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4-particle Amplituhedronics for 3-5 loops

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Abstract: Following the direction of 1712.09990 and 1712.09994, this article continues to excavate more interesting aspects of the 4-particle amplituhedron for a better understanding of the 4-particle integrand of planar $\mathcal{N}=4$ SYM to all loop orders, from the perspective of positive geometry. At 3-loop order, we introduce a much more refined dissection of the amplituhedron to understand its essential structure and maximally simplify its direct calculation, by fully utilizing its symmetry as well as the efficient Mondrian way for reorganizing all contributing pieces. Although significantly improved, this approach immediately encounters its technical bottleneck at 4-loop. Still, we manage to alleviate this difficulty by imitating the traditional (generalized) unitarity cuts, which is to use the so-called positive cuts. Given a basis of dual conformally invariant (DCI) loop integrals, we can figure out the coefficient of each DCI topology using its $d\log$ form via positivity conditions. Explicit examples include all 2+5 non-rung-rule topologies at 4- and 5-loop respectively. These results remarkably agree with previous knowledge, which confirms the validity of amplituhedron up to 5-loop and develops a new approach of determining the coefficient of each distinct DCI loop integral.

Keywords: Amplitudes, Loop integrands.

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Contents

1. The 3-loop Amplituhedron Revisited

The amplituhedron proposal for 4-particle integrand of planar $\mathcal{N}=4$ SYM [1, 2] has been fully understood up to 3-loop order [3] from which we have incidentally found an intriguing pattern valid at all loop orders for a special subset of dual conformally invariant (DCI) loop integrals: the Mondrian diagrammatics [4]. Even though there remain many unknown characteristics of the connection between this neat formalism and down-to-earth physics, to say the very least, it provides us a much more efficient way for reorganizing the 3-loop results via a direct calculation, by extensively using the properties of ordered subspaces which further refine the space spanned by loop variables x, y, z, w.

Now we would like to improve all these techniques to extract the essential structure of the 4-particle amplituhedron by fully utilizing the symmetry of (mutual) positivity conditions. Before this, let's briefly review the standard calculation, by taking the 2-loop case as a simplest nontrivial example below. For its only positivity condition

$$D_{12} = (x_2 - x_1)(z_1 - z_2) + (y_2 - y_1)(w_1 - w_2) > 0, \tag{1.1}$$

without loss of generality, we can fix the ordered subspace $X(12)$ in which $x_1 < x_2$, so it becomes

$$z_1 - z_2 + \frac{(y_2 - y_1)(w_1 - w_2)}{x_{21}} > 0, \tag{1.2}$$

where $x_{21} = x_2 - x_1$ is a positive variable. Then depending on the choice of ordered subspaces of y, w, there are 4 combinations to be considered, while the z-space is used for imposing $D_{12} > 0$. After that, we sum the result over all permutations of loop numbers, which are just $1, 2$ in the 2-loop case [2]. This has been used for the 3-loop case as well [3], while for the latter we have to deal with three intertwining conditions $D_{12}, D_{23}, D_{13} > 0$. Though such a straightforward approach successfully works for the first two nontrivial cases, it inevitably gets complicated by the tension between the simplicity of each contributing piece of a corresponding ordered subspace, and the number and variety of such building blocks. That is to say, the more refined each piece is, naturally, the simpler it looks, but there are more situations to be considered and hence their sum will be more involved, as one has to carefully ensure that all spurious poles brought by the subspace division must be wiped off after the summation. This disadvantage is due to overlooking the symmetry of positivity conditions. In the following, instead of picking subspace $X(123)$ at 3-loop, we will treat all x, y, z, w variables on the same footing.

To classify all possible positive configurations in a totally symmetric way, let's first define

$$D_{12} = X_{12} + Y_{12}, \quad D_{23} = X_{23} + Y_{23}, \quad D_{13} = X_{13} + Y_{13}, \tag{1.3}$$

with $X_{ij} = (x_j - x_i)(z_i - z_j)$ and $Y_{ij} = (y_j - y_i)(w_i - w_j)$, as what [4] has used. For each D_{ij}, there are three possible configurations: X_{ij} is positive while Y_{ij} is negative and the other way around, as well as both X_{ij} and Y_{ij} are positive. It goes without saying, the configuration of which both X_{ij} and Y_{ij} are negative must be excluded. We can use a convenient notation to precisely characterize each configuration, such as

$$\{(+-)_{12}, (+-)_{23}, (+-)_{13}\}, \tag{1.4}$$

which means X_{12}, X_{23}, X_{13} are positive and Y_{12}, Y_{23}, Y_{13} are negative. Since the positivity conditions are symmetric in combinations $12, 23, 13$, the counting of all possible configurations is given by a "generating function" which does not distinguish $12, 23, 13$, namely

$$(D+X+Y)^3 = D^3 + 3\,D^2(X+Y) + 3\,D\left(X^2+Y^2\right) + 6\,DXY + \left(X^3+Y^3\right) + 3\left(X^2Y+XY^2\right), \quad (1.5)$$

where D, X, Y stand for both X and Y are positive, only X is positive and only Y is positive respectively. Essentially there are only 6 distinct configurations, as we also treat X and Y on the same footing, which leads to switching $x, z \leftrightarrow y, w$. We see the coefficient 1, 3 or 6 above precisely represents the number of combinations within each distinct configuration. For example, for the 2nd term in the RHS above $3D^2X$ tells that X can be chosen to be X_{12}, X_{23} or X_{13}, and also for the 4th term there are $3!=6$ combinations of $12, 23, 13$ for D, X, Y. Moreover, we can count the number of ordered subspaces for each configuration and sum them as

$$36 + 24\times 6 + 24\times 6 + 16\times 6 + 36\times 2 + 16\times 6 = 588, \quad (1.6)$$

where each number in the sum will be explained in a detailed analysis of its corresponding configuration. On the other hand, the total number of ordered subspaces of x, y, z, w is $(3!)^4 = 1296$, so we see that the contributing pieces take up $49/108$ of all subspaces. By this more refined dissection, we immediately get rid of more than half of all subspaces which do not contribute, since they violate positivity conditions. In contrast, the standard way used in [3] has implicitly taken all non-contributing subspaces into account so it naturally looks more involved and contains more repetitive calculation. Using notations of (1.4), we select one representative for each of the 6 distinct configurations above for further calculation, as summarized in the following list:

$$\begin{aligned}
&\{(++)_{12},(++)_{23},(++)_{13}\}, \quad \{(++)_{12},(++)_{23},(+-)_{13}\}, \quad \{(++)_{12},(+-)_{23},(+-)_{13}\},\\
&\{(++)_{12},(+-)_{23},(-+)_{13}\}, \quad \{(+-)_{12},(+-)_{23},(+-)_{13}\}, \quad \{(+-)_{12},(+-)_{23},(-+)_{13}\}.
\end{aligned} \quad (1.7)$$

Note that after we obtain the $d\log$ forms of these 6 configurations, the multiplicity in (1.5) must be taken into account for correctly summing all relevant terms. Now we start to analyze them one by one.

1.1 Configuration $\{(++)_{12},(++)_{23},(++)_{13}\}$

For the simplest configuration $\{(++)_{12},(++)_{23},(++)_{13}\}$, since it is totally positive for all X_{ij}'s and Y_{ij}'s, there is no multiplicity, as its coefficient in (1.5) is simply 1. This corresponds to the collection of ordered subspaces (here $\otimes$ is used for separating X, Z and Y, W only, it is equivalent to the ordinary product)

$$X(\sigma_1\sigma_2\sigma_3)\,Z(\sigma_3\sigma_2\sigma_1) \otimes Y(\tau_1\tau_2\tau_3)\,W(\tau_3\tau_2\tau_1), \quad (1.8)$$

which means the orderings of x_1, x_2, x_3 are always opposite to those of z_1, z_2, z_3 and the same for y_1, y_2, y_3 and w_1, w_2, w_3. For x- and z-space there are $3!=6$ combinations, so there are in total 36 ordered subspaces in this collection, and it matches the counting in (1.6). Since for each D_{ij}, both X_{ij} and Y_{ij} are positive, the positivity of D_{ij} is trivial, which leads to the *proper numerator*

$$N = D_{12}D_{23}D_{13} \tag{1.9}$$

in the $d\log$ form (of any subspace in this collection)

$$\frac{1}{x_{\sigma_1}x_{\sigma_2\sigma_1}x_{\sigma_3\sigma_2}}\,\frac{1}{z_{\sigma_3}z_{\sigma_2\sigma_3}z_{\sigma_1\sigma_2}}\,\frac{1}{y_{\tau_1}y_{\tau_2\tau_1}y_{\tau_3\tau_2}}\,\frac{1}{w_{\tau_3}w_{\tau_2\tau_3}w_{\tau_1\tau_2}}\,\frac{N}{D_{12}D_{23}D_{13}}. \tag{1.10}$$

To make use of the Mondrian diagrammatics, we pick an explicit subspace $X(123)Z(321)\otimes Y(123)W(321)$ as a representative to separate its contributing and spurious parts. As extensively discussed in [3, 4], the identity

$$D_{12}D_{23}D_{13} = X_{12}X_{23}D_{13} + Y_{12}Y_{23}D_{13} + X_{13}X_{23}Y_{12} + X_{12}X_{13}Y_{23} + X_{12}Y_{13}Y_{23} + Y_{12}Y_{13}X_{23} \tag{1.11}$$

results in a vanishing spurious part, denoted by $S=0$. The relevant Mondrian seed diagrams are given in figure 1, corresponding to the six terms in the RHS above. This separation has significantly simplified the summation as we only need to check whether the final sum of all spurious parts vanishes.

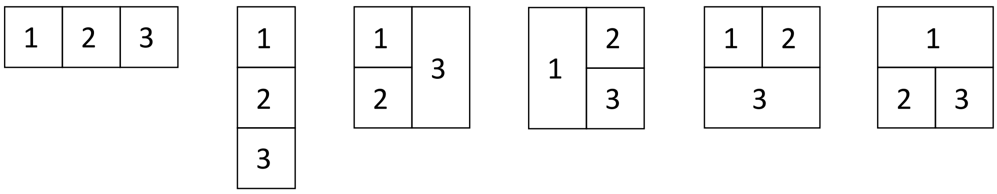

Figure 1: Mondrian seed diagrams in subspace $X(123)Z(321)\otimes Y(123)W(321)$.

1.2 Configuration $\{(++)_{12}, (++)_{23}, (+-)_{13}\}$

If we flip one plus into a minus in the former case, we obtain the configuration $\{(++)_{12}, (++)_{23}, (+-)_{13}\}$. Here Y_{13} is chosen to be negative but of course, the negative quantity can be Y_{12}, Y_{23}, X_{12}, X_{23} or X_{13} as well, which explains the multiplicity of $3D^2(X+Y)$ in (1.5). This corresponds to the collection of ordered subspaces

$$X(\sigma_1\sigma_2\sigma_3)\,Z(\sigma_3\sigma_2\sigma_1)\otimes Y(\cdot\cdot 2)W(2\cdot\cdot), \tag{1.12}$$

where

$$\begin{aligned} Y(\cdot\cdot 2)W(2\cdot\cdot) &\equiv Y(132)W(213) + Y(231)W(312) + (Y \leftrightarrow W) \\ &= Y(132)W(213) + Y(231)W(312) + Y(213)W(132) + Y(312)W(231) \end{aligned} \tag{1.13}$$

is the part satisfying $Y_{12}, Y_{23} > 0$ and $Y_{13} < 0$. It is clear that there are in total $6\times 4 = 24$ ordered subspaces in this collection. With the extra multiplicity 3×2, this matches the counting 24×6 in (1.6). To calculate the proper numerator, we observe that since only Y_{13} is negative, the 2-loop analysis for loop numbers 1,3 already suffices. Therefore we have

$$N = D_{12}D_{23}X_{13}. \tag{1.14}$$

Then as usual, we pick some explicit representative subspaces to separate their contributing and spurious parts, which include $X(123)Z(321)$, $X(132)Z(231)$ and $X(213)Z(312)$ among $X(\sigma_1\sigma_2\sigma_3)Z(\sigma_3\sigma_2\sigma_1)$ as we can get the rest three by reversing the orderings of loop numbers in all parentheses or switching $X \leftrightarrow Z$, and similarly $Y(132)W(213)$ among $Y(\cdot\cdot 2)W(2\cdot\cdot)$. The relevant Mondrian seed diagrams of these three subspaces are given in figure 2.

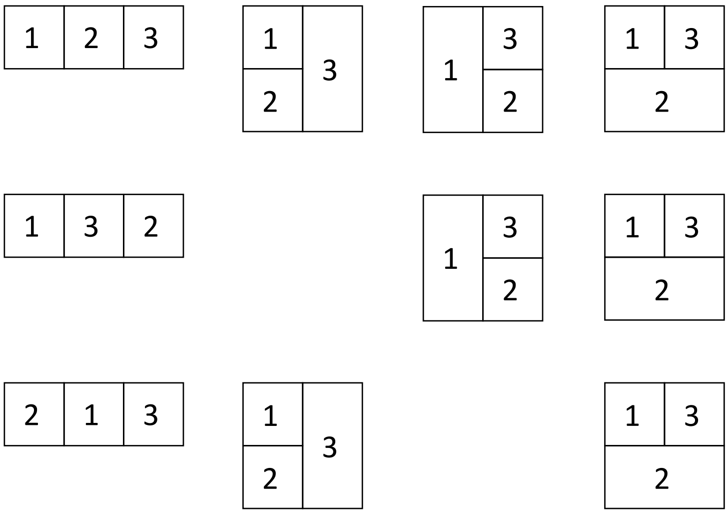

Figure 2: Mondrian seed diagrams in subspaces $X(123)Z(321) \otimes Y(132)W(213)$, $X(132)Z(231) \otimes Y(132)W(213)$ and $X(213)Z(312) \otimes Y(132)W(213)$. Each row corresponds to one subspace respectively.

Among these three cases, the only one with a nonzero spurious part is $X(123)Z(321)\otimes Y(132)W(213)$ with (recall that it is the difference between the proper numerator and Mondrian factors)

$$S = D_{12}D_{23}X_{13} - X_{12}X_{23}D_{13} - X_{13}X_{23}Y_{12} - X_{13}X_{12}Y_{23} - X_{13}Y_{12}Y_{23} = -X_{12}X_{23}Y_{13}. \tag{1.15}$$

To collect all spurious parts of this configuration, we need to permutate $13, 23, 12$ and switch $x, z \leftrightarrow y, w$. For compactness, we can consider those associated with $X(123)$ only [3], so the relevant terms are

$$X(123)Z(321) \otimes Y(\cdot\cdot 2)W(2\cdot\cdot): \quad -X_{12}X_{23}Y_{13}, \tag{1.16}$$

as well as

$$\begin{aligned} &[Y(132)W(231) + Y(231)W(132)] \otimes X(123)Z(312): \quad -Y_{13}Y_{23}X_{12}, \\ &[Y(213)W(312) + Y(312)W(213)] \otimes X(123)Z(231): \quad -Y_{12}Y_{13}X_{23}. \end{aligned} \tag{1.17}$$

These results will be summed over the corresponding forms of ordered subspaces for proving all spurious parts finally cancel.

1.3 Configuration $\{(++)_{12},(+-)_{23},(+-)_{13}\}$

If we flip one more plus into minus at the same side in the former case, we get $\{(++)_{12},(+-)_{23},(+-)_{13}\}$. Its multiplicity is similar to that of $\{(++)_{12},(++)_{23},(+-)_{13}\}$ as can be seen from (1.5). This corresponds to the collection of ordered subspaces

$$X(\sigma_1\sigma_2\sigma_3)\,Z(\sigma_3\sigma_2\sigma_1)\otimes Y(\cdot\cdot 3)W(\cdot\cdot 3), \tag{1.18}$$

where

$$\begin{aligned} Y(\cdot\cdot 3)W(\cdot\cdot 3) &\equiv Y(123)W(213)+Y(321)W(312)+(Y\leftrightarrow W)\\ &= Y(123)W(213)+Y(321)W(312)+Y(213)W(123)+Y(312)W(321) \end{aligned} \tag{1.19}$$

is the part satisfying $Y_{12}>0$ and $Y_{23},Y_{13}<0$. Similarly, there are in total $6\times 4=24$ ordered subspaces in this collection. This matches the counting 24×6 in (1.6) with the extra multiplicity 3×2. In this case, to calculate the proper numerator is nontrivial and we can again pick some explicit representative subspaces to analyze, which similarly include $X(123)Z(321)$, $X(132)Z(231)$, $X(213)Z(312)$ and also $Y(123)W(213)$. Note that $X(213)Z(312)\otimes Y(123)W(213)$ is identical to $X(123)Z(321)\otimes Y(123)W(213)$ if we switch $1\leftrightarrow 2$ and $Y\leftrightarrow W$, so there are only two distinct cases under consideration.

For $X(123)Z(321)\otimes Y(123)W(213)$, D_{12} is trivially positive, so we need to impose

$$D_{23}=x_{32}z_{23}-y_{32}(w_{31}+w_{12})>0,\quad D_{13}=(x_{32}+x_{21})(z_{12}+z_{23})-(y_{32}+y_{21})w_{31}>0. \tag{1.20}$$

For D_{23} let's define

$$z'_{23}\equiv z_{23}-\frac{y_{32}(w_{31}+w_{12})}{x_{32}}>0, \tag{1.21}$$

and its $d\log$ form is simply (for later convenience we multiply it by z_{23} to make a dimensionless ratio)

$$\frac{z_{23}}{z'_{23}}=\frac{X_{23}}{D_{23}}. \tag{1.22}$$

Next, for D_{13} we have

$$\begin{aligned} z_{12}+z_{23}-\frac{(y_{32}+y_{21})w_{31}}{x_{32}+x_{21}} &= z_{12}+z'_{23}+\frac{y_{32}(w_{31}+w_{12})}{x_{32}}-\frac{(y_{32}+y_{21})w_{31}}{x_{32}+x_{21}}\\ &= z_{12}+z'_{23}+\frac{y_{32}}{x_{32}}\left(w_{12}+w_{31}\frac{x_{21}}{x_{32}+x_{21}}\right)-\frac{y_{21}w_{31}}{x_{32}+x_{21}}>0, \end{aligned} \tag{1.23}$$

we can focus on z_{12}, z'_{23} and y_{32}, so its $d\log$ form is simply (omitting z_{12}, z'_{23} and y_{32} in the denominator to make a dimensionless ratio, the form of $x_1+\ldots+x_n>a$ can be referred in [3])

$$\begin{aligned} &\left[z_{12}+z'_{23}+\frac{y_{32}}{x_{32}}\left(w_{12}+w_{31}\frac{x_{21}}{x_{32}+x_{21}}\right)\right]\Big/\left[z_{12}+z'_{23}+\frac{y_{32}}{x_{32}}\left(w_{12}+w_{31}\frac{x_{21}}{x_{32}+x_{21}}\right)-\frac{y_{21}w_{31}}{x_{32}+x_{21}}\right]\\ &=\frac{D_{13}+y_{21}w_{31}}{D_{13}}. \end{aligned} \tag{1.24}$$

Collecting all three dimensionless ratios in the $d\log$ forms gives

$$\frac{D_{12}}{D_{12}}\frac{X_{23}}{D_{23}}\frac{D_{13}+y_{21}w_{31}}{D_{13}}, \tag{1.25}$$

the proper numerator is then $N=D_{12}X_{23}(D_{13}+y_{21}w_{31})$. The relevant Mondrian seed diagrams of this subspace are given in the 1st row of figure 3, and its spurious part is given by

$$S = D_{12}X_{23}(D_{13}+y_{21}w_{31}) - X_{12}X_{23}D_{13} - X_{13}X_{23}Y_{12} = X_{23}(Y_{12}Y_{13}+D_{12}\,y_{21}w_{31}). \tag{1.26}$$

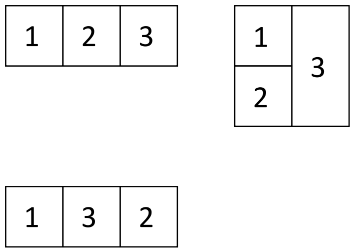

Figure 3: Mondrian seed diagrams in subspaces $X(123)Z(321)\otimes Y(123)W(213)$ and $X(132)Z(231)\otimes Y(123)W(213)$.

For $X(132)Z(231)\otimes Y(123)W(213)$, similarly we need to impose

$$D_{23} = x_{23}z_{32} - y_{32}(w_{31}+w_{12}) > 0, \quad D_{13} = x_{31}z_{13} - (y_{32}+y_{21})w_{31} > 0. \tag{1.27}$$

If we focus on x_{23} and x_{31}, we find these two conditions in fact "decouple". Then the dimensionless ratios are simply

$$\frac{D_{12}}{D_{12}}\frac{X_{23}}{D_{23}}\frac{X_{13}}{D_{13}}, \tag{1.28}$$

with the proper numerator $N=D_{12}X_{23}X_{13}$. The relevant Mondrian seed diagram is given in the 2nd row of figure 3, and its spurious part is obviously $S=0$.

To collect all spurious parts of this configuration, we again permutate $13, 23, 12$ and switch $x, z \leftrightarrow y, w$ for $X(123)Z(321)\otimes Y(123)W(213)$ and its derivative subspaces by reversing the orderings of loop numbers and/or switching $Y\leftrightarrow W$. Fixing $X(123)$, the relevant terms are

$$\begin{aligned} X(123)Z(321)\otimes Y(123)W(213):&\quad X_{23}(Y_{12}Y_{13}+D_{12}\,y_{21}w_{31}),\\ \ldots\otimes Y(321)W(312):&\quad X_{23}(Y_{12}Y_{13}+D_{12}\,y_{12}w_{13}),\\ \ldots\otimes Y(213)W(123):&\quad X_{23}(Y_{12}Y_{13}+D_{12}\,w_{21}y_{31}),\\ \ldots\otimes Y(312)W(321):&\quad X_{23}(Y_{12}Y_{13}+D_{12}\,w_{12}y_{13}), \end{aligned} \tag{1.29}$$

$$\begin{aligned} X(123)Z(321)\otimes Y(321)W(231):&\quad X_{12}(Y_{23}Y_{13}+D_{23}\,y_{23}w_{13}),\\ \ldots\otimes Y(123)W(132):&\quad X_{12}(Y_{23}Y_{13}+D_{23}\,y_{32}w_{31}),\\ \ldots\otimes Y(231)W(321):&\quad X_{12}(Y_{23}Y_{13}+D_{23}\,w_{23}y_{13}),\\ \ldots\otimes Y(132)W(123):&\quad X_{12}(Y_{23}Y_{13}+D_{23}\,w_{32}y_{31}), \end{aligned} \tag{1.30}$$

where $\ldots$ stands for the repetitive subspace (and similar below), as well as

$$\begin{aligned}
&[Y(123)W(321)+Y(321)W(123)]\otimes X(123)Z(213): && Y_{23}(X_{12}X_{13}+D_{12}\,x_{21}z_{31}),\\
&[Y(213)W(312)+Y(312)W(213)]\otimes \ldots && : \quad Y_{13}(X_{12}X_{23}+D_{12}\,z_{12}x_{32}),
\end{aligned} \tag{1.31}$$

$$\begin{aligned}
&[Y(321)W(123)+Y(123)W(321)]\otimes X(123)Z(132): && Y_{12}(X_{23}X_{13}+D_{23}\,x_{32}z_{31}),\\
&[Y(231)W(132)+Y(132)W(231)]\otimes \ldots && : \quad Y_{13}(X_{23}X_{12}+D_{23}\,z_{23}x_{21}).
\end{aligned} \tag{1.32}$$

These results will be used for proving all spurious parts finally cancel.

1.4 Configuration $\{(++)_{12},(+-)_{23},(-+)_{13}\}$

If we replace $(+-)_{13}$ by $(-+)_{13}$ in the former case, we get $\{(++)_{12},(+-)_{23},(-+)_{13}\}$. Now its multiplicity becomes 6 as can be seen from (1.5). This corresponds to the collection of ordered subspaces

$$X(\cdot\cdot 2)Z(2\cdot\cdot)\otimes Y(\cdot\cdot 1)W(1\cdot\cdot), \tag{1.33}$$

where $X(\cdot\cdot 2)Z(2\cdot\cdot)$ and $Y(\cdot\cdot 1)W(1\cdot\cdot)$ are similarly defined by (1.13). There are in total $4^2=16$ ordered subspaces in this collection, which matches the counting 16×6 in (1.6). To get the proper numerator, we again pick a representative subspace $X(132)Z(213)\otimes Y(231)W(123)$ to analyze.

Since D_{12} is trivially positive, we need to impose

$$D_{23}=x_{23}(z_{31}+z_{12})-y_{32}w_{32}>0, \quad D_{13}=-x_{31}z_{31}+y_{13}(w_{32}+w_{21})>0. \tag{1.34}$$

Focusing on x_{23} and x_{31}, we find these two conditions decouple. Then the dimensionless ratios are

$$\frac{D_{12}}{D_{12}}\frac{X_{23}}{D_{23}}\frac{Y_{13}}{D_{13}}, \tag{1.35}$$

with the proper numerator $N=D_{12}X_{23}Y_{13}$. The relevant Mondrian seed diagrams are given in figure 4, and its spurious part is obviously $S=0$. Therefore, similar to configuration $\{(++)_{12},(++)_{23},(++)_{13}\}$, in this case there is no spurious part to be collected.

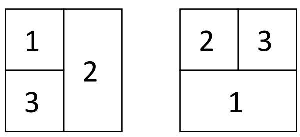

Figure 4: Mondrian seed diagrams in subspace $X(132)Z(213)\otimes Y(231)W(123)$.

1.5 Configuration $\{(+-)_{12},(+-)_{23},(+-)_{13}\}$

For this configuration, we have three minus signs at the same side. Its multiplicity is 2, due to switching $X\leftrightarrow Y$ in (1.5). This corresponds to the collection of ordered subspaces

$$X(\sigma_1\sigma_2\sigma_3)\,Z(\sigma_3\sigma_2\sigma_1)\otimes Y(\tau_1\tau_2\tau_3)\,W(\tau_1\tau_2\tau_3). \tag{1.36}$$

Similar to (1.8) there are in total 36 ordered subspaces in this collection, which matches the counting $36{\times}2$ in (1.6). We again pick some representative subspaces to analyze, in fact there are only two distinct types: $X(123)Z(321){\otimes}Y(123)W(123)$ and $X(123)Z(321){\otimes}Y(132)W(132)$.

For $X(123)Z(321){\otimes}Y(123)W(123)$, we need to impose

$$\begin{aligned}&D_{12}=x_{21}z_{12}-y_{21}w_{21}>0,\quad D_{23}=x_{32}z_{23}-y_{32}w_{32}>0,\\&D_{13}=(x_{32}+x_{21})(z_{12}+z_{23})-(y_{32}+y_{21})(w_{32}+w_{21})>0.\end{aligned} \tag{1.37}$$

For D_{12} and D_{23} let's define

$$z'_{12}\equiv z_{12}-\frac{y_{21}w_{21}}{x_{21}}>0,\quad z'_{23}\equiv z_{23}-\frac{y_{32}w_{32}}{x_{32}}>0, \tag{1.38}$$

next, for D_{13} we have

$$\begin{aligned}&z'_{12}+z'_{23}-\left(\frac{(y_{32}+y_{21})(w_{32}+w_{21})}{x_{32}+x_{21}}-\frac{y_{21}w_{21}}{x_{21}}-\frac{y_{32}w_{32}}{x_{32}}\right)\\=\,&z'_{12}+z'_{23}-\frac{x_{21}}{x_{32}(x_{32}+x_{21})}\left(y_{32}-y_{21}\,\frac{x_{32}}{x_{21}}\right)\left(\frac{x_{32}}{x_{21}}\,w_{21}-w_{32}\right)>0,\end{aligned} \tag{1.39}$$

this condition is only nontrivial when

$$a\equiv\frac{x_{21}}{x_{32}(x_{32}+x_{21})}\left(y_{32}-y_{21}\,\frac{x_{32}}{x_{21}}\right)\left(\frac{x_{32}}{x_{21}}\,w_{21}-w_{32}\right)>0, \tag{1.40}$$

so its $d\log$ form is (omitting z'_{12} and z'_{23} in the denominator as usual)

$$\begin{aligned}&\left[\frac{1}{y_{32}-y_{21}x_{32}/x_{21}}\left(\frac{1}{w_{32}}-\frac{1}{w_{32}-w_{21}x_{32}/x_{21}}\right)+\left(\frac{1}{y_{32}}-\frac{1}{y_{32}-y_{21}x_{32}/x_{21}}\right)\frac{1}{w_{32}-w_{21}x_{32}/x_{21}}\right]\frac{z'_{12}+z'_{23}}{z'_{12}+z'_{23}-a}\\&+\left[\frac{1}{y_{32}-y_{21}x_{32}/x_{21}}\frac{1}{w_{32}-w_{21}x_{32}/x_{21}}+\left(\frac{1}{y_{32}}-\frac{1}{y_{32}-y_{21}x_{32}/x_{21}}\right)\left(\frac{1}{w_{32}}-\frac{1}{w_{32}-w_{21}x_{32}/x_{21}}\right)\right]\\=\,&\frac{D_{13}+y_{32}w_{21}+y_{21}w_{32}}{y_{32}w_{32}D_{13}}.\end{aligned} \tag{1.41}$$

Collecting all three dimensionless ratios gives

$$\frac{X_{12}}{D_{12}}\frac{X_{23}}{D_{23}}\frac{D_{13}+y_{32}w_{21}+y_{21}w_{32}}{D_{13}}, \tag{1.42}$$

with the proper numerator $N=X_{12}X_{23}(D_{13}{+}y_{32}w_{21}{+}y_{21}w_{32})$. The relevant Mondrian seed diagram is given in figure 5, and its spurious part is obviously $S=X_{12}X_{23}(y_{32}w_{21}{+}y_{21}w_{32})$.

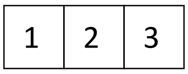

Figure 5: Mondrian seed diagram in subspaces $X(123)Z(321)\otimes Y(123)W(123)$ and $X(123)Z(321)\otimes Y(132)W(132)$.

For $X(123)Z(321)\otimes Y(132)W(132)$, similarly we need to impose

$$\begin{aligned} &D_{12} = x_{21}z_{12} - (y_{23}+y_{31})(w_{23}+w_{31}) > 0, \quad D_{23} = x_{32}z_{23} - y_{23}w_{23} > 0, \\ &D_{13} = (x_{32}+x_{21})(z_{12}+z_{23}) - y_{31}w_{31} > 0. \end{aligned} \tag{1.43}$$

Focusing on z_{12} and z_{23}, we find $D_{12}>0$ and $D_{23}>0$ decouple, and $D_{12}>0$ can trivialize $D_{13}>0$. Then the dimensionless ratios are

$$\frac{X_{12}}{D_{12}}\frac{X_{23}}{D_{23}}\frac{D_{13}}{D_{13}}, \tag{1.44}$$

with the proper numerator $N=X_{12}X_{23}D_{13}$. The relevant Mondrian seed diagram is identical to that of $X(123)Z(321)\otimes Y(123)W(123)$ given in figure 5, and its spurious part is obviously $S=0$.

To collect all spurious parts of this configuration, we again permutate $13, 23, 12$ and switch $x, z \leftrightarrow y, w$ for $X(123)Z(321)\otimes Y(123)W(123)$ and its derivative subspaces. Fixing $X(123)$, the relevant terms are

$$\begin{aligned} X(123)Z(321)\otimes Y(123)W(123): &\quad X_{12}X_{23}(y_{32}w_{21}+y_{21}w_{32}), \\ \ldots \otimes Y(321)W(321): &\quad X_{12}X_{23}(y_{23}w_{12}+y_{12}w_{23}), \end{aligned} \tag{1.45}$$

as well as

$$\begin{aligned} Y(123)W(321)\otimes X(123)Z(123): &\quad Y_{12}Y_{23}(x_{32}z_{21}+x_{21}z_{32}), \\ Y(321)W(123)\otimes \ldots \qquad\qquad\quad : &\quad Y_{12}Y_{23}(x_{32}z_{21}+x_{21}z_{32}). \end{aligned} \tag{1.46}$$

These results will be used for proving all spurious parts finally cancel.

1.6 Configuration $\{(+-)_{12}, (+-)_{23}, (-+)_{13}\}$

If we replace $(+-)_{13}$ by $(-+)_{13}$ in the former case, we get $\{(+-)_{12}, (+-)_{23}, (-+)_{13}\}$. Its multiplicity is 3×2, due to choosing one of $12, 23, 13$ to assign $(-+)$ and switching $X\leftrightarrow Y$ in (1.5). This corresponds to the collection of ordered subspaces

$$X(\cdot\cdot 2)Z(2\cdot\cdot)\otimes Y(\cdot\cdot 2)W(\cdot\cdot 2). \tag{1.47}$$

There are in total $4^2=16$ ordered subspaces in this collection, which matches the counting 16×6 in (1.6). To get the proper numerator, we again pick a representative subspace $X(132)Z(213)\otimes Y(132)W(312)$ to analyze, for which we need to impose

$$\begin{aligned} &D_{12} = (x_{23}+x_{31})z_{12} - (y_{23}+y_{31})w_{21} \equiv (x_{23}+x_{31})z'_{12} > 0, \\ &D_{23} = x_{23}(z_{31}+z_{12}) - y_{23}(w_{21}+w_{13}) > 0, \\ &D_{13} = -x_{31}z_{31} + y_{31}w_{13} \equiv y_{31}w'_{13} > 0, \end{aligned} \tag{1.48}$$

where similarly z'_{12} and w'_{13} are positive variables, so that for D_{23} we have

$$z_{31}\left(1-\frac{y_{23}}{x_{23}}\frac{x_{31}}{y_{31}}\right)+z'_{12}+\left(\frac{y_{23}+y_{31}}{x_{23}+x_{31}}-\frac{y_{23}}{x_{23}}\right)w_{21}-\frac{y_{23}}{x_{23}}\,w'_{13}>0, \tag{1.49}$$

note that

$$\frac{y_{23}}{x_{23}}\leqslant\frac{y_{31}}{x_{31}}\Longrightarrow\frac{y_{23}}{x_{23}}\leqslant\frac{y_{23}+y_{31}}{x_{23}+x_{31}}\leqslant\frac{y_{31}}{x_{31}}, \tag{1.50}$$

which determines signs of the factors of z_{31} and w_{21}, so its $d\log$ form is (omitting z_{31}, z'_{12} and w_{21} in the denominator)

$$\begin{aligned}&\frac{1}{y_{31}-y_{23}\,x_{31}/x_{23}}\left[z_{31}\left(1-\frac{y_{23}}{x_{23}}\frac{x_{31}}{y_{31}}\right)+z'_{12}+\left(\frac{y_{23}+y_{31}}{x_{23}+x_{31}}-\frac{y_{23}}{x_{23}}\right)w_{21}\right]\frac{x_{23}}{D_{23}}\\&+\left(\frac{1}{y_{31}}-\frac{1}{y_{31}-y_{23}\,x_{31}/x_{23}}\right)\frac{z'_{12}\,x_{23}}{D_{23}}\\=&\frac{1}{y_{31}D_{23}}\left(x_{23}(z_{31}+z_{12})-\frac{x_{23}}{x_{23}+x_{31}}\,y_{23}w_{21}\right).\end{aligned} \tag{1.51}$$

Collecting all three dimensionless ratios gives

$$\frac{X_{12}}{D_{12}}\frac{Y_{13}}{D_{13}}\frac{1}{D_{23}}\left(X_{23}-\frac{x_{23}}{x_{23}+x_{31}}\,y_{23}w_{21}\right), \tag{1.52}$$

with the proper numerator $N=X_{12}Y_{13}(X_{23}-y_{23}w_{21}x_{23}/(x_{23}+x_{31}))$. The relevant Mondrian seed diagram is given in figure 6, and its spurious part is obviously $S=X_{12}Y_{13}(-y_{23}w_{21}x_{23}/(x_{23}+x_{31}))$.

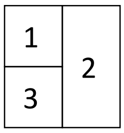

Figure 6: Mondrian seed diagram in subspace $X(132)Z(213)\otimes Y(132)W(312)$.

To collect all spurious parts of this configuration, we again permutate $13, 23, 12$ and switch $x,z\leftrightarrow y,w$ for $X(132)Z(213)\otimes Y(132)W(312)$ and its derivative subspaces. Fixing $X(123)$, the relevant terms are

$$\begin{aligned}X(123)Z(312)\otimes Y(123)W(213):&\quad X_{13}Y_{12}\left(-\frac{x_{32}}{x_{32}+x_{21}}\,y_{32}w_{31}\right),\\ \ldots\otimes Y(321)W(312):&\quad X_{13}Y_{12}\left(-\frac{x_{32}}{x_{32}+x_{21}}\,y_{23}w_{13}\right),\\ \ldots\otimes Y(213)W(123):&\quad X_{13}Y_{12}\left(-\frac{x_{32}}{x_{32}+x_{21}}\,w_{32}y_{31}\right),\\ \ldots\otimes Y(312)W(321):&\quad X_{13}Y_{12}\left(-\frac{x_{32}}{x_{32}+x_{21}}\,w_{23}y_{13}\right),\end{aligned} \tag{1.53}$$

$$
\begin{aligned}
X(123)Z(231)\otimes Y(231)W(321):&\quad X_{12}Y_{23}\left(-\frac{z_{13}}{z_{13}+z_{32}}\,y_{13}w_{12}\right),\\
\ldots\otimes Y(132)W(123):&\quad X_{12}Y_{23}\left(-\frac{z_{13}}{z_{13}+z_{32}}\,y_{31}w_{21}\right),\\
\ldots\otimes Y(321)W(231):&\quad X_{12}Y_{23}\left(-\frac{z_{13}}{z_{13}+z_{32}}\,w_{13}y_{12}\right),\\
\ldots\otimes Y(123)W(132):&\quad X_{12}Y_{23}\left(-\frac{z_{13}}{z_{13}+z_{32}}\,w_{31}y_{21}\right),
\end{aligned}
\tag{1.54}
$$

as well as

$$
\begin{aligned}
Y(123)W(312)\otimes X(123)Z(213):&\quad Y_{13}X_{12}\left(-\frac{y_{32}}{y_{32}+y_{21}}\,x_{32}z_{31}\right),\\
Y(321)W(213)\otimes\ldots\qquad\qquad\quad :&\quad Y_{13}X_{12}\left(-\frac{y_{23}}{y_{12}+y_{23}}\,x_{32}z_{31}\right),\\
Y(312)W(123)\otimes\ldots\qquad\qquad\quad :&\quad Y_{13}X_{12}\left(-\frac{w_{32}}{w_{32}+w_{21}}\,x_{32}z_{31}\right),\\
Y(213)W(321)\otimes\ldots\qquad\qquad\quad :&\quad Y_{13}X_{12}\left(-\frac{w_{23}}{w_{12}+w_{23}}\,x_{32}z_{31}\right),
\end{aligned}
\tag{1.55}
$$

$$
\begin{aligned}
Y(231)W(123)\otimes X(123)Z(132):&\quad Y_{12}X_{23}\left(-\frac{y_{13}}{y_{13}+y_{32}}\,z_{31}x_{21}\right),\\
Y(132)W(321)\otimes\ldots\qquad\qquad\quad :&\quad Y_{12}X_{23}\left(-\frac{y_{31}}{y_{23}+y_{31}}\,z_{31}x_{21}\right),\\
Y(123)W(231)\otimes\ldots\qquad\qquad\quad :&\quad Y_{12}X_{23}\left(-\frac{w_{13}}{w_{13}+w_{32}}\,z_{31}x_{21}\right),\\
Y(321)W(132)\otimes\ldots\qquad\qquad\quad :&\quad Y_{12}X_{23}\left(-\frac{w_{31}}{w_{23}+w_{31}}\,z_{31}x_{21}\right).
\end{aligned}
\tag{1.56}
$$

These results will be used for proving all spurious parts finally cancel.

1.7 Final sum of all spurious parts

One might notice that, even though we treat all x, y, z, w variables on the same footing and preserve the symmetry in combinations $12, 23, 13$, we can still consider terms associated with $X(123)$ only because we would like to confirm the sum of all spurious parts in subspace $X(123)$ matches the result in [3].

Explicitly, we collect those nonzero spurious parts in configurations $\{(++)_{12}, (++)_{23}, (+-)_{13}\}$, $\{(++)_{12}, (+-)_{23}, (+-)_{13}\}$, $\{(+-)_{12}, (+-)_{23}, (+-)_{13}\}$ and $\{(+-)_{12}, (+-)_{23}, (-+)_{13}\}$ then sum them over the corresponding forms of ordered subspaces, which gives the proper numerator

$$
S_{123} = x_{21}(-2\,z_1y_2y_3w_2w_3 - z_1y_1w_1(y_2w_3+y_3w_2) + z_2y_3w_3(y_1w_2+y_2w_1) + z_3y_2w_2(y_1w_3+y_3w_1)), \tag{1.57}
$$

and hence the final sum over permutations of loop numbers

$$
S_{123}X(123) + (5 \text{ permutations of 1,2,3}) = 0. \tag{1.58}
$$

In fact, this vanishing result can be further refined as $S_{123}X(123)+S_{132}X(132)=0$, which has not been noticed in [3].

1.8 Technical bottleneck at 4-loop

Completing the 3-loop proof, it is appealing to continue this approach at 4-loop. We can have a glance at the variety of its positive configurations via the generating function, as a generalization of (1.5):

$$\begin{aligned}(D+X+Y)^6=&\,D^6+6\,D^5(X+Y)+15\,D^4\left(X^2+Y^2\right)+30\,D^4XY+20\,D^3\left(X^3+Y^3\right)+60\,D^3\left(X^2Y+XY^2\right)\\&+15\,D^2\left(X^4+Y^4\right)+60\,D^2\left(X^3Y+XY^3\right)+90\,D^2X^2Y^2+6\,D\left(X^5+Y^5\right)+30\,D\left(X^4Y+XY^4\right)\\&+60\,D\left(X^3Y^2+X^2Y^3\right)+\left(X^6+Y^6\right)+6\left(X^5Y+XY^5\right)+15\left(X^4Y^2+X^2Y^4\right)+20\,X^3Y^3,\end{aligned}\tag{1.59}$$

so there are 16 distinct configurations. Taking X^6 as one of the most nontrivial examples, or equivalently, the configuration in terms of plus and minus signs

$$\{(+-)_{12},(+-)_{23},(+-)_{34},(+-)_{13},(+-)_{24},(+-)_{14}\},\tag{1.60}$$

we can pick the representative subspace $X(1234)Z(4321)\otimes Y(1234)W(1234)$ to analyze, for which we need to impose

$$\begin{aligned}&D_{12}=x_{21}z_{12}-y_{21}w_{21}>0,\quad D_{23}=x_{32}z_{23}-y_{32}w_{32}>0,\quad D_{34}=x_{43}z_{34}-y_{43}w_{43}>0,\\&D_{13}=(x_{32}+x_{21})(z_{12}+z_{23})-(y_{32}+y_{21})(w_{32}+w_{21})>0,\\&D_{24}=(x_{43}+x_{32})(z_{23}+z_{34})-(y_{43}+y_{32})(w_{43}+w_{32})>0,\\&D_{14}=(x_{43}+x_{32}+x_{21})(z_{12}+z_{23}+z_{34})-(y_{43}+y_{32}+y_{21})(w_{43}+w_{32}+w_{21})>0.\end{aligned}\tag{1.61}$$

For D_{12}, D_{23} and D_{34} let's define

$$z'_{12}\equiv z_{12}-\frac{y_{21}w_{21}}{x_{21}}>0,\quad z'_{23}\equiv z_{23}-\frac{y_{32}w_{32}}{x_{32}}>0,\quad z'_{34}\equiv z_{34}-\frac{y_{43}w_{43}}{x_{43}}>0,\tag{1.62}$$

then for D_{13}, D_{24} and D_{14} we have

$$\begin{aligned}(x_{32}+x_{21})(z'_{12}+z'_{23})-x_{32}\,x_{21}\left(\frac{y_{32}}{x_{32}}-\frac{y_{21}}{x_{21}}\right)\left(\frac{w_{21}}{x_{21}}-\frac{w_{32}}{x_{32}}\right)&>0,\\(x_{43}+x_{32})(z'_{23}+z'_{34})-x_{43}\,x_{32}\left(\frac{y_{43}}{x_{43}}-\frac{y_{32}}{x_{32}}\right)\left(\frac{w_{32}}{x_{32}}-\frac{w_{43}}{x_{43}}\right)&>0,\\(x_{43}+x_{32}+x_{21})(z'_{12}+z'_{23}+z'_{34})-x_{32}\,x_{21}\left(\frac{y_{32}}{x_{32}}-\frac{y_{21}}{x_{21}}\right)\left(\frac{w_{21}}{x_{21}}-\frac{w_{32}}{x_{32}}\right)&\\-\,x_{43}\,x_{32}\left(\frac{y_{43}}{x_{43}}-\frac{y_{32}}{x_{32}}\right)\left(\frac{w_{32}}{x_{32}}-\frac{w_{43}}{x_{43}}\right)-x_{43}\,x_{21}\left(\frac{y_{43}}{x_{43}}-\frac{y_{21}}{x_{21}}\right)\left(\frac{w_{21}}{x_{21}}-\frac{w_{43}}{x_{43}}\right)&>0.\end{aligned}\tag{1.63}$$

Note that this smallest sector of the 4-loop amplituhedron almost has the complexity of the entire 3-loop case already! As the loop order increases, the calculational complexity grows explosively. This advises us to stop at 4-loop even though we have a maximally refined recipe to dissect the amplituhedron iceberg.

2. Positive Cuts at 4-loop

For the 4-loop case besides continuing a direct derivation, we will also alleviate the calculational difficulty by imitating the traditional (generalized) unitarity cuts, which is to use the positive cuts. In this way, we can peel off the unnecessary flesh of the amplituhedron and concentrate on its essential skeleton – the pole structure. Given a basis of DCI loop integrals, we can first assign each DCI topology with an undetermined coefficient. Then after imposing as many positive cuts as possible for various pole structures, in general we obtain a set of equations by equating each resulting $d\log$ form via positivity conditions, and the deformed integrand as a sum of all non-vanishing DCI diagrams under the corresponding cuts. These equations will be complete for determining all coefficients.

However, as a simplified demonstration, below we will focus on the non-rung-rule topologies at 4-loop (of course, it is an interesting and challenging problem to prove the rung rule preserves coefficients of DCI topologies while increasing the number of loops, using the amplituhedron approach). First, we enumerate all eight distinct DCI topologies at 4-loop in figure 7, among which the cross and the only non-Mondrian topologies are of the non-rung-rule type, while the other six rung-rule (and also Mondrian) topologies are all associated with the coefficient +1. It is important to recall that, the term 'DCI topology' includes the numerator part as this matters for dual conformal invariance [4], but for convenience we will not draw the extra numerators explicitly as they can be inferred from the rung rule, as long as there is no ambiguity in the choices of DCI numerator. Then we assign the cross and non-Mondrian topologies with coefficients s_1 and s_2 respectively, and consider a particular diagram of the latter type given in figure 8.

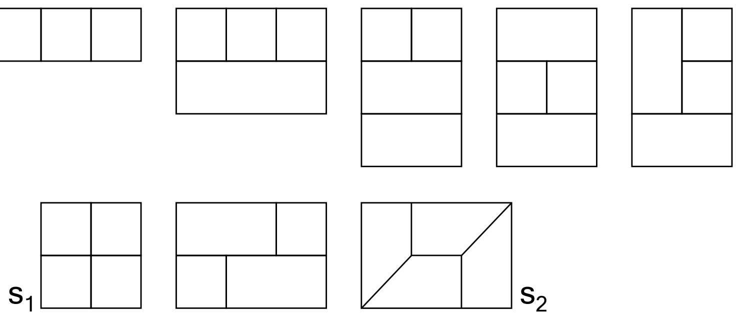

Figure 7: All eight distinct DCI topologies at 4-loop. s_1 and s_2 are coefficients of two non-rung-rule topologies.

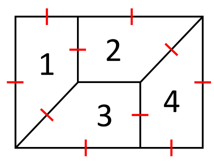

Figure 8: A particular diagram of the non-Mondrian topology at 4-loop with 6 external and 4 internal cuts.

For this diagram, we can first maximally impose all 6 available external cuts, as indicated by the red segments around its rim. Following the convention of external face variables in [3, 4], these 6 cuts result in $x_1 = y_1 = y_2 = z_4 = w_4 = w_3 = 0$, which can simplify the six D's as

$$
\begin{aligned}
D_{12} &= x_2(z_1 - z_2), \\
D_{34} &= z_3(x_4 - x_3), \\
D_{13} &= x_3(z_1 - z_3) + y_3 w_1, \\
D_{24} &= z_2(x_4 - x_2) + y_4 w_2, \\
D_{23} &= (x_3 - x_2)(z_2 - z_3) + y_3 w_2, \\
D_{14} &= x_4 z_1 + y_4 w_1.
\end{aligned}
\tag{2.1}
$$

Now for part of these D's as internal propagators, we can either cut them or impose their positivity. Note that there is no way to further cut D_{14} by fixing one variable, as discussed in [2], but since it is manifestly positive already, there is no positivity condition to be imposed. By tentatively setting

$$
z_1 = z_2, \quad x_4 = x_3, \quad z_3 = z_2 + \frac{y_3 w_1}{x_3} \equiv \hat{z}_3, \quad x_2 = x_3 + \frac{y_4 w_2}{z_2} \equiv \hat{x}_2, \tag{2.2}
$$

we can turn off $D_{12}, D_{34}, D_{13}, D_{24}$, and incidentally we have

$$
D_{23} = y_3 w_2 \left(1 + \frac{y_4 w_1}{x_3 z_2} \right), \tag{2.3}
$$

which is also manifestly positive, therefore we are done with this further simplification. Note the solutions of $D_{12} = D_{34} = D_{13} = D_{24} = 0$, namely (2.2), are also manifestly positive. In contrast, solutions that involve relative minus signs, such as $z_3 = z_2 - y_3 w_1 / x_3$, are clearly not, since we also have to impose $z_2 > y_3 w_1 / x_3$. Such a category of manifestly positive solutions will be named as the *positive cuts*.

The further 4 internal cuts are also drawn in figure 8, and besides this diagram, other diagrams of all topologies, orientations and configurations of loop numbers at 4-loop that survive these 10 cuts, are given in figure 9, as can be enumerated from the topologies in figure 7 then picked out by identifying all 10 poles $x_1, y_1, y_2, z_4, w_4, w_3, D_{12}, D_{34}, D_{13}, D_{24}$. Let's define the sum of these 9 surviving diagrams as a function of x, y, z, w (we only sum their proper numerators as usual)

$$
\begin{aligned}
& S\,(x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2, x_3, y_3, z_3, w_3, x_4, y_4, z_4, w_4) \\
&= x_2 x_3 x_4 z_1 z_2 z_3\, y_3 w_2\, D_{14}(s_2\, y_4 w_1 + D_{14}) + x_2 x_4 z_1 z_3\, y_3 w_2\, D_{14}(x_4 z_2\, y_3 w_1 + x_3 z_1\, y_4 w_2) \\
&\quad + x_2 x_4 z_1 z_3\, y_3 y_4 w_1 w_2\, (y_3 w_2 D_{14} + x_2 z_3 D_{14} + y_4 w_1 D_{23} + x_4 z_1 D_{23} + s_1 D_{14} D_{23}),
\end{aligned}
\tag{2.4}
$$

where s_1 and s_2 are coefficients to be determined. Since the cross diagram in figure 9 can survive these 10 cuts like the non-Mondrian one in figure 7, we can fix both s_1 and s_2 in only one equation. In contrast, if we impose all 8 external cuts available for the cross diagram, the non-Mondrian one cannot survive these cuts and hence s_2 will disappear in this equation, then one more equation that involves s_2 is needed. This explains why to determine s_1 and s_2 in one equation, we choose a set of external cuts in the non-Mondrian diagram which has less available external cuts than the cross diagram, as it is a general trick to minimize the number of equations needed for determining all coefficients.

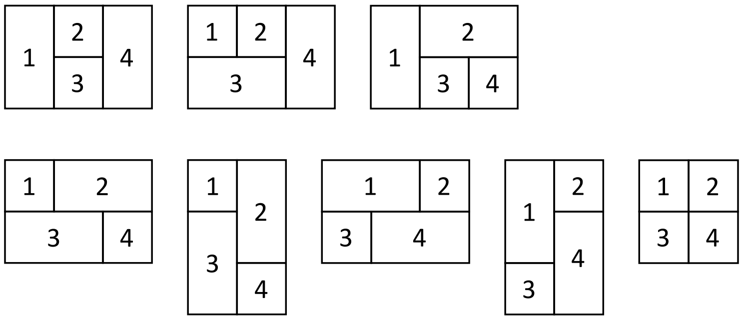

Figure 9: All other 8 diagrams that survive the 10 cuts $x_1 = y_1 = y_2 = z_4 = w_4 = w_3 = D_{12} = D_{34} = D_{13} = D_{24} = 0$.

On the other hand, from the positivity conditions of the amplituhedron we have the following dimensionless ratios with respect to $D_{12}, D_{34}, D_{13}, D_{24}$:

$$\begin{aligned} \frac{z_1}{z_1 - z_2} &= \frac{x_2 z_1}{D_{12}} \to \frac{\hat{x}_2 z_2}{D_{12}}, \\ \frac{x_4}{x_4 - x_3} &= \frac{z_3 x_4}{D_{34}} \to \frac{\hat{z}_3 x_3}{D_{34}}, \\ z_3 \left(\frac{1}{z_3} - \frac{1}{z_3 - \hat{z}_3} \right) &= \frac{x_3 \hat{z}_3}{D_{13}}, \\ x_2 \left(\frac{1}{x_2} - \frac{1}{x_2 - \hat{x}_2} \right) &= \frac{z_2 \hat{x}_2}{D_{24}}, \end{aligned} \tag{2.5}$$

where $\hat{x}_2$ and $\hat{z}_3$ are defined in (2.2), and $\to$ denotes some variables are replaced by the solutions of cuts. Since D_{14} and D_{23} are trivially positive, we get the proper numerator

$$(\hat{x}_2 x_3 z_2 \hat{z}_3)^2 D_{14} D_{23} = \left[\left(x_3 + \frac{y_4 w_2}{z_2} \right) \left(z_2 + \frac{y_3 w_1}{x_3} \right) x_3 z_2 \right]^2 y_3 w_2 \left(1 + \frac{y_4 w_1}{x_3 z_2} \right) (x_3 z_2 + y_4 w_1). \tag{2.6}$$

Now the critical step is to equate the deformed S defined in (2.4) on the 10 cuts and the quantity above, or consider their difference

$$\begin{aligned} & S\left(0\,,\, 0\,, z_2, w_1, x_3 + \frac{y_4 w_2}{z_2},\, 0\,, z_2, w_2, x_3, y_3, z_2 + \frac{y_3 w_1}{x_3},\, 0\,, x_3, y_4,\, 0\,,\, 0 \right) \\ & - \left[\left(x_3 + \frac{y_4 w_2}{z_2} \right) \left(z_2 + \frac{y_3 w_1}{x_3} \right) x_3 z_2 \right]^2 y_3 w_2 \left(1 + \frac{y_4 w_1}{x_3 z_2} \right) (x_3 z_2 + y_4 w_1) \\ = {} & y_3 y_4 w_1 w_2 \left(1 + \frac{y_4 w_1}{x_3 z_2} \right) (x_3 z_2 + y_3 w_1)(x_3 z_2 + y_4 w_2) \left[(1 + s_1) y_3 w_2 (x_3 z_2 + y_4 w_1) + (1 + s_2) x_3^2 z_2^2 \right], \end{aligned} \tag{2.7}$$

then it is clear that to make this difference vanish, we must take $s_1 = s_2 = -1$, which agrees with [5]. For this 4-loop case, we see the analysis and calculation are very simple, due to there is in fact no positivity condition to be imposed – all D's are either cut or manifestly positive. But in general this simplicity does not always occur, as immediately at 5-loop we will encounter some quite nontrivial and hence much more complicated examples. Still, with the aid of positive cuts, our calculational capability is greatly enhanced so that unlike the hopeless case study of (1.63), we manage to tackle all 5-loop examples.

3. Positive Cuts at 5-loop

For the 5-loop application of positive cuts, there is nothing new in its principle but we will see much more complexity in various techniques, as well as its miraculous agreement with previous knowledge. As usual, we first enumerate all 34 distinct DCI topologies at 5-loop: figure 10 lists all 24 Mondrian DCI topologies labelled by $T_1, \ldots, T_{24}$, as indicated by the red subscripts, and figure 11 all 10 non-Mondrian ones labelled by $T_{25}, \ldots, T_{34}$ similarly.

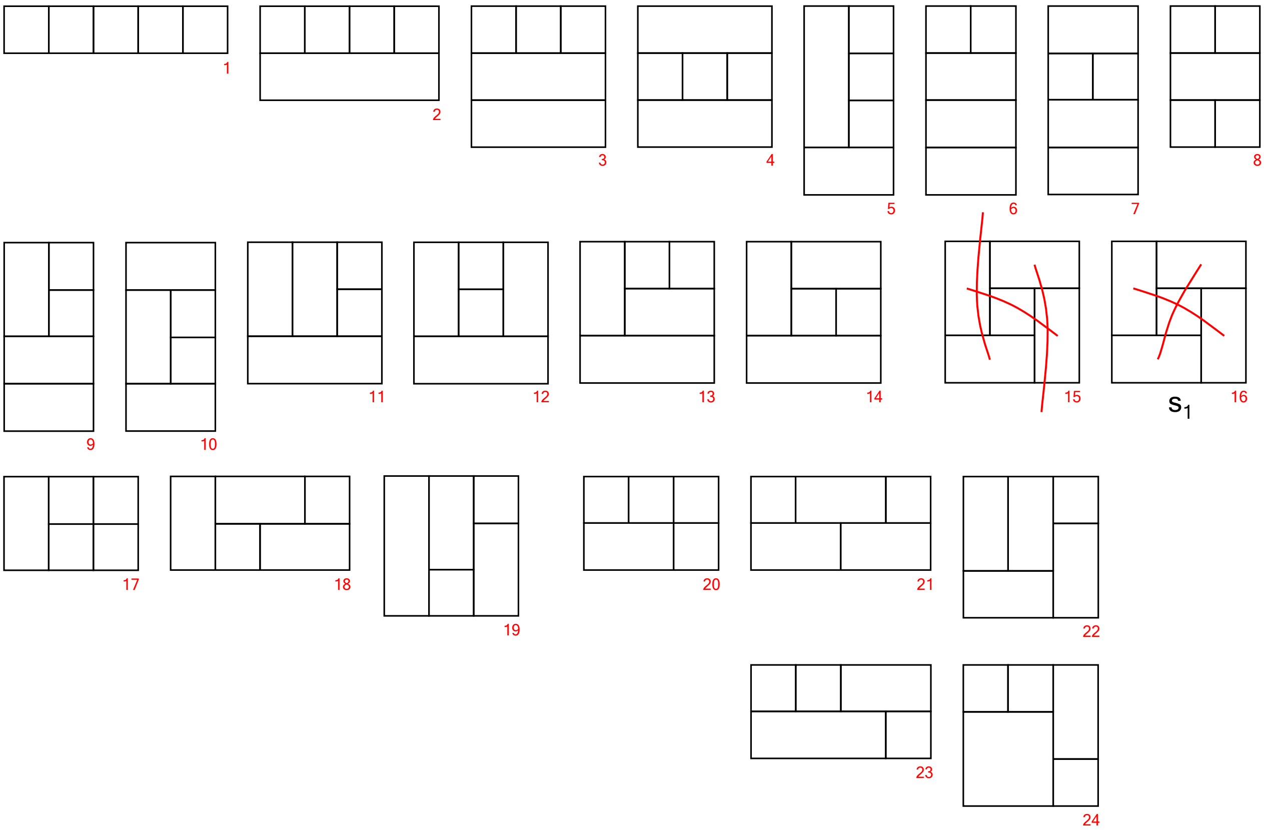

Figure 10: Mondrian DCI topologies $T_1, \ldots, T_{24}$ at 5-loop. T_{16} assigned with s_1 is a non-rung-rule topology (it is generated by the substitution rule).

Note that there exist two distinct choices of DCI numerator for the pinwheel's pole structure, namely T_{15} and T_{16} given in figure 10, we have to explicitly draw their numerators while we suppress those of the rest Mondrian topologies as they can be uniquely inferred from the rung rule. And for non-Mondrian ones in figure 11, we draw all numerators explicitly since the rung rule cannot account for all of them. Among all these 34 topologies, T_{16}, T_{30} are generated by applying the substitution rule to the 4-loop counterparts in figure 7, which also preserves coefficients [6], while the rules for T_{32}, T_{33}, T_{34} are unknown, and the rest are generated by the rung rule. As a simplified demonstration, we focus on non-rung-rule topologies only, so $T_{16}, T_{30}, T_{32}, T_{33}, T_{34}$ assigned with coefficients s_1, s_2, s_3, s_4, s_5 respectively are of our concern. Let's now determine these coefficients one by one using the amplituhedron approach.

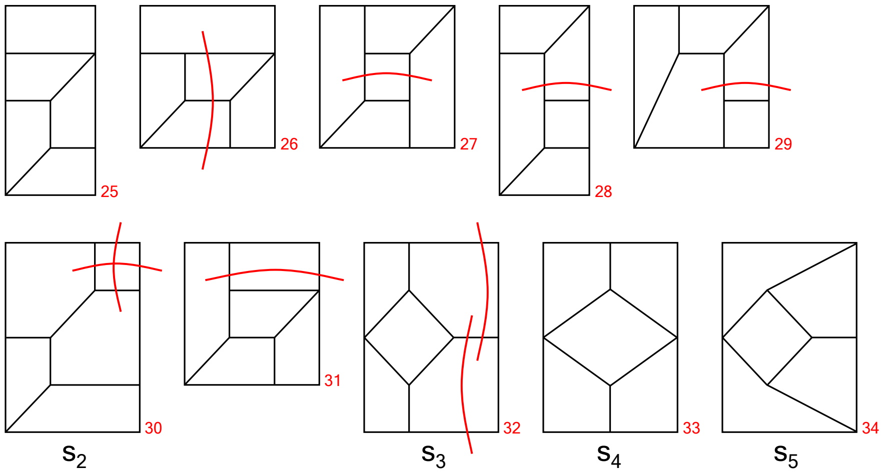

Figure 11: Non-Mondrian DCI topologies $T_{25}, \ldots, T_{34}$ at 5-loop. $T_{30}, T_{32}, T_{33}, T_{34}$ assigned with s_2, s_3, s_4, s_5 respectively are non-rung-rule topologies (T_{30} is generated by the substitution rule while T_{32}, T_{33}, T_{34} are neither generated by the rung nor substitution rule).

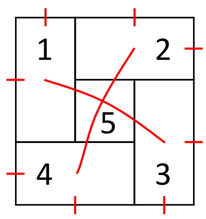

Figure 12: A particular diagram of T_{16} at 5-loop with 8 external cuts.

3.1 Determination of s_1

To determine s_1, let's consider a particular diagram of DCI topology T_{16} given in figure 12. As usual, we can maximally impose all 8 available external cuts, as indicated by the red segments. These 8 cuts result in $x_1\!=\!y_1\!=\!y_2\!=\!z_2\!=\!z_3\!=\!w_3\!=\!w_4\!=\!x_4\!=\!0$, which can simplify the ten D's as

$$\begin{aligned} &D_{12} = x_2 z_1, \quad D_{23} = y_3 w_2, \quad D_{34} = x_3 z_4, \quad D_{14} = y_4 w_1, \\ &D_{13} = x_3 z_1 + y_3 w_1, \quad D_{24} = x_2 z_4 + y_4 w_2, \end{aligned} \tag{3.1}$$

as well as

$$\begin{aligned} D_{15} &= x_5 z_1 + y_5 w_1 - x_5 z_5 - y_5 w_5, \\ D_{25} &= z_5 x_2 + y_5 w_2 - x_5 z_5 - y_5 w_5, \\ D_{35} &= z_5 x_3 + w_5 y_3 - x_5 z_5 - y_5 w_5, \\ D_{45} &= x_5 z_4 + w_5 y_4 - x_5 z_5 - y_5 w_5. \end{aligned} \tag{3.2}$$

Since $D_{12}, D_{23}, D_{34}, D_{14}, D_{13}, D_{24}$ are manifestly positive, we only need to either cut $D_{15}, D_{25}, D_{35}, D_{45}$ or impose their positivity. However, there is no straightforward positive cut for positivity conditions of the form $x\!+\!y\!>\!a$ in this case – the discussion can be rather complicated. Therefore let's keep their positivity and see what happens next, in fact, $D_{15}, D_{25}, D_{35}, D_{45}$ totally decouple partly due to the symmetry of the 8 external cuts in figure 12, so that we can impose the positivity for each D_{i5} individually. This leads to the simple proper numerator

$$\begin{aligned} N &= (x_5 z_1 + y_5 w_1)(z_5 x_2 + y_5 w_2)(z_5 x_3 + w_5 y_3)(x_5 z_4 + w_5 y_4) D_{12} D_{23} D_{34} D_{14} D_{13} D_{24} \\ &= (x_5 z_1 + y_5 w_1)(z_5 x_2 + y_5 w_2)(z_5 x_3 + w_5 y_3)(x_5 z_4 + w_5 y_4)\, x_2 x_3 z_1 z_4\, y_3 y_4 w_1 w_2\, (x_3 z_1 + y_3 w_1)(x_2 z_4 + y_4 w_2). \end{aligned} \tag{3.3}$$

On the other hand, diagrams of all topologies, orientations and configurations of loop numbers at 5-loop that survive these 8 cuts are summarized below

T_8	T_{15}	T_{16}	T_{20}	T_{21}	T_{22}	T_{23}	T_{24}	T_{32}	T_{33}
2	4	1	8	4	8	8	8	4	2

(3.4)

where all orientations generated by dihedral symmetry of these topologies contribute and each orientation exactly contributes one configuration of loop numbers, as given by the numbers of contributing diagrams of each T_i above. It is easy to enumerate all of them, and the sum of their proper numerators is

$$\begin{aligned} &S\,(x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2, x_3, y_3, z_3, w_3, x_4, y_4, z_4, w_4, x_5, y_5, z_5, w_5) \\ &= x_2 x_3 x_5 z_1 z_4 z_5\, y_3 y_4 y_5 w_1 w_2 w_5\, (S_8 + S_{15-16} + S_{20} + S_{21} + S_{22} + S_{23} + S_{24} + S_{32} + S_{33}), \end{aligned} \tag{3.5}$$

where for compactness we have factored out a common factor, and each piece in the sum is given by

$$S_8 = \frac{y_5 w_5}{x_5 z_5} D_{13} D_{14} D_{23} D_{24} + \frac{x_5 z_5}{y_5 w_5} D_{12} D_{13} D_{24} D_{34}, \tag{3.6}$$

$$S_{15-16} = D_{13} D_{24} (x_3 z_1 D_{24} + y_3 w_1 D_{24} + x_2 z_4 D_{13} + y_4 w_2 D_{13} + s_1 D_{13} D_{24}), \tag{3.7}$$

$$\begin{aligned} S_{20} = &- \frac{y_4}{y_5} D_{12} D_{13} D_{24} D_{35} - \frac{y_3}{y_5} D_{12} D_{13} D_{24} D_{45} - \frac{w_1}{w_5} D_{13} D_{24} D_{25} D_{34} - \frac{w_2}{w_5} D_{13} D_{15} D_{24} D_{34} \\ &- \frac{z_4}{z_5} D_{13} D_{15} D_{23} D_{24} - \frac{x_3}{x_5} D_{13} D_{14} D_{24} D_{25} - \frac{z_1}{z_5} D_{13} D_{23} D_{24} D_{45} - \frac{x_2}{x_5} D_{13} D_{14} D_{24} D_{35}, \end{aligned} \tag{3.8}$$

$$S_{21} = \frac{y_3 y_4 w_5}{y_5} D_{12} D_{13} D_{24} + \frac{y_5 w_1 w_2}{w_5} D_{13} D_{24} D_{34} + \frac{x_2 x_3 z_5}{x_5} D_{13} D_{14} D_{24} + \frac{x_5 z_1 z_4}{z_5} D_{13} D_{23} D_{24}, \tag{3.9}$$

$$\begin{aligned} S_{22} = &\frac{x_3 z_5 y_4}{y_5} D_{12} D_{13} D_{24} + \frac{x_5 z_4 y_3}{y_5} D_{12} D_{13} D_{24} + \frac{x_2 z_5 w_1}{w_5} D_{13} D_{24} D_{34} + \frac{x_5 z_1 w_2}{w_5} D_{13} D_{24} D_{34} \\ &+ \frac{x_2 y_3 w_5}{x_5} D_{13} D_{14} D_{24} + \frac{z_1 y_4 w_5}{z_5} D_{13} D_{23} D_{24} + \frac{x_3 y_5 w_2}{x_5} D_{13} D_{14} D_{24} + \frac{z_4 y_5 w_1}{z_5} D_{13} D_{23} D_{24}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} S_{23} = &\frac{y_4^2 w_2}{y_5} D_{12} D_{13} D_{35} + \frac{y_4 w_2^2}{w_5} D_{13} D_{15} D_{34} + \frac{y_3^2 w_1}{y_5} D_{12} D_{24} D_{45} + \frac{y_3 w_1^2}{w_5} D_{24} D_{25} D_{34} \\ &+ \frac{x_2^2 z_4}{x_5} D_{13} D_{14} D_{35} + \frac{x_2 z_4^2}{z_5} D_{13} D_{15} D_{23} + \frac{x_3 z_1^2}{z_5} D_{23} D_{24} D_{45} + \frac{x_3^2 z_1}{x_5} D_{14} D_{24} D_{25}, \end{aligned} \tag{3.11}$$

$$\begin{aligned} S_{24} = &\frac{x_2 z_4 y_4}{y_5} D_{12} D_{13} D_{35} + \frac{x_3 z_1 y_3}{y_5} D_{12} D_{24} D_{45} + \frac{x_3 z_1 w_1}{w_5} D_{24} D_{25} D_{34} + \frac{x_2 z_4 w_2}{w_5} D_{13} D_{15} D_{34} \\ &+ \frac{x_2 y_4 w_2}{x_5} D_{13} D_{14} D_{35} + \frac{z_1 y_3 w_1}{z_5} D_{23} D_{24} D_{45} + \frac{x_3 y_3 w_1}{x_5} D_{14} D_{24} D_{25} + \frac{z_4 y_4 w_2}{z_5} D_{13} D_{15} D_{23}, \end{aligned} \tag{3.12}$$

$$S_{32} = s_3 (y_3 w_2 D_{13} D_{14} D_{24} + y_4 w_1 D_{13} D_{23} D_{24} + x_3 z_4 D_{12} D_{13} D_{24} + x_2 z_1 D_{13} D_{24} D_{34}), \tag{3.13}$$

$$S_{33} = s_4 (D_{13} D_{14} D_{23} D_{24} + D_{12} D_{13} D_{24} D_{34}). \tag{3.14}$$

The difference between the deformed S on the 8 cuts and the proper numerator from positivity conditions is then

$$\begin{aligned} &S\,(0\,,\,0\,,z_1,w_1,x_2,\,0\,,\,0\,,w_2,x_3,y_3,\,0\,,\,0\,,\,0\,,y_4,z_4,\,0\,,x_5,y_5,z_5,w_5) \\ &-(x_5 z_1 + y_5 w_1)(z_5 x_2 + y_5 w_2)(z_5 x_3 + w_5 y_3)(x_5 z_4 + w_5 y_4)\, x_2 x_3 z_1 z_4\, y_3 y_4 w_1 w_2\, (x_3 z_1 + y_3 w_1)(x_2 z_4 + y_4 w_2) \\ =\ &x_2 x_3 x_5 z_1 z_4 z_5\, y_3 y_4 y_5 w_1 w_2 w_5\, (x_3 z_1 + y_3 w_1)(x_2 z_4 + y_4 w_2) \\ &\times [(1+s_1)(x_3 z_1\, y_4 w_2 + x_2 z_4\, y_3 w_1) + (2 + s_1 + 2 s_3 + s_4)(x_3 x_2 z_1 z_4 + y_3 y_4 w_1 w_2)], \end{aligned} \tag{3.15}$$

to make this difference vanish we must take $s_1 = -1$ which agrees with [5], and $1+2s_3+s_4=0$. Even though s_3 and s_4 cannot be determined by these 8 external cuts yet, we can determine one with the aid of further cuts then get the other via relation $1+2s_3+s_4=0$.

3.2 Determination of s_2, s_3, s_4

To figure out s_3 or s_4, we have to disentangle T_{32} and T_{33}, otherwise combination $(1+2s_3+s_4)$ will always obstruct our intention. Since T_{32} has one internal propagator more than T_{33} while their other topological features are identical, it is feasible to impose further internal cuts to kill T_{33} but let T_{32} survive so that s_3 can be isolated then determined. If we consider a particular diagram of T_{32} given in figure 13, a simplest choice is to impose $D_{12} = D_{23} = 0$, as one can easily check that none of the diagrams of T_{33} can survive it regardless of orientations and number configurations (we also maintain the 8 external cuts in figure 12).

However, since $D_{12} = x_2 z_1$ and $D_{23} = y_3 w_2$, setting $D_{12} = D_{23} = 0$ will force two external propagators which do not belong to the diagram in figure 12 to vanish. This involves a technical subtlety of composite

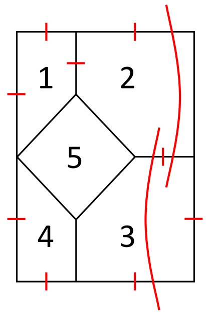

Figure 13: A particular diagram of T_{32} at 5-loop with 7 external and 2 internal cuts. The external cut $z_2=0$ is traded for two internal cuts $D_{12}=D_{23}=0$ that are free of the subtlety of composite residues.

residues, although there is no problem in this way after some clarification, we prefer to avoid this subtlety for the moment. Therefore, a simplest alternative is to relax one external cut, which is chosen to be z_2.

In summary, upon the 7 external cuts $x_1=y_1=y_2=z_3=w_3=w_4=x_4=0$, we can further impose

$$z_1 = z_2, \quad x_2 = x_3 + \frac{y_3 w_2}{z_2} \equiv \hat{x}_2, \tag{3.16}$$

so these 7+2 cuts can simplify the ten D's as

$$D_{12} = D_{23} = 0, \quad D_{34} = x_3 z_4, \quad D_{14} = y_4 w_1, \quad D_{13} = x_3 z_2 + y_3 w_1, \tag{3.17}$$

which are either zero or manifestly positive, as well as

$$\begin{aligned}
D_{15} &= x_5 z_2 + y_5 w_1 - x_5 z_5 - y_5 w_5, \\
D_{45} &= x_5 z_4 + w_5 y_4 - x_5 z_5 - y_5 w_5, \\
D_{35} &= z_5 x_3 + w_5 y_3 - x_5 z_5 - y_5 w_5, \\
D_{24} &= (x_3 z_2 + y_3 w_2) \left(\frac{z_4}{z_2} + \frac{y_4}{y_3 + x_3 z_2 / w_2} - 1 \right), \\
D_{25} &= (z_5 - z_2) \left(x_3 + y_3 \frac{w_2}{z_2} - x_5 \right) + y_5 (w_2 - w_5),
\end{aligned} \tag{3.18}$$

again there is no straightforward positive cut for any of these five positivity conditions, so it is better to keep their positivity. In this case, $D_{15}, D_{45}, D_{35}, D_{24}, D_{25}$ do not trivially decouple, as we can see it more clearly after the following reorganization

$$\begin{aligned}
&\frac{z_2}{z_5 + y_5 w_5 / x_5} + \frac{w_1}{w_5 + x_5 z_5 / y_5} > 1, \\
&\frac{x_3}{x_5 + y_5 w_5 / z_5} + \frac{y_3}{y_5 + x_5 z_5 / w_5} > 1, \quad (z_5 - z_2) \left(x_3 + y_3 \frac{w_2}{z_2} - x_5 \right) + y_5 (w_2 - w_5) > 0, \\
&\frac{z_4}{z_5 + y_5 w_5 / x_5} + \frac{y_4}{y_5 + x_5 z_5 / w_5} > 1, \quad \frac{z_4}{z_2} + \frac{y_4}{y_3 + x_3 z_2 / w_2} > 1.
\end{aligned} \tag{3.19}$$

In the first line we focus on z_2, w_1, in the second x_3, y_3 and in the third z_4, y_4. For the latter two lines, the discussion of imposing positivity is nontrivial, since we need to choose one condition (or both) as the relations between several variables vary. Explicitly, the second line's discussion depends on how z_2 varies in the first line and the third line's discussion depends on how x_3, y_3 vary in the second line. Its technical details is elaborated in appendix A, and below we just present the resulting $d\log$ form after analyzing all possible situations of variables $z_2, w_1, w_2, y_3, x_3, z_4, y_4$:

$$\frac{M}{z_2^3 w_1 w_2 y_3 x_3 z_4 y_4\, D_{15} D_{35} D_{25} D_{45} D_{24}} \equiv \frac{R}{z_2 w_1 w_2 y_3 x_3 z_4 y_4}, \tag{3.20}$$

where the expression of M can be referred in figure 14, as the result simplified by Mathematica, and R is the desired dimensionless ratio.

```
M = w1 y5 (w5 x5 y3 z2^2 (w5 (y4 - y5) + x5 z4) (w2 y4 z2 + w2 y3 z4 + x3 z2 z4) +
     (w2 w5 y4 z2 (w2 w5 y3^2 (y4 - y5) + x3 (w5 y3 y4 + w2 (-y3 + y4) y5 - w5
              (y3 + y4) y5) z2 - (x5^2 y3 + x3^2 y5) z2^2) + (w2^2 w5^2 y3^3 y4 +
           w2 w5 y3 (w5 x3 y4 (2 y3 - y5) + w2 (x5 y3 (-y3 + y4) + x3 y4 y5)) z2 +
           (w5 y3 (-w2 x5 (2 x3 + x5) y3 + x3 (w5 x3 + w2 x5) y4) + x3
              (w2 w5 x5 y3 + (w2 - w5) (w5 x3 + w2 x5) y4) y5) z2^2 -
           w5 x3 x5 ((x3 + x5) y3 - x3 y5) z2^3) z4 + x5 (w2 y3 + x3 z2)
         (w2 w5 y3^2 + x3 (w5 (y3 - y5) + w2 y5) z2) z4^2) z5 +
     (w2 w5 y4 z2 (w2 y3 (-x5 y3 + x3 y4) + x3 (x3 y4 - x5 (y3 + y4)) z2) +
        x3 (w2 y3 + (x3 - x5) z2)
          (x3 z2 (w5 y4 - x5 z2) + w2 (w5 y3 y4 + x5 (-y3 + y4) z2)) z4 +
        x3 x5 (w2 y3 + x3 z2) (w2 y3 + (x3 - x5) z2) z4^2) z5^2) +
   x5 z2^2 (w2^2 x3 y5 (w5 y4 (-y3 + y4) z2 + w5 y3 (y4 - y5) z4 +
        x5 z4 (y4 z2 + y3 (z4 - z5))) z5 -
     w5 x3 y3 z2 z4 (w5 (y4 - y5) + x5 (z4 - z5)) (w5 y5 + x5 (-z2 + z5)) +
     w2 (-w5^3 y3 (y4 - y5) y5 (y4 z2 + y3 z4) +
        w5^2 x5 y3 (y4 z2 + y3 z4) (-y5 (z2 + z4 - 2 z5) + y4 (z2 - z5)) +
        x3^2 x5 y5 z2 z4 (z4 - z5) z5 + w5 (x5^2 y3 (y4 z2 + y3 z4) (z2 - z5) (z4 - z5) -
           x3^2 y5 z2 (y4 z2 - y4 z4 + y5 z4) z5)));
```

Figure 14: Numerator M simplified by Mathematica.

To get the overall dimensionless ratio, we also need

$$\begin{aligned} \frac{z_1}{z_1 - z_2} &= \frac{x_2 z_1}{D_{12}} \to \frac{\hat{x}_2 z_2}{D_{12}}, \\ x_2 \left(\frac{1}{x_2} - \frac{1}{x_2 - \hat{x}_2} \right) &= \frac{z_2 \hat{x}_2}{D_{23}}, \end{aligned} \tag{3.21}$$

where $\hat{x}_2$ is defined in (3.16), and since the positivity of D_{34}, D_{14}, D_{13} is trivial, we finally obtain

$$\frac{\hat{x}_2 z_2}{D_{12}} \frac{z_2 \hat{x}_2}{D_{23}} \frac{D_{34} D_{14} D_{13}}{D_{34} D_{14} D_{13}} R = \frac{(\hat{x}_2 z_2)^2 D_{34} D_{14} D_{13}}{D_{12} D_{23} D_{34} D_{14} D_{13}} \frac{1}{D_{15} D_{35} D_{25} D_{45} D_{24}} \frac{M}{z_2^2}, \tag{3.22}$$

therefore the proper numerator is

$$N = \hat{x}_2^2 \, D_{34} D_{14} D_{13} \, M = \left(x_3 + \frac{y_3 w_2}{z_2} \right)^2 x_3 z_4 \, y_4 w_1 \, (x_3 z_2 + y_3 w_1) \, M. \tag{3.23}$$

On the other hand, diagrams of all topologies, orientations and configurations of loop numbers at 5-loop that survive these 7+2 cuts are summarized below

T_3	T_8	T_9	T_{11}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}	T_{18}	T_{19}	T_{20}	T_{21}	T_{22}	T_{23}	T_{24}	T_{30}	T_{31}	T_{32}
						4	1				4	2	4	4	4			2
1	1	1	1	1	1			1	2	2	1	1	2	1	1	1	1	

(3.24)

where the first line denotes a subset of diagrams among (3.5), and the second line the additional surviving contribution due to relaxing $z_2 = 0$. Again, each orientation of T_i can at most contribute one configuration of loop numbers. The sum of their proper numerators is

$$\begin{aligned} & S\left(x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2, x_3, y_3, z_3, w_3, x_4, y_4, z_4, w_4, x_5, y_5, z_5, w_5\right) \\ = \; & x_2 x_3 x_5 z_1 z_4 z_5 \, y_3 y_4 y_5 w_1 w_2 w_5 \left(S_{15-16} + S_{20} + S_{21} + S_{22} + S_{23} + S_{24} + S_{32}\right) \\ & + S_3 + S_8 + S_9 + S_{11} + S_{13} + S_{14} + S_{17-19} + S_{20-24} + S_{30} + S_{31}, \end{aligned} \tag{3.25}$$

where each piece in the sum is given by

$$S_{15-16} = D_{13} D_{24} (x_3 z_1 D_{24} + y_3 w_1 D_{24} + x_2 z_4 D_{13} + y_4 w_2 D_{13} + s_1 D_{13} D_{24}), \tag{3.26}$$

$$S_{20} = -0 - 0 - \frac{w_1}{w_5} D_{13} D_{24} D_{25} D_{34} - \frac{w_2}{w_5} D_{13} D_{15} D_{24} D_{34} - 0 - \frac{x_3}{x_5} D_{13} D_{14} D_{24} D_{25} - 0 - \frac{x_2}{x_5} D_{13} D_{14} D_{24} D_{35}, \tag{3.27}$$

$$S_{21} = 0 + \frac{y_5 w_1 w_2}{w_5} D_{13} D_{24} D_{34} + \frac{x_2 x_3 z_5}{x_5} D_{13} D_{14} D_{24} + 0, \tag{3.28}$$

$$S_{22} = 0 + 0 + \frac{x_2 z_5 w_1}{w_5} D_{13} D_{24} D_{34} + \frac{x_5 z_1 w_2}{w_5} D_{13} D_{24} D_{34} + \frac{x_2 y_3 w_5}{x_5} D_{13} D_{14} D_{24} + 0 + \frac{x_3 y_5 w_2}{x_5} D_{13} D_{14} D_{24} + 0, \tag{3.29}$$

$$S_{23} = 0 + \frac{y_4 w_2^2}{w_5} D_{13} D_{15} D_{34} + 0 + \frac{y_3 w_1^2}{w_5} D_{24} D_{25} D_{34} + \frac{x_2^2 z_4}{x_5} D_{13} D_{14} D_{35} + 0 + 0 + \frac{x_3^2 z_1}{x_5} D_{14} D_{24} D_{25}, \tag{3.30}$$

$$S_{24} = 0 + 0 + \frac{x_3 z_1 w_1}{w_5} D_{24} D_{25} D_{34} + \frac{x_2 z_4 w_2}{w_5} D_{13} D_{15} D_{34} + \frac{x_2 y_4 w_2}{x_5} D_{13} D_{14} D_{35} + 0 + \frac{x_3 y_3 w_1}{x_5} D_{14} D_{24} D_{25} + 0, \tag{3.31}$$

$$S_{32} = s_3 (y_3 w_2 D_{13} D_{14} D_{24} + 0 + 0 + x_2 z_1 D_{13} D_{24} D_{34}), \tag{3.32}$$

for the subset among (3.5) and the zeros denote diagrams killed by $D_{12} = D_{23} = 0$, as well as

$$S_3 = x_2^3 x_3 z_1 z_2 z_4 z_5 \, y_4 y_5 w_1 w_5 D_{13} D_{14} D_{34} D_{35}, \tag{3.33}$$

$$S_8 = x_2^2 x_3 x_5 z_1 z_2^2 z_4\, y_3 y_4 w_1 w_5 D_{13} D_{15} D_{34} D_{45}, \tag{3.34}$$

$$S_9 = x_2^2 x_3 x_5 z_1 z_2 z_4 z_5\, y_4 y_5 w_1^2 D_{13} D_{24} D_{34} D_{35}, \tag{3.35}$$

$$S_{11} = x_2 x_3^3 z_1 z_2 z_4 z_5\, y_4 y_5^2 w_1 w_2 w_5 D_{13} D_{14} D_{24}, \tag{3.36}$$

$$S_{13} = x_2^2 x_3^2 z_1 z_2 z_4 z_5\, y_4^2 y_5 w_1 w_2 w_5 D_{13} D_{14} D_{35}, \tag{3.37}$$

$$S_{14} = x_2 x_3^2 x_5 z_1 z_2 z_4 z_5\, y_4^2 y_5 w_1 w_2 w_5 D_{13}^2 D_{24}, \tag{3.38}$$

$$\begin{aligned} S_{17-19} = {} & x_2 x_3 x_5 z_1 z_2 z_4 z_5\, y_4 y_5 w_1 w_2 D_{13} D_{34} \\ & \times (-x_3 D_{15} D_{24} + x_3\, y_4 w_2 D_{15} + x_3\, y_5 w_1 D_{24} + x_2 D_{15} D_{34} + x_5 D_{13} D_{24}), \end{aligned} \tag{3.39}$$

$$\begin{aligned} S_{20-24} = {} & x_2 x_3 x_5 z_1 z_2 z_4\, y_3^2 y_4 w_1 w_2 w_5 D_{15} D_{45} \\ & \times \left(- D_{13} D_{24} + y_4 w_2 D_{13} + \frac{y_4}{y_3}\, x_3 z_2 D_{13} + x_2 z_4 D_{13} + y_3 w_1 D_{24} + x_3 z_1 D_{24} \right), \end{aligned} \tag{3.40}$$

$$S_{30} = s_2\, x_2 x_3 x_5^2 z_1 z_2 z_4 z_5\, y_3 y_4 y_5 w_1^2 w_2 D_{13} D_{24} D_{34}, \tag{3.41}$$

$$S_{31} = -\, x_2 x_3^2 x_5 z_1 z_2 z_4 z_5\, y_3 y_4 y_5 w_1 w_2 w_5 D_{13} D_{14} D_{24}, \tag{3.42}$$

for the additional surviving contribution. The difference between the deformed S on the 7+2 cuts and the proper numerator is then

$$\begin{aligned} & S\left(0\,,\,0\,,z_2,w_1,x_3+\frac{y_3 w_2}{z_2},\,0\,,z_2,w_2,x_3,y_3,\,0\,,\,0\,,\,0\,,y_4,z_4,\,0\,,x_5,y_5,z_5,w_5\right) \\ & - \left(x_3 + \frac{y_3 w_2}{z_2}\right)^2 x_3 z_4\, y_4 w_1\, (x_3 z_2 + y_3 w_1)\, M \\ = {} & x_3 x_5 z_4 z_5\, y_3 y_4 y_5 w_1 w_2\, (x_3 z_2 + y_3 w_1)(x_3 z_2 + y_3 w_2) \left[(x_3 z_2 + y_3 w_2) \left(\frac{z_4}{z_2} - 1 \right) +\, y_4 w_2 \right] \\ & \times [(1+s_2) x_3 x_5 z_2 z_4\, w_1 + (1+s_3) w_5 (x_3 z_4 (x_3 z_2 + y_3 w_2) + y_3 y_4 w_1 w_2)], \end{aligned} \tag{3.43}$$

to make this difference vanish we must take $s_2 = s_3 = -1$, so via $1+2s_3+s_4 = 0$ we also obtain $s_4 = +1$, all of which agree with [5]. We see that determining s_2 is a byproduct of determining s_3.

It is worth noticing the complexity of 5-loop topologies which have a purely internal loop: the simple case of T_{16} with 8 symmetric external cuts is clearly rather rare, as merely relaxing one cut results in five positivity conditions that do not trivially decouple. In general, the more external cuts a topology has, the easier its calculation may be. We will see how valid this qualitative criterion is from the case of T_{34}, which merely has two external cuts less than T_{16} but becomes extremely complicated, even compared to the case of T_{32} which is already very nontrivial.

3.3 Determination of s_5

To determine s_5, the coefficient of T_{34}, turns out to be the most difficult case at 5-loop. We again consider a particular diagram given in figure 15, in which all 6 available external cuts are imposed, now let's again impose internal cuts $D_{12}=D_{23}=0$ upon $x_1=y_1=z_2=z_3=w_4=x_4=0$. Even though this diagram has only one external cut less than that in figure 13, it is very different from the latter. In fact, the structure and complexity of the simplified positivity conditions are very sensitive to the choice of cuts.

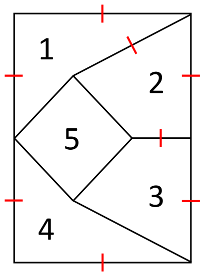

Figure 15: A particular diagram of T_{34} at 5-loop with 6 external and 2 internal cuts.

Explicitly, for the two internal cuts we can impose

$$w_2 = w_3 = w_1 + \frac{x_2 z_1}{y_2} \equiv \hat{w}_3, \tag{3.44}$$

so the ten D's can be simplified as

$$D_{12} = D_{23} = 0, \quad D_{14} = y_4 w_1, \tag{3.45}$$

which are either zero or manifestly positive, as well as

$$\begin{aligned}
D_{13} &= z_1 \left(x_3 - x_2 \frac{y_3}{y_2} \right) \equiv z_1 x_3', \\
D_{15} &= x_5 z_1 + y_5 w_1 - x_5 z_5 - y_5 w_5, \\
D_{45} &= x_5 z_4 + w_5 y_4 - x_5 z_5 - y_5 w_5, \\
D_{24} &= (x_2 z_1 + y_2 w_1) \left(\frac{z_4}{z_1 + w_1 y_2/x_2} + \frac{y_4}{y_2} - 1 \right), \\
D_{34} &= (x_2 z_1 + y_2 w_1) \frac{y_3}{y_2} \left(\frac{x_3}{y_3} \frac{z_4}{z_1 x_2/y_2 + w_1} + \frac{y_4}{y_3} - 1 \right), \\
D_{25} &= (y_5 - y_2) \left(w_1 + z_1 \frac{x_2}{y_2} - w_5 \right) + z_5 (x_2 - x_5), \\
D_{35} &= (y_5 - y_3) \left(w_1 + z_1 \frac{x_2}{y_2} - w_5 \right) + z_5 (x_3 - x_5),
\end{aligned} \tag{3.46}$$

where x_3' is defined to trivialize $D_{13}>0$, and the rest six conditions can be analyzed more clearly after the following reorganization

$$\begin{aligned}&w_1+z_1\frac{x_5}{y_5}>w_5+z_5\frac{x_5}{y_5},\quad (y_5-y_2)\left(w_1+z_1\frac{x_2}{y_2}-w_5\right)+z_5(x_2-x_5)>0,\quad (y_5-y_3)\left(w_1+z_1\frac{x_2}{y_2}-w_5\right)+z_5(x_3-x_5)>0,\\&\frac{z_4}{z_5+y_5w_5/x_5}+\frac{y_4}{y_5+x_5z_5/w_5}>1,\quad \frac{z_4}{z_1+w_1y_2/x_2}+\frac{y_4}{y_2}>1,\quad \frac{z_4}{k(z_1+w_1y_2/x_2)}+\frac{y_4}{y_3}>1,\end{aligned} \tag{3.47}$$

where $k=y_3x_2/(x_3y_2)<1$ due to $D_{13}>0$. In the first line we focus on w_1,z_1 and in the second z_4,y_4, as the second line's discussion depends on how w_1,z_1 vary in the first line, and its technical details is briefly given in appendix B. Below we just present the resulting $d\log$ form after analyzing all possible situations of variables $y_2,y_3,y_5,x_5,x_3',w_5,z_1,w_1,y_4,z_4$:

$$\begin{aligned}&\frac{1}{y_2y_3y_5x_5x_3'w_5z_1w_1y_4z_4\,D_{15}D_{25}D_{35}D_{45}D_{24}D_{34}}\,\frac{\hat{w}_3}{D_{23}}\,\frac{y_2(M_1y_2D_{34})+y_3M_2}{y_2^4}\\ \equiv\;&\frac{R}{y_2y_3y_5x_5x_3w_5z_1w_1y_4z_4},\end{aligned} \tag{3.48}$$

where the expressions of M_1 and M_2 simplified by Mathematica can be referred in appendix B, and R is the desired dimensionless ratio, which is explicitly given by

$$\begin{aligned}R&=\frac{x_3}{x_3'}\,\frac{\hat{w}_3}{D_{15}D_{25}D_{35}D_{45}D_{24}D_{34}D_{23}}\,\frac{y_2(M_1y_2D_{34})+y_3M_2}{y_2^4}\\&=\frac{x_3z_1\hat{w}_3}{D_{13}D_{15}D_{25}D_{35}D_{45}D_{24}D_{34}D_{23}}\,\frac{y_2(M_1y_2D_{34})+y_3M_2}{y_2^4}.\end{aligned} \tag{3.49}$$

To get the overall dimensionless ratio, we also need

$$w_2\left(\frac{1}{w_2}-\frac{1}{w_2-\hat{w}_3}\right)=\frac{y_2\hat{w}_3}{D_{12}}, \tag{3.50}$$

where $\hat{w}_3$ is defined in (3.44), and since the positivity of D_{14} is trivial, we finally obtain

$$\frac{y_2\hat{w}_3}{D_{12}}\,\frac{D_{14}}{D_{14}}\,R=\frac{y_2\hat{w}_3D_{14}}{D_{12}D_{14}}\,\frac{x_3z_1\hat{w}_3}{D_{13}D_{15}D_{25}D_{35}D_{45}D_{24}D_{34}D_{23}}\,\frac{y_2(M_1y_2D_{34})+y_3M_2}{y_2^4}, \tag{3.51}$$

therefore the proper numerator is

$$N=\hat{w}_3^2\,D_{14}\,x_3z_1\,\frac{y_2(M_1y_2D_{34})+y_3M_2}{y_2^3}=\left(w_1+\frac{x_2z_1}{y_2}\right)^2y_4w_1\,x_3z_1\,\frac{y_2(M_1y_2D_{34})+y_3M_2}{y_2^3}. \tag{3.52}$$

On the other hand, diagrams of all topologies, orientations and configurations of loop numbers at 5-loop that survive these 6+2 cuts are summarized below

T_1	T_3	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{13}	T_{14}	T_{15}	T_{16}	T_{17}	T_{18}	T_{19}	T_{20}	T_{21}	T_{22}	T_{23}	T_{24}	T_{25}	T_{30}	T_{31}	T_{32}	T_{34}
											4	1				4	2	4	4	4				2	
1	2	(4)+1	(3)+1	2	(3)	(4)+3	1	1	2	2	4	1	2	(2)+3	(2)+3	(3)+1	(3)	(3)+3	(4)+4	(4)+4	2	2	1		1

(3.53)

where the first line denotes a subset of diagrams among (3.5) which are identical to those given in (3.25), and the second line the additional surviving contribution. Now for some T_i's, a particular orientation can contribute more than one configuration of loop numbers, as the numbers in parentheses above denote this kind of multiplicity. An explicit example is (4)+1 for T_5 corresponding to the diagrams given in figure 16, of which the first four with different number configurations share the same orientation.

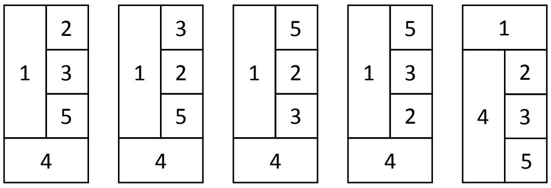

Figure 16: The (4)+1 multiplicity of T_5.

The sum of their proper numerators is

$$
\begin{aligned}
&S\left(x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2, x_3, y_3, z_3, w_3, x_4, y_4, z_4, w_4, x_5, y_5, z_5, w_5\right) \\
&= x_2 x_3 x_5 z_1 z_4 z_5\, y_3 y_4 y_5 w_1 w_2 w_5 \left(S_{15-16}+S_{20}+S_{21}+S_{22}+S_{23}+S_{24}+S_{32}+S_{34}\right) \\
&\quad +S_1+S_3+S_5+S_6+S_7+S_8+S_9+S_{10}+S_{11}+S_{13}+S_{14}+S'_{15-16}+S_{17-19}+S'_{20-24}+S_{25}+S_{30}+S_{31}.
\end{aligned}
\tag{3.54}
$$

Recall that $S_{15-16}, S_{20}, S_{21}, S_{22}, S_{23}, S_{24}, S_{32}$ are already given in (3.25), while

$$
S_{34} = s_5\, y_2 w_3 D_{13} D_{14} D_{24} \tag{3.55}
$$

is the extra term in the second line above, and each piece in the third line is given by

$$
S_1 = y_2 y_3 y_4 y_5 w_1 w_2 w_3 w_5 D_{13} D_{14} D_{15} D_{24} D_{25} D_{34}, \tag{3.56}
$$

$$
S_3 = x_3 x_5 z_4 z_5\, y_2 y_3 y_4 y_5 w_1 w_2^3 D_{13} D_{14} D_{15} D_{34} + x_3 x_5 z_1 z_5\, y_2^3 y_4 w_1 w_2 w_3 w_5 D_{13} D_{14} D_{34} D_{45}, \tag{3.57}
$$

$$
\begin{aligned}
S_5 = {} & x_2 x_3 x_5 z_1^3\, y_4^2 w_1 w_2 w_3 w_5 (y_3 y_5 D_{24} D_{25} D_{34} + y_2 y_5 D_{24} D_{34} D_{35} + y_2 y_3 D_{24} D_{35} D_{45} + y_2 y_3 D_{25} D_{34} D_{45}) \\
& + x_2 x_3 x_5 z_4^3\, y_2 y_3 y_4 y_5 w_1^2 w_2 w_3 D_{13} D_{15} D_{25},
\end{aligned}
\tag{3.58}
$$

$$
\begin{aligned}
S_6 = {} & z_1\, y_4 w_1 w_2 w_3 w_5 D_{14} D_{24} (x_2\, y_3^2 y_5 D_{15} D_{25} D_{34} + x_3\, y_2^2 y_5 D_{15} D_{34} D_{35} + x_5\, y_2^2 y_3 D_{13} D_{35} D_{45}) \\
& + x_5 z_4\, y_2 y_3 y_4 y_5 w_1 w_2 w_3^2 D_{13} D_{14} D_{15} D_{24} D_{25},
\end{aligned}
\tag{3.59}
$$

$$
S_7 = z_5\, y_2 y_3 y_4 y_5 w_1 w_2 w_3 w_5 D_{13} D_{14} D_{24} (x_2 D_{13} D_{45} + x_3 D_{15} D_{24}), \tag{3.60}
$$

$$
S_8 = x_5 z_1 z_4\, y_4 w_1 w_2 D_{14} (x_3\, y_2^2 y_3 w_2 w_5 D_{13} D_{35} D_{45} + x_2\, y_3^2 y_5 w_3^2 D_{15} D_{24} D_{25} + x_3\, y_2^2 y_5 w_2 w_3 D_{15} D_{34} D_{35}), \tag{3.61}
$$

$$
\begin{aligned}
S_9 = {} & z_1^2\, y_4 w_1 w_2 w_3 w_5 D_{14} (x_2 x_3\, y_3 y_5^2 D_{24} D_{25} D_{34} + x_2 x_3\, y_2 y_5^2 D_{24} D_{34} D_{35} + x_2 x_5\, y_2 y_3^2 D_{24} D_{35} D_{45} + x_3 x_5\, y_2^2 y_3 D_{25} D_{34} D_{45}) \\
& + x_2^2 z_1 z_5\, y_3^2 y_4 y_5 w_1 w_2 w_3 w_5 D_{13} D_{14} D_{24} D_{45} + x_3 z_4\, y_2 y_3 y_4 y_5 w_1 w_2^2 D_{13} D_{14} D_{15} (x_3 z_5\, w_5 D_{24} + x_5 z_4\, w_3 D_{25}),
\end{aligned}
\tag{3.62}
$$

$$S_{10} = x_2x_3z_5^2\, y_2y_3y_4y_5w_1w_2w_3w_5D_{13}D_{14}^2D_{24}, \tag{3.63}$$

$$S_{11} = x_2x_3x_5z_4z_5^2\, y_2y_3y_4y_5w_1^3w_2D_{13}D_{24}D_{34}, \tag{3.64}$$

$$S_{13} = x_2x_3x_5z_4^2z_5\, y_2y_3y_4y_5w_1^2w_2^2D_{13}D_{15}D_{34} + x_2x_3x_5z_1^2z_5\, y_2^2y_4^2w_1w_2w_3w_5D_{13}D_{34}D_{45}, \tag{3.65}$$

$$S_{14} = x_2x_3x_5z_5\, y_3y_4y_5w_1w_2w_5D_{13}D_{24}(z_1^2\, y_4w_3D_{24} + z_4^2\, y_2w_1D_{13}), \tag{3.66}$$

$$S'_{15-16} = x_2x_3x_5z_1z_4z_5\, y_2y_4y_5w_1w_3w_5D_{12}D_{34}(x_2z_1D_{34}+y_2w_1D_{34}+x_3z_4D_{12}+y_4w_3D_{12}+s_1D_{12}D_{34}), \tag{3.67}$$

$$\begin{aligned} S_{17-19} = {}& x_2x_3z_4z_5\, y_2y_3y_4y_5w_1w_2w_5D_{13}D_{14}(-\,w_1D_{24}D_{35} + w_1x_2z_4D_{35} + w_1x_3z_5D_{24} + w_2D_{14}D_{35} + w_5D_{13}D_{24}) \\ &+ x_2x_3z_1z_5\, y_3y_4y_5w_1w_2w_3w_5D_{14}D_{24}(-\,y_4D_{13}D_{25} + x_3z_1y_4D_{25} + x_2z_5\, y_4D_{13} + y_3D_{14}D_{25} + y_5D_{13}D_{24}) \\ &+ x_2x_3z_1z_5\, y_2y_4y_5w_1w_2w_3w_5D_{14}D_{34}D_{35}(x_2z_1\, y_4 + y_2D_{14}), \end{aligned} \tag{3.68}$$

$$\begin{aligned} S'_{20-24} = {}& (x_2x_3x_5z_1z_4z_5\, y_2y_4y_5w_1^2w_3D_{24}D_{34}D_{35}+x_2^2x_3z_1z_4z_5\, y_2y_4y_5w_1w_3w_5D_{14}D_{34}D_{35} \\ &+x_2x_3x_5z_1z_4z_5\, y_2^2y_4w_1w_3w_5D_{13}D_{34}D_{45})(x_2z_1+y_2w_1) \\ &+x_2x_3x_5z_1^2z_4\, y_2y_3y_4w_1w_2w_5D_{35}D_{45}\left(-D_{13}D_{24}+x_2z_4D_{13}+y_4w_2D_{13}+\frac{z_4}{z_1}\, y_2w_1D_{13}+x_3z_1D_{24}+y_3w_1D_{24}\right) \\ &+x_2x_3x_5z_1^2z_4\, y_2y_3y_4w_1w_3w_5D_{25}D_{34}D_{45}(x_2z_1+y_2w_1) \\ &+x_2x_3x_5z_1^2z_4\, y_3y_4y_5w_1w_2w_3D_{24}D_{25}\left(-D_{15}D_{34}-\frac{z_4}{z_1}D_{13}D_{15}+x_3z_4D_{15}+y_4w_3D_{15}+\frac{z_4}{z_1}\, y_3w_1D_{15}+x_5z_1D_{34}+y_5w_1D_{34}\right) \\ &+x_2x_3x_5z_1z_4^2\, y_3y_4y_5w_1w_2w_3D_{13}D_{15}D_{25}(x_2z_4+y_4w_2) \\ &+x_2x_3x_5z_1^2z_4\, y_2y_4y_5w_1w_2w_3D_{34}D_{35}\left(-D_{15}D_{24}+x_2z_4D_{15}+y_4w_2D_{15}+\frac{z_4}{z_1}\, y_2w_1D_{15}+x_5z_1D_{24}+y_5w_1D_{24}\right), \end{aligned} \tag{3.69}$$

$$S_{25} = -\, x_2x_3z_5\, y_2y_3y_4y_5w_1w_2w_3w_5D_{13}D_{14}D_{24}(z_4D_{15} + z_1D_{45}), \tag{3.70}$$

$$S_{30} = s_2\, x_2x_3z_1z_4z_5\, y_3y_4y_5w_1w_2w_5D_{13}D_{14}D_{24}(x_2\, y_5w_3 + x_3\, y_2w_5), \tag{3.71}$$

$$S_{31} = -\, x_2x_3x_5z_1z_4z_5\, y_2y_3y_4y_5w_1^2w_2w_5D_{13}D_{24}D_{34}. \tag{3.72}$$

The difference between the deformed S on the 6+2 cuts and the proper numerator is then

$$\begin{aligned} & S\left(0\,,\,0\,,z_1,w_1,x_2,y_2,\,0\,,w_1+\frac{x_2z_1}{y_2},x_3,y_3,\,0\,,w_1+\frac{x_2z_1}{y_2},\,0\,,y_4,z_4,\,0\,,x_5,y_5,z_5,w_5\right) \\ & -\left(w_1+\frac{x_2z_1}{y_2}\right)^2 y_4w_1\,x_3z_1\,\frac{y_2(M_1y_2D_{34})+y_3M_2}{y_2^3} \\ = {}& x_2x_3x_5z_1^2z_4z_5\, y_3y_4^2y_5w_1^2w_5\,(x_3y_2-x_2y_3)\left(w_1+\frac{x_2z_1}{y_2}\right)^2\left[x_2z_4+(y_4-y_2)\left(w_1+\frac{x_2z_1}{y_2}\right)\right](s_5-1), \end{aligned} \tag{3.73}$$

to make this difference vanish we must take $s_5 = +1$ which agrees with [6].

Now we complete the determination of s_1, s_2, s_3, s_4, s_5 for all five non-rung-rule topologies at 5-loop.

4. Beyond 5-loop Order?

It is clear that for the 4- and 5-loop 4-particle amplituhedra we are no longer using the Mondrian diagrammatics, instead we use the purely amplituhedronic way to obtain the $d\log$ forms from positivity conditions simplified by external and internal cuts, which looks more like the traditional unitarity cuts. As discussed in the end of [4], it is appealing to generalize the Mondrian diagrammatics to include the non-Mondrian complexity. In [7] there is some kind of evidence about how the Mondrian DCI topologies can be related to non-Mondrian ones, and it would be interesting to prove those rules which determine the coefficients of non-rung-rule topologies from the amplituhedronic perspective. All the effort on discovering new rules and patterns should finally help us go beyond the current understanding of the 5-loop case, such as to explain the coefficient $+2$ of a special 6-loop DCI topology in [8] since we believe a simple integer coefficient must have a simple origin. The brute-force calculation merely using positivity conditions might be significantly simplified by clever observations, as what we have witnessed from the Mondrian diagrammatics at 3-loop and the positive cuts at 4- and 5-loop. After extracting sufficient deeper features of positivity conditions, we might even have a purely combinatoric understanding of the amplituhedron.

Still, the standard geometric way has a lot to be excavated beyond the current primitive level. When we use positive cuts to determine the coefficient of a particular DCI topology, this looks like "projecting" the entire amplituhedron onto a subspace that contains a subset of all boundaries, we then would like to get more intuition of its geometric interpretation. And why the DCI topologies must be planar, as a basis in what sense they are complete, how this completeness is related to the triangulation of amplituhedron, as well as what role dual conformal invariance plays in the geometric picture, are very vague so far while we believe clarification of these questions will be a significant progress. When searching for various novel formalisms and connections to mathematics, we will always pay most attention to the new features which can better aid the practical calculation of physical integrands at higher loop orders.

A. Details of the $d\log$ Form for Determining s_2, s_3, s_4

Below we derive the $d\log$ form for determining s_2, s_3, s_4, with respect to positivity conditions

$$\begin{aligned}
&\frac{z_2}{z_5+y_5w_5/x_5}+\frac{w_1}{w_5+x_5z_5/y_5}>1,\\
&\frac{x_3}{x_5+y_5w_5/z_5}+\frac{y_3}{y_5+x_5z_5/w_5}>1,\quad (z_5-z_2)\left(x_3+y_3\frac{w_2}{z_2}-x_5\right)+y_5(w_2-w_5)>0,\\
&\frac{z_4}{z_5+y_5w_5/x_5}+\frac{y_4}{y_5+x_5z_5/w_5}>1,\quad \frac{z_4}{z_2}+\frac{y_4}{y_3+x_3z_2/w_2}>1.
\end{aligned}\tag{A.1}$$

For later convenience, we define quantities

$$\begin{aligned}
&n_3=x_3+y_3\frac{w_5}{z_5}-x_5-\frac{y_5w_5}{z_5},\quad n_5=x_3+y_3\frac{w_2}{z_2}-x_5-y_5\frac{w_5-w_2}{z_5-z_2},\\
&p_3=y_5+\frac{x_5z_5}{w_5},\quad p_5=\frac{z_2}{w_2}\left(x_5+y_5\frac{w_5-w_2}{z_5-z_2}\right),\quad p_{35}=y_5\frac{z_2}{z_2-z_5},\\
&n_{24}=x_3-\frac{w_2}{z_2}\left(y_5+\frac{x_5z_5}{w_5}-y_3\right),
\end{aligned}\tag{A.2}$$

for the discussion involving y_3, x_3, as well as

$$\begin{aligned}
&a_2=z_5+\frac{y_5w_5}{x_5},\quad b_2=y_5+\frac{x_5z_5}{w_5},\quad a_4=z_2,\quad b_4=y_3+\frac{x_3z_2}{w_2},\quad z_4^*=\frac{b_4-b_2}{b_4/a_4-b_2/a_2},\\
&n_2=z_4\frac{b_2}{a_2}+y_4-b_2,\quad n_4=z_4\frac{b_4}{a_4}+y_4-b_4,\\
&A=\left(\frac{1}{z_4}-\frac{1}{z_4-z_4^*}\right)\frac{1}{n_4}+\left(\frac{1}{z_4-z_4^*}-\frac{1}{z_4-a_2}\right)\frac{1}{n_2}+\frac{1}{z_4-a_2}\frac{1}{y_4},\quad B=\frac{1}{z_4y_4}\frac{n_2+b_2}{n_2},\\
&F=\frac{1}{z_4y_4}\frac{n_4+b_4}{n_4},\quad G=\left(\frac{1}{z_4}-\frac{1}{z_4-z_4^*}\right)\frac{1}{n_2}+\left(\frac{1}{z_4-z_4^*}-\frac{1}{z_4-a_4}\right)\frac{1}{n_4}+\frac{1}{z_4-a_4}\frac{1}{y_4},
\end{aligned}\tag{A.3}$$

for the discussion involving z_4, y_4. We will also use identities

$$\begin{aligned}
&\frac{w_5}{z_5}-\frac{w_5-w_2}{z_5-z_2}=\frac{z_2}{z_2-z_5}\left(\frac{w_5}{z_5}-\frac{w_2}{z_2}\right),\\
&p_3-p_5=\frac{z_2z_5}{w_2w_5}\frac{x_5}{z_2-z_5}\left(\frac{w_5}{z_5}-\frac{w_2}{z_2}\right)\left(z_5+\frac{y_5w_5}{x_5}-z_2\right).
\end{aligned}\tag{A.4}$$

Now let's analyze all possible situations of variables $z_2, w_1, w_2, y_3, x_3, z_4, y_4$, by first separating situations $z_2<z_5$, $z_5<z_2<z_5+y_5w_5/w_5$ and $z_2>z_5+y_5w_5/w_5$.

A.1 $z_2<z_5$

For $z_2<z_5$, the 1st line of (A.1) in terms of w_1 is nontrivial. The 2nd condition in its 2nd line becomes

$$x_3+y_3\frac{w_2}{z_2}>x_5+y_5\frac{w_5-w_2}{z_5-z_2},\tag{A.5}$$

and for comparison we can rewrite the 1st condition in the same line as

$$x_3 + y_3 \frac{w_5}{z_5} > x_5 + y_5 \frac{w_5}{z_5}, \tag{A.6}$$

using the 1st identity in (A.4), for $w_2 < w_5 z_2/z_5$ we find

$$w_2 < w_5 \frac{z_2}{z_5} \Longrightarrow \frac{w_5}{z_5} < \frac{w_5 - w_2}{z_5 - z_2}. \tag{A.7}$$

For these two conditions in the 2nd line of (A.1), in terms of n_3 and n_5 defined in (A.2), we have a clear picture in the y_3-x_3 plane: the x_3-intercept of $n_3 = 0$ is less than that of $n_5 = 0$, while its slope is greater than that of $n_5 = 0$, therefore $n_5 > 0$ already implies $n_3 > 0$ in the 1st quadrant.

For the two conditions in the 3rd line of (A.1), in terms of n_2 and n_4 defined in (A.3), since $z_2 < z_5 < z_5 + y_5 w_5/w_5$ and

$$y_3 + x_3 \frac{z_2}{w_2} > y_3 + x_3 \frac{z_5}{w_5} > y_5 + x_5 \frac{z_5}{w_5}, \tag{A.8}$$

in the z_4-y_4 plane the y_4-intercept of $n_4 = 0$ is greater than that of $n_2 = 0$ while its z_4-intercept is less than that of $n_2 = 0$, so they intercept at $z_4 = z_4^*$ in the 1st quadrant. Its $d\log$ form is given by A, where z_4^* and A are defined in (A.3), of which the geometric picture is given in figure 17.

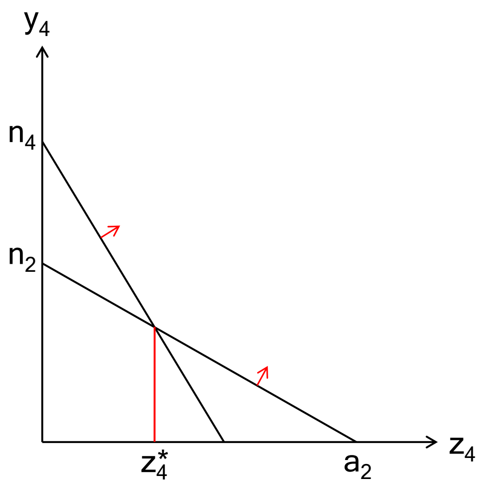

Figure 17: Geometric picture of the $d\log$ form A.

Now for $w_2 > w_5 z_2/z_5$, similarly we have

$$w_2 > w_5 \frac{z_2}{z_5} \Longrightarrow \frac{w_5}{z_5} > \frac{w_5 - w_2}{z_5 - z_2}, \tag{A.9}$$

therefore $n_3 > 0$ already implies $n_5 > 0$. Since

$$y_3 + x_3 \frac{z_2}{w_2} < y_3 + x_3 \frac{z_5}{w_5}, \tag{A.10}$$

we need n_{24} defined in (A.2) for comparing $y_3+x_3z_2/w_2$ and $y_5+x_5z_5/w_5$. If $y_3+x_3z_2/w_2<y_5+x_5z_5/w_5$, $n_2>0$ already implies $n_4>0$ in the z_4-y_4 plane, A will be replaced by B defined in (A.3), which involves n_2 only. This bifurcation divides the region of $n_3>0$ in the y_3-x_3 plane as shown in figure 18, in which p_3 defined in (A.2) is the y_3-intercept of both $n_3=0$ and $n_{24}=0$.

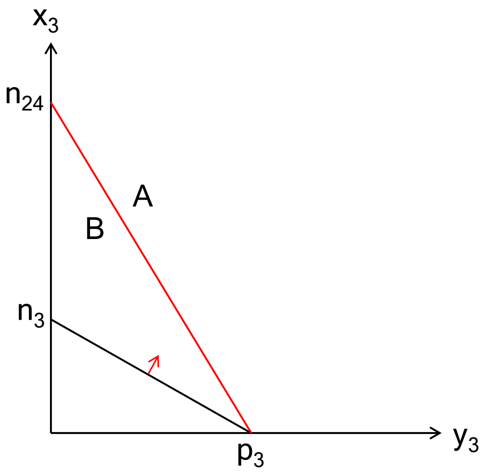

Figure 18: Bifurcation of $y_3+x_3z_2/w_2\lessgtr y_5+x_5z_5/w_5$ in the y_3-x_3 plane.

In summary, the $d\log$ form for $z_2<z_5$ is given by (omitting the part of z_2,w_1 for the moment)

$$\begin{aligned} S_1 =& \left(\frac{1}{w_2}-\frac{1}{w_2-w_5z_2/z_5}\right)\frac{1}{y_3x_3}\frac{x_3+y_3w_2/z_2}{n_5}A \\ &+\frac{1}{w_2-w_5z_2/z_5}\left[\left(\frac{1}{y_3}-\frac{1}{y_3-p_3}\right)\left(\left(\frac{1}{n_3}-\frac{1}{n_{24}}\right)B+\frac{1}{n_{24}}A\right)+\frac{1}{y_3-p_3}\frac{1}{x_3}A\right]. \end{aligned} \tag{A.11}$$

A.2 $z_5<z_2<z_5+y_5w_5/w_5$

For $z_5<z_2<z_5+y_5w_5/w_5$, the 1st line of (A.1) remains nontrivial. Its 2nd line becomes

$$x_3+y_3\frac{w_5}{z_5}>x_5+y_5\frac{w_5}{z_5}, \quad x_3+y_3\frac{w_2}{z_2}<x_5+y_5\frac{w_2-w_5}{z_2-z_5}, \tag{A.12}$$

using both identities in (A.4) we find (below p_5 defined in (A.2) is the y_3-intercept of $n_5=0$)

$$\begin{aligned} w_2\lessgtr w_5\frac{z_2}{z_5} &\Longrightarrow \frac{w_5}{z_5}\gtrless\frac{w_2-w_5}{z_2-z_5} \\ &\Longrightarrow p_3\gtrless p_5. \end{aligned} \tag{A.13}$$

If $w_2<w_5z_2/z_5$, both the x_3- and y_3-intercept of $n_3=0$ are greater than that of $n_5=0$, the regions of $n_3>0$ and $n_5<0$ have no overlap. Therefore only the $w_2>w_5z_2/z_5$ part contributes, for which both the x_3- and y_3-intercept of $n_3=0$ are less than that of $n_5=0$ as shown in figure 19. In this case, we again need n_{24} to divide its region, as the slope of $n_{24}=0$ is greater than that of $n_3=0$ ($n_{24}=0$ is parallel to $n_5=0$).

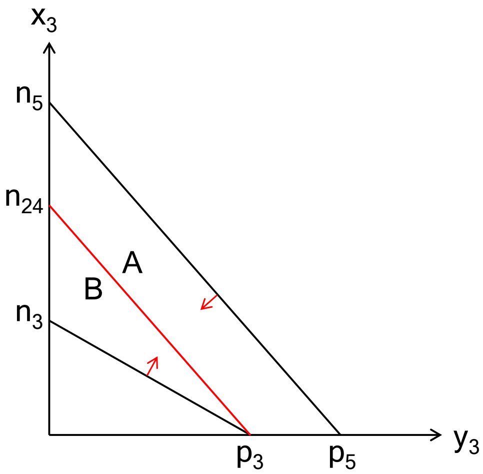

Figure 19: The only contributing part for $z_5 < z_2 < z_5 + y_5 w_5/w_5$, of which $w_2 > w_5 z_2/z_5$.

In summary, the $d\log$ form for $z_5 < z_2 < z_5 + y_5 w_5/w_5$ is given by

$$S_2 = \frac{1}{w_2 - w_5 z_2/z_5} \left[\left(\frac{1}{y_3} - \frac{1}{y_3 - p_3} \right) \left(\left(\frac{1}{n_3} - \frac{1}{n_{24}} \right) B + \left(\frac{1}{n_{24}} - \frac{1}{n_5} \right) A \right) + \left(\frac{1}{y_3 - p_3} - \frac{1}{y_3 - p_5} \right) \left(\frac{1}{x_3} - \frac{1}{n_5} \right) A \right]. \tag{A.14}$$

A.3 $z_2 > z_5 + y_5 w_5/w_5$

For $z_2 > z_5 + y_5 w_5/w_5$, the 1st line of (A.1) now becomes trivial. Its 2nd line remains the same as that for $z_5 < z_2 < z_5 + y_5 w_5/w_5$, but there is a slight difference in the 2nd identity in (A.4) as

$$\begin{aligned} w_2 \lessgtr w_5 \frac{z_2}{z_5} &\implies \frac{w_5}{z_5} \gtrless \frac{w_2 - w_5}{z_2 - z_5} \\ &\implies p_3 \lessgtr p_5, \end{aligned} \tag{A.15}$$

so that $n_3 = 0$ and $n_5 = 0$ always intercept, and its geometric pictures are given in figures 20 and 21 with respect to $w_2 \lessgtr w_5 z_2/z_5$. For $w_2 < w_5 z_2/z_5$ we again have

$$y_3 + x_3 \frac{z_2}{w_2} > y_3 + x_3 \frac{z_5}{w_5} > y_5 + x_5 \frac{z_5}{w_5}, \tag{A.16}$$

and since $z_2 > z_5 + y_5 w_5/w_5$, $n_4 > 0$ already implies $n_2 > 0$ in the z_4-y_4 plane. Its $d\log$ form is given by F defined in (A.3), which involves n_4 only. For $w_2 > w_5 z_2/z_5$, since $n_{24} = 0$ intercept $n_3 = 0$ at p_3 with $p_3 > p_5$ and $n_{24} = 0$ is parallel to $n_5 = 0$, $n_5 < 0$ already implies $n_{24} < 0$, which means

$$y_3 + x_3 \frac{z_2}{w_2} < y_5 + x_5 \frac{z_5}{w_5}, \tag{A.17}$$

and hence F will be replaced by G defined in (A.3), as it can be obtained from A by switching $n_2, a_2, b_2 \leftrightarrow n_4, a_4, b_4$.

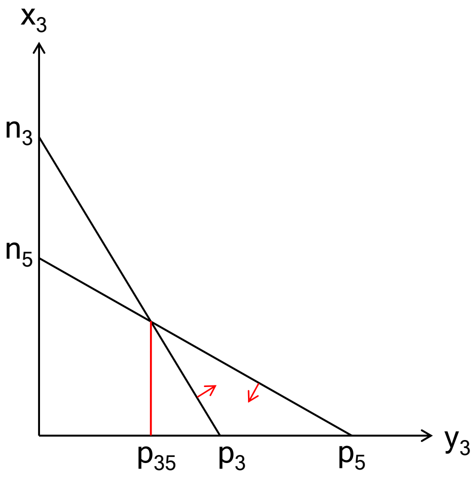

Figure 20: $n_3 = 0$ and $n_5 = 0$ intercept of which $w_2 < w_5 z_2/z_5$.

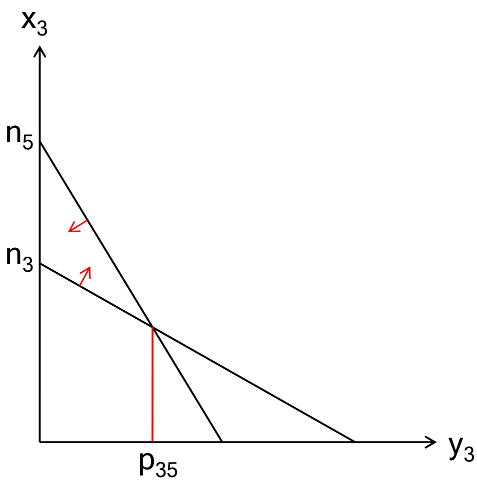

Figure 21: $n_3 = 0$ and $n_5 = 0$ intercept of which $w_2 > w_5 z_2/z_5$.

In summary, the $d\log$ form for $z_2 > z_5 + y_5 w_5/w_5$ is given by

$$S_3 = \left(\frac{1}{w_2} - \frac{1}{w_2 - w_5 z_2/z_5}\right)\left[\left(\frac{1}{y_3 - p_{35}} - \frac{1}{y_3 - p_3}\right)\left(\frac{1}{n_3} - \frac{1}{n_5}\right) + \left(\frac{1}{y_3 - p_3} - \frac{1}{y_3 - p_5}\right)\left(\frac{1}{x_3} - \frac{1}{n_5}\right)\right] F$$
$$+ \frac{1}{w_2 - w_5 z_2/z_5}\left(\frac{1}{y_3} - \frac{1}{y_3 - p_{35}}\right)\left(\frac{1}{n_3} - \frac{1}{n_5}\right) G. \tag{A.18}$$

Collecting S_1, S_2, S_3, the overall $d\log$ form is then

$$\begin{aligned}&\left[\left(\frac{1}{z_2}-\frac{1}{z_2-z_5}\right)S_1+\left(\frac{1}{z_2-z_5}-\frac{1}{z_2-z_5-y_5w_5/x_5}\right)S_2\right]\frac{1}{x_5z_2/y_5+w_1-x_5z_5/y_5-w_5}\\&+\frac{1}{z_2-z_5-y_5w_5/x_5}\frac{1}{w_1}S_3=\frac{M}{z_2^3w_1w_2y_3x_3z_4y_4\,D_{15}D_{35}D_{25}D_{45}D_{24}},\end{aligned}\tag{A.19}$$

where M is the numerator simplified by MATHEMATICA given in figure 14.

B. Details of the $d\log$ Form for Determining s_5

Below we present the $d\log$ form for determining s_5 with a brief description of its derivation, with respect to positivity conditions

$$\begin{aligned}&w_1+z_1\frac{x_5}{y_5}>w_5+z_5\frac{x_5}{y_5},\quad (y_5-y_2)\left(w_1+z_1\frac{x_2}{y_2}-w_5\right)+z_5(x_2-x_5)>0,\quad (y_5-y_3)\left(w_1+z_1\frac{x_2}{y_2}-w_5\right)+z_5(x_3-x_5)>0,\\&\frac{z_4}{z_5+y_5w_5/x_5}+\frac{y_4}{y_5+x_5z_5/w_5}>1,\quad \frac{z_4}{z_1+w_1y_2/x_2}+\frac{y_4}{y_2}>1,\quad \frac{z_4}{k(z_1+w_1y_2/x_2)}+\frac{y_4}{y_3}>1,\end{aligned}\tag{B.1}$$

where $k=y_3x_2/(x_3y_2)<1$. Recall that we focus on w_1, z_1 in the first line and z_4, y_4 in the second, so that the discussions can be done within two planes: the z_1-w_1 and the y_4-z_4 plane. For a clear picture, we can rewrite the 2nd and 3rd conditions in the 1st line as

$$\begin{aligned}w_1+z_1\frac{x_2}{y_2}&>w_5+z_5\frac{x_5-x_2}{y_5-y_2}\quad\text{for } y_2<y_5\\&<w_5+z_5\frac{x_2-x_5}{y_2-y_5}\quad\text{for } y_2>y_5,\end{aligned}\tag{B.2}$$

$$\begin{aligned}w_1+z_1\frac{x_2}{y_2}&>w_5+z_5\frac{x_5-x_3}{y_5-y_3}\quad\text{for } y_3<y_5\\&<w_5+z_5\frac{x_3-x_5}{y_3-y_5}\quad\text{for } y_3>y_5.\end{aligned}\tag{B.3}$$

We also have noticed that since $k<1$, if $y_3<y_2$ the 2nd condition in the 2nd line already implies the 3rd, which explains the factor D_{34} in the numerator of (3.48). There is another tricky issue depending on the relation between y_2 and y_3 as well, namely before we impose $w_2=w_3$ for setting $D_{23}=0$, we have

$$D_{23}=(y_3-y_2)(w_2-w_3),\tag{B.4}$$

so there is a bifurcation of $y_3\lessgtr y_2$ in the relevant dimensionless ratio

$$\frac{y_2}{y_2-y_3}\frac{w_3}{w_3-w_2}R_1+\frac{y_3}{y_3-y_2}\frac{w_2}{w_2-w_3}R_2\to\frac{\hat{w}_3}{D_{23}}(y_2R_1+y_3R_2)\tag{B.5}$$

after imposing $w_2=w_3=\hat{w}_3$, where R_1 and R_2 are proportional to M_1 and M_2 in (3.48) respectively which are the numerators simplified by MATHEMATICA as given in figures 22, 23, 24, 25, 26, 27, 28, 29 and 30.

As indicated above, it is better to separately consider situations $y_3 < y_2 < y_5$, $y_3 < y_5 < y_2$, $y_5 < y_3 < y_2$, $y_2 < y_3 < y_5$, $y_2 < y_5 < y_3$ and $y_5 < y_2 < y_3$ first, then depending on each case we may need to discuss various situations involving x_5, x'_3, w_5 as well. For example, to compare x_5/y_5 and x_2/y_2 involves x_5. And in the identity below which will be frequently used in the relevant discussions

$$\frac{x_5 - x_2}{y_5 - y_2} - \frac{x_5 - x_3}{y_5 - y_3} = \frac{y_2 - y_3}{(y_5 - y_2)(y_5 - y_3)} \left(x_5 + x'_3 \frac{y_5 - y_2}{y_2 - y_3} - x_2 \frac{y_5}{y_2} \right), \tag{B.6}$$

both x_5 and x'_3 are involved. Finally in the 2nd line of (B.1), to compare $y_5 + x_5 z_5 / w_5$, y_2 and y_3 may also involve w_5 given a fixed order of y_2, y_3, y_5.

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```
M1 = w1^4 y2^3 y4 (y3 - y5) y5^2 (w5 (y2 - y4) - x5 z4) +
  w1^3 y2^2 y5 (-2 w5^2 y2 (y2 - y4) y4 (y3 - y5) y5 +
    x5 z4 (x3 y2 y4 y5 z5 - x2 (y3 - y5) (3 y4 y5 z1 + y2 y5 z4 - y2^2 z5 + y2 y4 z5) +
      x5 y2 y4 (y5 (z1 - 2 z5) + y3 (-z1 + z5))) + w5 y4
     (x3 y2 (-y2 + y4) y5 z5 - x2 (y3 - y5) (3 y4 y5 z1 + y2 (y5 (-3 z1 + z4) + y4 z5)) +
      x5 y2 (y5 (y4 z1 - 2 y5 z4 - 2 y4 z5) + y3 (-y4 z1 + 2 y5 z4 + y4 z5) +
        y2 (y3 z1 - y5 z1 - y3 z5 + 2 y5 z5)))) -
  x2 x5 z1 (w5^3 y2^2 y3 (y2 - y5) (-y4 + y5) (y4 z1 + y2 z4) + x2 x5 y5 z1 z4
     (y4 z1 + y2 (z4 - z5)) (x2 (y3 - y5) z1 + (-x3 + x5) y2 z5) +
    w5^2 y2 (x2 z1 (y4 y5 (y4 y5 + y3 (-2 y4 + y5)) z1 + y2^2 y3 (y4 - y5) z4 +
        y2 (-y5^2 (y4 z1 - y4 z4 + y5 z4) + y3 (y4^2 z1 - 2 y4 y5 z4 + 2 y5^2 z4))) -
      y2 (y4 z1 + y2 z4) (x3 y2 (y4 - y5) z5 - x5 y3
         (y5 z4 + y4 z5 - 2 y5 z5 + y2 (-z4 + z5)))) +
    w5 (-x2^2 (y3 - y5) y5 z1^2 (-y4^2 z1 + y2 (y4 (z1 - z4) + y5 z4)) +
      x5 y2^2 (-x3 y2 + x5 y3) (y4 z1 + y2 z4) (z4 - z5) z5 + x2 y2 z1
       (x3 y5 (-y4^2 z1 + y2 (y4 z1 - y4 z4 + y5 z4)) z5 + x5 (y2^2 y3 z4 (z4 - z5) + y4
            y5 z1 (y5 z4 + y4 z5 + y3 (-2 z4 + z5)) + y2 (y3 (y4 z1 - 2 y5 z4)
              (z4 - z5) + y5 (y5 z4 (z4 - 2 z5) + y4 (-z1 + z4) z5)))))) +
  w1 (w5^3 y2^2 y4 (x5 y2 y3 (y2 - y5) (y4 - y5) z1 - x2 (y3 - y5) y5^2
       (y4 z1 + y2 (-z1 + z4))) + x2 x5 y5 z4
     (-x2^2 (y3 - y5) z1 (y4 z1 (y5 z1 + y2 z5) + y2 (y5 z1 z4 + y2 (-z1 + z4) z5)) -
      (x3 - x5) x5 y2^2 z5 (y4 z1 (-2 z1 + z5) + y2 (z4 z5 + z1 (-z4 + z5))) +
      x2 y2 (x3 z5 (y4 z1 (y5 z1 + y2 z5) + y2 (y5 z1 z4 + y2 (-z1 + z4) z5)) +
        x5 (y4 z1 (y5 z1 (3 z1 - 2 z5) - y2 z5^2) + y2
           (y2 (z1 - z4) z5^2 + 2 y5 z1 (z1 z4 - z1 z5 - z4 z5)) + y3 z1
           (y4 z1 (-3 z1 + z5) + y2 (-2 z1 z4 + 2 z1 z5 + z4 z5))))) -
    w5 (x5^2 y2^3 (-x3 y2 + x5 y3) y4 z1 (z4 - z5) z5 + x2^3 y4 (y3 - y5) y5
       z1 (y4 y5 z1^2 + y2^2 z4 z5 + y2 z1 (y5 (-z1 + z4) + y4 z5)) -
      x2^2 y2 y5 (x3 y4 z5 (y4 y5 z1^2 + y2^2 z4 z5 + y2 z1 (y5 (-z1 + z4) + y4 z5)) -
        x5 (y2^2 z4 z5 (y3 (z1 - z4) + y5 (-z1 + z4) + y4 z5) + y4 z1^2 (y3 (3 y4 z1 -
               2 y5 z4 - y4 z5) + y5 (-3 y4 z1 + 2 y5 z4 + 2 y4 z5)) + y2 z1
           (2 y5^2 z4 (z1 + z4) + y4^2 z5^2 + y4 y5 (3 z1^2 + 3 z4 z5 - 2 z1 (z4 + z5)) +
            y3 (-2 y5 z4 (z1 + z4) + y4 (-3 z1^2 + 2 z1 z4 + z1 z5 - 2 z4 z5))))) +
      x2 x5 y2^2 (x3 y5 z5 (y4 z1 (y5 z4 + y4 (-2 z1 + z5)) + y2
           (y5 z4 (z1 + z4) + y4 (2 z1^2 + z4 z5 - z1 (z4 + z5)))) +
        x5 (y2^2 y3 z1 z4 (z4 - z5) + y4 y5 z1 (2 y5 z4 (z1 - z5) + y4 (2 z1 - z5) z5 +
            y3 (-4 z1 z4 + 2 z1 z5 + z4 z5)) + y2 (y3 (2 y4 z1^2 (z4 - z5) +
              y5 z4 (-2 z1 z4 + 2 z1 z5 + z4 z5)) + y5 (y5 z4 (z1 (z4 - 2 z5) -
                2 z4 z5) + y4 z5 (-2 z1^2 - z4 z5 + z1 (z4 + z5))))))) -
    w5^2 y2 (-x2^2 y4 (y3 - y5) y5 (2 y4 y5 z1^2 + y2^2 z4 z5 + y2 z1
         (-2 y5 z1 + 2 y5 z4 + y4 z5)) + x5 y2^2 y4 z1
       (x3 y2 (-y4 + y5) z5 + x5 y3 (y5 z4 + y4 z5 - 2 y5 z5 + y2 (-z4 + z5))) +
      x2 y2 (x3 y4 y5^2 (y4 z1 + y2 (-z1 + z4)) z5 +
        x5 (y2^2 y3 (y4 - y5) z1 z4 + y4 y5 z1 (y3 (-4 y4 z1 + 2 y5 z1 +
               y5 z4 + y4 z5) - y5 (-2 y4 z1 + y5 z4 + 2 y4 z5)) + y2
```

Figure 22: Numerator M_1 simplified by MATHEMATICA: part 1/2.

```
                  (y3 (2 y4^2 z1^2 + y5^2 z4 (2 z1 + z4) + y4 y5 (z4 z5 - z1 (2 z4 + z5))) -
                    y5^2 (y5 z4 (z1 + z4) + y4 (2 z1^2 + 2 z4 z5 - z1 (z4 + 2 z5))))))) +
  w1^2 y2 (w5^3 y2^2 (y2 - y4) y4 (y3 - y5) y5^2 - w5^2 y2 y4 (y5 (x3 y2 (-y2 + y4) y5 z5 -
          x2 (y3 - y5) (4 y4 y5 z1 + y2 (-4 y5 z1 + 2 y5 z4 + y4 z5))) +
        x5 y2 (y2 (-y5^2 (z1 - 2 z5) + y3 (y4 z1 - y5 z5)) +
          y5 (y5 (-y5 z4 + y4 (z1 - 2 z5)) + y3 (y5 (z1 + z4) + y4 (-2 z1 + z5))))) +
    x5 y5 z4 ((x3 - x5) x5 y2^2 y4 (z1 - z5) z5 - x2^2 (y3 - y5)
         (y4 z1 (3 y5 z1 + 2 y2 z5) + y2 (2 y5 z1 z4 + y2 (-2 z1 + z4) z5)) +
       x2 y2 (x3 z5 (2 y4 y5 z1 + y2 y5 z4 - y2^2 z5 + y2 y4 z5) + x5 (y4 (y5 z1
                 (3 z1 - 4 z5) - y2 z5^2) + y2 (y2 z5^2 + y5 (z1 z4 - z1 z5 - 2 z4 z5)) +
             y3 (y4 z1 (-3 z1 + 2 z5) + y2 (-z1 z4 + z1 z5 + z4 z5))))) -
    w5 (x2^2 y4 (y3 - y5) y5 (3 y4 y5 z1^2 + y2^2 z4 z5 + y2 z1 (-3 y5 z1 + 2 y5 z4 +
             2 y4 z5)) + x5 y2^2 y4 (x3 y5 z5 (y5 z4 + y2 (z1 - z5) + y4 (-z1 + z5)) +
          x5 (y2 (y3 z1 (z4 - z5) + y5 z5 (-z1 + z5)) + y5 (y5 z4 (z1 - 2 z5) +
                y4 (z1 - z5) z5 + y3 (z4 z5 + z1 (-2 z4 + z5))))) -
       x2 y2 y5 (x3 y4 z5 (2 y4 y5 z1 + y2 (y5 (-2 z1 + z4) + y4 z5)) -
          x5 (y2^2 (y3 - y5) z4 z5 + y4 z1 (y3 (3 y4 z1 - 4 y5 z4 - 2 y4 z5) +
                y5 (-3 y4 z1 + 4 y5 z4 + 4 y4 z5)) + y2
              (y5^2 z4 (z1 + 2 z4) + y4^2 z5^2 + y4 y5 (3 z1^2 + 3 z4 z5 - z1 (z4 + 4 z5)) -
                y3 (y5 z4 (z1 + 2 z4) + y4 (3 z1^2 + 2 z4 z5 - z1 (z4 + 2 z5)))))))));
```

Figure 23: Numerator M_1 simplified by MATHEMATICA: part 2/2.

```
M2 = w1^5 y2^4 (y2 - y4) y4 (y2 - y5) y5^2 (w5 (-y3 + y4) + x5 z4) +
  w1^4 y2^3 y5 (2 w5^2 y2 (y2 - y4) (y3 - y4) y4 (y2 - y5) y5 +
    w5 y4 (x3 y2 (y2 - y4) y5 (-y5 z4 - y4 z5 + y2 (z4 + z5)) +
       x5 y2 (y2 - y4) (y5 (-y4 z1 + 2 y5 z4 + y3 (z1 - 2 z5) + 2 y4 z5) +
          y2 (y4 z1 - 2 y5 z4 - y4 z5 + y3 (-z1 + z5))) + x2 (4 y4 (-y3 + y4) y5^2 z1 +
          y2^2 (y4 y5 (4 z1 - z4 - z5) + y3 (-4 y5 z1 + y5 z4 + y4 z5)) +
          y2 (-y3 (-4 y4 y5 z1 + y5^2 (-4 z1 + z4) + y4^2 z5) + y4
              y5 (y5 (-4 z1 + z4) + y4 (-4 z1 + z5))))) +
    x5 z4 (x3 y2 y4 y5 (-y2 z4 + y5 z4 - y3 z5 + y4 z5) + x5 y2 (y2 - y4)
         y4 (y2 z1 - y5 z1 - y2 z5 + 2 y5 z5) +
       x2 (4 y4^2 y5^2 z1 + y2^2 (y3 (y5 z4 - y3 z5) + y4 (4 y5 z1 - y5 z4 + y3 z5 - y5 z5)) +
          y2 (y3 y5 (-y5 z4 + y3 z5) + y4 (y5^2 (-4 z1 + z4) + y3^2 z5 - y3 y5 z5) +
             y4^2 (-y3 z5 + y5 (-4 z1 + z5)))))) +
  x2 x5 z1 (-w5^3 y2^3 (y3 - y5) (-y4 + y5) (x3 y2 z4 (y4 z1 + y2 z4) +
       x2 z1 (y4^2 z1 - y2 (y4 z1 + y3 z4 - y4 z4))) -
    x2 x5 y5 z1 z4 (-x3 (x3 - x5) y2^2 z4 (y4 z1 + y2 (z4 - z5)) z5 +
       x2^2 z1 (-y2^2 (y4 (z1 - z4) + y3 (z4 - z5)) (z1 - z5) + y2 (y4^2 z1 (z1 - z5) + y4
               (z1 - z4) (y5 z1 - y3 z5) + y3 (z4 - z5) (y5 z1 - y3 z5)) +
          y4 z1 (y3 (-y3 + y5) z5 + y4 (-y5 z1 + y3 z5))) + x2 y2 (x3 (y4 z1 + y2
               (z4 - z5)) (-y5 z1 z4 + y2 z4 (z1 - z5) + (-y4 z1 + y3 (z1 + z4)) z5) +
          x5 z1 z5 (y4 (y4 z1 - y3 z5) + y2 (y4 (-z1 + z4) + y3 (-z4 + z5))))) -
    w5 (-x3 (x3 - x5) x5 y2^4 z4 (y4 z1 + y2 z4) (z4 - z5) z5 +
       x2 y2^2 z1 (x3^2 y5 z4 (-y4^2 z1 + y2 (y4 (z1 - z4) + y5 z4)) z5 -
```

Figure 24: Numerator M_2 simplified by MATHEMATICA: part 1/7.

```
x5^2 y2 (-y4^2 z1 + y2 (y4 (z1 - z4) + y3 z4)) (z4 - z5) z5 +
x3 x5 (y4 y5 z1 z4 (y5 z4 + y4 z5) + y2 (y5^2 z4^2 (z4 - 2 z5) + y3 y4 z1 z4
    (z4 - z5) + y4 y5 z4^2 (-2 z1 + z5) + y4^2 z1 z5 (-z4 + z5)) + y2^2
  (z4 - z5) (-2 y5 z4^2 + y4 (z1 - z4) z5 + y3 z4 (z4 + z5)))) + x2^3 y5 z1^2
(y2^2 (y3 (y4 z1 (z1 - z4) + y5 z4 (z1 - z5)) - y4^2 (z1 - z4) (z1 - z5)) +
  y4 z1 (-y4^2 y5 z1 + y3 (y4 y5 z1 + y5^2 z4 + y4^2 z5)) +
  y2 (y4^2 z1 (y5 (z1 - z4) + y4 (z1 - z5)) + y3^2 y5 z4 z5 - y3
    (y4 y5 z1^2 + y5^2 z1 z4 + y4^2 (z1^2 + z1 z5 - z4 z5)))) +
x2^2 y2 z1 (x5 z1 (y2^2 (-y3 (y4 z1 - y4 z4 - 2 y5 z4) (z4 - z5) +
    y4 y5 (z1 - z4) (2 z4 - z5) + y3^2 z4 (-z4 + z5)) + y4
    y5 (y4^2 z1 z5 - 2 y3 y5 z4 z5 + y4 z1 (y5 z4 - y3 z5)) + y2
    (y3 (-y5^2 z4 (z4 - 2 z5) + y4^2 z1 (z4 - z5) + y4 y5 z1 z5) +
     y4 y5 z4 (y5 (-z1 + z4) + y4 (-2 z1 + z5)))) +
  x3 y5 (-y4^2 z1 (y5 z1 z4 + y4 z1 z5 - y3 z4 z5) - y2^2
    (y5 z4^2 (z1 - z5) + y4 (z1 - z4) (-z4 z5 + z1 (z4 + z5))) + y2
    (y5 z4 (y5 z1 z4 - y3 (z1 + z4) z5) + y4 z4 (y3 (-z1 + z4) z5 +
       y5 z1 (z1 - z4 + z5)) + y4^2 z1 (-2 z4 z5 + z1 (z4 + 2 z5)))))) +
w5^2 y2 (x2^2 z1^2 (y4 y5^2 (y3 y4 z1 - y4^2 z1 + y3 y5 z4) + y2^2 (-y4 (2 y4 - y5) y5
    (z1 - z4) + y3^2 (y4 - y5) z4 + y3 (y4^2 (z1 - z4) - y4 y5 z4 + 2 y5^2 z4)) -
  y2 (y4^2 y5 (-2 y4 z1 + y5 z4) + y3 (y4^3 z1 + y4 y5^2 z1 + y5^3 z4))) -
 x3 y2^3 z4 (y4 z1 + y2 z4) (x3 (-y4 + y5) z5 +
  x5 (y5 z4 + y4 z5 - 2 y5 z5 + y3 (-z4 + z5))) -
 x2 y2 z1 (-x5 y2 (-y4^2 z1 + y2 (y4 (z1 - z4) + y3 z4)) (y5 (z4 - 2 z5) +
    y4 z5 + y3 (-z4 + z5)) + x3 (y4^2 y5^2 z1 z4 + y2 (y3 y4 (y4 - y5) z1 z4 +
     y4 y5^2 z4^2 - y5^3 z4^2 - y4^3 z1 z5 + y4^2 y5 z1 (-2 z4 + z5)) +
    y2^2 (y4^2 (z1 - z4) z5 + y5 z4 (2 y5 z4 - y3 (z4 + z5)) +
     y4 (y3 z4 (z4 + z5) + y5 (z1 (z4 - z5) + z4 (-2 z4 + z5))))))) +
w1^3 y2^2 (w5^3 y2^2 (y2 - y4) y4 (-y3 + y4) (y2 - y5) y5^2 + w5^2 y2 y4
 (x5 y2 (y2 - y4) (-y5^2 (-y4 z1 + y5 z4 + y3 (z1 - 2 z5) + 2 y4 z5) +
   y2 (y3 (y4 z1 - y5 z5) + y5 (y5 (z1 + z4) + y4 (-2 z1 + z5)))) -
  y5 (x3 y2 (y2 - y4) y5 (-2 y5 z4 - y4 z5 + y2 (2 z4 + z5)) +
   x2 (6 y4 (-y3 + y4) y5^2 z1 + y2^2 (y4 y5 (6 z1 - 2 z4 - z5) + y3
       (-6 y5 z1 + 2 y5 z4 + y4 z5)) + y2 (y4 y5 (-6 y4 z1 - 6 y5 z1 + 2 y5
         z4 + y4 z5) + y3 (6 y4 y5 z1 + 6 y5^2 z1 - 2 y5^2 z4 - y4^2 z5))))) +
 x5 y5 z4 (y2^2 y4 (x3^2 y5 z4 z5 + x5^2 (y2 - y4) (z1 - z5) z5 + x3 x5
     (y5 z4 (z1 - 2 z5) - (y3 - y4) (z1 - z5) z5 + y2 z4 (-z1 + z5))) +
  x2^2 (6 y4^2 y5^2 z1^2 - 3 y2 z1 (y3 y5 (y5 z4 - y3 z5) + y4^2
      (2 y5 z1 + y3 z5 - y5 z5) + y4 (y5^2 (2 z1 - z4) - y3^2 z5 + y3 y5 z5)) +
   y2^2 (y3 (y5 z4 (3 z1 - z5) + y3 (-3 z1 + z4) z5) + y4
      (y3 (3 z1 - z4) z5 + y5 (6 z1^2 + z4 z5 - 3 z1 (z4 + z5))))) +
  x2 y2 (x3 (3 y4 y5 z1 (y5 z4 + (-y3 + y4) z5) + y2^2 (-y5 z4^2 +
       z5 (-y4 z5 + y3 (z4 + z5))) + y2 (y4^2 z5^2 + y5 z4 (y5 z4 - 2 y3 z5) -
       y4 (y5 z4 (3 z1 - 2 z5) + y3 z5 (z4 + z5)))) +
   x5 (y4 z1 (y3 (y3 - y5) z5 + y4 (4 y5 z1 - y3 z5 - 6 y5 z5)) +
     y2^2 (y3 (z1 (z4 - z5) - z4 z5) +
       y4 (4 z1^2 + z5 (z4 + z5) - z1 (z4 + 4 z5))) + y2 (-y4^2 (-2 z1 + z5)^2 +
       y4 (y3 z5 (z1 + z5) + y5 (-4 z1^2 + z1 z4 + 6 z1 z5 - 2 z4 z5)) +
```

Figure 25: Numerator M_2 simplified by MATHEMATICA: part 2/7.

$$
\begin{aligned}
&y3\,(-y3\,z5^{2}+y5\,(-z1\,z4+z1\,z5+2\,z4\,z5)))))\,+\\
&w5\,(x2^{2}\,y4\,y5\,(6\,y4\,(-y3+y4)\,y5^{2}\,z1^{2}+3\,y2\,z1\,(-y3\,(-2\,y4\,y5\,z1+y5^{2}\,(-2\\
&z1+z4)+y4^{2}\,z5)+y4\,y5\,(y5\,(-2\,z1+z4)+y4\,(-2\,z1+z5)))\,+\\
&y2^{2}\,(y3^{2}\,z4\,z5+y4\,y5\,(6\,z1^{2}+z4\,z5-3\,z1\,(z4+z5))-y3\\
&(y4\,(-3\,z1+z4)\,z5+y5\,(6\,z1^{2}-3\,z1\,z4+z4\,z5))))\,+\\
&y2^{2}\,y4\,(x3^{2}\,(-y2+y4)\,y5^{2}\,z4\,z5+x5^{2}\,(y2-y4)\\
&(y5\,(y5\,z4\,(z1-2\,z5)-(y3-y4)\,(z1-z5)\,z5)+y2\\
&(y3\,z1\,(z4-z5)+y5\,(-2\,z1\,z4+z1\,z5+z4\,z5)))\,+\\
&x3\,x5\,y5\,(y4\,y5\,z4\,(z1-3\,z5)+y4^{2}\,(z1-z5)\,z5+y2^{2}\,(z1-z5)\\
&(z4+z5)+y5\,z4\,(-2\,y5\,z4+y3\,z5)+y2\\
&(-y4\,(z1-z5)\,(z4+2\,z5)+y5\,z4\,(-z1+2\,(z4+z5)))))\,-\\
&x2\,y2\,y5\,(x3\,y4\,(-3\,y4\,y5\,z1\,(y5\,z4+y4\,z5)+y2^{2}\,(y4\,z5^{2}\,-\\
&y5\,(3\,z1-z4)\,(z4+z5))+y2\,(y5^{2}\,(3\,z1-z4)\,z4-y4^{2}\,z5^{2}\,+\\
&y4\,(3\,y5\,z1\,z4+6\,y5\,z1\,z5+y3\,z4\,z5-2\,y5\,z4\,z5)))\,+\\
&x5\,(y4\,z1\,(2\,y4\,y5\,(-2\,y4\,z1+3\,y5\,z4+3\,y4\,z5)+y3\,(4\,y4\,y5\,z1\,+\\
&y5^{2}\,z4+y4^{2}\,z5-6\,y4\,y5\,z5))+y2\,(y3^{2}\,(y4+y5)\,z4\,z5\,-\\
&y3\,(y5^{2}\,z4\,(z1+2\,z4)+y4^{2}\,(4\,z1^{2}-2\,z1\,z5+z4\,z5)\,+\\
&y4\,y5\,(4\,z1^{2}-6\,z1\,z5+3\,z4\,z5))+y4\,(2\,y5^{2}\,z4\,(-3\,z1+z4)\,+\\
&y4^{2}\,(-2\,z1+z5)^{2}+y4\,y5\,(4\,z1^{2}-7\,z1\,z4-6\,z1\,z5+3\,z4\,z5)))\,-\\
&y2^{2}\,(y3^{2}\,z4\,z5-y3\,(y5\,z4\,(z1+2\,z4)+y4\,(4\,z1^{2}+2\,z4\,z5\,-\\
&z1\,(z4+3\,z5)))+y4\,(y5\,z4\,(-6\,z1+2\,z4+z5)\,+\\
&y4\,(4\,z1^{2}+z5\,(z4+z5)-z1\,(z4+4\,z5))))))))\,+\\
&w1^{2}\,y2\,(w5^{3}\,y2^{2}\,y4\,(x5\,y2^{2}\,(y2-y4)\,(y4-y5)\,(-y3+y5)\,z1+(y2-y5)\,y5^{2}\\
&(x3\,y2\,(y2-y4)\,z4+x2\,(y3-y4)\,(2\,y4\,z1+y2\,(-2\,z1+z4))))\,+\\
&x5\,y5\,z4\,(x3\,(x3-x5)\,x5\,y2^{3}\,y4\,z4\,(z1-z5)\,z5+x2^{3}\,z1\\
&(4\,y4^{2}\,y5^{2}\,z1^{2}+y2\,z1\,(3\,y3\,y5\,(-y5\,z4+y3\,z5)+y4^{2}\,(-4\,y5\,z1-3\,y3\,z5\,+\\
&3\,y5\,z5)+y4\,(y5^{2}\,(-4\,z1+3\,z4)+3\,y3^{2}\,z5-3\,y3\,y5\,z5))\,+\\
&y2^{2}\,(y3\,(y5\,z4\,(3\,z1-2\,z5)+y3\,(-3\,z1+2\,z4)\,z5)+y4\\
&(y3\,(3\,z1-2\,z4)\,z5+y5\,(4\,z1^{2}+2\,z4\,z5-3\,z1\,(z4+z5)))))\,+\\
&x2\,y2^{2}\,(x3^{2}\,z4\,z5\,(2\,y4\,y5\,z1+y2\,y5\,z4-y2^{2}\,z5+y2\,y4\,z5)\,-\\
&x3\,x5\,(y2^{2}\,z4\,(z1\,(z4-z5)-z4\,z5)+y4\,z1\\
&(z5\,(-3\,y4\,z1+y3\,(3\,z1+z4-2\,z5)+2\,y4\,z5)+y5\,(-3\,z1\,z4\,+\\
&4\,z4\,z5))+y2\,(y4\,(3\,z1^{2}\,z4+2\,z4\,z5^{2}+z1\,z5\,(-4\,z4+z5))\,+\\
&(y5\,z4-y3\,z5)\,(2\,z4\,z5+z1\,(-z4+z5))))+x5^{2}\,z5\\
&(y4\,z1\,(-3\,y4\,z1+y3\,z5+2\,y4\,z5)+y2\,(y3\,(z1\,(z4-z5)-z4\,z5)\,+\\
&y4\,(3\,z1^{2}+z4\,z5-z1\,(z4+2\,z5)))))\,+\\
&x2^{2}\,y2\,(x3\,(3\,y4\,y5\,z1^{2}\,(y5\,z4+(-y3+y4)\,z5)+y2^{2}\,(y5\,z4^{2}\,(-2\,z1+z5)\,+\\
&(2\,z1-z4)\,z5\,(-y4\,z5+y3\,(z4+z5)))+y2\,z1\,(2\,y4^{2}\,z5^{2}+2\,y5\,z4\\
&(y5\,z4-2\,y3\,z5)-y4\,(y5\,z4\,(3\,z1-4\,z5)+2\,y3\,z5\,(z4+z5))))\,+\\
&x5\,(3\,y4\,z1^{2}\,(y3\,(y3-y5)\,z5+y4\,(2\,y5\,z1-y3\,z5-2\,y5\,z5))\,+\\
&y2^{2}\,(y4\,(2\,z1-z4)\,(3\,z1^{2}-3\,z1\,z5+z5^{2})\,+\\
&y3\,(3\,z1^{2}\,(z4-z5)+z4\,z5^{2}+z1\,z5\,(-3\,z4+z5)))+y2\,z1\\
&(-2\,y4^{2}\,(3\,z1^{2}-3\,z1\,z5+z5^{2})+y4\,(y3\,z5\,(3\,z1-z4+2\,z5)\,+\\
&y5\,(-6\,z1^{2}+3\,z1\,z4+6\,z1\,z5-4\,z4\,z5))\,+\\
&y3\,(y3\,(z4-3\,z5)\,z5+y5\,(-3\,z1\,z4+3\,z1\,z5+4\,z4\,z5)))))\,-\\
&w5^{2}\,y2\,(x2^{2}\,y4\,y5\,(6\,y4\,(-y3+y4)\,y5^{2}\,z1^{2}+2\,y2\,z1\,(y3\,(3\,y4\,y5\,z1+y5^{2}\,(3\,z1\,-\\
&2\,z4)-y4^{2}\,z5)+y4\,y5\,(-3\,y5\,z1+2\,y5\,z4+y4\,(-3\,z1+z5)))\,+
\end{aligned}
$$

Figure 26: Numerator M_2 simplified by MATHEMATICA: part 3/7.

```
        y2^2 (y3^2 z4 z5 + y4 y5 (6 z1^2 + z4 z5 - 2 z1 (2 z4 + z5)) - y3
          (y4 (-2 z1 + z4) z5 + y5 (6 z1^2 - 4 z1 z4 + z4 z5)))) +
      y2^2 y4 (x3^2 (-y2 + y4) y5^2 z4 z5 - x5^2 y2 (y2 - y4) z1 (y5 (z4 - 2 z5) +
            y4 z5 + y3 (-z4 + z5)) + x3 x5 (y5^2 z4 (-y5 z4 + y4 (z1 - 2 z5)) + y2^2
            (y4 z1 z5 + y5 (z1 (z4 - z5) - z4 z5)) + y2 (y3 (y4 - y5) z1 z4 -
              y4^2 z1 z5 + y5^2 z4 (z4 + 2 z5) + y4 y5 (-2 z1 z4 + z1 z5 + z4 z5)))) +
      x2 y2 (x5 (y4 y5^2 z1 (y4 (3 y4 z1 - 2 y5 z4 - 4 y4 z5) - y3 (3 y4 z1 + y5 z4 - 4
                    y4 z5)) + y2 (y4 y5 (y5^2 (2 z1 - z4) z4 + 2 y4^2 z1 (-3 z1 + z5) +
                  y4 y5 (3 z1 z4 + 4 z1 z5 - 2 z4 z5)) + y3 (3 y4^3 z1^2 + y5^3 z4 (z1 +
                      z4) - 2 y4^2 y5 z1 z5 + y4 y5^2 (3 z1^2 - 4 z1 z5 + 2 z4 z5))) +
            y2^2 (y3^2 (-y4 + y5) z1 z4 + y3 (y4^2 z1 (-3 z1 + z4) - y5^2 z4
                  (2 z1 + z4) + y4 y5 (z1 z4 + 2 z1 z5 - z4 z5)) + y4 y5
                (y5 (-3 z1^2 - z1 z4 + z4^2) + y4 (6 z1^2 + z4 z5 - 2 z1 (z4 + z5))))) +
        x3 y4 y5 (y2^2 y5 (2 z1 - z4) (2 z4 + z5) + 2 y4 y5 z1
          (2 y5 z4 + y4 z5) - y2 (2 y5^2 (2 z1 - z4) z4 +
            y4 (y3 z4 z5 + y5 (-2 z4 z5 + 4 z1 (z4 + z5))))))) +
   w5 (x2^3 y4 y5 z1 (4 y4 (-y3 + y4) y5^2 z1^2 + y2 z1 (y4 y5 (-4 y4 z1 - 4 y5 z1 +
            3 y5 z4 + 3 y4 z5) + y3 (4 y4 y5 z1 + y5^2 (4 z1 - 3 z4) - 3 y4^2 z5)) +
        y2^2 (2 y3^2 z4 z5 + y4 y5 (4 z1^2 + 2 z4 z5 - 3 z1 (z4 + z5)) + y3
          (y4 (3 z1 - 2 z4) z5 + y5 (-4 z1^2 + 3 z1 z4 - 2 z4 z5)))) +
      x5 y2^3 y4 (x5^2 y2 (y2 - y4) z1 (z4 - z5) z5 - x3^2 y5 z4 z5
         (y5 z4 + y2 (z1 - z5) + y4 (-z1 + z5)) - x3 x5 (y2^2 z1 (z4 - z5) z5 + y5
          z4 (y5 z4 (z1 - 2 z5) + y4 (z1 - z5) z5) + y2 (y3 z1 z4 (z4 - z5) +
            y4 z1 z5 (-z4 + z5) + y5 z4 (-2 z1 z4 + z5 (z4 + z5))))) +
      x2^2 y2 y5 (x3 y4 (3 y4 y5 z1^2 (y5 z4 + y4 z5) + y2 z1 (y5^2 z4 (-3 z1 + 2 z4) +
            2 y4^2 z5^2 - y4 (3 y5 z1 z4 + 6 y5 z1 z5 + 2 y3 z4 z5 - 4 y5 z4 z5)) +
          y2^2 (-z5 (y4 (2 z1 - z4) z5 + y3 z4 (z4 + z5)) +
            y5 (z4^2 z5 + 3 z1^2 (z4 + z5) - 2 z1 z4 (z4 + z5)))) +
        x5 (-3 y4 z1^2 (y3 (y5^2 z4 + 2 y4 y5 (z1 - z5) + y4^2 z5) + 2 y4 y5
              (y5 z4 + y4 (-z1 + z5))) + y2 z1 (-y3^2 (2 y4 + 3 y5) z4 z5 +
            y3 (y5^2 z4 (3 z1 + 4 z4) + y4^2 (6 z1^2 + z4 z5) + 6 y4 y5 (z1^2 - z1 z5 +
                  z4 z5)) + y4 (2 y5^2 (3 z1 - 2 z4) z4 + 3 y4 y5 (-2 z1^2 +
                  3 z1 z4 + 2 z1 z5 - 2 z4 z5) - 2 y4^2 (3 z1^2 - 3 z1 z5 + z5^2))) +
          y2^2 (y3^2 (2 z1 - z4) z4 z5 + y4 (y5 z4 (-6 z1^2 + 4 z1 z4 +
                  2 z1 z5 - z4 z5) + y4 (2 z1 - z4) (3 z1^2 - 3 z1 z5 + z5^2)) +
            y3 (y5 z4 (-3 z1^2 + z4 z5 + z1 (-4 z4 + z5)) + y4
               (-6 z1^3 - 4 z1 z4 z5 + 3 z1^2 (z4 + z5) + z4 z5 (z4 + z5)))))) +
      x2 y2^2 (x3^2 y4 y5 z4 z5 (2 y4 y5 z1 + y2 (y5 (-2 z1 + z4) + y4 z5)) +
        x3 x5 y5 (y4 z1 (y4^2 (3 z1 - 2 z5) z5 + 2 y5 z4 (-2 y5 z4 + y3 z5) +
            y4 z4 (3 y5 z1 - y3 z5 - 6 y5 z5)) + y2 (-y5 z4 (z1 + 2 z4) (y5 z4 -
              y3 z5) + y4^2 (-2 z4 z5^2 + 4 z1 z5 (z4 + z5) - 3 z1^2 (z4 + 2 z5)) +
            y4 z4 (y3 (z1 + z4) z5 + y5 (-3 z1^2 + 5 z1 z4 + 3 z1 z5 - 4 z4 z5))) +
          y2^2 (z4^2 (y5 (z1 + 2 z4) - y3 z5) + y4 (3 z1^2 (z4 + z5) +
              z4 z5 (z4 + z5) - z1 (z4^2 + 4 z4 z5 + 2 z5^2)))) +
        x5^2 (y4 y5 z1 (2 y3 y5 z4 z5 + y4^2 z5 (-3 z1 + 2 z5) + y4 (-3 y5 z1 z4 +
                3 y3 z1 z5 + 4 y5 z4 z5 - 2 y3 z5^2)) + y2^2 (y3^2 z1 z4 (z4 - z5) +
            y4 y5 (-z4^2 z5 + z1 z4 (2 z4 + z5) + z1^2 (-6 z4 + 3 z5)) + y3
```

Figure 27: Numerator M_2 simplified by MATHEMATICA: part 4/7.

```
              (y4 z1 (3 z1 - z4) (z4 - z5) + y5 z4 (-2 z1 z4 + 2 z1 z5 + z4 z5))) -
           y2 (y3 (3 y4^2 z1^2 (z4 - z5) + y4 y5 z5 (3 z1^2 - 2 z1 z5 + z4 z5) +
                 y5^2 z4 (-z1 z4 + 2 z1 z5 + 2 z4 z5)) +
              y4 y5 (y4 (-6 z1^2 z4 - z4 z5^2 + z1 z5 (3 z4 + 2 z5)) +
                 y5 z4 (-3 z1^2 - 2 z4 z5 + z1 (z4 + 4 z5)))))))) -
w1 (w5^3 y2^2 (x3 x5 y2^3 y4 (y3 - y5) (-y4 + y5) z1 z4 - x2^2 (y3 - y4) y4
        (y2 - y5) y5^2 z1 (y4 z1 + y2 (-z1 + z4)) +
      x2 y2 (x3 y4 (y2 - y5) y5^2 z4 (y4 z1 + y2 (-z1 + z4)) +
         x5 y2 (y3 - y5) (y4 - y5) z1 (-2 y4^2 z1 + y2 (2 y4 z1 + y3 z4 - y4 z4)))) +
   x2 x5 y5 z4 (x3 (x3 - x5) x5 y2^3 z4 z5 (y4 z1 (-2 z1 + z5) +
         y2 (z4 z5 + z1 (-z4 + z5))) + x2^3 z1^2
        (-y4^2 y5^2 z1^2 + y2 z1 (y3 y5 (y5 z4 - y3 z5) + y4^2 (y5 (z1 - z5) + y3 z5) +
           y4 (y5^2 (z1 - z4) - y3^2 z5 + y3 y5 z5)) + y2^2 (-y4 (z1 - z4)
             (y5 (z1 - z5) + y3 z5) + y3 (y3 (z1 - z4) z5 + y5 z4 (-z1 + z5)))) +
      x2 y2^2 (-x3^2 z4 z5 (y4 z1 (y5 z1 + y2 z5) + y2 (y5 z1 z4 + y2 (-z1 + z4) z5)) +
         x5^2 z1 z5 (y4 z1 (3 y4 z1 - 2 y3 z5 - y4 z5) + y2 (y4 (-3 z1^2 + 2 z1 z4 +
                 z1 z5 - z4 z5) + y3 (-2 z1 z4 + 2 z1 z5 + z4 z5))) +
         x3 x5 (y2^2 z4 (2 z1^2 (z4 - z5) + z4 z5^2 + z1 z5 (-2 z4 + z5)) +
            y4 z1^2 (y5 (-3 z1 z4 + 2 z4 z5) +
               z5 (y3 (3 z1 + 2 z4 - z5) + y4 (-3 z1 + z5))) + y2 z1
             (y3 z5 (2 z1 z4 + z4^2 - 2 z1 z5 - 3 z4 z5) + 2 y5 z4 (-z1 z4 + z1 z5 +
                 z4 z5) + y4 (3 z1^2 z4 + 2 z4 z5^2 + z1 z5 (-5 z4 + 2 z5))))) -
      x2^2 y2 z1 (x3 (y4 y5 z1^2 (y5 z4 + (-y3 + y4) z5) + y2^2 (y5 z4^2 (-z1 + z5) +
               (z1 - z4) z5 (-y4 z5 + y3 (z4 + z5))) - y2 z1 (-y4^2 z5^2 + y5 z4
                (-y5 z4 + 2 y3 z5) + y4 (y5 z4 (z1 - 2 z5) + y3 z5 (z4 + z5)))) +
         x5 (y4 z1^2 (3 y3 (y3 - y5) z5 + y4 (4 y5 z1 - 3 y3 z5 - 2 y5 z5)) +
            y2^2 (y3 (3 z1^2 (z4 - z5) + z4 z5^2 + z1 z5 (-3 z4 + 2 z5)) +
               y4 (4 z1^3 - z4 z5^2 + z1 z5 (3 z4 + z5) - z1^2 (3 z4 + 4 z5))) +
            y2 z1 (-y4^2 (-2 z1 + z5)^2 + y4 (y3 z5 (3 z1 - 2 z4 + z5) +
                  y5 (-4 z1^2 + 3 z1 z4 + 2 z1 z5 - 2 z4 z5)) +
               y3 (y3 (2 z4 - 3 z5) z5 + y5 (-3 z1 z4 + 3 z1 z5 + 2 z4 z5)))))) +
   w5^2 y2 (x3 x5 y2^4 y4 z1 z4 (x3 (-y4 + y5) z5 + x5 (y5 z4 + y4 z5 - 2
              y5 z5 + y3 (-z4 + z5))) + x2^3 y4 y5 z1
        (2 y4 (-y3 + y4) y5^2 z1^2 + y2 z1 (y3 (2 y4 y5 z1 + 2 y5^2 (z1 - z4) -
                 y4^2 z5) + y4 y5 (2 y5 (-z1 + z4) + y4 (-2 z1 + z5))) +
           y2^2 (y4 y5 (z1 - z4) (2 z1 - z5) + y3^2 z4 z5 - y3
                (y4 (-z1 + z4) z5 + y5 (2 z1^2 - 2 z1 z4 + z4 z5)))) +
      x2 y2^2 (x3^2 y4 y5^2 z4 (y4 z1 + y2 (-z1 + z4)) z5 - x5^2 y2 z1
          (-2 y4^2 z1 + y2 (2 y4 z1 + y3 z4 - y4 z4))
          (y5 (z4 - 2 z5) + y4 z5 + y3 (-z4 + z5)) +
         x3 x5 (y4 y5^2 z1 z4 (-y5 z4 + 2 y4 (z1 - z5)) + y2
             (2 y3 y4 (y4 - y5) z1^2 z4 - y5^3 z4^2 (z1 + z4) -
               2 y4^3 z1^2 z5 + y4^2 y5 z1 (-4 z1 z4 + 2 z1 z5 + z4 z5) +
               2 y4 y5^2 z4 (-z4 z5 + z1 (z4 + z5))) + y2^2 (y4^2 z1 (2 z1 - z4) z5 +
               y5 z4 (y5 z4 (2 z1 + z4) - y3 z1 (z4 + z5)) + y4 (y3 z1 z4 (z4 + z5) +
                  y5 (-2 z1 z4^2 + 2 z1^2 (z4 - z5) + z4^2 z5))))) +
      x2^2 y2 (x3 y4 y5 (y4 y5 z1^2 (2 y5 z4 + y4 z5) + y2^2 (-y3 z4^2 z5 +
```

Figure 28: Numerator M_2 simplified by MATHEMATICA: part 5/7.

```
y5 (z4^2 z5 + z1^2 (2 z4 + z5) - z1 z4 (2 z4 + z5))) - y2 z1
(2 y5^2 (z1 - z4) z4 + y4 (y3 z4 z5 + 2 y5 (-z4 z5 + z1 (z4 + z5))))) +
x5 z1 (y4 y5^2 z1 (y4 (3 y4 z1 - y5 z4 - 2 y4 z5) + y3
(-3 y4 z1 - 2 y5 z4 + 2 y4 z5)) + y2 (y4 y5 (y5^2 (z1 - z4) z4 +
y4^2 z1 (-6 z1 + z5) + y4 y5 (3 z1 z4 + 2 z1 z5 - 2 z4 z5)) +
y3 (3 y4^3 z1^2 + y5^3 z4 (2 z1 + z4) - y4^2 y5 z1 z5 + y4 y5^2
(3 z1^2 - 2 z1 z5 + 2 z4 z5))) + y2^2 (2 y3^2 (-y4 + y5) z1 z4 -
y3 (y4^2 z1 (3 z1 - 2 z4) + y5^2 z4 (4 z1 + z4) + y4 y5
(z4 z5 - z1 (2 z4 + z5))) + y4 y5 (y5 (-3 z1^2 + z1 z4 + z4^2) +
y4 (6 z1^2 + z4 z5 - z1 (4 z4 + z5))))))) +
w5 (-x3 (x3 - x5) x5^2 y2^5 y4 z1 z4 (z4 - z5) z5 + x2^4 y4 y5 z1^2
((y3 - y4) y4 y5^2 z1^2 + y2 z1 (y4 y5 (y5 (z1 - z4) + y4 (z1 - z5)) +
y3 (-y4 y5 z1 + y5^2 (-z1 + z4) + y4^2 z5)) +
y2^2 (-y4 y5 (z1 - z4) (z1 - z5) - y3^2 z4 z5 + y3
(y4 (-z1 + z4) z5 + y5 (z1^2 - z1 z4 + z4 z5)))) +
x2 x5 y2^3 (-x5^2 y2 z1 (-2 y4^2 z1 + y2 (2 y4 z1 + y3 z4 - y4 z4))
(z4 - z5) z5 + x3^2 y5 z4 z5 (y4 z1 (y5 z4 + y4 (-2 z1 + z5)) + y2
(y5 z4 (z1 + z4) + y4 (2 z1^2 + z4 z5 - z1 (z4 + z5)))) +
x3 x5 (y4 y5 z1 z4 (2 y5 z4 (z1 - z5) + y4 (2 z1 - z5) z5) + y2^2
(y4 z1 (2 z1 - z4) (z4 - z5) z5 + y5 z4^2 (-2 z1 z4 + 2 z1 z5 + z4 z5) +
y3 z1 z4 (z4^2 - z5^2)) + y2 (2 y3 y4 z1^2 z4 (z4 - z5) +
2 y4^2 z1^2 z5 (-z4 + z5) + y5^2 z4^2 (z1 (z4 - 2 z5) - 2 z4 z5) +
y4 y5 z4 (-4 z1^2 z4 - z4 z5^2 + z1 z5 (2 z4 + z5))))) + x2^2 y2^2
(-x3^2 y4 y5 z4 z5 (y4 y5 z1^2 + y2^2 z4 z5 + y2 z1 (y5 (-z1 + z4) + y4 z5)) +
x3 x5 y5 (y4 z1^2 (y4^2 z5 (-3 z1 + z5) + y5 z4 (2 y5 z4 - y3 z5) +
y4 z4 (-3 y5 z1 + 2 y3 z5 + 3 y5 z5)) + y2
z1 (y5 z4 (2 y5 z4 (z1 + z4) - y3 (2 z1 + 3 z4) z5) +
y4^2 (2 z4 z5^2 + 3 z1^2 (z4 + 2 z5) - z1 z5 (5 z4 + 2 z5)) +
y4 z4 (-2 y3 z1 z5 + y5 (3 z1^2 - 4 z1 z4 + 4 z4 z5))) +
y2^2 (z4^2 (-y5 (z1 + z4) (2 z1 - z5) + y3 (z1 - z4) z5) +
y4 (z4^2 z5^2 - 3 z1^3 (z4 + z5) - z1 z4 z5 (2 z4 + z5) +
z1^2 (2 z4^2 + 5 z4 z5 + z5^2)))) +
x5^2 z1 (y4 y5 z1 (-4 y3 y5 z4 z5 + y4^2 (3 z1 - z5) z5 + y4
(y5 z4 (3 z1 - 2 z5) + y3 z5 (-3 z1 + z5))) + y2^2
(2 y3^2 z1 z4 (-z4 + z5) + y4 y5 (z1^2 (6 z4 - 3 z5) + z4^2 z5 +
z1 z4 (-4 z4 + z5)) - y3 (y4 z1 (3 z1 - 2 z4) (z4 - z5) +
y5 z4 (-4 z1 z4 + 4 z1 z5 + z4 z5))) + y2
(y3 (3 y4^2 z1^2 (z4 - z5) + y4 y5 z5 (3 z1^2 - z1 z5 + z4 z5) +
2 y5^2 z4 (-z1 z4 + 2 z1 z5 + z4 z5)) +
y4 y5 (y5 z4 (-3 z1^2 - 2 z4 z5 + 2 z1 (z4 + z5)) +
y4 (-6 z1^2 z4 - z4 z5^2 + z1 z5 (3 z4 + z5)))))) +
x2^3 y2 y5 z1 (x3 y4 (-y4 y5 z1^2 (y5 z4 + y4 z5) + y2 z1 (y5^2 (z1 - z4) z4 -
y4^2 z5^2 + y4 (y5 z1 z4 + 2 y5 z1 z5 + y3 z4 z5 - 2 y5 z4 z5)) +
y2^2 (z5 (y4 (z1 - z4) z5 + y3 z4 (z4 + z5)) -
y5 (z4^2 z5 + z1^2 (z4 + z5) - z1 z4 (z4 + z5)))) +
x5 (y4 z1^2 (2 y4 y5 (-2 y4 z1 + y5 z4 + y4 z5) + y3 (4 y4 y5 z1 +
3 y5^2 z4 + 3 y4^2 z5 - 2 y4 y5 z5)) + y2 z1 (y3^2 (y4 + 3 y5) z4 z5 -
```

Figure 29: Numerator M_2 simplified by MATHEMATICA: part 6/7.

```
        y3 (y5^2 z4 (3 z1 + 2 z4) + y4^2 (4 z1^2 + 2 z1 z5 - z4 z5) +
           y4 y5 (4 z1^2 - 2 z1 z5 + 3 z4 z5)) + y4 (2 y5^2 z4 (-z1 + z4) +
           y4^2 (-2 z1 + z5)^2 + y4 y5 (4 z1^2 - 5 z1 z4 - 2 z1 z5 + 3 z4 z5))) -
    y2^2 (y3^2 (z1 - z4) z4 z5 + y3 (y5 z4 (-3 z1^2 - 2 z1 z4 + 2 z1 z5 + z4 z5) +
           y4 (-4 z1^3 - 2 z1 z4 z5 + z4 z5 (z4 + z5) + z1^2 (3 z4 + z5))) +
        y4 (-y5 (z1 - z4) z4 (2 z1 - z5) + y4 (4 z1^3 - z4 z5^2 +
               z1 z5 (3 z4 + z5) - z1^2 (3 z4 + 4 z5)))))));
```

Figure 30: Numerator M_2 simplified by MATHEMATICA: part 7/7.