Classification of Matrix Product States with a Local (Gauge) Symmetry

Ilya Kull

München 2017

Classification of Matrix Product States with a Local (Gauge) Symmetry

Ilya Kull

Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

of

Master of Science

Within the Theoretical and Mathematical Physics Elite Master Program

Ludwig–Maximilians–Universität München

Technische Universität München

Written at

Max–Planck–Institute Für Quantenoptik

Supervised by:

Prof. Dr. J. Ignacio Cirac

Dr. Erez Zohar

Submitted August 28, 2017

Abstract

Matrix Product States (MPS) are a particular type of one dimensional tensor network states, that have been applied to the study of numerous quantum many body problems. One of their key features is the possibility to describe and encode symmetries on the level of a single building block (tensor), and hence they provide a natural playground for the study of symmetric systems. In particular, recent works have proposed to use MPS (and higher dimensional tensor networks) for the study of systems with local symmetry that appear in the context of gauge theories. In this work we classify MPS which exhibit local invariance under arbitrary gauge groups. We study the respective tensors and their structure, revealing known constructions that follow known gauging procedures, as well as different, other types of possible gauge invariant states.

Contents

Chapter 1

Introduction

Gauge theories play a paramount role in modern physics. Through the gauge principle, the theories describing the fundamental interactions in the standard model of particle physics are obtained by lifting the global symmetries of the interaction-free matter theories to be local symmetries, minimally coupled [1] to a gauge field. Moreover, gauge theories also emerge as effective low-energy descriptions in several condensed matter scenarios [2]. Historically, the gauging procedure was first conceived as a transformation of a Lagrangian or Hamiltonian describing a physical system; however, it can be performed on the level of quantum states as well, irrespective of dynamics associated to a specific theory [3].

In spite of their central role in the standard model, non-Abelian gauge theories still involve puzzles to be solved. Their complete understanding still poses a significant challenge due to non-perturbative phenomena (e.g. low energy quantum chromodynamics). Among the various approaches proposed to tackle the strongly coupled regime, a particularly general and successful one is lattice gauge theory [4]. Monte Carlo sampling of Wilson's Euclidean lattice version of gauge theories has so far been the most successful method of numerical simulation, nevertheless, it suffers from its own drawbacks. The sign problem [5] prevents application to systems with large fermionic densities, and the use of Euclidean time does not allow to study real time evolution and non-equilibrium phenomena in general scenarios. In order to describe real-time evolution of such theories, one is forced to abandon the Monte Carlo approach, and search for other methods. In this context, the Hamiltonian formulation of Kogut and Susskind [6] has been receiving renewed interest, with two recent approaches coming from the quantum information and quantum optics community: quantum simulation, using optical, atomic or solid-state systems [7, 8], and tensor network states.

The representation of quantum many-body states as tensor networks is connected to White's density-matrix renormalization group [9], and in the case of one dimensional spin lattices is known as matrix product states (MPS) [10]. Tensor networks provide a way of significantly reducing the dimension of the space of states one considers when describing a many body quantum system. By restricting the allowed coefficients appearing in the state's expansion to ones that are given by contracting a network of tensors, with a geometric structure that mimics the structure of the physical system, one can efficiently approximate physical states (i.e., ground states and their low-energy excitations) while keeping the overall number of parameters needed for the description low enough to make numerical tasks feasible [10,11]. Among many useful properties of tensor networks, one which makes them well suited to the description of states with symmetries, is the ability to encode the symmetry on the level of a single tensor (or a few) describing the state. In the case of global symmetries, both for MPS and for certain classes of PEPS in 2D (Projected Entangled Pair States - the generalization of MPS to higher dimensional lattices), the relation between the symmetry of the state and the properties of the tensor is well understood [12].

Tensor networks studies of lattice gauge theories have so far included numerical works (e.g., mass spectra, thermal states, real time dynamics and string breaking, phase diagrams etc. for the Schwinger model and others) [13–32], furthermore, several theoretical formulations of classes of gauge invariant tensor network states have been proposed [3, 33–36]. In all of the latter the construction method follows the ones common to conventional gauge theory formulations: symmetric tensors are used to describe the matter degree of freedom, and later on a gauge field degree of freedom is added, or, alternatively - a pure gauge field theory is considered. As was shown in [12, 37, 38], a tensor giving rise to a MPS with a global symmetry must have a particular structure; it translates (intertwines) the physical symmetry operation to a group action on each one of the virtual matrix indices (or tensor indices in the PEPS case). When copies of the tensor are contracted, since all the physical symmetry operators act with the same group element, the virtual group actions cancel out. This fact was used in [35] as a starting point for constructing gauge invariant PEPS. There, the tensor describing the gauge field degree of freedom is constructed such that it translates two different symmetry operators, right and left ones, to virtual group actions which locally cancel out the virtual transformations arising from the symmetry operator acting on the matter.

While the usefulness of tensor networks in lattice gauge theories has certainly been demonstrated by the above mentioned works, so far there were few attempts (e.g. [15]) to generally classify tensor network states with local symmetry. As shown in the classification of globally symmetric MPS [37] (and is known in group theory as the Wigner-Eckart theorem [39]), the requirement that a tensor should act as an intertwiner of an irreducible representation with a tensor product of two irreducible representations defines it up to a constant. This suggests that the construction presented in [35] might be the only way to obtain PEPS with a local symmetry.

In this thesis, starting from the assumption of a local symmetry, we find necessary and sufficient conditions to be satisfied by the tensors encoding a MPS. Similar work was done in [15] for MPS with local U(1) symmetry and with open boundary conditions. We focus on translation-invariant MPS, and deal with arbitrary finite or compact Lie groups. Clearly, one could come up with arbitrarily complicated constructions of states with a local symmetry (e.g. by using many kinds of symmetric tensors). Our analysis is therefore limited to three physically meaningful settings corresponding to: states describing matter, pure gauge field states and states of both matter and gauge field. In our analysis the matter degrees of freedom are represented by "spins"; this could in principle be extended to fermionic systems.

For states describing only matter we find that local symmetries can only be trivial, and show how to gauge such states by adding another degree of freedom. When investigating pure gauge states we show that local symmetry in MPS requires a specific structure of the Hilbert space describing the gauge field degree of freedom. In Wilson's lattice gauge theories, in order to obtain minimal coupling in a continuum limit, the gauge field degree of freedom is set as a group element in the same representation as the one acting on the matter [4]. In the Hamiltonian formulation, the corresponding Hilbert space is isomorphic to $L^2(G)$, equipped with the left and right regular representations [40], and is referred to by Kogut and Susskind as "the rigid rotator" (in the $SU(2)$ case) [6]. The structure that we find for the gauge field Hilbert space is more general and contains the rotator-like space introduced by Kogut and Susskind as a particular case.

In the combined matter and gauge field setting we show that, similarly to the case of MPS with a global symmetry, the tensor describing the matter degree of freedom is a (generalized) vector operator, and its structure is therefore determined by the Wigner-Eckart theorem; the gauge field tensor's structure is simpler: it is an intertwining map that translates the physical symmetry operators into the same group action on the virtual (bond) spaces. This is a one dimensional version of the construction principle used in [35]; as expected, we find this construction method is unique (in the 1D case) and describe the available structural and parametric freedom in choosing the tensors. However, the structure we derive allows for more general gauge invariant MPS, namely, ones that do not arise as a result of gauging a global symmetry or coupling matter to a pure gauge field. We construct examples of such states: while possessing a local symmetry when coupled to each other, the matter and gauge field degrees of freedom do not retain their individual symmetries when separated. Finally, we discuss mutual implications between the condition of local symmetry of the pure gauge field and the condition of global symmetry of the matter when the two can be coupled to each other to produce a MPS with local symmetry.

Chapter 2

Gauge Theories

We shall begin this chapter with a broad review of the role symmetries play in modern physics. In this context we shall then introduce the notion of gauge theories and describe the principles which placed them at the forefront of our current understanding of elementary particles. We shall give a rudimentary example of such a theory and use it as a stepping stone to introduce Wilson's lattice gauge theories, their Hamiltonian formulation by Kogut and Susskind and, within that framework, the local Hilbert spaces describing the matter and gauge field degrees of freedom.

2.1 Gauge Symmetry in Physics

Noether's theorem states that to every continuous symmetry of a physical system corresponds a conserved quantity. Invariance of the mathematical description of a system under space translations of the coordinate frame of reference leads to the conservation of momentum; invariance with respect to time translation leads to conservation of energy; invariance under spatial rotations implies the conservation of angular momentum. These are all geometric transformations of space-time, and respectively - geometric symmetries.

However, physical theories could be described in terms of quantities that depend on coordinate systems other than space-time (e.g. flavor or color). The degrees of freedom described by these coordinates are called internal. Such theories could be invariant under transformations of these internal coordinates, they are said to have an internal symmetry. The conserved quantities corresponding to such symmetries are called charges.

In quantum mechanical systems, equivalent to the existence of a continuous symmetry is the existence of observables - the generators of the group of transformations - which commute with the Hamiltonian that describes the dynamics of the system. Let G be a continuous symmetry group generated by self-adjoint charge operators Q^a such that

$$\Theta(g) = \exp(i \sum_a Q^a \phi^a(g)) \ , g \in G \ ,$$

where $\phi^a(g)$ are real parameters. Let a Hamiltonian H be invariant under all group transformations:

$$\Theta(g) H \Theta(g)^\dagger = H \ , \ \forall g \in G.$$

Differentiating with respect to any of the parameters ϕ^a we obtain:

$$Q^a H - H Q^a = 0 \ .$$

Fix one value of a. Since Q^a commutes with H they can be diagonalized simultaneously, their eigenvectors labeled by two quantum numbers: Let $|\phi_\lambda\rangle$ be an eigenvector of H with eigenvalue λ, $Q^a|\phi_\lambda\rangle$ is also an eigenvector with the same eigenvalue:

$$H Q^a |\phi_\lambda\rangle = Q^a H |\phi_\lambda\rangle = \lambda Q^a |\phi_\lambda\rangle \ .$$

Q^a therefore preserves each λ eigenspace of H and can be diagonalized in each such subspace. Label the mutual eigenvectors $|\varphi_{\lambda,q^a}\rangle$ such that

$$H|\varphi_{\lambda,q^a}\rangle = \lambda|\varphi_{\lambda,q^a}\rangle \quad ; \ Q^a|\varphi_{\lambda,q^a}\rangle = q^a|\varphi_{\lambda,q^a}\rangle \ .$$

We see that a system with a symmetry exhibits multiplets, i.e. degeneracies of its energy eigenstates: states with the same energy but with different eigenvalues q^a of the generator Q^a. Furthermore, a symmetry with respect to a non-Abelian Lie group G gives information about the structure of the energy eigenspaces. In this case the generators are representations of elements of $\mathfrak{g}$, the Lie algebra of G. As demonstrated above, the energy eigenspaces are preserved by the generators and are therefore representation spaces of $\mathfrak{g}$. As such, they can be decomposed into irreducible representation spaces which can be distinguished from one another by the value of yet another observable: the quadratic Casimir operator [41], e.g. the total angular momentum operator in the case of $SO(3)$ or $SU(2)$.

In addition, since an observable that commutes with the Hamiltonian also commutes with the time evolution operator

$$[Q^a, H] = 0 \quad \Leftrightarrow \quad [Q^a, \exp(itH)] = 0 \; \forall t \in \mathbb{R} \; ,$$

the dynamics of the system preserves the eigenvalues of the generator. These are so-called selection rules: transitions between certain states (ones with different eigenvalues of the generator) are not allowed by the dynamics.

Given a description of a physical system (Lagrangian or Hamiltonian) one can find all the groups of transformations under which it is invariant, apply Noether's theorem to deduce the conserved quantities, and predict the multiplets and selection rules. The latter could then be verified experimentally. Historically, once the relation between symmetries and conservation laws has been understood, particle physicists reversed the above reasoning: given the experimentally observed complex structure of multiplets and selection rules, they looked for candidate symmetry groups that would give rise to such a structure, and constructed theories observing those symmetries [42].

Since internal coordinates describe degrees of freedom independent of space-time, we can consider two kinds of symmetry groups of their transformations. One distinguishes between *global* symmetry groups, when the description of the system is invariant under an application of the same transformation at each point in space-time, and *local* symmetry groups or *gauge* symmetry groups, when the transformation applied to the internal coordinate system depends on the position in space-time. When considering the latter kind of symmetries, one quickly runs into a problem with an elementary notion, indispensable in any physical theory, namely, that of rates of change of quantities in space-time. If the quantity in question is determined up to an arbitrary "rotation" which differs between space-time points, what is then the meaning of the difference between its values at different points? Consequently, one is forced to redefine the notion of a derivative to one which is *covariant* with respect to the allowed group of transformations [1]. This procedure involves introducing an additional degree of freedom, one which has a defined transformation law under the gauge group. A theory which is invariant with respect to such a gauge group of transformations is called a gauge theory.

Starting with a theory described by a Lagrangian invariant under a given global symmetry (associated to it - a certain charge), one can gauge this symmetry, i.e. promote it to be a local one by the above procedure. This results in an interaction term in the Lagrangian, between the original (matter) degree of freedom and the additional one (gauge field). Adding the simplest local gauge invariant term which involves only the gauge field to the Lagrangian results in a theory where the gauge field degree of freedom is interpreted as a massless particle that mediates the interaction corresponding to that charge. The most well known, and historically defining example of this procedure, is the case of quantum electrodynamics (QED) described in detail by numerous authors (e.g. [1]).

We start from the Dirac Lagrangian density

$$\mathcal{L}_0(x) = \overline{\psi}(x)\,(i\gamma^\mu\partial_\mu - m)\,\psi(x) \ ,$$

where $\psi(x)$ is the free electron field, $\overline{\psi}(x) = \psi^\dagger(x)\gamma^0$ the conjugate field and γ^μ the Dirac matrices. Let $\alpha \in \mathbb{R}$, $\mathcal{L}_0(x)$ is invariant under the global phase transformation:

$$\psi(x) \mapsto \exp(i\alpha)\psi(x) \ \ , \overline{\psi}(x) \mapsto \exp(-i\alpha)\overline{\psi}(x) \ . \tag{2.1}$$

This invariance leads to the conservation of an additive quantum number, which is identified with the total electric charge. The invariance no longer holds if the constant α is replaced with a function $\alpha(x)$, since the derivative ∂_μ acts both on $\psi(x)$ and on $\alpha(x)$ via the product rule. In order to restore the invariance, the derivative is replaced by the covariant derivative D_μ

$$D_\mu\psi(x) = \left[\partial_\mu - ieA_\mu(x)\right]\psi(x) \ ,$$

where $-e$ is the electron charge and $A_\mu(x)$ a vector field which transforms according to

$$A_\mu(x) \mapsto A_\mu(x) + e^{-1}\partial_\mu\alpha(x) \ .$$

This substitution of the derivative introduces an interaction term in the Lagrangian

$$\mathcal{L}(x) = \mathcal{L}_0(x) + e\overline{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) \ ,$$

where the electron field is coupled to the vector field $A_\mu(x)$, which is interpreted as the vector potential. In order to complete the theory, one needs to add a dynamical term for the vector potential. The simplest local terms that are at most quadratic in the field A_μ and are invariant under both Lorentz and gauge transformations are proportional to $F_{\mu\nu}(x)F^{\mu\nu}(x)$ where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the electromagnetic field tensor. The complete, gauge invariant Lagrangian density is therefore:

$$\mathcal{L}(x) = \overline{\psi}(x)\left(i\gamma^\mu\partial_\mu - m\right)\psi(x) + e\overline{\psi}(x)\gamma^\mu\psi(x)A_\mu(x) - \frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) \ . \tag{2.2}$$

The fact that by performing this gauging procedure on the free electronic Lagrangian, one obtains the same theory as the one obtained by quantizing the classical electrodynamics Lagrangian (which classically includes the vector potential), led to the establishment of the *gauge principle* which can be summarized as follows: the fundamental interactions of nature originate from gauging global symmetries of the free theory. In the QED Lagrangian (Eq. (2.2)) the electron current is coupled to one A^μ term and not a more complicated expression involving the vector potential. This is referred to as *minimal coupling* [1].

The construction just outlined can be done in a much more general setting, with the group $U(1)$ replaced by a more general Lie group. In the case of groups from the family $SU(n)$, the obtained theories are called Yang-Mills theories, named after Yang and Mills who first proposed the construction of an $SU(2)$ gauge invariant theory. Such theories involve a field $\Psi(x)$ which is an n-tuple of fields $\psi(x) = (\psi^1(x), \psi^2(x), \ldots, \psi^n(x))$ which transforms under $V \in SU(n)$ according to

$$\Psi(x) \mapsto V\Psi(x) \ ,$$

where the product on the RHS is simply matrix multiplication, i.e. $(V\Psi)^i = \sum_j V_{i,j}\psi^j$. Starting from a Lagrangian invariant under such global transformations $\mathcal{L}_0(V\Psi, V\partial\Psi) = \mathcal{L}_0(\Psi, \partial\Psi)$, a gauge invariant Lagrangian is obtained by introducing the covariant derivative $D_\mu = \partial_\mu + iA_\mu(x)$, where A_μ is now an $\mathfrak{su}(n)$ valued function (which transforms under the group action in the appropriate way)

$$\mathcal{L}(\Psi, \partial\Psi) := \mathcal{L}_0(\Psi, D\Psi) \ .$$

To this a dynamic term for the gauge field is added $1/4\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$, where $F_{\mu\nu}$ is also an $\mathfrak{su}(n)$ valued function given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \ . \tag{2.3}$$

The full Lagrangian is therefore

$$\mathcal{L}(\Psi, \partial\Psi, A, \partial A) := \mathcal{L}_0(\Psi, D\Psi) + \frac{1}{4}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \ .$$

The resulting theory is is one in which the n fields ψ^i are coupled with $N = dim(\mathfrak{su}(n))$ gauge fields A^k_μ, the components of the $\mathfrak{su}(\mathfrak{n})$ valued function A_μ. Similarly to the Abelian case, the transformation rule obeyed by the gauge fields does not allow the incorporation of their mass terms in the Lagrangian. A novel feature of non-Abelian gauge theories is the fact that the gauge fields interact with each other directly (unlike photons in QED) due to the terms $[A_\mu, A_\nu]$ in Eq. (2.3).

Next, we present Wilson's action [4], which is a discretized version of the Dirac action in imaginary time, describing fermionic fields in a $3+1$ dimensional lattice with spacing a. One could try to construct such an action by performing the following substitutions

$$\int dx \mathcal{L}(x) \to a^4 \sum_{n\in\mathbb{Z}^4} \mathcal{L}(an)$$
$$\partial_\mu \psi(an) \to \frac{1}{2a}\left[\psi_{n+\hat{\mu}} - \psi_{n-\hat{\mu}}\right] \ ,$$

where $\psi_n := \psi(an)$ and $\hat{\mu}$ is a unit vector in the μ direction. However, this results in an action that recovers gauge invariance only in the limit $a \to 0$. Instead, Wilson's action is constructed such that gauge invariance is preserved for any value of a, and is given by[1]:

$$S_W = ma^4 \sum_n \overline{\psi}_n \psi_n + \frac{a^3}{2} \sum_{n,\hat{\mu}} \left[\overline{\psi}_n \gamma^\mu U_{n,\mu} \psi_{n+\hat{\mu}} - \overline{\psi}_{n+\hat{\mu}} \gamma^\mu U^\dagger_{n,\mu} \psi_n\right] + \frac{1}{g^2} \sum_n \sum_{\hat{\mu},\hat{\nu}} \left[U_{n,\mu} U_{n+\hat{\mu},\nu} U^\dagger_{n+\hat{\nu},\mu} U^\dagger_{n,\nu} + h.c.\right] \ ,$$

where g is a coupling constant and $U_{n,\mu}$ is a *connector*, residing on the link between the vertices n and $n+\hat{\mu}$ which is introduced in order to preserve gauge invariance and transforms

[1] This action results in the problem of fermion doubling when taking the continuum limit. This can be solved, e.g. by introducing staggered fermions [43], which we shall not do here.

according to

$$U_{n,\mu} \mapsto V_n U_{n,\mu} V^{\dagger}_{n+\hat{\mu}} \ ,$$

where $V_n := e^{i\alpha(an)}$ such that Eq. (2.1) becomes $\psi_n \mapsto V_n \psi_n$. The terms $U_{n,\mu}U_{n+\hat{\mu},\nu}U^{\dagger}_{n+\hat{\nu},\mu}U^{\dagger}_{n,\nu}$ are the lattice gauge invariant approximation of the gauge field dynamic term $F_{\mu\nu}(x)F^{\mu\nu}(x)$, each corresponds to a product of the connectors along the closed loop defined by the vertices $(n, n+\hat{\mu}, n+\hat{\mu}+\hat{\nu}, n+\hat{\nu})$, i.e. a plaquette.

2.2 Hamiltonian formulation of Lattice gauge theory

The Hamiltonian formulation of Wilson's lattice gauge theories by Kogut and Susskind recovers the Yang-Mills Hamiltonian in the continuum limit [6]. We review the construction of the theory and describe the associated local Hilbert spaces, the structure of which (as we are interested in studying gauge invariant *states*) will be in the focus of our attention.

We start from a Hubbard-like Hamiltonian describing fermions on a one dimensional lattice with sites numbered by the position index x:

$$H = M \sum_x \psi^{\dagger}_x \cdot \psi_x + \epsilon \sum_x \left(\psi^{\dagger}_x \cdot \psi_{x+1} + h.c. \right) \ ,$$

where M is the mass, ϵ is the hopping rate, $\psi^{\dagger}_x \cdot \psi_x := \sum_i (\psi^{\dagger}_x)^i \psi^i_x$ and the field ψ is a vector of fields which transforms with respect to a finite dimensional unitary representation of a compact Lie group G as follows:

$$\psi_x \mapsto V(g)\psi_x \ , \quad \psi^{\dagger}_x \mapsto \psi^{\dagger}_x V(g)^{\dagger} \ ,$$

where $(V(g)\psi_x)^i = \sum_j V(g)_{i,i'} \psi^{i'}_x$. The Hamiltonian consists of mass terms and nearest neighbor hopping terms. This Hamiltonian has a global symmetry with respect to the representation $V(g)$ (as all the group indices are contracted in the dot products). This symmetry corresponds to a conservation of the total number of fermions, which is also clear from inspection of H. Next, we promote the symmetry to be local. We now allow the group element to depend on the lattice site:

$$\psi_x \mapsto V(g_x)\psi_x \ , \quad \psi_x^\dagger \mapsto \psi_x^\dagger V(g_x)^\dagger \ .$$

The mass terms in the Hamiltonian are still invariant because they are local (products of fields at the same site); the hopping terms, however are not. As expected, the problem comes from the kinetic term, where the derivatives would appear in a continuum theory. As before, we can restore the symmetry by introducing a *connection* matrix U_x, associated with the links between the fermionic sites x and $x+1$ which transforms under the group action as:

$$U_x \mapsto V(g_x)U_x V(g_{x+1})^\dagger \ .$$

Since V is a finite dimensional representation of a compact Lie group it is given as a direct sum of irreducible representations $V = \oplus_j D^j$. Corresponding to the structure of V, the connection matrix is of the form $U = \oplus_j U^j$. The modified, gauge invariant Hamiltonian is given by:

$$H_G = M\sum_x \psi_x^\dagger \cdot \psi_x + \epsilon \sum_x \left(\psi_x^\dagger \cdot U_x \psi_{x+1} + h.c.\right) \ ,$$

Note that here we do not consider the dynamical term for the gauge field, as it is not necessary for our discussion. We denote the generators of the gauge transformation around lattice site n (when only $g_x \neq e$ and $e \in G$ is the trivial element) by $\{G_x^a\}_{a=1\ldots dimG}$ (there are as many generators as the dimension of G as a manifold). The gauge invariance of the Hamiltonian implies that all G_x^a commute with the Hamiltonian, which in turn implies that for any eigenvector of the Hamiltonian $|\phi_\lambda\rangle$ with eigenvalue λ, the generators G_x^a satisfy

$$HG_x^a|\phi_\lambda\rangle = G_x^a H|\phi_\lambda\rangle = \lambda G_x^a|\phi_\lambda\rangle \ ,$$

which means they preserve the λ eigenspace. The energy eigenspaces therefore consist of multiplets, i.e. they are representation spaces of the Lie algebra of G. Restricting ourselves to the singlet case, we have Gauss' law which defines the physical, gauge invariant subspace of states:

$$G_x^a|\phi\rangle = 0 \;\; \forall x, a \; . \tag{2.4}$$

2.3 Matter and Gauge Field Hilbert spaces

We proceed with the above example. Consider a single lattice site, the states in the associated Hilbert space transform according to:

$$|jm\rangle := \left(\psi^\dagger\right)^{j,m}|0\rangle \mapsto \left(\psi^\dagger V(g)^\dagger\right)^{j,m}|0\rangle = \sum_m D^j(g^{-1})_{m'm}|jm'\rangle \;, \tag{2.5}$$

where $|0\rangle$ is the one site Fock space vacuum and the fields composing $\psi^\dagger$ were relabeled by the irreducible representation index corresponding to the decomposition of V: $\psi^i \to \psi^{j,m}$. We denote the generators of these transformations $\{Q_x^a\}_{a=1...dimG}$.

The local Hilbert space on the link, describing the gauge field degree of freedom is isomorphic to $L^2(G)$. Let D^j be irreducible representation matrices of G, where for every j D^j is a unique representative of an equivalence class of irreducible representations. According to the Peter-Weyl theorem the matrix elements of the representations matrices D^j_{mn} are orthogonal to each other in $L^2(G)$ and span the entire Hilbert space [44]. We denote the elements of the orthonormal basis in ket notation $|jmn\rangle = \sqrt{dim(j)}D^j_{mn}$ where, similarly to the notation for wave-functions in QM $\langle x \mid \phi\rangle \equiv \phi(x)$, for any $g \in G$ $\langle g \mid jmn\rangle = \sqrt{dim(j)}D^j_{mn}(g)$. Each matrix element of U_x: $(U_x)^j_{mn}$ acts as a multiplication operator by the function $D^j_{m,n}$ on the Hilbert space $L^2(G)$ at link x (between lattice sites x and $x+1$). Its action on the singlet state $|000\rangle$, i.e. the constant function $\langle g \mid 000\rangle \equiv 1$ (we assume the Haar measure is normalized such that $|G| = 1$), is given by $\sqrt{dim(j)}(U_x)^j_{mn}|000\rangle_x = |jmn\rangle_x$.

If we consider the action of a gauge transformation corresponding to a trivial action in all sites except one ($\forall x \neq x_0 \; g_x = e$, where $e \in G$ is the trivial element), we see that the group action on the link Hilbert space is given by two transformations corresponding to $g_{x_0} \neq e$ and $g_{x_0+1} \neq e$:

$$\begin{aligned}|jmn\rangle_{x_0} = \sqrt{dim(j)}(U_{x_0})^j_{mn}|000\rangle_{x_0} \mapsto & \sqrt{dim(j)}\left(V(g_{x_0})U_{x_0}V(e)^\dagger\right)^j_{mn}|000\rangle_{x_0} \\ = & \sqrt{dim(j)}\sum_{m',n'} D^j(g_{x_0})_{mm'}(U_{x_0})^j_{m'n'}\delta_{n',n}|000\rangle_{x_0} \\ = & \sum_{m'} D^j(g_{x_0})_{mm'}|jm'n\rangle_{x_0}\end{aligned}$$

$$\begin{aligned}|jmn\rangle_{x_0} = \sqrt{dim(j)}(U_{x_0})^j_{mn}|000\rangle_{x_0} \mapsto & \sqrt{dim(j)}\left(V(e)U_{x_0}V(g_{x_0+1})^\dagger\right)^j_{mn}|000\rangle_{x_0} \\ = & \sqrt{dim(j)}\sum_{m',n'} \delta_{m,m'}(U_{x_0})^j_{m'n'}D^j(g^{-1}_{x_0+1})_{n'n}|000\rangle_{x_0} \\ = & \sum_{n'} D^j(g^{-1}_{x_0+1}))_{n'n}|jmn'\rangle_{x_0} \ .\end{aligned}$$

These correspond to the left and right regular representations of G on $L^2(G)$ [44], acting, however, with the inverse group element. This can by fixed by interchanging V and $V^\dagger$ in the definitions of the transformations of the fields. Equation (2.5) then also assumes the form of a proper group action. Respectively, these transformations are generated by the left and right generators of the group $\{L^a_{x_0}\}_{a=1...dimG}$ and $\{R^a_{x_0}\}_{a=1...dimG}$ [35].

In terms of the generators of the transformations on the local Hilbert spaces Eq. (2.4) reads:

$$R^a_x + Q^a_x + L^a_{x+1}|\phi\rangle = 0$$

For Abelian groups $R = -L$ and, therefore, $R^a_x + L^a_{x+1}$ is the lattice divergence of L; identifying L with the electric field and Q with the charge, when taking a continuum limit Eq. (2.4) becomes the familiar Gauss law.

In Wilson's lattice gauge theory, in order to recover the covariant derivative when taking a continuum limit, the connector U_x is taken to be an element in the representation of the group and not of its Lie algebra (as described in the previous section) [4]. However, one does not have to consider a continuum limit in order to obtain such a structure. The properties of the degree of freedom on the link, which has two parts - left and right - each one of which transforms independently, and thereby compensate for the local transformation of the matter field, seem like an intuitive solution to the problem of restoring invariance once the symmetry operation is allowed to be local. Once these properties are imposed as constraints on the gauge field degree of freedom, not many options remain for choosing it. It will indeed be a central theme in this work, the fact that the behavior of objects under group transformations determines their structure. To show the uniqueness of the gauging procedure described in our example is beyond the scope of this work, however, in the MPS framework we will demonstrate just that. One can therefore recover the minimal coupling rule in the continuum theory by finding the simplest (i.e. local) structure that connects two mass degrees of freedom on the lattice in a gauge invariant way.

Chapter 3

MPS Background

In this chapter we shall introduce the MPS formalism and the notation used throughout this work. We shall then present essential definitions and results from the theory of MPS, building up to the fundamental theorem of MPS. Next we shall proceed with a brief review of background in representation theory, in particular, projective representations, Schur's lemma and the Wigner-Eckart theorem. Finally we shall present the classification of MPS with a global symmetry, which will be utilized when we will prove our results in the next chapter.

3.1 Matrix product vectors

We consider matrix product vectors (MPV) rather than states (MPS). The distinction is emphasized because MPV can refer to unnormalized MPS as well as to matrix product operators, to which our results can also be applied. Moreover, in the following we shall define symmetries in terms of equalities between vectors and not states, i.e. we shall not allow a phase difference. For a comprehensive introduction to MPS we refer the reader to [10, 45, 46]. In the following we shall review the basic definitions, and quote essential results.

Let $\mathcal{H}$ be a d-dimensional Hilbert space. A matrix product vector (MPV) is a vector $|\psi_A^N\rangle \in \mathcal{H}^{\otimes N}$ given by

$$|\psi_A^N\rangle = \sum_{\{i\}} \mathrm{Tr}\left(A^{i_1}A^{i_2}\dots A^{i_N}\right)|i_1 i_2 \dots i_N\rangle \ , \tag{3.1}$$

where $\{A^i | i = 1,\dots,d\}$ are $D \times D$ matrices and $\{|i\rangle | i = 1,\dots,d\}$ is an orthonormal basis in $\mathcal{H}$. The dimension of the matrices, D, is called the bond dimension of A. We say that the tensor A, which consists of the matrices A^i, generates the MPV $|\psi_A^N\rangle$; in fact, it generates a family of vectors: $\left\{|\psi_A^N\rangle | N \in \mathbb{N}\right\}$. We refer to the entire family of vectors as the MPV generated by A. Vectors of this form will be at the focus of our study; they are typically used to describe translationally invariant (TI) states of N spins on a 1 dimensional lattice with periodic boundary conditions, where each spin is described by the Hilbert space $\mathcal{H}$, and the order of the terms in the tensor product corresponds to the positions of the spins on the lattice. Similarly, it is possible to describe states of a spin lattice with open boundary conditions, and that are not TI, with a different tensor $A^{[k]}$ associated with each site number k, and where two arbitrary vectors $|\alpha_0\rangle$, $|\beta_0\rangle \in \mathbb{C}^D$ encode the boundary conditions:

$$|\tilde{\psi}_A^N\rangle = \sum_{\{i\}} \langle\alpha_0|A^{[1]i_1}A^{[2]i_2}\dots A^{[N]i_N}|\beta_0\rangle |i_1 i_2 \dots i_N\rangle \ . \tag{3.2}$$

The number of parameters needed in order to specify a general state of the N body spin chain is $dim(\mathcal{H}^{\otimes N}) = d^N$. On the other hand, the vector $|\tilde{\psi}_A^N\rangle$ is specified by the entries of the tensors $A^{[k]}$, i.e. $D^2 \times d \times N$ parameters, which grows linearly with the system size N, rather than exponentially. It was shown in [47] that any state of a spin chain can be written in the form Eq. (3.2), and that the minimal D required is determined by the maximum rank of the reduced density matrix over all partitions of the chain of spins into two subsystems $[1, 2, \dots, k][k+1, \dots, N]$:

$$D \geq \max\{rank(\rho_k) | k = 1, \dots N\} \ ,$$

which is a measure of the maximum entanglement between two parts of the system.

An intuitive explanation for this is provided by the following so-called valence bond construction: To each spin on the lattice associate two auxiliary D dimensional Hilbert spaces (virtual spins), a left one and a right one. Initialize the auxiliary system so that each pair of neighboring virtual spins corresponding to different sites on the lattice are in an (unnormalized) maximally entangled state $\sum_{\alpha=1}^{D} |\alpha\rangle \otimes |\alpha\rangle$ (and the rightmost and leftmost virtual spins in some states $|\alpha_0\rangle$ and $|\beta_0\rangle$). Next, at each site, apply a linear map $\mathcal{A}^{[k]}$ from the virtual pair associated to that site to the physical Hilbert space:

$$\mathcal{A}^{[k]} := \sum_{i=1}^{d} \sum_{\alpha,\beta=1}^{D} A_{\alpha,\beta}^{[k]i} |i\rangle\langle\alpha,\beta| \ .$$

The resulting vector is the MPV in Eq. (3.2). Throughout this thesis we shall consider only TI-MPV.

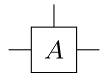
Example 3.1.1. We demonstrate the valence bond construction for $N = 2$ with periodic boundary conditions. The initial configuration of the virtual spins is $\sum_{\gamma,\kappa} |\gamma\rangle_{1,L}|\kappa\rangle_{1,R}|\kappa\rangle_{2,L}|\gamma\rangle_{2,R}$. Apply the map $\mathcal{A}$ to each virtual pair to obtain:

$$\begin{aligned}
&\sum_{i=1}^{d} \sum_{\alpha,\beta=1}^{D} \sum_{\gamma,\kappa=1}^{D} A^j_{\alpha',\beta'} A^i_{\alpha,\beta} \left\langle \alpha' \mid \gamma\right\rangle_{1,L} \left\langle \beta' \mid \kappa\right\rangle_{1,R} \left\langle \alpha \mid \kappa\right\rangle_{2,L} \left\langle \beta \mid \gamma\right\rangle_{2,R} |j\rangle_1 |i\rangle_2 \\
=&\sum_{i=1}^{d} \sum_{\gamma,\kappa=1}^{D} A^j_{\gamma,\kappa} A^i_{\kappa,\gamma} |j\rangle_1 |i\rangle_2 \\
=&\sum_{i=1}^{d} \mathrm{Tr}\left(A^j A^i\right) |j\rangle_1 |i\rangle_2 \ ,
\end{aligned}$$

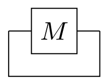
which is exactly Eq. (3.1).

In order to avoid cumbersome notation involving many indices, we will use the graphical notation commonly used in tensor networks. Each tensor is denoted by a rectangle with lines connected to it. Each line corresponds to an index of the tensor. For example, the tensor A

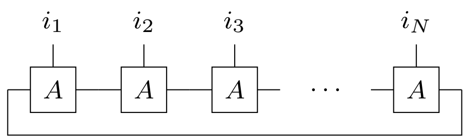
generating the MPV $|\psi_A^N\rangle$ in Eq. (3.1) is represented as:



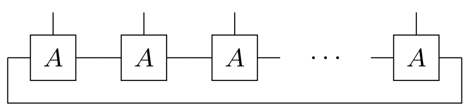
where the top line corresponds to the physical index: $i = 1, \ldots, d$, and the horizontal lines - to the ("virtual" or "bond") matrix indices: $\alpha = 1, \ldots, D$. Contraction of tensor indices is indicated by connecting the respective lines. If M is a square matrix, i.e. a rank 2 tensor, then $\mathrm{Tr}(M)$ is denoted by:



The coefficient corresponding to the $|i_1 i_2 \ldots i_N\rangle$ basis element of the MPV $|\psi_A^N\rangle$ in Eq. (3.1) is denoted by:



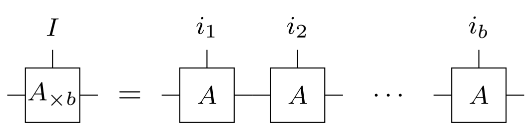
where we specified the values of the physical indices. We identify the MPV of length N generated by A with the set of its coefficients and denote the MPV as:



Definition 3.1.1. Let A be a tensor composed of matrices $\{A^i\}$. Blocking of b copies of A defines a new tensor denoted by $A_{\times b}$, which is composed of the matrices given by the b-fold products of A^i, and are numbered by an index $I := (i_1, i_2, \ldots, i_b)$:

$$\left\{ (A_{\times b})^I = A^{i_1} A^{i_2} \ldots A^{i_b} \mid \; i_1, i_2, \ldots, i_b = 1, \ldots, d \right\} \; .$$

The new index I corresponds to the basis $\{|I\rangle := |i_1\rangle \otimes |i_2\rangle \otimes, \ldots, \otimes |i_b\rangle\}$ of $\mathcal{H}^{\otimes b}$. Graphically:



The MPV of length N generated by $A_{\times b}$ is $|\psi_{A_{\times b}}^N\rangle \in \left(\mathcal{H}^{\otimes b}\right)^{\otimes N}$.

Definition 3.1.2 (Injective tensor)**.** A tensor A consisting of $D \times D$ matrices $\{A^i\}_{i=1}^d$ is injective if

$$span \left\{A^i \mid i = 1, \ldots, d\right\} = \mathcal{M}_{D\times D} \ ,$$

where $\mathcal{M}_{D\times D}$ is the algebra of $D \times D$ matrices.

Definition 3.1.3. Let A be a tensor consisting of matrices $\{A^i\}_{i=1}^d$. The completely positive (CP) map associated with A is defined by:

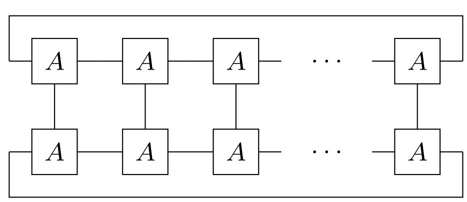
$$E_A(\cdot) = \sum_{i=1}^{D} A^i \cdot A^{i\dagger} \ ,$$

i.e., the matrices $\{A^i\}$ are the Kraus operators of E_A [48].

This map arises when computing the squared norm of $|\psi_A^N\rangle$:

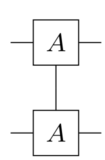
$$\left\langle \psi_A^N \mid \psi_A^N \right\rangle = \mathrm{Tr}\left[\left(\hat{E}_A\right)^N\right] \ ,$$

where $\hat{E}_A = \sum_i \overline{A_i} \otimes A_i$ is the matrix representation of E_A [48]. This can be seen by direct computation, or alternatively, by the graphical representation of $\left\langle \psi_A^N \mid \psi_A^N \right\rangle$:



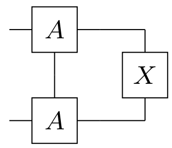
.

Instead of first contracting the row of As (horizontal indices) and then taking the inner product (vertical indices), we can first contract the vertical indices and obtain a trace of N powers of the map E_A, which is represented by:



,

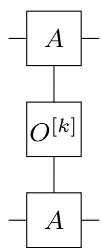
(when the tensor is drawn upside down, a complex conjugation is assumed, see [33] for more detailed notation convention) and acts from right to left mapping a matrix X to $E_A(X) = \sum_{i=1}^{D} A^i X A^{i\dagger}$:



Similar expressions are obtained when computing expectation values of local operators of the form $O = O^{[1]} \otimes O^{[2]} \otimes \ldots \otimes O^{[N]}$:

$$\langle \psi_A^N | O | \psi_A^N \rangle = \mathrm{Tr}\left[\prod_{k=1}^{N} \hat{E}_A^{(O^{[k]})} \right] ,$$

where $\hat{E}_A^{O^{[k]}} = \sum_{i,j} \langle i | O^{[k]} | j \rangle \overline{A_i} \otimes A_j$, and is represented graphically by:



It is clear that many different tensors give rise to the same vector, e.g. for any invertible matrix X, the tensor composed of the matrices $\tilde{A}^i := X^{-1} A^i X$ generates the same MPV as A. Furthermore, it could happen that the matrices A^i are all of an upper block diagonal structure:

$$A^i = \begin{pmatrix} A^i_{1,1} & A^i_{1,2} \\ 0 & A^i_{2,2} \end{pmatrix} .$$

In this case the MPV $|\psi_A^N\rangle$ does not depend on $\{A^i_{1,2}\}$. These two observations motivate the definition of a canonical form of a tensor generating a given MPV, which will be introduced shortly.

Definition 3.1.4 (Normal tensor)**.** a tensor A, consisting of $D \times D$ matrices $\{A^i\}_{i=1}^d$, is normal if there exists $L \in \mathbb{N}$ such that:

$$span\left\{A^{i_1}A^{i_2}\dots A^{i_L} \mid i_1, i_2, \dots, i_L = 1, \dots, d\right\} = \mathcal{M}_{D\times D} \ ,$$

where $\mathcal{M}_{D\times D}$ is the algebra of $D \times D$ matrices. That is, A is normal if it becomes injective after blocking a sufficient number of its copies. In addition we require that the spectral radius of the CP map E_A is equal to 1.

Remark 3.1.1. If a tensor becomes injective after blocking L_0 copies, it is also injective when blocking any number $L \geq L_0$ of copies. There is an upper bound on the minimal number of copies of a normal tensor needed to be blocked in order for the blocked tensor to be injective, which depends only on its bond dimension [49].

Proposition 3.1.1. *A tensor is normal (Definition 3.1.4) iff the CP map associated with it is primitive (irreducible and non-periodic).* [48]

Definition 3.1.5 (Canonical form)**.** A tensor A is in CF if the matrices A^i are block diagonal and have the following structure:

$$A^i = \oplus_{k=1}^n \nu_k A_k^i \ , \tag{3.3}$$

where $\{A_k\}$ are normal tensors and ν_k are constants.

Definition 3.1.6 (Canonical form II)**.** A is in CFII if in addition to being in CF, for any k appearing in Eq. (3.3) the CP map E_{A_k} is trace preserving, and has a positive full rank diagonal fixed point $\Lambda_k > 0$.

Proposition 3.1.2. *Let $|\psi_A^N\rangle$ be the MPV generated by a tensor A. If the CP map E_A has no periodic irreducible blocks, then there exists a tensor $\tilde{A}$ in CF (or CFII) such that:*

$$|\psi_A^N\rangle = |\psi_{\tilde{A}}^N\rangle \ , \forall N \in \mathbb{N} \ .$$

If E_A does have periodic blocks, then there exist a tensor $\tilde{A}$ in CF (of CFII) and $b \in \mathbb{N}$ such that:

$$|\psi^N_{A_{\times b}}\rangle = |\psi^N_{\tilde{A}}\rangle \ , \forall N \in \mathbb{N} \ ,$$

where $A^{\times b}$ is the tensor obtained by blocking b copies of A (Definition 3.1.1). [46]

Definition 3.1.7 (Basis of normal tensors)**.** Let A be a tensor in CF. A set of tensors $\{\hat{A}_j\}$ is said to be a basis of normal tensors (BNT) of A if $\hat{A}_j$ are normal tensors, and for every A_k appearing in A's expansion (Eq. (3.3)) there exists a unique $\hat{A}_j$, an invertible matrix V and a phase $e^{i\phi}$ such that $A_k = e^{i\phi} V^{-1} \hat{A}_j V$.

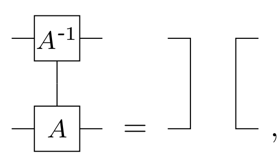
From now on whenever we consider a tensor A in CF we shall write it in terms of a BNT $\{A_j\}_{j=1}^m$:

$$A^i = \oplus_{j=1}^m \oplus_{q=1}^{r_j} \mu_{j,q} V_{j,q}^{-1} A_j^i V_{j,q} \ . \tag{3.4}$$

The MPV of length N generated by such a tensor A takes the form:

$$|\psi^N_A\rangle = \sum_{j=1}^{m} \sum_{q=1}^{r_j} (\mu_{j,q})^N \, |\psi^N_{A_j}\rangle \ .$$

Proposition 3.1.3. *Definition 3.1.2 is equivalent to the existence of a one-sided inverse tensor A^{-1} which satisfies:*



that is:

$$\sum_i A^i_{\alpha\beta} (A^{-1})^i_{\alpha\beta} = \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}$$

Definition 3.1.8 (Span of matrix products)**.** For a tensor A with bond dimension D let $S_L \subseteq \mathcal{M}_{D\times D}$ be the space spanned by all possible L-fold matrix products:

$$\mathcal{S}_L := span \left\{ A^{i_1} A^{i_2} \dots A^{i_L} \mid i_1, i_2, \dots, i_L = 1, \dots, d \right\} \ .$$

Definition 3.1.9. Let $\Gamma_A^L : \mathcal{M}_{D\times D} \to \mathcal{H}^{\otimes L}$ be defined by:

$$\Gamma_A^L(X) = \sum \mathrm{Tr}\left(XA^{i_1}A^{i_2}\dots A^{i_L}\right)|i_1 i_2 \dots i_L\rangle\ .$$

For a normal tensor, according to Definition 3.1.4, for L large enough, $S_L = \mathcal{M}_{D\times D}$. For tensors in CF the following holds:

Proposition 3.1.4 (Span property of BNT)**.** *Let A be in CF with each block being a unique element of its BNT, i.e. there is no q summation in Eq.* (3.4)*. Then for L large enough, $\mathcal{S}_L$ is the entire matrix algebra $\mathcal{M} := \oplus_{j=1}^m \mathcal{M}_{D_j\times D_j}$ where $\mathcal{M}_{D_j\times D_j}$ is the algebra of $D_j \times D_j$ matrices and D_j is the bond dimension of A_j.* [45]

Proposition 3.1.5. *Let A be a tensor consisting of block diagonal matrices: $A^i \in \mathcal{M} := \oplus_j^m \mathcal{M}_{D_j\times D_j}$, and let $\mathcal{S}_L$ and Γ_A^L be as in Definition 3.1.8 and Definition 3.1.9 respectively. Then $\mathcal{S}_L = \mathcal{M}$ iff $\Gamma_A^L|_{\mathcal{M}}$ is injective.*

Proof. Assume injectivity of $\Gamma_A^L|_{\mathcal{M}}$, then any element $X \in \mathcal{S}^\perp \cap \mathcal{M}$ satisfies $\Gamma_A^L(X^\dagger) = 0$ because the coefficients of the the vector $\Gamma_A^L(X^\dagger)$ are inner products of X with elements in $\mathcal{S}$. This implies $X = 0$. If $\mathcal{S} = \mathcal{M}$, then for every nonzero $X \in \mathcal{M}$, $X^\dagger$ has a non vanishing inner product with at least one element $A^{i_1}A^{i_2}\dots A^{i_L}$, and therefore $\Gamma_A^L(X)$ is non zero. □

Proposition 3.1.6. *For a tensor A in CF as in Eq.* (3.4)*, for L large enough the space $\mathcal{S}_L$ (Definition 3.1.8) has the form:*

$$\mathcal{S}_L = \left\{\oplus_{j=1}^m \oplus_{q=1}^{r_j} \mu_{j,q}^L V_{j,q}^{-1} M_j V_{j,q} \mid M_j \in \mathcal{M}_{D_j\times D_j}\right\} \tag{3.5}$$

Proof. Consider a tensor $\tilde{A}$ which consists of the BNT of A without multiplicities (as in Proposition 3.1.4). An element $S = \oplus_{j=1}^m \oplus_{q=1}^{r_j} \mu_{j,q}^L V_{j,q}^{-1} M_j V_{j,q}$ in $\mathcal{S}_L$ is obtained by taking the same linear combination of the matrix products $A^{i_1}A^{i_2}\dots A^{i_L}$ as the one which generates $\tilde{S} = \oplus_{j=1}^m M_j$ from the matrix products $\tilde{A}^{i_1}\tilde{A}^{i_2}\dots\tilde{A}^{i_L}$. □

Proposition 3.1.7. *Let* $\{A_j\}_{j=1}^m$ *be a BNT of* A*, and let each* A_j *appear in* A *with no multiplicities, i.e.* $A^i = \oplus_{j=1}^m \nu_j A_j^i$*. For* L *large enough the image of the algebra of block diagonal matrices* $\mathcal{M} := \oplus_{j=1}^m \mathcal{M}_{D_j \times D_j}$*, where* D_j *is the bond dimension of* A_j*, under the map* Γ_A^L *is a direct sum:*

$$\Gamma_A^L(\mathcal{M}) := \left\{\Gamma_A^L(X) \mid X \in \mathcal{M}\right\} = \bigoplus_{j=1}^m \Gamma_{A_j}^L(\mathcal{M}_{D_j \times D_j}) \ .$$

In particular $\sum_{j=1}^m \Gamma_{A_j}^L(X_j) = 0$ *implies* $X_j = 0 \ \ \forall j = 1, \ldots, m$. [45]

Proposition 3.1.7 allows us to prove the following lemma:

Lemma 3.1.1. *Let* A *be a tensor in CF with BNT* $\{A_j\}$*, and let* S *and* T *be tensors with the exact same block structure as* A*:*

$$A^i = \oplus_{j=1}^m \oplus_{q=1}^{r_j} \mu_{jq} V_{j,q}^{-1} A_j^i V_{j,q}$$

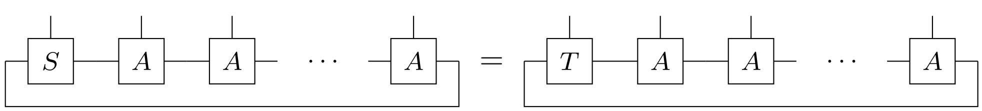
$$S^i = \oplus_{j=1}^m \oplus_{q=1}^{r_j} \mu_{jq} V_{j,q}^{-1} S_j^i V_{j,q}$$

$$T^i = \oplus_{j=1}^m \oplus_{q=1}^{r_j} \mu_{jq} V_{j,q}^{-1} T_j^i V_{j,q} \ .$$

If the following equality holds for any length N*:*

$$\sum_{\{i\}} \mathrm{Tr}\left(S^{i_1} A^{i_2} \ldots A^{i_N}\right) |i_1, i_2, \ldots, i_N\rangle = \sum_{\{i\}} \mathrm{Tr}\left(T^{i_1} A^{i_2} \ldots A^{i_N}\right) |i_1, i_2, \ldots, i_N\rangle \ , \tag{3.6}$$

which in tensor notation reads:



,

then $S = T$.

Proof. Plugging in the block structure of the tensors into Eq. (3.6) we obtain:

$$\begin{aligned} 0 &= \sum_{\{i\}} \mathrm{Tr}\left(\oplus_{j=1}^m \oplus_{q=1}^{r_j} \mu_{j,q}^N \left[T_j^{i_1} - S_j^{i_1}\right] A_j^{i_2} \ldots A_j^{i_N}\right) |i_1, i_2, \ldots, i_N\rangle \\ &= \sum_{j=1}^m \sum_{q=1}^{r_j} \mu_{j,q}^N \sum_{\{i\}} \mathrm{Tr}\left(\left[T_j^{i_1} - S_j^{i_1}\right] A_j^{i_2} \ldots A_j^{i_N}\right) |i_1, i_2, \ldots, i_N\rangle \ . \end{aligned}$$

Plugging in the definition of the map Γ_A (Definition 3.1.9)

$$\sum_{j=1}^{m}\sum_{i_1}\Gamma_{A_j}^{N-1}\left(\sum_{q=1}^{r_j}\mu_{j,q}^{N}\left[T_j^{i_1}-S_j^{i_1}\right]\right)\otimes|i_1\rangle=0\ .$$

According to Proposition 3.1.7, for N large enough ($\geq L_0$) we have for all i_1 and all j

$$\sum_{q=1}^{r_j}\mu_{j,q}^{N}\left[T_j^{i_1}-S_j^{i_1}\right]=0\ .$$

For all j, since $\{\mu_{j,q}\}_{q=1}^{r_j}$ are nonzero, there exists an $N \geq L_0$ such that $\sum_{q=1}^{r_j}\mu_{j,q}^{N}\neq 0$. Therefore for all j we have:

$$T_j^i = S_j^i\ . \tag{3.7}$$

□

We review the fundamental theorem of MPV [46].

Proposition 3.1.8. [46] *Let A and B be tensors in CF (Eq. (3.4)) with BNT $\{A_j\}_{j=1}^{g_a}$ and $\{B_k\}_{k=1}^{g_b}$ respectively. If for all N the tensors A and B generate MPVs proportional to each other, then $g_a = g_b$ and for every j there is a unique $k(j)$, a unitary matrix X_j and a phase $e^{i\phi_j}$ such that:*

$$A_j^i = e^{i\phi_j}X_j^{-1}B_{k(j)}^iX_j\ .$$

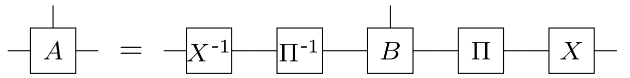
Remark 3.1.2. Note that X_j are determined up to a phase.

Proposition 3.1.8 was proved in [46] and was used to prove the following:

Theorem 3.1.1 (The Fundamental Theorem of MPV)**.** *Let two tensors A and B in CF (CFII) generate the same MPV for all N. Then they have the same block structure, and there exists an invertible (unitary) matrix X:*

$$X = \oplus_{j=1}^{m}\oplus_{q=1}^{r_j}X_j\ , \tag{3.8}$$

which is block diagonal, with the same block structure as A, and a permutation matrix Π *between the blocks, such that:*



Before we proceed to apply the this result to MPV with a global symmetry, we shall pause in order to review basic notions from representation theory.

3.2 Representation theory

In this section we introduce projective representations. We review basic facts from representation theory, stated in the more general setting of projective representation, following [41, 50]. Next, we describe how the general setting of a MPV with a symmetry with respect to a finite dimensional representation $\Theta(g)$, can be simplified by writing the MPV in a form compatible with the decomposition of $\Theta(g)$ into irreducible representations. Finally, we quote two theorems: Schur's lemma and the Wigner-Eckart theorem, that will allow us to classify the tensors generating symmetric MPVs.

Projective representations

Let $\mathcal{H}$ be a finite dimensional Hilbert space. Denote by $\mathsf{U}(\mathcal{H})$ the group of unitary operators on $\mathcal{H}$. Throughout the thesis, unless explicitly stated otherwise, G will always refer to a finite group or a compact Lie group.

Definition 3.2.1. A function $\gamma : G \times G \to U(1)$ satisfying:

$$\gamma(g,h)\gamma(gh,f) = \gamma(g,hf)\gamma(h,f), \;\; \forall g,h,f \in G$$

$$\gamma(g,e) = \gamma(e,g) = 1, \;\; \forall g \in G \;,$$

where $e \in G$ is the trivial element, is called a multiplier of G. For compact Lie groups we require γ to be continuous.

Definition 3.2.2. A projective unitary representation of a group G on $\mathcal{H}$ is a map $\Theta : G \to \mathsf{U}(\mathcal{H})$ such that for all $g, h \in G$ $\Theta(g)\Theta(h) = \gamma(g,h)\Theta(gh)$, where γ is a multiplier of G.

That is, projective unitary representations are unitary representations up to a phase factor. Throughout this thesis all representations will be assumed to be unitary and finite dimensional. From this point on, *unitary representation* shall be used to emphasize that it is not projective. *Projective representations* can refer to both, as unitary representations are a particular case of projective representations, namely, they are the ones with the trivial multiplier.

Two projective representations $(\Theta, \mathcal{H})$ and $(\Theta', \mathcal{H}')$ with multipliers γ and γ' are *equivalent in the sense of projective representations* if there exist an isomorphism $\phi : \mathcal{H} \to \mathcal{H}'$ and a function $\mu : G \to U(1)$ such that $\Theta'(g)\phi = \mu(g)\phi\Theta(g)$ for all $g \in G$. Their multipliers then satisfy:

$$\gamma'(g,h) = \gamma(g,h)\mu(g)\mu(h)\mu(gh)^{-1} \ . \tag{3.9}$$

Equation (3.9) defines an equivalence relation on the group of multipliers of G. The quotient of the subgroup of multipliers of the form $\gamma(g,h) = \mu(g)\mu(h)\mu(gh)^{-1}$ in the group of all multipliers is the second cohomology group $H^2(G, U(1))$ of G over $U(1)$ [50]. When two projective representations Θ and Θ' have multipliers related by Eq. (3.9), for some function $\mu : G \to U(1)$ we say they are in the same cohomology class.

Definition 3.2.3. Two projective representations $(\Theta, \mathcal{H})$ and $(\Theta', \mathcal{H}')$ with the same multiplier γ are *equivalent* if there exists an isomorphism $\phi : \mathcal{H} \to \mathcal{H}'$ such that $\Theta'(g)\phi = \phi\Theta(g)$ for all $g \in G$. We denote $\Theta'(g) \cong \Theta(g)$.

Complete reducibility

Fix a choice of representatives from the equivalence classes (Definition 3.2.3) of irreducible projective representations of G with multiplier γ; denote them by $D^j_\gamma : G \to \mathsf{U}(\mathcal{H}_j)$. Fixing a basis $\{|i\rangle\}$ in $\mathcal{H}_j$ for every j defines the irreducible projective representation matrices: $D^j_\gamma(g) = \sum_{m,n} D^j_\gamma(g)_{m,n}|m\rangle\langle n|$. These generalize the $SU(2)$ Wigner matrices to projective representations of arbitrary groups.

Let $\mathcal{H}$ be a finite dimensional Hilbert space, and let $\Theta : g \mapsto \Theta(g)$ be a projective representation of G with multiplier γ. For finite and compact groups any finite dimensional projective representation is fully reducible and is equivalent to a direct sum of irreducible projective representations $\oplus_j D^j_\gamma(g)$ with the same multiplier, i.e., there exists a basis $\{|j,m\rangle\}$ of $\mathcal{H}$ such that:

$$\Theta(g)|j,m\rangle = \sum_n D^j_\gamma(g)_{n,m}|j,n\rangle \ . \tag{3.10}$$

We refer to such a basis as the irreducible representation basis of $\Theta(g)$ (in general it is not unique, e.g., when an irreducible representation appears multiple times [51]; we shall assume a choice of such a basis).

When considering a representation acting on a MPV, it is convenient to write the MPV in the irreducible representation basis. In the following we describe how this is achieved, and show that it does not interfere with CF properties of the tensor generating the MPV.

Remark 3.2.1. A change of basis of the physical space from $\{|i\rangle\}$ to the irreducible representation basis $\{|j,m\rangle\}$ (Eq. (3.10)), involves a transformation of the tensor generating the MPV: $A \mapsto \tilde{A}$, where $\tilde{A}$ consists of the matrices $\{\tilde{A}^{j,m} = \sum_i \langle j,m \mid i\rangle\, A^i\}$. This is easily seen by inserting an identity operator $\sum_{j,m} |j,m\rangle\langle j,m|$ for every copy of $\mathcal{H}$ in the definition of $|\psi^N_A\rangle$ (Eq. (3.1)).

Proposition 3.2.1. *Let $\{A^i\}_{i=1}^d$ be the Kraus operators defining a CP map E_A. For any unitary $d \times d$ matrix U the matrices $\{\sum_j U_{i,j} A^j\}_{i=1}^d$ define the same CP map.* [48]

Corollary 3.2.1. *Let A be a tensor in CF (CFII) composed of the matrices $\{A^i\}$ corresponding to the basis $\{|i\rangle\}$ of $\mathcal{H}$. Then the tensor $\tilde{A}$, composed of the matrices $\{\tilde{A}^{j,m} = \sum_i A^i \langle j,m \mid i\rangle\}$ as in Remark 3.2.1, is also in CF (CFII).*

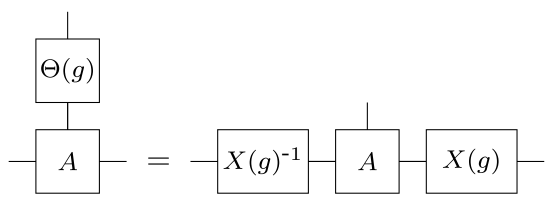
Proof. $\tilde{A}$ has the same block structure as A (Eq. (3.3)):

$$\tilde{A}^{j,m} = \oplus_{k=1}^{n} \nu_k \tilde{A}_k^{j,m} = \oplus_{k=1}^{n} \nu_k \sum_i \langle j, m \mid i \rangle \, A_k^i \ .$$

According to Proposition 3.1.1, the normality and CFII properties of each block $\tilde{A}_k$ are defined by the CP map associated to it. Proposition 3.2.1 says this maps is not affected by the transformation $A_k \mapsto \tilde{A}_k$ because $\{\langle j, m \mid i \rangle\}$ are the entries of a unitary matrix. Each block $\tilde{A}_k$ is therefore a normal tensor (and in CFII). □

Intertwining relations

It was shown in [37, 38] that an injective tensor A which generates a MPV with a global symmetry with respect to a representation Θ_g, satisfies:



, (3.11)

i.e., for all $i = 1, \ldots, d$: $\sum_{i'} \Theta(g)_{ii'} A^{i'} = X(g)^{-1} A^i X(g)$, where $X(g)$ is a projective representation of G. While we will make the precise statement and derive this result later, we now point out that in Eq. (3.11) the tensor A translates the action of $\Theta(g)$ on the physical space into a group action on the virtual space.

In the following, we quote two theorems: Schur's lemma and the Wigner-Eckart theorem, which can be used to classify tensors satisfying such intertwining relations.

Definition 3.2.4 (Intertwining map)**.** Let (η, V) and (π, W) be projective representations of a group G with the same multiplier. A linear map $T : V \to W$ is called an intertwining map if $\pi(g)T = T\eta(g), \ \ \forall g \in G$.

Lemma 3.2.1 (Schur's lemma)**.** *An intertwining map between irreducible projective representations with the same multiplier is zero if they are inequivalent, and proportional to the identity if they are equal.* [50]

The tensor product of two irreducible projective representations with multipliers γ and γ' is a projective representation with multiplier $\gamma\gamma'$ ($\gamma\gamma' : (g,h) \mapsto \gamma(g,h)\gamma'(g,h)$), and is generally a reducible one. The unitary map that realizes the decomposition of $D^j_\gamma(g) \otimes D^l_{\gamma'}(g)$ into a direct sum of irreducible representations $\oplus_{J\in\mathfrak{J}} D^J_{\gamma\gamma'}$ is the Clebsch-Gordan map whose matrix elements are the Clebsch-Gordan coefficients $\langle j, m; l, n \mid J, M\rangle$, which are determined by the choice of the representation matrices D^j_γ (for a discussion of their uniqueness having fixed the representation matrices see [51]).

The following is a generalization of the $SO(3)$ vector operators, well known in quantum mechanics [41].

Definition 3.2.5 (Vector operator)**.** Let (η, V), (π, W) and $(\kappa, \mathcal{H})$ be projective representations of G with $dim(\mathcal{H}) = d$. A vector operator with respect to (κ, π, η) is a d-tuple of linear operators $\vec{A} = (A^1, A^2, \ldots, A^d)$, $A^i : V \mapsto W$ which, for all $g \in G$ and all $\vec{v} \in \mathcal{H}$, satisfies:

$$(\kappa(g)\vec{v}) \cdot \vec{A} = \pi(g) \left(\vec{v} \cdot \vec{A}\right) \eta(g)^{-1} \tag{3.12}$$

where $\vec{v} \cdot \vec{A} := \sum_i v^i A^i$.

It was shown in [37] that Eq. (3.11) can be used to determine the tensor A satisfying it, and that it consists of Clebsch-Gordan coefficients. We will derive the same result using a generalized version of the well known Wigner-Eckart theorem, using the fact that Eq. (3.11) resembles a vector operator relation for A (Definition 3.2.5).

Theorem 3.2.1 (Wigner-Eckart)**.** *Let $D^{J_0}_\gamma(g)$, $D^j_{\gamma'}(g)$ and $D^l_{\gamma''}(g)$ be irreducible projective representations. Let $\vec{A}$ be a vector operator with respect to ($\kappa := D^{J_0}_\gamma$, $\pi := D^j_{\gamma'}$, $\eta := D^l_{\gamma''}$).*

If $\gamma\gamma'' \neq \gamma'$, then $A = 0$. Otherwise (if $\gamma\gamma'' = \gamma'$), then $\{A^M | M = 1, \ldots, dim(J_0)\}$ are of the form:

$$A^M = \sum_{J\in\mathfrak{J}:D^J=D^{J_0}} \alpha_J \sum_{m,n} \left\langle j, m; \bar{l}, n \mid J, M \right\rangle |m\rangle\langle n| \ , \tag{3.13}$$

where $\mathfrak{J}$ is the set of irreducible projective representation indices appearing in the decomposition of $D^j_{\gamma'}(g) \otimes \overline{D^l_{\gamma''}(g)}$, $\left\langle j, m; \bar{l}, n \mid J, M \right\rangle$ are the Clebsch-Gordan coefficients of this decomposition, $\overline{D^l_{\gamma''}(g)}$ is the complex conjugate representation to $D^l_{\gamma''}(g)$, $\{|m\rangle\}$ and $\{|n\rangle\}$ are the irreducible representation bases: $\pi(g)|m\rangle = \sum_{m'} D^j_{\gamma'}(g)_{m',m}|m'\rangle$, $\eta(g)|n\rangle = \sum_{n'} D^l_{\gamma''}(g)_{n',n}|n'\rangle$ and α_J are arbitrary constants.

For a proof of the theorem in the familiar $SO(3)$ setting, we refer the reader to [41]; for a proof in the the setting of projective representations see [39].

Remark 3.2.2. Apart from the freedom of choosing the constants $\{\alpha_J\}$ in Eq. (3.13), there is an additional freedom which comes from the fact that the the Clebsch-Gordan coefficients are not uniquely determined by the irreducible representation matrices [51].

Remark 3.2.3. The multiplier of the complex conjugate projective representation $\overline{D^l_\gamma(g)}$ is γ^{-1}. We will always use Theorem 3.2.1 with $\gamma \equiv 1$, then $A = 0$ unless $\gamma' = \gamma''$.

Remark 3.2.4. We assume a choice of a unique representative in each equivalence class of irreducible projective representations of G, so any two are either inequivalent or are represented by the same matrices.

Remark 3.2.5. A is zero if $D^{J_0}_\gamma(g)$ does not appear in the decomposition of $D^j_{\gamma'}(g) \otimes \overline{D^l_{\gamma''}(g)}$. There is a J summation in Eq. (3.13) because in general the same irreducible representation could appear multiple times in the decomposition of the tensor product of two irreducible representations.

3.3 Classification of MPV with a global symmetry

We review the derivation of of the classification of MPVs with a global symmetry, originally shown in [37]. In order for this work to be self contained, we derive the result from the fundamental theorem of MPV (Theorem 3.1.1), following [46] and references therein.

Let $\mathcal{H}_A$ be a d_A dimensional Hilbert space corresponding to a single degree of freedom ("spin"). Consider N such "spins" positioned on a one dimensional lattice, with periodic boundary conditions. A tensor A consisting of square matrices $\{A^i\}_{i=1}^{d_A}$ generates a TI-MPV that describes a state of what, in the next chapter, we will refer to as the chain of matter "spins". Let Θ be a unitary representation of G on $\mathcal{H}_A$, $\Theta : g \mapsto \Theta(g)$.

Definition I (Global Symmetry for matter MPV)**.** A MPV $|\psi_A^N\rangle$ has a global symmetry with respect to $\Theta(g)$ if for all $N \in \mathbb{N}$:

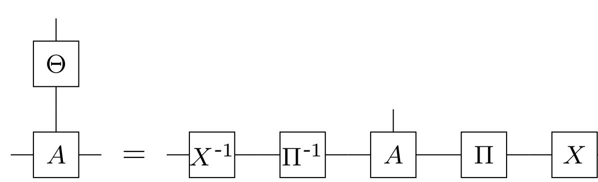
$$\Theta_g \otimes \Theta_g \otimes \ldots \otimes \Theta_g |\psi_A^N\rangle = |\psi_A^N\rangle, \ \ \forall g \in G \ .$$

We first apply the fundamental theorem of MPV to the case when a MPV generated by a tensor A in CFII is invariant under the action of the same unitary operator on every site (later we shall consider the case when it is invariant with respect to a unitary representation, i.e. when it has a global symmetry as in Definition I):

Corollary 3.3.1. *Let A be a tensor in CFII (Eq.* (3.4)*) generating a MPV with a global invariance under a unitary Θ:*

$$\Theta^{\otimes N} |\psi_A^N\rangle = |\psi_A^N\rangle \ ,$$

then A transforms under the unitary matrix as:



,

where X is a unitary matrix with the same block structure as A, and is unitary in each block (Eq. (3.8)), and Π is a permutation between the j blocks of A (it does not permute the q blocks).

Proof. The tensor $\tilde{A}$ consisting of the matrices $\tilde{A}^i := \sum_{i'} \Theta_{i,i'} A^{i'}$ generates $\Theta^{\otimes N}|\psi_A^N\rangle$. Before finishing the proof, we shall now prove the following lemma:

Lemma 3.3.1. *Let $\{A_j\}$ be the BNT of A, then the tensors $\{\tilde{A}_j\}$ composed of the matrices $\tilde{A}_j^i = \sum_{i'} \Theta_{i,i'} A_j^{i'}$ form a BNT of $\tilde{A}$, and $\tilde{A}$ is in CFII.*

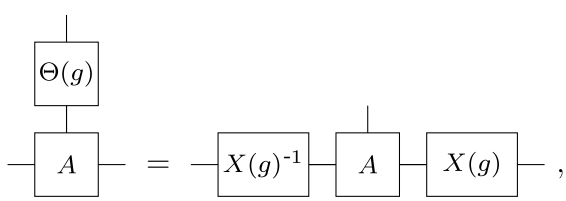
Proof: Lemma 3.3.1. $\tilde{A}_j$ are normal tensors and in CFII because a unitary mixture of the Kraus operators gives the same CP map (Proposition 3.2.1), and they are a basis because $\{A_j\}$ is. □

We can now apply the fundamental theorem of MPV to A and $\tilde{A}$. In this case, however, because the coefficients $\mu_{j,q}$ in Eq. (3.4) are the same for A and $\tilde{A}$, Π permutes only between j blocks. □

Next we apply the above to a MPV with a global symmetry as in Definition I:

$$\Theta(g)^{\otimes N}|\psi_A^N\rangle = |\psi_A^N\rangle \ .$$

Theorem I. *A tensor A in CFII which generates a MPV with a global symmetry with respect to a representation $\Theta(g)$ of a connected Lie group G (Definition I), transforms under the representation matrix as:*



(3.14)

where $X(g)$ has the same block structure as A:

$$X(g) = \oplus_{j=1}^{m} \oplus_{q=1}^{r_j} X_j(g) \ , \tag{3.15}$$

and where each block $X_j(g)$ is a projective representation, in the general case, for different j values $X_j(g)$ belong to different cohomology classes.

Proof. According to Corollary 3.3.1, for every $g \in G$ we have:

$$\sum_{i'} \Theta(g)_{i,i'} A^{i'} = X(g)^{-1}\Pi(g)^{-1}A^{i}\Pi(g)X(g) \ . \tag{3.16}$$

Consider the action of the group element $gh \in G$ in two ways using Eq. (3.16):

$$\begin{aligned} X(gh)^{-1}\Pi(gh)^{-1}A^{i}\Pi(gh)X(gh) =& \sum_{i'} \Theta(gh)_{i,i'} A^{i'} \\ =& \sum_{i',k} \Theta(g)_{i,k}\Theta(h)_{k,i'} A^{i'} \\ =& \sum_{k} \Theta(g)_{i,k} X(h)^{-1}\Pi(h)^{-1}A^{k}\Pi(h)X(h) \\ =& X(h)^{-1}\Pi(h)^{-1}X(g)^{-1}\Pi(g)^{-1}A^{i}\Pi(g)X(g)\Pi(h)X(h) \ . \end{aligned}$$

Taking the L-fold product of the LHS and RHS for different indices $i_1, i_2, \ldots, i_L$ we obtain:

$$\begin{aligned} &X(gh)^{-1}\Pi(gh)^{-1}\left(A^{i_1}A^{i_2}\ldots A^{i_L}\right)\Pi(gh)X(gh) = \\ &X(h)^{-1}\Pi(h)^{-1}X(g)^{-1}\Pi(g)^{-1}\left(A^{i_1}A^{i_2}\ldots A^{i_L}\right)\Pi(g)X(g)\Pi(h)X(h) \ . \end{aligned} \tag{3.17}$$

We shall now prove the following lemma, and then continue with the proof.

Lemma 3.3.2. $\Pi(g)$ *is a representation of G and is therefore the trivial one.*

Proof: Lemma 3.3.2. According to Proposition 3.1.6, by taking appropriate linear combinations of Eq. (3.17) we can obtain:

$$X(gh)^{-1}\Pi(gh)^{-1}\left(\Delta[j]\right)\Pi(gh)X(gh) = X(h)^{-1}\Pi(h)^{-1}X(g)^{-1}\Pi(g)^{-1}\left(\Delta[j]\right)\Pi(g)X(g)\Pi(h)X(h) \ , \tag{3.18}$$

where $\Delta[j_0]$ is a matrix consisting of multiples of $\mathbb{I}$ in the j_0 block and zero in all the rest: $\Delta[j_0] := \oplus_{j=1}^{m} \oplus_{q=1}^{r_j} \mu_{j,q}^{L} \delta_{j,j_0} \mathbb{I}_{D_j \times D_j}$. This is achieved by setting $M_j = \delta_{j,j_0}\mathbb{I}$ in Eq. (3.5). Denote by $g(j)$ the image of the block j under the permutation $\Pi(g)$, then $\Pi(g)^{-1}\Delta[j]\Pi(g) = \Delta[g^{-1}(j)]$. Plugging this into Eq. (3.18) we get:

$$\begin{aligned} LHS =& X(gh)^{-1}\left(\Delta[(gh)^{-1}(j)]\right)X(gh) \\ =& \Delta[(gh)^{-1}(j)] = \\ RHS =& X(h)^{-1}\Pi(h)^{-1}X(g)^{-1}\left(\Delta[g^{-1}(j)]\right)X(g)\Pi(h)X(h) \\ =& X(h)^{-1}\Pi(h)^{-1}\left(\Delta[g^{-1}(j)]\right)\Pi(h)X(h) \\ =& \Delta[h^{-1}(g^{-1}(j))] \ , \end{aligned}$$

where in each step the Xs commute with the Δs because they have the same block structure and the Δs are proportional to $\mathbb{I}$ in each block. We conclude that $(gh)^{-1}(j)$ and $h^{-1}(g^{-1}(j))$ are the same block number and therefore $\Pi(g)$ is a group homomorphism. It remains to show that $\Pi(g)$ depends on g smoothly. From Eq. (3.16) we obtain:

$$X(g)^{-1}\Pi(g)^{-1}A^{i_1}A^{i_2}\dots A^{i_L}\Pi(g)X(g) = \sum_{\{i'\}}\left(\Theta(g)_{i_1,i'_1}A^{i'_1}\right)\left(\Theta(g)_{i_2,i'_2}A^{i'_2}\right)\dots\left(\Theta(g)_{i_L,i'_L}A^{i'_L}\right) \ . \tag{3.19}$$

As above, we can take a linear combination of the As to get a $\Delta[j]$ between the permutations in the LHS. Knowing how the permutation acts on each $\Delta[j]$ determines $\Pi(g)$ completely. The Xs on the LHS commute with all $\Delta[j]$ as before. The RHS will then be a linear combination of $\{\Theta(g)A\}$, and will thus depend on g smoothly. Since we assumed G is a connected Lie group we conclude that $\Pi(g) \equiv \mathbb{I}$. □

We now repeat the step leading to Eq. (3.18) but this time with an arbitrary matrix M in the j block: $\Delta_{j_0}^{M} := \oplus_{j=1}^{m} \oplus_{q=1}^{r_j} \delta_{j,j_0}\mu_{j,q}^{L} M$. Equation Eq. (3.18) becomes:

$$X(gh)^{-1}\left(\Delta_j^M\right)X(gh) = X(h)^{-1}X(g)^{-1}\left(\Delta_j^M\right)X(g)X(h) \ .$$

This means that for any j block we have:

$$\oplus_{q=1}^{r_j}\mu_{j,q}^{L}X_j(gh)^{-1}MX_j(gh) = \oplus_{q=1}^{r_j}\mu_{j,q}^{L}X_j(h)^{-1}X_j(g)^{-1}MX_j(g)X_j(h)\ .$$

We see that $X_j(g)X_j(h)(X_j)^{-1}(gh)$ commutes with every matrix M and is therefore proportional to the identity. $X_j(g)$ is therefore a projective representation.

□

Remark 3.3.1. Note that different blocks of $X(g)$ can belong to different equivalence classes of projective representations. We could construct such an example by taking the direct sum of two normal tensors A and $\tilde{A}$, which transform under a given representation $\Theta(g)$ with $X(g)$ and $\tilde{X}(g)$, projective representations from different cohomology classes. $X(g) \oplus \tilde{X}(g)$ is then not a projective representation because $X(gh) \oplus \tilde{X}(gh)$ differs from $X(g)X(h) \oplus \tilde{X}(g)\tilde{X}(h)$ by a diagonal matrix and not a scalar one.

Chapter 4

Classification of MPVs with a local symmetry

Gauge theories involve the dynamics of two kinds of degrees of freedom: *matter* and *gauge field*. Given those two ingredients, one can consider three types of states: states of only matter degrees of freedom, states of only gauge field degrees of freedom and states of both matter and gauge field. These correspond to non-interacting theories, pure gauge theories and interacting gauge theories respectively (where interactions are understood as those between matter and gauge degrees of freedom).

When constructing a gauge theory one usually starts from an interaction-free theory of the matter degree of freedom which is invariant with respect to a group of global transformations, i.e., the same group element acting in each point in space (or space-time). Adding an additional degree of freedom - the gauge field - with its own transformation law with respect to the group, allows to define local symmetry operators which act on both the matter and the gauge field degrees of freedom. These operators commute with the transformed (gauged) Hamiltonian, and the subspace of states which is invariant under all such operators is considered as the space of physical states. The generators of such local symmetry operators are the so-called Gauss law operators. They correspond to locally conserved quantities (charges), i.e., associated to each point in space (or space-time).

Conversely, one could start from a pure gauge field theory with a local symmetry and couple a matter degree of freedom to it, once again resulting in a system with local symmetry. Finally one could have matter and gauge field coupled in such a way that the combined state has a local symmetry but neither the mass state nor the gauge field state have a symmetry on their own.

4.1 Three Settings of Gauge Invariant MPVs

We shall now describe the three types of MPVs considered in this thesis, corresponding to the above mentioned types of states, and for each one of them define the symmetries which will be investigated in subsequent sections.

4.1.1 Matter MPV

The setting of the matter MPV was introduced in Section 3.3, when we discussed global symmetries. We repeat in here for convenience. Let $\mathcal{H}_A$ be a d_A dimensional Hilbert space corresponding to a single degree of freedom ("spin"). Consider N such "spins" positioned on a one dimensional lattice, with periodic boundary conditions. A tensor A consisting of square matrices $\{A^i\}_{i=1}^{d_A}$ generates a TI-MPV that describes a state of the chain of matter "spins". Let Θ be a unitary representation of G on $\mathcal{H}_A$, $\Theta : g \mapsto \Theta(g)$.

It is well known that in order to lift a global symmetry to be a local one, an additional degree of freedom must be introduced [1]. When investigating the possibility of a local symmetry for a matter MPV, we will find this statement reaffirmed (see Theorem II). We define the setting of the theorem in the following:

Definition II (Local Symmetry for matter MPV)**.** A MPV $|\psi_A^N\rangle$ has a local symmetry with respect to $\Theta(g)$ if for all $N \in \mathbb{N}$:

$$\Theta_{g_1} \otimes \Theta_{g_2} \otimes \ldots \otimes \Theta_{g_N} |\psi_A^N\rangle = |\psi_A^N\rangle, \;\; \forall g_1, g_2, \ldots, g_N \in G \; .$$

Remark 4.1.1*.* The condition of a local symmetry (Definition II) is equivalent to invariance under any single-site group action (all $g_i = e$ except one). For TI-MPV it is therefore sufficient to consider only $g_1 \neq e$.

4.1.2 Gauge field MPV

Next we shall consider a case in which the local transformations act on two neighboring sites of a TI-MPV, which will be eventually seen as the pure gauge case.

Let $\mathcal{H}_B$ be a d_B dimensional Hilbert space corresponding to a single "spin". Consider N such spins positioned on a one dimensional lattice, with periodic boundary conditions. A tensor B consisting of square matrices $\{B^i\}_{i=1}^{d_B}$ generates a TI-MPV that describes a state of the chain of gauge field "spins".

Definition III (Local Symmetry for gauge field MPV)**.** Let $\mathcal{R}, \mathcal{L}$ be two projective representations of G on $\mathcal{H}_B$, $\mathcal{R} : g \mapsto \mathcal{R}(g)$, $\mathcal{L} : g \mapsto \mathcal{L}(g)$ with multipliers γ and γ^{-1}, so that the tensor product $\mathcal{R}(g) \otimes \mathcal{L}(g)$ is a unitary representation. A MPV $|\psi_B^N\rangle$ has a local symmetry with respect to $\mathcal{R}(g) \otimes \mathcal{L}(g)$ if for all $N \in \mathbb{N}$ and for any two neighboring lattice sites K and $K+1$:

$$\mathcal{R}_g^{[K]} \otimes \mathcal{L}_g^{[K+1]} |\psi_B^N\rangle = |\psi_B^N\rangle, \;\; \forall g \in G \; .$$

4.1.3 Matter and gauge field MPV

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be as in Section 4.1.1 and Section 4.1.2 respectively. Consider a lattice of length $2N$ with matter and gauge field spins alternating among sites. Tensors A and B, consisting of $D_1 \times D_2$ matrices $\{A^i\}_{i=1}^{d_A}$ and $D_2 \times D_1$ matrices $\{B^j\}_{j=1}^{d_B}$ respectively, generate a TI-MPV (in the sense of translating two sites) that describes a state of the chain of matter and gauge field "spins". The MPV, generated by a tensor we denote AB, takes the form:

$$|\psi_{AB}^N\rangle = \sum_{\{i\},\{j\}} \mathrm{Tr}\left(A^{i_1}B^{j_1}A^{i_2}B^{j_2}\dots A^{i_N}B^{j_N}\right)|i_1 j_1 i_2 j_2 \dots i_N j_N\rangle \ .$$

In lattice gauge theories, the matter degrees of freedom are located on the sites of a lattice whereas the gauge field degrees of freedom - on the links connecting adjacent sites [4]. In the one dimensional case, our setting differs from this structure only in notation, e.g., we could have chosen to call the even numbered sites "links".

Let $\Theta(g)$ and $\mathcal{R}(g)$, $\mathcal{L}(g)$ be as in Section 4.1.1 and Section 4.1.2 respectively.

Definition IV (Local Symmetry for both matter and gauge field MPV)**.** A MPV $|\psi_{AB}^N\rangle$ has a local symmetry with respect to $\mathcal{R}(g) \otimes \Theta(g) \otimes \mathcal{L}(g)$ if for all $N \in \mathbb{N}$ and for any three neighboring lattice sites numbered $2K, 2K+1$ and $2K+2$ (corresponding to $\mathcal{H}_B \otimes \mathcal{H}_A \otimes \mathcal{H}_B$):

$$\mathcal{R}(g)^{[2K]} \otimes \Theta(g)^{[2K+1]} \otimes \mathcal{L}(g)^{[2K+2]}|\psi_{AB}^N\rangle = |\psi_{AB}^N\rangle, \ \ \forall g \in G \ .$$

4.2 Generators and Gauss' law

In the previous section we defined the symmetries in terms of representations of a group G. For matrix Lie groups it is often the case that one could describe the same symmetry in terms of representations of the Lie algebra $\mathfrak{g}$ of G. While the two descriptions are mathematically equivalent, it is precisely the elements of the Lie algebra representation that correspond to observables in physical theories. Such observables are conserved by the dynamics in a theory which respects the symmetry, and are therefore of great importance.

To each scenario described above (Section 4.1.1, Section 4.1.2 and Section 4.1.3) correspond different such observables, and physical theories corresponding to the different settings - matter, gauge field or matter and gauge field - observe different conservation laws. In the following we describe the relation of those settings to physical lattice gauge theories [40].

When G is a compact and connected Lie group, e.g. $U(1)$ or $SU(N)$, the exponential map $\exp : \mathfrak{g} \to G$ is surjective. Thus every group element can be written as an exponential of an element in the Lie algebra $\mathfrak{g}$ [52]. Let $\mathcal{R}(g)$, $\mathcal{L}(g)$ and $\Theta(g)$ be representations on $\mathcal{H}_B$ and $\mathcal{H}_A$ respectively (for $SU(N)$ we can always choose $\mathcal{R}(g)$ and $\mathcal{L}(g)$ to be unitary representations keeping $\mathcal{R}(g) \otimes \mathcal{L}(g)$ unchanged [41]), and let $|\psi^N_{AB}\rangle$ be as defined in Section 4.1.3. We can express the physical representations as exponentials of generators:

$$\begin{aligned}
\Theta(g) =& \exp\left(i \sum_a Q_a \varphi_a(g)\right) \\
\mathcal{R}(g) =& \exp\left(i \sum_a R_a \varphi_a(g)\right) \\
\mathcal{L}(g) =& \exp\left(i \sum_a L_a \varphi_a(g)\right) \quad ,
\end{aligned}$$

where $\{\varphi_a(g)\}_{a=1}^{dim(\mathfrak{g})}$ are real parameters and $\{R_a\}_{a=1}^{dim(\mathfrak{g})}$, $\{L_a\}_{a=1}^{dim(\mathfrak{g})}$ and $\{Q_a\}_{a=1}^{dim(\mathfrak{g})}$ are Hermitian operators on $\mathcal{H}_B$ and $\mathcal{H}_A$ respectively such that $\{iR_a\}, \{iL_a\}$ and $\{iQ_a\}$ are bases of the respective Lie algebras. In the Hamiltonian formulation of lattice gauge theories [6, 40] $\{R_a\}$ and $\{L_a\}$ satisfy the Lie algebra relations:

$$\begin{aligned}
[R_a, R_b] =& i f_{abc} R_c \\
[L_a, L_b] =& i f_{abc} R_c \\
[R_a, L_b] =& 0 \ ,
\end{aligned}$$

where f_{abc} are the structure constants of the Lie algebra $\mathfrak{g}$. $\{Q_a\}$ satisfy the relations:

$$[Q_a, Q_b] = i f_{abc} Q_c \ .$$

The local symmetry transformations appearing in the matter and gauge field MPV scenario (Definition IV):

$$\mathcal{R}^{[2K]}(g) \otimes \Theta^{[2K+1]}(g) \otimes \mathcal{L}^{[2K+2]}(g)|\psi_{AB}^N\rangle = |\psi_{AB}^N\rangle \ , \tag{4.1}$$

are generated by the operators:

$$G_a^{[2K+1]} := \left(R_a^{[2K]} + Q_a^{[2K+1]} + L_a^{[2K+2]}\right) \ .$$

Differentiating Eq. (4.1) with respect to any of the parameters φ_a we obtain:

$$\left(R_a^{[2K]} + Q_a^{[2K+1]} + L_a^{[2K+2]}\right)|\psi_{AB}^N\rangle = G_a^{[2K+1]}|\psi_{AB}^N\rangle = 0 \ . \tag{4.2}$$

This is the lattice version of Gauss' law. In physical theories, states $|\psi_A\rangle$ have a global symmetry generated by $\{Q_a\}$ - the $SU(N)$ charge operators. In the $U(1)$ case there is one generator Q - the electric charge operator; furthermore, for Abelian groups $L = -R$. In that case Eq. (4.2) says that at each lattice site corresponding to matter, the charge is equal to the difference between the values of L on the right and on the left of it (the 1D lattice divergence of L). This becomes Gauss' law when taking a continuum limit. L is therefore identified as the electric field. Analogously, in the $SU(N)$ case $\{R_a\}$ and $\{L_a\}$ are identified with right and left electric fields respectively [40].

The same kind of equation can be obtained for the case of a gauge field MPV with a local symmetry (Definition III):

$$\left(R_a^{[K]} + L_a^{[K+1]}\right)|\psi_B^N\rangle = 0 \ .$$

In the case of a global symmetry for a matter MPV, differentiating the symmetry relation (Definition I), we obtain a global operator - the total charge:

$$\sum_K Q_a^{[K]}|\psi_A^N\rangle = 0 \ .$$

4.3 Results

We summarize the results presented in this thesis, first stating the main results of each of the cases presented above, and then turning to a more detailed and formal description. The detailed proofs will be given in the subsequent chapter. For each one of the settings introduced in the previous section, we shall first show that the symmetry condition implies a transformation relation satisfied by the tensor(s) generating the MPV. Second, we shall show that those transformation relations determine the structure of the tensor(s). For each setting we shall then discuss implications of the derived tensor structures.

4.3.1 Matter MPV with local symmetry

We show that a MPV with one degree of freedom - the mass "spins" - can have a local symmetry as in Definition II, only if it is the trivial one. This is consistent with the way gauge invariant states are usually constructed in lattice gauge theories, as well as with the construction of continuum gauge theories, where an additional degree of freedom is introduced. The first observation is a general one, not restricted to MPVs:

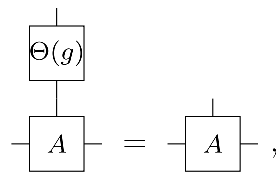
Proposition I. *Let $\mathcal{H}$ be a finite dimensional Hilbert space and let $\Theta : g \mapsto \Theta(g)$ be a representation on $\mathcal{H}$. Let $|\psi^N\rangle \in \mathcal{H}^{\otimes N}$ be a vector with a local symmetry, i.e.*

$$\Theta(g_1) \otimes \Theta(g_2) \otimes \dots \Theta(g_N)|\psi^N\rangle = |\psi^N\rangle, \quad \forall g_1, g_2, \dots, g_N \in G \ .$$

Then $|\psi^N\rangle \in {\mathcal{H}_0}^{\otimes N}$, where $\mathcal{H}_0 \subset \mathcal{H}$ is the subspace on which $\Theta(g)$ acts trivially.

In the following we show that for MPVs a similar statement to Proposition I can be made for the tensor generating the MPV. Let $|\psi_A\rangle$ and $\Theta(g)$ be as in Section 4.1.1. According to Proposition 3.1.2, given an arbitrary tensor A generating $|\psi_A\rangle$, one can obtain a tensor in CF which generates the same state, (possibly after blocking A). We therefore assume A to be in CF.

Theorem II. *Let A be a tensor in CF generating a MPV with a local symmetry with respect to a representation $\Theta(g)$ (Definition II). Then for all $g \in G$ the tensor A satisfies:*



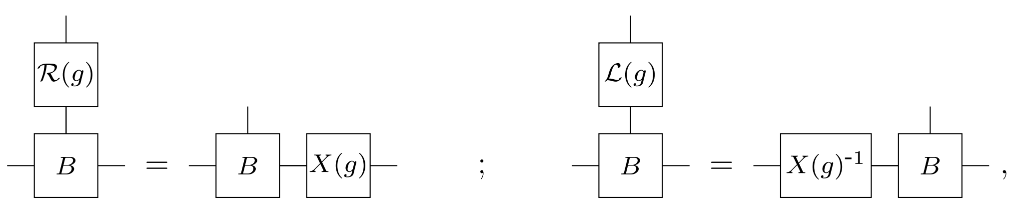
i.e., for all $i = 1, \ldots, d_A$: $\sum_{i'} \Theta(g)_{ii'} A^{i'} = A^i$.

According to Remark 3.2.1, the MPV generated by A can be written in terms of a tensor $\tilde{A}$, composed of the matrices $\{\tilde{A}^{j,m}\}$, corresponding to the irreducible representation basis $\{|j,m\rangle\}$ on which $\Theta(g)$ acts as $\Theta(g)|j,m\rangle = \sum_n D^j(g)_{n,m}|j,n\rangle$. According to Corollary 3.2.1, $\tilde{A}$ is also in CF. Applying Theorem II to $\tilde{A}$ leads to the following:

Corollary I. *The matrices $\tilde{A}^{j,m}$ are non-zero only for j such that $D^j(g) \equiv \mathbb{I}_{1\times 1}$.*

4.3.2 Gauge field MPV

We show that a local symmetry for a gauge field MPV $|\psi_B^N\rangle$ generated by a tensor B (in CFII) (as defined in Section 4.1.2), implies the following transformation relations for B:



(4.3)

where $X(g)$ is a projective representation with the same multiplier as that of $\mathcal{R}(g)$. This transformation relation allows to determine the structure of the physical Hilbert space of the gauge field degree of freedom. We find that the gauge field "spins" are composed of right and left parts:

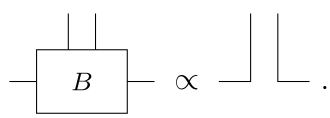
$$\mathcal{H}_B = \bigoplus_k \mathcal{H}_{l_k} \otimes \mathcal{H}_{r_k} \ ,$$

where $\mathcal{H}_{r_k}$ are irreducible representation spaces of G. The physical representations $\mathcal{R}(g)$ and $\mathcal{L}(g)$ take the forms: $\mathcal{R}(g) = \oplus_k(\mathbb{I} \otimes D^{r_k}_{\gamma}(g))$, $\mathcal{L}(g) = \oplus_k(D^{l_k}_{\gamma^{-1}}(g) \otimes \mathbb{I})$, and act on the right and left parts of $\mathcal{H}_B$ respectively.

The transformation relation Eq. (4.3) also determines the structure of the tensor B. Decompose $X(g)$ into its constituent irreducible representations and project Eq. (4.3) to the corresponding irreducible subspaces (virtual and physical). The obtained blocks of B intertwine irreducible representations, and their structure is therefore determined by Schur's lemma (Lemma 3.2.1). When the irreducible representations in Eq. (4.3) match, the corresponding elementary block of B is proportional to the tensor composed of the matrices:

$$B^{m,n} = |m\rangle\langle n| \ ,$$

so that B, when represented in graphical notation, takes the form:



Otherwise, if the irreducible representations do not match, that block of B is zero.

The tensor B is composed out of such elementary building blocks multiplied by constants - free parameters. Finally, we show that for any B generating a gauge field MPV with a local symmetry, one can always find a tensor A, describing a matter degree of freedom, such that the matter and gauge field MPV generated by A and B has a local symmetry.

We shall now describe these results in detail, and state the relevant theorems.

Let $|\psi_B\rangle$ be a MPV generated by a tensor B and let $\mathcal{R}(g)$, $\mathcal{L}(g)$ be projective representations as defined in Section 4.1.2. As in the case of a matter MPV above, according to Proposition 3.1.2 we can assume B is in CFII and write it in terms of its BNT:

$$B^i = \oplus_{j=1}^{n} \oplus_{q=1}^{r_j} \mu_{j,q} B^i_j \ , \tag{4.4}$$

where $\{B_j\}$ are normal tensors in CFII forming a BNT of B (Definition 3.1.7) and $\mu_{j,q}$ are constants.

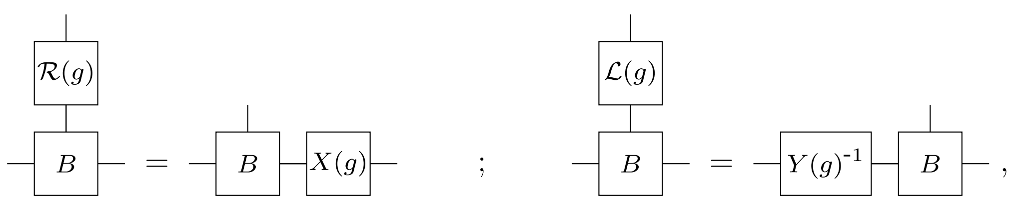
Theorem III (Gauge field MPV with a local symmetry)**.** *A tensor B in CFII which generates a MPV that has a local symmetry with respect to $\mathcal{R}(g) \otimes \mathcal{L}(g)$ where $\mathcal{R}(g)$ and $\mathcal{L}(g)$ are projective representations with inverse multipliers (Definition III), transforms under the representation matrices as:*

(4.5)

where $X(g)$ is a projective representation of G with the same multiplier as $\mathcal{R}(g)$ and with the same block structure as B (Eq. (4.4)*):*

$$X(g) = \oplus_{j=1}^{m} \oplus_{q=1}^{r_j} X_j(g) \ . \tag{4.6}$$

When considering matter and gauge field MPVs in the next section, we will show that in that setting, a more general relation than Eq. (4.5) is satisfied by the tensor B. Namely:



(4.7)

where $X(g)$ and $Y(g)$ are different projective representations (in the case when B is composed of non-square matrices they are of different dimensions). We shall now present results which follow from the more general relation (Eq. (4.7)), as they will be relevant also in the next section. Then we will apply them to the case at hand - Eq. (4.5) (i.e., when $X(g) = Y(g)$ and B is composed out of square matrices).

Equation (4.7) allows us to determine the structure of the Hilbert space of the gauge field degree of freedom. The fact that the action of $\mathcal{R}(g)$ is translated to a matrix multiplication from the right, and that of $\mathcal{L}(g)$ - to multiplication from the left implies that their actions on the "spin" representing the gauge field are independent, consequently the "spin" must be composed of right and left parts:

Proposition II (Structure of $\mathcal{H}_B$)**.** *Given a tensor B, projective representations $\mathcal{R}(g)$, $\mathcal{L}(g)$ with inverse multipliers γ and γ^{-1} (as defined in Section 4.1.2) and matrices $X(g)$ and $Y(g)$ which satisfy Eq.* (4.7)*, the Hilbert space $\mathcal{H}_B$ can be restricted to a representation space of $G \times G$ and thus decomposes into a direct sum of tensor products of irreducible representation spaces of G:*

$$\mathcal{H}_B = \bigoplus_{k=1}^{M} \mathcal{H}_{l_k} \otimes \mathcal{H}_{r_k} \ ,$$

where r_k and l_k are irreducible representation labels.

The structure of $\mathcal{H}_B$ described in [35] is a particular case of this Hilbert space. There:

$$\mathcal{H}_B = \bigoplus_{k=1}^{M} \mathcal{H}_{\overline{r_k}} \otimes \mathcal{H}_{r_k} \ , \tag{4.8}$$

where $\overline{r_k}$ indicates the complex conjugate representation to r_k. Equation (4.8) is a truncated version of the K-S Hilbert space, which allows to regain the whole space if M is increased such that all the irreducible representations are included. Each k sector in Eq. (4.8): $\mathcal{H}_{\overline{r_k}} \otimes \mathcal{H}_{r_k}$ is isomorphic to the function space spanned by

$$\left\{ D^{r_k}_{m,n} : g \mapsto D^{r_k}_{m,n}(g) \,|\, m, n = 1, \ldots, dim(r_k) \right\} \subset L^2(G) \ ,$$

with $\mathcal{R}(g)$ and $\mathcal{L}(g)$ equivalent to the right and left translations [44].

Remark 4.3.1. The group transformations $\mathcal{R}(g)$ and $\mathcal{L}(g)$ are equivalent, according to Proposition II, to $\oplus_k(\mathbb{I} \otimes D^{r_k}_{\gamma}(g))$ and $\oplus_k(D^{l_k}_{\gamma^{-1}}(g) \otimes \mathbb{I})$ respectively, where $D^{j}_{\gamma}(g)$ are irreducible projective representations. Changing the basis of the physical Hilbert space (as in Remark 3.2.1) to $\{|l_k, m\rangle \otimes |r_k, n\rangle\}$ in which the representations take this block diagonal form, involves transforming B into $\tilde{B}$ given by the matrices: $\tilde{B}^{k,m,n} = \sum_i B^i \langle l_k, m; r_k, n \mid i\rangle$. According to Corollary 3.2.1 $\tilde{B}$ is also in CFII. Equation (4.7) holds for the new tensor under the action of the transformed operators: $\tilde{\mathcal{R}}(g) = \oplus_k(\mathbb{I} \otimes D^{r_k}_{\gamma}(g))$ and $\tilde{\mathcal{L}}(g) = \oplus_k(D^{l_k}_{\gamma^{-1}}(g) \otimes \mathbb{I})$. We shall always assume B, $\mathcal{L}(g)$ and $\mathcal{R}(g)$ are in these forms.

Remark 4.3.2. The simplest case of Eq. (4.7) one could consider is when $\mathcal{R}(g) = \mathbb{I} \otimes D^r(g)$ and $\mathcal{L}(g) = D^l(g) \otimes \mathbb{I}$, for irreducible projective representations $D^r_{\gamma}(g)$ and $D^l_{\gamma^{-1}}(g)$. To these corresponds the basis $\{|m\rangle \otimes |n\rangle \mid m = 1, \dots, dim(l), n = 1, \dots, dim(r)\}$, and the matrices composing the tensor B are numbered by two indices:

$$B^{m,n} = \sum_{\alpha,\beta} B^{m,n}_{\alpha,\beta} |\alpha\rangle\langle\beta| \ .$$

B transforms under $\mathcal{R}(g)$ and $\mathcal{L}(g)$ in the following manner:

$$\mathcal{R}(g) : B^{m,n} \mapsto \sum_{n'} D^r_{\gamma}(g)_{n,n'} B^{m,n'} = B^{m,n} X(g)$$
$$\mathcal{L}(g) : B^{m,n} \mapsto \sum_{m'} D^l_{\gamma^{-1}}(g)_{m,m'} B^{m',n} = Y(g)^{-1} B^{m,n} \ .$$

We have seen in Remark 3.2.1 how to change the basis of the physical Hilbert space in order to bring the physical representations to block diagonal form. We would like to do the same for the virtual projective representation $X(g)$ appearing in Eq. (4.5). This can be achieved by a different transformation of the tensor B described in the following:

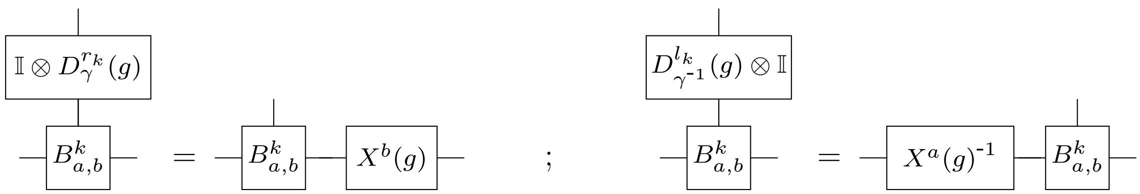
Remark 4.3.3. Given $B, \mathcal{R}(g)$, $\mathcal{L}(g)$ and $X(g)$ that satisfy Eq. (4.5), redefine B:

$$B^{k;m,n} \mapsto \tilde{B}^{k;m,n} = V^{-1} B^{k;m,n} V \ ,$$

with any invertible matrix V. The new tensor $\tilde{B}$ generates the same MPV and transform as in Eq. (4.5) with $X(g)$ replaced by $\tilde{X}(g) = V^{-1} X(g) V$.

Remark 4.3.4. Note that the transformation described in Remark 4.3.3 may ruin the CF property of B, as V does not in general preserve B's block structure (Eq. (4.4)). We shall therefore take care to use this freedom of choosing the basis of $X(g)$ only when we no longer intend to use the CF property.

Remark 4.3.3 allows us to assume without loss of generality $X(g)$ takes the form $\oplus_a X^a(g)$, where $X^a(g)$ are irreducible projective representations. Next we project Eq. (4.5) to the k sector of the physical Hilbert space (Remark 4.3.1) and to the (a,b) block in the virtual space, since the representations are block diagonal they commute with the projection operators for every group element $g \in G$. We therefore obtain:



, (4.9)

where $B^k_{a,b}$ is the tensor that consists of the (a,b) blocks of the matrices $B^{k;m,n}$.

The reduction procedure described above motivates the following definition of an elementary B block. Next we shall show that the irreducible representations appearing in Eq. (4.9) determine such blocks up to a constant.

Definition 4.3.1. An elementary block of the tensor B is one which satisfies Eq. (4.7), where $\mathcal{R}(g) = \mathbb{I} \otimes D^r_{\gamma}(g)$, $\mathcal{L}(g) = D^l_{\gamma^{-1}}(g) \otimes \mathbb{I}$ and $X(g)$, $Y(g)$, $D^r_{\gamma}(g)$ and $D^l_{\gamma^{-1}}(g)$ are irreducible projective representations (both $X(g)$ and $Y(g)$ have multiplier γ).

Proposition III (Structure of an elementary B block)**.** *Let B be an elementary B block (Definition 4.3.1). If $X(g) = D^r_{\gamma}(g)$ and $\overline{Y(g)} = D^l_{\gamma^{-1}}(g)$, then B is proportional to the tensor composed of the matrices*

$$B^{m,n} = |m\rangle\langle n| \ , m = 1, \ldots, dim(l), n = 1, \ldots, dim(r) \ .$$

Otherwise $B = 0$.

We have thus classified all tensors B that satisfy Eq. (4.5). There is however more information to be extracted from Theorem III. According to Proposition III, when projected to sectors corresponding to inequivalent representations, the tensor B is zero. This result, combined with the assumption that B is in CF imposes relations between the irreducible representations that comprise $\mathcal{R}(g)$, $\mathcal{L}(g)$ and $X(g)$:

Proposition IV. *Let $B, \mathcal{R}(g), \mathcal{L}(g)$ and $X(g)$ be as in Theorem III. Let $X_j(g) = \oplus_a X_j^a(g)$ be a block of $X(g)$ appearing in Eq.* (4.6)*, consisting of irreducible projective representations $X_j^a(g)$. Let $\mathcal{R}(g) = \oplus_k(\mathbb{I} \otimes D_\gamma^{r_k}(g))$ and $\mathcal{L}(g) = \oplus_k(D_{\gamma^{-1}}^{l_k}(g) \otimes \mathbb{I})$, where $D_\gamma^{r_k}$ and $D_{\gamma^{-1}}^{l_k}$ are irreducible projective representations. Then the following hold:*

1. *For all k either there exist a and b such that $X_j^b(g) = D_\gamma^{r_k}(g)$ and $\overline{X_j^a(g)} = D_{\gamma^{-1}}^{l_k}(g)$, or the projection of the corresponding tensor B_j (a BNT element of B) to the sector k of the physical space is zero.*
2. *$\forall a\ \exists k$ such that $\overline{X_j^a(g)} = D_{\gamma^{-1}}^{l_k}(g)$.*
3. *$\forall a\ \exists k$ such that $X_j^a(g) = D_\gamma^{r_k}(g)$.*

The elementary block of B described in Proposition III is the same as the one used in [35]. Note that even in lattices of higher dimensionality each gauge field degree of freedom still connects two lattice sites. There:

$$B^{j;m,n} = \beta_j |j, m\rangle\langle j, n| \ , \tag{4.10}$$

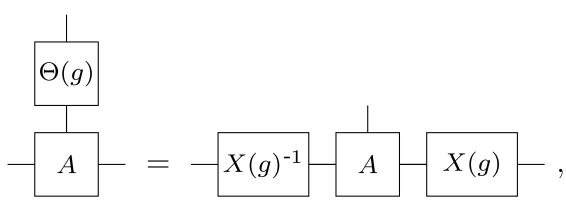
where β_j are arbitrary constants. The overall structure of the B tensor derived above admits more general structures than Eq. (4.10); these structures are recovered if for example, all blocks $X_j(g)$ appearing in $X(g)$ (Eq. (4.6)) are irreducible representations. In this case (since in Proposition IV the index a can assume only one value), for all k $D_{\gamma^{-1}}^{l_k}(g) = \overline{D_\gamma^{r_k}(g)}$ and $\mathcal{H}_B$ takes the K-S form, as in Eq. (4.8).

In the following two propositions we consider adding a matter degree of freedom to a gauge field MPV with a local symmetry. We show that it is always possible to find a tensor A and a unitary representation $\Theta(g)$ (non-trivial ones) that couple to it:

Proposition V. *Let B be in CFII and let $|\psi_B^N\rangle$ have a local symmetry with respect to $\mathcal{R}(g)\otimes\mathcal{L}(g)$ (as in Theorem III). It is always possible to find a tensor A and a representation $\Theta(g)$ such that the corresponding matter and gauge field MPV $|\psi_{AB}^N\rangle$ has a local symmetry with respect to $\mathcal{R}(g)\otimes\Theta(g)\otimes\mathcal{L}(g)$ (Definition IV). In addition, the corresponding matter MPV - $|\psi_A^N\rangle$ - has a global symmetry with respect to $\Theta(g)$.*

For a restricted class of B tensors, *any* A and $\Theta(g)$ that couple to it (satisfy Definition IV) will have a global symmetry:

Proposition VI. *Let B, $\mathcal{R}(g)$ and $\mathcal{L}(g)$ be as in Theorem III and in addition let span$\{B^{k;m,n} \mid k,m,n\}$ contain the identity matrix (e.g. Eq. (4.10)). Let A and $\Theta(g)$ be such that the MPV generated by AB has a local symmetry with respect to $\mathcal{R}(g)\otimes\Theta(g)\otimes\mathcal{L}(g)$ (Definition IV). Then $|\psi_A^N\rangle$ has a global symmetry with respect to $\Theta(g)$. If in addition A is in CF with the same block structure as B (Eq. (4.4)), then A transforms as:*

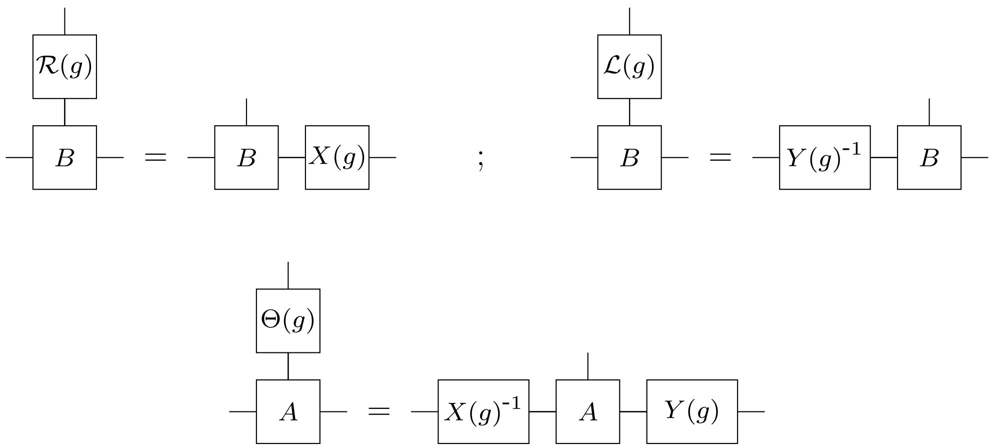


with the same $X(g)$ from Theorem III.

The MPVs described above may be combined in a way that allows coupling matter and gauge fields such that each of them could be invariant on its own, as in the conventional well known scenarios of gauge theories. However, as we shall demonstrate in the next section, this is not the most general setting of a local symmetry involving these two building blocks.

4.3.3 Matter and gauge field MPV

We show that a local symmetry for a combined matter and gauge field MPV $|\psi^N_{AB}\rangle$ (defined in Section 4.1.3) generated by tensors A and B (in an appropriate form), implies the following transformation relations for A and B:



(4.11)

where $X(g)$ and $Y(g)$ are projective representations from the same cohomology class. As described in the previous section, the relation for B allows to infer the structure of the Hilbert space $\mathcal{H}_B$ associated with the gauge field degree of freedom. As before, $\mathcal{H}_B$ splits into right and left parts. The structure of the tensor B can be derived in the same way as in the previous section. Each elementary block of the tensor A, obtained by projecting Eq. (4.11) to irreducible representation spaces, satisfies a vector operator relation, and is therefore determined by the Wigner-Eckart theorem (Theorem 3.2.1).

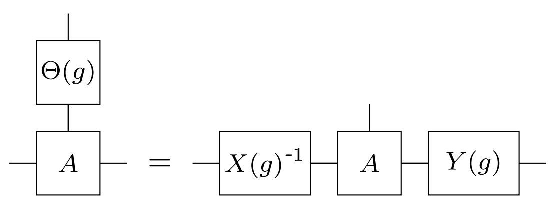
In the general case, the structure described in this section allows for “unconventional” gauge symmetries where a local symmetry exists for the matter and gauge field MPV but none of the constituents has a symmetry on its own, i.e., the gauge field MPV does not have a local symmetry and the matter MPV does not have a global one. We construct an explicit example of such a case (see Proposition XI).

Finally we use the known results about global symmetries in MPV [37] to find a class of matter MPVs with a global symmetry that can be gauged by adding a gauge field degree of freedom. We shall now state the above results in detail.

Let $|\psi_{AB}^N\rangle$ be a MPV generated by tensors A and B and let $\mathcal{R}(g)$, $\Theta(g)$ and $\mathcal{L}(g)$ be as defined in Section 4.1.3.

Theorem IV (Matter and gauge field MPV with a local symmetry)**.** *Let both BA and AB be normal tensors in CFII and let $\Theta(g)$ and $\mathcal{R}(g), \mathcal{L}(g)$ be unitary and projective representations (with inverse multipliers) of a group G respectively. Let $|\psi_{AB}^N\rangle$ be a MPV with a local symmetry with respect to $\mathcal{R}(g) \otimes \Theta(g) \otimes \mathcal{L}(g)$ (Definition IV). Then there exist projective representations $X(g)$ and $Y(g)$ on $\mathbb{C}^{D_1}$ and $\mathbb{C}^{D_2}$ respectively, such that $X(g)$ has the same multiplier as $\mathcal{R}(g)$, and $Y(g)$ - the inverse multiplier to that of $\mathcal{L}(g)$. The tensors A and B transform as follows:*

(4.12)



(4.13)

In the following proposition we show that given arbitrary tensors A and B, generating a MPV $|\psi_{AB}^N\rangle$, it is possible to describe the same MPV as a linear combination of MPVs that satisfy the normality condition in Theorem IV:

Proposition VII. *Let $|\psi_{AB}^N\rangle$ be a MPV generated by arbitrary tensors A and B. Then there exist tensors $\{A_\chi\}$ and $\{B_\chi\}$, and there exists $b \in \mathbb{N}$ such that for all χ both $A_\chi B_\chi$ and $B_\chi A_\chi$ are normal tensors and $\forall N \in \mathbb{N}$ $|\psi_{AB_{\times b}}^N\rangle = \sum_\chi \mu_\chi^N |\psi_{A_\chi B_\chi}^N\rangle$, where μ_χ are constants and $AB_{\times b}$ is the tensor obtained by blocking b copies of the tensor AB.*

Next we show that if $|\psi^N_{AB}\rangle = \sum_\chi \mu^N_\chi |\psi^N_{A_\chi B_\chi}\rangle$ has a local symmetry with respect to $\mathcal{R}(g) \otimes \Theta(g) \otimes \mathcal{L}(g)$, then every normal component $|\psi^N_{A_\chi B_\chi}\rangle$ must have the same symmetry. We can then apply Theorem IV to each of the components.

Proposition VIII. *Let $|\psi^N_{AB}\rangle = \sum_\chi \mu^N_\chi |\psi^N_{A_\chi B_\chi}\rangle$ where both $A_\chi B_\chi$ and $B_\chi A_\chi$ are normal tensors. Let O be a local operator acting on a fixed number of adjacent sites. If $\forall N$ O leaves the MPV invariant:*

$$O \otimes \mathbb{I}|_{rest} |\psi^N_{AB}\rangle = |\psi^N_{AB}\rangle \ ,$$

then O leaves every component invariant:

$$O \otimes \mathbb{I}|_{rest} |\psi^N_{A_\chi B_\chi}\rangle = |\psi^N_{A_\chi B_\chi}\rangle \, \forall \chi \ .$$

Having derived Eq. (4.12), Proposition II can be applied to determine the structure of the Hilbert space $\mathcal{H}_B$. As in the case of a gauge field MPV discussed in the previous section, we are free to assume $X(g)$ and $Y(g)$ are block diagonal in irreducible representations:

Remark 4.3.5. In Theorem IV we are free to choose similarity transformations for $X(g)$ and $Y(g)$ independently. Given $A, B, \mathcal{R}(g), \Theta(g), \mathcal{L}(g), X(g)$ and $Y(g)$ that satisfy Eq. (4.12) and Eq. (4.13) we can redefine A and B:

$$A^{j,m} \mapsto \tilde{A}^{j,m} = U^{-1} A^{j,m} V \ , \quad B^{k;m,n} \mapsto \tilde{B}^{k;m,n} = V^{-1} B^{k;m,n} U \ ,$$

with any invertible matrices U and V of fitting dimensions. The new tensors generate the same MPV $|\psi^N_{AB}\rangle$ and transform as in Theorem IV with $X(g)$ and $Y(g)$ replaced by $\tilde{X}(g) = U^{-1} X(g) U$ and $\tilde{Y}(g) = V^{-1} Y(g) V$.

Definition 4.3.2 (Elementary A block)**.** An elementary block of the tensor A is one which satisfies Eq. (4.13), where $\Theta(g)$, $X(g)$ and $Y(g)$ are all irreducible projective representations.

By bringing all of the representations appearing in Eq. (4.12) and Eq. (4.13) to block diagonal form (using Remark 3.2.1 on the physical representations and Remark 4.3.5 on the virtual ones), and projecting Eq. (4.12) and Eq. (4.13) to irreducible sectors of the physical and virtual Hilbert spaces (as explained in Section 4.3.2), we may reduce Eq. (4.12) and Eq. (4.13) to the cases of elementary blocks of B and of A respectively.

We have seen in Section 4.3.2 that Eq. (4.12) determines the tensor B given $\mathcal{R}(g)$, $\mathcal{L}(g)$, $X(g)$ and $Y(g)$ (Proposition IV). We now show that Eq. (4.13) determines the tensor A given $\Theta(g)$, $X(g)$ and $Y(g)$.

Proposition IX. *Let A be an elementary block (Definition 4.3.2), with $\Theta(g) = D^{J_0}(g)$, $X(g) = D^j_\gamma(g)$ and $Y(g) = D^l_{\gamma^{-1}}(g)$. Then A is built out of Clebsch-Gordan coefficients and has the form:*

$$A^M = \sum_{J \in \mathfrak{J} : D^J = D^{J_0}} \alpha_J \sum_{m,n} \left\langle J, M \mid \bar{j}, m; l, n \right\rangle |m\rangle\langle n| \ ,$$

where $\mathfrak{J}$ is the set of irreducible representation indices appearing in the decomposition of $\overline{D^j_\gamma(g)} \otimes D^l_{\gamma^{-1}}(g)$ into irreducible representations, $\left\langle \bar{j}, m : l, n \mid J, M \right\rangle$ are the Clebsch-Gordan coefficients of the decomposition, $\overline{D^j_\gamma(g)}$ is the complex conjugate representation to $D^j_\gamma(g)$ and α_J are arbitrary constants.

Proposition IX was shown in [37] in the context of MPS with a global symmetry.

The relation between the irreducible projective representations appearing in $\mathcal{R}(g)$ ($\mathcal{L}(g)$) and $X(g)$ ($Y(g)$) is characterized by the following:

Proposition X. *Let AB and BA be normal tensors and let B satisfy Eq. (4.12) with $\mathcal{R}(g) = \oplus_k(\mathbb{I} \otimes D^{r_k}_\gamma(g))$, $\mathcal{L}(g) = \oplus_k(D^{l_k}_{\gamma^{-1}}(g) \otimes \mathbb{I})$, $Y(g) = \oplus_a Y^a(g)$ and $X(g) = \oplus_b X^b(g)$, where $D^{r_k}_\gamma$, $D^{l_k}_{\gamma^{-1}}$, Y^a and X^b are irreducible projective representations, then*

1. *For all k either there exist a and b such that $X^b(g) = D^{r_k}_\gamma(g)$ and $\overline{Y^a(g)} = D^{l_k}_{\gamma^{-1}}(g)$ or the projection of the tensor B to the sector k of the physical space is zero (and it can*

be discarded).

2. $\forall a\ \exists k$ *such that* $\overline{Y^a(g)} = D^{l_k}_{\gamma^{-1}}(g)$.

3. $\forall b\ \exists k$ *such that* $X^b(g) = D^{r_k}_{\gamma}(g)$.

By constructing tensors A and B that transform as in Theorem IV with $X(g) \neq Y(g)$ we show the existence of matter and gauge field MPVs which have a local symmetry but for which the corresponding matter MPV does not have a global symmetry, nor does the gauge field MPV have a local one:

Proposition XI. *There exist tensors A and B such that $|\psi_{AB}\rangle$ has a local symmetry with respect to $\mathcal{R}(g)\otimes\Theta(g)\otimes\mathcal{L}(g)$, but $|\psi_A\rangle$ does not have a global symmetry with respect to $\Theta(g)$. In addition $\mathcal{R}(g)\otimes\mathcal{L}(g)|\psi_B\rangle \neq |\psi_B\rangle$.*

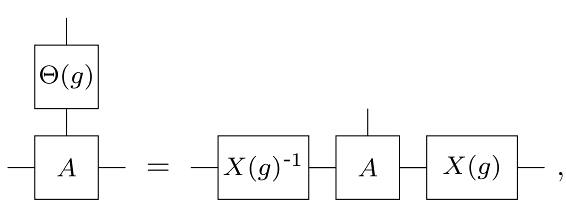
We review known results about MPV with global symmetry [37]. Let A be a tensor in CFII:

$$A^i = \oplus_{j=1}^{n} \oplus_{q=1}^{r_j} \mu_{j,q} A^i_j \ , \tag{4.14}$$

where $\{A_j\}$ are normal tensors in CFII forming a BNT of A (Definition 3.1.7) and $\mu_{j,q}$ are constants.

Recall Theorem I:

Theorem I. *A tensor A in CFII which generates a MPV with a global symmetry with respect to a representation $\Theta(g)$ of a connected Lie group G (Definition I), transforms under the representation matrix as:*



(3.14)

where $X(g)$ has the same block structure as A:

$$X(g) = \oplus_{j=1}^{m} \oplus_{q=1}^{r_j} X_j(g) \ , \tag{3.15}$$

and where each block $X_j(g)$ is a projective representation, in the general case, for different j values $X_j(g)$ belong to different cohomology classes.

In the case when all $X_j(g)$ obtained in Theorem I are from the same cohomology class, we can find a gauge field tensor B and projective representations $\mathcal{R}(g)$ and $\mathcal{L}(g)$ that gauge the symmetry:

Proposition XII. *Let A be a tensor in CFII generating a MPV with a global symmetry i.e., satisfying Theorem I. Let $X(g)$ (in Eq.* (3.14)*) be a projective representation (i.e. all $X_j(g)$ in Eq.* (3.15) *are in the same cohomology class). Then there exist a tensor B and projective representations $\mathcal{R}(g)$ and $\mathcal{L}(g)$ with inverse multipliers such that both local symmetries: Definition IV for $|\psi_{AB}^N\rangle$ and Definition III for $|\psi_B^N\rangle$ are satisfied.*

Chapter 5

Derivation and Proofs of the Results

In this chapter we prove the theorems stated in Section 4.3.

5.1 Matter MPV with local symmetry

Proposition I. *Let $\mathcal{H}$ be a finite dimensional Hilbert space and let $\Theta : g \mapsto \Theta(g)$ be a representation on $\mathcal{H}$. Let $|\psi^N\rangle \in \mathcal{H}^{\otimes N}$ be a vector with a local symmetry, i.e.*

$$\Theta(g_1) \otimes \Theta(g_2) \otimes \dots \Theta(g_N)|\psi^N\rangle = |\psi^N\rangle, \quad \forall g_1, g_2, \dots, g_N \in G \ .$$

Then $|\psi^N\rangle \in {\mathcal{H}_0}^{\otimes N}$, where $\mathcal{H}_0 \subset \mathcal{H}$ is the subspace on which $\Theta(g)$ acts trivially.

Proof. Write $|\psi^N\rangle$ in the irreducible representation basis which satisfies:

$$\Theta(g)|j, m\rangle = \sum_n D^j(g)_{n,m}|j, n\rangle \ ,$$

where $D^j(g)$ are irreducible representation matrices.

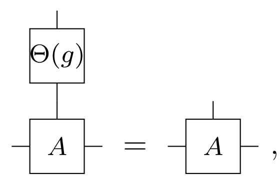
$$|\psi^N\rangle = \sum c_{j_1,m_1,\dots,j_N,m_N}|j_1, m_1, \dots, j_N, m_N\rangle \ .$$

The local symmetry condition implies:

$$\sum_{n_1} D^{j_1}(g)_{m_1,n_1} c_{j_1,n_1,\ldots,j_N,m_N} = c_{j_1,m_1,\ldots,j_N,m_N} \ ,$$

which means that the vector of coefficients $\overrightarrow{c}_{j_1,(\cdot),\ldots,j_N,m_N}$ is either zero or an invariant subspace of $D^{j_1}(g)$, in which case $D^{j_1}(g)$ is the trivial representation. This implies that the coefficients $c_{j_1,m_1,\ldots,j_N,m_N}$ are zero whenever any one of the j_ks corresponds to a non trivial representation. $\square$

Theorem II. *Let A be a tensor in CF generating a MPV with a local symmetry with respect to a representation $\Theta(g)$ (Definition II). Then for all $g \in G$ the tensor A satisfies:*



i.e., for all $i = 1, \ldots, d_A$: $\sum_{i'} \Theta(g)_{ii'} A^{i'} = A^i$.

Proof. We apply Lemma 3.1.1 with $S^i := \sum_{i'} \Theta(g)_{ii'} A^{i'}$ and $T^i := A^i$.

$\square$

Remark 5.1.1. We have never used any properties of $\Theta(g)$ as a representation. The same proof is valid for any operator Θ.

According to Remark 3.2.1, the MPV generated by A can be written in terms of a tensor $\tilde{A}$, composed of the matrices $\{\tilde{A}^{j,m}\}$, corresponding to the irreducible representation basis $\{|j,m\rangle\}$ on which $\Theta(g)$ acts as $\Theta(g)|j,m\rangle = \sum_n D^j(g)_{n,m}|j,n\rangle$. According to Corollary 3.2.1, $\tilde{A}$ is also in CF. Applying Theorem II to $\tilde{A}$ leads to the following:

Corollary I. *The matrices $\tilde{A}^{j,m}$ are non-zero only for j such that $D^j(g) \equiv \mathbb{I}_{1\times 1}$.*

Proof. From Theorem II we deduce that each vector of matrix elements of A: $\vec{A}^j_{\alpha,\beta} = \left(A^{j,1}_{\alpha,\beta}, A^{j,2}_{\alpha,\beta}, \ldots, A^{j,dim(j)}_{\alpha,\beta}\right)^T$ is invariant under $D^j(g)$ for all $g \in G$. This implies that either $\vec{A}^j_{\alpha,\beta}$ is zero or that $D^j(g)$ is the one dimensional trivial representation. $\square$

5.2 Pure gauge field MPV

In order to prove Theorem III we shall proceed as in Section 3.3: we shall first prove a lemma which describes the case when $\mathcal{R}$ and $\mathcal{L}$ are just unitary operators, and later use that to prove the case when they are representations.

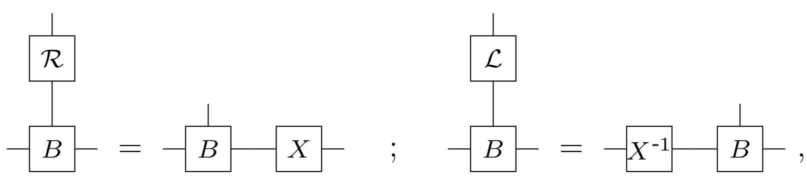
Lemma 5.2.1. *Let B be a tensor in CFII:*

$$B^i = \oplus_{j=1}^{m} \oplus_{q=1}^{r_j} \mu_{j,q} B_j^i \ ,$$

and let $\mathcal{R}$ and $\mathcal{L}$ be two unitary operators such that for all K

$$\mathcal{R}^{[K]}\mathcal{L}^{[K+1]}|\psi_B^N\rangle = |\psi_B^N\rangle \ .$$

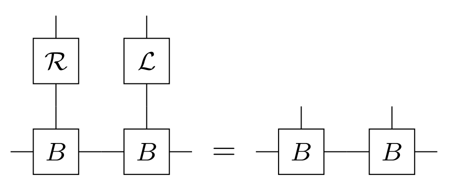
Then B transforms under the unitary matrices as follows:



(5.1)

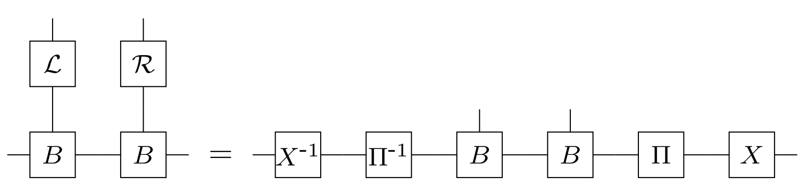
where X is a unitary matrix with the same block structure as B^i, as in Eq. (3.8).

Proof. Applying Theorem II (recall Remark 5.1.1) to the tensor BB and the unitary $\mathcal{R}\otimes\mathcal{L}$ (BB is in CF if B is in CF), we obtain:



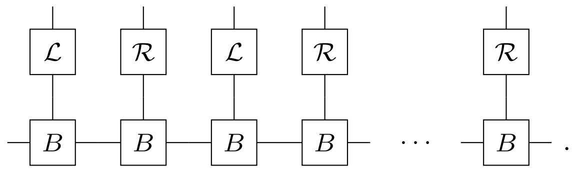
(5.2)

Applying the pair of operators to every site on the chain (for even N) we conclude that the MPV is invariant under the global application of the operators in reversed order: $(\mathcal{L}\otimes\mathcal{R})^{\otimes N}|\psi_B^{2N}\rangle = |\psi^2 N_B\rangle$. Using Corollary 3.3.1 we obtain:



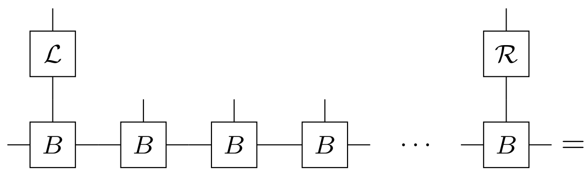
(5.3)

where X is unitary and Π is a permutation, as in Corollary 3.3.1. Next consider the following tensor:



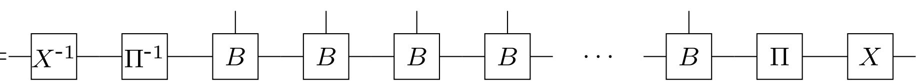
According to Eq. (5.2) this tensor is equal to the LHS of the following, and according to Eq. (5.3) - to the RHS:

$LHS =$

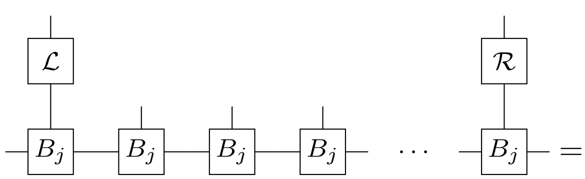


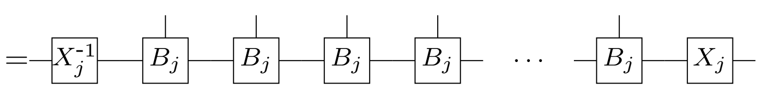
(5.4)

$RHS =$



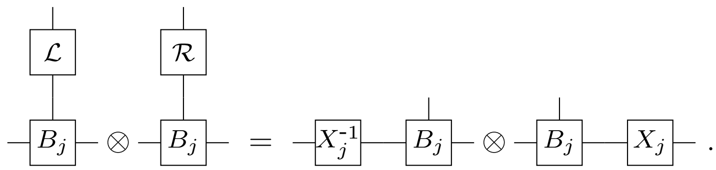
Using the same argument as in equation Eq. (3.18), we show that the permutation must act trivially: use Proposition 3.1.6 on the string of consecutive Bs, excluding the extreme right and left ones, to obtain multiples of $\mathbb{I}$ in a single j block and zeros elsewhere. Note that $\mathcal{R}$ and $\mathcal{L}$ do not change the block structure of the tensors they act on. Now compare the RHS with the LHS block-wise, if Π acts non trivially on a block j, then we get that $B_j B_j$ is zero, which is a contradiction to B_j being normal. Next, having eliminated the possibility of a permutation, project Eq. (5.4) to any (j, q) block to obtain:



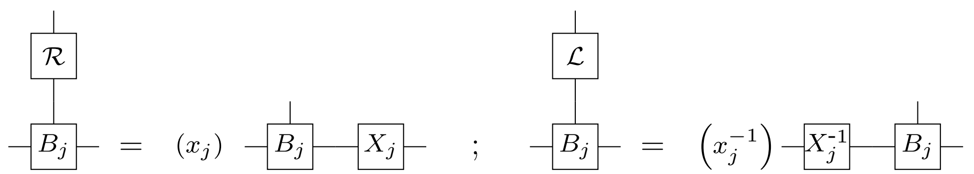


where B_j is a normal tensor by assumption. We can now apply the inverse on the string of

Bs in the middle (BB is normal if B is normal) to obtain:

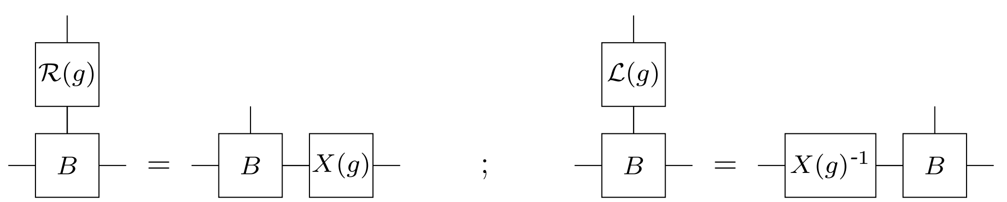


According to Remark 3.1.2, the matrices X_j are determined up to a constant. We now choose a representative from the projective unitary class of X_j. The above implies that for any such choice there is a constant x_j such that:



Therefore the desired X is $X = \oplus_{j=1}^{m} \oplus_{q=1}^{r_j} x_j X_j$. □

Theorem III (Gauge field MPV with a local symmetry)**.** *A tensor B in CFII which generates a MPV that has a local symmetry with respect to $\mathcal{R}(g) \otimes \mathcal{L}(g)$ where $\mathcal{R}(g)$ and $\mathcal{L}(g)$ are projective representations with inverse multipliers (Definition III), transforms under the representation matrices as:*

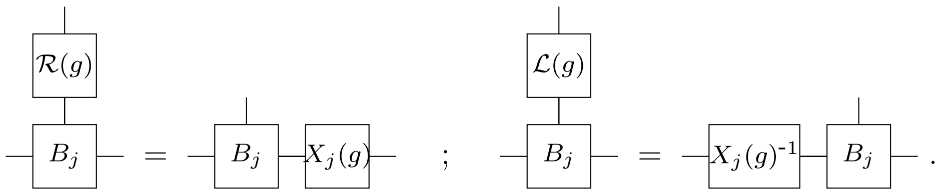


(4.5)

where $X(g)$ is a projective representation of G with the same multiplier as $\mathcal{R}(g)$ and with the same block structure as B (Eq. (4.4)):

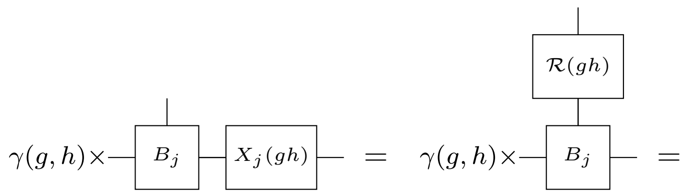
$$X(g) = \oplus_{j=1}^{m} \oplus_{q=1}^{r_j} X_j(g) \ . \tag{4.6}$$

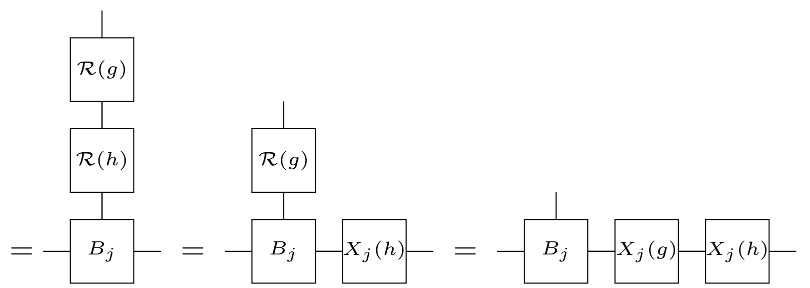
Proof. As we have seen in the proof of Lemma 5.2.1, Eq. (5.1) holds for each block of B, so for every group element $g \in G$ we have:



(5.5)

We write the action of the group element $\mathcal{R}(gh)$ on B in two ways:





Now by contracting with the tensor $B_j B_j \ldots B_j$ from the left, and taking the appropriate linear combination which results in the identity matrix (B_j is normal), we obtain $\gamma(g,h)X_j(gh) = X_j(g)X_j(h)$. This means that for all j $X_j(g)$ is a projective representation with the same multiplier as $\mathcal{R}(g)$ (γ). Therefore $X(g)$ is a projective representation. □

Proposition II (Structure of $\mathcal{H}_B$)**.** *Given a tensor B, projective representations $\mathcal{R}(g)$, $\mathcal{L}(g)$ with inverse multipliers γ and γ^{-1} (as defined in Section 4.1.2) and matrices $X(g)$ and $Y(g)$ which satisfy Eq.* (4.7)*, the Hilbert space $\mathcal{H}_B$ can be restricted to a representation space of $G \times G$ and thus decomposes into a direct sum of tensor products of irreducible representation spaces of G:*

$$\mathcal{H}_B = \bigoplus_{k=1}^{M} \mathcal{H}_{l_k} \otimes \mathcal{H}_{r_k} \ ,$$

where r_k and l_k are irreducible representation labels.

Proof. Even though $|\psi_B\rangle$ is defined in terms of the basis $\{|j\rangle\}$ in $\mathcal{H}_B$, it is sufficient to consider only vectors of the form:

$$|\phi_{\alpha,\beta}\rangle = \sum_i \langle\alpha|B^i|\beta\rangle|i\rangle \ \in \mathcal{H}_B \ .$$

Let $\mathcal{H} := span\{|\phi_{\alpha,\beta}\rangle\}_{\alpha,\beta}$. The group transformations $\mathcal{L}(g)$ and $\mathcal{R}(g)$ preserve $\mathcal{H}$:

$$\mathcal{R}(g)|\phi_{\alpha,\beta}\rangle = \sum_i \langle\alpha|B^i X(g)|\beta\rangle|i\rangle = \sum_{i,\gamma} \langle\alpha|B^i|\gamma\rangle\langle\gamma|X(g)|\beta\rangle|i\rangle = \sum_\gamma \langle\gamma|X(g)|\beta\rangle|\phi_{\alpha,\gamma}\rangle$$

$$\mathcal{L}(g)|\phi_{\alpha,\beta}\rangle = \sum_i \langle\alpha|Y(g)^{-1}B^i|\beta\rangle|i\rangle = \sum_{i,\gamma} \langle\alpha|Y(g)^{-1}|\gamma\rangle\langle\gamma|B^i|\beta\rangle|i\rangle = \sum_\gamma \langle\alpha|Y(g)^{-1}|\gamma\rangle|\phi_{\gamma,\beta}\rangle \ ,$$

where Eq. (4.7) was used. Performing a Schmidt decomposition of $|\psi_{AB}\rangle$ (or $|\psi_B\rangle$, the argument is the same) with respect to any partition where one gauge field Hilbert space is split off from the rest of the system:

$$\begin{aligned} |\psi_{AB}\rangle &= \sum_{\{i\},\{j\},\alpha,\beta} \left(\langle\alpha|B^{j_1}|\beta\rangle \ \langle\beta|A^{i_2}B^{j_2}\dots A^{i_N}B^{j_N}A^{i_1}|\alpha\rangle\right) \ |i_1\rangle\otimes|j_1\rangle\otimes|i_2\dots i_N j_N\rangle \\ &= \sum_{\alpha,\beta} |\phi_{\alpha,\beta}\rangle_{[2]}|\psi_{\beta,\alpha}\rangle_{[3,\dots,2N,1]} \ , \end{aligned}$$

we see that only vectors from $\mathcal{H}$ appear. Therefore it is sufficient to restrict ourselves to $\mathcal{H}_B = \mathcal{H}$. Next we show that $\mathcal{H}$ has a representation space structure. Equation (4.7) implies that $\mathcal{R}(g)$ and $\mathcal{L}(h)$ commute on $\mathcal{H}$:

$$\mathcal{L}(g)\mathcal{R}(h)|\phi_{\alpha,\beta}\rangle = \sum_i \langle\alpha|Y(g)^{-1}B^i X(h)|\beta\rangle|i\rangle = \mathcal{R}(h)\mathcal{L}(g)|\phi_{\alpha,\beta}\rangle \ .$$

Thus $\mathcal{H}$ forms a projective representation space of $G\times G$ with the projective representation map $(g,h)\mapsto\mathcal{L}(g)\mathcal{R}(h)$ with multiplier $\gamma^{-1}\times\gamma$ of $G\times G$ defined by $\gamma^{-1}\times\gamma : ((g,h),(g',h'))\mapsto \gamma^{-1}(g,g')\gamma(h,h')$ [50]:

$$\mathcal{L}(g)\mathcal{R}(h)\mathcal{L}(g')\mathcal{R}(h')|_{\mathcal{H}} = \mathcal{L}(g)\mathcal{L}(g')\mathcal{R}(h)\mathcal{R}(h')|_{\mathcal{H}} = \gamma^{-1}(g,g')\gamma(h,h')\mathcal{L}(gg')\mathcal{R}(hh')|_{\mathcal{H}} \ ,$$

where we used the fact that $\mathcal{L}(g)$ and $\mathcal{R}(h)$ commute and preserve $\mathcal{H}$; . For finite or compact groups $\mathcal{H}$ decomposes into a direct sum of irreducible projective representations of $G\times G$

with multiplier $\gamma^{-1} \times \gamma$, each one of which is equivalent to a projective representation of the form $(g,h) \mapsto D^l_{\gamma^{-1}}(g) \otimes D^r_\gamma(h)$ [50], which proves the proposition. □

Recall the definition of an elementary B block:

Definition 4.3.1. An elementary block of the tensor B is one which satisfies Eq. (4.7), where $\mathcal{R}(g) = \mathbb{I} \otimes D^r_\gamma(g)$, $\mathcal{L}(g) = D^l_{\gamma^{-1}}(g) \otimes \mathbb{I}$ and $X(g)$, $Y(g)$, $D^r_\gamma(g)$ and $D^l_{\gamma^{-1}}(g)$ are irreducible projective representations (both $X(g)$ and $Y(g)$ have multiplier γ).

Proposition III (Structure of an elementary B block)**.** *Let B be an elementary B block (Definition 4.3.1). If $X(g) = D^r_\gamma(g)$ and $\overline{Y(g)} = D^l_{\gamma^{-1}}(g)$, then B is proportional to the tensor composed of the matrices*

$$B^{m,n} = |m\rangle\langle n| \ , m = 1, \ldots, dim(l), n = 1, \ldots, dim(r) \ .$$

Otherwise $B = 0$.

Proof. Write B as a map $B : \mathbb{C}^{D_2} \to \mathbb{C}^{D_1} \otimes \mathcal{H}_B$:

$$B = \sum_{m,n} B^{m,n} \otimes |m\rangle|n\rangle = \sum_{m,n,\alpha,\beta} B^{m,n}_{\alpha,\beta} |\alpha\rangle\langle\beta| \otimes |m\rangle|n\rangle$$

By hypothesis B satisfies (Eq. (4.7)):

$$[\mathbb{I} \otimes (\mathcal{R}(g)\mathcal{L}(h))]\, B = \left[\mathbb{I} \otimes \left(D^l_{\gamma^{-1}}(h) \otimes D^r_\gamma(g)\right)\right] B = \left[Y(h)^{-1} \otimes \mathbb{I}\right] B \left[X(g) \otimes \mathbb{I}\right] \ .$$

Write the above equality explicitly (repeated indices are summed over):

$$\begin{aligned} LHS = & \sum B^{m,n}_{\alpha,\beta} |\alpha\rangle\langle\beta| \otimes D^l_{\gamma^{-1}}(h)|m\rangle D^r_\gamma(g)|n\rangle = \\ & \sum B^{m,n}_{\alpha,\beta} |\alpha\rangle\langle\beta| \otimes D^l_{\gamma^{-1}}(h)_{m',m} |m'\rangle D^r_\gamma(g)_{n',n} |n'\rangle = \\ RHS = & \sum B^{m,n}_{\alpha,\beta} Y(h)^{-1} |\alpha\rangle\langle\beta| X(g) \otimes |m\rangle|n\rangle = \\ & \sum B^{m,n}_{\alpha,\beta} \overline{Y(h)_{\alpha,\alpha'}} |\alpha'\rangle\langle\beta'| X(g)_{\beta,\beta'} \otimes |m\rangle|n\rangle \ . \end{aligned}$$

Projecting both LHS and RHS to $|\hat{\alpha}\rangle\langle\hat{\beta}| \otimes |\hat{m}\rangle|\hat{n}\rangle$ we obtain

$$\sum_{m,n} D^l_{\gamma^{-1}}(h)_{\hat{m},m} D^r_{\gamma}(g)_{\hat{n},n} B^{m,n}_{\hat{\alpha},\hat{\beta}} = \sum_{\alpha,\beta} B^{\hat{m},\hat{n}}_{\alpha,\beta} \overline{Y(h)_{\alpha,\hat{\alpha}}} X(g)_{\beta,\hat{\beta}} \ .$$

The LHS is a multiplication from the left (summing the indices m, n) of the matrix $\mathbf{B}$, with entries $\mathbf{B}_{(m,n),(\alpha,\beta)} := B^{m,n}_{\alpha,\beta}$, with the matrix $D^l_{\gamma^{-1}}(h) \otimes D^r_{\gamma}(g)$, which is an irreducible projective representation of $G \times G$. The RHS is a multiplication of $\mathbf{B}$ from the right (summing the indices α, β) with the matrix $\overline{Y(h)} \otimes X(g)$, which is also an irreducible projective representation of $G \times G$ (with the same multiplier). By Schur's lemma (Lemma 3.2.1) $\mathbf{B} \propto \mathbb{I}$ (i.e. $B^{m,n}_{\alpha,\beta} \propto \delta_{\alpha,m}\delta_{\beta,n}$) if $D^l_{\gamma^{-1}}(h) \otimes D^r_{\gamma}(g) = \overline{Y(h)} \otimes X(g)$, and zero otherwise. □

Proposition IV. *Let $B, \mathcal{R}(g), \mathcal{L}(g)$ and $X(g)$ be as in Theorem III. Let $X_j(g) = \oplus_a X^a_j(g)$ be a block of $X(g)$ appearing in Eq.* (4.6)*, consisting of irreducible projective representations $X^a_j(g)$. Let $\mathcal{R}(g) = \oplus_k (\mathbb{I} \otimes D^{r_k}_{\gamma}(g))$ and $\mathcal{L}(g) = \oplus_k (D^{l_k}_{\gamma^{-1}}(g) \otimes \mathbb{I})$, where $D^{r_k}_{\gamma}$ and $D^{l_k}_{\gamma^{-1}}$ are irreducible projective representations. Then the following hold:*

1. *For all k either there exist a and b such that $X^b_j(g) = D^{r_k}_{\gamma}(g)$ and $\overline{X^a_j(g)} = D^{l_k}_{\gamma^{-1}}(g)$, or the projection of the corresponding tensor B_j (a BNT element of B) to the sector k of the physical space is zero.*

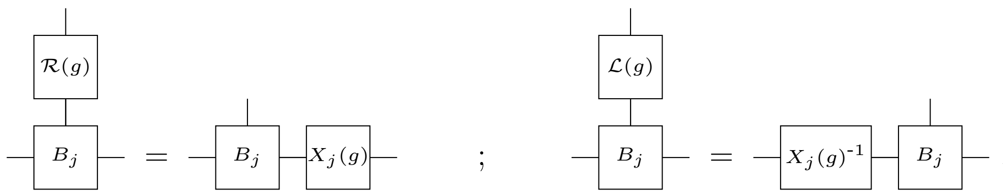
2. *$\forall a$ $\exists k$ such that $\overline{X^a_j(g)} = D^{l_k}_{\gamma^{-1}}(g)$.*

3. *$\forall a$ $\exists k$ such that $X^a_j(g) = D^{r_k}_{\gamma}(g)$.*

Proof. Recall the structure of the tensor B and the projective representation $X(g)$:

$$B^{k;m,n} = \oplus_{j=1}^{m} \oplus_{q=1}^{r_j} \mu_{j,q} B^{k;m,n}_j$$
$$X(g) = \oplus_{j=1}^{m} \oplus_{q=1}^{r_j} X_j(g) \ ,$$

where $\{B_j\}$ are normal tensors. Project Eq. (4.5) to a block j, q of the virtual space to obtain:



Let $X_j(g) = \oplus_a X_j^a(g)$ be a block of $X(g)$. We shall prove each item in the statement:

1. Let B_j^k be the projection of the tensor B_j to the k sector of the physical Hilbert space. If for a certain k there exist no a and b such that $X_j^b(g) = D_\gamma^{r_k}(g)$ and $\overline{X_j^a(g)} = D_{\gamma^{-1}}^{l_k}(g)$, then according to Proposition III, for all a, b the a, b block of B_j^k, consisting of the matrices $B_{j,a,b}^{k,m,n}$, is zero. This means B_j^k is zero.

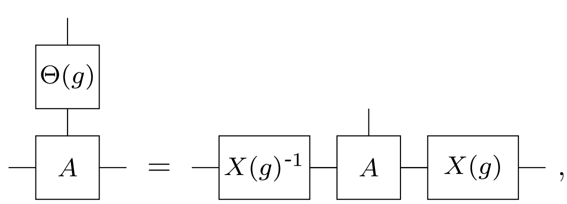
2. If there is a $Y^a(g)$ for which there is no appropriate k then according to Proposition III, $B_j^{k,m,n}$ all have a zero row which is a contradiction to the normality of B_j.

3. As in Item 2, $B_j^{k,m,n}$ now would have a zero column, which contradicts the normality of B_j.

□

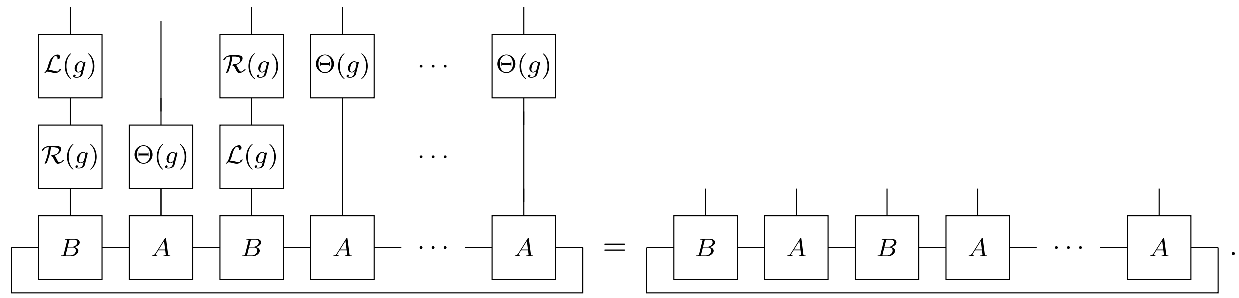
The proof of Proposition V will be presented in the next section after we derive the structure of the symmetric matter tensor A.

Proposition VI. *Let B, $\mathcal{R}(g)$ and $\mathcal{L}(g)$ be as in Theorem III and in addition let span$\{B^{k;m,n} \mid k, m, n\}$ contain the identity matrix (e.g. Eq. (4.10)). Let A and $\Theta(g)$ be such that the MPV generated by AB has a local symmetry with respect to $\mathcal{R}(g) \otimes \Theta(g) \otimes \mathcal{L}(g)$ (Definition IV). Then $|\psi_A^N\rangle$ has a global symmetry with respect to $\Theta(g)$. If in addition A is in CF with the same block structure as B (Eq. (4.4)), then A transforms as:*

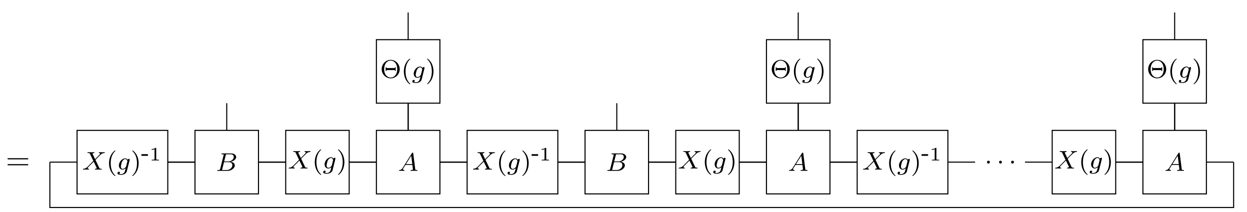


with the same $X(g)$ *from Theorem III.*

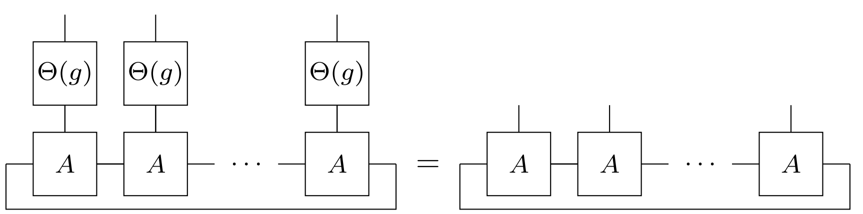
Proof. We use the local symmetry condition around every A:



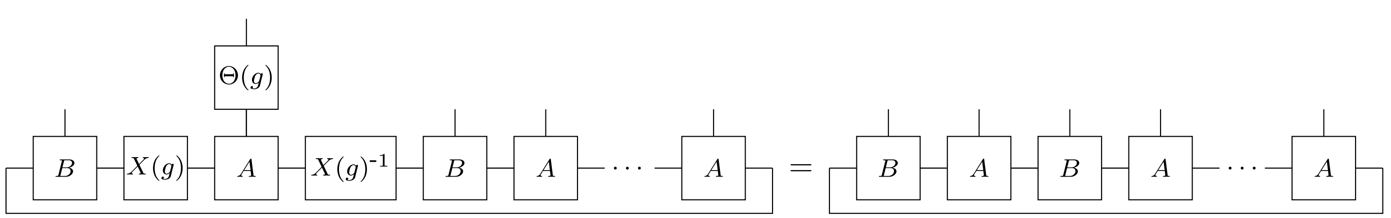
According to the transformation laws for B, the LHS of the above equals:



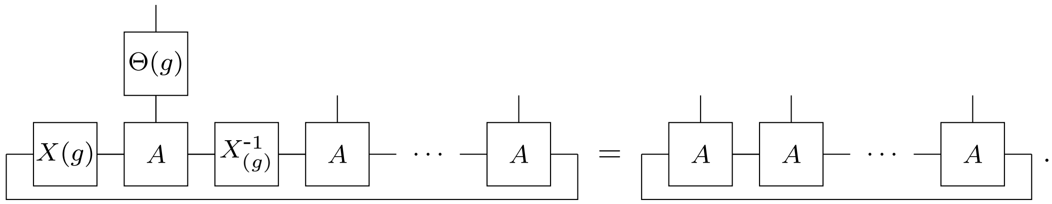
We can now use the assumption $\mathbb{I} \in span\{B^{k;m,n}\}$ to eliminate the Bs from the equation, the Xs then cancel out and we obtain the desired global symmetry:



If in addition A is in CF, we can apply Theorem I to obtain transformation relations for A. To show the rest of the claim (if A in addition has the block structure of B) we write the symmetry condition and again use the transformation rules for B:



We eliminate all Bs as before and are left with:

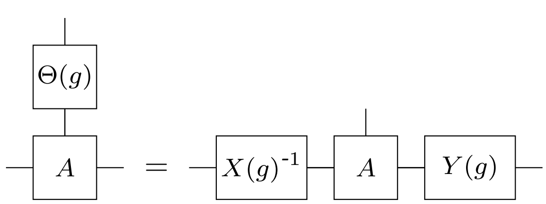


We can now use Lemma 3.1.1 with $S^i = A^i$ and $T^i = X(g)\sum_{i'}\Theta(g)_{ii'}A^{i'}X(g)^{-1}$ to finish the proof (this is where we use the assumption about the block structure of A, the crucial thing is that $X(g)$ is compatible with A's blocks as in Lemma 3.1.1). □

5.3 Matter and gauge field MPV

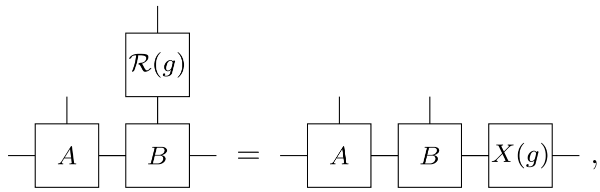
Theorem IV (Matter and gauge field MPV with a local symmetry)**.** *Let both BA and AB be normal tensors in CFII and let $\Theta(g)$ and $\mathcal{R}(g), \mathcal{L}(g)$ be unitary and projective representations (with inverse multipliers) of a group G respectively. Let $|\psi_{AB}^N\rangle$ be a MPV with a local symmetry with respect to $\mathcal{R}(g)\otimes\Theta(g)\otimes\mathcal{L}(g)$ (Definition IV). Then there exist projective representations $X(g)$ and $Y(g)$ on $\mathbb{C}^{D_1}$ and $\mathbb{C}^{D_2}$ respectively, such that $X(g)$ has the same multiplier as $\mathcal{R}(g)$, and $Y(g)$ - the inverse multiplier to that of $\mathcal{L}(g)$. The tensors A and B transform as follows:*

(4.12)



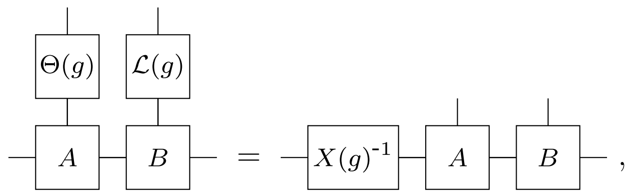
(4.13)

Proof. Apply Theorem III on the tensor AB and the representations $\tilde{\mathcal{R}}(g) := \mathbb{I} \otimes \mathcal{R}(g)$ and $\tilde{\mathcal{L}}(g) := \Theta(g) \otimes \mathcal{L}(g)$ to obtain:



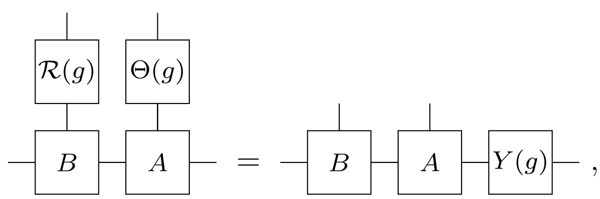
(5.6)

and



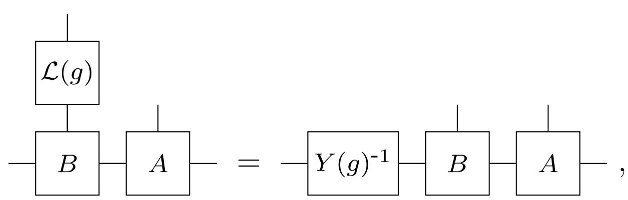
(5.7)

where $X(g)$ is a projective representation with the same multiplier as $\mathcal{R}(g)$. Apply Theorem III once more, this time on the tensor BA and the representations $\tilde{\mathcal{R}}(g) := \mathcal{R}(g) \otimes \Theta(g)$ and $\tilde{\mathcal{L}}(g) := \mathcal{L}(g) \otimes \mathbb{I}$ to obtain:



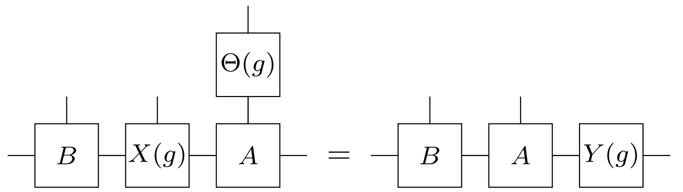
(5.8)

and



(5.9)

where $Y(g)$ is a projective representation with inverse multiplier to $\mathcal{L}(g)$. By contracting Eq. (5.6) from the left with the tensor $BA \ldots B$, and taking the appropriate linear combination to obtain the identity matrix out of the tensor $BA \ldots BA$ (using the normality of BA), we eliminate the the A in Eq. (5.6)). By contracting Eq. (5.9) with $BA \ldots B$ from the right - we eliminate the A in Eq. (5.9) (using the normality of AB). This proves the transformation rule for B - Eq. (4.12). Next plug in the transformation rules of B under $\mathcal{R}(g)$ into Eq. (5.8) to obtain:



(5.10)

Finally, eliminate the B from the equation as in the previous steps to obtain the transformation rule for A and finish the proof. □

Proposition VII. *Let $|\psi^N_{AB}\rangle$ be a MPV generated by arbitrary tensors A and B. Then there exist tensors $\{A_\chi\}$ and $\{B_\chi\}$, and there exists $b \in \mathbb{N}$ such that for all χ both $A_\chi B_\chi$ and $B_\chi A_\chi$ are normal tensors and $\forall N \in \mathbb{N}$ $|\psi^N_{AB_{\times b}}\rangle = \sum_\chi \mu_\chi^N |\psi^N_{A_\chi B_\chi}\rangle$, where μ_χ are constants and $AB_{\times b}$ is the tensor obtained by blocking b copies of the tensor AB.*

Proof. We argue similarly to [46] where it is described how to obtain, from an arbitrary tensor, a tensor in CF generating the same MPV. Begin by finding all of AB's minimal invariant subspaces S_α, such that $A^iB^jP_\alpha = P_\alpha A^iB^jP_\alpha$ for all i and j, where P_α is the orthogonal projection to S_α. Let $\hat{P}_\alpha$ be the partial isometry $\hat{P}_\alpha : \mathbb{C}^{D_1} \to S_\alpha$ such that $\hat{P}_\alpha^\dagger \hat{P}_\alpha = P_\alpha$ and $\hat{P}_\alpha \hat{P}_\alpha^\dagger = \mathbb{I}|_{S_\alpha}$. Define $A^i_\alpha := \hat{P}_\alpha A^i$ and $B^j_\alpha := B^j \hat{P}_\alpha^\dagger$. Then

$$\begin{aligned}
|\psi^N_{AB}\rangle &= \sum_{\{i\},\{j\}} \mathrm{Tr}\left(A^{i_1}B^{j_1}\dots A^{i_N}B^{j_N}\right)|i_1j_1\dots i_Nj_N\rangle \\
&= \sum_{\{i\},\{j\},\alpha} \mathrm{Tr}\left(P_\alpha A^{i_1}B^{j_1}\dots A^{i_N}B^{j_N}P_\alpha\right)|i_1j_1\dots i_Nj_N\rangle \\
&= \sum_{\{i\},\{j\},\alpha} \mathrm{Tr}\left(P_\alpha A^{i_1}B^{j_1}P_\alpha\dots P_\alpha A^{i_N}B^{j_N}P_\alpha\right)|i_1j_1\dots i_Nj_N\rangle \\
&= \sum_{\{i\},\{j\},\alpha} \mathrm{Tr}\left(\hat{P}_\alpha A^{i_1}B^{j_1}\hat{P}_\alpha^\dagger\hat{P}_\alpha\dots \hat{P}_\alpha^\dagger\hat{P}_\alpha A^{i_N}B^{j_N}\hat{P}_\alpha^\dagger\right)|i_1j_1\dots i_Nj_N\rangle \\
&= \sum_\alpha |\psi^N_{A_\alpha B_\alpha}\rangle \ .
\end{aligned}$$

Note that the bond dimension of the tensor $A_\alpha B_\alpha$ is $dim(S_\alpha)$ which is smaller than the original bond dimension D_2. Now $A_\alpha B_\alpha$ has no invariant subspaces but $B_\alpha A_\alpha$ might, therefore, perform the same for $B_\alpha A_\alpha$ - for each α find all minimal invariant subspaces $T_{\alpha\beta}$ of $B_\alpha A_\alpha$. Let $Q_{\alpha\beta}$ be the orthogonal projections to the invariant subspaces and $\hat{Q}_{\alpha\beta}$ the partial isometries. Define $A^i_{\alpha\beta} := A^i_\alpha \hat{Q}^\dagger_{\alpha\beta} = \hat{P}_\alpha A^i \hat{Q}^\dagger_{\alpha\beta}$, and $B^j_{\alpha\beta} := \hat{Q}_{\alpha\beta} B^j_\alpha = \hat{Q}_{\alpha\beta} B^j \hat{P}^\dagger_\alpha$. For each α we have

$$|\psi^N_{A_\alpha B_\alpha}\rangle = \sum_\beta |\psi^N_{A_{\alpha\beta} B_{\alpha\beta}}\rangle \ ,$$

and thus

$$|\psi^N_{AB}\rangle = \sum_\alpha |\psi^N_{A_\alpha B_\alpha}\rangle = \sum_{\alpha\beta} |\psi^N_{A_{\alpha\beta} B_{\alpha\beta}}\rangle \ .$$

Now each $A_{\alpha\beta} B_{\alpha\beta}$ might be reducible. Continue iterating this decomposition, once for AB and once for BA. Since the bond dimension of the tensors obtained at each step decreases, this procedure is bound to end after a finite number of steps. In the final step, we obtain the tensors $A^i_\chi = \hat{P}_\chi A^i \hat{Q}^\dagger_\chi$ and $B^j_\chi = \hat{Q}_\chi B^j \hat{P}^\dagger_\chi$, where χ incorporates all the previous indices, such that both $A_\chi B_\chi$ and $B_\chi A_\chi$ have no non trivial invariant subspaces. We can then perform the second step (as in [46]) which involves blocking the tensors in order to eliminate the periodicity of the associated CP maps. The blocking scheme is the following: $\tilde{A}^{ijk} := A^i B^j A^k$ and $\tilde{B}^{lmn} := B^l A^m B^n$. We can find the least common multiple of the length needed to eliminate the periodicity of all CP maps, and perform step 1 again if needed (after blocking the CP maps again become reducible [48]). We can repeat these steps as many times as needed. The process terminates at some point because the bond dimension decreases at each step. Finally, rescale the matrices $A_\chi B_\chi$ by a constant μ_χ to make the spectral radius of $E_{A_\chi B_\chi}$ and $E_{B_\chi A_\chi}$ equal to 1. The following lemma is required:

Lemma 5.3.1. *$E_{A_\chi B_\chi}$ and $E_{B_\chi A_\chi}$ have the same spectral radius.*

Proof. Let X be an eigenvector of $E_{A_\chi B_\chi}$ with eigenvalue λ: $E_{A_\chi B_\chi}(X) = E_{A_\chi} E_{B_\chi}(X) = \lambda X$.

Apply E_{B_χ} to both sides to obtain $E_{B_\chi A_\chi} E_{B_\chi}(X) = \lambda E_{B_\chi}(X)$, i.e., $E_{B_\chi}(X)$ is an eigenvector of $E_{B_\chi A_\chi}$ with eigenvalue λ. Interchanging A and B we obtain that $E_{A_\chi B_\chi}$ and $E_{B_\chi A_\chi}$ have the same spectrum, and therefore the same spectral radius. □

□

Remark 5.3.1 (Blocking of the symmetry operators). In the blocking scheme described in Proposition VII, if we start out with a MPV with a local symmetry under the operators $\mathcal{R}(g) \otimes \Theta(g) \otimes \mathcal{L}(g)$, after blocking we need to redefine the operators to act on the blocked degrees of freedom as follows: $\tilde{\mathcal{R}}(g) := \mathcal{R}(g) \otimes \Theta(g) \otimes (\mathcal{L}(g)\mathcal{R}(g))$, $\tilde{\Theta}(g) := \Theta(g) \otimes (\mathcal{L}(g)\mathcal{R}(g)) \otimes \Theta(g)$ and $\tilde{\mathcal{L}}(g) := (\mathcal{L}(g)\mathcal{R}(g)) \otimes \Theta(g) \otimes \mathcal{L}(g)$.

Proposition VIII. *Let* $|\psi^N_{AB}\rangle = \sum_\chi \mu_\chi^N |\psi^N_{A_\chi B_\chi}\rangle$ *where both* $A_\chi B_\chi$ *and* $B_\chi A_\chi$ *are normal tensors. Let* O *be a local operator acting on a fixed number of adjacent sites. If* $\forall N$ O *leaves the MPV invariant:*

$$O \otimes \mathbb{I}|_{rest} |\psi^N_{AB}\rangle = |\psi^N_{AB}\rangle \ ,$$

then O *leaves every component invariant:*

$$O \otimes \mathbb{I}|_{rest} |\psi^N_{A_\chi B_\chi}\rangle = |\psi^N_{A_\chi B_\chi}\rangle \, \forall \chi \ .$$

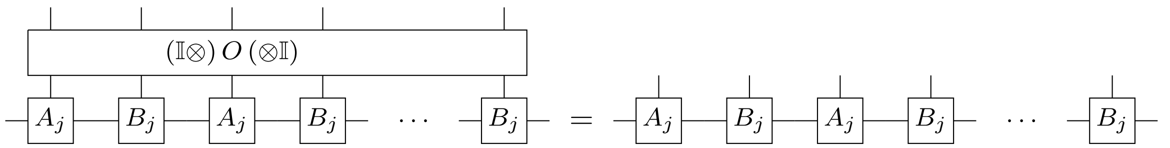
Proof. Pick a BNT $\{A_j B_j\}$ out of the normal tensors $\{A_\chi B_\chi\}$ and construct a new tensor C by blocking the tensors $\{A_\chi B_\chi\}$ diagonally (possibly changing the order of the blocks):

$$C^{ii'} = \oplus_\chi \mu_\chi A^i_\chi B^{i'}_\chi = \oplus_j \oplus_q \mu_{j,q} V^{-1}_{j,q} A^i_j B^{i'}_j V_{j,q} \ ,$$

where for every χ there is a j and a q such that $\mu_\chi A_\chi B_\chi = \mu_{j,q} V^{-1}_{j,q} A^i_j B^{i'}_j V_{j,q}$. Now C is in CF and generates the same MPV as AB. We have

$$O|\psi^N_C\rangle = O|\psi^N_{AB}\rangle = |\psi^N_{AB}\rangle = |\psi^N_C\rangle \ .$$

We can now use Lemma 3.1.1 (use Eq. (3.7) from the proof of the lemma) for the tensor $C = AB$ to obtain



where the operator in the box contains O (we need to extend it by at most one $\otimes\mathbb{I}$ from the right and from the left in order to occupy a full $AB \dots AB$ block). Finally, we have

$$O|\psi^N_{A_\chi B_\chi}\rangle = O|\psi^N_{V_{j,q}^{-1}A_j B_j V_{j,q}}\rangle = |\psi^N_{A_j B_j}\rangle = |\psi^N_{A_\chi B_\chi}\rangle$$

□

Recall the definition of an elementary A block:

Definition 4.3.2 (Elementary A block)**.** An elementary block of the tensor A is one which satisfies Eq. (4.13), where $\Theta(g)$, $X(g)$ and $Y(g)$ are all irreducible projective representations.

Proposition IX. *Let A be an elementary block (Definition 4.3.2), with $\Theta(g) = D^{J_0}(g)$, $X(g) = D^j_\gamma(g)$ and $Y(g) = D^l_{\gamma^{-1}}(g)$. Then A is built out of Clebsch-Gordan coefficients and has the form:*

$$A^M = \sum_{J\in\mathfrak{J}:D^J=D^{J_0}} \alpha_J \sum_{m,n} \left\langle J, M \mid \bar{j}, m; l, n \right\rangle |m\rangle\langle n| \ ,$$

where $\mathfrak{J}$ is the set of irreducible representation indices appearing in the decomposition of $\overline{D^j_\gamma(g)} \otimes D^l_{\gamma^{-1}}(g)$ into irreducible representations, $\left\langle \bar{j}, m : l, n \mid J, M \right\rangle$ are the Clebsch-Gordan coefficients of the decomposition, $\overline{D^j_\gamma(g)}$ is the complex conjugate representation to $D^j_\gamma(g)$ and α_J are arbitrary constants.

Proof. Write out Eq. (4.13):

$$\sum_{i'} \Theta(g)_{ii'} A^{i'} = X(g)^{-1} A^i Y(g) \ .$$

Taking the complex conjugate of both sides

$$\sum_{i'} \Theta(g^{-1})_{i'i} \overline{A^{i'}} = \overline{X(g)^{-1} A^i Y(g)}$$

we see that $\overline{\vec{A}}$ satisfies Eq. (3.12) for $\vec{v} = \vec{e^i}$ and the group element g^{-1}, with $\kappa = \Theta(g), \pi = \overline{X(g)}$ and $\eta = \overline{Y(g)}$. Therefore $\overline{\vec{A}}$ is a vector operator with respect to the above representations. In the case when $\Theta(g) = D^{J_0}(g), X(g) = D^j_\gamma(g)$ and $Y(g) = D^l_{\gamma^{-1}}(g)$ are irreducible representations, according to Theorem 3.2.1 $\overline{A}$ is of the form:

$$\overline{A^M} = \sum_{J: D^j(g) = D^{J_0}(g)} \alpha_J \sum_{m,n} \left\langle \bar{j}, m; l, n \mid J, M \right\rangle |m\rangle\langle n| \ ,$$

taking the complex conjugate, we find the desired form of A. □

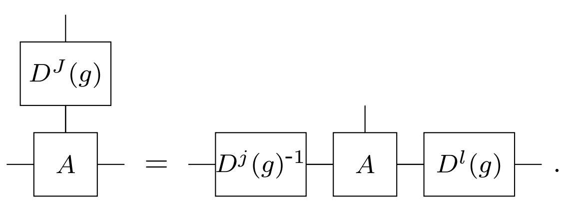
Example 5.3.1. A direct calculation using the Clebsch-Gordan series [51]:

$$D^j(g)_{m,m'} D^l(g)_{n,n'} = \sum_{L,N,N'} \langle j, m; l, n \mid L, N\rangle \langle L, N' \mid j, m'; l, n'\rangle D^l(g)_{N,N'}$$

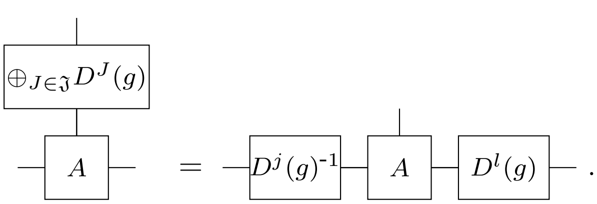
shows that the tensor composed of the matrices

$$A^{J,M} = \sum_{m,n} \left\langle J, M \mid \bar{j}, m; l, n \right\rangle |m\rangle\langle n| \ ,$$

for a fixed value of J, satisfies



Consequently, the tensor composed out of all matrices $\{A^{J,M}\}_{J\in\mathfrak{J},M}$ (all J appearing in the decomposition $\overline{D^j(g)} \otimes D^l(g) = \oplus_{J\in\mathfrak{J}} D^J(g)$) satisfies:

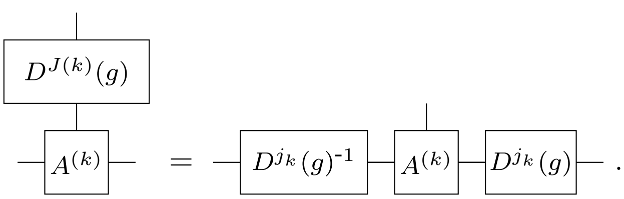


In addition to being a symmetric tensor, this tensor is always injective: let $D := dim(j) = dim(l)$. Due to the fact that the C-G coefficients are the entries of a unitary matrix, the matrices $A^{J,M}$ satisfy $\text{Tr}\left(A^{J,M\dagger}A^{J',M'}\right) = \delta_{J,J'}\delta_{M,M'}$. Since there are $D \times D$ of them, they form an ONB of the space of $D \times D$ matrices.

We can now prove the following proposition, the proof of which we postponed in the previous section.

Proposition V. *Let B be in CFII and let $|\psi_B^N\rangle$ have a local symmetry with respect to $\mathcal{R}(g) \otimes \mathcal{L}(g)$ (as in Theorem III). It is always possible to find a tensor A and a representation $\Theta(g)$ such that the corresponding matter and gauge field MPV $|\psi_{AB}^N\rangle$ has a local symmetry with respect to $\mathcal{R}(g) \otimes \Theta(g) \otimes \mathcal{L}(g)$ (Definition IV). In addition, the corresponding matter MPV - $|\psi_A^N\rangle$ - has a global symmetry with respect to $\Theta(g)$.*

Proof. For each $D_\gamma^{j_k}(g)$ appearing in $X(g) = \oplus_{k=1}^{s} D_\gamma^{j_k}(g)$, let $J(k)$ be an irreducible representation index appearing in the decomposition of $\overline{D_\gamma^{j_k}(g)} \otimes D_\gamma^{j_k}(g)$. Let $A^{(k)}$ be the tensor presented in Example 5.3.1, satisfying



Let the matter Hilbert space be $\mathcal{H}_A := \oplus_k \mathcal{H}_{J(k)}$. Let the tensor A in each sector $J(k)$ of the physical space be zero except for in the k,k virtual block, such that:

$$\left[X^{-1}(g)A^{J_k,M}X(g)\right]_{l,l'} = \delta(l,k)\delta(l',k)D^{J_k}_{M,M'}(g)A^{(k)J_k,M'} \ .$$

□

Proposition X. *Let AB and BA be normal tensors and let B satisfy Eq. (4.12) with $\mathcal{R}(g) = \oplus_k(\mathbb{I} \otimes D_\gamma^{r_k}(g))$, $\mathcal{L}(g) = \oplus_k(D_{\gamma^{-1}}^{l_k}(g) \otimes \mathbb{I})$, $Y(g) = \oplus_a Y^a(g)$ and $X(g) = \oplus_b X^b(g)$, where $D_\gamma^{r_k}$, $D_{\gamma^{-1}}^{l_k}$, Y^a and X^b are irreducible projective representations, then*

1. *For all k either there exist a and b such that $X^b(g) = D_{\gamma}^{r_k}(g)$ and $\overline{Y^a(g)} = D_{\gamma^{-1}}^{l_k}(g)$ or the projection of the tensor B to the sector k of the physical space is zero (and it can be discarded).*

2. *$\forall a\ \exists k$ such that $\overline{Y^a(g)} = D_{\gamma^{-1}}^{l_k}(g)$.*

3. *$\forall b\ \exists k$ such that $X^b(g) = D_{\gamma}^{r_k}(g)$.*

Proof. 1. Assume the contrary is true, then according to Proposition III, $B^{k,m,n}$ are all zero and this value of k does not contribute to the MPV.

2. If there is a $Y^a(g)$ for which there is not an appropriate k then according to Proposition III, $B^{k,m,n}$ all have a zero row which is a contradiction to the normality of AB.

3. As in Item 2, $B^{k,m,n}$ now would have a zero column and would contradict normality of BA.

□

Proposition XI. *There exist tensors A and B such that $|\psi_{AB}\rangle$ has a local symmetry with respect to $\mathcal{R}(g)\otimes\Theta(g)\otimes\mathcal{L}(g)$, but $|\psi_A\rangle$ does not have a global symmetry with respect to $\Theta(g)$. In addition $\mathcal{R}(g)\otimes\mathcal{L}(g)|\psi_B\rangle \neq |\psi_B\rangle$.*

The proof is given by the following example:

Example 5.3.2. Let $G = D_{10}$ the dihedral group of order 10. It is the group generated by two elements: r and s satisfying $r^5 = s^2 = (sr)^2 = e$. D_{10} has two inequivalent two dimensional irreducible representations ρ_1 and ρ_2 generated by:

$$\rho_1 : r \mapsto R_1 := \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$
$$s \mapsto S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\rho_2 : r \mapsto R_2 := \begin{pmatrix} e^{i2\theta} & 0 \\ 0 & e^{-i2\theta} \end{pmatrix}$$
$$s \mapsto S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

where $\theta = 2\pi/5$. The tensor product $\overline{\rho_1} \otimes \rho_2$ decomposes into $\rho_1 \oplus \rho_2$:

$$\overline{\rho_1} \otimes \rho_2 : r \mapsto R_1 \otimes R_2 = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{-i3\theta} & 0 & 0 \\ 0 & 0 & e^{i3\theta} & 0 \\ 0 & 0 & 0 & e^{-i\theta} \end{pmatrix}$$
$$s \mapsto S \otimes S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

It is clear from inspection of the above 4×4 matrices that the unitary transformation realizing the direct sum decomposition is a permutation of the basis elements, the non zero

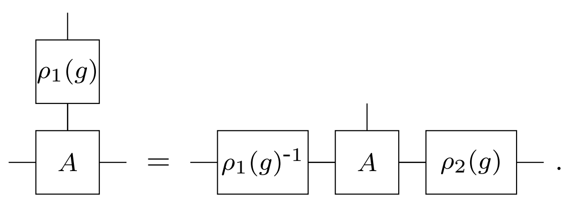
Clebsch-Gordan coefficients are:

$$\begin{aligned}
\langle \rho_1, 1 \,|\, \overline{\rho_1}, 1; \rho_2, 1\rangle &= 1\\
\langle \rho_1, 2 \,|\, \overline{\rho_1}, 2; \rho_2, 2\rangle &= 1\\
\langle \rho_2, 1 \,|\, \overline{\rho_1}, 1; \rho_2, 2\rangle &= 1\\
\langle \rho_2, 2 \,|\, \overline{\rho_1}, 2; \rho_2, 1\rangle &= 1 \ .
\end{aligned}$$

Following Example 5.3.1, and using these coefficients, define the tensor A:

$$A^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

A satisfies:



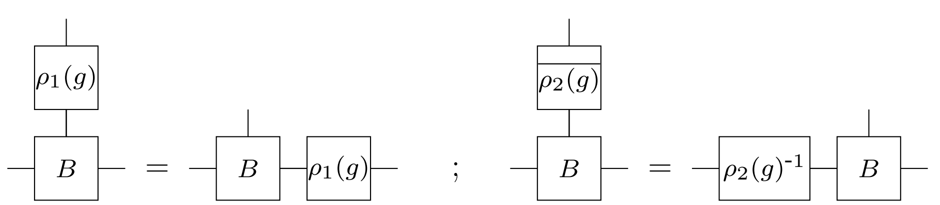
(5.11)

According to Proposition III the following tensor B:

$$B^{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad B^{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$B^{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad B^{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

satisfies:



(5.12)

Eq. (5.11)) and Eq. (5.12) are easily verified for the generators of the group, r and s, and therefore hold for any group element. From these equations it follows that $|\psi_{AB}^N\rangle$ has a local symmetry (Definition IV with $\mathcal{R}(g) = \rho_1(g)$, $\Theta(g) = \rho_1(g)$ and $\mathcal{L}(g) = \overline{\rho_2(g)}$); however, ρ_1 is not a global symmetry for $|\psi_A^N\rangle$, as is easily verified for a MPV of length 1. Similarly, a direct computation shows $\mathcal{R}(g) \otimes \mathcal{L}(g)|\psi_B^2\rangle \neq |\psi_B^2\rangle$.

Proposition XII. *Let A be a tensor in CFII generating a MPV with a global symmetry i.e., satisfying Theorem I. Let $X(g)$ (in Eq.* (3.14)*) be a projective representation (i.e. all $X_j(g)$ in Eq.* (3.15) *are in the same cohomology class). Then there exist a tensor B and projective representations $\mathcal{R}(g)$ and $\mathcal{L}(g)$ with inverse multipliers such that both local symmetries: Definition IV for $|\psi_{AB}^N\rangle$ and Definition III for $|\psi_B^N\rangle$ are satisfied.*

Proof. As $X(g)$ appears in Eq. (3.14) together with its inverse, it is defined only up to a phase. As we assumed all $X_j(g)$ are from the same cohomology class, we can lift each one of them to be projective representations with the same multiplier γ. We can assume without loss of generality (same argument as in Remark 4.3.3) that each $X_j(g)$ is block diagonal: $X(g) = \oplus_j \oplus_q \oplus_{a_j} D_\gamma^{a_j}(g)$. Set $\mathcal{R}(g) = X(g)$, $\mathcal{L}(g) = \overline{X(g)}$ and let B be completely block diagonal:

$$B^{j,q,a_j;m,n} = |j,q,a_j;m\rangle\langle j,q,a_j;n| \ ,$$

i.e., for each irreducible block of $X(g)$ there is a corresponding sector in $\mathcal{H}_B$:

$$\mathcal{H}_B = \oplus_j \oplus_q \oplus_{a_j} \mathcal{H}_{\overline{a_j}} \otimes \mathcal{H}_{a_j} \ ,$$

where $\overline{a_j}$ is the complex conjugate representation to a_j. □

Example 5.3.3 (An $SU(2)$ gauge invariant MPV)**.** For $G = SU(2)$ we demonstrate the construction of a general locally invariant MPV emphasizing the constituents of physical theories and relating our setting and notation to [35,40]. Write the irreducible representations $D^j(g)$ in terms of their generators:

$$D^j(g) = \exp\left(i \sum_a \tau_a^j \varphi_a(g)\right), \ \forall g \in SU(2),$$

where $\{\varphi_a(g)\}_{a=1}^3$ are real parameters and $\{\tau_a^j\}_{a=1}^3$ are Hermitian $(2j+1)\times(2j+1)$ matrices satisfying the $\mathfrak{su}(2)$ Lie algebra relations:

$$\left[\tau_a^j, \tau_b^j\right] = i\varepsilon_{abc}\tau_c^j \ ,$$

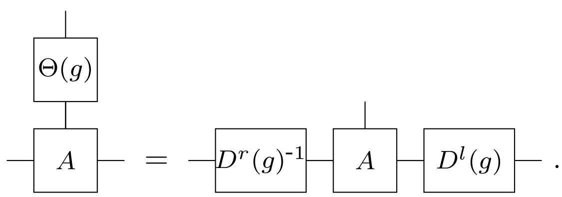
where ε_{abc} is the totally antisymmetric tensor. Let D^r and D^l be two irreducible representations of $SU(2)$ and let $\mathfrak{J}_0$ be the set of irreducible representation indices appearing in the decomposition of the tensor product: $\overline{D^r(g)} \otimes D^l(g) \cong \oplus_{J\in\mathfrak{J}_0} D^J(g)$. Let $\mathfrak{J} \subseteq \mathfrak{J}_0$. Define the representation $\Theta(g)$ as generated by $\{Q_a := \bigoplus_{J\in\mathfrak{J}} \tau_a^J\}_{a=1}^3$:

$$\Theta(g) = \bigoplus_{J\in\mathfrak{J}} D^J(g) = \bigoplus_{J\in\mathfrak{J}} \exp\left(i\sum_a \tau_a^J \varphi_a(g)\right) = \exp\left(i\sum_a Q_a\varphi_a(g)\right) \ .$$

As in Example 5.3.1, the tensor A, defined by the matrices:

$$A^{J,M} = \sum_{m,n} \alpha_J \left\langle J, M \mid \overline{r}, m; l, n\right\rangle |m\rangle\langle n| \ , J \in \mathfrak{J}, M = 1, \ldots, dim(J) \tag{5.13}$$

satisfies:



This relation, written in terms of the generators, reads:

$$\sum_{M'} \left[\exp\left(i\sum_a \tau_a^J\varphi_a(g)\right)\right]_{M,M'} A^{J,M'} = \exp\left(-i\sum_a \tau_a^r\varphi_a(g)\right) A^{J,M} \exp\left(i\sum_a \tau_a^l\varphi_a(g)\right) \ .$$

Differentiating this equation with respect to any one of the group parameters φ_a we obtain the "virtual Gauss law" satisfied by A:

$$Q_a : A^{J,M} \mapsto \sum_{M'} \left[\tau_a^J\right]_{M,M'} A^{J,M'} = -\tau_a^r A^{J,M} + A^{J,M}\tau_a^l \ .$$

Next, add a gauge field degree of freedom to the matter MPV, described by a tensor: $B^{m,n} = |m\rangle\langle n|$, and define the transformations:

$$\mathcal{R}(g) = \mathbb{I} \otimes D^r(g) \quad ; \quad \mathcal{L}(g) = \overline{D^l(g)} \otimes \mathbb{I} \ .$$

The action of $\mathcal{L}(g)$ on the gauge field Hilbert space is given by:

$$\mathcal{L}(g)|m,n\rangle = (\overline{D^l(g)}\otimes \mathbb{I})|m,n\rangle = \sum_{m'} \overline{D^l(g)}_{m',m}|m',n\rangle = \sum_{m'} D^l(g^{-1})_{m,m'}|m',n\rangle \ ;$$

whereas $\mathcal{R}(g)$ acts as:

$$\mathcal{R}(g)|m,n\rangle = \sum_{n'} D^r(g)_{n',n}|m,n'\rangle \ .$$

$\mathcal{R}(g)$ and $\mathcal{L}(g)$ can be defined in terms of right and left generators $\{R_a\}_{a=1}^3$ and $\{L_a\}_{a=1}^3$, as described in Section 4.2:

$$\mathcal{R}(g) = \exp\left(i\sum_a R_a\varphi_a(g)\right)$$
$$\mathcal{L}(g) = \exp\left(i\sum_a L_a\varphi_a(g)\right) \ .$$

In our case R_a is simply given by $\mathbb{I}\otimes\tau_a^r$ but in general R_a and L_a can have a block diagonal structure. Define the generators of the local gauge transformation around lattice site $2K+1$:

$$G_a^{[2K+1]} := \left(R_a^{[2K]} + Q_a^{[2K+1]} + L_a^{[2K+2]}\right) \ .$$

From our construction it follows that for all $g\in G$ and for all lattice sites K:

$$\mathcal{R}^{[2K]}(g)\otimes\Theta^{[2K+1]}(g)\otimes\mathcal{L}^{[2K+2]}(g)|\psi_{AB}^N\rangle = |\psi_{AB}^N\rangle \ .$$

Once again, differentiating with respect to the group parameters φ_a we obtain:

$$\left(R_a^{[2K]} + Q_a^{[2K+1]} + L_a^{[2K+2]}\right)|\psi_{AB}^N\rangle = G_a^{[2K+1]}|\psi_{AB}^N\rangle = 0 \ . \tag{5.14}$$

This is the lattice version of Gauss' law. In physical theories $D^l = \overline{D^r}$ and thus states $|\psi_A\rangle$ have a global symmetry generated by $\{Q_a\}$ - the $SU(2)$ charge operators. R_a and L_a are identified with right and left electric fields respectively [40].

One could generalize the above construction for

$$\mathcal{R}(g) = \oplus_k\left(\mathbb{I}\otimes D^{r_k}(g)\right) \quad ; \quad \mathcal{L}(g) = \oplus_k\left(\overline{D^{l_k}(g)}\otimes\mathbb{I}\right)$$

by constructing A and B as above for each k sector and combining them together block diagonally (in both physical and virtual dimensions). Duplicating the virtual representations while keeping the physical ones fixed can be achieved by $B^{m,n} \mapsto (B^{m,n} \oplus B^{m,n})$, $A^{J,M} \mapsto (A_1^{J,M} \oplus A_2^{J,M})$. This can be used to enlarge the number of variational parameters. The tensors A_1 and A_2 must both have the same structure (Eq. (5.13)) but can have different parameters α_J. The generalization to of the above to $G = SU(N)$ is straightforward.

Chapter 6

Summary

In this work, we studied and classified translationally invariant MPVs with a local (gauge) symmetry under arbitrary groups. The states we classified may involve two types of building blocks, A and B tensors, which represent matter and gauge fields respectively. We studied three physically relevant settings: mass field, pure gauge field and combined mass and gauge field states. In each one of the settings the analysis method was the same, and can be summarized as follows: first, we identified the form of the tensors appropriate for the setting in question (e.g. canonical form); next, we derived transformation relations satisfied by the tensors; finally, we used the transformation relations to derive the structure of the tensors. Note that while the second step relied mostly on established MPS theory, the third step was almost entirely group theoretical.

We showed that matter-only MPVs may only have a local symmetry, when one transforms a single site, if they are trivial (composed of products of invariant states at each site). This result, although expected, motivates the introduction of an additional degree of freedom, or alternatively, the inspection of the setting where there are two distinct operators (right and left ones) that act on the same degree of freedom.

Consequently, we studied and classified pure gauge states, which involve only B tensors and have local invariance when one transforms two neighboring sites. The B tensor is composed of elementary blocks (intertwining irreducible representations), each of which intertwining the left (right) physical transformation with a group action on the virtual space to the left (right) of the tensor. This property defines the structure of such elementary blocks up to a constant. The structure of well-known physical states involving only gauge fields is a particular case of the general structure that we found. We showed how to construct tensors A describing matter fields that can be coupled to the general B tensor and result in an overall gauge invariant state. We further showed that any matter field that can be coupled (in the same sense) to a gauge field described by a B tensor which corresponds to the well-known physical case, must have a global symmetry. So far our findings described generalizations of constructions common in conventional lattice gauge theory settings, where one starts from one degree of freedom with a symmetry (either matter field with a global symmetry or gauge field with a local symmetry) and to it couples the other one. These constructions, however, do not cover all possible gauge invariant states involving both types of degrees of freedom.

In the combined matter and gauge field setting we found familiar structures for A and for B: the structure of A resembles the one known from the classification of MPV with a global symmetry; the structure of B is similar to the pure gauge field case; however, in the general case A and B intertwine representations in such a way that symmetry is only observed when they are coupled together. When considered on their own, the matter field does not posses a global symmetry, and the gauge field does not have a local one. In this sense we expanded the class of gauge invariant states, and classified the structure of such MPVs as well. We have shown an example of such a state, which, aside from providing a proof of existence of such cases, provides a demonstration of how the tensors involved are constructed. We also showed how, and under which conditions, a global symmetry of a matter MPV can be gauged by adding a gauge field degree of freedom.

In both settings involving the gauge field, we found that the Hilbert space describing the gauge field degree of freedom must have a specific structure: it is a direct sum of spaces, each one of which is a tensor product of a left part and a right part. This structure is the only one which allows the gauge field to have two different group representations act on it independently, as required in order to have local symmetry of a MPV. The well-known Kogut-Susskind Hilbert space is a particular case of this more general structure.

Further work shall include a generalization to more dimensions, i.e. using PEPS. In our work we were able to connect some of the results to the symmetry properties and structure of previous gauge invariant PEPS constructions [3,33,35] when the space dimension is reduced to one, and therefore higher dimensional generalizations in the spirit of the current work should be possible. In particular, the tensor describing the gauge field, as it resides on the links of a lattice, is a one dimensional object for any spatial dimension, and has shown, in some particular cases, properties known from previous PEPS studies. In the 2D case, as a classification of global symmetry for injective PEPS is available [12], we suspect that the same methods used in this work can be applied with few modifications in deriving transformation relations for the tensors; from the group theoretical aspect, the question of the uniqueness of the A tensor constructed in [35] might require a generalization of the Wigner-Eckart theorem to higher rank tensors. Another important generalization one should consider is a fermionic representation of the matter, combining the spirit of this work with previous works on fermionic PEPS with gauge symmetry [34,36] or with global symmetry [53,54]. From the physical point of view, a physical study aiming at understanding the new classes of gauge invariant states introduced in this work, in which the matter and gauge field do not posses separate symmetries, may also potentially unfold new physical phenomena and phases.

Acknowledgements

The author would like to thank Andras Molnar for many fruitful discussions and for proof-reading; and is grateful to Erez Zohar and J. Ignacio Cirac for their guidance, patience, advice and suggestions.

The author acknowledges the support of the DAAD.

Bibliography

[1] M. E. Peskin, D. V. Schroeder, An Introduction to Quantum Field Theory; 1995 ed., Westview, Boulder, CO, 1995.
URL https://cds.cern.ch/record/257493

[2] E. Fradkin, Field Theories of Condensed Matter Physics, 2nd Edition, Cambridge University Press, 2013.

[3] J. Haegeman, K. Van Acoleyen, N. Schuch, J. I. Cirac, F. Verstraete, Gauging quantum states: from global to local symmetries in many-body systems, Phys. Rev. X5 (1) (2015) 011024. arXiv:1407.1025, doi:10.1103/PhysRevX.5.011024.

[4] K. G. Wilson, Confinement of Quarks, Phys. Rev. D10 (1974) 2445–2459, [,45(1974)]. doi:10.1103/PhysRevD.10.2445.

[5] M. Troyer, U.-J. Wiese, Computational complexity and fundamental limitations to fermionic quantum Monte Carlo simulations, Phys. Rev. Lett. 94 (2005) 170201. arXiv:cond-mat/0408370, doi:10.1103/PhysRevLett.94.170201.

[6] J. Kogut, L. Susskind, Hamiltonian formulation of wilson's lattice gauge theories, Phys. Rev. D 11 (1975) 395–408. doi:10.1103/PhysRevD.11.395.
URL https://link.aps.org/doi/10.1103/PhysRevD.11.395

[7] E. Zohar, J. I. Cirac, B. Reznik, Quantum Simulations of Lattice Gauge Theories using Ultracold Atoms in Optical Lattices, Rept. Prog. Phys. 79 (1) (2016) 014401. arXiv:1503.02312, doi:10.1088/0034-4885/79/1/014401.

[8] U.-J. Wiese, Ultracold Quantum Gases and Lattice Systems: Quantum Simulation of Lattice Gauge Theories, Annalen Phys. 525 (2013) 777–796. arXiv:1305.1602, doi:10.1002/andp.201300104.

[9] S. R. White, Density matrix formulation for quantum renormalization groups, Physical Review Letters 69 (19) (1992) 2863–2866. doi:10.1103/PhysRevLett.69.2863.
URL http://link.aps.org/doi/10.1103/PhysRevLett.69.2863

[10] F. Verstraete, V. Murg, J. I. Cirac, Matrix product states, projected entangled pair states, and variational renormalization group methods for quantum spin systems 57 (2) 143–224. arXiv:0907.2796, doi:10.1080/14789940801912366.
URL http://arxiv.org/abs/0907.2796

[11] F. Verstraete, J. I. Cirac, Matrix product states represent ground states faithfully 73 094423. arXiv:0910.1130, doi:10.1103/PhysRevB.73.094423.
URL http://arxiv.org/abs/cond-mat/0505140v6

[12] D. Pérez-García, M. Sanz, C. E. Gonzalez-Guillen, M. M. Wolf, J. I. Cirac, A canonical form for Projected Entangled Pair States and applications 12 025010. doi:10.1088/1367-2630/12/2/025010.
URL http://arxiv.org/abs/0908.1674v1;http://arxiv.org/pdf/0908.1674v1

[13] M. Bañuls, K. Cichy, J. Cirac, K. Jansen, The mass spectrum of the schwinger model with matrix product states, Journal of High Energy Physics 2013 (11) (2013) 158.

doi:10.1007/JHEP11(2013)158.
URL https://doi.org/10.1007/JHEP11(2013)158

[14] Y. Shimizu, Y. Kuramashi, Grassmann tensor renormalization group approach to one-flavor lattice schwinger model, Phys. Rev. D 90 (2014) 014508. doi:10.1103/PhysRevD.90.014508.
URL https://link.aps.org/doi/10.1103/PhysRevD.90.014508

[15] B. Buyens, J. Haegeman, K. Van Acoleyen, H. Verschelde, F. Verstraete, Matrix product states for gauge field theories, Phys. Rev. Lett. 113 (2014) 091601. doi:10.1103/PhysRevLett.113.091601.
URL https://link.aps.org/doi/10.1103/PhysRevLett.113.091601

[16] P. Silvi, E. Rico, T. Calarco, S. Montangero, Lattice gauge tensor networks, New Journal of Physics 16 (10) (2014) 103015.
URL http://stacks.iop.org/1367-2630/16/i=10/a=103015

[17] B. Buyens, K. Van Acoleyen, J. Haegeman, F. Verstraete, Matrix product states for Hamiltonian lattice gauge theories, ArXiv e-printsarXiv:1411.0020.

[18] E. Rico, T. Pichler, M. Dalmonte, P. Zoller, S. Montangero, Tensor networks for lattice gauge theories and atomic quantum simulation, Phys. Rev. Lett. 112 (2014) 201601. doi:10.1103/PhysRevLett.112.201601.
URL https://link.aps.org/doi/10.1103/PhysRevLett.112.201601

[19] H. Saito, M. C. Bañuls, K. Cichy, J. I. Cirac, K. Jansen, The temperature dependence of the chiral condensate in the Schwinger model with Matrix Product States, PoS LATTICE2014 (2014) 302. arXiv:1412.0596.

[20] S. Kühn, E. Zohar, J. I. Cirac, M. C. Bañuls, Non-abelian string breaking phenomena with matrix product states, Journal of High Energy Physics 2015 (7) (2015) 130. doi:10.1007/JHEP07(2015)130.
URL https://doi.org/10.1007/JHEP07(2015)130

[21] M. C. Bañuls, K. Cichy, J. I. Cirac, K. Jansen, H. Saito, Thermal evolution of the schwinger model with matrix product operators, Phys. Rev. D 92 (2015) 034519. doi:10.1103/PhysRevD.92.034519.
URL https://link.aps.org/doi/10.1103/PhysRevD.92.034519

[22] M. C. Bañuls, K. Cichy, K. Jansen, H. Saito, Chiral condensate in the schwinger model with matrix product operators, Phys. Rev. D 93 (2016) 094512. doi:10.1103/PhysRevD.93.094512.
URL https://link.aps.org/doi/10.1103/PhysRevD.93.094512

[23] T. Pichler, M. Dalmonte, E. Rico, P. Zoller, S. Montangero, Real-time dynamics in u(1) lattice gauge theories with tensor networks, Phys. Rev. X 6 (2016) 011023. doi:10.1103/PhysRevX.6.011023.
URL https://link.aps.org/doi/10.1103/PhysRevX.6.011023

[24] B. Buyens, J. Haegeman, F. Hebenstreit, F. Verstraete, K. Van Acoleyen, Real-time simulation of the Schwinger effect with Matrix Product StatesarXiv:1612.00739.

[25] P. Silvi, E. Rico, M. Dalmonte, F. Tschirsich, S. Montangero, Finite-density phase diagram of a (1+1)-d non-abelian lattice gauge theory with tensor networksarXiv:1606.05510, doi:10.22331/q-2017-04-25-9.

[26] A. Milsted, Matrix product states and the non-abelian rotor model, Phys. Rev. D 93 (2016) 085012. doi:10.1103/PhysRevD.93.085012.
URL https://link.aps.org/doi/10.1103/PhysRevD.93.085012

[27] M. C. Bañuls, K. Cichy, J. I. Cirac, K. Jansen, S. Kühn, H. Saito, The multi-flavor Schwinger model with chemical potential - Overcoming the sign problem with Matrix Product States, PoS LATTICE2016 (2016) 316. arXiv:1611.01458.

[28] B. Buyens, F. Verstraete, K. Van Acoleyen, Hamiltonian simulation of the schwinger model at finite temperature, Phys. Rev. D 94 (2016) 085018. doi:10.1103/PhysRevD.94.085018.
URL https://link.aps.org/doi/10.1103/PhysRevD.94.085018

[29] H. Saito, M. C. Bañuls, K. Cichy, J. I. Cirac, K. Jansen, Thermal evolution of the one-flavour Schwinger model with using Matrix Product States, PoS LATTICE2015 (2016) 283. arXiv:1511.00794.

[30] M. C. Bañuls, K. Cichy, J. I. Cirac, K. Jansen, S. Kühn, Density induced phase transitions in the schwinger model: A study with matrix product states, Phys. Rev. Lett. 118 (2017) 071601. doi:10.1103/PhysRevLett.118.071601.
URL https://link.aps.org/doi/10.1103/PhysRevLett.118.071601

[31] M. C. Bañuls, K. Cichy, J. Ignacio Cirac, K. Jansen, S. Kühn, H. Saito, Towards overcoming the Monte Carlo sign problem with tensor networks, in: European Physical Journal Web of Conferences, Vol. 137 of European Physical Journal Web of Conferences, 2017, p. 04001. arXiv:1611.04791, doi:10.1051/epjconf/201713704001.

[32] M. C. Bañuls, K. Cichy, J. I. Cirac, K. Jansen, S. Kühn, Efficient basis formulation for 1+1 dimensional SU(2) lattice gauge theory: Spectral calculations with matrix product statesarXiv:1707.06434.

[33] L. Tagliacozzo, A. Celi, M. Lewenstein, Tensor networks for lattice gauge theories with continuous groups, Phys. Rev. X 4 (2014) 041024. doi:10.1103/PhysRevX.4.041024. URL https://link.aps.org/doi/10.1103/PhysRevX.4.041024

[34] E. Zohar, M. Burrello, T. Wahl, J. I. Cirac, Fermionic Projected Entangled Pair States and Local U(1) Gauge Theories, Annals Phys. 363 (2015) 385–439. arXiv:1507.08837, doi:10.1016/j.aop.2015.10.009.

[35] E. Zohar, M. Burrello, Building Projected Entangled Pair States with a Local Gauge Symmetry, New J. Phys. 18 (4) (2016) 043008. arXiv:1511.08426, doi:10.1088/1367-2630/18/4/043008.

[36] E. Zohar, T. B. Wahl, M. Burrello, J. I. Cirac, Projected Entangled Pair States with non-Abelian gauge symmetries: an SU(2) study, Annals Phys. 374 (2016) 84–137. arXiv:1607.08115, doi:10.1016/j.aop.2016.08.008.

[37] M. Sanz, M. M. Wolf, D. Pérez-García, J. I. Cirac, Matrix product states: Symmetries and two-body hamiltonians, Phys. Rev. A 79 (2009) 042308. doi:10.1103/PhysRevA.79.042308.
URL https://link.aps.org/doi/10.1103/PhysRevA.79.042308

[38] F. Pollmann, A. M. Turner, E. Berg, M. Oshikawa, Entanglement spectrum of a topological phase in one dimension, prb 81 (6) (2010) 064439. arXiv:0910.1811, doi:10.1103/PhysRevB.81.064439.

[39] V. K. Agrawala, Wigner–Eckart theorem for an arbitrary group or lie algebra, Journal of Mathematical Physics 21 (7) (1980) 1562–1565. `arXiv:http://dx.doi.org/10.1063/1.524639`, `doi:10.1063/1.524639`.
URL `http://dx.doi.org/10.1063/1.524639`

[40] E. Zohar, M. Burrello, Formulation of lattice gauge theories for quantum simulations, Phys. Rev. D91 (5) (2015) 054506. `arXiv:1409.3085`, `doi:10.1103/PhysRevD.91.054506`.

[41] B. C. Hall, Quantum Theory for Mathematicians., Vol. 267 of Graduate Texts in Mathematics, Springer New York, 2013.

[42] G. Costa, G. Fogli, Symmetries and Group Theory in Particle Physics, Springer-Verlag Berlin Heidelberg, 2012.

[43] L. Susskind, Lattice fermions, Phys. Rev. D 16 (1977) 3031–3039. `doi:10.1103/PhysRevD.16.3031`.
URL `https://link.aps.org/doi/10.1103/PhysRevD.16.3031`

[44] G. B. Folland, A Course in Abstract Harmonic Analysis, CRC Press, 1994, theorem 7.25.

[45] D. Pérez-García, F. Verstraete, M. M. Wolf, J. I. Cirac, Matrix Product State Representations 7 401.
URL `http://arxiv.org/abs/quant-ph/0608197v2;http://arxiv.org/pdf/quant-ph/0608197v2`

[46] J. I. Cirac, D. Perez-Garcia, N. Schuch, F. Verstraete, Matrix product density operators: Renormalization fixed points and boundary theories 378 100–149. `arXiv:1606.00608`,

doi:10.1016/j.aop.2016.12.030.
URL http://arxiv.org/abs/1606.00608

[47] G. Vidal, Efficient Classical Simulation of Slightly Entangled Quantum Computations 91 (14) 147902. doi:10.1103/PhysRevLett.91.147902.
URL http://link.aps.org/doi/10.1103/PhysRevLett.91.147902

[48] M. Wolf, Quantum channels and operations guided tour, https://www-m5.ma.tum.de/foswiki/pub/M5/Allgemeines/MichaelWolf/QChannelLecture.pdf.

[49] M. Sanz, D. Perez-Garcia, M. M. Wolf, J. I. Cirac, A quantum version of wielandt's inequality 56 (9) 4668–4673. arXiv:0909.5347, doi:10.1109/TIT.2010.2054552.
URL http://arxiv.org/abs/0909.5347

[50] C. CHENG, A character theory for projective representations of finite group.
URL http://maths.nju.edu.cn/~ccheng/Writings/preps-notes.pdf

[51] N. Vilenkin, A. Klimyk, Representation of Lie Groups and Special Functions: Recent Advances, Springer, 1995, section 4.3.3.

[52] B. C. Hall, Lie groups, Lie algebras and representations , an elementary introduction, 2nd Edition, Springer, 2015.

[53] N. Bultinck, D. J. Williamson, J. Haegeman, F. Verstraete, Fermionic projected entangled-pair states and topological phases, ArXiv e-printsarXiv:1707.00470.

[54] N. Bultinck, D. J. Williamson, J. Haegeman, F. Verstraete, Fermionic matrix product states and one-dimensional topological phases, Phys. Rev. B95 (7) (2017) 075108. arXiv:1610.07849, doi:10.1103/PhysRevB.95.075108.

Declaration of Authorship

I hereby declare that this thesis has been composed by myself. The results presented in Chapter 4 and Chapter 5 are based entirely on my own work unless clearly stated otherwise.

Ilya Kull, August 28, 2017, München