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PAPER

Quantum simulation of expanding space–time with tunnel-coupled condensates

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Abstract

We consider two weakly interacting quasi-1D condensates of cold bosonic atoms. It turns out that a time-dependent variation of the tunnel-coupling between those condensates is equivalent to the spatial expansion of a one-dimensional toy-Universe, with regard to the dynamics of the relative phase field. The dynamics of this field is governed by the quantum sine-Gordon equation. Thus, this analogy could be used to 'quantum simulate' the dynamics of a scalar, interacting quantum field on an expanding background. We discuss how to observe the 'freezing' of quantum fluctuations during an accelerating expansion in a possible experiment. We also analyze an experimental protocol to study the formation of sine-Gordon breathers in the relative phase field, seeded by quantum fluctuations.

1. Introduction

The recent progress in coherently controlling systems of cold atoms (e.g., [1–5]), stimulated a lot of research concerned with employing these experimental systems to 'quantum simulate' prototypical quantum many-body models and quantum field theories (e.g., [6, 7]). This includes dynamics that is relevant for the early Universe, e.g. the initial acoustic density oscillations (Sakharov oscillations) that determine the structure observed in the cosmic microwave background radiation and which have recently been simulated using a quench in a cloud of cold atoms [8]. Particularly fascinating are ideas concerned with simulating quantum many-body physics on *curved* space–times ('analog gravity') (see [9–20], as well as references therein) connecting concepts and techniques from cosmology and condensed matter.

During the past few years, a very versatile platform for analog quantum simulations with cold atoms has been established in the Schmiedmayer group at Vienna: an atom chip holding one or two quasi-1D bosonic condensates [3, 21–23]. In this setup, it is possible to tune the trapping potential in a time-dependent fashion, which includes the possibility to modify the tunnel-coupling between two nearby condensates. That has been used experimentally to probe quench physics [24–28] and to control non-equilibrium dynamics [29]. After releasing the trap, the expanding clouds give rise to matter-wave interference that reveals the relative phase between the two condensates. The spatially resolved phase field can be measured for each run of the experiment, and the full statistics [22, 30] as well as higher-order correlators [28] have been extracted from the data of many runs. A recent comprehensive overview of this set of experiments can be found in [31]. The opportunities offered by this platform have resulted in a number of theoretical proposals for future experiments, e.g. [32–34].

In the present work, we argue that a pair of tunnel-coupled, quasi-1D, bosonic condensates can be employed for simulating an interacting, scalar quantum field on top of an expanding $1 + 1$ dimensional space–time (figures 1(a) and (c)). This scalar field is represented by the relative phase field between the condensates. As argued in [35], at low energies, its dynamics is described by the quantum sine-Gordon model. It turns out that in this setup, one can simulate the expansion of the 1D toy-Universe simply by varying the tunnel-amplitude according to a suitable protocol (figure 1(b)). In the experiments, the tunnel-amplitude itself can be tuned significantly and the field dynamics can be directly visualized by means of matter-wave interferometry. The 'quantumness' of the relative phase-field dynamics depends on the interaction strength within each condensate

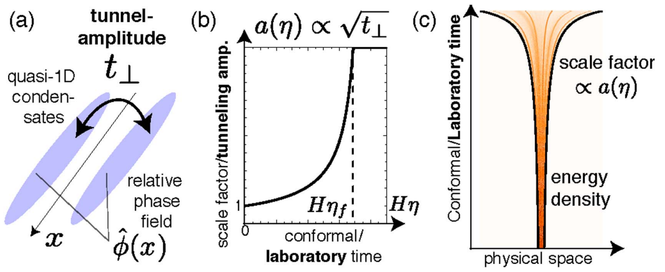

Figure 1. Quantum simulator. (a) Sketch of the quantum simulator consisting of a pair of tunnel-coupled 1D bosonic condensates. The tunnel amplitude $t_\perp$ is modulated time-dependently. It can be related to the scale factor a (b) describing the (exponential) expansion of a $1+1$ dimensional 'Universe'. The laboratory time (condensate coordinate x) is identified with the conformal time η (comoving coordinate). The relative phase field between the condensates $\hat{\phi}$ represents the scalar quantum field to be simulated. (c) Semiclassical evaluation of the physical energy density of the simulated quantum field (here displayed in physical coordinates $x_{\rm ph} = ax$). One can observe the structure formation out of zero-point quantum fluctuations during the expansion.

(which is tunable, e.g., by adapting the 1D condensate density). Here, we consider the weakly interacting 'semiclassical' limit (e.g., [32, 36–39]) and employ the truncated Wigner approximation (TWA) (e.g., [36]). In contrast, the proposed quantum simulator would also allow to explore the deep quantum regime—in fact, this is its main purpose.

In the following, we start with deriving an effective Hamiltonian description of an interacting quantum field on an expanding $1+1$ dimensional space time. In a second step, we introduce the experimental 'quantum simulator'—the tunnel-coupled condensates—and eventually discuss two of the possible effects, which could be explored.

The first effect is the well known 'freezing' of quantum fluctuations and the related 'cosmological particle production' of massive phonons during an accelerating expansion. This purely linear, though most fundamental effect is made responsible for the structure formation in the very early Universe [40]. This mode freezing also manifests itself in the spatial fluctuation spectrum of the field, which can be directly extracted experimentally. Although, strictly speaking, there is no need for a quantum simulation of the exactly solvable linear dynamics, observing the freezing of quantum fluctuations in the experiment would nevertheless be exciting (see, e.g., [10, 12]) and would constitute an important check on the setup. As shown in [17], the dynamics of phonons in an expanding $1+1$ dimensional space–time can be alternatively investigated by considering the phase-field of a single, physically expanding condensate.

The second feature is the generation of localized structures during the expansion out of quantum fluctuations. This pattern formation involves the full nonlinearity of the underlying sine-Gordon field theory and was also observed in the static case [32]. In contrast to [32], for an exponential expansion this happens only for small enough expansion rates. At large expansion times, these patterns seem to turn into standing sine-Gordon breathers simply drifting apart from each other. We discuss how to detect signatures of this 'Hubble' drift experimentally. In cosmology, excitations like these, e.g., in the scalar inflaton field, are sometimes denoted as 'oscillons' (e.g., [41–46]). A full experimental quantum simulation would allow for investigating the formation and persistence of these excitations on an expanding background, even in regimes where quantum effects become very important for the dynamics.

2. Quantum field on curved space–time

The space–time action of a classical, scalar field theory in $1+1$ dimensions is given by [40] ($c=1, \hbar=1$)

$$\mathcal{S} = \frac{1}{2}\int {\rm d}x^2\sqrt{-\det g_{\mu\nu}}\left[\left(\partial_\mu\chi\right)g^{\mu\nu}\left(\partial_\nu\chi\right) - 2V(\chi)\right], \tag{1}$$

where $g_{\mu\nu}$ denotes the metric, $\partial_\mu\chi = \partial\chi/\partial x^\mu$ and $V(\chi)$ is an arbitrary potential. We are interested in an homogeneous, spatially expanding space–time described by the Friedmann–Robertson–Walker (FRW) metric

$${\rm d}s^2 = {\rm d}\tau^2 - a^2(\tau){\rm d}x^2. \tag{2}$$

Here, x denotes comoving coordinates which are related to physical coordinates $x_{\rm ph}$ via the scale factor $a(\tau)$ as $x_{\rm ph} = ax$. Here, we treat $a(\tau)$ as a given function of time to be specified below.

At present, the nature of the field that drives cosmological inflation is still unknown, which has led to sustained activity in exploring various models [47, 48]. A nonlinear scalar field χ, such as the one whose dynamics will be analyzed here, would represent one of the many possibilities that are studied as candidates for inflation.

In $1 + 1$ dimensions, the Lagrangian corresponding to equation (1) takes an intriguingly simple form using the so-called conformal time η with $d\eta = d\tau/a$:

$$L = \frac{1}{2}\int dx\left[\left(\partial_\eta \chi\right)^2 - \left(\partial_x \chi\right)^2 - 2a^2(\eta)V(\chi)\right]. \tag{3}$$

The quantization of the field theory follows the standard prescription [40]. First, we identify the canonical momentum field $\Pi_\chi(x, \eta) \equiv \frac{\delta L}{\delta \partial_\eta \chi} = \partial_\eta \chi$. In a second step, we promote the fields to operators demanding that $[\hat{\chi}(x, \eta), \hat{\Pi}_\chi(x', \eta)] = i\delta(x - x')$. Eventually, we can switch to the Hamiltonian formulation introducing the time-dependent Hamiltonian

$$\hat{H}(\eta) = \frac{1}{2}\int dx\left[\hat{\Pi}_\chi^2 + \left(\partial_x \hat{\chi}\right)^2 + 2a^2(\eta)V\left(\hat{\chi}\right)\right]. \tag{4}$$

Note that all effects of the expanding space–time are now encoded in the time-dependence of $\hat{H}(\eta)$ (see [11, 49, 50]).

3. Tunnel-coupled condensates as quantum simulator

Here, we propose a quantum simulation of the field $\hat{\chi}(x, \eta)$ for the special case of a sine-Gordon potential $V = -m_0^2\beta^{-2}\cos(\beta\hat{\chi})$. This potential has several interesting properties: the corresponding field theory is interacting and integrable. Second, the sine-Gordon potential appears in the so-called 'natural inflation' scenario [47]. Third, the sine-Gordon potential supports the formation of 'quasibreathers' [32] (in the cosmology literature denoted as 'oscillons' e.g., [43, 44, 46, 51]).

The quantum simulator (figure 1(a)) consists of two tunnel-coupled quasi-1D condensates of cold, bosonic atoms [3, 21–23] with a time-dependent tunnel amplitude $t_\perp$. The laboratory time is identified with the conformal time η. At low energies, the dynamics of the relative phase field $\beta\hat{\phi}(x, t)/\sqrt{2} = (\hat{\phi}_1 - \hat{\phi}_2)/\sqrt{2}$ can be described by the quantum sine-Gordon model [35] (the sound velocity $v_s = 1$)

$$\hat{H}_{SG} = \frac{1}{2}\int_0^L dx\left[\hat{\Pi}^2 + \left(\partial_x\hat{\phi}\right)^2 - \frac{2m^2(\eta)}{\beta^2}\cos\beta\hat{\phi}\right]. \tag{5}$$

The relative phase field and the relative density variations $\sqrt{2}\beta^{-1}\hat{\Pi} \equiv \frac{\hat{n}_1 - \hat{n}_2}{\sqrt{2}}$ form a canonical pair. The tunnel amplitude $t_\perp(\eta)$ enters the mass term $m^2(\eta) = 2\beta^2\rho_0 t_\perp(\eta)$, where ρ_0 is the mean density per condensate. The Luttinger liquid description should be reliable as long as the typical length scale of equation (5), set by $\sqrt{\hbar v_s/m}$, is much larger than the healing length of the condensates ξ_h [32, 35]. This can always be achieved by choosing a sufficiently small tunnel-amplitude. The parameter β is related to the Luttinger parameter K as $\beta = \sqrt{2\pi/K}$ (for weak interactions $\beta \ll 1$). It can be shown that β plays the role of Planck's constant [36] and $\beta \ll 1$ corresponds to the considered semiclassical limit of the quantum sine-Gordon model (e.g., [52]). However, in the experiment one can go deep into the quantum regime corresponding to larger β (a rather broad range of values up to $K \sim 50$ is realizable, e.g., [53]).

The following identifications connect the quantum simulator and the quantum field theory on an expanding background. Identifying the fields $\hat{\chi} \leftrightarrow \hat{\phi}$ and $m^2(\eta) = m_0^2 a^2(\eta)$ (and thus $a(\eta) \leftrightarrow \sqrt{t_\perp(\eta)/t_\perp(0)}$), the dynamics of the relative phase field simulates $\hat{\chi}$ in conformal time and comoving coordinates. In the remainder, we will always argue in terms of the field $\hat{\phi}$, i.e., we analyze equation (5).

We note that the original model for two tunnel-coupled bosonic condensates also includes features not present in the low-energy approximation (5) adopted here. Specifically, these are amplitude fluctuations and the possible cross-coupling between the relative phase field (considered here) and the 'symmetric mode' (i.e. the sum of the phases). For more details on these issues, we refer the reader to [32, 54, 55].

4. Scale factor and initial state

We consider an exponential expansion $a(\tau) = e^{H\tau}$ with the 'Hubble constant' $H = \dot{a}(\tau)/a(\tau)$. Choosing $a(0) = 1$ and after a shift by $1/H$, we obtain $\eta = H^{-1}[1 - e^{-H\tau}]$. Correspondingly we find $a(\eta) = [1 - H\eta]^{-1}$ (see figure 1(b)) with $\eta \in [0, \eta_f]$ and $\eta_f < 1/H$. At η_f, the expansion ends and $a(\eta > \eta_f) = a_f$. The dimensionless parameter H/m_0 compares the expansion rate and the typical internal timescale (at short times) of the system and plays a crucial role.

At $\eta = 0$, we start in the ground state of massive phonons with a small mass m_0. In particular, m_0 is chosen much smaller than the UV cutoff $1 \sim 1/\xi_h$. All TWA simulations are performed on a lattice with lattice constant

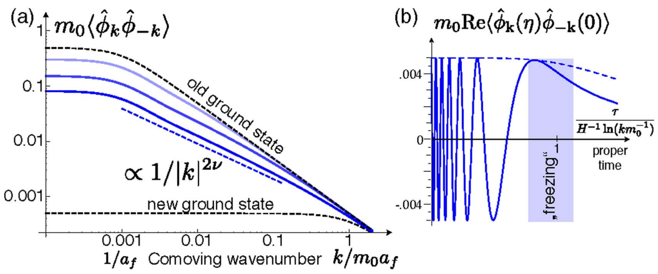

Figure 2. Mode 'freezing'. (a) Plot of the spectrum at $a = 19.6, 166.8, 1000$ (blue lines, top to bottom). Here, $H/m_0 = 2.6$ and $m_0 = 0.05$. One observes the spectral tilt (dashed, blue line) resulting from the freezing of modes. Dashed lines show the corresponding ground state spectra for $a = 1$ and $a = a_f$. (b) 'Freezing' of modes (here $k/a_f m_0 = 0.1$). While the suppression of the 1D field fluctuations at later times is non-universal (e.g., it depends on the spatial dimension), the mechanism of the expansion-induced mode freezing is the same as in higher dimensions.

set to one, while keeping $m_0 a(\eta) \ll 1$ throughout the whole simulation. Before, the expansion starts, the center-of-mass (COM) mode $\hat\Phi(\eta) \equiv \frac{1}{L}\int_0^L dx\ \hat\phi(x, \eta)$ is tuned to some value $\Phi(0) = \Phi_0$ with $\beta\Phi_0 \in [0, \pi]$ ($\Phi'(0) = 0$). As argued in [32], such an initial state can be fairly well achieved by slowly splitting a single condensate followed by applying a potential gradient between the condensates to tune Φ_0 (see also [56]).

5. Freezing of quantum fluctuations

One of the fascinating results of modern cosmology is that the structure formation in the very early Universe seems to have been seeded by quantum fluctuations [40]. This result is truly amazing, as on cosmological scales zero-point fluctuations are tiny. However, it seems that an exponential expansion of the very early Universe (inflationary stage) led to a freezing of quantum fluctuations and stretched them to cosmological scales. One can reformulate this basic mechanism in a condensed matter language (e.g., [11]). In this terminology, the inflationary expansion corresponds to a rapid, non-adiabatic 'quench'(see, e.g., [57]) producing a large number of excitations ('cosmological particle creation' see for instance [12, 18]).

A very similar effect should be observable in the considered $1 + 1$ dimensional toy-Universe. For this purpose, we consider the case $\Phi_0 = 0$. For small enough β and finite system size, one can safely expand the cosine-potential to lowest order $m^2(\eta)\beta^{-2}(1 - \cos\beta\hat\phi) \approx \frac{m^2(\eta)}{2}\hat\phi^2$ yielding a theory of massive phonons (see [16, 17, 38, 39, 57, 58]) on an expanding $1 + 1$ FRW space–time. According to equation (5), the dynamics of the modes $\hat\phi_k\,[\hat\phi(x, \eta) \equiv \hat\Phi + \frac{1}{\sqrt{L}}\sum_{k\neq 0} e^{ikx}\hat\phi_k(\eta)]$ is determined by

$$\hat\phi_k''(\eta) + \left[k^2 + m_0^2 a^2(\eta)\right]\hat\phi_k(\eta) \approx 0. \tag{6}$$

The general solution of equation (6) is given in the supplement (see appendix for details). Note that there exist two different regimes. Apparently, modes with $|k| \gg m_0 a(\eta)$ oscillate freely. In contrast, the expansion strongly impacts modes with $|k| \ll m_0 a(\eta)$. These modes stop oscillating and undergo an overdamped evolution algebraically in time, i.e., they are 'frozen' (see figure 2(b)). In contrast to the $3 + 1$ dimensional case, in $1 + 1$ dimensions this happens only for finite masses with $1/m_0$ playing the role of a 'horizon'. Most importantly, this freezing also manifests in the spectrum $\langle\hat\phi_k(\eta)\hat\phi_{-k}(\eta)\rangle$ (see [10, 11, 40]), which deviates from the instantaneous ground state spectrum indicating the expansion induced production of excitations. This allows to observe the effect with a single measurement of the relative phase field $\hat\phi(x, \eta)$ per run. For the considered protocol (restricting to $\eta < \eta_f$), we obtain for all modes with $|k| \gg m_0$ in the limit $|k|/[m_0 a(\eta)] \to 0$ (see figure 2(a))

$$\left\langle\hat\phi_k(\eta)\hat\phi_{-k}(\eta)\right\rangle \propto \frac{1}{a^{1-2\nu}}\frac{H^{2\nu-1}}{|k|^{2\nu}}, \tag{7}$$

where $\nu = \frac{1}{2}\sqrt{1 - 4m_0^2/H^2}$. A detailed derivation of this (and related) results for the linear dynamics is presented in the appendix.

Modes with $|k|/a(\eta) \gg m_0$ remain close to the initial ground state of massive phonons with mass m_0 yielding $\langle\hat\phi_k(\eta)\hat\phi_{-k}(\eta)\rangle \simeq 1/2\ |k|$. Here, we consider a fast expansion with $H/m_0 > 2$. In the limit $H/m_0 \to \infty$, ν approaches $1/2$ corresponding to a 'sudden quench' which leaves the spectrum unchanged. In a possible experiment, one could detect the different power-laws by probing the longitudinal phase coherence [59] of the condensates on different length-scales (similar to [22]). It can be easily checked (see footnote 1) that the finite $a'(0)$ does not influence the 'particle production' which happens during the evolution.

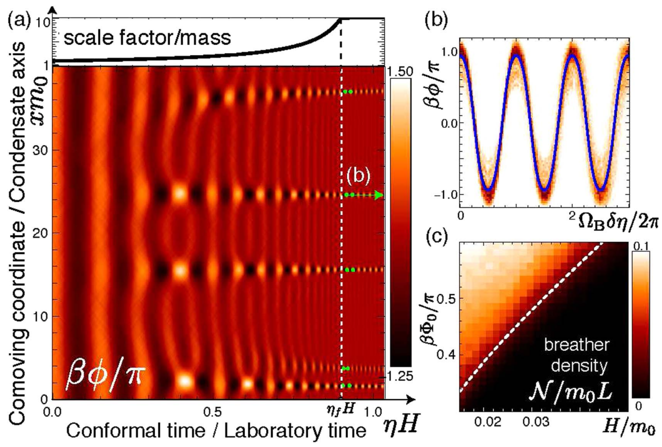

Figure 3. Emergence of breathers. (a) Plot of a single TWA run in comoving coordinates and conformal time ($\beta = 0.1, \beta\Phi_0/\pi = 0.65$, $H/m_0 = 0.02$). One observes the damped Josephson oscillations of the COM mode and the emergence of breather-like excitations out of parametrically amplified phonons. As the COM mode is damped away due to the expansion, standing sine-Gordon breathers are released. These can be tracked numerically (green dots). (b) Time-evolution of these breathers tracked at $\delta\eta = 0$ denoting the time of tracking ($\eta \geqslant \eta_f$) as a function of $\Omega_B\delta\eta$ (Ω_B is the numerically obtained breather frequency). The spatial center of the breathers is shown. Color code: distribution function; Blue, solid line shows the temporal evolution of a standing breather (corresponding to the mean breather frequency). (c) Density of created breathers ($\mathcal{N}$ is the number of tracked breathers) obtained by tracking breathers at $a_f = 10$. Here, $L = 1000$, $m_0 = 0.075$ and $\beta = 0.1$. The white, dashed line indicates where $\max_\eta\sigma = 0.2$ (see main text).

6. Formation of breathers out of quantum fluctuations

While the freezing out of quantum fluctuations is a purely linear effect, we now discuss a feature, which heavily relies on *interactions*. From now on, we consider a slow expansion $H/m_0 \ll 1$ and finite $0 < \Phi_0 \sim \pi/\beta$ (see figure 3(a)). At short times $\eta > 0$, the global relative phase $\Phi(\eta)$ performs Josephson oscillations. However, it is well known that these are parametrically unstable against small, spatial fluctuations (e.g., [32, 60, 61]). Linearizing around Φ, one finds $\hat{\phi}_k'' + [k^2 + m_0^2 a^2(\eta)\cos\beta\Phi]\hat{\phi}_k \simeq 0$. In the static case, this parametric drive leads to $\hat{\phi}_k(\eta) \sim e^{\Gamma(k,\Phi_0)\eta}$ with $\Gamma \simeq \frac{|k|}{2}\sqrt{\sin^2(\beta\Phi_0/2) - k^2/m^2}$ [32, 61] (displayed for $\beta\Phi_0 \lesssim 1$). In turn the fluctuations $\langle\delta\hat{\phi}^2\rangle$ grow and at some point the linearization breaks down. In the semiclassical limit $\beta \ll 1$, it was demonstrated that the nonlinearity of the sine-Gordon equations leads to the formation of localized patterns in the field $\hat{\phi}(x, \eta)$ [32]. These patterns were identified with 'quasibreather'-solutions of the classical sine-Gordon equation [32], which in contrast to usual breathers have a finite lifetime and could also explain the formation of these excitations out of phononic quantum fluctuations.

Here, for a slow, exponential expansion (in proper time τ), one observes the formation of similar excitations only for m_0/H exceeding a rather sharp threshold (figure 3(c), see also [46]), which depends on β and Φ_0. This can be understood realizing that as long as the linearization applies, the expansion causes a suppression $\sim 1/\sqrt{a}$ of long-wavelength modes with k up to the order $\mathcal{O}(m_0)$, including the COM mode. Furthermore, in the course of time, resonant modes can get shifted out of resonance as a consequence of the expansion. Replacing $k \to k/a$ and $\Phi_0 \to \Phi_0/\sqrt{a}$ (see [46, 61]), one finds that $\langle\hat{\phi}_k(\eta)\hat{\phi}_{-k}(\eta)\rangle \propto a^{-1}\exp\left[2\int_0^{\eta} d\eta' a \,\mathrm{Re}\,\Gamma\left(\frac{k}{a}, \frac{\Phi_0}{\sqrt{a}}\right)\right]$. The nonlinearity and thus the formation of localized patterns kicks in only if the fluctuations $\beta\sqrt{\langle\delta\hat{\phi}^2(\eta)\rangle} \approx \sigma(\eta)$, where $\sigma = [\frac{\beta^2}{a(\eta)}\int_{2\pi/L}^{k_c}\frac{dk}{2\pi}\frac{\exp[2\int_0^{\eta} d\eta' a \,\mathrm{Re}\,\Gamma(\frac{k}{a}, \frac{\Phi_0}{\sqrt{a}})]}{\sqrt{k^2 + m_0^2}}]^{1/2}$, exceed a certain value for some η (see figure 3(c)). Numerically, we find that this value is of the order $\mathcal{O}(10^{-1})$.

Close to the creation threshold, once created, these breather-like excitations persist at the position, where they were 'born' out of quantum fluctuations. This is in contrast to the 'quasibreathers' observed in the static case [32] (see also [51]). In the long-time limit, the homogenous part of the field Φ is damped away $\propto 1/\sqrt{a}$. We find good numerical evidence that the localized excitations, however, are robust against the expansion and can be well described as standing (classical) sine-Gordon breathers ϕ_B (see figure 3(b)). Their typical distance is set by the maximally amplified wavelength before the nonlinearity sets in, ending the parametric amplification. It seems that at late times ($a \gg 1$), the only effect of the adiabatic expansion on breathers is a trivial shrinking of the breather period and width (both $\propto 1/m_0 a(\eta)$) in comoving coordinates, while their amplitude stays approximately constant. This can be understood realizing that the amplitude of a classical sine-Gordon breather $\beta\phi_B$ is solely determined by the breather parameter $\varphi \in [0, \pi/2]$($\max\beta\phi_B = (2\pi - 4\varphi)$, see e.g., [62]). In the quantum sine-Gordon model, this parameter gets quantized [63], i.e., it is promoted to a quantum number.

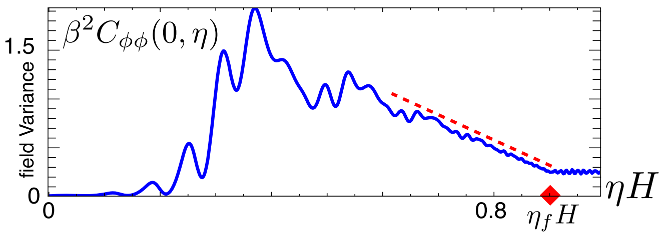

Figure 4. Plot of the correlation function $C_{\phi\phi}(0, \eta)$. Clearly, one observes the exponentially growing fluctuations (parametric resonance). At some point the nonlinearity of the sine-Gordon model leads to the formation of breather-like structures (around the main peak, see figure 3(a)). While the COM mode is damped by the expansion, these structures turn into standing sine-Gordon breathers with a constant amplitude (see figure 3(b)), which, in physical coordinates simple drift away from each other. This 'Hubble expansion' is mirrored by the linear decay of the correlation function as $\propto(1 - \eta H)$ at late times. Once the expansion stops at η_f, the correlation function becomes almost constant. Here: $\beta = 0.1$, $H/m_0 = 0.02$, $m_0 = 0.05$, $L = 800$ and $\beta\Phi_0 = 0.7\pi$.

However, it is well known that for a slow change of system parameters (by slow, here, we understand $\Omega_B^{-2}\partial\Omega_B/\partial\eta \ll 1$, where $\Omega_B = m_0 a(\eta)\sin\varphi$ is the instantaneous breather frequency), quantum numbers (and thus the breather amplitude) are approximatively preserved (see [64]). While for large $a \gg 1$ the number and amplitude of breathers remain constant (per experimental run), they simply move apart from each other in physical coordinates. This 'Hubble expansion' e.g., can be observed in the (experimentally accessible) equal time correlation $C_{\phi\phi}(x, \eta) = \langle\hat{\phi}(x, \eta)\hat{\phi}(0, \eta)\rangle - \langle\hat{\phi}(0, \eta)\rangle^2$ (figure 4). Under the assumption that at late times, the field $\hat{\phi}$ can be described as a set of independent (standing) sine-Gordon breathers with fixed amplitudes, one obtains that

$$C_{\phi\phi}(0, \eta) \propto 1 - \eta H. \tag{8}$$

The linear suppression of $C_{\phi\phi}$ is a direct consequence of the decreasing breather density in physical coordinates. From a condensed matter point of view the observation that a suitable protocol for the tunnel-amplitude prepares a state consisting of independent, standing sine-Gordon breathers is interesting by itself. While the analysis here is based on semiclassical considerations (numerically on the TWA) reliable for $\beta \ll 1$, the proposed quantum simulator could for instance test the stability of classical sine-Gordon breathers against quantum fluctuations for larger $\beta \sim 1$ (see [65]). Furthermore, a quantum simulation could give insight in the excitation of 'oscillonic' patterns in a scalar quantum field (such as the inflaton field) during spatial expansion. Such oscillonic patterns have been discussed extensively in the cosmology community, e.g., [44–46, 51], and also recently for coupled BECs [34]. Other variations seem possible as well, such as extending the proposals [66, 67] on cold-atom simulations of 'false vacuum' decay to situations with spatial expansion.

7. Conclusions

We have shown that by tuning the tunnel-amplitude between a pair of tunnel-coupled 1D condensates, one can simulate an interacting quantum field on an expanding 1 + 1 space–time. The proposed 'quantum simulator' should be realizable with present cold atom setups. We have discussed two examples of quantum-many body physics on an expanding background: the freezing of phonon modes and the creation of sine-Gordon breathers out of quantum fluctuations during an exponential FRW-expansion. While the discussion here is restricted to the semiclassical limit of the underlying quantum sine-Gordon model, any experimental realization of this 'quantum simulator' is ultimately meant to explore the deep quantum regime where additional surprising features might be encountered.

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Appendix

Here we provide a detailed derivation of the expansion-induced 'particle production' in a 1 + 1 dimensional toy-Universe, which could be observed in the proposed experiment as mentioned in the main text. In contrast to higher dimensions, in one spatial dimension this happens only for finite masses. Nevertheless, it is shown here that the production of field excitations in the considered $1 + 1$ dimensional Universe can be analogously understood as 'freezing' of modes with physical wavelengths exceeding a characteristic length scale.

In the main text, as a first example, we discuss a simple though fundamental feature of quantum field theories in an expanding space–time, namely the expansion induced production of field excitations (see section 5). In the case of $3 + 1$ dimensions this effect is known as 'cosmological particle creation'. The experimental observation of this linear effect would prove that the 'quantum simulator' is operating as expected. For this purpose, the global relative phase is tuned to zero before ramping up the tunnel-amplitude, i.e., before simulating the spatial expansion. In this case, for the considered and experimentally relevant parameters (see main text), the relative phase field can be described in terms of massive phonons. Starting in the massive ground state before the expansion, it turns out that in close analogy to the $3 + 1$ dimensional case, a fast, accelerating expansion leads to the excitation of massive phonons. In contrast to higher dimensions, in one spatial dimension this happens only for finite masses. In the following, we provide a detailed derivation of the time dependent fluctuation spectrum in one spatial dimension and compare the results qualitatively to the $3 + 1$ dimensional case. For this purpose, we closely follow [48] and perform all calculations directly for the 'cosmological' field $\hat{\chi}$ to be simulated experimentally.

Limit of massive 'phonons'—linear dynamics

As mentioned above, here it is assumed that the $k = 0$ mode is tuned to zero, such that the effective, classical Hamiltonian in conformal time and comoving coordinates is given by (with the canonical momentum $\Pi_\chi = \partial_\eta \chi$)

$$H_{\rm eff}(\eta) = \frac{1}{2}\int_0^L {\rm d}x\Big[\Pi_\chi^2 + \big(\partial_x\chi\big)^2 + m_0^2 a^2(\eta)\chi^2\Big]. \tag{9}$$

Demanding that $[\hat{\chi}(x,\eta), \hat{\Pi}_\chi(x',\eta)] = {\rm i}\delta(x-x')$, the field theory gets quantized. In order to evaluate the time evolution it is suitable to introduce the mode expansion:

$$\hat{\chi}(x,\eta) = \frac{1}{\sqrt{2L}}\sum_k\Big(\hat{a}_k v_k^*(\eta){\rm e}^{{\rm i}kx} + \hat{a}_k^\dagger v_k(\eta){\rm e}^{-{\rm i}kx}\Big), \tag{10}$$

$$\equiv \frac{1}{\sqrt{L}}\sum_k {\rm e}^{{\rm i}kx}\hat{\chi}_k(\eta), \tag{11}$$

with v_k fulfilling

$$v_k'' + \Big[k^2 + m_0^2 a^2(\eta)\Big]v_k = 0. \tag{12}$$

For convenience, we choose time such that $a(\eta) = -1/\eta H$ and $\eta \in\,]-\infty, 0[$ (this corresponds to a shift by $1/H$ compared to the main text). Furthermore, the canonical commutation relation puts an extra restriction on the mode functions: ${\rm Im} v_k^* v'_k = 1$ for all k and η ($[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}$). In the main publication, we consider the situation, where initially the field is prepared the instantaneous ground state of equation (9) for all modes with $k \neq 0$. Note that in the cosmological context, the situation is slightly more subtle. Only for modes with physical wavelengths much smaller than $1/H$, the influence of the expansion before inflation sets in can be neglected. Fortunately, it seems that precisely these modes are crucial for the large-scale structure of the observable Universe [40, 48]. In the cosmological context, these modes are assumed to start in the instantaneous ground state. This partial vacuum initial state is better known as the Bunch–Davies vacuum [48].

In our case, starting in the instantaneous ground state $|0\rangle$ ($\hat{a}_k|0\rangle = 0$) at time η_i, the initial conditions for $v_k = v_{-k}$ can be found by minimizing the expectation value of the Hamiltonian under the constraint ${\rm Im} v_k^* v_k' = 1$ yielding $v_k(\eta_i) = \frac{1}{\sqrt{\omega_k(\eta_i)}}$ and $v_k'(\eta_i) = {\rm i}\sqrt{\omega_k(\eta_i)}$ with the massive dispersion $\omega_k(\eta) = \sqrt{k^2 + m_0^2 a^2(\eta)}$.

Fluctuation spectrum—formal solution

We are interested in the fluctuation spectrum, i.e., $\langle \hat{\chi}_k^\dagger \hat{\chi}_k\rangle = \frac{|v_k(\eta)|^2}{2}$. In order to construct the explicit solutions of v_k, we pick two real and linearly independent solutions of equation (12), i.e., with a non-vanishing Wronskian $W[f_{1,k}, f_{2,k}] = f_{1,k}f'_{2,k} - f'_{1,k}f_{2,k} \neq 0$:

$$f_{1,2;k} = \frac{1}{\sqrt{a}}J_{\pm\nu}\left(\frac{|k|}{aH}\right), \tag{13}$$

where $\nu = \frac{1}{2}\sqrt{1 - \frac{4m_0^2}{H^2}}$ and we restrict to a fast expansion with $H > 2m_0$. The general solution is given by $v_k = \sum_{j=1,2} C_{j,k} f_{j,k}$. Note that $f_{\pm,k}(a > 0) \in \mathbb{R}$. One obtains

$$C_{1,k} = \frac{W\big[v_k(\eta_i), f_{2,k}(\eta_i)\big]}{W\big[f_{1,k}(\eta_i), f_{2,k}(\eta_i)\big]}, \tag{14}$$

$$C_{2,k} = \frac{W\left[f_{1,k}(\eta_i),\, v_k(\eta_i)\right]}{W\left[f_{1,k}(\eta_i),\, f_{2,k}(\eta_i)\right]}. \tag{15}$$

Indeed, these solutions are linearly independent as

$$W\left[f_{1,k}(\eta_i),\, f_{2,k}(\eta_i)\right] = -\frac{2H}{\pi}\sin\pi\nu \tag{16}$$

is non zero for all $\frac{1}{2} \geqslant \nu > 0$. Thus, we obtain for the fluctuation spectrum introducing the abbreviations $g_{\pm\nu} = \frac{\mathrm{d}}{\mathrm{d}\eta}\left[\frac{J_{\pm\nu}\left(\frac{|k|}{aH}\right)}{\sqrt{a}}\right]\Big|_{\eta_i}$ and $G_{\pm\nu} = \frac{J_{\pm\nu}\left(\frac{|k|}{aH}\right)}{\sqrt{a}}\Big|_{\eta_i}$:

$$\langle 0|\, \hat{\chi}_k^\dagger(\eta)\hat{\chi}_k(\eta)|0\rangle = \frac{1}{a}\left|\frac{C_{1,k}}{\sqrt{2}}J_\nu\left(\frac{|k|}{aH}\right) + \frac{C_{2,k}}{\sqrt{2}}J_{-\nu}\left(\frac{|k|}{aH}\right)\right|^2, \tag{17}$$

$$C_{1,k} = \frac{-\frac{1}{\sqrt{\omega_k(\eta_i)}}g_{-\nu} + \mathrm{i}\sqrt{\omega_k(\eta_i)}\,G_{-\nu}}{\frac{2H}{\pi}\sin\pi\nu}, \tag{18}$$

$$C_{2,k} = \frac{\frac{1}{\sqrt{\omega_k(\eta_i)}}g_{\nu} - \mathrm{i}\sqrt{\omega_k(\eta_i)}\,G_{\nu}}{\frac{2H}{\pi}\sin\pi\nu}. \tag{19}$$

Explicitly, the spectrum reads

$$\langle 0|\, \hat{\chi}(\eta)_k^\dagger\hat{\chi}_k(\eta)|0\rangle = \frac{1}{a(\eta)} \times \frac{\frac{1}{2\omega_k(\eta_i)}\left[g_{-\nu}J_\nu\left(\frac{|k|}{aH}\right) - g_\nu J_{-\nu}\left(\frac{|k|}{aH}\right)\right]^2 + \frac{\omega_k(\eta_i)}{2}\left[G_{-\nu}J_\nu\left(\frac{|k|}{aH}\right) - G_\nu J_{-\nu}\left(\frac{|k|}{aH}\right)\right]^2}{\left[\frac{2H}{\pi}\sin\pi\nu\right]^2}. \tag{20}$$

Discussion of formal solution

The crucial feature of the differential equation (12) is the fact that there obviously exist two different regimes. A first estimate (neglecting the specific evolution of $a(\eta)$) indicates that modes with $k \gg m_0 a(\eta)$ oscillate freely, while modes with $k \ll m_0 a$ undergo an overdamped evolution, i.e., they are 'frozen'. Thus the inverse mass $1/m_0$ sets the critical length scale for this crossover to happen. Defining $\eta_k = \frac{m_0}{H}\frac{1}{|k|}$, one finds

$$v_k(\eta) = \begin{cases} A_k^+ \mathrm{e}^{\mathrm{i}|k|\eta} + A_k^- \mathrm{e}^{-\mathrm{i}|k|\eta}, & \eta \ll \eta_k \\ B_k^+\,|\eta|^{\frac{1}{2}+\nu} + B_k^-\,|\eta|^{\frac{1}{2}-\nu}, & \eta \gg \eta_k. \end{cases} \tag{21}$$

This behavior can also be extracted from the exact solution equation (20) by making use of the limits

$$\lim_{x/\left|\nu^2-\frac{1}{4}\right|\to\infty} J_\nu(|x|) = -\sqrt{\frac{2}{\pi x}}\sin\left(\frac{\pi\nu}{2} - x - \frac{\pi}{4}\right),$$
$$\lim_{x\to 0} J_\nu(|x|) = \frac{|x|^\nu}{2^\nu\Gamma(\nu+1)}. \tag{22}$$

The first limit corresponds to $\frac{1}{m_0}\left(\frac{H}{m_0}\right) \gg \lambda_{\mathrm{ph}}$ ($\lambda_{\mathrm{ph}} = 2\pi a/|k|$) and the second to $\frac{\frac{1}{m_0}\left(\frac{m_0}{H}\right)}{\lambda_{\mathrm{ph}}} \to 0$. Let us first consider the limit $\eta_i \to -\infty$ for a finite and fixed k, such that these modes do not experience the expansion, initially. One obtains

$$\lim_{\eta_i\to-\infty} g_\nu = -\sqrt{\frac{2\,|k|\,H}{\pi}}\cos\left(\frac{\pi\nu}{2} + |k|\,\eta_i - \frac{\pi}{4}\right),$$
$$\lim_{\eta_i\to-\infty} G_\nu = -\sqrt{\frac{2H}{\pi\,|k|}}\sin\left(\frac{\pi\nu}{2} + |k|\,\eta_i - \frac{\pi}{4}\right).$$

At late times with $\frac{\frac{1}{m_0}\left(\frac{m_0}{H}\right)}{\lambda_{\mathrm{ph}}} \to 0$, these modes freeze as a result of the expansion crossing the scale essentially set by $1/m_0$. According to the limit equation (22), the $-\nu$-contributions dominate the spectrum leading to the long-time spectrum (see equation (7) in the main publication)

$$\left\langle \hat{\chi}_k^\dagger \hat{\chi}_k \right\rangle \to \frac{\pi}{[Ha(\eta)]^{1-2\nu}} \frac{4^{-1+\nu}}{\sin^2(\pi\nu)} \frac{1}{\Gamma^2(1-\nu)} \frac{1}{|k|^{2\nu}}. \tag{23}$$

Comparing this spectrum to the instantaneous ground state spectrum $\langle \hat{\chi}_k^\dagger \hat{\chi}_k \rangle = \frac{1}{2\sqrt{k^2 + a^2 m_0^2}}$, it turns out that indeed, as a result of the expansion, the field strongly deviates from the ground state. In particular, the expansion induced excitation of the field leads to a shift of the power-law exponent in the fluctuation spectrum from $1/|k| \to 1/|k|^{2\nu}$. Note that by taking the limit $\eta_i \to -\infty$ for a fixed and finite k, we exclude the possibility that the finite derivative $a'(\eta_i) = Ha^2(\eta_i) \to 0$ is responsible for the excitation of modes.

'Freezing' of modes

In the previous section, it was shown that the expansion transforms the $1/|k|$ zero-point spectrum into $1/|k|^{2\nu}$. In the following, we provide a simple estimate emphasizing that this spectral 'tilt' can be directly related to the 'freezing' of quantum fluctuations after crossing the physical lengthscale $1/m_0$.

As long as $\frac{1}{m_0}\left(\frac{H}{m_0}\right) \gg \lambda_{\text{ph}}$, the fluctuations in these modes are not affected by the expansion, i.e., $\langle \hat{\chi}_k^\dagger \hat{\chi}_k \rangle \approx \frac{1}{2|k|}$. This changes, as soon as the modes enter the regime II of the differential equation (12) at times $\eta = \mathcal{O}(\eta_k)$, i.e., when their physical wavelength crosses the scale $1/m_0$. For time $\eta \gg \eta_k$, these modes undergo an overdamped dynamics, i.e., they are 'frozen' and decay monotonically according to the second line in equation (21).

As $\eta \to 0$, the part of the solution with the exponent $\frac{1}{2} - \nu$ clearly dominates the long-time behavior, such that modes are suppressed as $\sim(\eta/\eta_k)^{1/2 - \nu}$. A rough estimate thus yields

$$\begin{aligned} &\left\langle \hat{\chi}_k^\dagger(\eta_k) \hat{\chi}_k(\eta_k) \right\rangle \left[\frac{\eta}{\eta_k}\right]^{1-2\nu} \\ &\propto \frac{1}{a^{1-2\nu}} \frac{1}{|k|^{2\nu}} \end{aligned} \tag{24}$$

and thus the power-law shift obtained in equation (23).

In $3 + 1$ dimensions, mode freezing and thus particle production during an exponential expansion happens even in massless theories. Furthermore, the crucial lengthscale for this to happen is given by the inverse Hubble constant, in contrast to $1 + 1$ dimensions, where the relevant scale is essentially set by $1/m_0$. Despite these differences, the expansion induced production of excitations is very similar to the $3 + 1$ dimensional analogue, as in both cases the particle production relies on the fact that the underlying differential equation qualitatively changes at a mode-specific time [48]. At early times modes evolve freely, but undergo an overdamped dynamics at late times. In other words, different modes freeze at different times resulting effectively in a change of the power-law exponent of the fluctuation spectrum, i.e., $1/|k| \to 1/|k|^{\gamma}$. Here γ is an exponent which depends on the specifics of the underlying model.

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