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Supplementary Materials for

Cavity quantum-electrodynamical polaritonically enhanced electron-phonon coupling and its influence on superconductivity

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Section S1. Relevant photon modes in cavity

In this work we consider a 2D material on a dielectric substrate in a nanocavity. We impose reflecting mirror boundary conditions with $\vec{n} \cdot \vec{B} = 0$ and $\vec{n} \times \vec{E} = 0$ for the magnetic $\vec{B}$ and electric $\vec{E}$ components of the photonic field, and $\vec{n} = \hat{z}$ the surface normal. The size of the cavity in z direction is L_z. If the dielectric substrate has a very high dielectric constant, such as for $SrTiO_3$ at low temperature, it can be considered almost metallic and L_z is reduced accordingly in our effective description.

Assuming periodic boundary conditions in the $x - y$ plane, we obtain for example for the vacuum electric field, obeying the wave equation $\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$ with c the speed of light

$$E_x(x, y, z, t) = E_1 \exp(ik_x x) \exp(ik_y y) \sin(k_z z) \exp(-i\omega_{\text{phot}}(\vec{k})t) \tag{S1}$$

$$E_y(x, y, z, t) = E_2 \exp(ik_x x) \exp(ik_y y) \sin(k_z z) \exp(-i\omega_{\text{phot}}(\vec{k})t) \tag{S2}$$

$$E_z(x, y, z, t) = E_3 \exp(ik_x x) \exp(ik_y y) \cos(k_z z) \exp(-i\omega_{\text{phot}}(\vec{k})t) \tag{S3}$$

with $\omega_{\text{phot}}(\vec{k}) = c|\vec{k}|$, and

$$k_x = \frac{2\pi l}{L_x}, \ l \in \mathbb{N}_0 \tag{S4}$$

$$k_y = \frac{2\pi m}{L_y}, \ m \in \mathbb{N}_0 \tag{S5}$$

$$k_z = \frac{\pi n}{L_z} \ n \in \mathbb{N}_0 \tag{S6}$$

We assume L_x and L_y to be large to obtain a fine momentum grid in the $x - y$ plane. By contrast L_z is assumed to be small ($L_z \ll L_x, L_y$), implying that for $n = 1$ the photon energy is at least $c\frac{\pi}{L_z}$ well above typical phonon energy scales and thus irrelevant to the problem of our interest. We retain only the $n = 0$, $k_z = 0$ component that has constant mode amplitude along the z direction. Thus we will use only one mode for each in-plane momentum $\vec{q} = (q_x, q_y)$ with

$$E_x(x, y, z, t) = 0 \tag{S7}$$

$$E_y(x, y, z, t) = 0 \tag{S8}$$

$$E_z(x, y, z, t) = E_3 \exp(iq_x x) \exp(iq_y y) \exp(-i\omega_{\text{phot}}(\vec{k})t) \tag{S9}$$

Section S2. **Phonon-photon Hamiltonian**

We consider the generic Hamiltonian for phonon-photon coupling (*41*)

$$H_{\text{phon-phot}} = H_0 + H' \tag{S10}$$

$$H_0 = \Omega \sum_{\vec{q}} b^\dagger_{\vec{q}} b_{\vec{q}} + \sum_{\vec{q}} \omega_{\text{phot}}(\vec{q}) a^\dagger_{\vec{q}} a_{\vec{q}} \tag{S11}$$

$$H' = -\frac{e}{Mc} \sum_j \vec{P}_j \cdot \vec{A}(\vec{R}_j) + \frac{e^2}{2Mc^2} \sum_j \vec{A}(\vec{R}_j) \cdot \vec{A}(\vec{R}_j) \tag{S12}$$

Throughout we approximate the phonon dispersion relevant for FeSe/$SrTiO_3$ with a dispersionless $\Omega = 92$ meV (*29*). Here $\vec{q}$ summations are over the first Brillouin zone $[-\pi, \pi)^2$ in the 2D square lattice with lattice constant $a = 1$, implying a high-frequency cutoff to the photons, which is irrelevant to the electron-boson physics happening at much lower energy. For the photon, we take only the mode polarized along the $\hat{z}$ direction parallel to the phonon dipoles, and restrict it to the lowest branch $q_z = 0$ due to cavity confinement as discussed above, implying $\omega_{\text{phot}}(\vec{q}) = c|\vec{q}| = c\sqrt{q_x^2 + q_y^2}$.

We write the phononic dipole current operator via bosonic operators

$$\vec{J}_j \equiv \frac{e}{M}\vec{P}_j = ie \sum_{\vec{q}} \left(\frac{\Omega}{2NM}\right)^{1/2} \hat{\xi}_{\vec{q}} \left(b^\dagger_{\vec{q}} - b_{-\vec{q}}\right) e^{-i\vec{q}\vec{R}_j} \equiv \sum_{\vec{q}} \frac{1}{\sqrt{N}} \vec{J}(\vec{q}) e^{-i\vec{q}\vec{R}_j} \tag{S13}$$

with polarization vector $\hat{\xi}_{\vec{q}} = \hat{z}$, and similarly for the relevant z component of the photonic vector potential

$$A_z(\vec{R}_j) \equiv \sum_{\vec{q}} \left(\frac{2\pi c^2}{\omega_{\text{phot}}(\vec{q})\nu_0}\right)^{1/2} \left(a^\dagger_{\vec{q}} + a_{-\vec{q}}\right) e^{-i\vec{q}\vec{R}_j} \equiv \sum_{\vec{q}} \frac{c}{\sqrt{\nu_0}} A_\mu(\vec{q}) e^{-i\vec{q}\vec{R}_j} \tag{S14}$$

assuming periodic boundary conditions inside the 2D plane. Here $b^\dagger_{\vec{q}}$ ($b_{\vec{q}}$) creates (annihilates) a phonon with wavevector $\vec{q}$; $a^\dagger_{\vec{q}}$ ($a_{\vec{q}}$) creates (annihilates) a cavity photon with wavevector $\vec{q}$. N is the number of unit cells, V the system volume, $\nu_0 \equiv V/N$ the unit cell volume, and e and M the ionic charge and reduced mass, respectively, related to the relative motion of positively and negatively charged ions in the optical phonon mode. In momentum space we have

$$J_z(\vec{q}) \equiv ie \left(\frac{\Omega}{2M}\right)^{1/2} \left(b^\dagger_{\vec{q}} - b_{-\vec{q}}\right) \tag{S15}$$

$$A_z(\vec{q}) \equiv \left(\frac{2\pi}{\omega_{\text{phot}}(\vec{q})}\right)^{1/2} \left(a^\dagger_{\vec{q}} + a_{-\vec{q}}\right) \tag{S16}$$

Now we first diagonalize the bare photon plus A^2 terms of the Hamiltonian

$$H_{0,\mathrm{phot}} = \sum_{\vec{q}} \omega_{\mathrm{phot}}(\vec{q}) a^\dagger_{\vec{q}} a_{\vec{q}} \tag{S17}$$

$$= \frac{1}{2} \sum_{\vec{q}} \left(P_{A,\vec{q}} P_{A,-\vec{q}} + \omega_{\mathrm{phot}}(\vec{q})^2 X_{A,\vec{q}} X_{A,-\vec{q}} \right) \tag{S18}$$

$$H_{A^2} = \frac{1}{2} \sum_{\vec{q}} \omega_{\mathrm{P}}^2 X_{A,\vec{q}} X_{A,-\vec{q}} \tag{S19}$$

Here we introduced canonical position and momentum operators for photon degrees of freedom

$$X_{A,\vec{q}} \equiv \sqrt{\frac{1}{2\omega_{\mathrm{phot}}(\vec{q})}} \left(a_{\vec{q}} + a^\dagger_{-\vec{q}} \right) \tag{S20}$$

$$P_{A,\vec{q}} \equiv -i \sqrt{\frac{\omega_{\mathrm{phot}}(\vec{q})}{2}} \left(a_{-\vec{q}} - a^\dagger_{\vec{q}} \right) \tag{S21}$$

We also defined the phononic plasma frequency

$$\omega_{\mathrm{P}} \equiv \sqrt{\frac{4\pi e^2}{M\nu_0}} = \sqrt{\frac{4\pi e^2}{M\nu_{0,2D} L_z}} \tag{S22}$$

which for the 2D system in the cavity is governed by the length of the vacuum inside the cavity in z direction, L_z, and the 2D unit cell area $\nu_{0,2D}$. The expressions above are given in cgs units. In the SI system, $\omega_{\mathrm{P}}^{\mathrm{SI}} = \sqrt{\frac{e^2}{M\epsilon_0 \nu_{0,2D} L_z}}$ with the vacuum permittivity ϵ_0.

The bilinear $J \cdot A$ coupling term is written as

$$H_{J\cdot A} = -\frac{1}{\sqrt{\nu_0}} \sum_{\vec{q}} \vec{J}(\vec{q}) \cdot \vec{A}(-\vec{q}) \tag{S23}$$

$$= -\sum_{\vec{q}} \omega_{\mathrm{P}} X_{A,\vec{q}} P_{B,\vec{q}} \tag{S24}$$

where it is convenient to introduce canonical position and momentum operators for the phonons

$$X_{B,\vec{q}} \equiv \sqrt{\frac{1}{2\Omega}} \left(b_{\vec{q}} + b^\dagger_{-\vec{q}} \right) \tag{S25}$$

$$P_{B,\vec{q}} \equiv -i \sqrt{\frac{\Omega}{2}} \left(b_{-\vec{q}} - b^\dagger_{\vec{q}} \right) \tag{S26}$$

Written in these operators, the bare phonon term $H_{0,\mathrm{phon}} \equiv \Omega \sum_{\vec{q}} b^\dagger_{\vec{q}} b_{\vec{q}}$ takes the form

$$H_{0,\mathrm{phon}} = \frac{1}{2} \sum_{\vec{q}} \left(P_{B,\vec{q}} P_{B,-\vec{q}} + \Omega^2 X_{B,\vec{q}} X_{B,-\vec{q}} \right) \tag{S27}$$

The total phonon-photon Hamiltonian is now written as pairs of coupled harmonic oscillators

$$H_{\text{phon-phot}} = H_{0,\text{phot}} + H_{0,\text{phon}} + H_{A^2} + H_{J\cdot A} \tag{S28}$$

$$= \frac{1}{2}\sum_{\vec{q}} \Big(P_{A,\vec{q}}P_{A,-\vec{q}} + P_{B,\vec{q}}P_{B,-\vec{q}} + (\omega_{\text{phot}}(\vec{q})^2 + \omega_{\text{P}}^2)X_{A,\vec{q}}X_{A,-\vec{q}} + + \Omega^2 X_{B,\vec{q}}X_{B,-\vec{q}} - 2\omega_{\text{P}}X_{A,\vec{q}}P_{B,\vec{q}}\Big) \tag{S29}$$

In order to diagonalize this Hamiltonian, we introduce a transformation

$$\tilde{P}_{B,\vec{q}} \equiv \Omega X_{B,\vec{q}}, \tag{S30}$$

$$\tilde{X}_{B,\vec{q}} \equiv -\Omega^{-1} P_{B,\vec{q}} \tag{S31}$$

which leaves the canonical commutator unchanged but interchanges position and momentum operators. The phonon-photon Hamiltonian is then compactly represented as

$$H_{\text{phon-phot}} = \frac{1}{2}\sum_{\vec{q}} \begin{bmatrix} P_{A,\vec{q}} \\ \tilde{P}_{B,\vec{q}} \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_{A,-\vec{q}} \\ \tilde{P}_{B,-\vec{q}} \end{bmatrix} + + \frac{1}{2}\sum_{\vec{q}} \begin{bmatrix} X_{A,\vec{q}} \\ \tilde{X}_{B,\vec{q}} \end{bmatrix}^T \begin{bmatrix} \omega_{\text{phot}}(\vec{q})^2 + \omega_{\text{P}}^2 & \Omega\omega_{\text{P}} \\ \Omega\omega_{\text{P}} & \Omega^2 \end{bmatrix} \begin{bmatrix} X_{A,-\vec{q}} \\ \tilde{X}_{B,-\vec{q}} \end{bmatrix} \tag{S32}$$

Diagonalization is now achieved with the following unitary transformation to polariton canonical position and momentum operators

$$\begin{bmatrix} X_{+,\vec{q}} \\ X_{-,\vec{q}} \end{bmatrix} = \begin{bmatrix} \cos(\theta_{\vec{q}}) & \sin(\theta_{\vec{q}}) \\ -\sin(\theta_{\vec{q}}) & \cos(\theta_{\vec{q}}) \end{bmatrix} \begin{bmatrix} X_{A,\vec{q}} \\ \tilde{X}_{B,\vec{q}} \end{bmatrix} \tag{S33}$$

$$\begin{bmatrix} P_{+,\vec{q}} \\ P_{-,\vec{q}} \end{bmatrix} = \begin{bmatrix} \cos(\theta_{\vec{q}}) & \sin(\theta_{\vec{q}}) \\ -\sin(\theta_{\vec{q}}) & \cos(\theta_{\vec{q}}) \end{bmatrix} \begin{bmatrix} P_{A,\vec{q}} \\ \tilde{P}_{B,\vec{q}} \end{bmatrix} \tag{S34}$$

which leaves canonical commutation relations intact. The resulting phonon-photon Hamiltonian expressed in polaritonic operators is

$$H_{\text{phon-phot}} = \frac{1}{2}\sum_{\vec{q}} \begin{bmatrix} P_{+,\vec{q}} \\ P_{-,\vec{q}} \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_{+,-\vec{q}} \\ P_{-,-\vec{q}} \end{bmatrix} + + \frac{1}{2}\sum_{\vec{q}} \begin{bmatrix} X_{+,\vec{q}} \\ X_{-,\vec{q}} \end{bmatrix}^T \begin{bmatrix} \omega_{+}(\vec{q})^2 & 0 \\ 0 & \omega_{-}(\vec{q})^2 \end{bmatrix} \begin{bmatrix} X_{+,-\vec{q}} \\ X_{-,-\vec{q}} \end{bmatrix} \tag{S35}$$

with polaritonic dispersions $\omega_{\pm}(\vec{q})$ fulfilling

$$\omega_{\pm}(\vec{q})^2 = \frac{1}{2}\left(\omega_{\text{phot}}(\vec{q})^2 + \omega_{\text{P}}^2 + \Omega^2 \pm \sqrt{(\omega_{\text{phot}}(\vec{q})^2 + \omega_{\text{P}}^2 + \Omega^2)^2 - 4\omega_{\text{phot}}(\vec{q})^2\Omega^2}\right) \tag{S36}$$

In particular, in the long-wavelength limit one obtains

$$\omega_{+}(\vec{q} \to 0) \to \sqrt{\Omega^2 + \omega_{\text{P}}^2} \tag{S37}$$

$$\omega_{-}(\vec{q} \to 0) \to 0 \tag{S38}$$

as shown for the semiclassical polariton dispersions in (*41*). The diagonalization condition is given by

$$\arctan(\theta_{\vec{q}}) = \frac{\omega_{\text{phot}}(\vec{q})^2 + \omega_{\text{P}}^2 - \Omega^2 + \sqrt{(\omega_{\text{phot}}(\vec{q})^2 + \omega_{\text{P}}^2 + \Omega^2)^2 - 4\omega_{\text{phot}}(\vec{q})^2\Omega^2}}{2\Omega\omega_{\text{P}}} \tag{S39}$$

Defining bosonic operators for the upper ($\lambda = +$) and lower ($\lambda = -$) polariton branches

$$X_{\lambda,\vec{q}} \equiv \sqrt{\frac{1}{2\omega_{\lambda}(\vec{q})}}\left(\alpha_{\vec{q},\lambda} + \alpha_{-\vec{q},\lambda}^{\dagger}\right) \tag{S40}$$

$$P_{\lambda,\vec{q}} \equiv -i\sqrt{\frac{\omega_{\lambda}(\vec{q})}{2}}\left(\alpha_{-\vec{q},\lambda} - \alpha_{\vec{q},\lambda}^{\dagger}\right) \tag{S41}$$

we rewrite the phonon-photon Hamiltonian in a very compact polaritonic form

$$H_{\text{phon-phot}} = \sum_{\vec{q},\lambda=\pm} \omega_{\lambda}(\vec{q})\alpha_{\vec{q},\lambda}^{\dagger}\,\alpha_{\vec{q},\lambda} \tag{S42}$$

The transformation from the initial phononic degrees of freedom to the final polaritonic ones is then given by

$$X_{B,\vec{q}} = \frac{1}{\Omega}(\sin(\theta_{\vec{q}})P_{+,\vec{q}} + \cos(\theta_{\vec{q}})P_{-,\vec{q}}) \tag{S43}$$

For the bosonic operators, this implies

$$b_{\vec{q}} + b_{-\vec{q}}^{\dagger} = -i\sin(\theta_{\vec{q}})\sqrt{\frac{\omega_{+}(\vec{q})}{\Omega}}(\alpha_{-\vec{q},+} - \alpha_{\vec{q},+}^{\dagger}) - i\cos(\theta_{\vec{q}})\sqrt{\frac{\omega_{-}(\vec{q})}{\Omega}}(\alpha_{-\vec{q},-} - \alpha_{\vec{q},-}^{\dagger})) \tag{S44}$$

which will give the transformation from electron-phonon to electron-polariton coupling in the following.

Section S3. **Electron-polariton Hamiltonian**

The electron-polariton model Hamiltonian for FeSe/$SrTiO_3$ inside the cavity reads

$$H = H_{e-\text{phon}} + H_{\text{phon-phot}} \tag{S45}$$

$$H_{e-\text{phon}} = \sum_{\vec{k},\sigma} \epsilon_{\vec{k}} c^{\dagger}_{\vec{k},\sigma} c_{\vec{k},\sigma} + \frac{1}{\sqrt{N}} \sum_{\vec{k},\vec{q},\sigma} g(\vec{k},\vec{q}) c^{\dagger}_{\vec{k}+\vec{q},\sigma} c_{\vec{k},\sigma} (b_{\vec{q}} + b^{\dagger}_{-\vec{q}}) \tag{S46}$$

Here, $c^{\dagger}_{\vec{k},\sigma}$ ($c_{\vec{k},\sigma}$) creates (annihilates) an electron with wavevector $\vec{k}$ and spin σ; $\epsilon_{\vec{k}}$ is the electronic band dispersion measured relative to the chemical potential μ; $g(\vec{k},\vec{q})$ is the momentum dependent electron-phonon coupling. The direct electron-photon coupling of electrons in the FeSe plane to the photon branch of interest is neglected, which amounts to the assumption that the paramagnetic electronic current density $\vec{j}$ inside the FeSe layer is perfectly two-dimensional, thus not coupling to the photonic vector potential $\vec{A}$ which points perpendicular to the plane, implying $\vec{j} \cdot \vec{A} \approx 0$.

Adopting the FeSe/$SrTiO_3$ single-band model from Rademaker *et al.* (*28*), we take an electronic band dispersion $\epsilon_{\vec{k}} = -2t[\cos(k_x a) + \cos(k_y a)] - \mu$, where a is the in-plane lattice constant. We set $t = 0.075$ eV and use as an initial guess $\mu = -0.235$ eV, which is adjusted during the self-consistent calculations (see below) to a fixed band filling $n_{\uparrow} = n_{\downarrow} = 0.07$ for each spin. We neglect the fermion momentum dependence in the electron-phonon coupling $g(\vec{k},\vec{q}) = g(\vec{q})$, where $\vec{q}$ is the momentum transfer, and use $g(\vec{q}) = g_0 \exp(-|\vec{q}|/q_0)$. Here, g_0 is adjusted to fix the total dimensionless coupling strength $\lambda \approx 0.18$ of the electron-phonon interaction in absence of the cavity coupling, and q_0 sets the range of the interaction in momentum space.

The electron-polariton expressed in polaritonic bosonic operators is obtained via Eq. (S44) as

$$H = \sum_{\vec{k},\sigma} \epsilon_{\vec{k}} c^{\dagger}_{\vec{k},\sigma} c_{\vec{k},\sigma} + \frac{1}{\sqrt{N}} \sum_{\vec{k},\vec{q},\sigma,\lambda=\pm} c^{\dagger}_{\vec{k}+\vec{q},\sigma} c_{\vec{k},\sigma} (g^{*}_{\lambda}(\vec{q}) \alpha^{\dagger}_{-\vec{q},\lambda} + g_{\lambda}(\vec{q}) \alpha_{\vec{q},\lambda}) + \sum_{\vec{q},\lambda=\pm} \omega_{\lambda}(\vec{q}) \alpha^{\dagger}_{\vec{q},\lambda} \alpha_{\vec{q},\lambda} \tag{S47}$$

where

$$g_{+}(\vec{q}) = i \sin(\theta_{\vec{q}}) \sqrt{\frac{\omega_{+}(\vec{q})}{\Omega}}\, g_0 \exp(-|\vec{q}|/q_0) \tag{S48}$$

$$g_{-}(\vec{q}) = i \cos(\theta_{\vec{q}}) \sqrt{\frac{\omega_{-}(\vec{q})}{\Omega}}\, g_0 \exp(-|\vec{q}|/q_0) \tag{S49}$$

The couplings are thus fully determined through Eqs. (S48, S49) in connection with Eqs. (S36) and (S39). The polariton branches and couplings to the electrons are shown in Fig. 1 in the main text.

Section S4. Migdal-Eliashberg simulations

The electronic self-energy in Migdal-Eliashberg theory on the Matsubara frequency axis employing Nambu notation reads (*28*)

$$\hat{\Sigma}(\vec{k}, i\omega_n) = i\omega_n[1 - Z(\vec{k}, i\omega_n)]\hat{\tau}_0 + \chi(\vec{k}, i\omega_n)\hat{\tau}_3 + \phi(\vec{k}, i\omega_n)\hat{\tau}_1 \tag{S50}$$

where $\hat{\tau}_i$ are the Pauli matrices, $Z(\vec{k}, i\omega_n)$ and $\chi(\vec{k}, i\omega_n)$ renormalize the electronic single-particle mass and band dispersion, respectively, and $\phi(\vec{k}, i\omega_n)$ is the anomalous self-energy, which vanishes in the normal state. In Migdal-Eliashberg theory, the self-energy corresponding to the Hamiltonian (S46) is computed by self-consistently evaluating

$$\hat{\Sigma}(\vec{k}, i\omega_n) = \frac{-1}{N\beta}\sum_{\vec{q},m} |g(\vec{q})|^2 D^{(0)}(\vec{q}, i\omega_n - i\omega_m)\hat{\tau}_3\hat{G}(\vec{k}+\vec{q}, i\omega_m)\hat{\tau}_3 \tag{S51}$$

where $D^{(0)}(\vec{q}, i\omega_\nu) = -\frac{2\Omega}{\Omega^2+\omega_\nu^2}$ is the bare phonon propagator, $\hat{G}^{-1}(\vec{k}, i\omega_n) = i\omega_n\hat{\tau}_0 - \epsilon_{\vec{k}}\hat{\tau}_3 - \hat{\Sigma}(\vec{k}, i\omega_n)$ is the dressed electron propagator, N is number of momentum grid points, and $\beta = 1/(k_\mathrm{B}T)$ is the inverse temperature.

Inside the cavity with $\omega_\mathrm{P} > 0$, these well-known equations are modified to account for the Hamiltonian (S47) by using polariton branches $\lambda = \pm$ instead of the phonon

$$\hat{\Sigma}(\vec{k}, i\omega_n) = \frac{-1}{N\beta}\sum_{\vec{q},m,\lambda=\pm} |g_\lambda(\vec{q})|^2 D_\lambda^{(0)}(\vec{q}, i\omega_n - i\omega_m)\hat{\tau}_3\hat{G}(\vec{k}+\vec{q}, i\omega_m)\hat{\tau}_3 \tag{S52}$$

where $D_\lambda^{(0)}(\vec{q}, i\omega_\nu) = -\frac{2\omega_\lambda(\vec{q})}{\omega_\lambda(\vec{q})^2+\omega_\nu^2}$ is the bare polariton propagator.

In practice, we use an initial guess of 0.007 eV for the anomalous self-energy and run the self-consistency until a convergence to better than 10^{-6} eV is achieved. The 2D momentum grid to sample the Brillouin zone is chosen as 2000×2000 and convergence checked by comparing against 4000×4000 grids in selected cases. For the patch around $q = 0$ we avoid the point $q = 0$ where the lower polariton branch becomes soft since the corresponding propagator diverges in the static $\omega_\nu = 0$ case. Under the q integral this divergence is cured. We therefore apply a q coarse graining by averaging $\frac{1}{N_\mathrm{small}}\tilde{\sum}_q |g(\vec{q})|^2 D^{(0)}(\vec{q}, i\omega_\nu)$ over N_small

small patches ($\tilde{\sum}_q$ is the sum inside the momentum patch around $q = 0$), and using this averaged function in lieu of $|g(0)|^2 D^{(0)}(0, i\omega_\nu)$, again checking convergence in the momentum grid. The momentum convolution in Equations (S51) and (S52) is performed by fast Fourier transforms to a real-space grid and products on the real-space grid. The Matsubara cutoff is 0.4 eV for the frequency summations, and convergence in this cutoff also checked.