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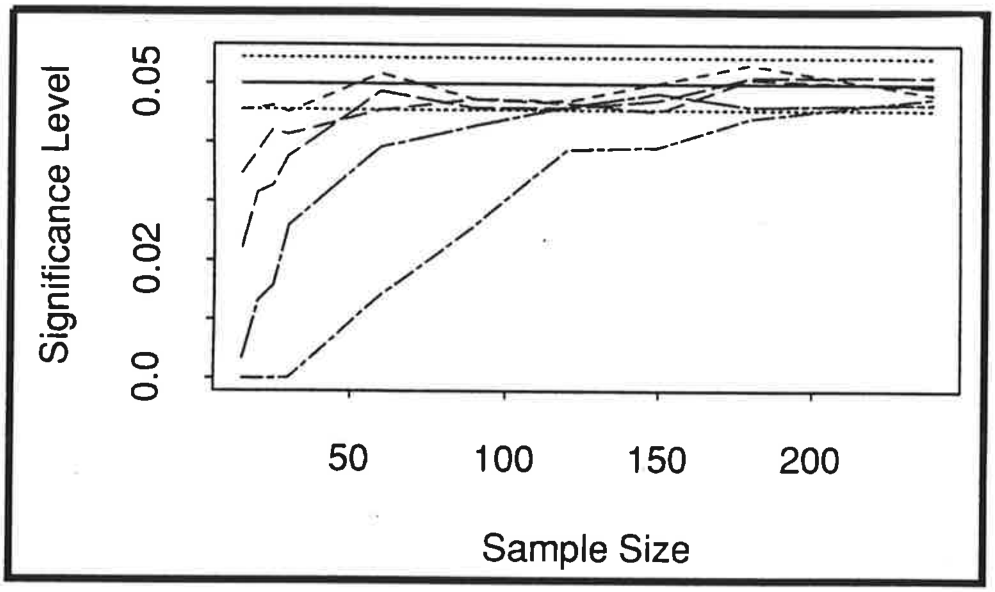

TAKING SERIAL CORRELATION INTO ACCOUNT IN TESTS OF THE MEAN

by

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Taking Serial Correlation into Account in Tests of the Mean

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Abstract

The comparison of means derived from samples of noisy data is a standard part of climatology. When the data are not serially correlated the appropriate statistical tool for this task is usually the conventional Student's t-test. However, data frequently are serially correlated in climatological applications with the result that the t-test in its standard form is not applicable. The usual solution to this problem is to scale the t-statistic by a factor which depends upon the equivalent sample size n_e.

We show, by means of simulations, that the revised t-test is often conservative (the actual significance level is smaller than the specified significance level) when the equivalent sample size is known. However, in most practical cases the equivalent sample size is not known. Then the test becomes liberal (the actual significance level is greater than the specified significance level). This systematic error becomes small when the true equivalent sample size is large (greater than approximately 30).

We re-examine the difficulties inherent in difference of means tests when there is serial dependence. We provide guidelines for the application of the "usual" t-test and propose two alternative tests which substantially improve upon the "usual" t-test when samples are small.

1 Introduction

Statistical comparisons of means are frequently conducted in climatology to inter-compare observed and/or simulated climates amongst themselves or against fixed reference values. These comparisons are conducted by employing a paradigm in which (i) a statistical model is imposed upon the samples of climate data, (ii) a *null hypothesis* H_0 which is to be tested is specified, (iii) an *alternate hypothesis* H_a which guides the interpretation of the test statistic is specified, and (iv) a test statistic is computed to determine how unusual the observed difference of means is in the context of the model and the null hypothesis.

It is well known that the classical method, which employs the *Student's t-test* (see, for example, Mood and Graybill, 1963 [14]) and assumes a statistical model in which climate observations are statistically independent, is sensitive to serial correlation within the samples. The effect of serial correlation is, usually, to make comparisons of means liberal. That is, "significant" differences are found more frequently than expected when there is no difference.

The purpose of this paper is to review existing methods for dealing with serial correlation, propose a new test for the difference of means which better takes the effects of serial correlation into account, and provide guidelines for application of comparison of means procedures.

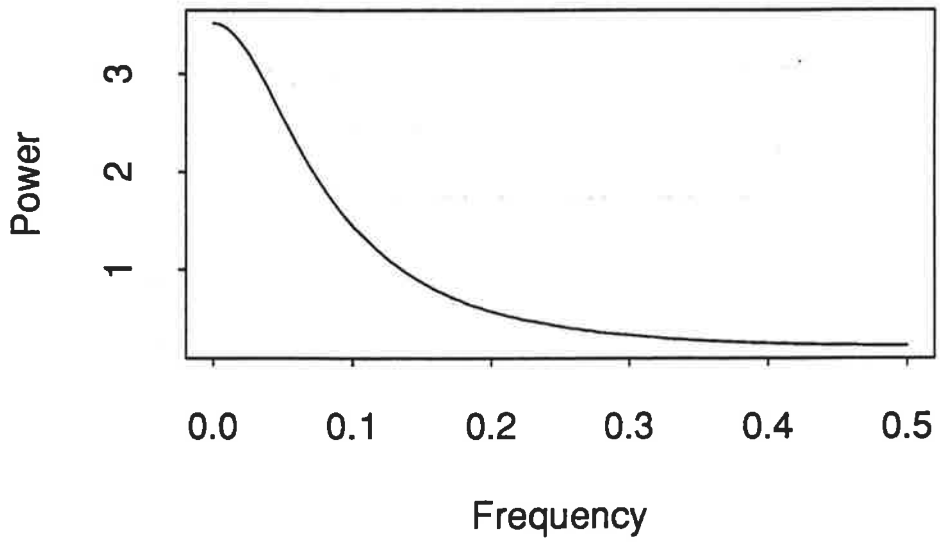
We will operate by replacing the independence assumption referred to above with a red noise assumption. That is, it will be assumed that the evolution of an observed climate process is governed, approximately, by a difference equation of the form[1]

$$\mathbf{X}_t - \mu_X = \rho_1(\mathbf{X}_{t-1} - \mu_X) + \epsilon_t \qquad (1)$$

where $\{\epsilon_t\}$ is a white noise process, ρ_1 is the lag-1 correlation coefficient, and μ_X is the long term mean. This simple difference equation model approximates observed climate behaviour when $0 \leq \rho_1 < 1$. In this case the power spectrum, an example of which is illustrated in Figure 1, has maximum energy at zero frequency and decreases smoothly with increasing frequency. This characteristic roughly approximates the behaviour of most thermo-dynamic variables in the free atmosphere, even if accurate representation of their stochastic behaviour requires the use of higher order models. The AR(1) model also arises naturally when one employs stochastic climate models (e.g., Hasselmann, 1976 [8]) to explain how the climate system can exhibit low

[1] Stochastic processes which satisfy (1) are frequently referred to as being auto-regressive of order 1 (AR(1)).

Figure 1: The power spectrum of an AR(1) process with lag-1 correlation coefficient $\rho_1 = 0.75$ and innovation variance $\sigma_\epsilon^2 = 1$.



frequency variability without resorting to either normal modes which operate at low frequencies, complex non-linear feedbacks, or external forcing. We therefore feel that the difficulties with the standard t-test can be usefully alleviated by approximating the stochastic structure of climate observations with an AR(1) model.

The remainder of this paper is organized with two groups of readers in mind. Those interested in the practical aspects of tests of the mean will find our final recommendations for one- and two-sample tests of the mean in Section 2. Readers interested in a detailed description of the testing problem should read the remainder of this section, skip Section 2 and return to it after reading the rest of the paper.

1.1 A pedagogical example.

A parochial and naive, but none the less instructive question is whether long-term mean winter temperatures at two locations such as Hamburg and Victoria are equal. To attempt to answer this question, suppose that we have at our disposal the daily observations at both locations for the winter of 1992/93. We treat the winter temperatures at both locations as random variables, say T_H and T_V. The "long term mean" winter temperatures at the two locations, denoted as μ_H and μ_V respectively, are parameters of the probability distributions of these random variables.

The statistical question we pose is: do the two samples of temperature observations contain sufficient evidence to reject the null hypothesis

$$H_0 : \mu_H - \mu_V = 0 \tag{2}$$

In this example, and in many applications in climate research, the assumption that the observations are statistically independent is not satisfied. Consequently, the Student's t-test tends to reject that null hypothesis on weaker evidence than is implied by the significance level[2] which is specified for the test. Consequently, the Student's t-test will reject the null hypothesis more frequently than expected when the null hypothesis is true.

1.2 Subsampling the data.

A relatively clean and simple minded solution to this problem is to form subsamples of independent observations. In the case of daily temperature data, one might argue that observations which are separated by, say, 5 days, are effectively independent of each other. If the number of samples, the sample means and standard deviations of these reduced data sets are denoted by n^*, $\bar{T}_H^*$, $\bar{T}_V^*$, S_H^* and S_V^* respectively, then the usual t-statistic

$$t = \frac{\bar{T}_H^* - \bar{T}_V^*}{\sqrt{({S_H^*}^2 + {S_V^*}^2)/n^*}} \tag{3}$$

has a Student's t-distribution with n^* degrees of freedom provided that the null hypothesis is true[3]. A test can be conducted at the specified significance level by comparing the value of (3) with the appropriate percentiles of this distribution.

Thus, appropriately sub-sampling the data will result in a test which operates as specified by the user. Unfortunately, this is achieved by throwing away much of the data and, presumably, at least part of the information contained in the samples.

1.3 The equivalent sample size.

The main reason for the liberal behaviour of the t-test when the data are not subsampled is that the denominator of (3) under estimates the sampling variance of the numerator (see Laurmann and Gates, 1974 [13], Chervin and Schneider, 1976 [5], Jones, 1975 [10] and Thiebaux and Zwiers, 1984 [17] (here after TZ) amongst others). To correct this problem, the sum of sample variances $S_H^2 + S_V^2$ must be scaled by the *equivalent sample size* n_e. The equivalent sample size is given by

$$n_e = n/[1 + 2\sum_{\tau=1}^{n-1}(1 - \frac{\tau}{n})\rho(\tau)] \tag{4}$$

where $\rho(\tau)$ is the correlation between temperature at time t and temperature at time $t+\tau$. When the observed processes are AR(1) with lag-1 correlation coefficient ρ_1, $\rho(\tau) = {\rho_1}^{|\tau|}$. Thus

[2] The significance level indicates the probability with which the null hypothesis will be rejected when it is true.
[3] Strictly speaking, this is true only if the standard deviations of T_H and T_V are equal.

$$n_e = n/[1 + 2\sum_{\tau=1}^{n-1}(1 - \frac{\tau}{n})\rho_1{}^{\tau}] \tag{5}$$

which may be approximated by

$$n_e \approx n\frac{(1-\rho_1)}{(1+\rho_1)} \tag{6}$$

when n is large.

As an (important) aside, we reiterate the point made in TZ that the equivalent sample size is not uniquely defined. The equivalent sample size arises because we interpret t as the distance between the sample means expressed in units of standard deviations of sample means. Many statistics have this basic form:

$$T = D/S_D \tag{7}$$

where D is some characteristic of the difference between two samples and S_D is an estimate of the standard error of D. In many cases, the standard error estimator which is appropriate when observations are independent needs to be scaled by some function of the sample size n when observations are serially correlated. The resulting expression for the equivalent sample size n_e or *integral time scale* n/n_e depends upon the definition of D.

Another way to think about the effective sample size is that it is a diagnostic which tells us something about the information loss due to serial correlation in a sample of size n. When n_e is defined as in (5), the interpretation is that samples of independent observations of size n_e contain as much information about the *difference of means* as samples of serially correlated observations of size n. In that context, we can think of n_e as the *number of effectively independent observations*. However, when our interest is in some other feature of the difference between two samples, the definition of information changes. Consequently, the definition of the equivalent sample size, or number of effectively independent observations, also changes. It is therefore impossible to interprete n_e as the number of effectively independent observations in an absolute sense.

1.4 Adjusting the t-statistic.

When samples are sufficiently large, the adjusted t statistic,

$$t = \frac{\bar{T}_H - \bar{T}_V}{\sqrt{(S_H^2 + S_V^2)/n_e}} \tag{8}$$

has a standard Gaussian, or normal, distribution N(0,1) with mean 0 and standard deviation 1 (Albers, 1978 [1]) under the null hypothesis. Thus one can conduct a test by comparing (8) to the percentiles of the standard Gaussian distribution. When samples are small it is often assumed that (8) will behave as Student's t with $n_e - 1$ degrees of freedom under the null hypothesis. While this assumption, which appears to have a heuristic basis, is asymptotically correct (Albers, 1978 [1], see Lemma 2.1), it is not correct for small samples (Katz, 1982 [12]; TZ).

The imprecision of the assumption that (8) is distributed Student's t is demonstrated with the following example. Suppose samples are obtained from a specified zero mean AR(1)-process. The exact equivalent sample size n_e is known because the AR(1) parameters are known. Samples of length n are randomly generated for various choices of n and AR(1) parameters and each sample is used to test the null hypothesis that $\mu_X = 0$ with t-statistic (8) at the 5% significance level. We find that the actual rejection rate (Figure 2) is notably smaller than the expected rate of 5% for $n_e \leq 30$.

Additional difficulties arise when the equivalent sample size must be estimated from the data. In this case the actual rejection rate of the t-test tends to be greater than the nominal rate (see Section 3). In some instances the actual significance level can be several times greater than the nominal significance level.

Although the effects of dependence on tests of the mean are well known in the statistical literature, it is relatively void of advice on how to counteract these effects. Albers (1978 [1]) considers the cost of making a large sample version of the t-test robust against against some kinds of dependence. Cressie (1980 [7]) contains a survey of the effects of various departures from the assumptions which are implicit in the t-test. Tubbs (1980 [18]) describes the effects of serial correlation on multivariate versions of the t-test. Katz (1982 [12]) describes an asymptotic test. Kabaila and Nelson (1985 [11]) describe univariate and multivariate versions of a competing asymptotic test. Both approaches were examined in TZ. Sutrahdar, et al. (1987 [16]) discuss one-way analysis of variance of experimental designs in which each "treatment" results in a sample taken from a time series. The comparison of the means of two relatively large samples is considered as a special case. None of these authors discuss the small sample case.

The remainder of this paper is organized as follows. Our final recommendations for one- and two-sample tests of the mean are contained in Section 2. Readers interested in a detailed description of the testing problem should skip this section and return to it after reading the rest of the paper. The "usual" t-test which is adjusted for serial correlation using the equivalent sample size is discussed in Section 3. The likelihood ratio (LR) test, a competing asymptotic test which is based on rigorous statistical principles, is described in Section 4. An empirically developed table look up test which has superior small sample properties is described in Section 5. A summary is presented in Section 6.

2 Recommended test procedures

2.1 Large Samples ($n_e \geq 30$)

We recommend the use of the "usual" t-test when the equivalent sample size n_e (one sample) or the sum of the equivalent sample sizes (two sample) is **known** to be greater than 30. The procedures are as follows.

- The One Sample Case:

  To test $H_0 : \mu = \mu_0$ using a sample of size n compute

$$t = \frac{(\bar{x} - \mu_0)}{s/\sqrt{\hat{n}'_e}} \tag{9}$$

  where $\bar{x}$ is the sample mean and s^2 is the sample variance. Compute the estimated equivalent sample size $\hat{n}'_e$ using

$$\hat{n}'_e = \begin{cases} 2 & \text{if } \hat{n}_e \leq 2, \\ \hat{n}_e & \text{if } 2 < \hat{n}_e \leq n, \\ n & \text{otherwise.} \end{cases} \tag{10}$$

  where $\hat{n}_e = n(1 - r_1)/(1 + r_1)$ and r_1 is the sample lag-1 correlation coefficient which is given by

$$r_1 = \frac{\sum_{t=2}^{n}(x_t - \bar{x})(x_{t-1} - \bar{x})}{\sum_{t=1}^{n}(x_t - \bar{x})^2} \tag{11}$$

  Compare the computed t-value with the appropriate critical values of the standard Gaussian distribution.

- The Two Sample Case:

  To test $H_0 : \mu_y = \mu_x$ using $\mathbf{Y}$ and $\mathbf{X}$ samples of size m and n respectively compute

$$t = \frac{\bar{y} - \bar{x}}{s(1/\sqrt{\hat{m}'_e} + 1/\sqrt{\hat{n}'_e})} \tag{12}$$

  where $\bar{y}$ and $\bar{x}$ are sample means and s^2 is the pooled sample variance. The latter is given by

$$s^2 = [\sum_{t=1}^{m}(x_t - \bar{x})^2 + \sum_{t=1}^{n}(y_t - \bar{y})^2]/(m + n - 2). \tag{13}$$

  The equivalent sample size estimates $\hat{m}'_e$ and $\hat{n}'_e$ are obtained by substituting a pooled estimate of the lag-1 correlation coefficient

$$r_1 = \frac{\sum_{t=2}^{m}(x_t - \bar{x})(x_{t-1} - \bar{x}) + \sum_{t=2}^{n}(y_t - \bar{y})(y_{t-1} - \bar{y})}{\sum_{t=1}^{m}(x_t - \bar{x})^2 + \sum_{t=1}^{n}(y_t - \bar{y})^2} \tag{14}$$

  into (10) for sample sizes m and n respectively. Compare the computed t-value with the appropriate critical values of the standard Gaussian distribution.

2.2 Small Samples ($n_e < 30$)

When n_e is **known** to be less than 30 a two stage procedure is required.

- First make a subjective estimate of n_e which you know to be no greater than the actual value. Use this estimate with the "usual" t-test (based on (9) or (12)) to make a preliminary decision. If the decision is to reject H_0 then the analysis can stop because evidence against H_0 has been found with a conservative test. Otherwise the analysis should continue with either the likelihood ratio test (LR test, see section 2.2.1) or table lookup test (see section 2.2.2).
- The LR test is suggested for the second stage when it is known that $n_e > 15$ (or $n_e > 15$ for both samples in the two sample case) . The LR test is preferable to the table look up test from a practical point of view because the required software is widely available and because the test does not rely upon specialized tables which are difficult to derive.
- The table look up test is suggested for the second stage when it is not known if $n_e > 15$ (or $n_e > 15$ for both samples in the 2 sample case). In this case it is prudent to utilize the protection against spurious reject decisions which is offered by the superior small sample properties of the table look up test.

Outlines of the LR and table lookup tests follow.

2.2.1 The LR test

Our recommendations for the LR test are made under the assumption that the user has access to computer codes which are able to make *exact* Gaussian maximum likelihood estimates of autoregressive time series models. We recommend Ansley's algorithm (Ansley, 1979 [2]) which is contained in the **Splus** *arima.mle* function (Splus, 1992 [15]; Becker, et. al., 1988 [3]). The **Splus** function also makes use of a transformation of the autoregressive parameters (in this case only ρ_1) which insures stationarity of the fitted model (Jones, 1980 [9]).

- The One Sample Case:
  1. Assume that $H_0 : \mu = \mu_0$ is true, compute the deviations $\mathbf{x}'_t = \mathbf{x}_t - \mu_0$, chose parameters ρ_1 and σ_ϵ to maximize the log likelihood l_r of the model $\mathbf{X}'_t = \rho_1 \mathbf{X}'_{t-1} + \epsilon_t$.
  2. Assume the $H_0 : \mu = \mu_0$ is false, chose parameters ρ_1, μ and σ_ϵ to maximize the log likelihood l_f of the model $(\mathbf{X}_t - \mu) = \rho_1(\mathbf{X}_{t-1} - \mu) + \epsilon_t$.
  3. Compute the difference between the log-likelihood of the *full* model (l_f) and that of the H_0 restricted model (l_r). Compare $(l_f - l_r)$ against the critical values of the χ^2 distribution with 1-df.
- The Two Sample Case:
  1. Assume that $H_0 : \mu_X = \mu_Y$ is true, compute the common mean $\hat{\mu} = [\sum_{t=1}^{m} \mathbf{x}_t + \sum_{t=1}^{n} \mathbf{y}_t]/(m+n)$ from both samples, compute the deviations $\mathbf{x}'_t = \mathbf{x}_t - \hat{\mu}$ and $\mathbf{y}'_s = \mathbf{y}_s - \hat{\mu}$, chose parameters $\rho_{X,1}$, σ_ϵ, $\rho_{Y,1}$ and σ_δ to maximize the log likelihoods of the models $\mathbf{X}'_t = \rho_1 \mathbf{X}'_{t-1} + \epsilon_t$ and $\mathbf{Y}'_s = \beta \mathbf{Y}'_{s-1} + \delta_s$ respectively, and set l_r to the sum of the log likelihoods.
  2. Assume that $H_0 : \mu_X = \mu_Y$ is false, chose parameters $\rho_{X,1}$, μ_X, σ_ϵ, $\rho_{Y,1}$, μ_Y and σ_δ to maximize the log likelihoods of the models $\mathbf{x}_t - \mu_x = \rho_1(\mathbf{x}_{t-1} - \mu_x) + \epsilon_t$ and $\mathbf{y}_s - \mu_y = \beta(\mathbf{y}_{s-1} - \mu_y) + \delta_s$ respectively, and set l_f to the sum of the log likelihoods.
  3. Compute the difference between the log-likelihood of the *full* model (l_f) and that of the H_0 restricted model (l_r). Compare $(l_f - l_r)$ with the critical values of the χ^2 distribution with 1-df.

2.2.2 The table look up test

- The One Sample Case:

  To test $H_0 : \mu = \mu_0$ using a sample of size n compute

$$t = \frac{(\bar{x} - \mu_0)}{s/\sqrt{n}} \tag{15}$$

  where $\bar{x}$ is the sample mean and s^2 is the sample variance. Compute the sample lag-1 correlation coefficient r_1 using (11). Use Tables 6-10[4] in Section 5 to determine the critical value for t which is appropriate to

[4] Electronic versions of the tables are available from the authors, either on DOS compatible floppy disk or via Internet e-mail (fzwiers@uvic.bc.doe.ca).

a sample of size n which has a lag-1 correlation coefficient r_1.

- The Two Sample Case:

  To test $H_0 : \mu_y = \mu_x$ using $\mathbf{Y}$ and $\mathbf{X}$ samples of size m and n respectively compute

$$t = \frac{\bar{y} - \bar{x}}{s\sqrt{\frac{1}{n} + \frac{1}{m}}} \tag{16}$$

  where $\bar{y}$ and $\bar{x}$ are sample means and s^2 is the pooled sample variance (13). Compute the pooled sample lag-1 correlation coefficient r_1 using (14). Use Tables 6-10 to determine the critical value for t which is appropriate to a sample of size $m + n$ which has a lag-1 correlation coefficient r_1.

3 The Usual Test

In this section we discuss the "usual" t-test in which the ordinary Student's t-statistic is scaled using an estimate of the equivalent sample size. For simplicity, the discussion will focus primarily on the one-sample test. We indicate how the results apply in the more usual two-sample context at the end of this section. As discussed in Section 1, the common approach used in climatology for testing the null hypothesis

$$H_0 : \mu = \mu_0 \tag{17}$$

is to

- compute a t-statistic as

$$t = \frac{(\bar{x} - \mu_0)}{s/\sqrt{\hat{n}_e}} \tag{18}$$

  where $\bar{x}$ is the sample mean, s^2 is the sample variance and $\hat{n}_e$ is an estimate of the equivalent sample size, and

- compare the computed t with critical values from the Student's t-distribution with $\hat{n}_e - 1$ df.

Variations in this approach are distinguished by the method used to estimate $\hat{n}_e$. In general, parsimonious methods (i.e., methods which rely upon the estimation of a small number of parameters) perform best.

The extreme antithesis of a parsimonious method is that which TZ called "DIRECT". In that method n_e is estimated by substituting the sample autocorrelation function into (4) directly. This results in highly variable estimates of n_e and by inference, very poor test performance. The variability is caused by sampling variability in the $n - 1$ estimated parameters which enter the calculation.

Substantial improvement can be obtained by truncating the sample autocorrelation function after a fixed number of lags (the method TZ call "DIRECT2"). Further improvement can be obtained by deriving the autocorrelation function from a parametric time series model fitted to the observations (the method TZ call "ARMA"; see also Katz, 1982 [12]).

We modify the ARMA estimator in two ways:

1. We fit only AR(1) models to the observations. We do this by estimating the lag-1 correlation coefficient of the observed time series with (11) and substituting this estimate of ρ_1 into (5) to provide an estimate $\hat{n}_e$ of n_e. Note that (6) may also be used when samples are large.

2. We note that sampling variability sometimes results in unrealistic values of $\hat{n}_e$. We therefore constrain the estimates to realistic values using (10).

The performance of this equivalent sample size estimator and the corresponding approximate t-test of the mean were examined by means of a simulation experiment. One thousand samples of length $n = 15, 30, 60, 90, 120,$ and 240 were taken from simulated AR(1) stochastic processes with mean zero and lag-1 correlation coefficients $\rho_1 = 0.3, 0.6,$ and 0.9. Thus a total of 18 combinations of sample length and persistence were considered. Each sample was used to estimate the equivalent sample size using the modified ARMA method described above, and each was used to conduct an approximate test of (17) with $\mu_0 = 0$.

The results for $\hat{n}_e'$ are summarized in Table 1. The modified ARMA estimator of the equivalent sample size shows improved performance compared to that reported by TZ. Both the bias and the variability of the estimates are reduced.

The results for a two-sided test of (17) with $\mu_0 = 0$ using the t-statistic (18) with n_e estimator $\hat{n}'_e$ and critical values appropriate to the 5% significance level are summarized in Table 2. Because H_0 is true, we would expect about 5% of the 1000 simulated tests to result in reject decisions if the test operates as anticipated.

We need to take the sampling variability of the observed rejection rates into account to determine whether they are significantly different from the anticipated rate. Test decisions are independent of each other because the samples were generated in such a way as to insure their mutual independence. Consequently, the number of reject decisions in each 1000-trial experiment should have a Binomial distribution in which the probability of a "success" (a rejection) on any trial will be 0.05 (von Storch, 1982 [19]). We therefore expect the observed proportion of reject decision to lie in the interval (0.0365, 0.0635) with probability 0.95.

Table 2 shows that the test generally rejects the null hypothesis too frequently. The effect is particularly dramatic when samples are small and the sampled time series is persistent. The test operates as designed only when samples are "large", that is, roughly when $n_e > 30$. Our conjecture is that at these sample sizes the variance of $\bar{x}$ is well enough estimated by $s^2/\hat{n}'_e$ that t has approximately a Gaussian distribution.

The fact that the test rejects H_0 too frequently at small sample sizes is due to the sampling variability of $\hat{n}'_e$. Table 3 and Figure 2 show that H_0 is not rejected frequently enough when the simulation is repeated with the known rather than estimated value of n_e.

Our advice then is that this simple and intuitively appealing test of the mean should be used either when samples are very large or when other knowledge about the problem can be used to infer something about the true value of n_e. As a very rough guideline, when n_e is estimated users should expect the actual level of significance of this test (when conducted at the nominal 5% level) to be about 10% when $n_e \approx 15$. The significance level will approach the nominal level when $n_e > 30$. If n_e cannot be reliably estimated a conservative *guestimate* (which is known to be no greater than the true n_e) can be used in place of $\hat{n}'_e$. This results in a safe (i.e. conservative) test which requires stronger evidence to reject H_0 than would be required by an optimal test.

A two sample test version of the test is constructed by analogy with the standard difference of means test which is described in Section 1. To apply the test it is necessary to assume that the two samples come from processes with the same variance and lag-1 correlation coefficient. This assumption is probably reasonable in most climate applications. The characteristics of the resulting two sample test, and our recommendations for its use, are the same as those for the one sample test except that they apply to the sum of the equivalent sample sizes for the two samples.

4 The Likelihood Ratio Test

A more formal approach to the problem of testing the mean of a time series is to base the inference on the likelihood ratio (LR) test (see Cox and Hinkley, 1974 [6]). The idea here is that the *likelihood* of the observations is maximized under two scenarios: one in which the null hypothesis is true and the other in which it is false. These likelihoods are compared by computing their ratio. Asymptotic theory provides a large sample reference distribution for the natural logarithm of the likelihood ratio and demonstrates that LR tests are asymptotically optimal.

Suppose that the sample $\mathbf{x}_1, \ldots, \mathbf{x}_n$ represents a realization of the random variables $\mathbf{X}_1, \ldots, \mathbf{X}_n$. Suppose also that our assumptions about the observed process $\{\mathbf{X}_t\}$ together with either the null of alternate hypotheses allows us to determine the joint density function $f(\mathbf{X}_1, \ldots, \mathbf{X}_n \mid \vec{\theta})$ of $\mathbf{X}_1, \ldots, \mathbf{X}_n$. The vector $\vec{\theta}$) represents parameters, such as the lag-1 correlation coefficient, which must be estimated. The *likelihood function* is then defined as

$$L(\vec{\theta} \mid \mathbf{x}_1, \ldots, \mathbf{x}_n) = f(\mathbf{x}_1, \ldots, \mathbf{x}_n \mid \vec{\theta}) \tag{19}$$

and the *maximum likelihood* estimate $\tilde{\vec{\theta}}$ of $\vec{\theta}$ is obtained determining the value of $\vec{\theta}$ which maximizes L.

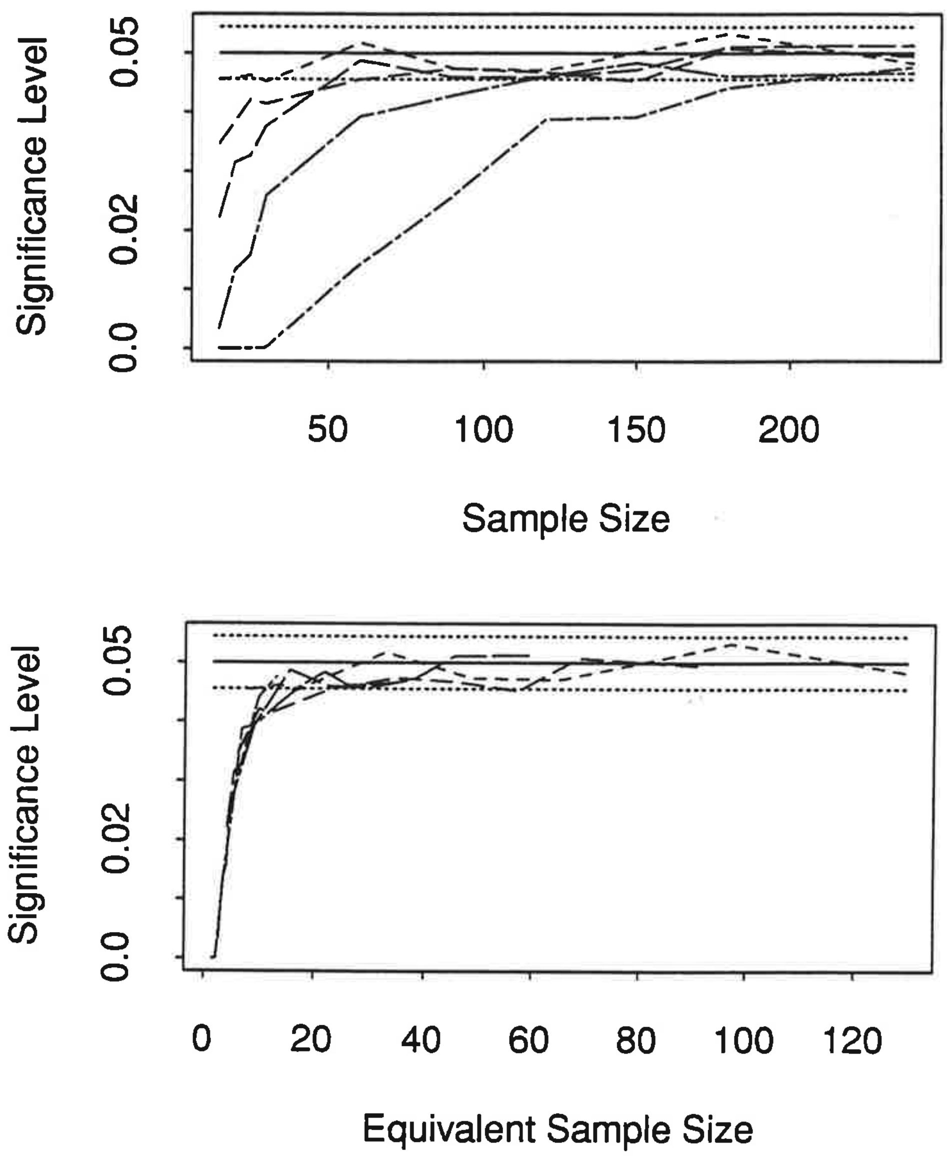
In the present context the two models which are fitted to the observed time series via the method of maximum likelihood are

$$\begin{aligned}
(\mathbf{X}_t - \mu_0) &= \rho_1(\mathbf{X}_{t-1} - \mu_0) + \epsilon_{\mathbf{t}} \quad \text{when } H_0 \text{ true.} && (20)\\
(\mathbf{X}_t - \mu) &= \rho_1(\mathbf{X}_{t-1} - \mu) + \epsilon_{\mathbf{t}} \quad \text{when } H_0 \text{ false.} && (21)
\end{aligned}$$

The noise, $\{\epsilon_{\mathbf{t}}\}$, is white. When H_0 is true the likelihood function depends upon ρ_1 and σ_ϵ^2, the variance of the noise. When H_0 is false, it depends upon ρ_1, σ_ϵ^2 and μ.

The likelihood functions for the *full* model (i.e., H_0 false) is given by:

Figure 2: Actual rejection rates of the t-test with n_e known in a series of simulation experiments. The upper panel shows the rejection rate as a function of sample size while the lower panel shows it as a function of equivalent sample size. Each panel contains 5 curves corresponding to data generated first order autoregressive processes $\mathbf{X}_t$ with zero mean and lag-1 correlation coefficients of 0.3 (short dashes), 0.45 (medium dashes), 0.6 (long dashes), 0.75 (long broken dashes) and 0.9 (medium broken dashes) respectively. The null hypothesis $\mathcal{E}(\mathbf{X}_t) = 0$ was tested at the 5% significance level in each of 1000 Monte Carlo trials conducted for each combination of sample size and lag-1 correlation coefficient. The solid horizontal line represents the nominal (5%) rejection rate and the dashed horizontal lines represent critical points at which the null hypothesis that the test operates at the nominal significance level is rejected.



$$L(\mu, \rho_1, \sigma_\epsilon|\bar{\mathbf{x}}) = (2\pi\sigma_\epsilon^2)^{-n/2}|\mathbf{M}|^{1/2}exp - \frac{(\bar{\mathbf{x}} - \vec{\mu})'\mathbf{M}(\bar{\mathbf{x}} - \vec{\mu})}{2\sigma_\epsilon^2} \tag{22}$$

where $\vec{\mu}$ is the $n \times 1$ vector $(\mu, \mu, \ldots, \mu)'$. The matrix $\mathbf{M}$ is given by $\mathbf{\Sigma} = \sigma_\epsilon^2\mathbf{M}^{-1}$ where $\mathbf{\Sigma}$ is the variance-covariance matrix of the vector of observations $\bar{\mathbf{x}}$. σ_ϵ^2 is the white noise variance. The (i, j) element of $\mathbf{\Sigma}$ is given by $\sigma_{i,j} = \sigma_\epsilon^2 {\rho_1}^{|i-j|}/(1 - {\rho_1}^2)$. The likelihood function for the H_0 restricted model is identical except that $\vec{\mu}_0 = (\mu_0, \mu_0, \ldots, \mu_0)'$ is substituted for $\vec{\mu}$. The intricacies of the exact likelihood function are described in detail in Box and Jenkins, 1976 [4].

The likelihood ratio (LR) test is conducted by computing the difference between the maximum log likelihood of the full (21) and partial (20) models. The resulting LR statistic is asymptotically distributed χ^2 with 1 df under the null hypothesis.

We conducted an experiment identical to that described in Section 3 to determine the actual significance level of the LR test when decisions are made at the nominal 5% significance level. Results of this experiment are summarized in Table 3.

The LR test clearly improves upon the "usual" test of the mean. Our advice is that this test should be used in preference to the "usual" test. Its mathematical and computational complexity should not be a deterrent because modern maximum likelihood estimation routines for Box-Jenkins models are readily available in a number of statistical packages. The LR test approach has desirable optimality properties and also has the advantage that it can be expanded to incorporate more sophisticated statistical representations of the stochastic nature of the observed climate. As a rough guideline, users should expect the actual level of significance of the LR test (when conducted at the nominal 5% level) to be about 10% for $n_e \approx 8$. The significance level will approach the nominal level when $n_e > 15$.

A two sample version of the LR test is constructed by analogy to the one sample test. The procedure is detailed in Section 2.2.1. Two sample LR tests can be constructed with or without the assumption of Section 3 that the two samples come from processes with the same variance and lag-1 correlation coefficient. We recommend the somewhat more general version which does not depend upon this assumption because the necessary computations can be done with existing software when the test is cast in this way. This generality comes at a cost in that larger samples are needed for the test to attain its asymptotic properties. The concept behind the test is that the *joint* log likelihood of samples $\mathbf{x}_1, \ldots, \mathbf{x}_m$ and $\mathbf{y}_1, \ldots, \mathbf{y}_n$ is maximized both with and without the restrictions imposed by the null hypotheses. The difference between the two log likelihoods is then computed and compared against the critical values of a χ^2-distribution. The number of degrees of freedom is found by computing the difference between the number of free parameters in the full model and the H_0 restricted model.

Our guidelines for use of the two sample test are the same as those for the one sample test except that they should be satisfied individually by the equivalent sample size of each sample. This cautious approach, which differs from that for the two sample test discussed in Section 3, is required because of the generality of the two sample LR test we advocate.

5 The Table Lookup Test

The "usual" t-test described in Section 3 works poorly because:

- The test statistic does not have a Student's t distribution under the null hypothesis.
- The denominator of the test statistic is highly variable because the equivalent sample size is poorly estimated.
- The critical value to which the test statistic is compared is subject to variability because the reference distribution is indexed by the same equivalent sample size estimator.

We reasoned that these problems could be ameliorated with an empirical test procedure based on the following ideas:

1. Base the test on a statistic which is not affected by the sampling variability which is present in equivalent sample size estimators. We chose to use the ordinary t-statistic which does not take serial correlation into account. The statistic (15) is given by

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}. \tag{23}$$

2. Index the critical values which are used to make test decisions with an indicator of the serial correlation which has low variability. We chose to use the sample correlation coefficient r_1 rather than $\hat{n}'_e$. The latter is highly variable because it is a non-linear function of the former.

3. Derive critical values appropriate to a particular value of r_1 via Monte Carlo simulation.

The last point deserves some comment.

When the true lag-1 correlation coefficient ρ_1 is known very good estimates of the critical values for the test of (17) based on (23) are easily obtained via Monte Carlo simulation. The approach is to generate a large number of samples of size n from the appropriate AR(1) process, compute t for each sample, compute the appropriate percentiles from the ensemble of simulated t's, and compare the observed t with the derived percentiles. The operating significance level of such as test will be very close to the nominal level provided one derives the critical values (i.e., percentiles) from a large ensemble of simulated t values.

When the true lag-1 correlation coefficient is not known critical values are more difficult to obtain because a wide range of true lag-1 correlation coefficients ρ_1 will be consistent with the sample coefficient r_1. The Bayesian approach to this problem is to use probability distributions to express the likelihood that ρ_1 has a particular value.

The *prior distribution* expresses our subjective beliefs about likely values of ρ_1 before observations are taken. In the absence of any other information, a reasonable choice of prior distribution on ρ_1 is the uninformative prior which places equal weight on all values in the interval $[0, 1)$. That is, we think any value of ρ_1 consistent with a red spectrum is as likely as any other. The *posterior distribution* expresses our revised opinion about likely values of ρ_1 after sampling is complete.

Given the posterior distribution, Monte Carlo methods can be used to derive critical values for the test of (17) based on (23). The approach is to generate a large number of ρ_1's from the posterior distribution, generate a sample of length n from each corresponding AR(1) process, compute t's from these samples, and then compute the appropriate percentiles from the ensemble of simulated t's.

We emulate the Bayesian approach by deriving critical values as follows:

1. Generate an ensemble of 240,000 lag-1 correlation coefficients ρ_1 randomly on the interval $(0, 1)$.

2. For each ρ_1 generate a sample of length n from the corresponding AR(1) process.

3. Compute r_1 and t from each sample.

4. Sort the resulting ensemble of 240,000 (r_1, t) pairs in order of increasing r_1.

5. Select 200 equally spaced points $r_{1,i}, i = 1, \ldots, 200$ between the minimum and maximum of the simulated r_1's.

6. At each of these 200 "base" points select the m (r_1, t) pairs with r_1 nearest the base value. We used $m = 4800\sqrt{240/n}$. This choice for m will be explained below.

7. Compute the 80'th, 90'th, 95'th, 98'th and 99'th quantiles of $|t|$ from each subset of selected t-values. We will refer to these quantiles as $t_{0.90,i}$, $t_{0.95,i}$, ..., $t_{0.995,i}$[5].

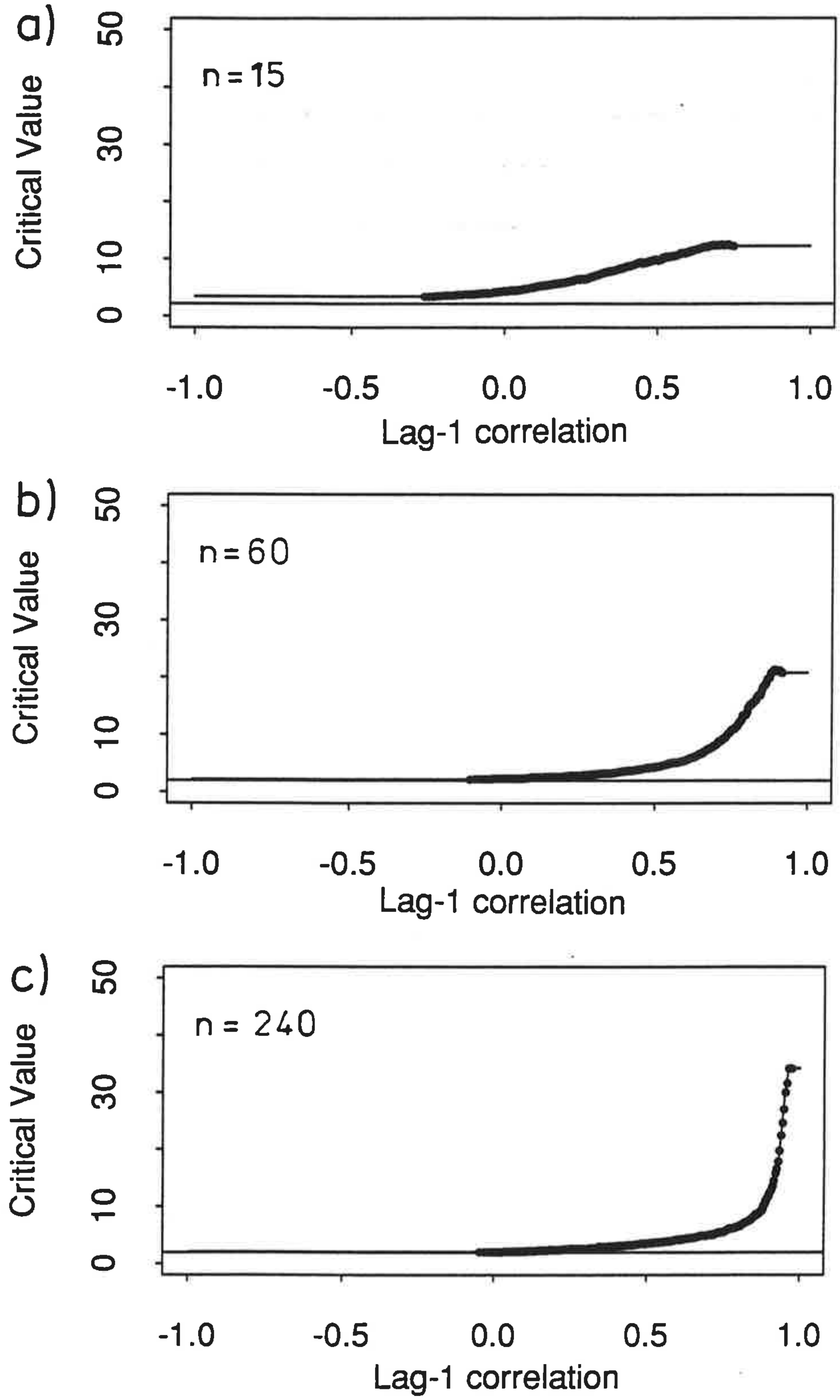
The result is 5 critical value tables (Tables 6-10) which are indexed by r_1 and are suitable for two sided (one sided) tests of (17) at the 20%, 10%, 5%, 2% or 1% (10%, 5%, 2.5%, 1% or 0.5%) level respectively.

These tables form the basis of our empirical test procedure. To conduct a test at a given significance level we first compute r_1 and t from the sample. The sample correlation r_1 is used to enter the appropriate critical value table. It may be necessary to interpolate between table entries which are nearest to r_1.

Examples of the tables of the quantiles appropriate for a 5% level two-sided test with samples of size $n = 15$, 60 and 240 are illustrated in Figure 3. Note that for the larger sample sizes the crossing point at $r_1 = 0$ corresponds closely to the critical value which would be used if we could assume that the data are not serially correlated. The latter (percentiles of the Student's t-distribution with $n-1$ degrees of freedom) are illustrated as horizontal lines. The discrepancy between the crossing point at $r_1 = 0$ and the Student's t critical value is large for small samples because the distribution of r_1 is widely dispersed in this case. Note that quantiles gradually increase as r_1 increases. Critical values for tests conducted at higher levels of significance, such as the 1% level, actually peak for r_1 near 1 and then decrease as r_1 approaches 1.

[5]Note that because of symmetry, the 80th (90th, ...) percentile of the simulated $|t|$'s is a better estimator of the 90th (95th, ...) percentile of t than the 90th (95th, ...) percentile of the simulated t's

Figure 3: Critical value curves for the standard t-statistic (21) indexed by the value of the estimated lag-1 correlation coefficient (16) for samples of size 15 (a), 60 (b) and 240 (c). The curves are appropriate for 5%-level two-sided or 2.5%-level one-sided tests of (14).



This empirical approach emulates the Bayesian approach in the following sense. The critical value which is referenced by a sample lag-1 correlation coefficient was derived as the corresponding quantile of a collection of m realizations of t. Each of these realizations was obtained from an AR(1) process with a different lag-1 correlation coefficient ρ_1. This collection of ρ_1's constitutes an empirical posterior distribution on ρ_1. The prior distribution in this instance is the uniform distribution on the interval $[0, 1)$.

The total number of simulated (r_1, t) pairs used to obtain the critical value table is large to reduce noise in the table. The total number of pairs (240,000) and the number of pairs used to determine an individual table entry ($m = 4800\sqrt{240/n}$) was chosen so that (r_1, t) pairs spanning only 2% of the r_1 range could be used to accurately determine the critical values for the largest sample size considered (240). A narrow span is necessary when the sample size is large because the critical value curve changes quickly in this case for values of r_1 near 1. The number of simulated (r_1, t) pairs used to determine critical value table entries is a function of $n^{-1/2}$ to compensate for the effects of sampling variability on r_1.

The operation of the test when the null hypothesis is true was examined in a simulation experiment analogous to those described in Sections 3 and 4. The rejection rate for the test under H_0 in a 1000 trial experiment is reported in Table 5. Note that there is some imprecision when sample sizes are small: the test is conservative when the lag-1 correlation coefficient is small and somewhat liberal when it is large. Otherwise, the test operates more or less as advertised - and it generally does so for smaller samples than either of the competing tests described previously. Our experimentation has shown us that the imprecision in the test at small sample sizes is caused by sampling variability in the lag-1 correlation coefficient.

The power of the table lookup test is contrasted with that of the LR test in Figure 4 for samples of size 90 for which the true lag-1 correlation is 0.3 and 0.6 and for samples of size 240 for which the true lag-1 correlation is 0.3, 0.6 and 0.9. The comparison can be made fairly for these combinations of sample size and lag-1 correlation because both tests appear to operate at the nominal 5% significance level in this circumstance. The power curves were obtained from 1000 trial experiments in which departures from the null hypothesis ranging between $\sigma_x/4$ and $5\sigma_x$ were prescribed. What we see is that there is little difference between the power of the likelihood-ratio test (which is known to be asymptotically optimal) and the table lookup test.

Therefore, the table lookup test is an attractive competitor to the "usual" t-test discussed in Section 3 and the likelihood ratio test. A significant efficiency penalty is apparently not imposed through the use of the table lookup test. Moreover, the table lookup test operates at near the nominal significance level with smaller samples than either the "usual" test or the likelihood ratio test.

A two sample version of the table lookup test is detailed in Section 2.2.2. The test is developed using the assumption that both sampled processes have the same variance and lag-1 correlation. The ingredients are virtually identical to those used in the one sample test: the ordinary Student's t-statistic for the difference of means is computed and an estimate is made of the common lag-1 correlation coefficient. As in the one sample case, the properties of the t-statistic depend upon the the true lag one correlation coefficient and the number of observations used to compute the standard deviation in the denominator of the t-statistic. The uncertainty in the estimated lag-1 one correlation coefficient also depends the number of observations used in the estimate. Thus the appropriate critical values are obtained by entering the critical value tables with the estimated lag-1 correlation coefficient and the sum of the two sample sizes. Our guidelines for the use of the two sample test are the same as those for the one sample test except that they apply to the sum of the sample sizes.

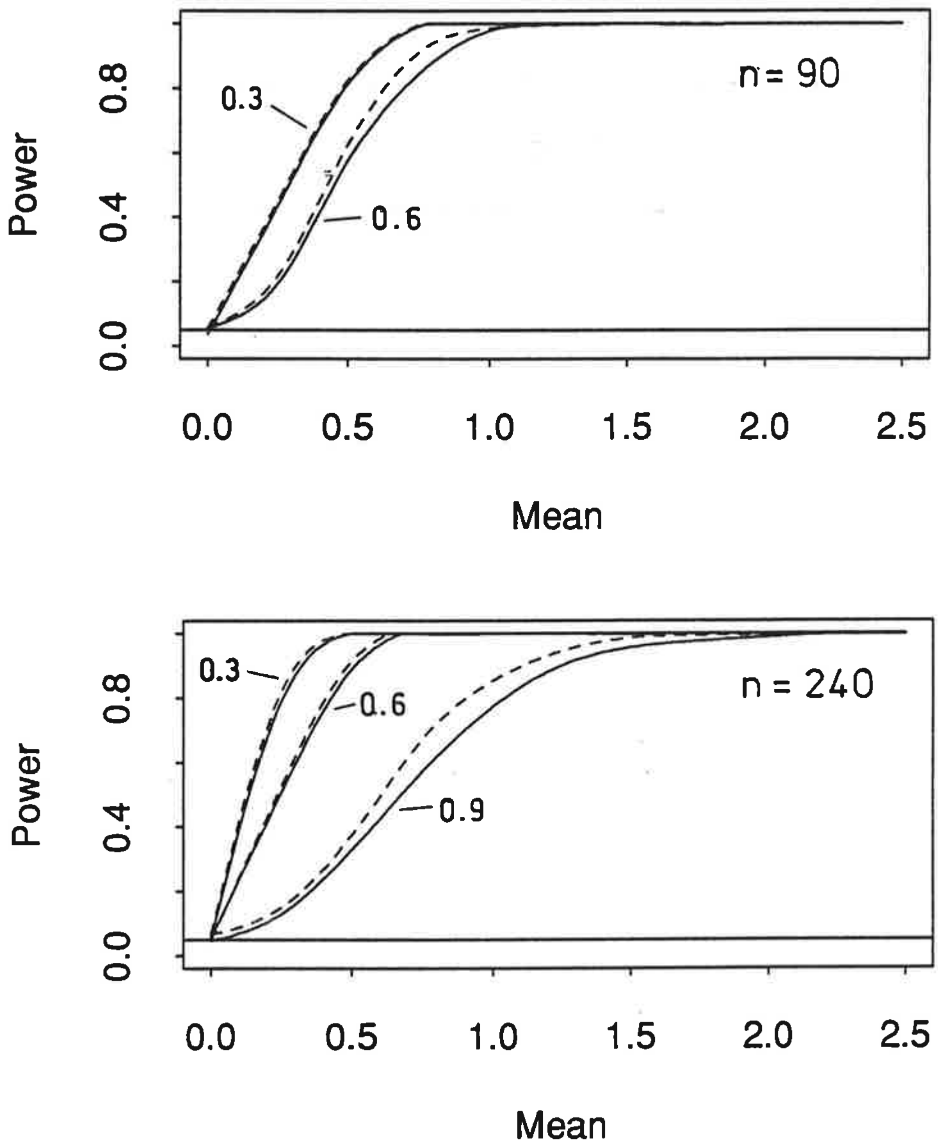
6 Summary

We described the ordinary t-test and the usual way in which it is adapted to climatological inference problems in Section 1. The usual test, which is adjusted by an estimate of the equivalent sample size, performs poorly because the equivalent sample size is poorly estimated and because it is incorrectly assumed that the adjusted statistic has a Student's t distribution under the null hypothesis.

We also reiterated the observation that the equivalent sample size (or equivalently, the integral time scale) is obtained simply by asking how a measure of the difference between two samples (or a sample and reference point) should be scaled so that the difference can be expressed in units of standard deviations. Different measures of discrepancy will therefore result in different scaling factors. Thus the equivalent sample size is only one of many scaling factors and has no intrinsic physical interpretation of its own.

Next, we carefully revisited the subject of equivalent sample size estimation and examined the properties of the "usual" serial correlation compensated t-test in section 3. We were able to improve the ARMA n_e estimator of TZ but found that this improvement had little effect on the operation of the test. We showed that the test is conservative when n_e is known or subjectively estimated conservatively. We also showed that the test is liberal when n_e is estimated objectively with the improved ARMA estimator. In both cases (n_e "known" or objectively

Figure 4: The estimated power of the table lookup test (solid line) and the likelihood ratio test (dashed line) for a range of alternatives to the null hypothesis. The power was estimated by generating 1000 samples from an AR(1) process with the specified mean and testing the null hypothesis on each sample. The upper panel shows the power of the test for samples of size 90 obtained from AR(1) processes with $\rho_1 = 0.3$ and 0.6. The lower panel displays corresponding results for samples of size 240 $\rho_1 = 0.3, 0.6$ and 0.9.



estimated) we found that the test has actual significance levels which are practically indistinguishable from the nominal levels when the true $n_e > 30$.

We then considered two competing tests of the mean: the likelihood ratio test and a table lookup test. Both are computationally intensive. The LR ratio test requires numerical minimization of a complex quadratic form while the table lookup test requires extensive simulation to develop the conditional reference distributions which are used in that test. The likelihood ratio test, which is an asymptotic test, operates at significance levels which are similar to the nominal levels when $n_e > 15$. The table look up test operates at significance levels which are close to the nominal levels for all but the smallest sample sizes tested. Moreover, the power of the two tests is virtually indistinguishable for samples large enough so that the actual significance level of the LR test is equal to its nominal level.

Comparison and analysis of the three competing tests was performed in the one sample setting in which comparisons are made between a sample mean and a fixed standard. The more usual setting requires the comparison of the means of two samples. Procedures for the application of the three tests in both the one- and two-sample setting were detailed in Section 2. Recommendations for when to use the various tests were also given in Section 2.

The working assumption in this paper has been that the stochastic behaviour of the atmosphere can be reasonably approximated by a Gaussian auto-regressive process of order 1 (AR(1) process - see (1)). The Gaussian part of the assumption is important. The tests discussed above may be compromised to a considerable extent if applied to non-Gaussian data (such as daily precipitation accumulations). For processes which are Gaussian, or nearly so (such as most thermo-dynamic variables of the free atmosphere) the details of the stochastic behaviour may be somewhat more complex than that an AR(1) process without compromising the tests. Most such processes will exhibit power spectra which have maxima at the origin and have decreasing power with increasing frequency. Such spectra can be reasonably well approximated by AR(1) processes.

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Table 1: Each cell contains a summary of 1000 simulated realizations of estimator $\hat{n}'_e$. Each estimate was computed from a sample of length n generated from an AR(1) stochastic process with lag-1 correlation α. The first entry in each cell is the known equivalent sample size obtained from (11). The second entry is the mean of 1000 realizations of $\hat{n}'_e$. The entries in parentheses indicate the *interquartile range* (IQR) of the 1000 realizations of $\hat{n}'_e$. The IQR contains the middle 50% of all $\hat{n}'_e$ realizations.

	lag-1 correlation α								
	0.3			0.6			0.9		
n	n_e	$\hat{n}'_e$	IQR	n_e	$\hat{n}'_e$	IQR	n_e	$\hat{n}'_e$	IQR
15	8.4	10.9	(8, 15)	4.3	7.8	(5, 10)	1.6	5.1	(3, 6)
30	16.5	19.7	(15, 24)	8.0	11.2	(8, 14)	2.3	5.7	(4, 7)
60	32.7	36.7	(30, 43)	15.5	18.6	(14, 22)	3.7	6.9	(5, 8)
90	48.8	51.7	(44, 58)	23.0	26.3	(21, 30)	5.3	8.2	(6, 10)
120	65.0	68.3	(59, 76)	30.5	33.6	(28, 38)	6.9	9.7	(7, 12)
240	129.6	134	(121,144)	60.5	63.3	(57, 69)	13.2	16.1	(13, 19)

Table 2: Each cell summarizes the results of 1000 tests of $H_0 : \mu = 0$. The test was conducted by computing t-statistic (15) and comparing it with the 5% critical points of the Student's t-distribution with $\hat{n}'_e - 1$ df. Bold face type indicates observed rejection rates under H_0 which are different from the nominal 5% at the 5% significance level.

	lag-1 correlation α		
n	0.3	0.6	0.9
15	**0.077**	**0.169**	**0.308**
30	**0.083**	**0.116**	**0.249**
60	**0.070**	**0.084**	**0.170**
90	0.060	**0.079**	**0.130**
120	0.051	0.062	**0.118**
240	0.052	0.060	**0.090**

Table 3: Each cell summarizes the results of 1000 tests of $H_0 : \mu = 0$. The test was conducted by computing t-statistic (15) using the known value of n_e and comparing it with the 5% critical points of the Student's t-distribution with $n_e - 1$ df. Bold face type indicates observed rejection rates under H_0 which are different from the nominal 5% at the 5% significance level.

	lag-1 correlation α		
n	0.3	0.6	0.9
15	0.046	**0.022**	**0.000**
30	0.045	0.038	**0.000**
60	0.051	0.049	**0.014**
90	0.047	0.046	**0.026**
120	0.047	0.046	0.039
240	0.048	0.051	0.048

Table 4: Each cell summarizes the results of 1000 LR tests of $H_0 : \mu = 0$. The test was conducted by computing the LR statistic and comparing it with the 5% critical point of the χ^2 distribution with 1 df. Bold face type indicates observed rejection rates under H_0 which are different from the nominal 5% rate at the 5% significance level.

	lag-1 correlation α		
n	0.3	0.6	0.9
15	**0.088**	**0.118**	**0.214**
30	0.057	**0.085**	**0.150**
60	0.045	**0.077**	**0.096**
90	0.063	0.062	**0.094**
120	0.049	0.056	**0.092**
240	0.055	0.056	0.064

Table 5: Each cell summarizes the results of 1000 table lookup tests of $H_0 : \mu = 0$. The test was conducted by computing the ordinary t-statistic and comparing it with an empirical distribution which is indexed by the sample lag-1 correlation coefficient (see text). Bold face type indicates observed rejection rates under H_0 which are different from the nominal 5% at the 5% significance level.

	lag-1 correlation α		
n	0.3	0.6	0.9
15	**0.003**	**0.030**	**0.128**
30	**0.027**	0.035	**0.085**
60	0.043	0.043	0.061
90	0.037	0.043	**0.034**
120	0.042	0.047	0.051
240	0.053	0.050	0.054

Table 6: Critical values for the table look up test appropriate for a two (one) sided test conducted at the 20% (10%) significance level. Dashes indicate that sample correlations of this magnitude were not observed in the simulations used to create the table.

α	n											
	10	15	20	25	30	45	60	75	90	120	180	240
-0.35	2.30	—	—	—	—	—	—	—	—	—	—	—
-0.30	2.34	1.87	—	—	—	—	—	—	—	—	—	—
-0.25	2.41	1.88	1.70	—	—	—	—	—	—	—	—	—
-0.20	2.46	1.92	1.71	1.62	1.54	—	—	—	—	—	—	—
-0.15	2.52	1.99	1.75	1.64	1.55	1.45	—	—	—	—	—	—
-0.10	2.61	2.04	1.80	1.67	1.59	1.47	1.44	1.42	1.39	1.38	—	—
-0.05	2.67	2.09	1.84	1.68	1.64	1.50	1.47	1.44	1.39	1.39	1.36	1.33
0.00	2.75	2.18	1.88	1.72	1.66	1.53	1.49	1.46	1.43	1.40	1.38	1.35
0.05	2.82	2.26	1.97	1.79	1.71	1.60	1.55	1.47	1.50	1.46	1.43	1.42
0.10	2.94	2.35	2.08	1.87	1.76	1.64	1.58	1.55	1.50	1.48	1.46	1.40
0.15	3.10	2.48	2.18	1.96	1.82	1.70	1.66	1.60	1.58	1.57	1.53	1.51
0.20	3.23	2.59	2.27	2.08	1.91	1.77	1.70	1.66	1.64	1.63	1.59	1.57
0.25	3.36	2.75	2.42	2.22	2.05	1.87	1.78	1.75	1.77	1.69	1.69	1.70
0.30	3.48	2.96	2.57	2.36	2.22	1.96	1.91	1.91	1.86	1.86	1.79	1.77
0.35	3.61	3.20	2.79	2.56	2.38	2.12	2.02	2.03	1.96	1.88	1.92	1.90
0.40	3.77	3.46	3.02	2.74	2.61	2.31	2.22	2.13	2.11	2.07	2.00	2.02
0.45	3.95	3.66	3.38	3.05	2.79	2.55	2.36	2.25	2.23	2.19	2.14	2.13
0.50	4.13	3.92	3.71	3.41	3.24	2.73	2.57	2.47	2.46	2.39	2.31	2.30
0.55	4.27	4.23	4.13	3.77	3.55	3.03	2.86	2.65	2.61	2.56	2.50	2.43
0.60	4.45	4.59	4.47	4.26	3.94	3.45	3.14	3.03	2.83	2.75	2.65	2.72
0.65	4.55	4.83	4.85	4.72	4.49	3.89	3.60	3.26	3.16	3.02	2.91	2.94
0.70	4.56	5.17	5.37	5.28	5.25	4.49	4.01	3.74	3.59	3.36	3.27	3.33
0.75	—	5.37	5.71	5.82	5.85	5.35	4.91	4.40	4.14	3.92	3.69	3.55
0.80	—	—	5.99	6.49	6.49	6.42	6.00	5.51	5.18	4.67	4.28	4.25
0.85	—	—	—	6.66	7.26	7.82	7.33	7.28	6.76	6.09	5.41	5.05
0.90	—	—	—	—	7.31	8.77	9.47	9.45	9.01	8.59	7.55	6.97
0.95	—	—	—	—	—	—	9.93	10.7	11.3	13.2	13.3	12.5

Table 7: Critical values for the table look up test appropriate for a two (one) sided test conducted at the 10% (5%) significance level. Dashes indicate that sample correlations of this magnitude were not observed in the simulations used to create the table.

α	n											
	10	15	20	25	30	45	60	75	90	120	180	240
-0.35	3.46	—	—	—	—	—	—	—	—	—	—	—
-0.30	3.55	2.57	—	—	—	—	—	—	—	—	—	—
-0.25	3.68	2.59	2.26	—	—	—	—	—	—	—	—	—
-0.20	3.80	2.66	2.28	2.15	2.03	—	—	—	—	—	—	—
-0.15	3.85	2.76	2.35	2.17	2.04	1.89	—	—	—	—	—	—
-0.10	4.03	2.89	2.44	2.22	2.09	1.92	1.85	1.83	1.81	1.78	—	—
-0.05	4.21	2.98	2.52	2.24	2.13	1.96	1.90	1.88	1.81	1.80	1.73	1.72
0.00	4.34	3.15	2.59	2.31	2.18	2.02	1.92	1.89	1.84	1.83	1.75	1.74
0.05	4.48	3.30	2.76	2.40	2.26	2.09	1.98	1.89	1.94	1.86	1.84	1.84
0.10	4.75	3.47	2.88	2.56	2.38	2.15	2.07	2.01	1.96	1.91	1.89	1.85
0.15	5.07	3.71	3.06	2.69	2.43	2.24	2.14	2.09	2.04	2.02	1.96	1.96
0.20	5.27	3.96	3.26	2.85	2.58	2.33	2.21	2.17	2.14	2.12	2.05	2.03
0.25	5.42	4.25	3.63	3.07	2.79	2.48	2.35	2.28	2.29	2.21	2.19	2.20
0.30	5.63	4.60	3.82	3.32	3.07	2.66	2.53	2.51	2.42	2.37	2.33	2.26
0.35	5.90	5.10	4.20	3.72	3.35	2.85	2.63	2.65	2.56	2.47	2.44	2.43
0.40	6.17	5.56	4.74	4.07	3.67	3.11	2.94	2.81	2.76	2.68	2.56	2.56
0.45	6.44	5.60	5.28	4.59	4.09	3.53	3.19	2.97	2.94	2.84	2.79	2.76
0.50	6.80	6.54	5.93	5.23	4.80	3.79	3.52	3.26	3.19	3.11	3.02	2.95
0.55	7.00	6.93	6.64	5.96	5.31	4.39	3.85	3.52	3.49	3.33	3.25	3.13
0.60	7.15	7.45	7.36	6.96	6.14	5.05	4.29	4.10	3.76	3.61	3.44	3.54
0.65	7.20	7.91	8.03	7.67	7.15	5.80	5.03	4.49	4.24	4.06	3.79	3.78
0.70	7.21	8.39	8.70	8.72	8.50	6.87	5.74	5.30	4.85	4.53	4.34	4.21
0.75	—	8.57	9.13	9.44	9.22	8.43	7.19	6.43	5.80	5.23	4.89	4.67
0.80	—	—	9.65	10.3	10.5	10.4	9.11	8.28	7.64	6.55	5.79	5.50
0.85	—	—	9.67	10.4	11.4	12.8	11.8	11.4	10.3	9.02	7.40	6.72
0.90	—	—	—	—	11.4	13.8	14.4	14.9	14.9	13.4	10.9	9.58
0.95	—	—	—	—	—	—	14.9	16.4	17.3	20.3	20.7	19.8

Table 8: Critical values for the table look up test appropriate for a two (one) sided test conducted at the 5% (2.5%) significance level. Dashes indicate that sample correlations of this magnitude were not observed in the simulations used to create the table.

α	n											
	10	15	20	25	30	45	60	75	90	120	180	240
-0.35	5.18	—	—	—	—	—	—	—	—	—	—	—
-0.30	5.32	3.44	—	—	—	—	—	—	—	—	—	—
-0.25	5.56	3.47	2.84	—	—	—	—	—	—	—	—	—
-0.20	5.71	3.61	2.86	2.65	2.50	—	—	—	—	—	—	—
-0.15	5.76	3.75	2.98	2.70	2.50	2.34	—	—	—	—	—	—
-0.10	6.25	3.88	3.11	2.76	2.56	2.34	2.22	2.23	2.19	2.10	—	—
-0.05	6.48	4.05	3.19	2.81	2.62	2.40	2.28	2.28	2.19	2.12	2.08	2.05
0.00	6.75	4.44	3.34	2.90	2.71	2.46	2.33	2.28	2.31	2.19	2.11	2.20
0.05	7.17	4.68	3.59	3.05	2.82	2.53	2.37	2.31	2.31	2.25	2.18	2.20
0.10	7.49	5.16	3.82	3.29	2.99	2.65	2.53	2.42	2.34	2.32	2.21	2.21
0.15	7.91	5.55	4.15	3.55	3.12	2.77	2.63	2.52	2.46	2.43	2.38	2.35
0.20	8.38	6.00	4.61	3.74	3.34	2.89	2.70	2.64	2.60	2.59	2.45	2.43
0.25	8.52	6.50	5.20	4.13	3.52	3.07	2.89	2.78	2.76	2.69	2.66	2.66
0.30	8.81	7.17	5.57	4.52	4.05	3.32	3.10	3.05	2.91	2.86	2.76	2.70
0.35	9.11	7.93	6.25	5.23	4.48	3.60	3.28	3.21	3.11	2.92	2.93	3.00
0.40	9.55	8.74	7.19	5.88	5.05	3.96	3.61	3.45	3.34	3.28	3.13	3.11
0.45	9.91	9.36	8.25	6.72	5.92	4.50	3.95	3.75	3.51	3.48	3.36	3.29
0.50	10.4	9.84	9.10	8.00	7.01	4.99	4.36	3.97	3.90	3.78	3.70	3.54
0.55	10.6	10.6	10.2	9.21	7.79	5.82	5.04	4.41	4.28	4.09	4.00	3.85
0.60	10.7	11.2	11.4	11.0	9.22	6.98	5.49	5.31	4.69	4.47	4.18	4.25
0.65	10.6	12.1	12.1	11.9	11.0	8.33	6.68	5.81	5.36	5.04	4.67	4.57
0.70	10.6	12.5	13.2	13.5	13.0	10.1	8.20	7.07	6.25	5.71	5.25	5.04
0.75	—	12.3	13.8	14.2	14.2	13.1	10.7	9.12	7.77	6.58	5.98	5.62
0.80	—	—	14.2	15.2	15.6	15.2	13.6	12.1	11.0	8.60	7.22	6.64
0.85	—	—	—	15.1	16.5	18.8	17.3	16.9	15.4	12.9	9.50	8.37
0.90	—	—	—	—	16.5	20.0	21.2	22.0	21.2	20.0	15.1	12.6
0.95	—	—	—	—	—	—	20.9	23.4	24.3	27.4	28.8	29.5

Table 9: Critical values for the table look up test appropriate for a two (one) sided test conducted at the 2% (1%) significance level. Dashes indicate that sample correlations of this magnitude were not observed in the simulations used to create the table.

α	n											
	10	15	20	25	30	45	60	75	90	120	180	240
-0.35	8.76	—	—	—	—	—	—	—	—	—	—	—
-0.30	9.03	5.02	—	—	—	—	—	—	—	—	—	—
-0.25	9.26	5.12	3.80	—	—	—	—	—	—	—	—	—
-0.20	9.98	5.43	3.89	3.34	3.07	—	—	—	—	—	—	—
-0.15	10.1	5.67	4.03	3.42	3.10	2.79	—	—	—	—	—	—
-0.10	10.9	5.87	4.21	3.52	3.19	2.82	2.68	2.68	2.63	2.51	—	—
-0.05	11.1	6.49	4.38	3.67	3.30	2.97	2.77	2.69	2.61	2.51	2.52	2.45
0.00	11.9	7.35	4.71	3.91	3.44	3.04	2.85	2.76	2.69	2.61	2.50	2.41
0.05	12.4	7.97	5.30	4.07	3.60	3.08	2.89	2.82	2.76	2.69	2.61	2.59
0.10	12.8	8.92	5.72	4.49	3.95	3.28	3.03	2.88	2.78	2.71	2.66	2.64
0.15	14.0	10.1	6.51	4.78	4.26	3.45	3.24	3.08	3.01	2.92	2.83	2.86
0.20	14.6	10.5	7.18	5.28	4.40	3.60	3.39	3.19	3.06	3.09	2.94	2.96
0.25	14.7	11.1	8.59	6.16	4.81	3.81	3.55	3.43	3.27	3.21	3.19	3.23
0.30	15.2	12.8	8.94	6.88	5.68	4.26	3.97	3.72	3.57	3.42	3.31	3.21
0.35	15.4	13.7	10.5	8.26	6.58	4.70	4.05	3.99	3.82	3.51	3.49	3.56
0.40	15.7	14.9	12.8	9.54	7.85	5.26	4.60	4.37	4.07	3.85	3.84	3.79
0.45	16.4	16.3	14.9	11.4	8.92	6.32	5.14	4.73	4.33	4.28	4.07	3.92
0.50	17.1	17.4	16.3	13.8	11.4	6.99	5.81	5.18	4.79	4.58	4.39	4.35
0.55	16.9	17.9	17.3	15.6	12.6	8.73	6.91	5.65	5.34	5.05	4.85	4.57
0.60	17.2	18.6	18.7	18.9	14.8	10.6	7.53	6.78	5.94	5.63	4.94	5.10
0.65	16.9	19.8	19.2	20.6	18.9	14.2	10.1	7.76	7.05	6.24	5.62	5.37
0.70	16.9	19.3	21.3	22.0	22.2	17.6	12.4	9.77	8.35	7.32	6.51	6.12
0.75	—	18.8	21.4	22.1	23.0	21.9	18.1	14.2	11.0	8.56	7.64	6.98
0.80	—	—	20.7	22.9	24.6	24.6	21.3	20.0	16.6	12.6	9.15	8.18
0.85	—	—	—	22.7	25.1	28.6	28.4	30.3	24.6	19.1	12.9	11.1
0.90	—	—	—	—	25.0	27.9	31.0	33.7	33.1	31.1	23.8	17.8
0.95	—	—	—	—	—	—	29.7	32.2	33.4	37.4	42.3	45.8

Table 10: Critical values for the table look up test appropriate for a two (one) sided test conducted at the 1% (0.5%) significance level. Dashes indicate that sample correlations of this magnitude were not observed in the simulations used to create the table.

α	n											
	10	15	20	25	30	45	60	75	90	120	180	240
-0.35	13.0	—	—	—	—	—	—	—	—	—	—	—
-0.30	13.3	6.82	—	—	—	—	—	—	—	—	—	—
-0.25	14.0	7.01	4.70	—	—	—	—	—	—	—	—	—
-0.20	15.1	7.42	4.83	3.87	3.58	—	—	—	—	—	—	—
-0.15	15.5	7.70	5.12	3.98	3.58	3.20	—	—	—	—	—	—
-0.10	16.2	8.41	5.26	4.16	3.69	3.23	3.02	2.96	2.97	2.82	—	—
-0.05	16.2	9.41	5.45	4.47	3.80	3.34	3.20	2.98	2.94	2.83	2.80	2.79
0.00	18.0	10.6	5.91	4.77	4.01	3.48	3.24	3.05	2.99	2.93	2.82	2.67
0.05	19.0	11.6	7.22	5.11	4.21	3.56	3.20	3.20	3.13	3.00	2.92	2.88
0.10	19.2	13.7	8.28	5.80	4.77	3.66	3.43	3.31	3.16	2.99	2.95	2.93
0.15	20.5	14.9	9.44	6.20	5.27	3.96	3.66	3.44	3.37	3.30	3.16	3.13
0.20	20.6	15.5	11.0	7.08	5.37	4.25	3.93	3.60	3.47	3.55	3.29	3.28
0.25	20.5	16.6	12.1	8.80	6.17	4.43	4.04	3.85	3.70	3.65	3.50	3.49
0.30	21.2	18.4	12.7	10.1	7.51	5.25	4.57	4.29	3.93	3.95	3.76	3.59
0.35	21.5	19.8	15.9	12.0	8.80	5.71	4.63	4.50	4.26	3.89	3.90	3.93
0.40	21.8	20.9	19.7	14.1	11.3	6.32	5.44	4.96	4.58	4.36	4.25	4.22
0.45	22.3	23.0	23.0	17.6	13.6	7.65	6.20	5.49	4.92	4.83	4.61	4.41
0.50	23.3	23.7	24.9	19.6	16.4	9.56	7.13	6.00	5.49	5.17	4.90	4.75
0.55	23.3	25.0	25.7	22.6	19.2	12.0	8.75	6.60	6.25	5.82	5.56	5.07
0.60	22.9	25.9	26.3	26.1	22.0	14.5	9.83	8.50	7.01	6.43	5.63	5.62
0.65	22.1	26.8	26.3	28.6	27.6	20.6	14.4	9.76	8.43	7.29	6.43	5.88
0.70	22.0	26.3	27.8	30.8	30.4	26.5	17.8	12.8	10.7	8.69	7.72	7.09
0.75	—	24.3	27.3	29.9	29.7	29.8	25.0	20.6	14.1	10.6	8.97	7.96
0.80	—	—	26.8	29.8	31.3	33.5	32.0	28.5	26.8	15.7	10.7	9.92
0.85	—	—	—	29.3	30.9	36.3	38.4	41.2	35.1	26.5	15.8	13.2
0.90	—	—	—	—	30.8	33.8	39.1	42.9	42.9	41.1	35.8	21.4
0.95	—	—	—	—	—	—	37.6	39.5	41.4	46.3	54.3	61.5