

Modeling and Stabilization of a Rotating Mechanical System with Elastic Plates

Alexander Zuyev* Julia Novikova**

* *Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg, Germany (e-mail: zuyev@mpi-magdeburg.mpg.de).*

** *Industrial Institute, Donetsk National Technical University, Pokrovsk, Ukraine (e-mail: yuliya.novikova.88@mail.ru)*

Abstract: A mechanical system consisting of a rigid body and attached Kirchhoff plates under the action of three independent controls torques is considered. The equations of motion of such model are derived in the form of a system of coupled nonlinear ordinary and partial differential equations. The operator form of this system is represented as an abstract differential equation in a Hilbert space. A feedback control law is constructed such that the corresponding infinitesimal generator is dissipative.

© 2018, IFAC (International Federation of Automatic Control) Hosting by Elsevier Ltd. All rights reserved.

Keywords: Distributed parameter systems, elastic structures, control design, Lyapunov methods, nonlinear control

1. INTRODUCTION

Problems of the aerospace industry and robotics stimulate the development of new methods for mathematical modeling and control design for complex mechanical systems with elastic elements. In particular, it is a well-known fact that the vibrations of flexible parts of satellites influence significantly their dynamics, so that a rigid body model is not acceptable in stability and control investigations for such distributed parameter systems (see, e.g., Brereton and Modi (1968); Lips and Modi (1980); Skaar et al. (1986)). This brings the motivation for studying the controllability and stabilization issues for infinite-dimensional mathematical models of flexible structures with strings, beams, and plates. Without pretending to be complete, we refer to the monographs in this area by Krabs (1992); Lagnese et al. (1994); Dáger and Zuazua (2006); Meurer (2013); Zuyev (2015).

The stabilization problem for a thin plate with boundary control has already received attention in Lagnese (1989); Lasiecka and Triggiani (1991); Horn and Lasiecka (1995); Guo and Zhou (2016). A mathematical model of a rigid body with the Kirchhoff plate has been considered in the paper by Zuyev (2010). It is assumed there that the body rotates around the fixed axis and its angular acceleration is taken as the control. The reachable sets for the linearized representation of such a system with modal coordinates have been analysed by Zuyev and Novikova (2015).

The purpose of our present paper is to derive a nonlinear model of a rotating rigid body with two Kirchhoff plates and propose a stabilizing feedback control for this model. We will consider spatial rotations of the system and treat the three independent torques, applied to the body, as control inputs. This framework is considered as a mathematical model of a satellite with solar panels controlled by jet thrusters or flywheels.

2. NONLINEAR MODEL OF THE ROTATIONAL MOTION

Consider a mechanical system that consists of a rigid body and two elastic plates (Fig. 1).

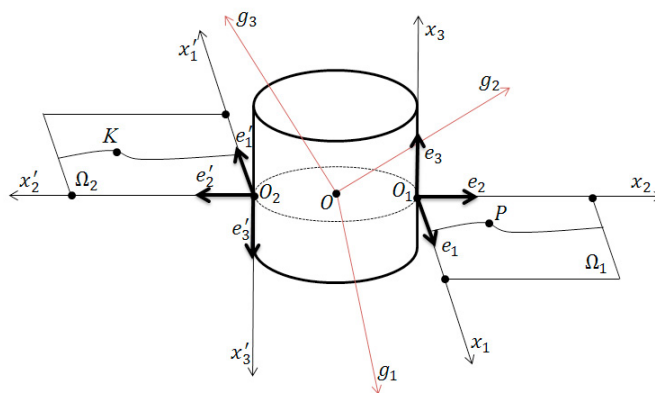


Fig. 1. Rigid body with elastic plates.

Let (g_1, g_2, g_3) be the unit vectors of a fixed Cartesian frame. Suppose that the rigid body is endowed with two Cartesian frames $O_1 x_1 x_2 x_3$ and $O_2 x'_1 x'_2 x'_3$, and their basis vectors (e_1, e_2, e_3) and (e'_1, e'_2, e'_3) , respectively, are related as $e'_1 = -e_1$, $e'_2 = -e_2$, and $e'_3 = -e_3$.

Let $M = f_1 e_1 + f_2 e_2 + f_3 e_3$ be the torque of external forces applied to the rigid body. We will treat the components $(f_1, f_2, f_3)^T \in \mathbb{R}^3$ as control inputs and consider the problem of defining a state feedback law in order to stabilize the moving frame (e_1, e_2, e_3) in the direction of (g_1, g_2, g_3) and to damp the vibrations of the plates. Note that a similar problem for an absolutely rigid body was solved in the book by Zubov (1975), and the problem of partial stabilization was considered by Zuyev (2001) and Kovalev et al. (2009).

In this paper, we assume that two rectangular plates are attached to the rigid body, so that in the undeformed state their median surfaces are located on the planes $O_1x_1x_2$ and $O_2x'_1x'_2$, respectively. At time t , the coordinates of a point P on the median surface of the first plate can be represented in the frame $O_1x_1x_2x_3$ as

$$P = (x_1, x_2, w_1(x_1, x_2, t)), \quad (x_1, x_2) \in \Omega_1 = [0, l_1] \times [0, l_2].$$

Similarly, the coordinates of a point K on the median surface of the second plate are as follows (in the frame $O_2x'_1x'_2x'_3$):

$$K = (x'_1, x'_2, w_2(x'_1, x'_2, t)), \quad (x'_1, x'_2) \in \Omega_2 = [0, l'_1] \times [0, l'_2].$$

Thus, the functions $w_1(x_1, x_2, t)$ and $w_2(x'_1, x'_2, t)$ define the transverse displacements for the case of small deformations of the plates.

In order to describe the motion of the considered mechanical system, we assume that the center of mass of the rigid body (point O) is fixed and expand the vectors OO_1 and OO_2 with respect to the moving frames:

$$OO_1 = d_1e_1 + d_2e_2 + d_3e_3, \quad OO_2 = d'_1e'_1 + d'_2e'_2 + d'_3e'_3.$$

Then the absolute velocities of the points P and K are, respectively,

$$v_P = \omega \times r_P + \dot{w}_1e_3, \quad v_K = \omega \times r_K - \dot{w}_2e_3, \quad (1)$$

where $\omega = \omega_1e_1 + \omega_2e_2 + \omega_3e_3$ is the angular velocity vector of the rigid body,

$$r_P = (x_1 + d_1)e_1 + (x_2 + d_2)e_2 + (w_1 + d_3)e_3, \quad (2)$$

$$r_K = -(x'_1 + d'_1)e_1 - (x'_2 + d'_2)e_2 - (w_2 + d'_3)e_3. \quad (3)$$

According to the Kirchhoff plate model (cf. Lagnese (1989)), we write the following partial differential equations with respect to w_1 and w_2 :

$$\ddot{w}_1 + a_1^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) w_1 = (x_1 + d_1)\dot{\omega}_2 - (x_2 + d_2)\dot{\omega}_1, \quad \text{for } (x_1, x_2) \in \Omega_1, \quad (4)$$

$$\ddot{w}_2 + a_2^2 \left(\frac{\partial^2}{\partial x_1'^2} + \frac{\partial^2}{\partial x_2'^2} \right) w_2 = (x'_1 + d'_1)\dot{\omega}_2 - (x'_2 + d'_2)\dot{\omega}_1, \quad \text{for } (x'_1, x'_2) \in \Omega_2, \quad (5)$$

where $a_j^2 = \frac{E_j h_j^3}{12\rho_j(1-\nu_j^2)}$ is the stiffness parameter of the j -th plate, E_j is Young's modulus, ν_j is Poisson's ratio, ρ_j is the area density, and h_j is the thickness of the j -th plate. The right-hand sides of (4) and (5) contains the inertia forces because of the rotational motion of the rigid body (cf. Lurie (2002)). In the differential equations (4) and (5), only linear terms with respect to the displacements, angular velocities, and derivatives of these quantities are taken into account (this is the linearized model of the plates' vibrations).

We assume that the plates are supported at their boundaries, which yields the following boundary conditions:

$$w_j|_{\partial\Omega_j} = 0, \quad \frac{\partial^2 w_j}{\partial n^2} \Big|_{\partial\Omega_j} = 0, \quad j = 1, 2. \quad (6)$$

Here $\frac{\partial w_j}{\partial n} \Big|_{\partial\Omega_j}$ is the normal derivative of w_j evaluated at the boundary of Ω_j .

To derive the equations of motion of the rigid body-carrier, we exploit the angular momentum equation with respect to the fixed point O (see, e.g., Lurie (2002)):

$$\dot{K} + \omega \times K = M, \quad (7)$$

where $K = K_1e_1 + K_2e_2 + K_3e_3$ is the angular momentum of the system, and \dot{K} stands for the local derivative of K in the moving frame (e_1, e_2, e_3) , i.e.

$$\dot{K} = \dot{K}_1e_1 + \dot{K}_2e_2 + \dot{K}_3e_3. \quad (8)$$

In the sequel, we use formulas (2) to express the angular momentum K for the mechanical system under consideration:

$$K = I\omega + K_{p1} + K_{p2}, \quad (9)$$

where I is the tensor of inertia of the rigid body, and

$$K_{p1} = \int_{\Omega_1} r_P \times v_P \rho_1 dx, \quad K_{p2} = \int_{\Omega_2} r_K \times v_K \rho_2 dx'.$$

We will assume that (e_1, e_2, e_3) are the principal axes of inertia of the rigid body to simplify computations, so that $I = \text{diag}(I_1, I_2, I_3)$. Thus, the angular momentum (9) can be rewritten as follows:

$$K = (I + J)\omega + \sum_{n=1}^2 \int_{\Omega_n} \dot{w}_n (x_2 + d_{2n}) \rho_n dx e_1 - \sum_{n=1}^2 \rho_n \int_{\Omega_n} \dot{w}_n (x_1 + d_{1n}) dx e_2 + K_\delta, \quad (10)$$

where $J = (J_{ij})$ is the tensor of inertia for the mechanical system with "frozen" plates (i.e. when the plates are considered as rigid bodies),

$$\begin{aligned} J_{11} &= \sum_{n=1}^2 \rho_n \int_{\Omega_n} ((x_2 + d_{2n})^2 + d_{3n}^2) dx, \\ J_{12} = J_{21} &= - \sum_{n=1}^2 \rho_n \int_{\Omega_n} (x_1 + d_{1n})(x_2 + d_{2n}) dx, \\ J_{22} &= \sum_{n=1}^2 \rho_n \int_{\Omega_n} ((x_1 + d_{1n})^2 + d_{3n}^2) dx, \\ J_{23} = J_{32} &= - \sum_{n=1}^2 \rho_n d_{3n} \int_{\Omega_n} (x_2 + d_{2n}) dx, \\ J_{33} &= \sum_{n=1}^2 \rho_n \int_{\Omega_n} ((x_1 + d_{1n})^2 + (x_2 + d_{2n})^2) dx, \\ J_{31} = J_{13} &= - \sum_{n=1}^2 \rho_n d_{3n} \int_{\Omega_n} (x_1 + d_{1n}) dx, \end{aligned}$$

and the term K_δ is of order $O(\|\omega\|(\|\dot{w}_1\| + \|\dot{w}_2\|))$ for small \dot{w}_j . By computing the local derivative (in the sense of (8)) for the angular momentum K given by formula (10), we get:

$$\dot{K}_1 = (J_{11} + I_1)\dot{\omega}_1 + J_{12}\dot{\omega}_2 + J_{13}\dot{\omega}_3 + \sum_{n=1}^2 \rho_n \int_{\Omega_n} \ddot{w}_n (x_2 + d_{2n}) dx,$$

$$\dot{K}_2 = J_{21}\dot{\omega}_1 + (J_{22} + I_2)\dot{\omega}_2 + J_{23}\dot{\omega}_3 - \sum_{n=1}^2 \rho_n \int_{\Omega_n} \ddot{w}_n(x_1 + d_{1n}) dx,$$

$$\dot{K}_3 = J_{31}\dot{\omega}_1 + J_{32}\dot{\omega}_2 + (J_{33} + I_3)\dot{\omega}_3, \quad (11)$$

where the nonlinear terms with respect to the derivatives of ω_j and w_n are omitted.

In order to rewrite the differential equations (7) in the normal form with respect to $\dot{\omega}_j$, we compute the inverse matrix $J^{-1} = \hat{J} = (\hat{J}_{ij})$:

$$\begin{aligned} \hat{J}_{11} &= \frac{(I_2 + \sum_{n=1}^2 \rho_n d_{3n}^2 l_{1n} l_{2n})(I_3 + J_{33}) - J_{23}^2}{D}, \\ \hat{J}_{12} &= \hat{J}_{21} = \frac{J_{13} J_{23}}{D}, \\ \hat{J}_{13} &= \hat{J}_{31} = -\frac{(I_2 + \sum_{n=1}^2 \rho_n d_{3n}^2 l_{1n} l_{2n}) J_{13}}{D}, \\ \hat{J}_{22} &= \frac{(I_1 + \sum_{n=1}^2 \rho_n d_{3n}^2 l_{1n} l_{2n})(I_3 + J_{33}) - J_{13}^2}{D}, \\ \hat{J}_{23} &= \hat{J}_{32} = -\frac{(I_1 + \sum_{n=1}^2 \rho_n d_{3n}^2 l_{1n} l_{2n}) J_{23}}{D}, \\ \hat{J}_{33} &= \frac{(I_1 + \sum_{n=1}^2 \rho_n d_{3n}^2 l_{1n} l_{2n})(I_2 + \sum_{n=1}^2 \rho_n d_{3n}^2 l_{1n} l_{2n})}{D}, \end{aligned} \quad (12)$$

$$\begin{aligned} D &= (I_1 + \sum_{n=1}^2 \rho_n d_{3n}^2 l_{1n} l_{2n})(I_2 + \sum_{n=1}^2 \rho_n d_{3n}^2 l_{1n} l_{2n})(I_3 + J_{33}) - \\ &- J_{13}^2 (I_2 + \sum_{n=1}^2 \rho_n d_{3n}^2 l_{1n} l_{2n}) - J_{23}^2 (I_1 + \sum_{n=1}^2 \rho_n d_{3n}^2 l_{1n} l_{2n}). \end{aligned}$$

Note that the denominator D in formulas (12) is strictly positive at least for sufficiently small moments of inertia of the plates J_{ik} compared with the moments of inertia of the carrier body I_i . In particular, this condition is satisfied for sufficiently small area densities ρ_j (i.e., for sufficiently thin plates). Thus, we assume that $D \neq 0$ in the sequel. Then (7) can be written in the form

$$\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} = \hat{J} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \quad (13)$$

with

$$\begin{aligned} \phi_1 &= f_1 + \omega_3 [J_{21}\omega_1 + (J_{22} + I_2)\omega_2 + J_{23}\omega_3] - \\ &- \omega_2 [J_{31}\omega_1 + J_{32}\omega_2 + (J_{33} + I_3)\omega_3] + \\ &+ \sum_{n=1}^2 \rho_n \int_{\Omega_n} \{a_n^2(x_2 + d_{2n})\Delta^2 w_n - \omega_3 \dot{w}_n(x_1 + d_{1n})\} dx, \\ \phi_2 &= f_2 + \omega_1 [J_{31}\omega_1 + J_{32}\omega_2 + (J_{33} + I_3)\omega_3] - \\ &- \omega_3 [(J_{11} + I_1)\omega_1 + J_{12}\omega_2 + J_{13}\omega_3] - \\ &- \sum_{n=1}^2 \rho_n \int_{\Omega_n} \{a_n^2(x_1 + d_{1n})\Delta^2 w_n + \omega_3 \dot{w}_n(x_2 + d_{2n})\} dx, \\ \phi_3 &= f_3 - \omega_1 [J_{21}\omega_1 + (J_{22} + I_2)\omega_2 + J_{23}\omega_3] + \\ &+ \omega_2 [(J_{11} + I_1)\omega_1 + J_{12}\omega_2 + J_{13}\omega_3] + \\ &+ \sum_{n=1}^2 \rho_n \int_{\Omega_n} \{(x_1 + d_{1n})\omega_1 + (x_2 + d_{2n})\omega_2\} \dot{w}_n dx. \end{aligned}$$

We write the Poisson kinematic equations to ensure the condition that the frame (g_1, g_2, g_3) is fixed in the inertial

space: $\dot{g}_i = -\omega \times g_i$, $i = \overline{1, 3}$. Let $g_i = g_{i1}e_1 + g_{i2}e_2 + g_{i3}e_3$, then the above Poisson equations take the following coordinate form:

$$\begin{aligned} \dot{g}_{i1} &= \omega_3 g_{i2} - \omega_2 g_{i3}, \quad \dot{g}_{i2} = \omega_1 g_{i3} - \omega_3 g_{i1}, \\ \dot{g}_{i3} &= \omega_2 g_{i1} - \omega_1 g_{i2}, \quad i = \overline{1, 3}. \end{aligned} \quad (14)$$

For the Cartesian frames (g_1, g_2, g_3) and (e_1, e_2, e_3) of the same orientation, the system of differential equations (4)–(6), (13), (14) has the following particular solution with $f_1 = f_2 = f_3 = 0$:

$$w(x, t) = 0, \quad \omega_i(t) = 0, \quad g_{ij}(t) = \delta_{ij}, \quad i, j = \overline{1, 3}, \quad (15)$$

where δ_{ij} is the Kronecker symbol.

To study the stabilization problem for the equilibrium (15), we introduce new variables $\tilde{g}_{ij}(t) = g_{ij}(t) - \delta_{ij}$ and consider the equations of perturbed motion for (14):

$$\begin{aligned} \dot{\tilde{g}}_{11} &= \omega_3 \tilde{g}_{12} - \omega_2 \tilde{g}_{13}, \quad \dot{\tilde{g}}_{12} = \omega_1 \tilde{g}_{13} - \omega_3 (\tilde{g}_{11} + 1), \\ \dot{\tilde{g}}_{13} &= \omega_2 (\tilde{g}_{11} + 1) - \omega_1 \tilde{g}_{12}, \\ \dot{\tilde{g}}_{21} &= \omega_3 (\tilde{g}_{22} + 1) - \omega_2 \tilde{g}_{23}, \quad \dot{\tilde{g}}_{22} = \omega_1 \tilde{g}_{23} - \omega_3 \tilde{g}_{21}, \\ \dot{\tilde{g}}_{23} &= \omega_2 \tilde{g}_{21} - \omega_1 (\tilde{g}_{22} + 1), \\ \dot{\tilde{g}}_{31} &= \omega_3 \tilde{g}_{32} - \omega_2 (\tilde{g}_{33} + 1), \quad \dot{\tilde{g}}_{32} = \omega_1 (\tilde{g}_{33} + 1) - \omega_3 \tilde{g}_{31}, \\ \dot{\tilde{g}}_{33} &= \omega_2 \tilde{g}_{31} - \omega_1 \tilde{g}_{32}. \end{aligned} \quad (16)$$

We consider a modified energy functional

$$\mathcal{E} = T + U + \frac{1}{2} \sum_{i,j=1}^3 \alpha_i \tilde{g}_{ij}^2 \quad (17)$$

with positive parameters α_i , where

$$T = \frac{1}{2} \left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 + \int_{\Omega_1} v_P^2 \rho_1 dx + \int_{\Omega_2} v_K^2 \rho_2 dx \right)$$

is the kinetic energy of the system, and

$$U = \frac{1}{2} \sum_{n=1}^2 \int_{\Omega_n} (\Delta w_n(x, t))^2 a_n^2 \rho_n dx$$

is the potential energy of elastic deformations according to the Kirchhoff model. Here $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplace operator. For future use, we introduce the Lyapunov functional V as a quadratic approximation of \mathcal{E} :

$$\begin{aligned} 2V &= (I_1 + J_{11})\omega_1^2 + (I_2 + J_{22})\omega_2^2 + (I_3 + J_{33})\omega_3^2 + 2J_{12}\omega_1\omega_2 + \\ &+ 2J_{13}\omega_1\omega_3 + 2J_{23}\omega_2\omega_3 + \sum_{i,j=1}^3 \alpha_i \tilde{g}_{ij}^2 + \sum_{n=1}^2 \rho_n \int_{\Omega_n} \dot{w}_n^2 dx + \\ &+ \sum_{n=1}^2 \rho_n \int_{\Omega_n} \{2\dot{w}_n[\omega_1(d_{2n} + x_2) - \omega_2(d_{1n} + x_1)] + a_n^2(\Delta w_n)^2\} dx. \end{aligned} \quad (18)$$

Let us compute the time derivative of the functional (17) along the trajectories of (4), (13), (16):

$$\begin{aligned} \dot{V} &= (\dot{K}_1 - \alpha_2 \tilde{g}_{23} + \alpha_3 \tilde{g}_{32}) \omega_1 + (\dot{K}_2 + \alpha_1 \tilde{g}_{13} - \alpha_3 \tilde{g}_{31}) \omega_2 + \\ &+ (\dot{K}_3 + \alpha_2 \tilde{g}_{21} - \alpha_1 \tilde{g}_{12}) \omega_3 + \end{aligned}$$

$$+ \sum_{n=1}^2 \int_{\Omega_n} \{ \Delta w_n \Delta \dot{w}_n - \dot{w}_n \Delta^2 w_n \} a_n^2 \rho_n dx, \quad (19)$$

where the expressions for \dot{K}_i are given by (11). If the partial derivatives of $w_i(x, t)$ of the fourth order in x and the first order in t are continuous and the boundary conditions (6) are satisfied, then the integration by parts in formula (19) leads to the following identities:

$$\int_{\Omega_n} \{ \Delta w_n \Delta \dot{w}_n - \dot{w}_n \Delta^2 w_n \} dx = 0, \quad n = 1, 2.$$

Using these identities and expressing \dot{K}_i from the equation (7), we rewrite formula (19) as

$$\begin{aligned} \dot{V} = & (f_1 - (\omega \times K, e_1) - \alpha_2 \tilde{g}_{23} + \alpha_3 \tilde{g}_{32}) \omega_1 + \\ & + (f_2 - (\omega \times K, e_2) + \alpha_1 \tilde{g}_{13} - \alpha_3 \tilde{g}_{31}) \omega_2 + \\ & + (f_3 - (\omega \times K, e_3) + \alpha_2 \tilde{g}_{21} - \alpha_1 \tilde{g}_{12}) \omega_3. \end{aligned}$$

To stabilize the trivial solution of the system (4)–(6), (13), (16), we define a feedback control from the condition

$$\dot{V} = -k(\omega_1^2 + \omega_2^2 + \omega_3^2) \leq 0, \quad (20)$$

where k is a positive constant. It is easy to see that condition (20) corresponds to the following choice of controls:

$$\begin{aligned} f_1 = & -k\omega_1 + (\omega \times K, e_1) + \alpha_2 \tilde{g}_{23} - \alpha_3 \tilde{g}_{32}, \\ f_2 = & -k\omega_2 + (\omega \times K, e_2) - \alpha_1 \tilde{g}_{13} + \alpha_3 \tilde{g}_{31}, \\ f_3 = & -k\omega_3 + (\omega \times K, e_3) + \alpha_1 \tilde{g}_{12} - \alpha_2 \tilde{g}_{21}. \end{aligned} \quad (21)$$

Note that the time-derivative (20) is not negative definite, and the finite-dimensional method described by Zuyev (2016) is not directly applicable to establish strictly decreasing behavior of V along the trajectories of the closed-loop system.

3. OPERATOR FORM OF THE DYNAMICAL EQUATIONS

To formulate our main result, we rewrite the equations of motion of the mechanical system under consideration in the operator form. We introduce the real linear space $H = \dot{H}^2(\Omega_1) \times \dot{H}^2(\Omega_2) \times L^2(\Omega_1) \times L^2(\Omega_2) \times \mathbb{R}^{12}$, whose elements are denoted as

$$\begin{aligned} \xi = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \omega \\ \tilde{g} \end{pmatrix} : u_n \in \dot{H}^2(\Omega_n), v_n \in L^2(\Omega_n), n = 1, 2, \\ \omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \in \mathbb{R}^3, \tilde{g} = \begin{pmatrix} \tilde{g}_{11} \\ \tilde{g}_{12} \\ \vdots \\ \tilde{g}_{33} \end{pmatrix} \in \mathbb{R}^9. \end{aligned}$$

Here $\dot{H}^2(\Omega_n)$ is the Sobolev space of the functions $u \in H^2(\Omega_n)$ having zero trace on $\partial\Omega_n$. The inner product of

$$\xi^1 = \begin{pmatrix} u_1^1 \\ v_1^1 \\ u_2^1 \\ v_2^1 \\ \omega^1 \\ \tilde{g}^1 \end{pmatrix} \in H \quad \text{and} \quad \xi^2 = \begin{pmatrix} u_1^2 \\ v_1^2 \\ u_2^2 \\ v_2^2 \\ \omega^2 \\ \tilde{g}^2 \end{pmatrix} \in H$$

is defined as

$$\begin{aligned} \langle \xi^1, \xi^2 \rangle_H = & \sum_{n=1}^2 \rho_n \int_{\Omega_n} \{ a_n^2 \Delta u_n^1(x) \Delta u_n^2(x) + v_n^1(x) v_n^2(x) + \\ & + (\omega_1^2 v_n^1(x) + \omega_1^1 v_n^2(x))(d_{2n} + x_2) - \\ & - (\omega_2^2 v_n^1(x) + \omega_2^1 v_n^2(x))(d_{1n} + x_1) \} dx + \\ & + \left((I + J) \omega^1, \omega^2 \right) + \sum_{i,j=1}^3 \alpha_i \tilde{g}_{ij}^1 \tilde{g}_{ij}^2. \end{aligned} \quad (22)$$

Using the Cauchy–Schwarz and Friedrichs’ inequalities, it can be shown that the norm $\|\xi\|_H = \sqrt{\langle \xi, \xi \rangle_H}$ is equivalent to the standard norm in $\dot{H}^2(\Omega_n) \times L^2(\Omega_n) \times \mathbb{R}^{12}$. Thus, $(H, \langle \cdot, \cdot \rangle_H)$ is a Hilbert space.

We define an unbounded operator $A : D(A) \rightarrow H$ and a bounded linear operator $B : \mathbb{R}^3 \rightarrow H$ in the following way:

$$A : \xi = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \omega \\ \tilde{g} \end{pmatrix} \mapsto A\xi = \begin{pmatrix} u_1^\xi \\ v_1^\xi \\ u_2^\xi \\ v_2^\xi \\ \omega^\xi \\ \tilde{g}^\xi \end{pmatrix} \in H, \quad (23)$$

$$B : f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \mapsto Bf = \begin{pmatrix} u_1^f \\ v_1^f \\ u_2^f \\ v_2^f \\ \omega^f \\ \tilde{g}^f \end{pmatrix} \in H, \quad (24)$$

where

$$\begin{aligned} \omega_i^\xi = & (\hat{J}_{i1} \omega_3 - \hat{J}_{i3} \omega_1) [J_{21} \omega_1 + (J_{22} + I_2) \omega_2 + J_{23} \omega_3] + \\ & + (\hat{J}_{i2} \omega_1 - \hat{J}_{i1} \omega_2) [J_{31} \omega_1 + J_{32} \omega_2 + (J_{33} + I_3) \omega_3] + \\ & + (\hat{J}_{i3} \omega_2 - \hat{J}_{i2} \omega_3) [(J_{11} + I_1) \omega_1 + J_{12} \omega_2 + J_{13} \omega_3] + \\ & + \sum_{n=1}^2 \rho_n \int_{\Omega_n} \left(\hat{J}_{i3} [(x_1 + d_{1n}) \omega_1 + (x_2 + d_{2n}) \omega_2] - \right. \\ & \left. - [\hat{J}_{i1} (x_1 + d_{1n}) + \hat{J}_{i2} (x_2 + d_{2n})] \omega_3 \right) v_n(x) dx + \\ & + \sum_{n=1}^2 \rho_n a_n^2 \int_{\Omega_n} \left(\hat{J}_{i1} (x_2 + d_{2n}) - \hat{J}_{i2} (x_1 + d_{1n}) \right) \Delta^2 u_n(x) dx, \\ v_n^\xi(x) = & -a_n^2 \Delta^2 u_n(x) + (x_1 + d_{1n}) \omega_2^\xi - (x_2 + d_{2n}) \omega_1^\xi, \\ u_n^\xi(x) = & v_n(x), \quad n = 1, 2, \\ \tilde{g}_{11}^\xi = & \omega_3 \tilde{g}_{12} - \omega_2 \tilde{g}_{13}, \tilde{g}_{12}^\xi = \omega_1 \tilde{g}_{13} - \omega_3 (\tilde{g}_{11} + 1), \\ \tilde{g}_{13}^\xi = & \omega_2 (\tilde{g}_{11} + 1) - \omega_1 \tilde{g}_{12}, \\ \tilde{g}_{21}^\xi = & \omega_3 (\tilde{g}_{22} + 1) - \omega_2 \tilde{g}_{23}, \tilde{g}_{22}^\xi = \omega_1 \tilde{g}_{23} - \omega_3 \tilde{g}_{21}, \\ \tilde{g}_{23}^\xi = & \omega_2 \tilde{g}_{21} - \omega_1 (\tilde{g}_{22} + 1), \\ \tilde{g}_{31}^\xi = & \omega_3 \tilde{g}_{32} - \omega_2 (\tilde{g}_{33} + 1), \tilde{g}_{32}^\xi = \omega_1 (\tilde{g}_{33} + 1) - \omega_3 \tilde{g}_{31}, \\ \tilde{g}_{33}^\xi = & \omega_2 \tilde{g}_{31} - \omega_1 \tilde{g}_{32}, \\ v_n^f(x) = & \sum_{k=1}^3 \left(\hat{J}_{2k} (x_1 + d_{1n}) - \hat{J}_{1k} (x_2 + d_{2n}) \right) f_k, \\ u_n^f(x) = & 0, \tilde{g}^f = 0, \omega_i^f = \sum_{k=1}^3 \hat{J}_{ik} f_k, \quad i = 1, 2, 3, \end{aligned}$$

and the coefficients \hat{J}_{ik} are given in (12). The domain of definition of the nonlinear operator A has the form

$$D(A) = \left\{ \xi \in H \left| \begin{array}{l} u_n \in \dot{H}^4(\Omega_n), v_n \in \dot{H}^2(\Omega_n), \\ \frac{\partial^2 u_n}{\partial x_1^2} \Big|_{\partial\Omega_n} = \frac{\partial^2 u_n}{\partial x_2^2} \Big|_{\partial\Omega_n} = 0, n = 1, 2 \end{array} \right. \right\}. \quad (25)$$

4. MAIN RESULT

We consider a nonlinear control system governed by the following abstract differential equation in H :

$$\frac{d}{dt}\xi(t) = A\xi(t) + Bf, \quad (26)$$

where $\xi(t) \in H$ is the state, $f \in \mathbb{R}^3$ is the control, and the operators A and B are given by (23) and (24). If the functions $w_1(x, t)$, $w_2(x, t)$, $\omega(t)$, $\tilde{g}(t)$ define a classical solution of system (4), (5), (6), (13), (16) with a control $f = f(t)$ on the interval $t \in \mathcal{I} = [t_0, T)$, $T \leq +\infty$, then by direct substitution we verify that the corresponding function

$$\xi(t) = \begin{pmatrix} w_1(\cdot, t) \\ \dot{w}_1(\cdot, t) \\ w_2(\cdot, t) \\ \dot{w}_2(\cdot, t) \\ \omega(t) \\ \tilde{g}(t) \end{pmatrix} \in D(A) \subset H \quad (27)$$

satisfies the equation (26) with $f = f(t)$ on $t \in \mathcal{I}$. Thus, we consider the differential equation (26) as a mathematical model of the considered mechanical system.

Let us represent the feedback control (21) by an operator $G : H \rightarrow \mathbb{R}^3$ defined on the state space of system (26):

$$G : \xi = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \omega \\ \tilde{g} \end{pmatrix} \mapsto f = G\xi = \begin{pmatrix} f_1^\xi \\ f_2^\xi \\ f_3^\xi \end{pmatrix}, \quad (28)$$

$$f_1^\xi = -\omega_3 (J_{21}\omega_1 + (J_{22} + I_2)\omega_2 + J_{23}\omega_3) - k\omega_1 + \alpha_2\tilde{g}_{23} - \alpha_3\tilde{g}_{32} + \omega_2 (J_{31}\omega_1 + J_{32}\omega_2 + (J_{33} + I_3)\omega_3) + \omega_3 \sum_{n=1}^2 \rho_n \int_{\Omega_n} (x_1 + d_{1n}) v_n(x) dx,$$

$$f_2^\xi = -\omega_1 (J_{31}\omega_1 + J_{32}\omega_2 + (J_{33} + I_3)\omega_3) - k\omega_2 - \alpha_1\tilde{g}_{13} + \alpha_3\tilde{g}_{31} + \omega_3 ((J_{11} + I_1)\omega_1 + J_{12}\omega_2 + J_{13}\omega_3) + \omega_3 \sum_{n=1}^2 \rho_n \int_{\Omega_n} (x_2 + d_{2n}) v_n(x) dx,$$

$$f_3^\xi = -\omega_2 ((J_{11} + I_1)\omega_1 + J_{12}\omega_2 + J_{13}\omega_3) - k\omega_3 + \alpha_1\tilde{g}_{12} - \alpha_2\tilde{g}_{21} + \omega_1 (J_{21}\omega_1 + (J_{22} + I_2)\omega_2 + J_{23}\omega_3) - \sum_{n=1}^2 \rho_n \int_{\Omega_n} [(x_1 + d_{1n})\omega_1 + (x_2 + d_{2n})\omega_2] v_n(x) dx,$$

where k and α_i are arbitrary positive constants. Then the closed-loop system (26) with $f = G\xi$ takes the form

$$\frac{d}{dt}\xi(t) = F\xi(t), \quad F = A + BG, \quad (29)$$

where the domain of definition of the unbounded nonlinear operator $F : D(F) \rightarrow H$ is dense in H , $D(F) = D(A)$.

As it was noted above, the classical solutions of system (4), (5), (6), (13), (16) correspond to the functions $\xi(t) \in D(A)$ according to the rule (27). Then the condition $\dot{V} \leq 0$ for the time derivative of the functional V along the trajectories of the closed-loop system can be rewritten as

$$\langle F\xi, \xi \rangle_H \leq 0 \quad (30)$$

for the corresponding element $\xi \in D(F) = D(A)$. This is a consequence of the definition of the functional V in (18) and the inner product $\langle \cdot, \cdot \rangle_H$ in (22). Thus, inequality (30) implies the following result concerning the operator F of the closed-loop system (29).

Theorem 1. *The operator $F : D(F) \rightarrow H$ is dissipative and $\overline{D(F)} = H$.*

We denote by I_H the identity operator on H . If F is closed and the image of $I_H - \lambda F$ coincides with H for $\lambda > 0$, then Theorem 1 implies that F is the infinitesimal generator of a strongly continuous semigroup of nonlinear operators $\{S(t)\}_{t \geq 0}$ on H because of the Crandall–Liggett theorem (cf. Barbu (1992)). Then the mild solution of the Cauchy problem for (29) with the initial condition $\xi(0) = \xi^0$ is defined by the formula

$$\xi(t) = S(t)\xi^0, \quad t \geq 0, \quad (31)$$

for any $\xi^0 \in H$. Under these assumptions, the trivial solution of the abstract differential equation (29) is stable in the sense of Lyapunov because of the dissipativity inequality $\langle F\xi, \xi \rangle_H \leq 0$ ($\dot{V} \leq 0$). Note that the mild solutions given by formula (31) are classical if $\xi^0 \in D(F)$.

5. NUMERICAL SIMULATIONS

In order to illustrate the transient behaviour of the proposed controller, we perform a numerical simulation of the closed-loop system (29). For this purpose we consider the case of identical rectangular domains $\Omega_1 = \Omega_2 = [0, l_1] \times [0, l_2]$ and introduce finite-dimensional approximations of the displacements $w_j(x_1, x_2, t) = q_j(t)W_1(x_1)W_2(x_2)$, $j = 1, 2$, for equations (4) and (5), respectively. Here $W_j(x) = \sin\left(\frac{\pi m_j x}{l_j}\right)$, $x \in [0, l_j]$, $m_j \in \mathbb{N}$, are taken as eigenfunctions of the Sturm–Liouville problem with the boundary conditions $W_j(0) = W_j(l_j) = 0$, $j = 1, 2$. Let us consider the first flexible mode only ($m_1 = m_2 = 1$) and apply the Ritz–Galerkin method (cf. Zuyev and Sawodny (2015)) for the nonlinear closed-loop system (4)–(6), (13), (16), (21) to derive its finite-dimensional approximation of the form

$$\dot{X}(t) = \Phi(X(t)), \quad X(t) \in \mathbb{R}^{16}, \quad (32)$$

whose state vector is

$$X = (\tilde{g}_{11}, \tilde{g}_{12}, \dots, \tilde{g}_{33}, \omega_1, \omega_2, \omega_3, q_1, q_2, \dot{q}_1, \dot{q}_2).$$

We choose the following initial data and parameters for the simulation (the dimensions of physical quantities are omitted to simplify notations):

$$X(0) = (0, 0, 0, 0, -1, 1, 0, -1, -1, 0, \dots, 0), \quad (33)$$

$$l_1 = \frac{l_2}{2} = 1, \quad d_1 = d'_1 = 0, \quad d_2 = d'_2 = 1, \quad d_3 = d'_3 = 0,$$

$$\rho_1 = \rho_2 = 1, \quad a_1 = a_2 = \frac{1}{2}, \quad J = I, \quad \alpha_1 = \alpha_2 = \alpha_3 = 1.$$

The above $X(0)$ corresponds to an equilibrium of the considered mechanical system with $g_1 = e_1$, $g_2 = e_3$, and $g_3 = -e_2$. In this case, the considered stabilization

problem (steering the closed-loop system to its trivial equilibrium) means rotation about the x_1 -axis by the angle $\pi/2$ with simultaneous damping of the vibration modes. Fig. 2 shows the behavior of solutions to the

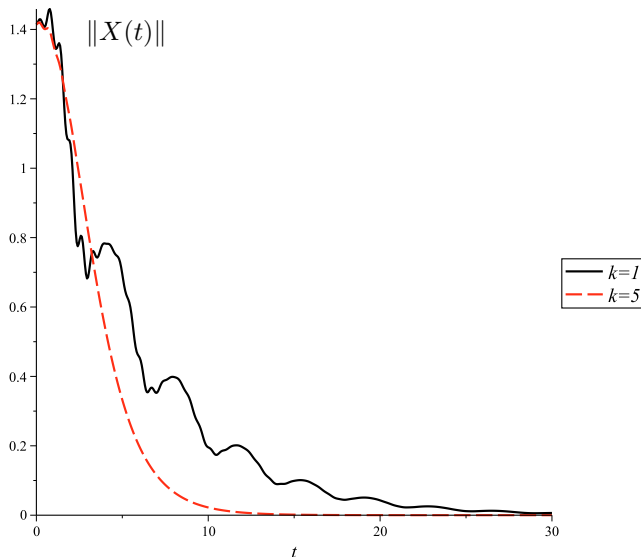


Fig. 2. Euclidean norm $\|X(t)\|$ of the solution of (32), (33).

Cauchy problem (32), (33) for different values of the tuning parameter $k > 0$ appearing in the feedback law (21). We see that the proposed controller can be used to steer the state of system (32) to zero as $t \rightarrow +\infty$, and the higher value of k leads to faster convergence of $X(t)$ to $X = 0$.

6. CONCLUSIONS

In this paper, a new mathematical model of a controlled mechanical system consisting of a rotating body and elastic Kirchhoff plates has been introduced. The model is described by the system of nonlinear ordinary and partial differential equations (4)–(6), (13), (16), or, equivalently, by the abstract differential equation (26) in the Hilbert space H . A state feedback control has been proposed explicitly in the form (21) to ensure that the time derivative of a Lyapunov functional is non-positive.

The main theoretical contribution of this work establishes the dissipativity property for the infinitesimal generator F in (29). Although numerical simulations illustrate the efficiency of the proposed controller, the question about *asymptotic stability* (or even *partial asymptotic stability* in the sense of Zuyev (2003)) remains open for the infinite-dimensional closed-loop dynamics.

REFERENCES

V. Barbu. *Analysis and Control of Nonlinear Infinite Dimensional Systems*. Academic Press, San Diego, 1992.

R.C. Brereton and V.J. Modi. Planar librational stability of a flexible satellite. *AIAA Journal*, 6:511–517, 1968.

R. Dáger and E. Zuazua. *Wave Propagation, Observation and Control in 1-d Flexible Multi-Structures*. Springer-Verlag, Berlin Heidelberg, 2006.

B.-Z. Guo and H.-C. Zhou. Output feedback stabilization for multi-dimensional Kirchhoff plate with general

corrupted boundary observation. *European Journal of Control*, 28: 38–48, 2016.

M.A. Horn and I. Lasiecka. Global stabilization of a dynamic von Kármán plate with nonlinear boundary feedback. *Applied mathematics & Optimization*, 31:57–84, 1995.

A.M. Kovalev, A.A. Martynyuk, O.A. Boichuk, A.G. Mazko, R.I. Petryshyn, V.Yu. Slyusarchuk, A.L. Zuyev, and V.I. Slyn'ko. Novel qualitative methods of nonlinear mechanics and their application to the analysis of multifrequency oscillations, stability, and control problems. *Nonlinear Dynamics and Systems Theory*, 9:117–145, 2009.

W. Krabs. *On Moment Theory and Controllability of One-Dimensional Vibrating Systems and Heating Processes*. Springer-Verlag, Berlin Heidelberg, 1992.

J.E. Lagnese. *Boundary Stabilization of Thin Plates*. SIAM, Philadelphia, 1989.

J.E. Lagnese, G. Leugering, and E.J.P.G. Schmidt. *Modeling, Analysis and Control of Dynamic Elastic Multi-Link Structures*. Springer, New York, 1994.

I. Lasiecka and R. Triggiani. Exact controllability of semi-linear abstract systems with application to waves and plates boundary control problems. *Applied Mathematics and Optimization*, 23:109–154, 1991.

K.W. Lips and V.J. Modi. General dynamics of a large class of flexible satellite systems. *Acta Astronautica*, 7: 1349–1360, 1980.

A.I. Lurie. *Analytical Mechanics*. Springer-Verlag, Berlin Heidelberg, 2002.

T. Meurer. *Control of Higher Dimensional PDEs: Flatness and Backstepping Designs*. Springer-Verlag, Berlin Heidelberg, 2013.

S.B. Skaar, L. Tang, and I. Yalda-Mooshabad. On-off attitude control of flexible satellites. *Journal of Guidance, Control, and Dynamics*, 9:507–510, 1986.

V.I. Zubov. *Lectures on Control Theory* (in Russian). Nauka, Moscow, 1975.

A.L. Zuyev. Application of control Lyapunov functions technique for partial stabilization. *Proc. IEEE Conference on Control Applications, Mexico*, 509–513, 2001.

A.L. Zuyev. Partial asymptotic stability and stabilization of nonlinear abstract differential equations. *Proc. 42nd IEEE Conference on Decision and Control, Maui*, 1321–1326, 2003.

A.L. Zuyev. Approximate controllability of a rotating Kirchhoff plate model. *Proc. 49th IEEE Conference on Decision and Control, Atlanta*, 944–948, 2010.

A.L. Zuyev. *Partial Stabilization and Control of Distributed Parameter Systems with Elastic Elements*. Springer, Cham, 2015.

A.L. Zuyev and Yu.V. Novikova. Estimation of the reachable set for the problem of vibrating Kirchhoff plate. *Ukrainian Mathematical Journal*, 66:1639–1653, 2015.

A. Zuyev and O. Sawodny. Modelling and control of a shell structure based on a finite dimensional variational formulation. *Mathematical and Computer Modelling of Dynamical Systems*, 21:591–612, 2015.

A. Zuyev. Exponential stabilization of nonholonomic systems by means of oscillating controls. *SIAM Journal on Control and Optimization*, 54:1678–1696, 2016.