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New Gramians for Switched Linear Systems: Reachability, Observability, and Model Reduction

Igor Pontes Duff, Sara Grundel and Peter Benner

***Abstract*—In this paper, we propose new algebraic Gramians for continuous-time switched linear systems, which satisfy generalized Lyapunov equations. The main contribution of this work is twofold. First, we show that the ranges of those Gramians encode the reachability and observability spaces of a switched linear system. As a consequence, a simple Gramian-based criterion for reachability and observability is established. Second, a balancing-based model order reduction technique is proposed and, under some sufficient conditions, stability preservation and an error bound are shown. Finally, the efficiency of the proposed method is illustrated by means of numerical examples.**

***Index Terms*—Model reduction; switched systems; balanced truncation; reachability and observability**

I. Introduction

We consider a continuous-time switched linear system (see [26], [35]) (abbreviated by SLS[1]) given by

$$\Sigma_{SLS} : \begin{cases} \dot{x}(t) = A_{q(t)}x(t) + B_{q(t)}u(t),\, x(0) = x_0, \\ y(t) = C_{q(t)}x(t), \end{cases} \tag{1}$$

where $\Omega = \{1, \dots, M\}$ is the set of different modes of Σ_{SLS}, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the controlled input, $y(t) \in \mathbb{R}^p$ is the measured output and $q(t)$ is the switching signal, i.e., a piecewise constant function taking values from the index set Ω. The system matrices $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$ and $C_j \in \mathbb{R}^{p\times n}$, where $j \in \Omega$, correspond to the linear system active in mode q, and x_0 is the initial state. Furthermore, let $x(t) = \phi(t, x_0, u, q)$ denote the state trajectory at time t of the SLS initialized at $x(0) = x_0 \in \mathbb{R}^n$, with input u and switching signal q. In what follows, we assume a zero initial condition, i.e., $x(0) = 0$ in (1) and, for $j \in \Omega$, the matrices A_j are Hurwitz. When these models are large-scale, modern analysis, simulation and optimization tools become drastically inefficient and thus, model order reduction (MOR) may become necessary.

In the context of linear time-invariant systems, several model reduction approaches have been efficiently developed since the 1960s (see the monograph [3] and the recent surveys [4], [9]). However, reliable MOR techniques for switched systems have been only studied in recent years. For discrete-time switched linear systems, see for instance [8] for reachability and observability reduction with constrained switching, [17], [18], [38] for $\mathcal{H}_\infty$-type reduction, and [16], [21], [33] for balancing-based methods. For continuous-time switched linear systems, see [6], [7], [22] for a class of moment-matching methods, [23], [27], [28], [32] for balancing-based methods and [31] for model reduction of systems affected by a low-rank switching. Also, [29] presents a theoretical analysis of the techniques proposed in [32] and [33] for continuous- and discrete-time SLS.

Besides [27], all of the balancing-based methods rely on Gramians satisfying Linear Matrix Inequalities (LMIs). Although LMIs provide a very flexible tool in control theory, they are costly to solve numerically in the large-scale setting. To overcome this, the current paper aims at providing new algebraic Gramians for SLS, denoted by $\mathcal{P}$ and $\mathcal{Q}$, respectively, which satisfy generalized Lyapunov equations. These Gramians are inspired by bilinear model reduction techniques, in which the generalized Lyapunov equation plays an important role, e.g., (see [13]). In addition, we prove that $\mathcal{P}$ and $\mathcal{Q}$ encode the reachability and observability spaces of the SLS (1), and their kernels correspond to the uncontrollable and unobservable spaces.

Once the proposed Gramians are computed, by means of a square root balancing approach (see [3]) and for a given state space dimension $r \ll n$, we are able to construct two projection matrices $V, W \in \mathbb{R}^{n\times r}$ such that $W^TV = I_r$, which allows us to determine the reduced-order SLS as

$$\hat{\Sigma}_{SLS} : \begin{cases} \dot{\hat{x}}(t) = \hat{A}_{q(t)}\hat{x}(t) + \hat{B}_{q(t)}u(t),\, \hat{x}(0) = 0, \\ \hat{y}(t) = \hat{C}_{q(t)}\hat{x}(t), \end{cases} \tag{2}$$

where

$$\hat{A}_j = W^TA_jV, \quad \hat{B}_j = W^TB_j, \quad \hat{C}_j = C_jV \tag{3}$$

for $j \in \Omega$. We call $V, W \in \mathbb{R}^{n\times r}$ global projection matrices because they are independent of the switched mode $j \in \Omega$ (see [15] for a discussion of local and global projection techniques). Readers should refer to [23] for a balancing-type method where the projection matrices $V_j, W_j \in \mathbb{R}^{n\times r}$ might depend on the mode $j \in \Omega$ and the authors consider a more general realization than (1).

Outline. The remaining parts of the paper are organized as follows. In Section 2, the SLS is formulated as a bilinear system. Inspired by this transformation, Gramians for SLS are proposed. In Section 3, we prove that those Gramians encode the reachability and observability spaces. Also, we propose a Gramian-based criterion to determine if an SLS is reachable and observable. In Section 4, the balanced truncation procedure based on these Gramians is introduced. Moreover,

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[1]Note that this class of systems is also referred to as Linear Switched Systems (or LSS) in the literature.

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under a certain assumption (see [29]), this procedure is shown to preserve quadratic stability and to have an error bound. Finally, numerical results are shown in Section 5, and Section 6 concludes the paper.

Notation. We denote by $\mathbb{N}$ the set of natural numbers including 0. Let $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{p\times q}$ be two real matrices, then $A \otimes B \in \mathbb{R}^{np\times mq}$ is the corresponding Kronecker product between A and B, and $\mathrm{vec}\,(A)$ is the vectorization of the matrix A formed by stacking the columns of A into a single column vector. If $\mathcal{P}$ and $\mathcal{Q}$ are two symmetric matrices, we write $\mathcal{P} < \mathcal{Q}$ (resp. $\mathcal{P} \leq \mathcal{Q}$) if the matrix $\mathcal{Q} - \mathcal{P}$ is positive definite (resp. semidefinite).

II. Bilinear formulation of an SLS and generalized Gramians

A. Bilinear realization

In this section, we rewrite the equations of an SLS to resemble a bilinear system. A very similar procedure was developed in [11] in the context of parametric systems.

To this aim, first, let us define the matrices

$$A = A_1 \quad D_j = A_j - A_1, \text{ for } j = 1, \ldots, M. \tag{4}$$

Notice, even though $D_1 = 0$, for simplicity, we are going to keep it in the equation. Now, let us replace the switching signal $q(t)$, which takes values in the mode set Ω, by M switching indicators $\{q_1(t), \ldots, q_M(t)\}$, taking binary values, i.e., $q_j(t) \in \{0, 1\}$ such that $\sum_{k=1}^{M} q_k(t) \equiv 1$. Therefore,

$$q(t) = k \Leftrightarrow q_k(t) = 1 \text{ and } q_j(t) = 0 \text{ for } j \neq k.$$

With this notion, the mode k is active when $q_k(t) = 1$ and $q_j = 0$ for $j \neq k$. The SLS from (1) can be expressed as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{j=1}^{M} q_j(t)D_jx(t) + q_j(t)B_ju(t), \\ y(t) &= \sum_{j=1}^{M} q_j(t)C_jx(t). \end{aligned} \tag{5}$$

Let us include the switching indicators as additional inputs, i.e.,

$$(\tilde{u}(t))^T = \begin{bmatrix} u(t)^T & q_1(t) & \ldots & q_M(t) \end{bmatrix} \in \mathbb{R}^{1\times(m+M)}$$

and $\tilde{B}_j = \begin{bmatrix} B_j & 0 \end{bmatrix}$ for $j \in \Omega$. Then,

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{j=1}^{M} \tilde{u}_{j+m}(t)D_jx(t) + \tilde{u}_{j+m}(t)\tilde{B}_j\tilde{u}(t), \\ y(t) &= \sum_{j=1}^{M} \tilde{u}_{j+m}(t)C_jx(t). \end{aligned} \tag{6}$$

The crucial observation is that the equations above are very similar to a bilinear system realization, which is usually given as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \sum_{j=1}^{M} u_j(t)N_jx(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \tag{7}$$

Indeed, if $B_j = B$ and $C_j = C$ for $j \in \Omega$, then, since $u_{j+m}(t) = q_j(t)$ and $\sum_{k=1}^{M} q_k(t) \equiv 1$, the terms of equation (6) can be written as

$$\sum_{j=1}^{M} \tilde{u}_{j+m}(t)\tilde{B}_j\tilde{u}(t) = \begin{bmatrix} B & 0 \end{bmatrix} \sum_{j=1}^{M} q_j(t)\tilde{u}(t) = Bu(t),$$

and, similarly,

$$\sum_{j=1}^{M} \tilde{u}_{j+m}(t)C_jx(t) = C\sum_{j=1}^{M} q_j(t)x(t) = Cx(t).$$

Hence, in this case, the realization of an SLS can be recast as a bilinear system. However, in the general case, when $B_j \neq B_k$ and $C_j \neq C_k$ for $j \neq k$, this is no longer true.

In what follows, we recall some results of model reduction of bilinear systems and, inspired by that, new Gramians for SLS are proposed.

B. Generalized Gramians for SLS

In the past years, model reduction of bilinear systems has been studied in the literature, see [5], [13], [30], [37] for more details. A bilinear system as (7) is associated to the reachability and observability Gramians

$$\mathcal{P}_B = \sum_{k=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} P_k(t_1, \ldots, t_k)P_k(t_1, \ldots, t_k)^T dt_1 \ldots dt_k, \tag{8a}$$

$$\mathcal{Q}_B = \sum_{k=1}^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} Q_k(t_1, \ldots, t_k)Q_k(t_1, \ldots, t_k)^T dt_1 \ldots dt_k, \tag{8b}$$

respectively, where

$$\begin{aligned} &P_1(t_1) = e^{At_1}B, \\ &Q_1(t_1) = e^{A^Tt_1}C^T, \\ &P_k(t_1, \ldots, t_k) = e^{At_k}\begin{bmatrix} N_1P_{k-1} & \ldots & N_MP_{k-1} \end{bmatrix}, \\ &Q_k(t_1, \ldots, t_k) = e^{A^Tt_k}\begin{bmatrix} N_1^TQ_{k-1} & \ldots & N_M^TQ_{k-1} \end{bmatrix}. \end{aligned}$$

Moreover, if the Gramians exist, i.e., the infinite sums converge, they satisfy the following generalized Lyapunov equations

$$A\mathcal{P}_B + \mathcal{P}_BA^T + \sum_{j=1}^{M} \left(N_j\mathcal{P}_BN_j^T\right) + BB^T = 0, \tag{9a}$$

$$A^T\mathcal{Q}_B + \mathcal{Q}_BA + \sum_{j=1}^{M} \left(N_j^T\mathcal{Q}_BN_j\right) + CC^T = 0. \tag{9b}$$

Those equations where proposed in [24] and used to construct minimal realizations and model reduction techniques based on balanced truncation of bilinear systems, see, e.g., [1], [2], [37] and [13]. As mentioned before, the realization of an SLS is not equivalent to a bilinear realization because $B_k \neq B_j$ and $C_k \neq C_j$ for $j \neq k$. However, inspired by those expressions, we propose the following Gramians to be associated to a given SLS.

Definition 1 (Generalized Gramians for SLS)**.** *Given an SLS as in* (1) *and the matrices D_j defined in* (4)*, let $\mathcal{P}, \mathcal{Q}$ be*

$$\mathcal{P} = \sum_{k=1}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} P_k(t_1, \dots, t_k) P_k(t_1, \dots, t_k)^T dt_1 \dots dt_k, \tag{10a}$$

$$\mathcal{Q} = \sum_{k=1}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} Q_k(t_1, \dots, t_k) Q_k(t_1, \dots, t_k)^T dt_1 \dots dt_k, \tag{10b}$$

where

$$\begin{aligned}
P_1(t_1) &= e^{At_1} \begin{bmatrix} B_1 & \dots & B_M \end{bmatrix}, \\
Q_1(t_1) &= e^{A^T t_1} \begin{bmatrix} C_1^T & \dots & C_M^T \end{bmatrix}, \\
P_k(t_1, \dots, t_k) &= e^{At_k} \begin{bmatrix} D_1 P_{k-1} & \dots & D_M P_{k-1} \end{bmatrix}, \\
Q_k(t_1, \dots, t_k) &= e^{A^T t_k} \begin{bmatrix} D_1^T Q_{k-1} & \dots & D_M^T Q_{k-1} \end{bmatrix}.
\end{aligned}$$

If they exist, $\mathcal{P}$ and $\mathcal{Q}$ will be called the reachabillity *and* observability Gramians *of the SLS* (1)*.*

As a consequence, if $\mathcal{P}, \mathcal{Q}$ exist, they are symmetric, positive semidefinite matrices which satisfy the following generalized Lyapunov equations

$$A\mathcal{P} + \mathcal{P}A^T + \sum_{j=1}^{M} \left(D_j \mathcal{P} D_j^T + B_j B_j^T \right) = 0, \tag{11a}$$

$$A^T\mathcal{Q} + \mathcal{Q}A + \sum_{j=1}^{M} \left(D_j^T \mathcal{Q} D_j + C_j^T C_j \right) = 0. \tag{11b}$$

Note that the name "Gramian" will be justified in Section III, where the connection between the newly proposed SLS Gramians and the reachability and observability sets is established.

The SLS Gramians can be computed using the Kronecker product, *i.e.*, let

$$\mathcal{M} = \left(A \otimes I_n + I_n \otimes A + \sum_{j=1}^{M} D_j \otimes D_j \right) \in \mathbb{R}^{n^2 \times n^2},$$

$$\mathcal{B} = \text{vec} \left(\sum_{k=1}^{M} B_j B_j^T \right) \quad \text{and} \quad \mathcal{C} = \text{vec} \left(\sum_{j=1}^{M} C_j^T C_j \right).$$

Then, the generalized reachability and observability Gramians are given by

$$\text{vec}\,(\mathcal{P}) = -\mathcal{M}^{-1}\mathcal{B} \quad \text{and} \quad \text{vec}\,(\mathcal{Q}) = -\mathcal{M}^{-T}\mathcal{C}.$$

However, in this Kronecker form, the solution of the generalized Lyapunov equation is determined by solving a set of $n(n+1)/2$ equations in $n(n+1)/2$ variables, whose cost is $O(n^6)$ operations. Fortunately, new efficient methodologies have been developed recently to determine low-rank solutions of these generalized Lyapunov equations (see [19], [12], [34] and [25]) which are suitable in the large-scale setting.

The following theorem, from [37], states a sufficient condition for existence and uniqueness of $\mathcal{P}$ and $\mathcal{Q}$.

Theorem 1 (Sufficient conditions for existence and uniqueness [37], Theorem 2)**.** *Let A, D_j, B_j, and C_j be given by the notation above. In addition, suppose that A is Hurwitz. Then, there exist real scalars $\beta > 0$ and $0 < \alpha \leq -\max_i(Re(\lambda_i(A)))$ such that*

$$\|e^{At}\| \leq \beta e^{-\alpha t}, \text{ for } t \geq 0.$$

Then, the reachabillity and observability Gramians satisfying (11a) *and* (11b) *exist if*

$$\left\| \sum_{j=1}^{M} D_j D_j^T \right\| < \frac{2\alpha}{\beta^2}. \tag{12}$$

Furthermore, under the conditions of Theorem 1, the symmetric positive semidefinite solutions $\mathcal{P}$ and $\mathcal{Q}$ of equation (11a) can be expressed as an infinite sum of symmetric positive semidefinite matrices $\mathcal{P}_k$ and $\mathcal{Q}_k$ (see [37] for more details), i.e.,

$$\mathcal{P} = \sum_{k=1}^{\infty} \mathcal{P}_k \text{ and } \mathcal{Q} = \sum_{k=1}^{\infty} \mathcal{Q}_k,$$

where

$$\begin{aligned}
A\mathcal{P}_1 + \mathcal{P}_1 A^T + \sum_{j=1}^{M} B_j B_j^T &= 0, \\
A^T \mathcal{Q}_1 + \mathcal{Q}_1 A + \sum_{j=1}^{M} C_j^T C_j &= 0,
\end{aligned}$$

and

$$\begin{aligned}
A\mathcal{P}_k + \mathcal{P}_k A^T + \sum_{j=1}^{M} D_j \mathcal{P}_{k-1} D_j^T &= 0, \\
A^T \mathcal{Q}_k + \mathcal{Q}_k A + \sum_{j=1}^{M} D_j^T \mathcal{Q}_{k-1} D_j &= 0.
\end{aligned}$$

Remark 1. *In [31], the authors also replace the SLS by a non-switched system with extended input and output vectors, which is able to reproduce the dynamical behavior of the original SLS by applying a certain feedback law. This approach is designed for systems with low-rank switching, i.e., the matrices D_j have a low-rank factorization. In contrast to [31], the approach presented in this section does not have any limitation with respect to low-rank switching.*

From here on, we assume the existence and uniqueness of positive semidefinite solutions to (11a) and (11b) and that the conditions of Theorem 1 hold. In what follows, we show that the proposed Gramians encode the reachability and observability sets of the SLS (1).

III. Gramians and reachability and observability sets

As previously mentioned, the main goal of this section is to show that the SLS Gramians encode the reachability and observability sets of the SLS (1).

First of all, let us recall the definition and properties of those sets in the context of SLS. The reader should refer to [36] and [35] for more details. Let us start with the notion of reachability and observability sets.

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Definition 2 (Reachable set)**.** *A state $x \in \mathbb{R}^n$ is* reachable*, if there exist a time instant t_f, a switching signal $q : [0, t_f] \to \Omega$, and an input $u : [0, t_f] \to \mathbb{R}^p$, such that $\phi(t_f, 0, u, q) = x$. The* reachable set *of the SLS* (1) *is denoted by $\mathcal{R}$, that is the set of states which are reachable.*

Definition 3 (Observability set)**.** *A state x is said to be* unobservable*, if for any switching signal q, there exists an input $u(t)$ such that*

$$C_{q(t)}\phi(t, x, u, q) = C_{q(t)}\phi(t, 0, u, q),\ \forall t \geq 0.$$

The unobservable set *of the SLS* (1)*, denoted by $\mathcal{UO}$, is the set of states which are unobservable. The* observable set *of the SLS, denoted by $\mathcal{O}$, is defined by $\mathcal{O} = (\mathcal{UO})^\perp$.*

In what follows, we recall the algebraic characterizations of $\mathcal{R}$ and $\mathcal{O}$ and we state the main result of this paper, i.e., the Gramian-based version of this result.

A. Characterization of the reachability and observability sets

The following result, from [35], describes the reachable and observable sets of the SLS (1) by algebraic conditions.

Theorem 2 (Algebraic conditions [35], Theorem 4.17)**.** *For the SLS* (1)*, the reachable and observable sets $\mathcal{R}$ and $\mathcal{O}$ are linear subspaces of $\mathbb{R}^n$ given by*

$$\mathcal{R} = \sum_{k=1}^{\infty} \left(\sum_{\substack{i_0,\dots,i_k \in \Omega \\ j_1,\dots,j_k \in \mathbb{N}}} A_{i_k}^{j_k} \dots A_{i_1}^{j_1} \operatorname{range}(B_{i_0}) \right),$$

and

$$\mathcal{O} = \sum_{k=1}^{\infty} \left(\sum_{\substack{i_0,\dots,i_k \in \Omega \\ j_1,\dots,j_k \in \mathbb{N}}} (A_{i_k}^{j_k})^T \dots (A_{i_1}^{j_1})^T \operatorname{range}\left(C_{i_0}^T\right) \right).$$

Theorem 2 generalizes the well-known reachability and observability criteria for LTI systems. In the context of LTI systems, the reachable set is a linear subspace of $\mathbb{R}^n$ given by $\mathcal{R} = \sum_{k=0}^{\infty} A^k \operatorname{range}(B)$, i.e., all possible combinations of polynomials in A multiplied by B. In the context of SLS, the reachable set is also a linear subspace of $\mathbb{R}^n$ given by all possible combinations of M-variate polynomials in $A_1, \dots, A_M$ multiplied by B_k. Moreover, this subspace can be seen as the smallest subspace of $\mathbb{R}^n$ that contains each $\operatorname{range}(B_i)$ and is invariant under each A_i, for $i \in \Omega$.

In what follows, we state the main result of this paper.

Theorem 3 (Gramian conditions)**.** *Let $\mathcal{P}, \mathcal{Q}$ be the solutions of the generalized Lyapunov equations* (11a) *and* (11b)*, respectively. Then, the reachable and observable spaces $\mathcal{R}$, $\mathcal{O}$ are given by*

$$\mathcal{R} = \operatorname{range}(\mathcal{P}) \quad \text{and} \quad \mathcal{O} = \operatorname{range}(\mathcal{Q}).$$

Proof. The proof of this theorem shows that the ranges of $\mathcal{P}$ and $\mathcal{Q}$ are given by the algebraic condition of Theorem 2. The complete proof is detailed in the appendix. □

Theorem 3 states a Gramian-based characterization of the reachable and observable sets. Moreover, the following reachability and observability criteria are corollaries of this result.

Corollary 1 (Reachability and observability criteria)**.** *Given Σ, an SLS as in* (1)*, and suppose that $\mathcal{P}, \mathcal{Q}$ are the unique solutions of the generalized Lyapunov equations* (11a) *and* (11b)*. Then,*

1) *Σ is completely reachable if and only if*

$$\operatorname{range}(\mathcal{P}) = \mathbb{R}^n.$$

2) *Σ is completely observable if and only if*

$$\operatorname{range}(\mathcal{Q}) = \mathbb{R}^n.$$

Proof. The SLS is completely reachable (respectively, observable) if and only if $\mathcal{R} = \mathbb{R}^n$ (respectively, $\mathcal{O} = \mathbb{R}^n$). Then the result is a straightforward application of Theorem 3. □

Corollary 1 provides simple criteria to determine if a given SLS is completely reachable and observable. This result is equivalent to verifying if the algebraic conditions given in Theorem 2 generate the entire space. However, to the best of the authors' knowledge, they have not been presented in this Gramian-based form.

To sum up, the Gramians $\mathcal{P}$ and $\mathcal{Q}$ proposed in Definition 1 encode the reachable and observable spaces of a given SLS (as stated in Theorem 3). As a consequence, Corollary 1 provides a simple way to verify if a given SLS is completely reachable and observable. In the next section, we present the procedure for model order reduction by balanced truncation using these Gramians.

IV. Model reduction for switched linear systems

In this section, we state the balancing procedure for model reduction of SLS and we state some sufficient conditions under which this procedure preserves stability, and provide an error bound.

A. Balanced truncation for SLS

As mentioned before, the Gramians $\mathcal{P}$ and $\mathcal{Q}$ encode the reachable and observable spaces. This can be rewritten as follows :

1) If a state x lies in $\ker(\mathcal{P})$, then it is unreachable.
2) If a state x lies in $\ker(\mathcal{Q})$, then it is unobservable.

Hence, the subspace $\ker(\mathcal{P}) \cap \ker(\mathcal{Q})$ is not important for the transfer between input and output and might be truncated. This motivates us to use the proposed Gramians to determine the reduced-order models. To guarantee that states which are hard to control and hard to observe will be truncated simultaneously, we need to find a transformation T, leading to a transformed switched system, whose controllability and observability Gramians are equal and diagonal, i.e.,

$$T^{-1}\mathcal{P}T^{-T} = T^T\mathcal{Q}T = \Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n),$$

with $\sigma_i \geq \sigma_{i+1}$. This balancing transformation exists if and only if $\mathcal{P}$ and $\mathcal{Q}$ are full rank matrices (see Chapter 7 of [3]).

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Next, we assume that the matrices of the balanced system are partitioned as

$$A_{j,\mathcal{B}} = \begin{bmatrix} A_j^{11} & A_j^{12} \\ A_j^{21} & A_j^{22} \end{bmatrix},\ B_{j,\mathcal{B}} = \begin{bmatrix} B_j^1 \\ B_j^2 \end{bmatrix},$$
$$C_{j,\mathcal{B}} = \begin{bmatrix} C_j^1 & C_j^2 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},$$

where $\Sigma_1 = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$ and $\Sigma_2 = \operatorname{diag}(\sigma_{r+1}, \ldots, \sigma_n)$. In the balancing basis, the truncation step is simply obtained by setting the ROM to be given by the matrices $\hat{A}_j = A_j^{11}, \hat{B}_j = B_j^1,\ \hat{C}_j = C_j^1$.

It is worth noticing that, even if the Gramians $\mathcal{P}$ and $\mathcal{Q}$ are not full rank matrices, balanced truncation can still be performed. In this case, as a consequence of Corollary 1, there exist some states that are either unreachable or unobservable. Analogous to the linear case, we do not need to compute the balancing transformation explicitly. Instead, one can construct two projection matrices V and W using the Cholesky factors of $\mathcal{P}$ and $\mathcal{Q}$, and the SVD of their product. This procedure is known as square-root balanced truncation (see Section 7.3 of [3]), and its version for SLS is presented in Algorithm 1.

Algorithm 1 Balanced truncation for SLS

Input: Matrices (A_j, B_j, C_j) for $j = 1, \ldots, M$ and reduced-order r.

Output: Reduced-order matrices $(\hat{A}_j, \hat{B}_j, \hat{C}_j)$ for $j = 1, \ldots, M$.

1: Let $A = A_1,\ D_j = A_j - A_1$.
2: Compute $\mathcal{P}$ and $\mathcal{Q}$ by solving the generalized Lyapunov equations (11a) and (11b).
3: Compute the Cholesky decomposition $\mathcal{P} = SS^T$ and $\mathcal{Q} = RR^T$.
4: Compute the SVD of $S^T R$ written as
$$S^T R = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \operatorname{diag}(\Sigma_1, \Sigma_2) \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T.$$
5: Construct the projection matrices $V = SU_1\Sigma_1^{-\frac{1}{2}}$ and $W = RV_1\Sigma_1^{\frac{1}{2}}$.
6: Construct $\hat{A}_j = W^T A_j V,\ \hat{B}_j = W^T B_j$ and $\hat{C}_j = C_j V$ for $j = 1, \ldots, M$.
7: **return** $\hat{A}_j, \hat{B}_j$ and $\hat{C}_j$.

One should notice that, if the matrix A is Hurwitz, the proposed procedure provides a matrix $\hat{A} = W^T A V$ which is also Hurwitz, provided $\sigma_r \neq \sigma_{r+1}$. This is a consequence of Theorem 2.3 from [14]. In a large-scale setting, a solution of the generalized Lyapunov equation is computed directly in the factorized form, i.e., one searches for the solution S as a low-rank factor such that $\mathcal{P} \approx SS^T$ (see [12], [34] and [25]). In this context, one can avoid constructing the full solutions $\mathcal{P}$, $\mathcal{Q}$, which is very costly with respect to memory consumption and computational resources.

Remark 2. *The reader should notice that, in equation* (4)*, we have made the choice of $A = A_1$, which plays an important role in the Gramians computation. However, by reordering the subsystems, other choices could be made. Hence, for different ordering, we might expect different Gramians and, consequently, different model reduction results. In our experience, the strategy to set A to be the matrix of the subsystem which corresponds to the first mode in the simulation of the SLS* (1) *is a good choice.*

Remark 3. *If the subsystems present very different dynamics, we might expect the Gramians to be very rich with respect to subspace information. As a consequence, very few states (or even zero) can be removed from the original system.*

In the next subsection, under some assumptions, we show some properties of the reduced-order models obtained by Algorithm 1.

B. Quadratic stability preservation and error bounds

We briefly review the definition of quadratic stability for SLS.

Definition 4 (Quadratic stability [29], Lemma 1)**.** *The SLS as in* (1) *is called* quadratically stable *if there exists a positive definite matrix $P > 0$ such that*

$$A_j^T P + P A_j < 0, \text{ for all } j \in \Omega.$$

Quadratic stability is a sufficient condition for exponential stability for all switching signals (see [26]). In what follows in this section, we employ the following assumption.

Assumption 1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be symmetric positive definite solutions of* (11a) *and* (11b)*. Let us assume that*

$$D_k \mathcal{P} + \mathcal{P} D_k^T \leq \sum_{j=1}^{M} D_j \mathcal{P} D_j^T + \sum_{j=1, j\neq k}^{M} B_j B_j^T, \quad \text{and} \tag{13a}$$
$$D_k^T \mathcal{Q} + \mathcal{Q} D_k \leq \sum_{j=1}^{M} D_j^T \mathcal{Q} D_j + \sum_{j=1, j\neq k}^{M} C_j^T C_j, \tag{13b}$$

for every $k = 2, \ldots, M$.

Reader should notice that Assumption 1 implies that

$$A_k \mathcal{P} + \mathcal{P} A_k^T + B_k B_k^T \leq 0, \tag{14a}$$
$$A_k^T \mathcal{Q} + \mathcal{Q} A_k + C_k^T C_k \leq 0, \tag{14b}$$

for every $k \in \Omega$. Hence, under Assumption 1, the Gramians proposed in this work are also Gramians in the sense of [29, Definition 10], i.e., symmetric positive definite matrices which satisfy the set of LMIs (14). As a consequence, under the Assumption 1, all of the results developed in [29] are also valid for the Gramians proposed in (10). Two are particularly important for model reduction, and we recall them in what follows.

Proposition 1 (Quadratic stability preservation [29], Lemma 12)**.** *Under Assumption 1, suppose that at least one of the above propositions holds:*

1) $A_k \mathcal{P} + \mathcal{P} A_k^T + B_k B_k^T < 0, \forall k \in \Omega$,
2) $A_k^T \mathcal{Q} + \mathcal{Q} A_k + C_k^T C_k < 0, \forall k \in \Omega$.

Then the reduced-order model constructed by Algorithm 1 is also quadratically stable.

Proposition 1 states a sufficient condition to preserve quadratic stability by model reduction using Algorithm 1. The following result provides an error bound between the original and the reduced-order model.

Theorem 4 (Error bound [29], Theorem 6)**.** *Under Assumption 1, the output error between the original model and the reduced-order model* (2)*, obtained by Algorithm 1, is bounded by*

$$\|y-\hat{y}\|_{L_2} \leq 2\left(\sum_{k=r+1}^{n}\sigma_k\right)\|u\|_{L_2} \tag{15}$$

for every switching signal $q(t)$, where σ_k are the neglected singular values.

Proposition 1 and Theorem 4 give some important properties of model reduction by balanced truncation using the Gramians from Definition 1, provided Assumption 1 holds. Since this assumption involves LMIs, it is hard to check in the large-scale setting. However, in the case where $D_k = 0$, for all $k \in \Omega$, i.e., where all of the subsystems have the same matrix A (a situation that might occur when only the actuator position varies with time), Assumption 1 holds. This can induce us to think that Assumption 1 might hold whenever the matrices D_k are small. We leave it as an open problem whether weaker assumptions exist such that similar results are also valid. It is worth noticing that the error bound for balanced truncation of bilinear systems was for a long time also an open problem. Recently, the paper [10] proposed a solution using an infinite-dimensional setting and Hilbert space techniques. We believe that this methodology might also be adapted to SLS and we leave this as future work.

If Assumption 1 is not satisfied, we can still use the proposed Gramians to remove states that are related to zero and very small singular values. This would be a balancing-like approach were the main information with respect to the important subspaces is still kept in the reduced-order model. Also, it is important to emphasize that the Gramians presented in [29] are solutions of LMIs and, in some cases, they might not exist. This happens typically when the original system is not quadratically stable (even if all of the subsystems are stable). In this case, we can still use the Gramians from (10) for model reduction.

In the next section, we apply the results derived in this paper to some numerical examples.

V. Numerical examples

This section is dedicated to the application of results proposed in Sections III and IV, namely the Gramian-based characterization (Theorem 3) of the reachable and observable spaces and the balanced truncation procedure (Algorithm 1). The results will be compared with the balancing method proposed in [27]. There, it has been shown that, if certain restrictive conditions are satisfied, a simultaneous balanced transformation can be constructed. When those conditions are not satisfied, the authors propose to use, instead, the so-called reachability and observability average Gramians given by

$$\mathcal{P}_{avg} = \frac{1}{M}\sum_{k=1}^{M}\mathcal{P}_k \quad \text{and} \quad \mathcal{Q}_{avg} = \frac{1}{M}\sum_{k=1}^{M}\mathcal{Q}_k, \tag{16}$$

where $\mathcal{P}_k$ and $\mathcal{Q}_k$ satisfy

$$\begin{aligned} A_k\mathcal{P}_k + \mathcal{P}_k A_k^T + B_k B_k^T &= 0, \\ A_k^T\mathcal{Q}_k + \mathcal{Q}_k A_k + C_k^T C_k &= 0. \end{aligned}$$

In what follows, we illustrate the Gramian-based characterization of the reachbility set using Theorem 3.

A. Example 1: Reachability set of SLS

Let us consider a 2-mode SLS Σ given by

$$A_1 = -I_8, A_2 = A_1 + D,$$

where $D \in \mathbb{R}^{8\times 8}$ satisfies $D_{21} = D_{32} = D_{43} = 1$ and $D_{jk} = 0$ elsewhere. In addition, $B_1^T = \begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}$, $B_2^T = \begin{bmatrix}0 & \dots & 0 & 1\end{bmatrix}$. Then, the reachability Gramian $\mathcal{P}$ given by equation (11a) is

$$\mathcal{P} = \operatorname{diag}\left(\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{16},0,0,0,\frac{1}{2}\right)$$

and the average reachability Gramian (proposed in [27]) is

$$\mathcal{P}_{avg} = \operatorname{diag}\left(\frac{1}{2},0,0,0,0,0,0,\frac{1}{2}\right).$$

As a consequence, since $\operatorname{rank}(\mathcal{P}) \neq 8$, Corollary 1 tells us that Σ is not completely reachable. In addition, according to Theorem 3, the reachable space of Σ is given by

$$\mathcal{R} = \operatorname{range}(\mathcal{P}) = \operatorname{span}(e_1,e_2,e_3,e_4,e_8)$$

Notice that the average Gramian $\mathcal{P}_{avg}$ does not encode the reachability space. More generally, one can show that

$$\operatorname{range}(\mathcal{P}_{avg}) \subset \operatorname{range}(\mathcal{P}).$$

In what follows, we use the Gramians to construct reduced-order models via Algorithm 1.

B. Example 2: Model reduction of a small scale SLS

Let us now consider a 2-mode SLS of order 12, with state matrices given by $A_1, A_2 \in \mathbb{R}^{12\times 12}$ such that

$$A_1(i,j) = \begin{cases} -1, & \text{if } i=j, \\ \frac{1}{2}, & \text{if } i-1=j, \\ 0, & \text{otherwise,}\end{cases} \text{ and } A_2(i,j) = \begin{cases} -2, & \text{if } i=j, \\ \frac{4}{5}, & \text{if } i-1=j, \\ -\frac{1}{5}, & \text{if } i+1=j, \\ 0, & \text{elsewhere,}\end{cases}$$

$B_1^T = C_1 = \begin{bmatrix}0 & \dots & 0 & 1\end{bmatrix}$ and $B_2^T = C_2 = \begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}$. Let $A = A_1$ and $D = A_2 - A_1$. The reader should notice that the matrix D has no low-rank factorization and the method proposed in [31] is not suitable for this example. For an order $r = 6$, we construct reduced-order models using generalized Gramians $\mathcal{P}$ and $\mathcal{Q}$ satisfying (11), and the average Gramians $\mathcal{P}_{avg}$ and $\mathcal{Q}_{avg}$ satisfying (16). We compare the time domain response of the original SLS with that of the reduced-order models. We consider a control input $u(t) = e^{-\frac{1}{2}t}$ and a switching signal

$$q(t) = \begin{cases} 1, & t\in[0,0.2)\cup[0.5,3)\cup[3.5,5), \\ 2, & t\in[0.2,0.5)\cup[3,3.5). \end{cases}$$

We depict the absolute error between the original system and the reduced-order models in Figure 1.

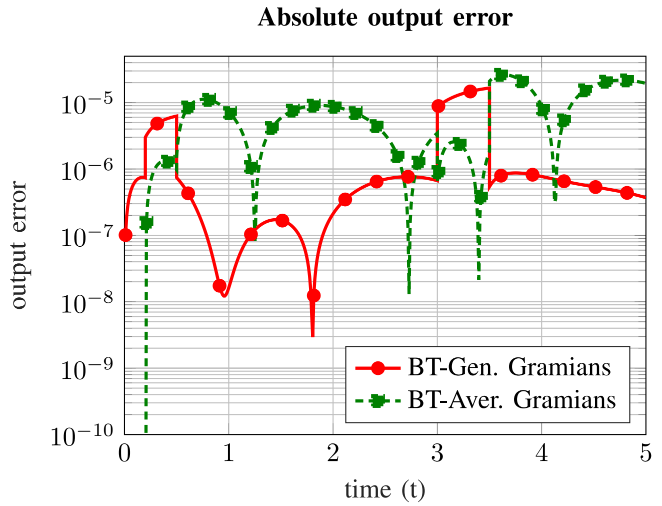

Fig. 1. Absolute output error between the original model and the reduced ones for Example 2 (generalized Gramians: red solid line, averaged Gramians: green dashed line).

By inspecting Figure 1, we conclude that both methods are able to follow the behavior of the original system with a certain precision. It is worth noticing that the proposed method performs slightly better than the averaged Gramians. Additionally, by numerical computation, we can verify that the following matrix inequalities

$$0 < D\mathcal{P}D^T + B_2B_2^T, \;\; 0 < D^T\mathcal{Q}D + C_2^TC_2,$$
$$D\mathcal{P} + \mathcal{P}D^T < D\mathcal{P}D^T + B_1B_1^T, \text{ and}$$
$$D^T\mathcal{Q} + \mathcal{Q}D < D^T\mathcal{Q}D + C_1^TC_1$$

are satisfied for the generalized Gramians. This implies that Assumption 1 holds. As a consequence, the original model and the reduced-order model obtained using the generalized Gramians are quadratically stable. Furthermore, the error bound (15)

$$\|y - \hat{y}\|_{L_2} \leq 2\left(\sum_{k=7}^{12} \sigma_k\right)\|u\|_{L_2} \approx 1.87 \cdot 10^{-2}$$

also holds. Indeed, one can compute by quadrature that $\|y - \hat{y}\|_{L_2} \approx 1.29 \cdot 10^{-4}$ for the above simulation.

In what follows, we apply the proposed method to a large-scale system.

C. Example 3: Model reduction of a large scale SLS

For the next experiment, let us consider a 2-mode SLS of order 1000, whose matrices are given by

$$A_1 = \begin{bmatrix} -2 & 1 & & \\ 0.1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 0.1 & -2 \end{bmatrix}, \; A_2 = \begin{bmatrix} -2 & 0.5 & & \\ 1 & -2 & 0.5 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2 \end{bmatrix},$$

$B_1^T = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$, $B_2^T = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \end{bmatrix}$ and $C_2 = \begin{bmatrix} 0 & \dots & 0 & 1 & 0 \end{bmatrix}$. Let $A = A_1$ and $D = A_2 - A_1$. Once again the matrix D has no low-rank factorization and the method proposed in [31] is not suitable for the example. We first compute the generalized Gramians $\mathcal{P}$ and $\mathcal{Q}$ satisfying (11), and the average Gramians $\mathcal{P}_{avg}$ and $\mathcal{Q}_{avg}$ satisfying (16). The normalized Hankel singular values are represented in Figure 2.

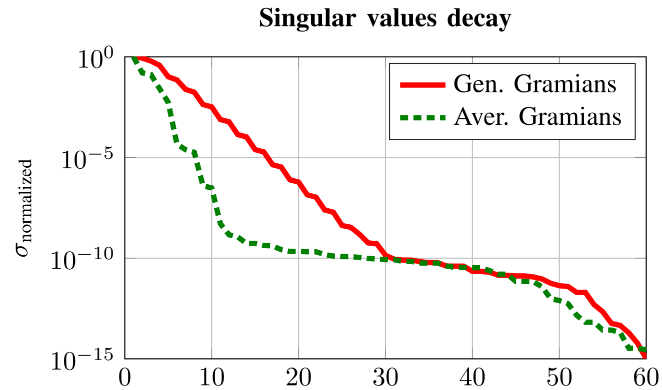

Fig. 2. Normalized Hankel singular values decay for Example 3 corresponding to the generalized Gramians (red solid line) and averaged Gramians (green dashed line).

Choose the truncation order $r = 15$ for the reduced SLS using both methods. We compare the time domain response of the original SLS against the ones corresponding to the two reduced models. For this, we use as the control input, $u(t) = 10\sin(30t)e^{-t}$ and as the switching signal

$$q(t) = \begin{cases} 1, & t \in [0, 0.5) \cup [2, 2.5) \cup [4, 5) \cup [5.5, 6), \\ 2, & t \in [0.5, 2) \cup [2.5, 4) \cup [5, 5.5). \end{cases}$$

The results are represented in Figure 3. The absolute errors are represented in Figure 4. Also, Assumption 1 does not hold for this system. Indeed, we have computed numerically the eigenvalues of $B_1B_1^T + D\mathcal{P}D^T - D\mathcal{P} - \mathcal{P}D^T$, and we have found negative eigenvalues.

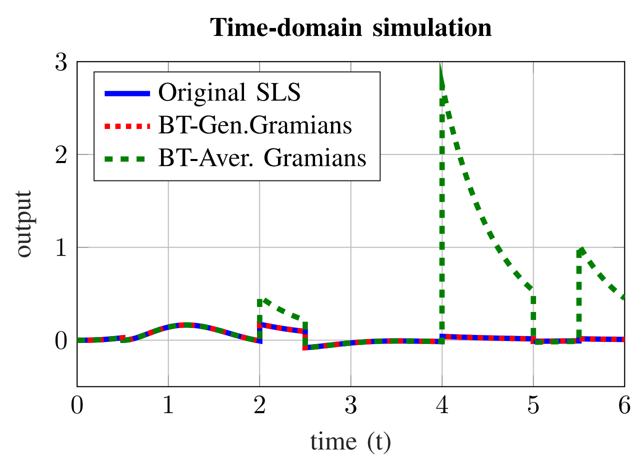

Fig. 3. Output corresponding to the time domain simulation of the original model (blue solid line), generalized Gramian ROM (red dotted line) and averaged Gramian (green dashed line) ROM for Example 3.

By inspecting the time-domain error between the original response and the two reduced-order models (Figure 4), we observe that the new proposed method generally produces better

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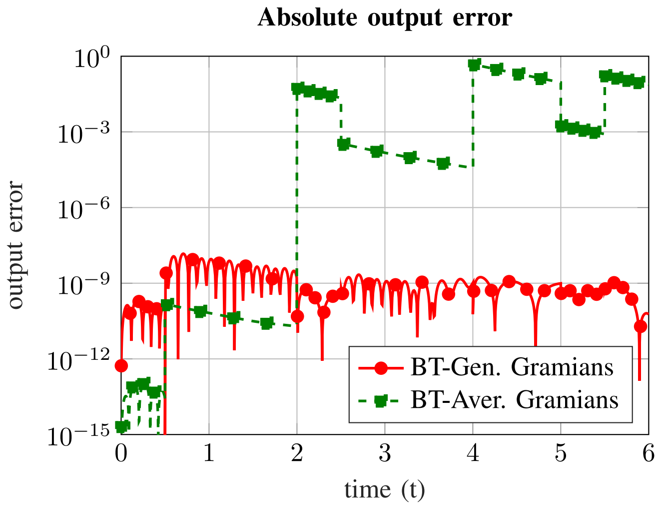

Fig. 4. Absolute output error between the original model and the reduced ones for Example 3 (generalized Gramians: red solid line, averaged Gramians: green dashed line).

results. Although Assumption 1 does not hold, we compute the error bound of Theorem 4 to inspect if it is also valid in this case. For this, notice that $\|u\|_{L_2} = 10\sqrt{\frac{225}{901}} \approx 4.997$, so that the error bound can be computed as $2\left(\sum_{k=r+1}^{n} \sigma_k\right)\|u\|_{L_2} = 5.033 \cdot 10^{-5}$. By numerical computing of the L_2-norm of the error between the original and the reduced-order model, one obtains $7.0 \cdot 10^{-9}$ for the system obtained using the proposed method and 0.27 for the one obtained using average Gramians. As a conclusion, even though Assumption 1 does not hold, the bounds presented in Theorem 4 are satisfied for the proposed method.

VI. Conclusion

In this paper, we have proposed new reachability and observability Gramians for SLS, satisfying generalized Lyapunov equations. Also, we prove that those Gramians encode the reachable and observable sets of the SLS. Based on these Gramians, a balancing-type procedure is proposed enabling to find global projectors V and W to construct a reduced-order model. Also, under certain assumptions, the proposed procedure is shown to preserve quadratic stability and to have an error bound. However, since those assumptions are difficult to check in the large-scale context, one possible future research axis is to find whether weaker assumptions exist such that similar results are also valid. Finally, the results are illustrated by some numerical examples.

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Appendix

In what follows, we present the proof of Theorem 3. We only prove that $\mathcal{R} = \operatorname{range}(\mathcal{P})$. The proof that $\mathcal{O} = \operatorname{range}(\mathcal{Q})$ follows analogously.

The complete proof of Theorem 3 requires the following propositions.

Proposition 2 ([20], Theorem 2.2)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a Hurwitz matrix and $B \in \mathbb{R}^{n\times m}$. Then, the Lyapunov equation*

$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0 \tag{17}$$

has a unique symmetric positive semidefinite solution $\mathcal{P}$ satisfying

$$\operatorname{range}(\mathcal{P}) = \sum_{l=0}^{\infty} A^l \operatorname{range}(B).$$

Proposition 3. *Let $A \in \mathbb{R}^{n\times n}$ be a Hurwitz matrix and $B_1, \ldots, B_M \in \mathbb{R}^{n\times m}$. Then, the Lyapunov equation*

$$A\mathcal{P} + \mathcal{P}A^T + \sum_{j=1}^{M} B_j B_j^T = 0 \tag{18}$$

has a unique symmetric positive semidefinite solution $\mathcal{P}$ satisfying

$$\operatorname{range}(\mathcal{P}) = \sum_{l=0}^{\infty}\sum_{j=1}^{M} A^l \operatorname{range}(B_j)$$

Proof. Let $\mathcal{P}_j$ be the unique solution of

$$A\mathcal{P}_j + \mathcal{P}_j A^T + B_j B_j^T = 0.$$

Then, by linearity, $\mathcal{P} = \sum_{j=1}^{M} \mathcal{P}_j$ is the solution of the Lyapunov equation (18). Moreover, since $\mathcal{P}_j$ is a symmetric positive semidefinite matrix, $\operatorname{range}(\mathcal{P}) = \sum_{j=1}^{m} \operatorname{range}(\mathcal{P}_j)$ and the result follows as a consequence of Proposition 2. □

Proposition 4. *Let $A \in \mathbb{R}^{n\times n}$ be a Hurwitz matrix and $\mathcal{P}_{k-1}$ be a symmetric positive semidefinite matrix. Then, the Lyapunov equation*

$$A\mathcal{P}_k + \mathcal{P}_k A^T + \sum_{j=1}^{M} D_j \mathcal{P}_{k-1} D_j^T = 0 \tag{19}$$

has a unique symmetric positive semidefinite solution $\mathcal{P}_k$ satisfying

$$\operatorname{range}(\mathcal{P}_k) = \sum_{l=0}^{\infty}\sum_{j=1}^{M} A^l D_j \operatorname{range}(\mathcal{P}_{k-1}).$$

Proof. Since $\mathcal{P}_{k-1}$ is symmetric positive semidefinite, it has a decomposition given by $\mathcal{P}_{k-1} = L_{k-1}L_{k-1}^T$. Then, equation (19) can be rewritten as

$$A\mathcal{P}_k + \mathcal{P}_k A^T + \sum_{j=1}^{M} D_j L_{k-1} L_{k-1}^T D_j^T = 0$$

If we rewrite $\tilde{B}_j = D_j L_{k-1}$, by applying Propostion 3 and using the fact that $\operatorname{range}(L_{k-1}) = \operatorname{range}(\mathcal{P}_{k-1})$, the result follows. □

Proposition 5. *Let $A \in \mathbb{R}^{n\times n}$ be a Hurwitz matrix. Suppose $\mathcal{P}$ is the unique solution of*

$$A\mathcal{P} + \mathcal{P}A^T + \sum_{j=1}^{M}\left(D_j \mathcal{P} D_j^T + B_j B_j^T\right) = 0.$$

To simplify the notation, let us denote $A = D_{M+1}$. Then,

$$\mathrm{range}\left(\mathcal{P}\right) = \sum_{k=1}^{\infty} \left(\sum_{\substack{i_0,\ldots,i_k \in \Omega \cup \{M+1\} \\ j_1,\ldots,j_k \in \mathbb{N}}} D_{i_k}^{j_k} \ldots D_{i_1}^{j_1} \,\mathrm{range}\left(B_{i_0}\right) \right).$$

Proof. As stated in Section 2, $\mathcal{P} = \sum_{k=1}^{\infty} \mathcal{P}_k$. Hence, since $\mathcal{P}_k$ are symmetric positive semidefinite matrices for all $k = 1, 2, \ldots,$ we must have

$$\mathrm{range}\left(\mathcal{P}\right) = \sum_{k=1}^{\infty} \mathrm{range}\left(\mathcal{P}_k\right).$$

The result follows from Proposition 3 and by recurrence using Proposition 4. □

Finally, Theorem 3 follows from Proposition 5 by the fact that

$$A_j \in \mathrm{span}\left(D_1, \ldots, D_M, D_{M+1}\right), \forall j \in \Omega, \quad A = D_{M+1},$$

and

$$D_j \in \mathrm{span}\left(A_1, \ldots, A_M\right), \forall j \in \Omega \cup \{M+1\},$$

so that the algebraic conditions given in Theorem 2 are equivalent to the algebraic condition given in Proposition 5.

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