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Entanglement entropy in (1+1)D CFTs with multiple local excitations

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Abstract: In this paper, we use the replica approach to study the Rényi entropy S_L of generic locally excited states in (1+1)D CFTs, which are constructed from the insertion of multiple product of local primary operators on vacuum. Alternatively, one can calculate the Rényi entropy S_R corresponding to the same states using Schmidt decomposition and operator product expansion, which reduces the multiple product of local primary operators to linear combination of operators. The equivalence $S_L = S_R$ translates into an identity in terms of the F symbols and quantum dimensions for rational CFT, and the latter can be proved algebraically. This, along with a series of papers, gives a complete picture of how the quantum information quantities and the intrinsic structure of (1+1)D CFTs are consistently related.

Keywords: Conformal Field Theory, Anyons, Field Theories in Lower Dimensions, Holography and condensed matter physics (AdS/CMT)

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1 Introduction

Information theory provides us with a new view on the structure of quantum field theory (QFT). Recently many attempts have given us more insights into the relations between the two, e.g., [1]–[13]. For example: the entropic g-function [7] for 1+1 dimensional quantum field theories can be derived from the relative entanglement entropy, the quantum null energy condition can be obtained [5, 6] from the inequalities of entanglement entropy, and authors of [8–13] use quantum information quantities to set up criterion of Eigenstate Thermalization Hypothesis (ETH) in order to classify the chaotic behaviors of CFTs.

Among all the quantum information quantities, we will be interested in the Rényi and entanglement entropies of locally excited states in (1+1)D conformal field theory(CFT). The n-th Rényi entanglement entropy for a subsystem A is defined by $S_A^{(n)} = \log \mathrm{Tr}[\rho_A^n]/(1-n)$, where ρ_A is the reduced density matrix of A.The subsystem A is chosen to be the half plane $x>0$ in this paper, for simplicity. The locally excited states are defined by inserting operators on the vacuum of the theory, in the form $\mathcal{O}\,|0\rangle$, where $\mathcal{O}$ can be a primary or descendant operator, or even the product or linear combination of different operators. The former cases have been extensively studied in the literature [9–11, 14–31], while the latter is the focus of the current paper.

We mainly study the variation of $S_A^{(n)}$ between the excited states and the ground states, where the excited states are obtained by acting general product of different primary operators or linear combination of different operators. That is the state $|\psi\rangle_m := \prod_i^m O_i(x_i)\,|0\rangle$, where $O_i(x_i)$ is a primary or descendant operator located at point x_i. We will consider the time evolution of the variation of n-th REE, denoted by $\Delta S_A^{(n)}$. In the limit $t \to \infty$, we will show that the variation of Rényi entropy of state $|\psi\rangle_m$ satisfies the following sum rule

$$\Delta S_A^{(n)}\big(\prod_i^m O_i(x_i)\,|0\rangle\big) \simeq \sum_i^m \Delta S_A^{(n)}(O_i(x_i)\,|0\rangle). \tag{1.1}$$

This sum rule tell us $\Delta S_A^{(n)}\big(\prod_i^m O_i(x_i)\,|0\rangle\big)$ depends only on individual state $O_i\,|0\rangle$.

For operators in CFT, we expect the following operator product expansion (OPE), or fusion rule $O_i \times O_j = \sum_k N_{ij}^k O_k$, where the entries of rank-three tensor N_{ij}^k are non-negative integers. For simplicity, we consider the case $m = 2$, the state $|\psi\rangle_L = O_1(x_1)O_2(x_2)\,|0\rangle$. In (1+1)D CFT, we can rewrite $O_1(x_1)O_2(x_2)$ as a linear combination of OPE blocks [35], i.e.,

$$O_1(x_1)O_2(x_2) = |x_1 - x_2|^{-2(h_1+h_2)} \sum_k C_{12k}\; \mathcal{O}_k(x_2; x_1), \tag{1.2}$$

where h_1, h_2 are conformal dimension of operator O_1, O_2, C_{12k} is the coupling constant for 3-point function, and $\mathcal{O}_k(x_2; x_1)$ is a non-local operator, in the sense that the two points x_1 and x_2 can have a nonlocal distance [36]. Here the sum is over all the possible fusion channels. So we can define an equivalent state to $|\psi\rangle_L$,

$$|\psi\rangle_R = |x_1 - x_2|^{-2(h_1+h_2)} \sum_k C_{12k}\mathcal{O}_k(x_2; x_1)\,|0\rangle\,. \tag{1.3}$$

The Rényi or entanglement entropy of state $|\psi\rangle_R$ is denoted by S_R. As a result, S_R depends on the operator $\mathcal{O}_k(x_2; x_1)$ and their linear combination coefficients explicitly. Due to eq. (1.2), the entanglement entropy S_L of the state $|\psi\rangle_L$ should be equal to S_R. Then the constraint $S_L = S_R$ provides a connection between different data of the theory.

For (1+1)D rational CFTs, S_L is only associated with the quantum dimension of operators O_1 and O_2 which has been obtained in [14], while S_R depends on the quantum dimension of O_k and the fusion coefficients. It is difficult to get the complete form S_R by replica trick. In this paper, we use the Schmidt decomposition approach to obtain the late time behavior of S_R. The constraint $S_L = S_R$ will then leads to an identity (eq. (3.47) in the main context), which can be proved using algebraic relations of F symbols and quantum dimensions. We examine Minimal models $\mathcal{M}(p, p')$ as typical examples.

The layout of this paper is as follows. In section 2, we will give the general set-up. For the locally excited state with many primary operators inserted, we prove the sum rule (1.1). For the case of linear combination of different operators, we also obtain the Rényi entropy by Schmidt decomposition. In section 3, we focus on the $S_L = S_R$ in rational CFTs and obtain the identity. Minimal model examples are discussed in detail. In section 4, we prove the identity. In section 5, we discuss the extension of the above analysis to large-c CFTs, and the relation with (2+1)-D topological orders.

2 Entanglement of locally excited states

As reviewed in the introduction, the locally excited states we will focus on are of the form

$$|\psi\rangle := \mathcal{O}\,|0\rangle\,, \tag{2.1}$$

where $|0\rangle$ is the vacuum of (1+1)D CFT, and $\mathcal{O}$ can be a primary operator, a descendant operator, or the products or linear combinations of different operators. The former two cases have been studied in papers [14, 16]. In this section we will study latter two more complicated situations:

1. $\mathcal{O}$ is the product of primary operators.

2. $\mathcal{O}$ is linear combination of different operators.

We will mainly focus on rational CFTs, for which the result is robust. The first case has already been studied in paper [23] in rational CFTs. We slightly generalize the result to other (1+1)D CFTs and give the sum rule. As far as we know the second case has not been discussed in literature.

2.1 Product of primary operator

Consider the state defined by

$$|\psi\rangle_m := \mathcal{N}(\epsilon; l_1, l_2, \ldots, l_m)e^{-\epsilon H}\prod_i^m O_i(l_i, 0)\,|0\rangle\,, \tag{2.2}$$

where $O_i(l_i, 0)$ are primary operators located at $x = -l_i$ ($l_i > 0$). We regularize the state by introducing a UV cut-off ϵ as usual, and $\mathcal{N}(\epsilon; l_1, l_2, \ldots, l_m)$ is the normalization constant. We shall further assume the distance between different operators $|l_i - l_j| \gg \epsilon$ ($i \neq j$). At time t, the state becomes

$$|\psi(t)\rangle_m = \mathcal{N}(\epsilon; l_1, l_2, \ldots, l_m)\prod_i^m O_i(w_i, \bar{w}_i)\,|0\rangle\,, \tag{2.3}$$

where $w_i = -l_i + t + i\epsilon$, $\bar{w}_i = -l_i - t - i\epsilon$. In the following we will first consider $m = 2$ and $O_1 = O_2 = O$, it will be straightforward to generalize to arbitrary m. We would like to study these locally excited states by calculating the entanglement entropy or Rényi entropy of the subsystem $A := \{x > 0\}$. By using the definition of Rényi entropy and the replica trick, we find the difference between the excited state $|\psi(t)\rangle_2$ and ground state as

$$\Delta S_A^{(n)}(|\psi(t)\rangle_2) = \frac{1}{1-n}\left(\log\frac{\langle\prod_s^n O^\dagger(w'_{s,2}, \bar{w}'_{s,2})O^\dagger(w'_{s,1}, \bar{w}'_{s,1})O(w_{s,1}, \bar{w}_{s,1})O(w_{s,2}, \bar{w}_{s,2})\rangle_{\mathcal{R}_n}}{\langle O^\dagger(w'_2, \bar{w}'_2)O^\dagger(w'_1, \bar{w}'_1)O(w_1, \bar{w}_1)O(w_2, \bar{w}_2)\rangle^n}\right), \tag{2.4}$$

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where

$$w_1=-l_1+t+i\epsilon,\ \bar{w}_1=-l_1-t-i\epsilon;\qquad w_2=-l_2+t+i\epsilon,\ \bar{w}_2=-l_2-t-i\epsilon,\qquad (2.5)$$
$$w_1'=-l_1+t-i\epsilon,\ \bar{w}_1'=-l_1-t+i\epsilon;\qquad w_2'=-l_2+t-i\epsilon,\ \bar{w}_2'=-l_2-t+i\epsilon,$$

and $(w_{s,i},\bar{w}_{s,i})$, $(w'_{s,i},\bar{w}'_{s,i})$ ($i=1,2$ and $s=1,2,\dots,n$) are the replica of $(w_i,\bar{w}_i)$ and $(w_i',\bar{w}_i')$ on the s-th sheet of $\mathcal{R}_n$. The denominator is the four point correlation function on complex plane C, which is related to normalization constant $\mathcal{N}(\epsilon;l_1,l_2)$. In the limit $\epsilon\to 0$, we have

$$\begin{aligned}&\langle O^\dagger(w_2',\bar{w}_2')O^\dagger(w_1',\bar{w}_1')O(w_1,\bar{w}_1)O(w_2,\bar{w}_2)\rangle\\&\simeq\langle O^\dagger(w_1',\bar{w}_1')O(w_1,\bar{w}_1)\rangle\langle O^\dagger(w_2',\bar{w}_2')O(w_2,\bar{w}_2)\rangle=\frac{1}{(2\epsilon)^{8\Delta_O}},\end{aligned}\qquad (2.6)$$

where Δ_O is the conformal dimension of operator O. Notice we have used the assumption $|l_1-l_2|\gg\epsilon$.

To calculate the correlators on $\mathcal{R}_n$ we could apply the conformal transformation $w=z^n$, which maps $\mathcal{R}_n$ to the complex plane C. The correlation function on $\mathcal{R}_n$ is mapped to

$$\begin{aligned}&\Big\langle\prod_s^n O^\dagger(w'_{s,2},\bar{w}'_{s,2})O^\dagger(w'_{s,1},\bar{w}'_{s,1})O(w_{s,1},\bar{w}_{s,1})O(w_{s,2},\bar{w}_{s,2})\Big\rangle_{\mathcal{R}_n}\\&=C_n\Big\langle\prod_s^n O^\dagger(z'_{s,2},\bar{z}'_{s,2})O^\dagger(z'_{s,1},\bar{z}'_{s,1})O(z_{s,1},\bar{z}_{s,1})O(z_{s,2},\bar{z}_{s,2})\Big\rangle,\end{aligned}\qquad (2.7)$$

where C_n is a constant of $O(1)$, and the coordinates $(w_{s,i},\bar{w}_{s,i})$, $(w'_{s,i},\bar{w}'_{s,i})$ are mapped to

$$\begin{aligned}z_{s,1}&=e^{2\pi is/n}(-l_1+t+i\epsilon)^{1/n},\ \bar{z}_{s,1}=e^{-2\pi is/n}(-l_1-t-i\epsilon)^{1/n},\\z'_{s,1}&=e^{2\pi is/n}(-l_1+t-i\epsilon)^{1/n},\ \bar{z}'_{s,1}=e^{-2\pi is/n}(-l_1-t+i\epsilon)^{1/n},\\z_{s,2}&=e^{2\pi is/n}(-l_2+t+i\epsilon)^{1/n},\ \bar{z}_{s,2}=e^{-2\pi is/n}(-l_2-t-i\epsilon)^{1/n},\\z'_{s,2}&=e^{2\pi is/n}(-l_2+t-i\epsilon)^{1/n},\ \bar{z}'_{s,2}=e^{-2\pi is/n}(-l_2-t+i\epsilon)^{1/n}.\end{aligned}\qquad (2.8)$$

In this paper we are mainly interested in the result in the late-time region $t\gg l_i$. We find

$$\begin{aligned}z_{s,1}-z'_{s-1,1}&\sim O(\epsilon),\qquad & z_{s,2}-z'_{s-1,2}&\sim O(\epsilon),\\\bar{z}_{s,1}-\bar{z}'_{s,1}&\sim O(\epsilon),\qquad & \bar{z}_{s,2}-\bar{z}'_{s,2}&\sim O(\epsilon).\end{aligned}\qquad (2.9)$$

As we can see from (2.6), the numerator of (2.4) is divergent of $O(1/\epsilon^{8n\Delta_O})$. Only the most divergent term in the numerator of (2.4) will contribute to the final result. From (2.8) we also find

$$|z_{s,i}-z_{t,j}|\sim O(1)\gg\epsilon,\quad |z'_{s,i}-z_{t,j}|\sim O(1)\gg\epsilon,\quad |z'_{s,i}-z'_{t,j}|\sim O(1)\gg\epsilon,$$

for $i\neq j$ $(i,j=1,2;s,t=1,2,\dots,n)$. Therefore, the most divergent term comes from the correlation between $O(z_{s,i},\bar{z}_{s,i})$ and $O(z'_{s,i},\bar{z}'_{s,i})$, which means

$$\begin{aligned}&\Big\langle\prod_s^n O^\dagger(z'_{s,2},\bar{z}'_{s,2})O^\dagger(z'_{s,1},\bar{z}'_{s,1})O(z_{s,1},\bar{z}_{s,1})O(z_{s,2},\bar{z}_{s,2})\Big\rangle\\&=\Big\langle\prod_s^n O^\dagger(z'_{s,2},\bar{z}'_{s,2})O(z_{s,2},\bar{z}_{s,2})\Big\rangle\Big\langle\prod_s^n O^\dagger(z'_{s,1},\bar{z}'_{s,1})O(z_{s,1},\bar{z}_{s,1})\Big\rangle+O(1).\end{aligned}\qquad (2.10)$$

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Taking the above expression into (2.4) by using (2.7), we immediately obtain a sum rule of Rényi entropy

$$\begin{aligned}
&\Delta S_A^{(n)}(|\psi(t)\rangle_2)\\
&=\frac{1}{1-n}\left(\log\frac{\langle\prod_s^n O^\dagger(w'_{s,2},\bar{w}'_{s,2})O(w_{s,2},\bar{w}_{s,2})\rangle_{\mathcal{R}_n}\langle\prod_t^n O^\dagger(w'_{t,1},\bar{w}'_{t,1})O(w_{t,1},\bar{w}_{t,1})\rangle_{\mathcal{R}_n}}{\langle O^\dagger(w'_2,\bar{w}'_2)O(w_2,\bar{w}_2)\rangle^n\langle O^\dagger(w'_1,\bar{w}'_1)O(w_1,\bar{w}_1)\rangle^n}+O(\epsilon^{8n\Delta_O})\right)\\
&\simeq\frac{1}{1-n}\left(\log\frac{\langle\prod_s^n O^\dagger(w'_{s,2},\bar{w}'_{s,2})O(w_{s,2},\bar{w}_{s,2})\rangle_{\mathcal{R}_n}}{\langle O^\dagger(w'_2,\bar{w}'_2)O(w_2,\bar{w}_2)\rangle^n}+\log\frac{\langle\prod_t^n O^\dagger(w'_{t,1},\bar{w}'_{t,1})O(w_{t,1},\bar{w}_{t,1})\rangle_{\mathcal{R}_n}}{\langle O^\dagger(w'_1,\bar{w}'_1)O(w_1,\bar{w}_1)\rangle^n}+\dots\right)\\
&=\Delta S_A^{(n)}(O(w_1,\bar{w}_1)\,|0\rangle)+\Delta S_A^{(n)}(O(w_2,\bar{w}_2)\,|0\rangle),
\end{aligned}\tag{2.11}$$

where $\Delta S_A^{(n)}(O(w_1,\bar{w}_1)\,|0\rangle)$ and $\Delta S_A^{(n)}(O(w_2,\bar{w}_2)\,|0\rangle)$ are the Rényi entropy of state $O(w_1,\bar{w}_1)\,|0\rangle$ and $O(w_2,\bar{w}_2)\,|0\rangle$. The above analysis works for general CFTs. Specifically in rational CFTs, by $2(n-1)$ times fusion transformation we could re-arrange the order of holomorphic coordinates $z_{s,i}$ into the order as follows,

$$\begin{aligned}
&(z'_{1,2},z'_{1,1},z_{1,1},z_{1,2})(z'_{2,2},z'_{2,1},z_{2,1},z_{2,2})\dots(z'_{n,2},z'_{n,1},z_{n,1},z_{n,2})\\
\to\ &(z'_{2,2},z'_{2,1},z_{1,1},z_{1,2})(z'_{3,2},z'_{3,1},z_{2,1},z_{2,2})\dots(z'_{1,2},z'_{1,1},z_{n,1},z_{n,2})
\end{aligned}\tag{2.12}$$

The correlation function would become

$$\begin{aligned}
&\Big\langle\prod_s^n O^\dagger(z'_{s,2},\bar{z}'_{s,2})O^\dagger(z'_{z,1},\bar{z}'_{s,1})O(z_{s,1},\bar{z}_{s,1})O(z_{s,2},\bar{z}_{s,2})\Big\rangle\\
&=F_{00}^{2(n-1)}\langle O^\dagger(z'_{2,2},\bar{z}'_{1,2})O(z_{1,2},\bar{z}_{1,2})\rangle\langle O^\dagger(z'_{2,1},\bar{z}'_{1,1})O(z_{1,1},\bar{z}_{1,1})\rangle\dots\\
&\quad\langle O^\dagger(z'_{1,2},\bar{z}_{n,2})O(z_{n,2},\bar{z}_{n,2})\rangle\langle O(z'_{1,1},\bar{z}_{n,1})O(z_{1,1},\bar{z}_{n,1})\rangle
\end{aligned}\tag{2.13}$$

Finally we could obtain the result

$$\Delta S_A^{(n)}=-2\log F_{00}=2\log d_O,\tag{2.14}$$

where d_O is the quantum dimension [32, 33] of operator O.

2.2 Linear combination of operators

In this subsection we would like to explore the entanglement properties of a linear combination of different operators. For a series of operators O_p, which could be primary or descendant operators, we further assume they are orthogonal to each other in the vacuum in the sense that $\langle 0|\,O_pO_{p'}\,|0\rangle=0$ if $p\neq p'$. The state we would like to explore is then

$$|\Psi\rangle\sim\sum_p O_p(x)\,|0\rangle\,,\tag{2.15}$$

where the state is local at point x. We follow the same regularization methods as before by defining

$$|\Psi\rangle=\mathcal{N}(\epsilon)\sum_p e^{-\epsilon H}O_p(x,0)\,|0\rangle\,,\tag{2.16}$$

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where ϵ is the cut-off, H is the Hamiltonian of CFT, and $\mathcal{N}(\epsilon)$ is the normalization constant. In (1+1)D CFTs, we assume $x = -l$. The normalization constant $\mathcal{N}(\epsilon)$ is

$$\mathcal{N}(\epsilon) = \frac{1}{\sqrt{\sum_p \langle O_p^\dagger(w_1, \bar{w}_1) O_p(w_2, \bar{w}_2)\rangle}}, \tag{2.17}$$

where $w_1 := -l + i\epsilon$, $\bar{w}_1 := -l - i\epsilon$, $w_2 := -l - i\epsilon$ and $\bar{w}_2 := -l + i\epsilon$.

One could consider the time evolution of state (2.16), $|\Psi(t)\rangle = e^{-iHt} |\Psi\rangle$. We expect the entanglement entropy of state $|\Psi(t)\rangle$ has the following form in large t limit:[1]

$$S_A = -\sum \log \lambda_p \log \lambda_p + \sum \lambda_p S_p, \tag{2.18}$$

where S_p is the entanglement entropy of A for state $O_p |0\rangle$, and λ_p is defined as

$$\lambda_p := \frac{\langle O_p^\dagger(w_1, \bar{w}_1) O_p(w_2, \bar{w}_2)\rangle}{\sum_q \langle O_q^\dagger(w_1, \bar{w}_1) O_q(w_2, \bar{w}_2)\rangle}. \tag{2.19}$$

This can be understood as the probability of state $|p\rangle$ in the superposition state (2.16).

To prove above formula, let's consider a general form like (2.16),

$$|\psi\rangle = \sum_p \sqrt{\lambda_p} |p\rangle , \tag{2.20}$$

where we normalize $\sum_p \lambda_p = 1$ and assume $\langle p| p'\rangle = \delta_{p,p'}$. Generally $|p\rangle$ is an entangled state if we divide the Hilbert space into two sub-Hilbert space $H_p \otimes \bar{H}_p$. By Schmidt decomposition we could write

$$|p\rangle = \sum_{i_p} \alpha^p_{i_p} |p_{i_p}\rangle \otimes |\bar{p}_{i_p}\rangle , \tag{2.21}$$

where $|p_{i_p}\rangle$ and $|\bar{p}_{i_p}\rangle$ are orthonormal basis of two Hilbert spaces, and α_{i_p} are the real coefficients. In this basis EE of $|p\rangle$ is

$$S_p := -\sum_{i_p} (\alpha^p_{i_p})^2 \log(\alpha^p_{i_p})^2. \tag{2.22}$$

One could calculate the reduced density matrix of state $|\psi\rangle \langle\psi|$,

$$\rho_H := tr_{\bar{H}} |\psi\rangle \langle\psi| = \sum_{\bar{q}, j_q} \langle \bar{q}_{j_q}| \psi\rangle \langle\psi |\bar{q}_{j_q}\rangle . \tag{2.23}$$

With some algebra, this becomes

$$\rho_H = \sum_{p, i_p} \lambda_p (\alpha^p_{i_p})^2 |p_{i_p}\rangle \langle p_{i_p}| . \tag{2.24}$$

The n-th Rényi entropy is

$$S^{(n)} := \frac{\log tr_{(\oplus_p H_p)} \rho_H^n}{1-n} = \frac{\log \sum_{p, i_p} \lambda_p^n (\alpha^p_{i_p})^{2n}}{1-n}, \tag{2.25}$$

[1]This expression has been used in paper [15] written by one of the authors without a proof.

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which can be expressed as

$$S^{(n)} = \frac{\log \sum_p \lambda_p^n e^{(1-n)S_p^{(n)}}}{1-n}, \tag{2.26}$$

where $S_p^{(n)}$ is the Rényi entropy of the state $|p\rangle$. Taking the limit $n \to 1$ of $S^{(n)}$ we will obtain the entanglement entropy (EE),

$$S = -\sum \lambda_p \log \lambda_p + \sum \lambda_p S_p. \tag{2.27}$$

We could write (2.16) as the form (2.20), $|\Psi\rangle = \sum \lambda_p |\psi_p\rangle$, with λ_p defined as (2.19),

$$|\psi_p\rangle := \mathcal{N}_p(\epsilon) e^{-\epsilon H} O_p(x,0) |0\rangle \,, \tag{2.28}$$

and $\mathcal{N}_p(\epsilon) := 1/\sqrt{\langle O_p^\dagger(w_1,\bar{w}_1) O_p(w_2,\bar{w}_2)\rangle}$.

3 Identity from the constraint

In this section we would like to discuss the constraint $S_L = S_R$ as we have mentioned in the introduction.

3.1 General discussion

Before we go on to the details of calculations, let's explain the idea behind the constraint $S_L = S_R$ and our motivations. We will study the time evolution of the state $|\psi\rangle_L := O_1(x_1)O_2(x_2)|0\rangle$, which is an excited state by inserting primary operators O_1 and O_2 at point x_1 and x_2. One could calculate the REE $S_L^{(n)}$ for a subsystem $A := \{x > 0\}$, the EE $S_L = \lim_{n\to 1} S_L^{(n)}$. $S_L^{(n)}$ depends on t, we expect it will approach to a constant in the large t limit. Using the sum rule we have derived in section 2.1, we only need to know the results for states $O_1(x_1)|0\rangle$ and $O_2(x_2)|0\rangle$.

On the other hand we could rewrite $O_1(x_1)O_2(x_2)$ OPE blocks (1.2). Note that (1.2) is an operator equality, so we may define a state $|\psi\rangle_R$ (1.3) by the OPE blocks. $|\psi\rangle_R$ and $|\psi\rangle_L$ can be seen as same states in the Hilbert space but with different basis. This fact immediately leads to the constraint $S_L^{(n)} = S_R^{(n)}$ as well as $S_L = S_R$. In the following we mainly focus on $S_L = S_R$. More importantly, $|\psi\rangle_R$ explicitly depends on the CFT data associated with the coupling constant C_{12k} for the three point function $\langle O_1 O_2 O_k\rangle$. $|\psi\rangle_R$ is like the form (2.15) we discuss in section 2.2, therefore the final expression (2.26) for S_R will depend on C_{12k}. However S_L is given by the sum of the REE for $O_1(x_1)|0\rangle$ and $O_2(x_2)|0\rangle$, which include different CFT data. The constraint $S_L = S_R$ actually can be seen as a bridge between different CFT data.

Of course this constraint should be consistent with other constraints imposed by symmetry, such as crossing symmetry, modular invariance on torus, since here we only use the OPE of local operators, which is expected to be true for CFTs.

In this section we will mainly focus on RCFTs. On the one hand, our calculations for $S_L = S_R$ can be seen as a check on the consistency of the replica method to calculate REE for locally excited states. On the other hand it may give us more insight on the

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physical explanation of local excitation. For RCFTs we know the REE is $\log d_O$ for the state $O|0\rangle$ [14]. But it is still not clear why the quantum dimension d_O appears. It is expected this should be related to the topological entanglement entropy for anyons in (2+1)D [32, 33]. Our results give more support on this. We will briefly discuss their relation in section 5.1.

3.2 The states

We continue discussing entanglement properties of the state

$$|\psi\rangle_L := \mathcal{N}(\epsilon)e^{-\epsilon H}O(w_1,\bar{w}_1)O(w_2,\bar{w}_2)\,|0\rangle\,, \tag{3.1}$$

with $w_1 = \bar{w}_1 = -l$ and $w_2 = \bar{w}_2 = 0$. We have shown in section 2.1 that the entanglement entropy for subsystem A ($x > 0$) in late time limit is $2S_A$. It is expected S_A is only related to the information of operator O. But on the other hand the operator $O(w_1,\bar{w}_1)O(w_2,\bar{w}_2)$ can be expanded as follows in (1+1)D CFTs,

$$O(w_1,\bar{w}_1)O(w_2,\bar{w}_2)=\sum_p C_p(w_1-w_2)^{h_p-2h}(\bar{w}_1-\bar{w}_2)^{\bar{h}_p-2\bar{h}}\mathcal{L}(w_1-w_2)\bar{\mathcal{L}}(\bar{w}_1-\bar{w}_2)O_p(w_2,\bar{w}_2), \tag{3.2}$$

with

$$\mathcal{L}(w_1,\bar{w}_2) := \sum_{\{k\}}(w_1 - w_2)^K \beta_p^{\{k\}} L_{-k_1}\dots L_{-k_N}\,, \tag{3.3}$$

where $K = \sum_{i=1}^N k_i$, L_{-k_i} are the Virasoro generators, and $\beta_p^{\{k\}}$ can be fixed with the help of Virasoro algebra. The right hand side of (3.3) seems complicated, but it should exhibit the same conformal properties as the left hand side [35]. Let's denote

$$\mathcal{O}_p(w_2,\bar{w}_2;w_1,\bar{w}_1) := C_p(w_1 - w_2)^{h_p-2h}(\bar{w}_1 - \bar{w}_2)^{\bar{h}_p-2\bar{h}}\mathcal{L}(w_1 - w_2)\bar{\mathcal{L}}(\bar{w}_1 - \bar{w}_2)O_p(w_2,\bar{w}_2). \tag{3.4}$$

Under conformal transformation $w = w(z)$, $\bar{w} = \bar{w}(\bar{z})$, the left hand side of (3.3) transforms as

$$O(z_1,\bar{z}_1)O(z_2,\bar{z}_2) = \left(\prod_{i=1,2}\frac{dw_i}{dz_i}\right)^h\left(\prod_{i=1,2}\frac{d\bar{w}_i}{d\bar{z}_i}\right)^{\bar{h}} O(w_1,\bar{w}_1)O(w_2,\bar{w}_2). \tag{3.5}$$

$\mathcal{O}_p(w_2,\bar{w}_2;w_1,\bar{w}_1)$ should transform by the same law as (3.5). We could define a state

$$|\psi\rangle_R := \mathcal{N}(\epsilon)\sum_p e^{-\epsilon H}\mathcal{O}_p(w_2,\bar{w}_2;w_1,\bar{w}_1)\,|0\rangle\,. \tag{3.6}$$

$|\psi\rangle_R$ can be seen as locally excited state created by a linear combination of primary and descendant operators, which are labeled by p. We have discussed the entanglement entropy of this kind state above. This state depends on the details of the fusion rule of $O\times O$ and the corresponding structure constants. Although the expression for entanglement entropies of $|\psi\rangle_L$ and $|\psi\rangle_R$ look different, they should be equal due to the consistency of OPE. This equality, as we will see later, leads to an algebraic identity.

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3.3 Normalization

Let's first discuss the normalization of state, which are closely associated with the entanglement entropy. From the definition (3.1) we obtain

$$\mathcal{N}(\epsilon) = \frac{1}{\sqrt{\langle O^\dagger(z_1,\bar{z}_1)O^\dagger(z_2,\bar{z}_2)O(z_3,\bar{z}_3)O(z_4,\bar{z}_4)\rangle}}, \tag{3.7}$$

where $z_1 := w_2 - i\epsilon$, $z_2 := w_1 - i\epsilon$, $z_3 := w_1 + i\epsilon$ and $z_4 = w_2 + i\epsilon$. Note that the cross ratio $z = z_{12}z_{34}/z_{13}z_{24} = 1 + O(\epsilon^2)$. Because of the form of OPE in (3.3), the four point appeared in the normalization constant $\mathcal{N}(\epsilon)$ can be written as sum of conformal blocks,

$$\langle O^\dagger(z_1,\bar{z}_1)O^\dagger(z_2,\bar{z}_2)O(z_3,\bar{z}_3)O(z_4,\bar{z}_4)\rangle = (z_{13}z_{24})^{-2h}(\bar{z}_{13}\bar{z}_{24})^{-2\bar{h}}G(z,\bar{z}), \tag{3.8}$$

with

$$G(z,\bar{z}) = \sum_p \mathcal{F}_p(z)\bar{\mathcal{F}}_p(\bar{z}). \tag{3.9}$$

For the state $|\psi\rangle_R$, we rewrite it in the standard form (2.20). One could check $\langle 0|\mathcal{O}_p^\dagger \mathcal{O}_{p'}|0\rangle \sim \delta_{pp'}$ and by definition

$$\langle 0|\, \mathcal{O}_p^\dagger \mathcal{O}_p \,|0\rangle = (z_{13}z_{24})^{-2h}(\bar{z}_{13}\bar{z}_{24})^{-2\bar{h}}\mathcal{F}_p(z)\bar{\mathcal{F}}_p(\bar{z}). \tag{3.10}$$

$|\psi\rangle_R$ can be rewritten as

$$|\psi\rangle_R = \sum_p \sqrt{\lambda_p}\,|p\rangle\,, \tag{3.11}$$

with

$$\sqrt{\lambda_p} = \frac{\mathcal{N}(\epsilon)}{\mathcal{N}_p(\epsilon)}, \quad |p\rangle := \mathcal{N}_p(\epsilon)e^{-\epsilon H}\mathcal{O}_p(w_2,\bar{w}_2;w_1,\bar{w}_1)\,|0\rangle\,, \tag{3.12}$$

where $\mathcal{N}_p(\epsilon)$ is the normalization constant of state $|p\rangle$. We can further simplify λ_p as

$$\lambda_p = \lim_{z,\bar{z}\to 1} \frac{\mathcal{F}_p(z)\bar{\mathcal{F}}_p(\bar{z})}{\sum_p \mathcal{F}_p(z)\bar{\mathcal{F}}_p(\bar{z})}, \tag{3.13}$$

where we take the limit $z,\bar{z} \to 1$ because we would finally take $\epsilon \to 0$ which leads to $z,\bar{z} \to 1$. λ_p will become a real number between 0 and 1, which can be interpreted as the probability.

3.4 Rényi entropy of the state $|p\rangle$

As we can see from (2.25), (2.27), to calculate the Rényi or entanglement entropy one need to know the $S_p^{(n)}$ besides λ_p. The state $|p\rangle$ can be considered as a locally excited state by the following descendant operators,

$$\tilde{O}(w,\bar{w}) := L^-\bar{L}^-O(w,\bar{w}), \tag{3.14}$$

with

$$L^- := \sum_k \alpha_k \prod_i L_{-k_i} \quad \text{and} \quad \bar{L}^- := \sum_{k'} \alpha'_{k'} \prod_{i'} \bar{L}_{-k'_i}, \tag{3.15}$$

where α_k and $\alpha'_{k'}$ are dimensional parameters. In paper [16] the authors have calculated the entanglement entropy of locally excited state by descendant operators for rational CFTs. However, they only consider linear combination of descendant operators with fixed conformal dimensions, i.e.,

$$[L_0 + \bar{L}_0, L^- \bar{L}^-] = (K + \bar{K}) L^- \bar{L}^-, \tag{3.16}$$

where $K := \sum_i k_i$ and $\bar{K} := \sum_{i'} k'_{i'}$ are some constant. By definition (3.4), the states[2] considered in this subsection is quite different from that in [16]. But $\mathcal{O}_p(w_2, \bar{w}_2; w_1, \bar{w}_1)$ is organized as a special form such that it should satisfy the transformation law (3.5). This allows us to use the replica trick as before to calculate the Rényi entanglement entropy.

$\mathcal{O}_p(w_2, \bar{w}_2; w_1, \bar{w}_1)$ can be seen as a non-local operator associated with the coordinates $(w_1, \bar{w}_1)$, $(w_2, \bar{w}_2)$. Consider the state $|p(t)\rangle = e^{-itH} |p\rangle$,

$$|p(t)\rangle = \mathcal{N}_p(\epsilon) \mathcal{O}_p(w_2, \bar{w}_2; w_1, \bar{w}_1) |0\rangle \,, \tag{3.17}$$

where $w_1 = -l_1 + t + i\epsilon$, $w_2 = -l_2 + t + i\epsilon$, $\bar{w}_1 = -l_1 - t - i\epsilon$ and $\bar{w}_2 = -l_2 - t - i\epsilon$. The normalization constat $\mathcal{N}_p$ is given by

$$\mathcal{N}_p(\epsilon) = \frac{1}{\sqrt{\langle 0| \mathcal{O}_p^\dagger(w'_2, \bar{w}'_2; w'_1, \bar{w}'_1) \mathcal{O}_p(w_2, \bar{w}_2; w_1, \bar{w}_1) |0\rangle}}, \tag{3.18}$$

where $w'_1 = -l_1 + t - i\epsilon$, $w'_2 = -l_2 + t - i\epsilon$, $\bar{w}'_1 = -l_1 - t + i\epsilon$ and $\bar{w}'_2 = -l_2 - t + i\epsilon$. From (3.10) we have

$$\langle \mathcal{O}_p^\dagger(w'_2, \bar{w}'_2; w'_1, \bar{w}'_1) \mathcal{O}_p(w_2, \bar{w}_2; w_1, \bar{w}_1) \rangle = |w'_2 - w_1|^{-4h} |w'_1 - w_2|^{-4h} \mathcal{F}_p(w) \bar{\mathcal{F}}_p(\bar{w}), \tag{3.19}$$

where

$$\begin{aligned} w &:= \frac{(w'_2 - w'_1)(w_1 - w_2)}{(w'_2 - w_1)(w'_1 - w_2)} \simeq 1 - \frac{4\epsilon^2}{(l_1 - l_2)^2}, \\ \bar{w} &:= \frac{(\bar{w}'_2 - \bar{w}'_1)(\bar{w}_1 - \bar{w}_2)}{(\bar{w}'_2 - \bar{w}_1)(\bar{w}'_1 - \bar{w}_2)} \simeq 1 + \frac{4\epsilon^2}{(l_1 - l_2)^2}. \end{aligned} \tag{3.20}$$

In the limit $\epsilon \to 0$, $w, \bar{w} \to 1$. In this limit we expect the conformal block $\mathcal{F}_p(w) \sim (1-w)^{-2h} \sim \epsilon^{-4h}$, where we only keep the most divergent term.[3] Now we could use the replica method to calculate the Rényi entropy for subsystem A, with $x > 0$. We could express the difference of Rényi entropy between state $|p(t)\rangle$ and vacuum state $\Delta S^{(n)}_{A,p}(|p(t)\rangle)$ as

$$\Delta S^{(n)}_{A,p}(|p(t)\rangle) = \frac{1}{1-n} \left(\log \frac{\langle \prod_s^n \mathcal{O}_p^\dagger(w'_{s,2}, \bar{w}'_{s,2}; w'_{s,1}, \bar{w}'_{s,1}) \mathcal{O}_p(w_{s,2}, \bar{w}_{s,2}; w_{s,1}, \bar{w}_{s,1}) \rangle_{\mathcal{R}_n}}{\langle \mathcal{O}_p^\dagger(w'_2, \bar{w}'_2; w'_1, \bar{w}'_1) \mathcal{O}_p(w_2, \bar{w}_2; w_1, \bar{w}_1) \rangle^n} \right), \tag{3.21}$$

[2] Here we consider the state is a summation of all possible descendant states.

[3] We will take some examples to illustrate this phenomenon in the following subsections. In rational CFTs $\mathcal{F}_p(w) = \sum_q F_{pq} \mathcal{F}_q(1-w)$, the leading contribution comes from $q = 0$, thus $\mathcal{F}_p(w) \simeq F_{p0}(1-w)^{-2h}$.

where $(w'_{s,i}, \bar{w}_{s,i})$ and $(w_{s,i}, \bar{w}_{s,i})$ $(i = 1, 2$ and $s = 1, \dots, n)$ are the replica coordinates on the s-th sheet of $\mathcal{R}_n$. We could make a conformal transformation $w = z^n$, so that $\mathcal{R}_n$ is mapped to the complex plane C. By using the transformation law of $\mathcal{O}_p$, which is same as (2.7), we have

$$\begin{aligned} &\Big\langle \prod_s^n \mathcal{O}_p^\dagger(w'_{s,2}, \bar{w}'_{s,2}; w'_{s,1}, \bar{w}'_{s,1}) \mathcal{O}_p(w_{s,2}, \bar{w}_{s,2}; w_{s,1}, \bar{w}_{s,1}) \Big\rangle_{\mathcal{R}_n} \\ &\quad = C_n \Big\langle \prod_s^n \mathcal{O}_p^\dagger(z'_{s,2}, \bar{z}'_{s,2}; z'_{s,1}, \bar{z}'_{s,1}) \mathcal{O}_p(z_{s,2}, \bar{z}_{s,2}; z_{s,1}, \bar{z}_{s,1}) \Big\rangle, \end{aligned} \tag{3.22}$$

where

$$\begin{aligned} C_n &= \prod_s^n \prod_{i=1,2} \left(\frac{dw_{s,i}}{dz_{s,i}}\right)^{-h} \left(\frac{dw'_{s,i}}{dz'_{s,i}}\right)^{-h} \left(\frac{d\bar{w}_{s,i}}{d\bar{z}_{s,i}}\right)^{-h} \left(\frac{d\bar{w}'_{s,i}}{d\bar{z}'_{s,i}}\right)^{-h} \\ &= \prod_{i=1,2} \left(\frac{1}{n^2(l_i^2 - t^2)}\right)^{2h} \prod_s^n (z_{s,i}\bar{z}_{s,i}z'_{s,i}\bar{z}'_{s,i})^h. \end{aligned} \tag{3.23}$$

Firstly, let's consider $t < l_i$, as we can see from (2.8),

$$\begin{aligned} z_{s,1} - z'_{s,1} &\simeq -\frac{2i\epsilon}{n(l-t)} z_{s,1}, & \bar{z}_{s,1} - \bar{z}'_{s,1} &\simeq \frac{2i\epsilon}{n(l-t)} \bar{z}_{s,1} \\ z_{s,2} - z'_{s,2} &\simeq -\frac{2i\epsilon}{n(l-t)} z_{s,2}, & \bar{z}_{s,2} - \bar{z}'_{s,2} &\simeq \frac{2i\epsilon}{n(l-t)} \bar{z}_{s,2}. \end{aligned} \tag{3.24}$$

Therefore, the leading contribution is given by

$$C_n \Big\langle \prod_s^n \mathcal{O}_p^\dagger(z'_{s,2}, \bar{z}'_{s,2}; z'_{s,1}, \bar{z}'_{s,1}) \mathcal{O}_p(z_{s,2}, \bar{z}_{s,2}; z_{s,1}, \bar{z}_{s,1}) \sim \epsilon^{-8nh}. \tag{3.25}$$

Taking the results into (3.21), we have $\Delta S_{A,p}^{(n)}(|p(t)\rangle) = 0$.

For $t > l_i$, the coordinates $(z_{s,i}, \bar{z}_{s,i})$ would have a different behavior (2.9). $\mathcal{O}_p$ can be taken as a linear combination of descendant states like the form (3.14). The correlation functions of descendant operators are associated with the correlation functions of primary operators by means of linear differential operators, i.e.,

$$\begin{aligned} &\Big\langle \prod_s^n \mathcal{O}_p^\dagger(z'_{s,2}, \bar{z}'_{s,2}; z'_{s,1}, \bar{z}'_{s,1}) \mathcal{O}_p(z_{s,2}, \bar{z}_{s,2}; z_{s,1}, \bar{z}_{s,1}) \Big\rangle \\ &\quad = \mathcal{L}\bar{\mathcal{L}} \Big\langle \prod_s O_p^\dagger(z'_{s,2}, \bar{z}'_{s,2}) O_p(z_{s,2}, \bar{z}_{s,2}) \Big\rangle. \end{aligned} \tag{3.26}$$

The $\mathcal{L}$ is a differential operator as a function $\mathcal{L}(z_{s,1} - z_{s,2}, z'_{s,1} - z'_{s,2})$ because of the form (3.4). The action of anti-holomorphic operator $\bar{\mathcal{L}}$ on the anti-holomorphic partial wave is the same as that of $\mathcal{L}$.

To simplify the notation let's consider $n = 2$,and the generalization to arbitrary n is straightforward. For $n = 2$, we have

$$
\begin{aligned}
&\langle \mathcal{O}_p^\dagger(z'_{1,2},\bar{z}'_{1,2};z'_{1,1},\bar{z}'_{1,1})\mathcal{O}_p(z_{1,2},\bar{z}_{1,2};z_{1,1},\bar{z}_{1,1})\mathcal{O}_p^\dagger(z'_{2,2},\bar{z}'_{2,2};z'_{2,1},\bar{z}'_{2,1})\mathcal{O}_p(z_{2,2},\bar{z}_{2,2};z_{2,1},\bar{z}_{2,1})\rangle\\
&=\mathcal{L}(z_{1,1}-z_{1,2},z_{2,1}-z_{2,2},z'_{1,1}-z'_{1,2},z'_{2,1}-z'_{2,2})\bar{\mathcal{L}}(\bar{z}_{1,1}-\bar{z}_{1,2},\bar{z}_{2,1}-\bar{z}_{2,2},\bar{z}'_{1,1}-\bar{z}'_{1,2},\bar{z}'_{2,1}-\bar{z}'_{2,2})\\
&\langle O_p^\dagger(z'_{1,2},\bar{z}'_{1,2})O_p(z_{1,2},\bar{z}_{1,2})O_p^\dagger(z'_{2,2},\bar{z}'_{2,2})O_p(z_{2,2},\bar{z}_{2,2})\rangle\\
&=\mathcal{L}(z_{1,1}-z_{1,2},z_{2,1}-z_{2,2},z'_{1,1}-z'_{1,2},z'_{2,1}-z'_{2,2})\bar{\mathcal{L}}(\bar{z}_{1,1}-\bar{z}_{1,2},\bar{z}_{2,1}-\bar{z}_{2,2},\bar{z}'_{1,1}-\bar{z}'_{1,2},\bar{z}'_{2,1}-\bar{z}'_{2,2})\\
&\sum_m\langle O_p^\dagger(z'_{1,2})O_p(z_{1,2})\,|m\rangle\,\langle m|\,O_p^\dagger(z'_{2,2})O_p(z_{2,2})\rangle\langle O_p^\dagger(\bar{z}'_{1,2})O_p(\bar{z}_{1,2})\,|m\rangle\,\langle m|\,O_p^\dagger(\bar{z}'_{2,2})O_p(\bar{z}_{2,2})\rangle\\
&=\mathcal{L}(z_{1,1}-z_{1,2},z_{2,1}-z_{2,2},z'_{1,1}-z'_{1,2},z'_{2,1}-z'_{2,2})\bar{\mathcal{L}}(\bar{z}_{1,1}-\bar{z}_{1,2},\bar{z}_{2,1}-\bar{z}_{2,2},\bar{z}'_{1,1}-\bar{z}'_{1,2},\bar{z}'_{2,1}-\bar{z}'_{2,2})\\
&\sum_m F^p_{mn}\langle O_p^\dagger(z'_{1,2})O_p(z_{2,2})\,|n\rangle\,\langle n|\,O_p^\dagger(z'_{2,2})O_p(z_{1,2})\rangle\langle O_p^\dagger(\bar{z}'_{1,2})O_p(\bar{z}_{1,2})\,|m\rangle\,\langle m|\,O_p^\dagger(\bar{z}'_{2,2})O_p(\bar{z}_{2,2})\rangle\\
&=\sum_m F^p_{mn}\langle\mathcal{O}_p^\dagger(z'_{1,2};z'_{1,1})\mathcal{O}_p(z_{2,2};z_{2,1})|n\rangle\langle n|\mathcal{O}_p^\dagger(z'_{2,2};z'_{2,1})\mathcal{O}_p(z_{1,2};z_{1,1})\rangle\langle\mathcal{O}_p^\dagger(\bar{z}'_{1,2};\bar{z}'_{1,1})\mathcal{O}_p(\bar{z}_{1,2};\bar{z}_{1,1})|m\rangle\\
&\langle m|\,\mathcal{O}_p^\dagger(\bar{z}'_{2,2};\bar{z}'_{2,1})\mathcal{O}_p(\bar{z}_{2,2};\bar{z}_{2,1})\rangle,\\
&\simeq F^p_{00}\langle\mathcal{O}_p^\dagger(z'_{1,2},\bar{z}'_{1,2};z'_{1,1},\bar{z}'_{1,1})\mathcal{O}_p(z_{2,2},\bar{z}_{1,2};z_{2,1},\bar{z}_{1,1})\rangle\\
&\quad\times\langle\mathcal{O}_p^\dagger(z'_{2,2},\bar{z}'_{2,2};z'_{2,1},\bar{z}'_{2,1})\mathcal{O}_p(z_{1,2},\bar{z}_{2,1};z_{1,1},\bar{z}_{2,1})\rangle
\end{aligned}\tag{3.27}
$$

We will explain the above statement more clearly. In the first equality, we write the correlation function of $\mathcal{O}_p$ as correlation function on primary operators O_p with some differential operator. In the second equality, we write the correlation function of O_p as conformal blocks, $|m\rangle$ denote the m-th Virasoro module. In the third equality, we transfer the expansion into t-channel. Here we assume the theory is a rational CFT, so that different expansion is related to each other by the fusion matrix F^p_{mn}. In the fourth equality, we act the differential operators on the correlator again. The operators appeared in the correlator are the corresponding descendant operators $\mathcal{O}_p$. Note that since we have changed the position of coordinates in the third equality, the descendant operators $\mathcal{O}_p$ will also change according the right order of coordinates. Finally in the fifth equality, we keep the leading contributions. Since we have the relation (2.9), only the identity channel gives the most dominant contributions. In the last step we rearrange the holomorphic and anti-holomorphic part together.

One could calculate the final quantity in (3.27) by (3.10). Taking the result into (3.21) we find $\Delta S^{(2)}_{A,p}(|p(t)\rangle) = -\log F^p_{00} = \log d_p$. It is straightforward to generalize the statement into arbitrary n.

3.5 The induced equality from entanglement entropy

Using the result (2.27), we obtain the entanglement entropy S_R for subsystem A $(x > 0)$ in late-time limit,

$$S_R = -\sum_p \lambda_p \log\lambda_p + \sum_p \lambda_p S_p, \tag{3.28}$$

where λ_p is defined as (3.13), S_p is the entanglement entropy of state $|p\rangle$. Since $|p\rangle$ is the locally excited state by descendant operators defined by eq. (3.14). We have shown in section 3.4 the entanglement entropy of this type of state is same as the primary state O_p. In the rational CFT we know $S_p = \log d_p$, where d_p is the quantum dimension of operator O_p. $S_L = \log d_O^2$ only depends on the quantum dimension d_O of operator O. So we have a constraint by $S_L = S_R$,

$$\log d_O^2 = -\sum_p \lambda_p \log \lambda_p + \sum_p \lambda_p \log d_p. \tag{3.29}$$

The solution of above equation is $\lambda_p = d_p/d_O^2$. Therefore we obtain the following identity:

$$\lim_{z,\bar{z}\to 1} \frac{\mathcal{F}_p(z)\bar{\mathcal{F}}_p(\bar{z})}{\sum_p \mathcal{F}_p(z)\bar{\mathcal{F}}_p(\bar{z})} = \frac{d_p}{d_O^2}. \tag{3.30}$$

Conformal blocks have the following transformation rule for rational CFTs,

$$\mathcal{F}_p(z) = \sum_q F_{pq}\mathcal{F}_q(1-z), \qquad \bar{\mathcal{F}}_p(\bar{z}) = \sum_q \bar{F}_{pq}\bar{\mathcal{F}}_q(1-\bar{z}) \tag{3.31}$$

where F_{qp} is the fusion matrix [37, 38].

In the limit $z, \bar{z} \to 1$, we have

$$\mathcal{F}_q(1-z) \simeq (1-z)^{h_q - 2h}. \tag{3.32}$$

The leading contribution is $q = 0$. Thus (3.30) can be further simplified to the relation between fusion matrixes and quantum dimensions of operators

$$\frac{F_{p0}\bar{F}_{p0}}{\sum_q F_{q0}\bar{F}_{q0}} = \frac{d_p}{d_O^2}. \tag{3.33}$$

The Rényi entropy $S_L^{(n)}$ of $|\psi\rangle_L$, which is independent of n in rational CFTs, equals to the entanglement entropy. Actually combing (2.26), the solution $\lambda_p = d_p/d_O^2$, and the fact $S_p^{(n)} = \log d_p$ in rational CFTs, we could obtain $S_R^{(n)}$,

$$S_R^{(n)} = \frac{\log \sum_p d_p/d_O^{2n}}{1-n} = \log d_O^2, \tag{3.34}$$

where we use the equality of quantum dimensions $\sum_p d_p = d_O^2$. Therefore, we again obtain a consistent result $S_L^{(n)} = S_R^{(n)}$.

3.6 More general cases

We have considered the product state $|\psi\rangle_2 = O_1 O_2 |0\rangle$ with $O_1 = O_2 = O$, it is not hard to generalize to the case $O_1 \neq O_2$. Define the state

$$|\phi(t)\rangle_L := \mathcal{N}(\epsilon; |\phi\rangle_L) e^{itH-\epsilon H} O_1(-l_1, 0) O_2(-l_2, 0) |0\rangle\,, \tag{3.35}$$

where we still assume $|l_1 - l_2| \gg \epsilon$. The sum rule will be still right, for $t > l_i$, we have

$$\Delta S_A^{(n)}(|\phi(t)_L\rangle) \simeq \Delta S_A^{(n)}(O_1|0\rangle) + \Delta S_A^{(n)}(O_2|0\rangle). \tag{3.36}$$

On the other hand we have OPE

$$O_1(w_1,\bar{w}_1)O_2(w_2,\bar{w}_2)=\sum_p \mathcal{O}_p^{12}(w_2,\bar{w}_2;w_1,\bar{w}_1), \tag{3.37}$$

with

$$\begin{aligned}&\mathcal{O}_p^{12}(w_2,\bar{w}_2;w_1,\bar{w}_1)\\ &:=\sum_p C_{12p}(w_1-w_2)^{h_p-h_1-h_2}(\bar{w}_1-\bar{w}_2)^{\bar{h}_p-h_1-h_2}\mathcal{L}(w_1-w_2)\bar{\mathcal{L}}(\bar{w}_1-\bar{w}_2)O_p(w_2,\bar{w}_2).\end{aligned} \tag{3.38}$$

Define the state

$$\begin{aligned}|\phi(t)\rangle_R &:= \mathcal{N}(\epsilon;|\phi\rangle_R)\sum_p e^{itH-\epsilon H}\mathcal{O}_p^{12}(l_2,l_2;l_1,l_1)\,|0\rangle\\ &= \mathcal{N}(\epsilon;|\phi\rangle_R)\sum_p \mathcal{O}_p^{12}(w_2,\bar{w}_2;w_1,\bar{w}_1)\,|0\rangle\,,\end{aligned} \tag{3.39}$$

where $w_1=-l_1+t+i\epsilon$, $w_2=-l_2+t+i\epsilon$, $\bar{w}_1=-l_1-t-i\epsilon$ and $\bar{w}_2=-l_2-t-i\epsilon$. The normalization constant $\mathcal{N}(\epsilon;|\phi\rangle_R)$ is same as $\mathcal{N}(\epsilon;|\phi\rangle_L)$, which is given by

$$\mathcal{N}(\epsilon;|\phi\rangle_L)=\frac{1}{\sqrt{\langle O_2^\dagger(w_2',\bar{w}_2')O_1^\dagger(w_1',\bar{w}_1')O_1(w_1,\bar{w}_1)O_2(w_2,\bar{w}_2)\rangle}}, \tag{3.40}$$

where $w_1'=-l_1+t-i\epsilon$, $w_2'=-l_2+t-i\epsilon$, $\bar{w}_1'=-l_1-t+i\epsilon$ and $\bar{w}_2'=-l_2-t+i\epsilon$. For the OPE block we have the normalization

$$\begin{aligned}&\langle 0|\,\mathcal{O}_p^{12\dagger}(w_2',\bar{w}_2';w_1',\bar{w}_1')\mathcal{O}_p^{12}(w_2,\bar{w}_2;w_1,\bar{w}_1)\,|0\rangle\\ &=|w_2'-w_1|^{-2(h_1+h_2)}|w_1'-w_2|^{-2(h_1+h_2)}|w_2'-w_2|^{2(h_1-h_2)}|w_1'-w_1|^{2(h_2-h_1)}\mathcal{F}_p^{12}(w)\bar{\mathcal{F}}_p^{12}(\bar{w})\end{aligned} \tag{3.41}$$

where $w:=(w_2'-w_1')(w_1-w_2)/(w_2'-w_1)(w_1'-w_2)$, $\mathcal{F}_p^{12}(w)$ is the conformal block. Define

$$\mathcal{N}_p(\epsilon;|\phi\rangle_R:=\frac{1}{\sqrt{\langle 0|\,\mathcal{O}_p^{12\dagger}(w_2',\bar{w}_2';w_1',\bar{w}_1')\mathcal{O}_p^{12}(w_2,\bar{w}_2;w_1,\bar{w}_1)\,|0\rangle}}. \tag{3.42}$$

We could rewrite $|\phi(t)\rangle_R$ as the standard form (2.16),

$$|\phi(t)\rangle_R=\sqrt{\lambda_p^{12}}\,|p\rangle^{12}\,, \tag{3.43}$$

with

$$\begin{aligned}|p\rangle^{12} &:= \mathcal{N}_p(\epsilon;|\phi\rangle_R)\mathcal{O}_p^{12}(w_2,\bar{w}_2;w_1,\bar{w}_1)\,|0\rangle\\ \lambda_p &:= \lim_{\epsilon\to 0}\frac{\mathcal{N}(\epsilon;|\phi\rangle_R)}{\mathcal{N}_p(\epsilon;|\phi\rangle_R)}.\end{aligned} \tag{3.44}$$

Using the similar argument in section 3.4, we have the Rényi entropy S_p^{12} of state $|p\rangle$ for $t>l_i$ is same as the locally excited state by primary operator O_p, which is $\log d_p$ in rational CFTs. Comparing with the entanglement entropy of $|\phi(t)\rangle_L$ and $|\phi(t)\rangle_R$, we have the constraint,

$$\log(d_1 d_2) = -\sum_p \lambda_p^{12} \log \lambda_p^{12} + \sum_p \lambda_p^{12} \log d_p, \tag{3.45}$$

which gives

$$\lambda_p^{12} = \frac{d_p}{d_1 d_2}. \tag{3.46}$$

λ_p^{12} is only related to conformal blocks. Finally we have

$$\lim_{w\to 1} \frac{\mathcal{F}_p^{12}(w)\bar{\mathcal{F}}_p^{12}(\bar{w})}{\sum_p \mathcal{F}_p^{12}(w)\bar{\mathcal{F}}_p^{12}(\bar{w})} = \frac{d_p}{d_1 d_2}. \tag{3.47}$$

It is also straightforward to generalize to the general product state (2.3).

3.7 Some examples

In this subsection we will show some examples to check the relation (3.30), (3.33) and (3.47).

3.7.1 Free massless scalar field

Consider the vertex operator $\mathcal{V}_\alpha = e^{i\alpha\phi}$, which has the fusion rule $\mathcal{V}_\alpha \times \mathcal{V}_\beta = \mathcal{V}_{\alpha+\beta}$. So there is only one fusion channel, the result is consistent with the fact the quantum dimension of $\mathcal{V}_\alpha$ is one.

For operator $\mathcal{O}_\alpha := \frac{1}{\sqrt{2}}(\mathcal{V}_\alpha + \mathcal{V}_{-\alpha})$, we have the fusion rule $\mathcal{O}_\alpha \times \mathcal{O}_\alpha = I + \mathcal{O}_{2\alpha}$. The four point correlation function of $\mathcal{O}_\alpha$,

$$\begin{aligned}&\langle \mathcal{O}_\alpha(z_1,\bar{z}_1)\mathcal{O}_\alpha(z_2,\bar{z}_2)\mathcal{O}_\alpha(z_3,\bar{z}_3)\mathcal{O}_\alpha(z_4,\bar{z}_4)\rangle \\ &= (|z_{12}||z_{34}|)^{-4h_{\mathcal{O}_\alpha}}\Big(|\mathcal{F}_I^1|^2 + |\mathcal{F}_I^2|^2 + |\mathcal{F}_{\mathcal{O}_{2\alpha}}|^2\Big),\end{aligned} \tag{3.48}$$

with

$$\begin{aligned}|\mathcal{F}_I^1|^2 &= |\mathcal{F}_I^2|^2 = |1-z|^{-4h_{\mathcal{O}_\alpha}} + |1-z|^{4h_{\mathcal{O}_\alpha}}, \\ |\mathcal{F}_{\mathcal{O}_{2\alpha}}|^2 &= 2|z|^{8h_{\mathcal{O}_\alpha}}|1-z|^{-2h_{\mathcal{O}_\alpha}},\end{aligned} \tag{3.49}$$

where the fusion channel of I has two possible ways, we label them as 1 and 2. We have

$$\begin{aligned}\lambda_I^1 &= \lambda_I^2 = \lim_{z,\bar{z}\to 1} \frac{|\mathcal{F}_I^1|^2}{|\mathcal{F}_I^1|^2 + |\mathcal{F}_I^2|^2 + |\mathcal{F}_{\mathcal{O}_{2\alpha}}|^2} = \frac{1}{4}, \\ \lambda_{\mathcal{O}_{2\alpha}}^1 &= \lim_{z,\bar{z}\to 1} \frac{|\mathcal{F}_{\mathcal{O}_{2\alpha}}^1|^2}{|\mathcal{F}_I^1|^2 + |\mathcal{F}_I^2|^2 + |\mathcal{F}_{\mathcal{O}_{2\alpha}}|^2} = \frac{1}{2}.\end{aligned} \tag{3.50}$$

This is consistent with $\lambda_I^1 = \lambda_I^2 = \frac{d_I}{d_{\mathcal{O}_\alpha}^2}$ and $\lambda_{\mathcal{O}_{2\alpha}} = \frac{d_{\mathcal{O}_{2\alpha}}}{d_{\mathcal{O}_\alpha}^2}$, where $d_I = 1$, $d_{\mathcal{O}_{2\alpha}} = d_{\mathcal{O}_\alpha} = 2$.

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3.7.2 Ising model or Minimal model $\mathcal{M}(p=4, p'=3)$

Ising model [39] at critical point has three primary operator I, ϵ and σ, which satisfy the fusion rule,

$$\epsilon \times \epsilon = I, \quad \sigma \times \sigma = I + \epsilon. \tag{3.51}$$

The quantum dimension of ϵ is 1, $\epsilon \times \epsilon$ has only one fusion channel, which is trivially consistent with the result (3.30). The four point correlation function [40]

$$\begin{aligned}&\langle \sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)\sigma(z_3,\bar{z}_3)\sigma(z_4,\bar{z}_4)\rangle \\ &= (|z_{12}z_{34}|)^{4h_\sigma}\left(|\mathcal{F}_I(z)|^2 + C_{\sigma\sigma\epsilon}|\mathcal{F}_\epsilon(z)|^2\right),\end{aligned} \tag{3.52}$$

with

$$\mathcal{F}_I(z) = \frac{(1-z)^{3/8}}{z^{1/8}\sqrt{2}}\left(\frac{1+\sqrt{1-z}}{1-z}\right)^{1/2}, \quad \mathcal{F}_\epsilon(z) = \sqrt{2}\frac{(1-z)^{3/8}}{z^{1/8}}\left(\frac{1-\sqrt{1-z}}{1-z}\right)^{1/2}, \tag{3.53}$$

One could check

$$\begin{aligned}\lambda_I &= \lim_{z,\bar{z}\to 1}\frac{\mathcal{F}_I(z)\bar{\mathcal{F}}_I(\bar{z})}{\mathcal{F}_I(z)\bar{\mathcal{F}}_I(\bar{z}) + C^2_{\sigma\sigma\epsilon}\mathcal{F}_\epsilon(z)\bar{\mathcal{F}}_\epsilon(\bar{z})} = \frac{1}{2}, \\ \lambda_\epsilon &= \lim_{z,\bar{z}\to 1}\frac{C^2_{\sigma\sigma\epsilon}\mathcal{F}_\epsilon(z)\bar{\mathcal{F}}_\epsilon(\bar{z})}{\mathcal{F}_I(z)\bar{\mathcal{F}}_I(\bar{z}) + C^2_{\sigma\sigma\epsilon}\mathcal{F}_\epsilon(z)\bar{\mathcal{F}}_\epsilon(\bar{z})} = \frac{1}{2},\end{aligned} \tag{3.54}$$

which is consistent with $\lambda_I = \frac{d_I}{d_\sigma^2} = \frac{1}{2}$ and $\lambda_\epsilon = \frac{d_\epsilon}{d_\sigma^2} = \frac{1}{2}$.

3.8 Operator $\phi_{(2,1)}\phi_{(r,s)}$ in Minimal model

In this subsection we consider an example which has product of different operators. We choose the operators $\phi_{(2,1)}$ and $\phi_{(r,s)}$, with the fusion rule

$$\phi_{(2,1)} \times \phi_{(r,s)} = \phi_{(r-1,s)} + \phi_{(r+1,s)}. \tag{3.55}$$

We will consider the state $\phi_{(2,1)}\phi_{(r,s)}|0\rangle$. The four point correlation function [35, 41, 42] is

$$\begin{aligned}&\langle \phi^\dagger_{(r,s)}(w_2',\bar{w}_2')\phi^\dagger_{(2,1)}(w_1',\bar{w}_1')\phi_{(2,1)}(w_1,\bar{w}_1)\phi_{(r,s)}(w_2,\bar{w}_2)\rangle \\ &\sim \left[\frac{\sin(b\pi)\sin(2b+a)\pi}{\sin(a+b)\pi}|I_1(w;a,b)|^2 + \frac{\sin(a\pi)\sin(b\pi)}{\sin(a+b)\pi}|I_2(w;a,b)|^2\right],\end{aligned} \tag{3.56}$$

with

$$\begin{aligned}I_1(w;a,b) &= \frac{\Gamma(-a-2b-1)\Gamma(b+1)}{\Gamma(-a-b)}\ {}_2F_1(-b,-a-2b-1,-a-b,w), \\ I_2(w;a,b) &= w^{1+a+2b}\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}\ {}_2F_1(-b,a+1,a+b+2,w),\end{aligned} \tag{3.57}$$

where $a := [p(1-r) - p'(1-s)]/p'$, $b = -p/p'$. The $|I_2(w)|^2$ part is associated with the conformal block of $\phi_{(r-1,s)}$,

$$\begin{aligned}\mathcal{F}_{\phi_{(r-1,s)}}\bar{\mathcal{F}}_{\phi_{(r-1,s)}} &\sim \frac{\sin(a\pi)\sin(b\pi)}{\sin(a+b)\pi}|I_2(w)|^2 \\ \mathcal{F}_{\phi_{(r+1,s)}}\bar{\mathcal{F}}_{\phi_{(r+1,s)}} &\sim \frac{\sin(b)\pi\sin(2b+a)\pi}{\sin(a+b)\pi}|I_1(w)|^2\end{aligned} \tag{3.58}$$

$I_{1(2)}(w)$ satisfies the following transformation relation,

$$\begin{aligned} I_1(w;a,b) &= \frac{\sin(a)\pi}{\sin(2b\pi)} I_1(1-w;b,a) - \frac{\sin(b\pi)}{\sin(2b\pi)} I_2(1-w;b,a) \\ I_2(w;a,b) &= -\frac{\sin(a+2b)\pi}{\sin(2b\pi)} I_1(1-w;b,a) - \frac{\sin(b\pi)}{\sin(2b\pi)} I_2(1-w;b,a). \end{aligned} \tag{3.59}$$

By using all the result we have

$$\lambda_{\phi_{(r+1,s)}} = \frac{\sin(2b+a)\pi}{\sin(a\pi)+\sin(2b+a)\pi}, \quad \lambda_{\phi_{(r-1,s)}} = \frac{\sin(a\pi)}{\sin(a\pi)+\sin(2b+a)\pi}, \tag{3.60}$$

which can be simplified to

$$\begin{aligned} \lambda_{\phi_{(r+1,s)}} &= \frac{\sin[(1+r)\pi p/p']\pi}{\sin[(1+r)\pi p/p']\pi + \sin[(r-1)\pi p/p']\pi}, \\ \lambda_{\phi_{(r-1,s)}} &= \frac{\sin[(r-1)\pi p/p']\pi}{\sin[(1+r)\pi p/p']\pi + \sin[(r-1)\pi p/p']\pi}, \end{aligned} \tag{3.61}$$

where we have used the fact s is an integer, so the result is independent on s. The quantum dimension of operator $\phi_{(r,s)}$ in Minimal model is defined by [38]

$$d_{\phi_{(r,s)}} = \frac{\mathcal{S}_{(1,1),(r,s)}}{\mathcal{S}_{(1,1),(1,1)}}, \tag{3.62}$$

where $\mathcal{S}_{(r_1,s_1),(r_2,s_2)}$ is the S-matrix of modular transformation. The S-matrix is given by

$$\mathcal{S}_{(r_1,s_1),(r_2,s_2)} = 2\sqrt{\frac{2}{pp'}}(-1)^{1+r_2 s_1 + r_1 s_2} \sin\left(\frac{\pi p}{p'} r_1 r_2\right) \sin\left(\frac{\pi p'}{p} s_1 s_2\right). \tag{3.63}$$

We have

$$d_{\phi_{(2,1)}} = -\frac{\sin(2\pi p/p')}{\sin(\pi p/p')}, \quad d_{\phi_{(r,s)}} = (-)^{r+s} \frac{\sin(r\pi p/p')\sin(s\pi p'/p)}{\sin(\pi p/p')\sin(\pi p'/p)}. \tag{3.64}$$

One could check the relations

$$\lambda_{\phi_{(r+1,s)}} = \frac{d_{\phi_{(r+1,s)}}}{d_{\phi_{(2,1)}} d_{\phi_{(r,s)}}}, \quad \lambda_{\phi_{(r-1,s)}} = \frac{d_{\phi_{(r-1,s)}}}{d_{\phi_{(2,1)}} d_{\phi_{(r,s)}}}. \tag{3.65}$$

4 Proof of identity (3.47)

In this section, we prove the identity shown in eqs. (3.31), (3.47) using the language of modular tensor category. We will see that only the "tensor" part of the category is involved. We start with reviewing some relevant concepts: a tensor category $\mathcal{C}$ is a set of data $\{\mathrm{Obj}(\mathcal{C}), d, N, F\}$ that satisfy some consistency conditions. The set $\mathrm{Obj}(\mathcal{C})$ consists of superselection sectors $a, b, c\cdots$. Quantum dimension d_a assigns a real number to each sector $a \in \mathrm{Obj}(\mathcal{C})$, and the rank-three tensor N_{ab}^c describes fusion rules between the sectors:

$$a \times b = \sum_c N_{ab}^c c. \tag{4.1}$$

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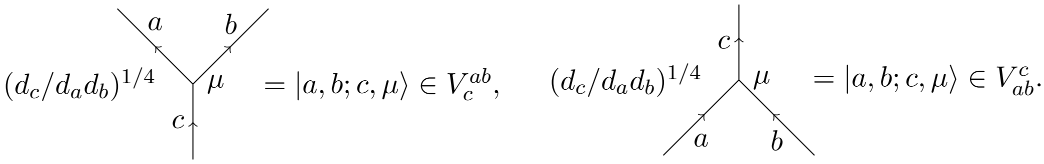

Figure 1. Graphical representation of fusion and splitting.

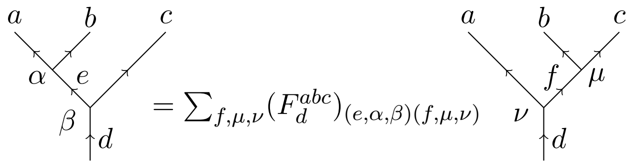

Figure 2. Graphical representation of F-move.

Each entry N^c_{ab} is a non-negative integer counting the number of different channels that a and b can be combined to produce the c. In rational CFTs, the fusion is finite which means $\sum_c N^c_{ab}$ is a finite integer.

The quantum dimensions are consistent with the fusion rules,

$$d_a d_b = \sum_c N^c_{ab} d_c. \tag{4.2}$$

Each fusion product $a \times b \to c$ has an associated vector space V^c_{ab} and its dual splitting space V^{ab}_c. The dimension of this vector space is $\dim V^c_{ab} = N^c_{ab}$. There are two different ways to fuse a, b and c into d, related by associativity in the form of the following isomorphism:

$$V^d_{abc} \cong \bigoplus_e V^e_{ab} \otimes V^d_{ec} \cong \bigoplus_f V^f_{bc} \otimes V^d_{af}, \tag{4.3}$$

In terms of N tensor, this simply leads to

$$\sum_e N^e_{ab} N^d_{ec} = \sum_f N^d_{af} N^f_{bc}. \tag{4.4}$$

Finally, we introduce the F tensor. We will use the following graphical representation in figure 1 for the basis in V^{ab}_c and V^c_{ab}, where $\mu = 1, \ldots, N^c_{ab}$:

The changing of basis in (4.3) are then realized through the F-moves in figure 2.

These F-moves are unitary transformations,

$$\left[\left(F^{abc}_d\right)^{-1}\right]_{(f,\mu,\nu)(e,\alpha,\beta)} = \left[\left(F^{abc}_d\right)^{\dagger}\right]_{(f,\mu,\nu)(e,\alpha,\beta)} = \left[F^{abc}_d\right]^*_{(e,\alpha,\beta)(f,\mu,\nu)}. \tag{4.5}$$

Additionally, we have the useful resolution of identity as shown in figure 3.

For a tensor category, we should further require the F-moves to satisfy the Pentagon equation corresponding to the associativity conditions involving five external legs in total. For a modular tensor category, a consistent braiding structure, the Hexagon identity and modularity of the S-matrix are required. We will omit the further details since they are not necessary for the proof.

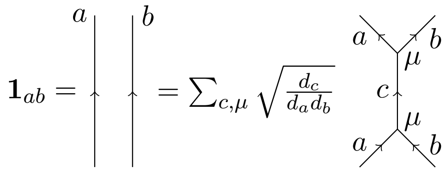

Figure 3. Resolution of identity.

Now we give the proof of the desired identity. For simplicity of narration, we assume the fusion rules are multiplicity-free, i.e. $N_{ab}^c \in \{0,1\}$, so that the indices $\alpha, \beta, \cdots$ on the vertices can be omitted. The most general case can be recovered straightforwardly by adding them back and perform summations over these indices when appropriate.

In figure 2, we observe that in order for the F symbol $[F_{\bar{O}}^{O\bar{O}O}]_{p0}$ to be nonzero, fusion N_{OO}^0 must be nonzero. This would indicate $O = \bar{O}$, so that we can omit the arrows in the diagrams and suppress the external leg indices of the F symbol: $[F_O^{OOO}]_{p0} = [F_{\bar{O}}^{O\bar{O}O}]_{p0} \equiv F_{p0}$. The identity to be proved can then be rewritten as

$$d_O^2 F_{p0}^2 = d_p N_{OO}^p \sum_q F_{q0}^2 N_{OO}^q. \tag{4.6}$$

We notice that in the graphical representations, one has freedom to add trivial lines 0 anywhere in any graph, as it has no physical consequence. Upon adding a trivial line 0 on the left hand side in the resolution of identity to connect the a and b lines, identifying $a = b = O$, $c = p$ and comparing with the definition of F symbols, one observes that the coefficients on the right hand side of the resolution of identity in figure 3 exactly gives $[F_O^{OOO}]_{0p}$:

$$F_{0p} = \sqrt{d_p/d_O^2} N_{OO}^p. \tag{4.7}$$

From the unitarity of the F symbols in eq. (4.5), we have $F_{0p}^{-1} = F_{0p}^\dagger$. Since the labels O are self-dual, one can rotate the external legs as in figure 4, leading to

$$F_{p0} = (F_{0p}^\dagger)^* = (F_{0p}^{-1})^* = F_{0p}^* = \sqrt{d_p/d_O^2} N_{OO}^p \tag{4.8}$$

Plugging in the above value for F_{p0} to both sides of the target identity (4.6), we obtain

$$\text{l.h.s.} = d_p N_{OO}^p, \quad \text{r.h.s.} = d_p N_{OO}^p \sum_q \frac{d_q}{d_O^2} N_{OO}^q. \tag{4.9}$$

Using (4.2) by identifying $a = b = O$ and $c = q$, one immediately observes that l.h.s.=r.h.s. in (4.9). A parallel proof will follow if one consider a slightly more general case where the four external legs are not all the same. The identity would have the form

$$d_a d_b [F_b^{aab}]_{0p}^2 = d_p N_{ab}^p \sum_q [F_b^{aab}]_{q0}^2 N_{ab}^q. \tag{4.10}$$

The main physics involved in proving the identity is the manifestation of the fusion rules in terms of quantum dimensions, (4.2). This should not come as a surprise: the

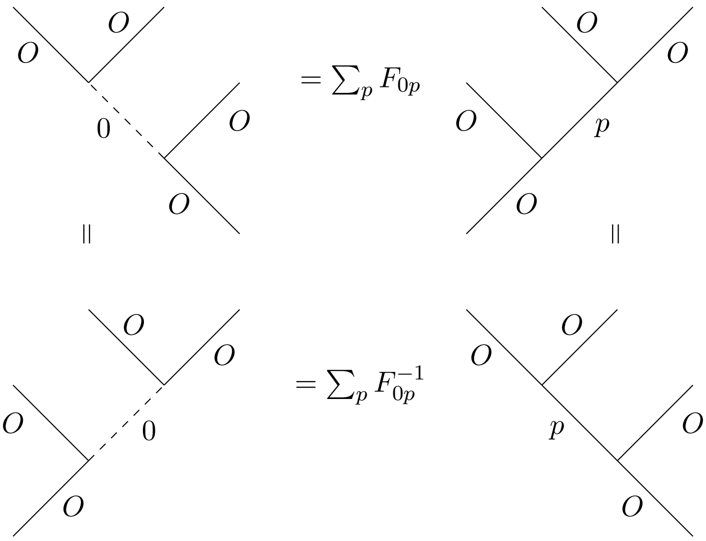

Figure 4. One identifies F_{0p} with F^{-1}_{0p} from the above figure.

identity to prove (4.6) is derived under the physical constraint $S_L = S_R$, namely the two procedures, doing OPE and calculating entanglement entropy, are interchangeable. In other words, the entanglement should be consistent with OPE. From the categorical point of view taken in this section, the entanglement stems from quantum dimensions, while the OPEs are fusion rules. Since the same algebraic structure is shared by anyons and quasi-particles (local operators) in RCFTs, we can use the language of anyon to prove (4.6). In this sense, we do show the quasi-particles of locally excited state in rational CFT follows the same rule as anyons. This can be seen as an example to realize anyons in RCFTs.

5 Conclusion and discussions

In this paper, we begin with same 1+1 dimensional setup with [14] and study the late time behavior $t \to \infty$ of Rényi entropy of the two equivalent locally excited states defined by l.h.s. and r.h.s. of eq. (1.2) and obtain the Rényi entropy of a subsystem $x > 0$ in (1+1)D CFTs. In the limit $t \to \infty$, we prove that S_L satisfies with a sum rule (1.1) by replica method and showed that S_L depends on the information of individual operator O_i in l.h.s. of eq. (1.2). In general, S_R is hard to obtain by replica method. In the late time limit, we derive S_R of the excited states involving in r.h.s. of eq. (1.2) by making use of Schmidt decomposition. It is associated with the fusion channels and conformal block presented in r.h.s. of eq. (1.2). The constraint $S_L = S_R$ leads to an identity in (1+1)D CFTs. We studied the S_L, S_R in rational CFTs as examples and proved the relation (3.30), (3.47).

From $S_L = S_R$ with late time limit in our setup, we indeed used crossing symmetry to obtain the entanglement entropy. Namely, we have made use of bootstrap equation from s channel conformal block to t channel conformal block.

5.1 Bulk-edge correspondence

We have seen that the modular tensor category language was used in section 4 to prove the identity 3.47. On the other hand, anyons in topological orders share the same algebraic structure of modular tensor categories, see for example [43, 44]. As noted in [47], the non-chiral rational CFT can be viewed as the edge theory of (2+1)-D chiral topological order in a strip [45, 46]. Insertion of operators in the rational CFT can be explained in the bulk theory. Roughly, inserting of a primary operators O_a at spacetime (x,t) in (1+1)D rational CFT corresponds to creating a pair of anyons labeled as $(a,\bar{a})$ at earlier time. The state $|\psi\rangle_2 = O_a(-l,0)O_a(0,0)\,|0\rangle$ can be viewed as creating two pairs of anyons in the bulk at some time $t<0$, and they pass the boundary at spacetime $(-l,0)$ and $(0,0)$. We can specify the possible values of the total charge of the two anyons by the fusion rule $a\times a = N^p_{aa}p$. In the CFT side this is just the OPE of two operators O_a.

Calculations of entanglement entropies with two pairs of anyons has been carried out in (2+1)D [48], where the result shares similar structure as above. It would be interesting to look at the general correspondence between the entanglement properties in the bulk and on the boundary.

5.2 Large-c CFTs

We mainly focus on the (1+1)D rational CFTs in previous sections. In rational CFTs, the spectrum and fusion rules are relatively simpler than the irrational ones, such as CFT with a gravity dual or Liouville CFT. In rational CFTs, we can analytically calculate the Rényi entropy of locally excited states and explain the evolution behavior by quasi-particles picture.

In this section, we would like to briefly discuss the constraint $S_L = S_R$ in the CFTs with a gravity dual, or large-c CFTs. Generally the time evolution of Rényi entropy can be very different from the rational ones [27, 28], see also the case for Liouville CFT [30]. The feature of such theory is a logarithmic growth in the intermediate time [26–28]. But we expect in the limit $t\to\infty$ the Rényi entropy or entanglement entropy to approach a constant [27]. In rational CFTs this constant is related to the quantum dimension of the inserted operator. However, for large c CFT the quantum dimension is not so well defined as rational CFT. As far as we know, this is still an unsolved problem at the moment.

In any CFT, the sum rule is still true for S_L, so one can obtain S_L as long as the result of locally excited states created by one primary operator is known. Two local operators can still be expanded as OPE blocks as in (3.2), consequently S_R (3.28) can similarly be calculated in large c CFTs, except that the sum over p may be replaced by an integration if the spectrum of the theory is continuous. By the definition of λ_p we know it is only associated with conformal blocks. In large c CFT, the details of the conformal blocks are known for few cases [49, 50]. One of them is the correlator

$$\langle O_L^\dagger(z_1,\bar{z}_1)O_L^\dagger(z_2,\bar{z}_2)O_L(z_3,\bar{z}_3)O_L(z_4,\bar{z}_4)\rangle \sim \sum_p \mathcal{F}_p(z)\mathcal{F}_p(\bar{z}), \tag{5.1}$$

for the primary operator O_L with conformal dimension h_L to be fixed in the limit $c\to\infty$. In this case, the Virasoro blocks reduce to representations of the global conformal group.

The holomorphic Virasoro blocks [50] are

$$\mathcal{F}_p(z) = z^{h_p}\ {}_2F_1(h_p, h_p, 2h_p; z), \tag{5.2}$$

where h_p is the conformal dimension of the intermediate operator, which is also assumed fixed in the limit $c \to \infty$. If we consider the locally excited state by two light operators $O_L(0,0)O_L(-l,0)\left|0\right\rangle$, the "probability " λ_p, as shown in eq. (2.20) is well defined in this case, since

$$\lim_{z\to 1} \frac{F_1(h_p, h_p, 2h_p; z)}{F_1(h_{p'}, h_{p'}, 2h_{'p}; z)} = C(h_p)/C(h_{p'}), \tag{5.3}$$

where $C(h_p), C(h_{p'})$ is only a constant depending on the conformal dimension h_p and $h_{p'}$ [29]. In rational CFTs, we know that the ratio $\lambda_p/\lambda_{p'}$ is associated with the the quantum dimension $d_p/d_{p'}$. In the present case the constant $C(h_p), C(h_{p'})$ may be an alternative of quantum dimension in large c CFT.

To check this claim, we will need to know the result of Rényi entropy for state $O_L\left|0\right\rangle$ in the limit $t \to \infty$. One more subtle problem is the entanglement entropy S_p of the state $\left|p\right\rangle$. For rational CFTs we show in section 3.4 that S_p is equal to the entanglement entropy of the state $O_p\left|0\right\rangle$. It is not straightforward to generalize the result to large-c CFTs, due to the lack of the simple fusion transformation.

Our setup depends on the leading behavior of OPE and success of the replica trick. The identity might break down due to the two facts. Firstly, the constraint should be modified for irrational CFTs, e.g. Liouville field theory. The spectrum of Liouville field theory is continuous and no vacuum exists in the Hilbert space. The OPE involves integration over continuous spectrum instead of discrete summation. Secondly, in the $(z, \bar{z}) \to (1, 1)$ limit, the dominant conformal block to the REE is no longer identity block in this limit. The author [30] have carefully studied the variation of REE of local excited states in late time by same bootstrap equation, showing that the late time of REE is associated with fusion matrix element instead of quantum dimensions.

It is also interesting to study the gravity dual of multiple local excitations, e.g., the bilocal quench can be associated with black hole creation in AdS_3 [51, 52].

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