alteri esse superposita, quamvis re ipsa non fiat illa superpositio, ut in propos. 4. lib. I. factum est: At in problematibus, in quibus magnitudinem quis alteri aequalem construere jubetur, licet mente cogitet magnitudinem propositam transferri in alium locum, non tamen propterea quicquam efficiet, cum re ipsa translatio nulla facta sit: Ut mirum sit, Peletarium sibi persuadere potuisse, propos. 2. & 3. lib. I. & alias pene omnes per superpositionem, sive translationem linearum, figurarumve posse demonstrari, si hoc modo argumentandi in Geometria uti liceret".

166 [Clavius, 1589, 24]: "In mathematico [problemate] vero, quamcunque quis partem elegerit, eam firma demonstratione, ita ut nihil omnino dubij sit reliquum, comprobabit. Si enim Geometra statuat ex puncto quolibet lineae rectae propositae lineam perpendicularem educere, efficiet utique hoc ipsum ratione constanti et evidenti: Eodem modo dicendum est, si ex eodem puncto velit educere lineam non perpendicularem".

167 [Pappus, 2009, Prop. 31, 132–133] ([Pappus, 1876–1878, I, 31, § 46–47, 254–256]): "Sporus [. . .] is with good reason displeased with it, on account of the following observations. [. . .] For how is it possible when two points start from B, that they move, the one along the straight line to A, the other along the arc to D, and come to a halt at their respective end points at the same time, unless the ratio of the straight line AB to the arc BED is known beforehand? For the velocities of the motions must be in this ratio, also. Also, how do they think that they come to a halt simultaneously, when they use indeterminate velocities, except that it might happen sometime by chance; and how is that not absurd? [. . .] Without this ratio being given, however, one must not, trusting in the opinion of the men who invented the line, accept it, since it is rather mechanical". See also [Bos, 2001, 40–43].

168 [Bos, 2001, 160–165].

169 On this issue, see [Molland, 1976], [Mancosu, 1996, 71–79], [Bos, 2001, 335–342], [Mancosu, 2008], [Domski, 2009] and [Crippa, 2014, 68–72, 100–103, 221–228, 238–247 and 255–263]. See also [Panza, 2011, 74–91]. Although one might see some similarity between Peletier and Descartes as to their use of the term "mechanical", since this term, in Peletier's commentary on Euclid as in Descartes's *Geometrie*, is opposed to "geometrical" and aims to qualify, in both contexts, a constructive process, or a motion, that is not rationally determinable (such a motion, for Descartes, would indeed not produce lines that may be measured with exactness, contrary to the rotations that generate circles, for example), it should however be noted that Descartes, in reinterpreting Pappus's classification of curves, also distinguished geometrical and mechanical curves by their algebraic expression (the former being expressible by a first- or second-degree polynomial equation, unlike the latter) [Domski, 2003, in part. 1114–1118]. Some of the criteria adopted by Descartes for his distinction between geometrical and mechanical processes are therefore quite different from those proposed by Peletier and offer, in this regard, a much more modern perspective on the issue. We thank R. Siegmund-Schultze for calling this point to our attention. We hope to offer a more in-depth comparison between the conceptions of Peletier, Descartes and other authors on the complementary notions of "geometrical" and "mechanical" at a later occasion.

perfectly allow one to distinguish it from any constructive procedure and prevent its being regarded as mechanical and thus excluded from geometry.

This being said, the manner in which Clavius described the mathematician's apprehension of superpo-sition would not forbid it from being represented spatially in the imagination and therefore from being, to a certain extent, depicted as a local motion of geometrical figures. As he expressed it then, in the demon-stration of the congruence of two figures which are assumed as possessing certain parts known to be equal, we can still "conceive with our soul one figure as superposed on another" (*figuram unam alteri superponi concipere animo*)[170] or "think in our mind that the proposed magnitude is transported to another place" (*mente cogitet magnitudinem propositam transferri in alium locum*).[171] Clavius certainly says here that the superposition of figures is thought or intellectually apprehended, but as shown in his commentary on the definitions, where the local motion of the point generating the line is characterised either as perceived, imagined, conceived or intellectually seized (*perceptus*, *imaginatus*, *conceptus*, *intellectus*),[172] there is no clear distinction between imagination and intellection in Clavius's commentary on the *Elements*. Hence, this mental representation or conception of a figure transported from one place and superposed on another figure can be understood as an imaginarily visualised and therefore spatialised motion of figures.

Which difference would there then be between the motion conceived in the frame of Prop. I.4 in relation to superposition and the motion involved in the generation or construction of figures, as displayed, for instance, in Prop. I.2 and I.3? If both these processes may be represented as a local displacement of figures, it remains that the actual execution and completion of the superposition of two given figures would not play, for Clavius, any role (other than didactic) in the demonstration of the theorem, contrary to the motion involved in the constructions required by problems. Indeed, in Prop. I.4, which aims to demonstrate a universal and necessary relation rather than to construct a particular geometrical object, the imaginary or concrete representation of the first triangle moving towards and placed onto the other or, could we say, its change of position with respect to place, would only help visualise the spatial properties and relations of triangles which are assumed to have two sides equal to two sides and the connecting angles equal, a relation which is then fundamentally deduced from the Common Notions, and notably from the axiom of congruence,[173] which, in Clavius's commentary on this common notion, is explicitly shown to imply the

170 [Clavius, 1586, 343] and [Clavius, 1589, 368]. Cf. *ibid*.: "if we conceive with our soul a magnitude as superposed on the other" (*si animo concipiatur unam magnitudinem alteri esse superposita*).

171 [Clavius, 1586, 343] and [Clavius, 1589, 369].

172 [Clavius, 1589, Df. I.2, 29]: "Mathematici quoque, ut nobis inculcent veram lineae intelligentiam, imaginantur punctum iam descriptum superiore definitione, è loco in locum moveri. Cum enim punctum sit prorsus individuum, relinquetur ex isto motu imaginario vestigium quoddam longum omnis expers latitudinis."; [Clavius, 1589, Df. I.5, 33]: "Mathematici verò, ut nobis eam ob oculos ponant, monent, ut intelligamus lineam aliquam in transversum moveri: Vestigium enim relictum ex ipso motu erit quidem longum, propter longitudinem lineae, latum quoque propter motum, qui in transversum est factus; nulla verò ratione profundum esse poterit, cùm linea ipsum describens omni careat profunditate; quare superficies dicetur."; [Clavius, 1589, Df. I.7, 34]: "Haec autem superficies sola erit ea, quam *imaginari*, et *intelligere* possumus describi ex motu lineae rectae in transversum, qui super duas alias lineas rectas conficitur."; [Clavius, 1589, Df. XI.1, 522]: "Quemadmodum Mathematici, ut recte *intelligamus* lineam, praecipiunt, ut *imaginemur* punctum aliquod e loco in locum moveri; hoc enim describit vestigium quoddam longum tantum, haec est, lineam, propterea quod punctum omnis sit magnitudinis expers; ut autem *percipiamus* superficiem, monent, ut *intelligamus* lineam aliquam in transversum moveri; haec enim describet vestigium longum et latum duntaxat; longum quidem propter longitudinem lineae, latum vero propter motum illum, qui in transversum est factus; carens autem profunditate, quod et linea illius sit expers: Ita quoque, ut *nobis ob oculos ponant* corpus, seu solidum, hoc est, quantitatem trina dimensione praeditum, consulunt, ut *concipiamus* superficiem aliquam aequaliter elevari, sive in transversum moveri; hac enim ratione describetur vestigium quoddam longum, latum, atque profundum; longum quidem et latum, ob superficiem, quae longa et lata existit; profundum vero seu crassum, propter elevationem illam, seu motum superficiei".

173 In Clavius's commentary, these are mainly C.N. 8: "Things which coincide with each other are equal" ([Clavius, 1589, 63]: "Et quae sibi mutuo congruunt, ea inter se sunt aequalia"), from which he reciprocally establishes that, in the case of magnitudes, "things which are equal coincide with each other, when they are superposed" ([Clavius, 1589, 63]: "Econtrario, quae inter se sunt

admissibility of superposition.[174] In this context, the fact for a magnitude *to be superposed* or *to be adjusted on another* simply implies or means that it *coincides with it* and that *it is therefore equal to it*.

In the generation or construction of a specific line or figure in the context of a problem, the motion which allows one to produce the figure would play an essential role, as it is also that which guarantees that the construction can be accomplished such as stated or required in the enunciation and in conformity with the principles of geometry. Through this motion, the construction procedure can be considered as having satisfied a practical request or as having enabled to obtain the effect or result that was required (the construction or finding of a certain point or magnitude), for which reason it may be considered as actually done and concluded, as is the case in Euclidean problems, by *quod erat faciendum*. This is certainly not the case of superposition in the framework of Prop. I.4, I.8 and III.24, propositions which only require that the manipulation and movement of magnitudes is intellectually assumed for the sake of the demonstration.[175]

As shown by Clavius's commentary on the definitions, and especially on Df. I.4, it would be the structure of the generative motion of the point that would determine the structure and the quality of the geometrical line produced in the imagination and that allows to define it as straight, circular, uniform or difform.

> But as mathematicians conceive the line to be described by the imaginary flow of the point, thus they seize the quality of the described line through the quality of the flow of the point. And indeed, if the point is conceived as flowing in a straight line through the shortest space, so that it does not deviate in one part or in another, but maintains an equal and unceasing motion, the described line will be straight; if however the flowing point is thought as vacillating in its motion and as staggering from here to there, the described line will be called mixed; if finally the flowing point does not vacillate in its motion, but is carried around in a circle according to a certain uniform motion and from a distance to a certain determined point, the described line will be called circular.[176]

Although Clavius describes here all types of lines, and not just the straight and circular lines to which Euclid appeals for his constructions, he relates, in his commentary on the Postulates, this description of the imaginary motion of the point as determining the quality of the line to the constructive procedures authorised by the first three Postulates. Indeed, as shown by the commentaries on Post. 1 and Post. 3,[177] it is the motion by which the straight line or the circle are conceived as generated (as expressed in particular through

aequalia, sibi muto congruent, si alterum alteri superponatur") and C.N. 14: "Two straight lines do not enclose a surface" ([Clavius, 1589, 68]: "Duæ rectæ lineæ spatium non comprehendunt").

174 [Clavius, 1589, 63]: "Et quae sibi mutuo congruunt, ea inter se sunt aequalia. Hoc est, duae quantitates, quarum una superposita alteri, neutra alteram excedit, sed ambae inter se congruunt, aequales erunt. Ut duae lineae rectae dicentur esse aequales, quando una alteri superposita, ea quae superponitur, alteri tota congruit, ita ut eam nec excedat, nec ab ea excedatur. Sic etiam duo anguli rectilinei aequales erunt, quando uno alteri superposito, is qui superponitur, alteram nec excedit, nec ab eo exceditur, sed lineae illius cum lineis hujus prorsus coincidunt: Ita enim erunt inclinationes linearum aequales, quamvis lineae interdum inter se inaequales existant."

175 This distinction does not mean that one may find, in Clavius's discourse, the opinion that constructions are aimed to prove the existence of geometrical objects as a preliminary step to the demonstration of their properties, thesis defended by [Zeuthen, 1896] and refuted by [Knorr, 1983], as, to him, being done *in fact* appears to simply mean that the construction, or the object produced through the construction, can be technically considered as accomplished or obtained.

176 [Clavius, 1589, Df. I.4, 31]: "Quemadmodum autem Mathematici per fluxum puncti imaginarium concipiunt describi lineam, ita per qualitatem fluxus puncti qualitatem lineae descriptae intelligunt. Si namque punctum rectà fluere concipiatur per brevissimum spatium, ita ut neque in hanc partem, neque in illam deflectat, sed aequabilem quendam motum, atque incessum teneat, dicetur linea illa descripta, Recta: Si vero punctum fluens cogitetur in motu vacillare, atque hinc inde titubare, appellabitur linea descripta, mixta: Si denique punctum fluens in suo motu non vacillet, sed in orbem feratur uniformi quodam motu, atque distantia à certo aliquo puncto, circa quod fertur, vocabitur descripta illa linea, circularis".

177 [Euclid, 1956, 1, Post. 1, 195]: "Let the following be postulated: To draw a straight line from any point to any point." and Post. 3 [Euclid, 1956, 1, Post. 1, 199]: "To describe a circle with any centre and distance".

the kinematic definitions of these objects) that enables their production and which therefore confirms that they have been brought about as required in the framework of geometrical constructions.

Post. 1: *Let there be postulated that to draw a straight line from any point to any point is conceded.* This first postulate is fully evident, if what we have written before about the line has been correctly considered. Because, since the line is the imaginary flow of a point and since furthermore the straight line is a flow progressing along an absolutely straight path, it turns out that, if we have conceived the point as something which is moved towards another straightly, the straight line will be correctly led from one point to another. [...] Post. 3: *To describe in the same manner a circle of any centre and interval.* However, if we conceive by the mind that a finite straight line of any quantity has been applied to any point by one of its two extremities and is carried about the point which remains fixed, until it has been brought back to where it started to be moved, the circle will be described and produced, which is what the third petition prescribes.[178]

In this framework, the effectiveness and concreteness of the generative motion by which straight lines and circles are produced is also marked by its ability to be carried out instrumentally (at least by the use of authorised instruments, namely, the compass and straightedge), which would in no way be conceivable for the superposition of figures, at least such as intended in Euclidean theorems. In his commentaries on the problems, at least in those that are followed by a section entitled *praxis*,[179] Clavius clearly sets forth the relation between the constructions presented by Euclid in these propositions and their realisation by the means of instruments, although for him the properties of geometrical objects must be properly apprehended independently from physical matter.[180]

For Clavius, the operation of superposing figures, such as intended by mathematicians and hence by Euclid, could not be carried out instrumentally or even be represented as such in the intellect, mainly because, as said, it does not aim, nor allow one, to produce any figure, but only to compare two figures, which represent a class of objects rather than particular figures. Superposition may also *not* be related to any constructive postulates (Post. I–III), unlike the constructions involved in Euclidean problems. It only seems relatable to Post. 4, which establishes that all right angles are equal to each other[181] and which does not enable any construction, but rather a comparison of angles, although Proclus suggested that the admission of superposition would make this postulate redundant.[182]

Peletier's discourse does not indicate whether he conceived superposition as a process which is or could be instrumentally carried out, contrary to Candale after him.[183] Certainly, the adjective "mechanical" could in principle be related to the use of instruments, as it is for instance in the *De usu geometriae*,[184] but it

[178] [Clavius, 1589, Post. 1, 57–58]: "*Postuletur, ut a quovis puncto in quodvis punctum, rectam lineam ducere concedatur.* Primum hoc postulatum planum admodum est, si recte considerentur ea, quae paulo ante de linea scripsimus. Nam cum linea sit fluxus quidam puncti imaginarius, atque adeo linea recta fluxus directo omnino itinere progrediens, fit ut si punctum quodpiam ad aliud directo moveri intellexerimus, ducta sane sit à puncto ad punctum recta linea." and [Clavius, 1589, Post. 3, 58]: "*Item quovis centro, et intervallo circulum describere*. Iam vero, si terminatam rectam lineam cuiuscunque quantitatis mente conceperimus applicatam esse secundum alterum extremum ad quodvis punctum, ipsamque circa hoc punctum fixum circumduci, donec ad eum revertatur locum, a quo dimoveri coepit; descriptus erit circulus, effectumque quod tertia petitio jubet".

[179] For a few examples, see [Clavius, 1589, Prop. I.1, 78; Prop. 9, 98; Prop. 10, 101; Prop. 11, 103; Prop. 12, 106 and Prop. 22, 130].

[180] See *supra*, n. 76.

[181] [Euclid, 1956, 1, Post. 4, 200]: "That all right angles are equal to one another".

[182] On this, see Vitrac, in [Euclid, 1990, 1, 294] and the relevant passage in [Proclus, 1873, 188–189]. See also Heath, [Euclid, 1956, 1, 200–201], who mentions Hilbert's deduction of Post. 4 from his congruence axioms ([Hilbert, 1903, 13]).

[183] See *supra*, n. 110: "mechanicorum usuum instrumenta"; "instrumento palpantes"; "instrumento ferè mechanico, nempe coaptata figura supra figuram".

[184] [Peletier, 1572, 7]: "Quo fit, ut hic etiam mechanica doceamus; usum scilicet Circini, Regulae, aliorùmque instrumentorum quae ad opus Geometricum accommodari solent."

does not seem to be the case, as most of the instrumental procedures represented in Peletier's practical geometry reflect the geometrical procedures appealed to in theoretical geometry. Rather, it seems that what prevailed, for him, in defining superposition as mechanical in the context of the *Elements* was its inability to correspond to a properly rational means of construction and comparison of figures, independently of its degree of abstraction. It is to be noted that Clavius's response to Peletier does not either reflect an interpretation of superposition as an instrumentally carried-out process. In any case, Clavius did not disagree with Peletier on the mechanical character of a superposition procedure that would be conceived as effectively accomplished on particular geometrical figures in a constructive aim.

5. Conclusion

In order to compare Peletier and Clavius's respective understanding of superposition and their respective arguments to prove or disprove the legitimacy of superposition as a geometrical means to establish the equality of figures, we have focused here on Peletier's comparison between Prop. I.2–I.3 and Prop. I.4 of the *Elements* and on Clavius's rebuttal of this comparison as a means to dismiss superposition as a valid geometrical procedure.

What first appears from Peletier's comparison between Prop. I.4 and the two propositions which immediately precede it in the first book of the *Elements*, and also from his considerations on the generation of the circle in Df. I.16, is that in characterising superposition as mechanical, his aim was not to attack the kinematic implications of the procedure. It rather pointed to the fact that the type of motion it entailed would not allow the geometer to rationally display the rigidity of the moved figure by setting forth, at each step of the motion, the identity of the moved lines or angles. This shows that Peletier did not consider the notion of rigid motion as implied and founded by the axiom of congruence, as Clavius, on the contrary, seems to have done. To the undetermined motion of figures implied by superposition, Peletier opposed the kinematic process by which the geometer places a given line at a given point by the means of circles, as in Prop. I.2 and I.3.

Since superposition would not be able to demonstrate the congruence of the moved figure with the figure on which it is superposed, the proof proposed in Euclid's Prop. I.4 would leave, for Peletier, the judgement of the congruence of figures to the senses or the imagination, rather than to the intellect or to the reason.Pointing to the absence in Euclid's *Elements* of a definition of the equality of angles, Peletier thus proposed to formulate Prop. I.4 as a principle, as an axiom or as a definition, intending thereby to eliminate the need to introduce superposition as a means to demonstrate the congruence of figures. The fact that Peletier did not add such a definition in his edition of the *Elements* and attempted rather to propose an alternative proof (which he however did not regard as fully satisfactory), would be due to the fact that such a notion would have been too complex to formulate as a principle. This is one of the reasons why, according to Peletier, Euclid resorted to superposition in Prop. I.4, I.8 and III.24, although, to him, the Greek mathematician would not himself have considered it as a valid means of proof.

Against Peletier, Clavius rejected the constructive understanding of superposition, calling upon the distinction between problems and theorems in Euclid's *Elements*. He therefore admitted superposition in Prop. I.4 as a hypothetically (as opposed to effectively or constructively) carried-out process. In this sense, this method was then not conceived as a kinematic process in a straightforward manner, but only as a means to express certain relations between the parts of the compared figures, which, by the intermediary of the Common Notions, would enable one to deduce the equality of the remaining parts. Hence, Clavius agreed with Peletier that, if it were intended as a constructive process, superposition would not be admissible as a geometrical means to determine the congruence of figures, and should be regarded as mechanical and therefore rejected from geometry. In other words, the main point of disagreement between them on this issue, which however impacted their respective interpretation of Euclid's use of superposition, is their understanding of the procedure of superposition. Clavius, contrary to Peletier, excluded it indeed from effective motions of figures, presenting its imaginary conception rather as a didactic device to help the visualisation of the relations of congruence and equality deduced from the Common Notions.

The consideration of this debate between Peletier and Clavius also shows that the epistemological status of superposition brought mathematicians to consider questions, such as the requirements for properly geometrical procedures in Euclidean geometry and the conditions and limits for the use of motion in geometry, that were crucial to the definition of the nature of geometrical knowledge and of geometrical practice in the sixteenth century, a time when the epistemological and institutional status of geometry, and of mathematics in general, was being reassessed in the light of the newly rediscovered Ancient Greek mathematical and philosophical sources and of the development of physical-mathematical sciences.

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References

Apollonius of Perga, 2009. Apollonius de Perge. Les Coniques, tome 4: Livres VI et VII. Walter de Gruyter, Berlin–New York. Edited and translated from the Arab by R. Rashed.

Archimedes, 1953a. On conoids and spheroids. In: The Works of Archimedes. Dover Publications, New York. Translated by T.L. Heath.

Archimedes, 1953b. On the equilibrium of planes. In: The Works of Archimedes. Dover Publications, New York. Translated by T.L. Heath.

Archimedes, 1953c. On spirals. In: The Works of Archimedes. Dover Publications, New York. Translated by T.L. Heath.

Axworthy, A., 2013. The ontological status of geometrical objects in the commentary on the Elements of Euclid of Jacques Peletier, du Mans (1517–1582). Fondation Maison des Sciences de l'Homme, Working Paper 41.

Axworthy,A., 2017. La notion géométrique de flux du point à la Renaissance et dans le commentaire des Éléments de Jacques Peletier du Mans. In: Grellard, C. (Ed.), Mélanges offerts à Joël Biard. Vrin, Paris, pp. 453–464. Baldi, B., 1998. Le Vite de' matematici. Annotated and edited by E. Nenci. Francoangeli, Milan.

Baldini, U., 1983. Christoph Clavius and the scientific scene in Rome. In: Coyne, G.V., Hoskin, M.A., Pederson, O. (Eds.), Gregorian Reform of the Calendar. Specola Vaticana, Pontifica academia scientiarum.

Bamforth, S., 1989. Peletier du Mans and 'scientific eloquence'. Renaissance Stud. 3, 202–211.

Barrow, I., 1683/1976. Lectiones mathematicae. Playford, London. Also in: The Mathematical Works of Isaac Barrow. Edited by W. Whewell. Cambridge University Press, Cambridge, 1860. Reprint: G. Olm Verlag, Hildesheim–New York, 1976.

Barrow, I., 1734/1970. The Usefulness of Mathematical Learning explained and demonstrated: being Mathematical Lectures read in the Publick Schools at the University of Cambridge. London. Reprint: Cass Publishing Company, London. Translated by J. Kirby, S. Austen, 1970.

Biancani, G., 1615. De mathematicarum natura dissertatio. B. Cochi, Bologna. Also as: A treatise on the nature of mathematics. Translated into English by G. Klima. In: P. Mancosu. Philosophy of Mathematics and Mathematical Practice. Oxford University Press, New York–Oxford, 1996, pp. 178–212.

Bien, R., 2007. Viète's controversy with Clavius over the truly Gregorian calendar. Arch. Hist. Exact Sci. 61, 39–66.

Borrel, J., 1559. Ioannis Buteonis Annotationum liber in errores Campanu, Zamberti, Orontij, Peletarij, Io. Penae interpretum Euclidis. In: Ioannis Buteonis De quadratura circuli libri duo, ubi multorum quadraturae confutantur, & ab monium impugnatione defenditur Archimedes. G. Rouillé, Lyon, pp. 207–267.

Borrel, J., 1562. Apologia adversus epistolam Iacobi Peletarii depravatoris Elementorum Euclidis. Ubi multa de Mathematicis cognitu non indigna tractantur. M. Jouve, Lyon.

Bos, H.J.M., 2001. Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction. Springer, New York.
Bosmans, H., 1907. L'Algèbre de J. Peletier du Mans départie en deux livres. Revue des questions scientifiques 61, 117–173.
Candale, F. (de Foix), 1566. Elementa Geometrica, Libris XV. Ad Germanam Geometriæ Intelligentiam è diversis lapsibus temporis iniuria contractis restituta, ad impletis praeter maiorum spem, quæ hactenus deerant, solidorum regularium conferentiis ac inscriptionibus. His acceßit decimus sextus liber, de solidorum regularium sibi inuicem inscriptorum collationibus. J. Royer, Paris.
Cavalieri, B., 1635. Geometria indivisibilibus continuorum nova quadam ratione promota. C. Ferroni, Bologna.
Cavalieri, B., 1647. Exercitationes geometricae sex. Giacomo Monti, Bologna.
Cifoletti, G., 1992. Mathematics and Rhetoric. Jacques Peletier, Guillaume Gosselin and the French Algebraic Tradition. Ph.D. Thesis. Princeton University, Princeton.
Cifoletti, G., 1994. Du français au latin: L'Algèbre de Jacques Peletier et ses projets pour une nouvelle langue des sciences. In: Roger, C., Pietro, C. (Eds.), Sciences et langues en Europe. Centre Alexandre Koyré, Paris, pp. 95–105.
Cifoletti, G., 1995. La question de l'algèbre: Mathématiques et rhétorique des hommes de droit dans la France du 16e siècle. Rhétorique et civilité 50, 1385–1416.
Cifoletti, G., 2004. The algebraic art of discourse. Algebraic dispositio, invention and imitation in sixteenth-century France. Boston Stud. Philos. Sci. 238, 123–135.
Claessens, G., 2009. Clavius, Proclus, and the limits of interpretation: snapshot-idealization versus projectionism. Hist. Sci. 47, 317–336.
Clavius, C., 1570. In Sphaeram Ioannis de Sacro Bosco Commentarius. V. Helianus, Rome.
Clavius, C., 1574. Euclidis elementorum libri XV: Accessit XVI. De solidorum regularium comparatione: omnes perspicuis demonstrationibus accuratisque scholijs illustrati. V. Accolto, Rome.
Clavius, C., 1586. Theodosii Tripolitæ Sphæricorum libri III a Christophoro Clavio Bambergensis Societatis Iesu perspicuis demonstrationibus ac scholijs illustrati. Item eiusdem Christophori Clavii Sinus, lineæ tangentes et secantes, triangula rectilinea atque sphærica. D. Basa, Rome.
Clavius, C., 1588. Novi calendarii romani apologia. Santi & C., Rome.
Clavius, C., 1589. Euclidis elementorum libri XV: Accessit XVI. De solidorum regularium comparatione: omnes perspicuis demonstrationibus accuratisque scholijs illustrati. B. Grassi, Rome.
Clavius, C., 1591. Euclidis elementorum libri XV: Accessit XVI. De solidorum regularium comparatione: omnes perspicuis demonstrationibus accuratisque scholijs illustrati. B. Ciotti, Cologne.
Clavius, C., 1603a. Euclidis elementorum libri XV: Accessit XVI. De solidorum regularium comparatione: omnes perspicuis demonstrationibus accuratisque scholijs illustrati. Rome.
Clavius, C., 1603b. Romani calendarii a Gregorio XIII P.M. restituti explicatio. L. Zannetti, Rome.
Clavius, C., 1604. Geometria practica. L. Zannetti, Rome.
Clavius, C., 1607a. Euclidis elementorum libri XV: Accessit XVI. De solidorum regularium comparatione: omnes perspicuis demonstrationibus accuratisque scholijs illustrati. N. Hoffmann, Frankfurt.
Clavius, C., 1607b. Euclidis elementorum libri XV: Accessit XVI. De solidorum regularium comparatione: omnes perspicuis demonstrationibus accuratisque scholijs illustrati. M. Cholinus, Cologne.
Clavius, C., 1607c. Tabulæ sinuum, tangentium et secantium, ad partes radij 10,000,000. & ad scrupula prima Quadrantis, et ad earum praxin brevis introductio. J. Albinus, Mainz.
Clavius, C., 1608. Algebra. B. Zannetti, Rome.
Clavius, C., 1611–1612. Opera omnia. Antonius Hierat, Mainz.
Commandino, F., 1572. Euclidis Elementorum libri XV. Unà cum Scholijs antiquis. A Federico Commandino Urbinate. Nuper in latinum conversi, commentarijsque quibusdam illustrati. C. Francischino, Pesaro.
Crippa, D., 2014. Impossibility results: from geometry to analysis. A study in early modern conceptions of impossibility. PhD thesis. Université Paris–Diderot/Sorbonne Paris Cité, Paris.
Davis, N.Z., 1960. Sixteenth-century French arithmetics on the business life. J. Hist. Ideas 21, 18–48.
De Pace, A., 1993. Le Matematiche e il mondo: Ricerche su un dibattito in Italia nella seconda metà del Cinquecento. Francoangeli, Milano.
De Risi, V., 2016. The development of Euclidean axiomatics. The systems of principles and the foundations of mathematics in editions of the Elements in the Early Modern Age. Arch. Hist. Exact Sci. 70, 591–676.

Djebbar, A., 2002. L'épître d'al-Khayyâm sur L'explication des prémisses problématiques du livre d'Euclide.
Farhang 14, 79–136.
Domski, M., 2003. The constructible and the intelligible in Newton's philosophy of geometry. In: Mitchell, Sandra
D. (Ed.), Proceedings of the 2002 Biennial Meeting of The Philosophy of Science Association. Part I: Contributed Papers. In: Philos. Sci., vol. 70, pp. 1114–1124.
Domski, M., 2009. The intelligibility of motion and construction: Descartes' early mathematics and metaphysics, 1619–1637. Stud. Hist. Philos. Sci. Part A 40, 119–130.
Dutka, J., 1988. On the Gregorian revision of the Julian calendar. Math. Intell. 10, 56–64.
Dye, G., Vitrac, B., 2009. Against the geometers by Sextus Empiricus: sources, targets, structure. Phronesis 54, 155–203.
Euclid, 1516. Geometricorum elementorum Libri XV. Comment. Giovanni Campano, Theon Alexandrinus, Hypsicles Alexandrinus. Trad. Bartolomeo Zamberto. Edited by J. Lefèvre d'Etaples. H. Estienne, Paris.
Euclid, 1956. The Thirteen Books of Euclid's Elements. Commented and translated by T.L. Heath. Dover publications, New York.
Euclid, 1990. Euclide, Les Éléments. Annotated and translated by B. Vitrac. Presse Universitaire de France, Paris.
Feldhay, R., 1998. The use and abuse of mathematical entities: Galileo and the Jesuits revisited. In: Machamer, P. (Ed.), Cambridge Companion to Galileo. Cambridge University Press, Cambridge, pp. 80–145.
Fine, O., 1536. In sex priores libros geometricorum elementorum Euclidis Megarensis Demonstrationes. Quibus ipsius Euclidis textus graecus, suis locis insertus est: unà cum interpratione latina Bartholomaei Zamberti Veneti, ad fidem geometricam per eundem Orontium recognita. S. de Colines, Paris.
Fine, O., 1544. In sex priores libros geometricorum elementorum Euclidis Megarensis Demonstrationes. S. de Colines, Paris.
Fine, O., 1551. In sex priores libros geometricorum elementorum Euclidis Megarensis Demonstrationes. R. Chaudière, Paris.
Gatto, R., 2006. Christoph Clavius' Ordo Servandus in Addiscendis Disciplinis Mathematicis and the teaching of mathematics in Jesuit colleges at the beginning of the Modern Era. Science & Education 15, 235–258. Goldstein, M., 1972. The historical development of group theoretical ideas in connection with Euclid's axiom of congruence. Notre Dame J. Form. Log. 13, 331–349.
Harari, O., 2003. The concept of existence and the role of constructions in Euclid's elements. Arch. Hist. Exact Sci. 57, 1–23.
Heath, T.L., 1921. A History of Greek Mathematics. Clarendon Press, Oxford.
Higashi, S., 2007. Giorgio Valla et Alessandro Piccolomini. Quelques aspects de la réception de Proclus à la Renaissance. In: Bull. Liberal Arts Educ. Center, vol. 27. Tokai University, pp. 31–52.
Hilbert, D., 1903. Grundlagen der Geometrie. B.G. Teubner, Leipzig.
Ingrami, G., 1904. Elementi di geometria ad uso dei licei. P. Cuppini, Bologna.
Jaouiche, K., 1986. La Théorie des parallèles en pays d'islam: contribution à la préhistoire des géométries non-euclidiennes. Vrin, Paris.
Jardine, N., 1979. The forging of modern realism: Clavius and Kepler against the sceptics. Stud. Hist. Philos. Sci. Part A 10, 141–173.
Killing, W., 1898. Einführung in die Grundlagen der Geometrie. F. Schöningh, Paderborn.
Knorr, W., 1981. On the early history of axiomatics. The interaction of mathematics and philosophy in Greek antiquity.
In: Hintikka, J., Gruender, D., Agazzi, E. (Eds.), Theory Change, Ancient Axiomatics, and Galileo's Methodology: Proceedings of the 1978 Pisa Conference on the History and Philosophy of Science. D. Reidel Publishing Company, Dordrecht–Boston, pp. 145–186.
Knorr, W., 1983. Construction as existence proof in ancient geometry. Ancient Philos. 3, 125–148.
Lattis, J.M., 1994. Between Copernicus and Galileo: Christoph Clavius and the Collapse of Ptolemaic Cosmology. University of Chicago Press, Chicago.
Levi, B., 2003 [1949]. En lisant Euclide [Leyendo a Euclides]. Agone. Marseille.
Loget, François, 2000. La Querelle de l'angle de contact (1554–1685). Constitution et autonomie de la communauté mathématique entre Renaissance et Âge baroque. Ph.D. Thesis. EHESS, Paris.
Loget, F., 2002. Wallis entre Hobbes et Newton. La question de l'angle de contact chez les anglais. Rev. Hist. Math. 8, 207–262.

Loget, F., 2012. Printers and algebraists in mid-16th century France. Philosophica 87, 85–116.
Lukacs, L. (Ed.), 1965–1992. Monumenta Pædagogica Societatis Jesu, vol. 7. Institutum historica Societatis Iesu, Rome.
Maierù, L., 1991. John Wallis: lettura della polemica fra Peletier e Clavio circa l'angolo di contatto. In: Galluzzi, M. (Ed.), Atti del Convegno "Giornate di storia della matematica". Editel, Rende, pp. 115–137.
Maierù, L., 1999. La diffusione di Proclo, commentatore di Euclide, nel Cinquecento. In: 11° Annuario del Liceo Scientifico "B.G. Scorza". Calabria Letteraria Editrice, Soveria Mannelli, pp. 49–68.
Mancosu, P., 1996. Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century. Oxford University Press, New York–Oxford.
Mancosu, P., 2008. Descartes and Mathematics. In: Broughton, J., Carriero, J. (Eds.), A Companion to Descartes. Blackwell Publishing, Malden, pp. 103–123.
Molland, G., 1976. Shifting the foundations: Descartes's transformation of ancient geometry. Hist. Math. 3, 21–49.
Mueller, I., 2006. Philosophy of Mathematics and Deductive Structure in Euclid's Elements. Dover Publications, Mineola, NY.
Mugler, C., 1959. Dictionnaire historique de la terminologie géométrique des Grecs. Gauthier-Villars, Paris. Murdoch, J.E., 1964. Superposition, congruence and continuity in the Middle Ages. In: Cohen, I.B., Taton, R. (Eds.), Mélanges Alexandre Koyré. In: L'Aventure de la science, vol. 1. Hermann, Paris, pp. 416–441.
Oñate Guillen, Carmelo, 2000. Christopher Clavius y el Calendario Gregoriano. Letras de Deusto 30, 55–70. Palmieri, P., 2009. Superposition: on Cavalieri's practice of mathematics. Arch. Hist. Exact Sci. 63, 471–495. Pantin, I., 2002. La représentation des mathématiques chez Jacques Peletier du Mans : Cosmos hiéroglyphique ou ordre rhétorique? Rhetorica 20, 375–389.
Panza, M., 2011. Rethinking geometrical exactness. Hist. Math. 38, 42–95.
Pappus of Alexandria, 1876–1878. Pappi Alexandrini Collectionis quae supersunt. Edited, translated and commented by F. Hultsch. Weidmann, Berlin.
Pappus of Alexandria, 2009. Book 4 of the Collection. Edited and translated by H. Sefrin-Weis. Springer, London.
Peletier, J., 1545. De arithmeticae practicae methodus facilis per Gemmam Frisium. Huc accesserunt Jacobi Peletarii, annotationes. G. Richard, Paris.
Peletier, J., 1547. Les oeuvres poétiques de Jacques Peletier du Mans. M. de Vascosan, Paris.
Peletier, J., 1549. L'Arithmetique de Jacques Peletier du Mans, departie an quatre livres. J. & E. de Marnef, Poitiers.
Peletier, J., 1550. Dialogue de l'ortografe e prononciation françoese departi an deus livres. J. & E. de Marnef, Poitiers.
Peletier, J., 1554. L'Algebre de Jaques Peletier du Mans, departie an deus livres. J. de Tournes, Lyon.
Peletier, J., 1557. In Euclidis Elementa Geometrica Demonstrationum Libri sex. J. de Tournes and G. Gazeau, Lyon.
Peletier, J., 1560. De occulta parte numerorum, quam algebram vocant, libri duo. G. Cavellat, Paris.
Peletier, J., 1563. De dimensione circuli. De contactu linearum commentarius. De duabus lineis in eodem plano neque parallelis, neque concurrentibus. In: Jacobi Peletarii Medici et Mathematici Commentarii tres. J. Oporinus, Basel.
Peletier, J., 1567. Disquisitiones Geometricae. J. de Tournes, Lyon.
Peletier, J., 1572. De usu geometriae. G. Gourbin, Paris.
Peletier, J., 1573. De l'usage de géométrie. G. Gourbin, Paris.
Peletier, J., 1579a. In Christophorum Clavium, De contactu linearum, Apologia. J. de Marnef, Paris.
Peletier, J., 1579b. Oratio Pictavii habita in praelectiones mathematicas. J. Bouchet, Poitiers. Also in: Laumonier, P., 1904. Un discours inconnu de Peletier du Mans. Revue de la Renaissance, 5, 281–303.
Peletier, J., 1581. Euvres poétiques intituléz Louanges. R. Coulombel, Paris.
Peletier, J., 1610. In Euclidis Elementa Geometrica Demonstrationum Libri sex. J. de Tournes, Lyon.
Peletier, J., 1611. Les six premiers livres des Elements geometriques d'Euclide, avec les demonstrations de Jaques Peletier du Mans. J. de Tournes, Geneva.
Peletier, J., 1628. Les six premiers livres des Elements geometriques d'Euclide, avec les demonstrations de Jaques Peletier du Mans. J. de Tournes, Geneva.
Proclus, 1873. Procli Diadochi in primum Euclidis Elementorum librum commentarij. Edited by G. Friedlein. B.G. Teubner, Leipzig.
Proclus, 1992. A Commentary on the First Book of Euclid's Elements. Translated by G. Morrow. Princeton University Press, Princeton.
Ramus, P., 1569. Arithmeticæ libri II, Geometriæ libri XXVII. E. Episcopius, Basel.

Rashed, R., 2013. Le mouvement en géométrie classique. Al-Mukhatabat 7, 58–68.
Romano, A., 1999. La Contre-réforme mathématique. Constitution et diffusion d'une culture mathématique jésuite à la Renaissance. École française de Rome, Rome.
Rommevaux, S., 2004. Les introductions aux éditions des Éléments d'Euclide au Moyen-Âge et à la Renaissance. In: Grellard, C. (Ed.), Méthodes et statut des sciences à la fin du Moyen Âge. Presses Universitaires du Septentrion, Lille, pp. 143–158.
Rommevaux, S., 2006. Un débat dans les mathématiques de la Renaissance : le statut de l'angle de contingence. J. Renaissance 4, 291–302.
Rommevaux, S., 2012. Christoph Clavius, un promoteur des mathématiques à la Renaissance. Seizième Siècle 8, 127–140.
Russell, B., 1938. The Principles Of Mathematics, 2nd ed. W.W. Norton & Company, New York.
Saito, K., 2009. Reading ancient Greek mathematics. In: Robson, E., Stedall, J. (Eds.), The Oxford Handbook of the History of Mathematics. Oxford University Press, Oxford, pp. 801–826.
Savile, H., 1621. Praelectiones tresdecim in principium elementorum Euclidis. J. Lichfield and J. Short, Oxford.
Schmidt, A.-M., 1938. La Poésie scientifique en France au seizième siècle. A. Michel, Paris.
Schopenhauer, A., 1988. Die Welt als Wille und Vorstellung, vol. 2. Sämtliche Werke, vol. III. Brockhaus, Mannheim.
Sidoli, N., Saito, K., 2009. The role of geometrical construction in Theodosius's Spherics. Arch. Hist. Exact Sci. 63, 581–609.
Smolarski, D.C., 2002. The Jesuit Ratio Studiorum, Christopher Clavius, and the study of mathematical sciences in universities. Science in Context 15, 447–457.
Tannery, P., 1884. Sur l'authenticité des axiomes d'Euclide. Bull. sci. math. astron. 2e sér. VIII, 162–175.
Vinel, N., 2010. La Rhusis mathématique. De l'ancien Pythagorisme à Proclus. In: Lernould, A. (Ed.), Études sur le Commentaire de Proclus au premier livre des Éléments d'Euclide. Presse Universitaire du Septentrion, Villeneuve d'Ascq.
Vitrac, B., 2005. Quelques remarques sur l'usage du mouvement en géométrie dans la tradition euclidienne : de Platon et Aristote à Omar Khayyâm. Fahrang. Q. J. Humanities Cult. Stud. 18, 1–56.
Wagner, R.J., 1983. Euclid's intended interpretation of superposition. Hist. Math. 10, 68–89.
Zeuthen, H.G., 1896. Die geometrische Konstruktion als 'Existenzbeweis' in der antiken Geometrie. Math. Ann. 47, 222–228.