New variational and multisymplectic formulations of the Euler-Poincaré equation on the Virasoro-Bott group using the inverse map

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Abstract

We derive a new variational principle and a new multisymplectic formulation for a family of equations defined on the Virasoro-Bott group using the inverse map (also called 'back-to-labels' map). This family contains as special cases the well-known Korteweg-de Vries, Camassa-Holm, and Hunter-Saxton equations.

1 Introduction

The family of equations

$$\alpha(u_t + 3uu_x) - \beta(u_{xxt} + 2u_x u_{xx} + uu_{xxx}) + au_{xxx} = 0, \tag{1.1}$$

where a, α, β are real nonnegative parameters, was introduced in [29] as equations of the geodesic flow associated to different right-invariant metrics on the Virasoro-Bott group (see also [30], [38]). Various hydrodynamical approximations are special cases of (1.1): for $\alpha = 1$ and $\beta = 0$ it becomes the Korteweg-de Vries equation ([31], [16])

$$u_t + 3uu_x + au_{xxx} = 0, \tag{1.2}$$

whereas for $\alpha = \beta = 1$ we obtain the Camassa-Holm equation ([8], [9], [23])

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + au_{xxx} = 0, \tag{1.3}$$

and for $\alpha = 0$ and $\beta = 1$ we get the Hunter-Saxton equation ([26], [27])

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$$u_{xxt} + 2u_x u_{xx} + u u_{xxx} - a u_{xxx} = 0. \tag{1.4}$$

The main goal of this paper is to derive a new canonical variational principle for the family of equations (1.1), and further determine their new multisymplectic formulation. By doing so, we obtain unified variational and multisymplectic characterizations of the well-known KdV, CH, and HS equations.

Variational principles have proved extremely useful in the study of nonlinear evolution PDEs. For instance, they often provide physical insights into the considered problem, facilitate finding conserved quantities by relating them to symmetries via Noether's theorem, allow one to determine approximate solutions to PDEs by minimizing the action functional over a class of test functions (see, e.g., [11]), and provide a way to construct a class of numerical methods called variational integrators (see [35], [36]). A canonical variational principle for the KdV equation expressed in terms of the velocity potential was first proposed by Whitham [47]; see also [11], [16], [28], [34]. In fact, there is an infinite family of such Lagrangians, as shown by Nutku [41]. Two canonical variational principles for the dispersionless CH equation ($a = 0$) were introduced in [12] and [32]. Two variational structures are also known for the HS equation with $a = 0$ (see [1], [26], [27]).

Multisymplectic structures of Hamiltonian PDEs were first considered by Bridges [5] as a natural generalization of the symplectic structure of Hamiltonian ODEs. Among other applications, multisymplectic formalism is useful for, e.g., the stability analysis of water waves (see [5], [6]) or construction of a class of numerical methods known as multisymplectic integrators (see [7], [35]). It was observed in the literature that similar to symplectic integrators for Hamiltonian ODEs, multisymplectic integrators demonstrate superior performance in capturing long time dynamics of PDEs (see [40]). To the best of our knowledge, only one multisymplectic formulation of the KdV equation has been considered so far (see [6], [50]). Four different multisymplectic formulations are known for the dispersionless CH equation (see [10], [12], [32]). Two multisymplectic structures for the HS equation with $a = 0$ were described in [39].

Main content The main content of the remainder of this paper is, as follows.

In Section 2 we review the Euler-Poincaré theory on the Virasoro-Bott group and then construct a new canonical variational principle in terms of the inverse map. The main results of this section are Theorem 2.2 and the variational principle (2.24).

In Section 3 we derive the multisymplectic form formula associated with our variational principle and then deduce a new multisymplectic formulation of the family of equations (1.1). The main result of this section is Theorem 3.1.

Section 4 contains the summary of our work and the discussion of the directions in which it can be extended.

2 The inverse map and Clebsch representation

Equation (1.1) was first introduced in the Lie-Poisson context (see [29], [30], [38]). In this section we take the Lagrangian point of view and formulate (1.1) as the Euler-Poincaré equation on the Virasoro-Bott group. Further, we construct a canonical variational principle that will later allow us to determine a multisymplectic formulation of (1.1).

2.1 Euler-Poincaré equation on the Virasoro-Bott group

Let $S^1 = \mathbb{R}/2\pi\mathbb{Z} = \{\theta \in [0, 2\pi)\}$ denote the circle group, and let $\mathrm{Diff}(S^1)$ be the diffeomorphism group of S^1. The tangent bundles can be identified as $TS^1 = S^1 \times \mathbb{R}$ and $T\mathrm{Diff}(S^1) = \mathrm{Diff}(S^1) \times \mathfrak{X}(S^1)$, where $\mathfrak{X}(S^1) = \{\chi : S^1 \longrightarrow \mathbb{R}\}$ is the set of all smooth vector fields on S^1. In particular, the Lie algebra of S^1 is $\mathbb{R}$, and the Lie algebra of $\mathrm{Diff}(S^1)$ is $\mathfrak{X}(S^1)$. The Virasoro-Bott group is the central extension $\widehat{\mathrm{Diff}}(S^1) = \mathrm{Diff}(S^1) \times S^1$ with the group operation

$$(\psi_1, \theta_1) \cdot (\psi_2, \theta_2) = (\psi_1 \circ \psi_2, B(\psi_1, \psi_2) + \theta_1 + \theta_2), \tag{2.1}$$

where the 2-cocycle $B(\psi_1, \psi_2)$ is given by

$$B(\psi_1, \psi_2) = \frac{1}{2} \int_{S^1} \log \frac{\partial(\psi_1 \circ \psi_2)}{\partial x} \, d \log \frac{\partial \psi_2}{\partial x}. \tag{2.2}$$

The tangent bundle of the Virasoro-Bott group is $T\widehat{\mathrm{Diff}}(S^1) = \widehat{\mathrm{Diff}}(S^1) \times \mathfrak{X}(S^1) \times \mathbb{R}$. The Virasoro algebra $\mathfrak{vir}$ is the Lie algebra of the Virasoro-Bott group and can be identified as $\mathfrak{vir} = \mathfrak{X}(S^1) \times \mathbb{R}$. The Lie algebra bracket (or adjoint action) on $\mathfrak{vir}$ is given by

$$\mathrm{ad}_{(u,a)}(v, b) = \big[(u, a), (v, b)\big] = \left(-uv_x + u_x v, \int_{S^1} u_x v_{xx} \, dx \right) \tag{2.3}$$

for $(u, a), (v, b) \in \mathfrak{vir}$. Identify the dual of $\mathfrak{vir}$ with $\mathfrak{vir}$ by the L^2 inner product

$$\big\langle (u, a), (v, b) \big\rangle = ab + \int_{S^1} uv \, dx. \tag{2.4}$$

With respect to this inner product the coadjoint action $\mathrm{ad}^*_{(u,a)} : \mathfrak{vir} \longrightarrow \mathfrak{vir}$ can be represented as

$$\mathrm{ad}^*_{(u,a)}(v, b) = (2vu_x + uv_x + bu_{xxx}, 0). \tag{2.5}$$

For more information on the Virasoro-Bott group and the Virasoro algebra we refer the reader to [30] and [34].

Suppose a Lagrangian system is defined on $T\widehat{\mathrm{Diff}}(S^1)$ by specifying the right-invariant Lagrangian $L : T\widehat{\mathrm{Diff}}(S^1) \longrightarrow \mathbb{R}$. Rather then on the full tangent bundle, the dynamics of such a system can be analyzed on the Lie algebra $\mathfrak{vir}$ via the process called Euler-Poincaré reduction (see [24], [34]). We consider the reduced Lagrangian $\ell : \mathfrak{vir} \longrightarrow \mathbb{R}$ defined by $\ell(u, a) = L(\mathrm{id}, 0, u, a)$ and the reduced variational principle

$$\delta \int_{t_a}^{t_b} \ell\big(u(t), a(t)\big) \, dt = 0, \tag{2.6}$$

using variations of the form $\delta(u, a) = \frac{\partial}{\partial t}(v, b) - \big[(u, a), (v, b)\big]$, where $(v(t), b(t))$ vanish at the endpoints. This variational principle leads to the Euler-Poincaré equation

$$\frac{d}{dt} \frac{\delta \ell}{\delta(u, a)} + \mathrm{ad}^*_{(u,a)} \frac{\delta \ell}{\delta(u, a)} = 0, \tag{2.7}$$

where the variational derivatives and the coadjoint action are computed with respect to the inner product (2.4). Below we demonstrate that (1.1) can be written as the Euler-Poincaré equation.

Theorem 2.1. *Let the reduced Lagrangian be defined as*

$$\ell(u,a) = \frac{1}{2}a^2 + \frac{1}{2}\int_{S^1}\left(\alpha u^2 + \beta u_x^2\right)dx, \tag{2.8}$$

where $\alpha, \beta \geq 0$. *Then the corresponding Euler-Poincaré equations take the form*

$$\begin{aligned} \frac{da}{dt} &= 0, \\ \alpha(u_t + 3uu_x) - \beta(u_{xxt} + 2u_x u_{xx} + uu_{xxx}) + au_{xxx} &= 0. \end{aligned} \tag{2.9}$$

Proof. The case $\alpha = 1$ and $\beta = 0$ is shown in [34]. The case $\alpha, \beta \geq 0$ is a straightforward generalization. □

The first equation in (2.9) implies $a = \text{const}$, and therefore the second equation is equivalent to (1.1).

2.2 Reconstruction equations and the inverse map

A solution $(u(t), a(t))$ of (2.7) describes the evolution of the (right-invariant) Lagrangian system in the Virasoro algebra $\mathfrak{vir}$. One can reconstruct the evolution on the whole Virasoro-Bott group by finding a curve $(\psi(t), \theta(t)) \in \widehat{\text{Diff}}(S^1)$ which right-translates its tangent vector back to $(u(t), a(t))$, i.e., in short-hand notation $(u(t), a(t)) = (\dot{\psi}(t), \dot{\theta}(t)) \cdot (\psi(t), \theta(t))^{-1}$. More precisely,

$$\big(u(t), a(t)\big) \cong \big(\text{id}, 0, u(t), a(t)\big) = T_{(\psi(t),\theta(t))}R_{(\psi^{-1}(t), -\theta(t))} \cdot \big(\psi(t), \theta(t), \dot{\psi}(t), \dot{\theta}(t)\big), \tag{2.10}$$

where R denotes right translation on the Virasoro-Bott group and TR its tangent lift (see [24], [34]). By using (2.1) and (2.2), we obtain the reconstruction equations

$$\begin{aligned} u(t) &= \dot{\psi}(t) \circ \psi^{-1}(t), \\ a(t) &= \dot{\theta}(t) + \frac{d}{ds}\bigg|_{s=t} B\big(\psi(s), \psi^{-1}(t)\big). \end{aligned} \tag{2.11}$$

In the context of incompressible fluid dynamics, a time-dependent diffeomorphism $\psi(t) \in \text{Diff}(S^1)$ maps some reference configuration to the fluid domain at each instant of time, i.e., $\psi(t, X)$ represents the position at time t of the fluid particle labeled by X. On the other hand, the inverse map $l(t) = \psi^{-1}(t)$ maps from the current configuration of the fluid to the reference configuration, i.e., $l(x,t)$ is the label of the fluid particle occupying the position x at time t. The Eulerian velocity field $u(x,t)$ gives the velocity of the fluid particle that occupies the position x at time t, i.e., $\dot{\psi}(X,t) = u(\psi(X,t), t)$. This is precisely the meaning of the first of the reconstruction equations in (2.11). It will be convenient for us to rewrite the reconstruction equations in terms of the inverse map. One can check via a straightforward calculation that the first equation in (2.11) is equivalent to

$$l_t + ul_x = 0. \tag{2.12}$$

Using the definition of the 2-cocycle (2.2), we further calculate

$$
\begin{aligned}
\frac{d}{ds}\bigg|_{s=t} B\big(\psi(s),\psi^{-1}(t)\big) &= \frac{1}{2}\int_{S^1} \frac{\partial\big(\dot{\psi}(t)\circ\psi^{-1}(t)\big)}{\partial x}\, d\log\frac{\partial\psi^{-1}(t)}{\partial x} \\
&= \frac{1}{2}\int_{S^1} u_x\, d\log l_x \\
&= \frac{1}{2}\int_{S^1} \frac{u_x l_{xx}}{l_x}\, dx,
\end{aligned}
\tag{2.13}
$$

where in the second equality we used the first reconstruction equation in (2.11) and the definition of the inverse map. Therefore, the reconstruction equations in terms of the inverse map take the form

$$
\begin{aligned}
l_t + u l_x &= 0, \\
a(t) &= \dot{\theta}(t) + \frac{1}{2}\int_{S^1} \frac{u_x l_{xx}}{l_x}\, dx.
\end{aligned}
\tag{2.14}
$$

Given a solution $(u(t), a(t))$ of (2.7), one can easily solve (2.14) for $l(x,t)$ and $\theta(t)$.

2.3 Clebsch variational principle

2.3.1 General reduced Lagrangian

As discussed in Section 2.1, Equation (1.1) has an underlying variational structure. However, the Euler-Poincaré variational principle (2.6) imposes constraints on the variations of the functions u and a, which may be inconvenient in some applications, for instance, when one is interested in deriving variational integrators, or determining the underlying multisymplectic structure, as is our goal in this work. One can circumvent this issue by considering an augmented action functional which includes the reconstruction equations as constraints. This idea was formalized in the context of variational Lie group integrators in [4]. The idea of using the inverse map $l(x,t)$ (also called 'back-to-labels' map) and the advection condition (2.12) appeared in [20], and was later used in [12] to construct multisymplectic formulations of a class of fluid dynamics equations. We extend these ideas to systems defined on the Virasoro-Bott group.

The Clebsch variational principle (also known as the Hamilton-Pontryagin principle) enforces stationarity of the action $S = \int \ell(u,a)\, dt$ under the constraint that the reconstruction equations (2.14) are satisfied. Define the augmented action functional

$$
S[u,a,l,\theta,\pi,\lambda] = \int_{t_a}^{t_b} \ell(u,a)\, dt + \int_{t_a}^{t_b}\int_{S^1} \pi(l_t + u l_x)\, dx dt + \int_{t_a}^{t_b} \lambda\left(\dot{\theta} - a + \frac{1}{2}\int_{S^1} \frac{u_x l_{xx}}{l_x}\, dx\right) dt, \tag{2.15}
$$

where $\pi = \pi(x,t)$ and $\lambda = \lambda(t)$ are Lagrange multipliers, and consider the variational principle

$$
\delta S = 0 \tag{2.16}
$$

with respect to arbitrary variations δu, δa, $\delta\pi$, $\delta\lambda$, and vanishing endpoint variations δl and $\delta\theta$, i.e., $\delta l(x,t_a) = \delta l(x,t_b) = \delta\theta(t_a) = \delta\theta(t_b) = 0$. The resulting Euler-Lagrange equations are

$$\delta\theta: \quad \dot{\lambda} = 0, \tag{2.17a}$$

$$\delta a: \quad \lambda = \frac{\partial \ell(u,a)}{\partial a}, \tag{2.17b}$$

$$\delta\lambda: \quad \dot{\theta} = a - \frac{1}{2}\int_{S^1} \frac{u_x l_{xx}}{l_x}\,dx, \tag{2.17c}$$

$$\delta u: \quad \frac{\delta \ell(u,a)}{\delta u} + \pi l_x = \frac{1}{2}\lambda \frac{\partial}{\partial x}\frac{l_{xx}}{l_x}, \tag{2.17d}$$

$$\delta\pi: \quad l_t + u l_x = 0, \tag{2.17e}$$

$$\delta l: \quad \pi_t + \frac{\partial}{\partial x}\left(\pi u - \frac{1}{2}\lambda\frac{u_{xx}}{l_x}\right) = 0, \tag{2.17f}$$

where $\frac{\delta\ell}{\delta(u,a)} = \left(\frac{\delta\ell}{\delta u}, \frac{\partial\ell}{\partial a}\right)$. We will now show that the dynamics generated by the system (2.17) are equivalent to the dynamics generated by the Euler-Poincaré equation (2.7).

Theorem 2.2. *Suppose the functions $u(x,t)$, $a(t)$, $l(x,t)$, $\theta(t)$, $\pi(x,t)$, and $\lambda(t)$ satisfy the Euler-Lagrange equations* (2.17)*. Then the functions $u(x,t)$ and $a(t)$ satisfy the Euler-Poincaré equation* (2.7)*.*

Proof. Let (w,c) be an arbitrary element of the Virasoro algebra $\mathfrak{vir}$. Let us calculate

$$\left\langle \frac{d}{dt}\frac{\delta\ell}{\delta(u,a)}, (w,c)\right\rangle = \int_{S^1}\left(\frac{\partial}{\partial t}\frac{\delta\ell}{\delta u}\right)\cdot w\,dx + \left(\frac{\partial}{\partial t}\frac{\partial\ell}{\partial a}\right)\cdot c, \tag{2.18}$$

where the inner product $\langle\cdot,\cdot\rangle$ was defined in (2.4). By using (2.17a), (2.17b), and (2.17c), we further have

$$\begin{aligned}\left\langle \frac{d}{dt}\frac{\delta\ell}{\delta(u,a)}, (w,c)\right\rangle &= \int_{S^1}\frac{\partial}{\partial t}\left(\frac{1}{2}\lambda\frac{\partial}{\partial x}\frac{l_{xx}}{l_x} - \pi l_x\right)\cdot w\,dx \\ &= \int_{S^1}\left(\frac{1}{2}\lambda\frac{\partial}{\partial x}\frac{l_{txx}l_x - l_{xx}l_{tx}}{l_x^2} - \pi_t l_x - \pi l_{tx}\right)\cdot w\,dx.\end{aligned} \tag{2.19}$$

We now use (2.17e) and (2.17f) to eliminate the time derivatives in the integrand, which yields

$$\left\langle \frac{d}{dt}\frac{\delta\ell}{\delta(u,a)}, (w,c)\right\rangle = \int_{S^1}\underbrace{\left[\frac{1}{2}\lambda\frac{\partial}{\partial x}\frac{-l_x\frac{\partial^2}{\partial x^2}(ul_x) + l_{xx}\frac{\partial}{\partial x}(ul_x)}{l_x^2} + l_x\frac{\partial}{\partial x}\left(\pi u - \frac{1}{2}\lambda\frac{u_{xx}}{l_x}\right) + \pi\frac{\partial}{\partial x}(ul_x)\right]}_{A}\cdot w\,dx. \tag{2.20}$$

Note that the expression A contains the functions u, l, π, λ, and their spatial derivatives. On the other hand we have

$$
\begin{aligned}
\left\langle \mathrm{ad}^*_{(u,a)}\frac{\delta\ell}{\delta(u,a)},(w,c)\right\rangle &= \int_{S^1}\left[2\frac{\delta\ell}{\delta u}u_x + u\frac{\partial}{\partial x}\frac{\delta\ell}{\delta u} + \frac{\partial\ell}{\partial a}u_{xxx}\right]\cdot w\,dx \\
&= \int_{S^1}\underbrace{\left[\left(\lambda\frac{\partial}{\partial x}\frac{l_{xx}}{l_x} - 2\pi l_x\right)u_x + u\frac{\partial}{\partial x}\left(\frac{1}{2}\lambda\frac{\partial}{\partial x}\frac{l_{xx}}{l_x} - \pi l_x\right) + \lambda u_{xxx}\right]}_{B}\cdot w\,dx,
\end{aligned}
\tag{2.21}
$$

where in the first equality we used (2.5), and in the second equality we used (2.17b) and (2.17d). Note that the expression B contains the functions u, l, π, λ, and their spatial derivatives. After rather tedious, albeit straightforward algebraic manipulations we find that $A + B = 0$. Therefore, we have that for all $(w,c) \in \mathfrak{vir}$

$$
\left\langle \frac{d}{dt}\frac{\delta\ell}{\delta(u,a)} + \mathrm{ad}^*_{(u,a)}\frac{\delta\ell}{\delta(u,a)},(w,c)\right\rangle = 0, \tag{2.22}
$$

which completes the proof, since the inner product is nondegenerate.

□

2.3.2 Separable reduced Lagrangian

The variational principle (2.16) simplifies significantly when one considers separable Lagrangians of the form

$$
\ell(u,a) = \frac{1}{2}a^2 + \bar{\ell}(u). \tag{2.23}
$$

In that case Equations (2.17a) and (2.17b) imply $\lambda = a = \text{const}$. Treating a as a constant, we can eliminate the variables θ and λ from the action functional (2.15). Consider the action functional

$$
S[u,l,\pi] = \int_{t_a}^{t_b}\left(\bar{\ell}(u) + \frac{a}{2}\int_{S^1}\frac{u_x l_{xx}}{l_x}\,dx\right)dt + \int_{t_a}^{t_b}\int_{S^1}\pi(l_t + u l_x)\,dxdt. \tag{2.24}
$$

The stationarity condition $\delta S = 0$ with respect to arbitrary variations δu, $\delta\pi$, and vanishing endpoint variations δl, yields the Euler-Lagrange equations

$$
\delta u: \quad \frac{\delta\bar{\ell}(u)}{\delta u} + \pi l_x = \frac{a}{2}\frac{\partial}{\partial x}\frac{l_{xx}}{l_x}, \tag{2.25a}
$$

$$
\delta\pi: \quad l_t + u l_x = 0, \tag{2.25b}
$$

$$
\delta l: \quad \pi_t + \frac{\partial}{\partial x}\left(\pi u - \frac{a}{2}\frac{u_{xx}}{l_x}\right) = 0. \tag{2.25c}
$$

It is straightforward to see that the system (2.17) reduces to (2.25) for Lagrangians of the form (2.23).

Remark. The action functional (2.24) provides a new variational formulation for Equation (1.1) when the Lagrangian (2.8) is considered. For $a = 0$ this action functional reduces to the action functional for the dispersionless CH equation ($\alpha = \beta = 1$) introduced in [12] and one of the action functionals for the HS equation ($\alpha = 0$ and $\beta = 1$) described in [27]. For $\alpha = 1$ and $\beta = a = 0$ we also get a variational principle for Burgers' equation.

3 Inverse map multisymplectic formulation

The action functional and variational principle introduced in Section 2.3.2 allow the identification and analysis of a new multisymplectic formulation of the family of equations (1.1). Multisymplectic geometry provides a covariant formalism for the study of field theories in which time and space are treated on equal footing. Multisymplectic formalism is useful for, e.g., the stability analysis of water waves (see [5], [6]) or construction of structure-preserving numerical algorithms (see [7], [35]). The multisymplectic form formula was first proved by Marsden & Patrick & Shkoller [35] and provides an intrinsic and covariant description of the conservation of symplecticity law, first introduced by Bridges [5] in the context of multisymplectic Hamiltonian PDEs. In Section 3.1 we review the multisymplectic geometry formalism and derive the multisymplectic form formula associated with (2.24). We further make a connection with Bridges' approach to multisymplecticity in Section 3.2 and determine a multisymplectic Hamiltonian form of the Euler-Lagrange equations (2.25).

3.1 Multisymplectic form formula and conservation of symplecticity

The multisymplectic form formula is the multisymplectic counterpart of the fact that in finite-dimensional mechanics, the flow of a mechanical system consists of symplectic maps. It was first proved for first-order field theories in [35], and later generalized to second-order field theories in [32]. Since the field theory described by the action functional (2.24) with the Lagrangian (2.8) is second-order, we follow the theory developed in [32]. For the convenience of the reader, below we briefly review multisymplectic geometry and jet bundle formalism necessary for our discussion.

Let $X = S^1 \times \mathbb{R}$ represent spacetime and denote the local coordinates by $(x^\mu) = (x^1, x^0)$, where $x^1 \equiv x$ is the spatial coordinate and $x^0 \equiv t$ is time. Define the configuration fiber bundle $\tau_{XY} : Y \longrightarrow X$ as $Y = X \times S^1 \times \mathbb{R} \times \mathbb{R}$. Denote the fiber coordinates by $(y^A) = (y^1, y^2, y^3)$ with $y^1 \equiv l$, $y^2 \equiv u$, and $y^3 \equiv \pi$. Physical fields are sections of the configuration bundle, that is, continuous maps $\phi : X \longrightarrow Y$ such that $\tau_{XY} \circ \phi = \mathrm{id}_X$. In the coordinates (x^μ, y^A) a field ϕ is represented as $\phi(x,t) = (x^\mu, \phi^A(x^\mu)) = (x, t, l(x,t), u(x,t), \pi(x,t))$.

For a k-th order field theory, the evolution of the field takes place on the k-th jet bundle J^kY. The first jet bundle J^1Y is the affine bundle over Y with the fibers J^1_yY defined as

$$J^1_yY = \left\{\vartheta : T_{(x,t)}X \to T_yY \;\middle|\; T\tau_{XY} \circ \vartheta = \mathrm{id}_{T_{(x,t)}X}\right\} \tag{3.1}$$

for $y \in Y_{(x,t)}$, where the linear maps ϑ represent the tangent mappings $T_{(x,t)}\phi$ for local sections ϕ such that $\phi(x,t) = y$. The local coordinates (x^μ, y^A) on Y induce the coordinates $(x^\mu, y^A, v^A{}_\mu)$ on J^1Y. Intuitively, the first jet bundle consists of the configuration bundle Y, and of the first partial derivatives of the field variables with respect to the independent variables. We can think of J^1Y as a fiber bundle over X. Given a section $\phi : X \longrightarrow Y$, we can define its first jet prolongation

$$j^1\phi : X \ni (x,t) \longrightarrow T_{(x,t)}\phi \in J^1Y, \tag{3.2}$$

in coordinates given by

$$j^1\phi(x^\mu) = \left(x^\mu, \phi^A(x^\nu), \frac{\partial \phi^A(x^\nu)}{\partial x^\mu}\right), \tag{3.3}$$

which is a section of the fiber bundle J^1Y over X. For higher-order field theories we consider higher-order jet bundles, defined iteratively by $J^{k+1}Y = J^1(J^kY)$. We denote the local coordinates on J^2Y by $(x^\mu, y^A, v^A{}_\mu, w^A{}_{\mu\nu})$. The second jet prolongation $j^2\phi : X \longrightarrow J^2Y$ is given in coordinates by $j^2\phi(x^\mu) = (x^\mu, \phi^A, \partial\phi^A/\partial x^\mu, \partial^2\phi^A/\partial x^\mu \partial x^\nu)$. Let $(x^\mu, y^A, v^A{}_\mu, w^A{}_{\mu\nu}, s^A{}_{\mu\nu\sigma})$ denote the coordinates on J^3Y. The third jet prolongation $j^3\phi$ is defined similar to $j^1\phi$ and $j^2\phi$. For more information about the geometry of jet bundles see [43] and [19].

In the jet bundle formalism introduced above, the action functional (2.24) with the reduced Lagrangian (2.8) can be written as

$$S[\phi] = \int_{\mathcal{U}} \mathcal{L}(j^2\phi)\, d^2x, \tag{3.4}$$

where $\mathcal{U} = S^1 \times [t_a, t_b]$, $d^2x = dx \wedge dt$, and the Lagrangian density $\mathcal{L} : J^2Y \longrightarrow \mathbb{R}$ is

$$\mathcal{L}(x^\mu, y^A, v^A{}_\mu, w^A{}_{\mu\nu}) = \frac{\alpha}{2}(y^2)^2 + \frac{\beta}{2}(v^2{}_1)^2 + \frac{a}{2}\frac{v^2{}_1 w^1{}_{11}}{v^1{}_1} + y^3(v^1{}_0 + y^2 v^1{}_1). \tag{3.5}$$

Hamilton's variational principle seeks fields $\phi(x,t)$ that extremize S, that is,

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} S[\eta^\lambda_Y \circ \phi] = 0 \tag{3.6}$$

for all η^λ_Y that keep the boundary conditions on $\partial\mathcal{U}$ fixed, where $\eta^\lambda_Y : Y \longrightarrow Y$ is the flow of a vertical vector field V on Y. This leads to the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial y^A}(j^2\phi) - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial v^A{}_\mu}(j^2\phi)\right) + \frac{\partial^2}{\partial x^\mu \partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial w^A{}_{\mu\nu}}(j^2\phi)\right) = 0, \tag{3.7}$$

where Einstein's summation convention is used. With the Lagrangian density (3.5), these Euler-Lagrange equations take the form (2.25). For more information on multisymplectic geometry and jet bundle setting of field theories see [18], [19], [32], and [35].

For a second-order field theory, the multisymplectic structure is defined on J^3Y (see [32]). Given the Lagrangian density $\mathcal{L}$ one can define the Cartan 2-form $\Theta_\mathcal{L}$ on J^3Y, in local coordinates given by

$$\begin{aligned}\Theta_\mathcal{L} = &\left(\frac{\partial \mathcal{L}}{\partial v^A{}_\mu} - D_\nu\left(\frac{\partial \mathcal{L}}{\partial w^A{}_{\mu\nu}}\right)\right) dy^A \wedge dx_\mu + \frac{\partial \mathcal{L}}{\partial w^A{}_{\nu\mu}} dv^A{}_\nu \wedge dx_\mu \\ &+ \left(\mathcal{L} - \frac{\partial \mathcal{L}}{\partial v^A{}_\mu} v^A{}_\mu + D_\nu\left(\frac{\partial \mathcal{L}}{\partial w^A{}_{\mu\nu}}\right) v^A{}_\mu - \frac{\partial \mathcal{L}}{\partial w^A{}_{\nu\mu}} w^A{}_{\nu\mu}\right) d^2x,\end{aligned} \tag{3.8}$$

where $dx_\mu = \partial_\mu \lrcorner\, d^2x$, i.e., $dx_0 = -dx$ and $dx_1 = dt$, and the *formal* partial derivative in the direction x^ν of a function $f : J^2Y \longrightarrow \mathbb{R}$ is defined in coordinates as

$$D_\nu f = \frac{\partial f}{\partial x^\nu} + \frac{\partial f}{\partial y^A} v^A{}_\nu + \frac{\partial f}{\partial v^A{}_\mu} w^A{}_{\mu\nu} + \frac{\partial f}{\partial w^A{}_{\sigma\mu}} s^A{}_{\sigma\mu\nu}. \tag{3.9}$$

For the Lagrangian density (3.5), the Cartan form is

$$\begin{aligned}\Theta_{\mathcal{L}} = &- y^3 dy^1 \wedge dx + \left(y^3 y^2 - \frac{a}{2}\frac{w^2{}_{11}}{v^1{}_1} \right) dy^1 \wedge dt + \left(\beta v^2{}_1 + \frac{a}{2}\frac{w^1{}_{11}}{v^1{}_1} \right) dy^2 \wedge dt \\ &+ \frac{a}{2}\frac{v^2{}_1}{v^1{}_1} dv^1{}_1 \wedge dt + \left(\frac{\alpha}{2}(y^2)^2 - \frac{\beta}{2}(v^2{}_1)^2 - \frac{a}{2}\frac{v^2{}_1 w^1{}_{11}}{v^1{}_1} + \frac{a}{2} w^2{}_{11} \right) dx \wedge dt. \end{aligned} \tag{3.10}$$

The multisymplectic 3-form $\Omega_{\mathcal{L}}$ is then defined as the exterior derivative of the Cartan form:

$$\begin{aligned}\Omega_{\mathcal{L}} = d\Theta_{\mathcal{L}} = \; & dy^1 \wedge dy^3 \wedge dx \; - \; y^3 dy^1 \wedge dy^2 \wedge dt \; - \; y^2 dy^1 \wedge dy^3 \wedge dt \; - \; \frac{a}{2}\frac{w^2{}_{11}}{(v^1{}_1)^2} dy^1 \wedge dv^1{}_1 \wedge dt \\ & + \; \frac{a}{2v^1{}_1} dy^1 \wedge dw^2{}_{11} \wedge dt \; - \; \beta dy^2 \wedge dv^2{}_1 \wedge dt \; + \; \frac{a}{2}\frac{w^1{}_{11}}{(v^1{}_1)^2} dy^2 \wedge dv^1{}_1 \wedge dt \\ & - \; \frac{a}{2v^1{}_1} dy^2 \wedge dw^1{}_{11} \wedge dt \; - \; \frac{a}{2v^1{}_1} dv^1{}_1 \wedge dv^2{}_1 \wedge dt \; + \; \alpha y^2 dy^2 \wedge dx \wedge dt \\ & - \; \left(\beta v^2{}_1 + \frac{a}{2}\frac{w^1{}_{11}}{v^1{}_1} \right) dv^2{}_1 \wedge dx \wedge dt \; + \; \frac{a}{2}\frac{v^2{}_1 w^1{}_{11}}{(v^1{}_1)^2} dv^1{}_1 \wedge dx \wedge dt \\ & - \; \frac{a}{2}\frac{v^2{}_1}{v^1{}_1} dw^1{}_{11} \wedge dx \wedge dt \; + \; \frac{a}{2} dw^2{}_{11} \wedge dx \wedge dt. \end{aligned} \tag{3.11}$$

Let $\mathcal{P}$ be the set of solutions of the Euler-Lagrange equations, that is, the set of sections ϕ satisfying (3.6) or (3.7). For a given $\phi \in \mathcal{P}$, let $\mathcal{F}$ be the set of first variations, that is, the set of vector fields V on Y such that $(x,t) \to \eta_Y^\epsilon \circ \phi(x,t)$ is also a solution, where η_Y^ϵ is the flow of V. The multisymplectic form formula for second-order field theories (see [32]) states that if $\phi \in \mathcal{P}$ then for all V and W in $\mathcal{F}$,

$$\int_{\partial \mathcal{U}} (j^3\phi)^* \big(j^3V \lrcorner\, j^3W \lrcorner\, \Omega_{\mathcal{L}} \big) = 0, \tag{3.12}$$

where $(j^3\phi)^*$ denotes the pull-back by the mapping $j^3\phi$, and j^3V is the third jet prolongation of V, that is, the vector field on J^3Y whose flow is the third jet prolongation of the flow η_Y^ϵ for V, i.e.,

$$j^3V = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} j^3\eta_Y^\epsilon. \tag{3.13}$$

Consider two arbitrary first variation vector fields V, W, in the local coordinates (x^μ, y^A) represented by $(V^\mu(x^\mu,y^A), V^A(x^\mu,y^A))$ and $(W^\mu(x^\mu,y^A), W^A(x^\mu,y^A))$, respectively. Let us work out the form of the formula (3.11) for τ_{XY}-vertical first variations, i.e., $V^\mu(x^\mu,y^A) = W^\mu(x^\mu,y^A) = 0$.

Denote the components of j^3V as $(0, V^A, V^A_\mu, V^A_{\mu\nu}, V^A_{\mu\nu\sigma})$, and similarly for j^3W. The multisymplectic form formula then becomes

$$\int_{\partial\mathcal{U}} -F(x,t)\,dx + G(x,t)\,dt = 0, \tag{3.14}$$

with

$$\begin{aligned}
F(x,t) &= -W^1V^3 + W^3V^1,\\
G(x,t) &= -\pi(W^1V^2 - W^2V^1) - u(W^1V^3 - W^3V^1) - \frac{a}{2}\frac{u_{xx}}{l_x^2}(W^1V^1{}_1 - W^1{}_1V^1)\\
&\quad + \frac{a}{2l_x}(W^1V^2{}_{11} - W^2{}_{11}V^1) - \beta(W^2V^2{}_1 - W^2{}_1V^2) + \frac{a}{2}\frac{l_{xx}}{l_x^2}(W^2V^1{}_1 - W^1{}_1V^2)\\
&\quad - \frac{a}{2l_x}(W^2V^1{}_{11} - W^1{}_{11}V^2) - \frac{a}{2l_x}(W^1{}_1V^2{}_1 - W^2{}_1V^1{}_1),
\end{aligned} \tag{3.15}$$

where the vector components are evaluated at $j^3\phi(x,t)$. By applying Stokes' theorem and using the fact that $\mathcal{U}$ is arbitrary, the multisymplectic form formula (3.14) can be rewritten equivalently as the conservation law

$$\frac{\partial}{\partial t}F(x,t) + \frac{\partial}{\partial x}G(x,t) = 0. \tag{3.16}$$

This kind of a conservation law was first considered by Bridges [5]. In Section 3.2 we make a further connection with Bridges' theory and find a multisymplectic PDE form of the Euler-Lagrange equations (2.25).

3.2 Multisymplectic Hamiltonian PDE formulation

Bridges [5] introduced the notion of multisymplecticity by generalizing the notion of Hamiltonian systems to Partial Differential Equations (PDEs). A multisymplectic structure $(\mathcal{M}, \omega, \kappa)$ consists of the phase space $\mathcal{M} = \mathbb{R}^n$, and pre-symplectic 2-forms ω and κ, where pre-symplectic means that the 2-forms are closed, but not necessarily nondegenerate. A multisymplectic Hamiltonian system is a PDE of the form

$$M(z)z_t + K(z)z_x = \nabla H(z), \tag{3.17}$$

where $z : X \ni (x,t) \longrightarrow z(x,t) \in \mathcal{M}$ is a function of the spacetime variables x and t, $H : \mathcal{M} \longrightarrow \mathbb{R}$ is the Hamiltonian, and $M(z)$, $K(z)$ are $n \times n$ antisymmetric matrices defined by

$$\omega(\bar{W}, \bar{V}) \equiv \langle M(z)\bar{W}, \bar{V}\rangle_{\mathcal{M}}, \qquad \kappa(\bar{W}, \bar{V}) \equiv \langle K(z)\bar{W}, \bar{V}\rangle_{\mathcal{M}}, \tag{3.18}$$

where $\bar{V}$, $\bar{W}$ are arbitrary vector fields on $\mathcal{M}$, and $\langle\cdot,\cdot\rangle_{\mathcal{M}}$ is the standard Euclidean inner product on $\mathcal{M} = \mathbb{R}^n$.

We will use the multisymplectic form formula (3.14) to deduce the multisymplectic Hamiltonian PDE form (3.17) of the Euler-Lagrange equations (2.25). We note that for $a > 0$ the vector components that appear in (3.15) only correspond to the 7 coordinate directions y^1, y^2, y^3, $v^1{}_1$,

${v^2}_1$, ${w^1}_{11}$, ${w^2}_{11}$ on J^3Y. We will therefore consider $\mathcal{M} = \mathbb{R}^7$ and denote the coordinates on $\mathcal{M}$ as $(l, u, \pi, \Delta, \Theta, \Xi, \Pi)$. Define the projection map

$$\mathbb{F}\mathcal{L} : J^3Y \ni (x^\mu, y^A, {v^A}_\mu, {w^A}_{\mu\nu}, {s^A}_{\mu\nu\sigma}) \longrightarrow (y^1, y^2, y^3, {v^1}_1, {v^2}_1, {w^1}_{11}, {w^2}_{11}) \in \mathcal{M}. \tag{3.19}$$

The suitable entries for the matrices $M(z)$ and $K(z)$ can be read off from (3.15) as

$$M = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K(z) = \begin{pmatrix} 0 & \pi & u & \frac{a}{2}\frac{\Pi}{\Delta^2} & 0 & 0 & -\frac{a}{2\Delta} \\ -\pi & 0 & 0 & -\frac{a}{2}\frac{\Xi}{\Delta^2} & \beta & \frac{a}{2\Delta} & 0 \\ -u & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{a}{2}\frac{\Pi}{\Delta^2} & \frac{a}{2}\frac{\Xi}{\Delta^2} & 0 & 0 & \frac{a}{2\Delta} & 0 & 0 \\ 0 & -\beta & 0 & -\frac{a}{2\Delta} & 0 & 0 & 0 \\ 0 & -\frac{a}{2\Delta} & 0 & 0 & 0 & 0 & 0 \\ \frac{a}{2\Delta} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.20}$$

With that choice, we have $F(x,t) = \omega(\bar{W}, \bar{V})$ and $G(x,t) = \kappa(\bar{W}, \bar{V})$, where $\bar{W} = T\mathbb{F}\mathcal{L} \cdot j^3W$ and $\bar{V} = T\mathbb{F}\mathcal{L} \cdot j^3V$. The Hamiltonian H can be read off from the $dx \wedge dt$ term in (3.10) as

$$H(z) = \frac{\alpha}{2}u^2 - \frac{\beta}{2}\Theta^2 - \frac{a}{2}\frac{\Theta\Xi}{\Delta} + \frac{a}{2}\Pi. \tag{3.21}$$

Below we show that the Euler-Lagrange equations (2.25) indeed can be given the multisymplectic structure (3.17).

Theorem 3.1. *Suppose $a > 0$. Then the Euler-Lagrange equations* (2.25) *with the Lagrangian* (2.8) *are equivalent to the multisymplectic Hamiltonian system* (3.17) *with the matrices* (3.20) *and the Hamiltonian* (3.21). *That is, if $\phi(x,t) = (x, t, l(x,t), u(x,t), \pi(x,t))$ is a solution of* (2.25)*, then $z(x,t) = \mathbb{F}\mathcal{L} \circ j^3\phi(x,t)$ is a solution of* (3.17)*, and conversely, if $z(x,t)$ is a solution of* (3.17)*, then $\phi(x,t) = (x, t, z_1(x,t), z_2(x,t), z_3(x,t)) = (x, t, l(x,t), u(x,t), \pi(x,t))$ is a solution of* (2.25).

Proof. Substituting (3.20) and (3.21) in (3.17) yields the system of equations

$$\pi_t + \pi u_x + u\pi_x + \frac{a}{2}\frac{\Pi}{\Delta^2}\Delta_x - \frac{a}{2\Delta}\Pi_x = 0, \tag{3.22a}$$

$$-\pi l_x - \frac{a}{2}\frac{\Xi}{\Delta^2}\Delta_x + \beta\Theta_x + \frac{a}{2\Delta}\Xi_x = \alpha u, \tag{3.22b}$$

$$-l_t - ul_x = 0, \tag{3.22c}$$

$$-\frac{a}{2}\frac{\Pi}{\Delta^2}l_x + \frac{a}{2}\frac{\Xi}{\Delta^2}u_x + \frac{a}{2\Delta}\Theta_x = \frac{a}{2}\frac{\Theta\Xi}{\Delta^2}, \tag{3.22d}$$

$$-\beta u_x - \frac{a}{2\Delta}\Delta_x = -\beta\Theta - \frac{a}{2}\frac{\Xi}{\Delta}, \tag{3.22e}$$

$$-\frac{a}{2\Delta}u_x = -\frac{a}{2}\frac{\Theta}{\Delta}, \tag{3.22f}$$

$$\frac{a}{2\Delta}l_x = \frac{a}{2}. \tag{3.22g}$$

Equation (3.22g) implies $\Delta = l_x$ and Equation (3.22f) implies $\Theta = u_x$. Then, Equations (3.22e) and (3.22d) imply $\Xi = l_{xx}$ and $\Pi = u_{xx}$, respectively. By substituting these identities in the remaining equations (3.22a)-(3.22c), we obtain a system equivalent to (2.25), which completes the proof. $\square$

Bridges [5] showed that the conservation of symplecticity law

$$\frac{\partial}{\partial t}\omega(\bar{W},\bar{V})+\frac{\partial}{\partial x}\kappa(\bar{W},\bar{V})=0 \tag{3.23}$$

is satisfied for solutions $z(x,t)$ of (3.17), where $\bar{W}$, $\bar{V}$ are arbitrary first variations of $z(x,t)$. This is an equivalent statement of (3.16), since if W and V are first variations for (3.7), then $\bar{W}=T\mathbb{F}\mathcal{L}\cdot j^3W$ and $\bar{V}=T\mathbb{F}\mathcal{L}\cdot j^3V$ are first variations for (3.17).

Remark. Equations (3.17), (3.20), and (3.21) provide a new multisymplectic formulation for the family of equations (1.1) with $a>0$. For $a=0$ several special cases can be obtained. If $\beta>0$, then Equations (3.22d), (3.22f), and (3.22g) become trivial, and it is enough to consider the variables $z=(l,u,\pi,\Theta)$. The matrices M and K then take the form

$$M=\begin{pmatrix}0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0\end{pmatrix},\qquad K(z)=\begin{pmatrix}0&\pi&u&0\\-\pi&0&0&\beta\\-u&0&0&0\\0&-\beta&0&0\end{pmatrix}, \tag{3.24}$$

and the Hamiltonian becomes $H(z)=\frac{\alpha}{2}u^2-\frac{\beta}{2}\Theta^2$. For $\alpha=\beta=1$ this reproduces the multisyplectic structure for the dispersionless CH equation found in [12], and for $\alpha=0$, $\beta=1$ we obtain a new multisymplectic formulation of the HS equation with $a=0$. If in addition $\beta=0$, then Equation (3.22e) also becomes trivial, and a further simplification is possible: we consider the variables $z=(l,u,\pi)$ with the matrices

$$M=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\qquad K(z)=\begin{pmatrix}0&\pi&u\\-\pi&0&0\\-u&0&0\end{pmatrix}, \tag{3.25}$$

and the Hamiltonian $H(z)=\frac{\alpha}{2}u^2$. This provides a multisymplectic formulation for Burgers' equation.

4 Summary and future work

In this paper we have introduced a new type of Clebsch representation that extends the previous general formulation for fluid dynamics in Cotter, Holm & Hydon [12] to the case when the group actions governing Lagrangian fluid paths include 2-cocycles. Physically, this means that linear dispersion with third order spatial derivatives can be included, as required for investigating the multisymplectic structures of the Korteweg-de Vries, Camassa-Holm, and Hunter-Saxton equations. Moreover, the multisymplectic form formula was shown to persist and derived explicitly for this important class of equations, by using our new type of Clebsch representation, derived as the momentum map associated with particle relabeling with group actions which include 2-cocycles. In addition, symplecticity was found to be conserved in this new class of flows. Consequently, new types of structure-preserving numerics for soliton equations with linear dispersion can now be developed. Multisymplectic integrators are methods that preserve a discrete version of the symplectic conservation law (3.23). There is numerical evidence that these schemes locally conserve energy and momentum remarkably well (see, e.g., [2], [3], [7], [10], [39], [46], [48], [49], [50]), which is a much stronger property than merely global conservation over the whole spatial domain (see [37]). Variational integrators are based on discrete variational principles and provide a natural framework for the discretization of Lagrangian systems (see, e.g., [33], [35], [36], [42], [44], [45]). A discrete action functional can be obtained by discretizing the functional (2.24) on a spacetime mesh. A variational numerical scheme is then derived by extremizing the discrete action with respect to the discrete set of the values of the fields l, u, and π. Variational integrators satisfy a discrete version of the multisymplectic form formula (3.12), and are therefore multisymplectic. Moreover, in the presence of a symmetry, they satisfy a discrete version of Noether's theorem, as a consequence of which they retain exactly some of the conservation laws of the continuous system.

Furthermore, the new type of Clebsch momentum map admits a new type of interplay among nonlinearity, dispersion and noise. This opens a new class of problems based on dynamics of 'wobbling' solitons governed by SPDEs with stochastic mass/label transport. Consider a stochastic deformation of the action functional (2.24) such that the velocity field u in the reconstruction equation (2.12) is replaced with $u + \xi(x) \circ \dot{W}(t)$, where $\dot{W}(t)$ denotes the white noise and the prescribed function $\xi(x)$ represents the spatial correlations of the noise. The action functional (2.24) then takes the form

$$S[u,l,\pi] = \int_{t_a}^{t_b} \bigg(\bar{\ell}(u) + \frac{a}{2}\int_{S^1} \frac{u_x l_{xx}}{l_x}\,dx \bigg)\,dt + \int_{S^1}\int_{t_a}^{t_b} \pi\Big(\circ\, dl + u l_x\,dt + \xi(x) l_x \circ dW(t)\Big)\,dx, \quad (4.1)$$

where $W(t)$ is the standard Wiener process, and $\circ$ denotes Stratonovich integration. This kind of stochastic deformations has been proposed for the Camassa-Holm equation, electromagnetic field equations, and fluid dynamics equations (see [13], [14], [15], [17], [21], [22], [25]), and appears to retain a number of the properties of the unperturbed equations, such as soliton solutions of the Camassa-Holm equation or the Kelvin circulation theorem for fluid dynamics. In particular, for certain functional forms of $\xi(x)$, the introduction of this type of noise can preserve the deterministic isospectral problem and thereby produce stochastic inverse scattering methods for determining the soliton solutions of SPDEs. It would therefore be of interest to investigate the effects of the stochastic term in (4.1) on the dynamics of the KdV, CH, and HS equations, and construct related stochastic variational and multisymplectic integrators.

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