Data-driven model order reduction of quadratic-bilinear systems

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Abstract

We introduce a data-driven model order reduction (MOR) approach which can be viewed as the generalization of the Loewner framework for quadratic-bilinear (QB) control systems.

For certain types of nonlinear systems, one can always find an equivalent QB model without performing any approximation.

Proceed with appropriately defining generalized higher order transfer functions for QB systems. These multi-variate rational functions play an important role in the MOR process. We construct reduced order systems for which the associated transfer functions match those corresponding to the original system at selected interpolations points.

The generalizations of the Loewner matrices can be directly computed by solving generalized Sylvester equations with quadratic terms.

The advantage is that the approach is data-driven since one would only need computed/measured samples to construct a reduced order QB system. We illustrate the practical applicability of the proposed method by means of several numerical experiments resulting from semi-discretized nonlinear partial differential equations.

1 Introduction

In broad terms, model order reduction (MOR) is used to replace large, complex models of time dependent processes into much smaller, simpler models that are still capable of accurately representing the behavior of the original process under a variety of conditions.

The motivation for MOR stems from the need for accurate modeling of physical phenomena that often leads to large-scale dynamical systems which require long simulation times and large data storage. The reduced order models can be efficiently used as surrogates for the original model, i.e., by replacing it as a component in large scale simulations.

Generally, large systems arise due to accuracy requirements on the spatial discretization of partial differential equations for fluids, structures, or in the context of lumped-circuit approximations of distributed circuit elements. For some applications, see [3, 7].

Model reduction methods can be classified in several broad categories, ranging from *SVD-based* (e.g. balanced truncation), *Krylov-based* or *moment matching* methods, proper orthogonal

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decomposition (POD) and reduced basis (RB) methods. Most of these methods are included in the broad family of *projection based* methods for which the internal state variable $\mathbf{x}$ is approximated by the projected variable $\hat{\mathbf{x}}$ into a particular subspace. For details we refer the reader to the book [3], the book chapter [4] and the surveys [5, 7, 13].

Many smooth nonlinear control systems can be rewritten as QB systems by taking derivatives and adding algebraic equations [21]. In this way, no approximation is performed, the transformation being exact. Nevertheless, this procedure generates a linear increase in the state dimension even before the MOR step. Applications range from the Burgers', Chafee-Infante and Navier-Stokes equations to nonlinear RC circuits.

In particular, we are going to focus on *data driven* model order reduction methods and specifically on the Loewner method (as was introduced in [27]).

Using rational functions, compute models that match (interpolate) given data sets of measurements. In the context of linear systems, we start from data sets that contain frequency response measurements and we seek reduced systems that models these measurements. This particular property will be generalized for the QB nonlinear systems.

Since the proposed approach is data-driven, one would only need samples of the so-called generalized transfer functions corresponding to the underlying system, to construct a reduced order QB system. This corresponds to the case for which an original state-space model is not available.

A main ingredient of the Loewner methodology is represented by the Loewner matrices $\mathbb{L}$ and $\mathbb{L}_s$. They are divided difference matrices which can be exclusively written in terms of the given measured/computed data. It turns out, that for linear systems, these matrices can be factored in terms of the $\mathbf{E}$ and $\mathbf{A}$ matrix corresponding to the underlying system.

The structure of the paper is described as; after the introduction in Section 1, we continue with a short background on QB systems and some general properties of the Kronecker product in Section 2. In Section 3 we introduce the generalization of the Loewner framework to QB systems by constructing system matrices using computed data (samples of specifically chosen higher order transfer functions). Then the theoretical discussions are illustrated in Section 4 via three numerical examples.

1.1 Literature overview

In the following we present a short historical account of some contributions made towards reduction of *quadratic-bilinear* systems by means of various methods (most of them projection-based).

• One of the first tries of reducing this class of dynamical systems was made by Chen. He adapted the Arnoldi algorithm for building one sided projectors which can be applied to the dimension reduction of QB systems by using a Krylov subspace generate from linearized analysis (see [18, 19]).

• Li and Pileggi introduced a compact nonlinear MOR method (known as NORM) suitable mostly to weakly nonlinear systems that can be well characterized by low-order Volterra functional series. It is based on moment matching of nonlinear transfer functions by projection of the original system onto a set of minimum Krylov sub-spaces (see [25]).

• Gu introduced the QLMOR framework which is basically a projection based moment matching MOR approach that uses the quadratic-linear representation of nonlinear systems. The method was proven to preserve local passivity and also to provide an upper bound on the number of quadratic DAE's derived from a polynomial system (see [21, 22, 23, 24]).

• Van Beeumen and Meerbergen adapted the widely use balanced truncation method to the class of linear systems with quadratic output ([8]).

• Benner and Breiten extended the results from Gu by introducing Rational Krylov-subspace based methods for quadratic-bilinear approximations of nonlinear systems. In particular, application to QB differential algebraic equations (see [9, 10, 15]). Their work is continued in [1].

• We mention the more recent publications that cover the MIMO case (see [29, 30]).

• In recent years, increased attention has been allocated to MOR by means of (symmetric) tensor decomposition. Such methods were applied for reducing QB systems (see [26, 20]).

• Recent breakthroughs were made by Benner, Goyal and collaborators who managed to adapt two very well established MOR techniques, i.e., balanced truncation as well as the IRKA method, to the class of QB systems (see [11, 12]). Also mention new results on reducing Stokes-type QB systems in descriptor format (see [2]).

2 Quadratic-bilinear systems

We analyze quadratic-bilinear control systems $\mathbf{\Sigma}_{QB} = (\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{Q}, \mathbf{N}, \mathbf{B})$ characterized by the following equations

$$\mathbf{\Sigma}_{\mathrm{QB}} : \quad \Big\{ \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{Q}\big(\mathbf{x}(t) \otimes \mathbf{x}(t)\big) + \mathbf{N}\mathbf{x}(t)\mathbf{u}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \tag{1}$$

where $\mathbf{E},\ \mathbf{A},\ \mathbf{N} \in \mathbb{R}^{n\times n}$, $\mathbf{Q} \in \mathbb{R}^{n\times n^2}$, $\mathbf{B}, \mathbf{C}^T \in \mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u}, \mathbf{y} \in \mathbb{R}$. We discuss the approximation of systems in (1), by constructing reduced-order models $\hat{\mathbf{\Sigma}}_{\mathrm{QB}} = (\hat{\mathbf{C}}, \hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{Q}}, \hat{\mathbf{N}}, \hat{\mathbf{B}})$, described by

$$\hat{\mathbf{\Sigma}}_{\mathrm{QB}} : \quad \Big\{ \hat{\mathbf{E}}\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{A}}\hat{\mathbf{x}}(t) + \hat{\mathbf{Q}}\big(\hat{\mathbf{x}}(t) \otimes \hat{\mathbf{x}}(t)\big) + \hat{\mathbf{N}}\hat{\mathbf{x}}(t)\mathbf{u}(t) + \hat{\mathbf{B}}\mathbf{u}(t), \quad \hat{\mathbf{y}}(t) = \hat{\mathbf{C}}\hat{\mathbf{x}}(t), \tag{2}$$

where $\hat{\mathbf{E}},\ \hat{\mathbf{A}},\ \hat{\mathbf{N}} \in \mathbb{R}^{n\times n}$, $\hat{\mathbf{Q}} \in \mathbb{R}^{n\times n^2}$, $\hat{\mathbf{B}}, \hat{\mathbf{C}}^T \in \mathbb{R}^k$ and $\hat{\mathbf{x}} \in \mathbb{R}^k$, $\hat{\mathbf{y}} \in \mathbb{R}$.

For simplicity of exposition, we will treat the single-input, single-output (SISO) case. The multi-input case is technically more involved but it is based on the same ideas.

By splitting the internal variable $\mathbf{x}(t)$ corresponding to the original system $\mathbf{\Sigma}_{\mathrm{QB}}$ in an additive manner, i.e. , $\mathbf{x}(t) = \sum_{i=1}^{\infty} \mathbf{x}_i(t)$, one can show that the differential equation in (1) can be equivalently written as infinitely many equations corresponding to a set of coupled pseudolinear sub-systems written in the following format

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}_1(t) = \mathbf{A}\mathbf{x}_1(t) + \mathbf{B}\mathbf{u}(t), \quad i = 1 \\ \mathbf{E}\dot{\mathbf{x}}_i(t) = \mathbf{A}\mathbf{x}_i(t) + \mathbf{Q}\big(\sum\limits_{k=1}^{i-1} \mathbf{x}_k \otimes \mathbf{x}_{i-k}\big) + \mathbf{N}\mathbf{x}_{i-1}(t)\mathbf{u}(t), \quad i \geqslant 2, \end{cases} \tag{3}$$

where the solution of the subsystem at level $i-1$ is used as an additional input for the subsystem at level i. This approach is widely known as the variational analysis approach and it was introduced in [28]. It is assumed that the system $\mathbf{\Sigma}_{\mathrm{QB}}$ consists of a series of homogeneous subsystems, which in turn follows that the solution to the differential equation when feeding the input $a\mathbf{u}(t)$ can be written as $\mathbf{x}_a(t) = \sum_{i=1}^{\infty} a^i \mathbf{x}_i(t)$ (where $a > 0$).

By explicitly computing the solution of the $(i-1)^{\mathrm{th}}$ equation in (3), and substituting it onto the i^{th} equation, the Volterra series expansion of $\mathbf{x}(t)$ is constructed (see [28]).

Next, generalized transfer functions can be computed by applying the multivariable Laplace transform to the generelized impulse responses or kernels that form the expansion of the output $y(t)$. For explicit derivations of the symmetric transfer functions, we refer the readers to ([10, 1]).

Out of the multitude of transfer functions that can be defined following the procedure described above, we select some that follow a set of specific rules. For that, we introduce the next definitions.

Definition 2.1 *Let* $\Upsilon = \{\mathbf{N}, \mathbf{Q}\}$ *and consider the functions* $c : \Upsilon \to \{1, 2\}$, $\mathbf{\Gamma} : \Upsilon \times \mathbb{R}^c \to \mathbb{R}^{n\times n}$*:*

$$c(w) = \begin{cases} 1, & \text{if } w = \mathbf{N} \\ 2, & \text{if } w = \mathbf{Q} \end{cases}, \quad \text{and} \quad \mathbf{\Gamma}(w, \mathbf{S}) = \begin{cases} \mathbf{\Phi}(s_1), & \text{if } w = \mathbf{N} \text{ and } \mathbf{S} = \{s_1\} \\ \mathbf{\Phi}(s_1)\mathbf{B} \otimes \mathbf{\Phi}(s_2), & \text{if } w = \mathbf{Q} \text{ and } \mathbf{S} = \{s_1, s_2\} \end{cases}.$$

where $\mathbf{\Phi}(x) = (x\mathbf{E} - \mathbf{A})^{-1}$ *is the resolvent of the pencil* $(\mathbf{A}, \mathbf{E})$.

Denote with Υ^ℓ the set of all tuples of length ℓ with entries from Υ, i.e., for all $\ell \geqslant 1$, write $\Upsilon^\ell = \{(w_1, w_2, \ldots, w_\ell) | w_k \in \Upsilon, 1 \leqslant k \leqslant \ell\}$. Moreover, let $\Upsilon^0 = \{\epsilon\}$ contain only the null symbol.

Definition 2.2 *Let* $\mathbf{w} \in \Upsilon^\ell, \ell \geqslant 0$. *Then introduce the following functions*

$$\mathbf{H}_\ell^{\mathbf{w}}(s_1, s_2, s_3, \ldots, s_h) = \begin{cases} \mathbf{C}\mathbf{\Phi}(s)\mathbf{B}, & \ell = 0, \ \mathbf{w} = \epsilon \\ \mathbf{C}\mathbf{\Phi}(s_1)w(1)\mathbf{\Gamma}(w(1), \mathbf{S}_1)\cdots w(\ell)\mathbf{\Gamma}(w(\ell), \mathbf{S}_\ell)\mathbf{B}, & \ell \geqslant 1 \end{cases} \tag{4}$$

These newly introduced functions are divided in sub-categories or levels. Each level contains a number of 2^k functions, for $k \geqslant 0$. The transfer function corresponding to the linear counterpart of $\mathbf{\Sigma}_{\mathrm{QB}}$ (level 0) is $\mathbf{H}_0^\epsilon(s) = \mathbf{C}\,\mathbf{\Phi}(s)\,\mathbf{B}$; then continue to level 1, for which we write

$$\mathbf{H}_1^{(\mathbf{N})}(s_1, s_2) = \mathbf{C}\,\mathbf{\Phi}(s_1)\,\mathbf{N}\,\mathbf{\Phi}(s_2)\mathbf{B}, \quad \mathbf{H}_1^{(\mathbf{Q})}(s_1, s_2, s_3) = \mathbf{C}\,\mathbf{\Phi}(s_1)\,\mathbf{Q}\,\big(\mathbf{\Phi}(s_2)\mathbf{B} \otimes \mathbf{\Phi}(s_3)\mathbf{B}\big).$$

Thus, in general a k^{th} level transfer function defined in (4) is a multivariate rational function depending on h variables $\{s_1, \cdots, s_h\}$, for $k + 1 \leqslant h \leqslant 2k + 1$.

Definition 2.3 *Let* $\mathbf{X}_k$ *be the* k^{th} *column of of the matrix* $\mathbf{X} \in \mathbb{R}^{m\times n}$. *The vectorization of* $\mathbf{X}$ *is represented by the mapping* $\mathrm{vec} : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn}$ *obtained by including all the columns of* $\mathbf{X}$ *into a column vector. It is represented by the mapping* $\mathrm{vec} : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn}$. *Additionally, introduce the inverse vectorization operation as* $\mathrm{vec}_{m,n}^{-1} : \mathbb{R}^{mn} \to \mathbb{R}^{m\times n}$

$$\mathrm{vec}(\mathbf{X}) = \begin{bmatrix} \mathbf{X}_1^T & \ldots & \mathbf{X}_n^T \end{bmatrix}^T, \quad \mathrm{vec}_{m,n}^{-1}\big(\mathrm{vec}(\mathbf{X})\big) = \mathbf{X}.$$

If $m = n$, *the notation* $\mathrm{vec}_n^{-1}(\mathbf{x})$ *is going to be used instead.*

Proposition 2.1 *Given the following matrices,* $\mathbf{X} \in \mathbb{R}^{m\times n}$, $\mathbf{Y} \in \mathbb{R}^{p\times q}$, $\mathbf{Z} \in \mathbb{R}^{n\times r}$, $\mathbf{V} \in \mathbb{R}^{q\times s}$ *and* $\mathbf{W} \in \mathbb{R}^{r\times o}$, *the identities hold*

1. $\mathrm{vec}(\mathbf{XZW}) = \big(\mathbf{W}^T \otimes \mathbf{X}\big)\mathrm{vec}(\mathbf{C})$,
2. $\big(\mathbf{X} \otimes \mathbf{Y}\big)\big(\mathbf{Z} \otimes \mathbf{V}\big) = \big(\mathbf{XZ}\big) \otimes \big(\mathbf{YV}\big)$,
3. $\big(\mathbf{X} \otimes \mathbf{Y}\big)\big(\mathbf{V}\big) = \mathbf{X} \otimes \big(\mathbf{YV}\big)$, *for* $\mathbf{X} \in \mathbb{R}^{m\times 1}$ *(for* $n = r = 1$ *and* $\mathbf{Z} = 1$ *in 2.),*
4. $\big(\mathbf{X} \otimes \mathbf{Y}\big)\big(\mathbf{Z}\big) = \big(\mathbf{XZ}\big) \otimes \mathbf{Y}$, *for* $\mathbf{Z} \in \mathbb{R}^{n\times 1}$ *(for* $q = s = 1$ *and* $\mathbf{V} = 1$ *in 2.).*

Some of the results stated in Proposition 2.1 were also mentioned in [14]. They will used to prove certain results in section 3.

Proposition 2.2 *The Kronecker product of two unit vectors is also an unit vector. When multiplying* $\mathbf{e}_{j,m} \in \mathbb{R}^m$ *and* $\mathbf{e}_{k,n} \in \mathbb{R}^n$*, the product will be an unit vector of size mn, i.e.,*

$$\mathbf{e}_{j,m} \otimes \mathbf{e}_{k,n} = \mathbf{e}_{(j-1)n+k,mn}. \tag{5}$$

The process of reshaping a 3-tensor $\mathcal{X} \in \mathbb{R}^{n\times n\times n}$ into a matrix $\mathbf{X}^{(\mu)} \in \mathbb{R}^{n\times n^2}$ is known as the matricization of the tensor $\mathcal{X}$. Depending on the mode-μ fibers that are used for the unfolding, there are three different ways to unfold the tensor (see [14]). This procedure is often called mode-μ matricization of the the tensor.

Definition 2.4 *Let* $\mathbf{X} \in \mathbb{R}^{n\times n^2}$ *be a matrix that scales the Kronecker product of the internal variable* $\mathbf{x}(t)$ *with itself for a certain QB system. Consider* $\mathbf{X}$ *to be the mode-1 matricization of a 3-tensor* $\mathcal{X} \in \mathbb{R}^{n\times n\times n}$*. The frontal slices* $\mathcal{X}_i \in \mathbb{R}^{n\times n}$*,* $i = \{1, 2, \dots, n\}$ *corresponding to this tensor are also a component of* $\mathbf{X}$*, i.e.,* $\mathbf{X} = \mathcal{X}^{(1)} = \begin{bmatrix} \mathcal{X}_1 & \mathcal{X}_2 & \dots & \mathcal{X}_n \end{bmatrix}$ *The mode-*μ *matricizations (for* $\mu \in \{2, 3\}$*) of tensor* $\mathcal{X}$ *are defined as follows,*

$$\mathcal{X}^{(2)} = \begin{bmatrix} (\mathcal{X}_1)^T & (\mathcal{X}_2)^T & \dots & (\mathcal{X}_n)^T \end{bmatrix}, \quad \mathcal{X}^{(3)} = \begin{bmatrix} \mathrm{vec}(\mathcal{X}_1) & \mathrm{vec}(\mathcal{X}_2) & \dots & \mathrm{vec}(\mathcal{X}_n) \end{bmatrix}^T.$$

Definition 2.5 *Given* $\mathbf{X} \in \mathbb{R}^{n\times n^2}$*, define the matrix* $\mathbf{X}^{(-1)} \in \mathbb{R}^{n\times n^2}$ *as follows*

$$\mathbf{X}^{(-1)} = \begin{bmatrix} \mathrm{vec}_n^{-1}\big(\mathbf{X}^T\mathbf{e}_1\big) & \mathrm{vec}_n^{-1}\big(\mathbf{X}^T\mathbf{e}_2\big) & \cdots & \mathrm{vec}_n^{-1}\big(\mathbf{X}^T\mathbf{e}_n\big) \end{bmatrix}. \tag{6}$$

Example 2.1 Let $\mathbf{Q} = [\mathcal{Q}_1 \quad \mathcal{Q}_2] \in \mathbb{R}^{2\times 4}$ where the frontal slices $\mathcal{Q}_k$, $k = 1, 2$ correspond to the tensor $\mathcal{Q} \in \mathbb{R}^{2\times 2\times 2}$.

$$\mathcal{Q}_1 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad \mathcal{Q}_2 = \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}.$$

Then it follows that

$$\mathcal{Q}^{(1)} = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}, \quad \mathcal{Q}^{(2)} = \begin{bmatrix} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \end{bmatrix}, \mathcal{Q}^{(3)} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}, \quad \mathbf{Q}^{(-1)} = \begin{bmatrix} 1 & 5 & 2 & 6 \\ 3 & 7 & 4 & 8 \end{bmatrix}.$$

Proposition 2.3 *Let* $\mathbf{Q} \in \mathbb{R}^{n\times n^2}$ *be a matrix and* $\mathbf{Q}^{(-1)}$ *be defined in terms of* $\mathbf{Q}$ *as in (6). Then, it follows that*

$$\mathbf{Q}^{(-1)}\big(\mathbf{I}_n \otimes \mathbf{v}\big) = \big(\mathbf{v}^T \otimes \mathbf{I}_n\big)\mathbf{Q}^T, \ \forall \ \ \mathbf{v} \in \mathbb{R}^n. \tag{7}$$

3 The Loewner framework for QB systems

As already stated, our goal is the generalization of the Loewner framework to quadratic-bilinear systems. This section presents the theoretical foundations of this approach while section 5.4 provides numerical simulations illustrating the theory.

3.1 The generalized controllability and observability matrices

Consider a quadratic-bilinear system $\mathbf{\Sigma}_{QB} = (\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{Q}, \mathbf{N}, \mathbf{B})$. Let $\mathcal{S} = \{\omega_1, \omega_2, \dots, \omega_m\}$ be the set of interpolation points. First partition this set into two disjoint sets corresponding to left and right points: $\mathcal{S} = \{\mu_1, \mu_2, \dots, \mu_q\} \cup \{\lambda_1, \lambda_2, \dots, \lambda_k\}$, where $q + k = m$. Moreover, consider that both q and k are multiples of 3.

Definition 3.1 *We define the* **nested right multi-tuples** *and the* **nested left multi-tuples**

$$\boldsymbol{\lambda} = \left\{\boldsymbol{\lambda}^{(1)}, \boldsymbol{\lambda}^{(2)}, \ldots, \boldsymbol{\lambda}^{(k^\dagger)}\right\}, \ \boldsymbol{\mu} = \left\{\boldsymbol{\mu}^{(1)}, \boldsymbol{\mu}^{(2)}, \ldots, \boldsymbol{\mu}^{(q^\dagger)}\right\}, \tag{8}$$

composed of the right i^{th} tuples and the left j^{th} tuples:

$$\boldsymbol{\lambda}^{(i)} = \begin{cases} (\lambda_1^{(i)}), \\ (\lambda_2^{(i)},\ \lambda_1^{(i)}), \\ (\lambda_3^{(i)},\ \lambda_1^{(i)},\ \lambda_1^{(i)}). \end{cases}, \ \boldsymbol{\mu}^{(j)} = \begin{cases} (\mu_1^{(j)}), \\ (\mu_1^{(j)},\ \mu_2^{(j)}), \\ (\mu_1^{(j)},\ \lambda_1^{(j)},\ \mu_3^{(j)}). \end{cases}, \tag{9}$$

where $\lambda_1^{(i)}, \lambda_2^{(i)}, \lambda_3^{(i)}, \mu_1^{(j)}, \mu_2^{(j)}, \mu_3^{(j)} \in \mathbb{C}$ and $3k^\dagger = k,\ 3q^\dagger = q$. Here we denote with $k^\dagger$ and $q^\dagger$ the number of 'ladder' structures corresponding to the right and left multi-tuples respectively.

Note that these indices satisfy a *nestedness* property, namely, each row in $\boldsymbol{\lambda}^{(i)}$ ($\boldsymbol{\mu}^{(j)}$) is contained in the subsequent ones. To these tuples the following matrices are associated

$$\mathcal{R}^{(i)} = \left[\boldsymbol{\Phi}(\lambda_1^{(i)})\,\mathbf{B}, \quad \boldsymbol{\Phi}(\lambda_2^{(i)})\,\mathbf{N}\,\boldsymbol{\Phi}(\lambda_1^{(i)})\,\mathbf{B}, \quad \boldsymbol{\Phi}(\lambda_3^{(i)})\mathbf{N}\boldsymbol{\Phi}(\lambda_2^{(i)})\,\mathbf{N}\,\boldsymbol{\Phi}(\lambda_1^{(i)})\,\mathbf{B}\right],$$

for $i = 1, \ldots, k^\dagger$ where $\mathcal{R}^{(i)} \in \mathbb{C}^{n\times 3}$ is attached to $\boldsymbol{\lambda}^{(i)}$. The matrix

$$\mathcal{R} = \left[\mathcal{R}^{(1)},\ \mathcal{R}^{(2)},\ \cdots,\ \mathcal{R}^{(k^\dagger)}\right] \in \mathbb{C}^{n\times k}, \tag{10}$$

is defined as the *generalized controllability matrix* of the bilinear system $\boldsymbol{\Sigma}$, associated with the right multi-tuple $\boldsymbol{\lambda}$. Similarly, to the left tuple we associate the matrices

$$\mathcal{O}^{(j)} = \begin{bmatrix} \mathbf{C}\,\boldsymbol{\Phi}(\mu_1^{(j)}) \\ \mathbf{C}\,\boldsymbol{\Phi}(\mu_1^{(j)})\,\mathbf{N}\,\boldsymbol{\Phi}(\mu_2^{(j)}) \\ \mathbf{C}\,\boldsymbol{\Phi}(\mu_1^{(j)})\,\mathbf{Q}\left(\boldsymbol{\Phi}(\lambda_1^{(j)})\mathbf{B}\otimes\boldsymbol{\Phi}(\mu_3^{(j)})\right) \end{bmatrix} \in \mathbb{C}^{3\times n},$$

and the *generalized observability matrix*, as

$$\mathcal{O} = \begin{bmatrix} \mathcal{O}^{(1)} \\ \vdots \\ \mathcal{O}^{(q^\dagger)} \end{bmatrix} \in \mathbb{C}^{q\times n}. \tag{11}$$

Example 3.1 *For instance, by taking $q = 6, q^\dagger = 2$, the corresponding $\mathcal{O}$ can be written:*

$$\mathcal{O} = \begin{bmatrix} \mathbf{C}\,\boldsymbol{\Phi}(\mu_1^{(1)}) \\ \mathbf{C}\,\boldsymbol{\Phi}(\mu_1^{(1)})\,\mathbf{N}\,\boldsymbol{\Phi}(\mu_2^{(1)}) \\ \mathbf{C}\,\boldsymbol{\Phi}(\mu_1^{(1)})\,\mathbf{Q}\left(\boldsymbol{\Phi}(\lambda_1^{(1)})\mathbf{B}\otimes\boldsymbol{\Phi}(\mu_3^{(1)})\right) \\ \mathbf{C}\,\boldsymbol{\Phi}(\mu_1^{(2)}) \\ \mathbf{C}\,\boldsymbol{\Phi}(\mu_1^{(2)})\,\mathbf{N}\,\boldsymbol{\Phi}(\mu_2^{(2)}) \\ \mathbf{C}\,\boldsymbol{\Phi}(\mu_1^{(2)})\,\mathbf{Q}\left(\boldsymbol{\Phi}(\lambda_1^{(2)})\mathbf{B}\otimes\boldsymbol{\Phi}(\mu_3^{(2)})\right) \end{bmatrix}$$

Definition 3.2 *Consider two tuples composed of elements (symbols) $\alpha_1, ..., \alpha_i$, and $\beta_1, ..., \beta_j$ that are part of the finite set $\boldsymbol{\Omega}$. Introduce the concatenation of such tuples as the mapping $\circledcirc : \boldsymbol{\Omega}^i \times \boldsymbol{\Omega}^j \rightarrow \boldsymbol{\Omega}^{i+j}$ with the following property*

$$\left(\alpha_1, \alpha_2, \ldots, \alpha_i\right) \circledcirc \left(\beta_1, \beta_2, \ldots, \beta_j\right) = \left(\alpha_1, \alpha_2, \ldots \alpha_i, \beta_1, \beta_2, \ldots \beta_j\right).$$

The following lemma extends the rational interpolation idea for linear systems approximation (see e.g. [3] Chapter 11.3) to the quadratic-bilinear case.

Lemma 3.1 *Interpolation of QB systems; let $\mathbf{\Sigma}_{QB} = (\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{Q}, \mathbf{N}, \mathbf{B})$ be a quadratic-bilinear system of order n and assume that $\mathbf{Q}$ is written in the format from Proposition 2.2.14. Consider that $\mathbf{\Sigma}_{QB}$ is projected to a k^{th} order system by means of $\mathbf{X} = \mathcal{R}$ and $\mathbf{Y}^T = \mathcal{O}$ (as defined in (10) and (11)). The reduced system $\hat{\mathbf{\Sigma}} = (\hat{\mathbf{C}}, \hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{Q}}, \hat{\mathbf{N}}, \hat{\mathbf{B}})$, of order k, where*

$$\hat{\mathbf{E}} = \mathbf{Y}^T\mathbf{E}\mathbf{X}, \quad \hat{\mathbf{A}} = \mathbf{Y}^T\mathbf{A}\mathbf{X}, \quad \hat{\mathbf{Q}} = \mathbf{Y}^T\mathbf{Q}\big(\mathbf{X}\otimes\mathbf{X}\big), \quad \hat{\mathbf{N}} = \mathbf{Y}^T\mathbf{N}\mathbf{X}, \quad \hat{\mathbf{B}} = \mathbf{Y}^T\mathbf{B}, \quad \hat{\mathbf{C}} = \mathbf{C}\mathbf{X},$$

satisfies the following interpolation conditions (where $\mathbf{w} = (\epsilon, \mathbf{N}, \mathbf{Q})$, $\hat{\mathbf{w}} = (\epsilon, \hat{\mathbf{N}}, \hat{\mathbf{Q}})$, $i, j \in \{1, 2, \ldots, k^\dagger\}$, $\ell, h, h_1, h_3 \in \{1, 2, 3\}$ with $h_1 \vee h_2 = 1$):

$$\underline{\mathbf{k}\ \textit{conditions:}}\ \begin{cases} \mathbf{H}_0^{\epsilon}(\boldsymbol{\mu}^{(j)}(1)) = \hat{\mathbf{H}}_0^{\epsilon}(\boldsymbol{\mu}^{(j)}(1)) \\ \mathbf{H}_1^{\mathbf{N}}(\boldsymbol{\mu}^{(j)}(2)) = \hat{\mathbf{H}}_1^{\mathbf{N}}(\boldsymbol{\mu}^{(j)}(2)) \\ \mathbf{H}_1^{\mathbf{Q}}(\boldsymbol{\mu}^{(j)}(3)) = \hat{\mathbf{H}}_1^{\mathbf{Q}}(\boldsymbol{\mu}^{(j)}(3)) \end{cases} \text{, or } \mathbf{H}_{|\mathbf{w}(\ell)|}^{\mathbf{w}(\ell)}(\boldsymbol{\mu}^{(j)}(\ell)) = \hat{\mathbf{H}}_{|\hat{\mathbf{w}}(\ell)|}^{\hat{\mathbf{w}}(\ell)}(\boldsymbol{\mu}^{(j)}(\ell)), \tag{12}$$

$$\underline{\mathbf{k}\ \textit{conditions:}}\ \begin{cases} \mathbf{H}_0^{\epsilon}(\boldsymbol{\lambda}^{(i)}(1)) = \hat{\mathbf{H}}_0^{\epsilon}(\boldsymbol{\lambda}^{(i)}(1)) \\ \mathbf{H}_1^{\mathbf{N}}(\boldsymbol{\lambda}^{(i)}(2)) = \hat{\mathbf{H}}_1^{\mathbf{N}}(\boldsymbol{\lambda}^{(i)}(2)) \\ \mathbf{H}_1^{\mathbf{Q}}(\boldsymbol{\lambda}^{(i)}(3)) = \hat{\mathbf{H}}_1^{\mathbf{Q}}(\lambda^{(i)}(3)) \end{cases} \text{, or } \mathbf{H}_{|\mathbf{w}(h)|}^{\mathbf{w}(h)}(\boldsymbol{\lambda}^{(i)}(h)) = \hat{\mathbf{H}}_{|\hat{\mathbf{w}}(h)|}^{\hat{\mathbf{w}}(h)}(\boldsymbol{\lambda}^{(i)}(h)), \tag{13}$$

$$\underline{\mathbf{k^2}\ \textit{conditions:}}\ \mathbf{H}_{|w(\ell)|+|v(h)|+1}^{w(\ell)\circledcirc\mathbf{N}\circledcirc\mathbf{v}(h)}(\boldsymbol{\mu}^{(j)}(\ell)\circledcirc\boldsymbol{\lambda}^{(i)}(h)) = \hat{\mathbf{H}}_{|\hat{\mathbf{w}}(\ell)|+|\hat{\mathbf{w}}(h)|+1}^{\hat{\mathbf{w}}(\ell)\circledcirc\hat{\mathbf{N}}\circledcirc\hat{\mathbf{w}(h)}}(\boldsymbol{\mu}^{(j)}(\ell)\circledcirc\boldsymbol{\lambda}^{(i)}(h)), \tag{14}$$

$$\underline{\delta(\mathbf{k})\ \textit{conditions:}}\ \mathbf{H}_{|\mathbf{w}(\ell)|+|\mathbf{w}(h_1)|+|\mathbf{w}(h_2)|+1}^{\mathbf{w}(\ell)\circledcirc\mathbf{Q}\circledcirc\mathbf{w}(h_1)\circledcirc\mathbf{w}(h_2)}(\boldsymbol{\mu}^{(j)}(\ell)\circledcirc\boldsymbol{\lambda}^{(i)}(h_1)\circledcirc\boldsymbol{\lambda}^{(i)}(h_2))$$
$$= \hat{\mathbf{H}}_{|\hat{\mathbf{w}}(\ell)|+|\hat{\mathbf{w}}(h_1)|+|\hat{\mathbf{w}}(h_2)|+1}^{\hat{\mathbf{w}}(\ell)\circledcirc\mathbf{Q}\circledcirc\hat{\mathbf{w}}(h_1)\circledcirc\hat{\mathbf{w}}(h_2)}(\boldsymbol{\mu}^{(j)}(\ell)\circledcirc\boldsymbol{\lambda}^{(i)}(h_1)\circledcirc\boldsymbol{\lambda}^{(i)}(h_2)), \tag{15}$$

where $\delta(k) = \frac{(k+1)k^2}{2} - \frac{(2k+3)k^2}{9}$. Additionally, a number of $\frac{(2k+3)k^2}{9}$ interpolation conditions are also satisfied which can not directly be written in terms of the functions introduced in (4). Thus, in total $2k + k^2 + \frac{(k+1)k^2}{2}$ moments (interpolation conditions) are matched.

Remark 3.1 Since the $\mathbf{Q}$ matrix satisfies the condition stated in Proposition 2.2.14, then the identities in (**??**) hold. Hence, the total number of moments matched should be less than the predicted number, i.e., $2k+k^2+k^3$. By excluding the moments which are counted twice, it follows that the total number of moments matched using this procedure is instead $2k + k^2 + \frac{(k+1)k^2}{2}$.

3.1.1 Sylvester equations for $\mathcal{O}$ and $\mathcal{R}$

The generalized controllabilty and observability matrices satisfy Sylvester equations. To state the corresponding result we first need to define some quantities. Introduce the matrices

$$\mathbf{R} = \begin{bmatrix} \mathbf{e}_{1,3}^T & \cdots & \mathbf{e}_{1,3}^T \end{bmatrix} \in \mathbb{R}^{1\times k}, \quad \mathbf{L}^T = \begin{bmatrix} \mathbf{e}_{1,3}^T & \cdots & \mathbf{e}_{1,3}^T \end{bmatrix} \in \mathbb{R}^{1\times q}, \tag{16}$$

and the block-shift matrices

$$\mathbf{Z_R} = \mathbf{I}_{k^\dagger} \otimes \mathbf{e}_{1,3} \otimes \mathbf{e}_{2,3}^T \in \mathbb{R}^{k\times k}, \qquad \mathbf{Z_L} = \mathbf{I}_{q^\dagger} \otimes \mathbf{e}_{1,3}^T \otimes \mathbf{e}_{2,3} \in \mathbb{R}^{q\times q}, \tag{17}$$

$$\mathbf{Y_R} = \sum_{j=1}^{k^\dagger} \mathbf{e}_{3j-2,k} \otimes \mathbf{e}_{3j-2,k} \otimes \mathbf{e}_{3j,k}^T \in \mathbb{R}^{k^2\times k}, \ \mathbf{Y_L} = \sum_{j=1}^{q^\dagger} \mathbf{e}_{3j-2,q}^T \otimes \mathbf{e}_{3j-2,q}^T \otimes \mathbf{e}_{3j,q} \in \mathbb{R}^{q\times q^2}. \tag{18}$$

as well as the matrices,

$$\mathbf{Y}_{\mathbf{R}}^{(j)} = \mathbf{e}_{3j-2,k} \otimes \mathbf{e}_{3j-2,k} \otimes \mathbf{e}_{3j,k}^T \in \mathbb{R}^{k^2 \times k}, \quad \mathbf{Y}_{\mathbf{L}}^{(j)} = \mathbf{e}_{3j-2,q}^T \otimes \mathbf{e}_{3j-2,q}^T \otimes \mathbf{e}_{3j,q} \in \mathbb{R}^{q \times q^2}, \tag{19}$$

and hence write $\mathbf{Y}_{\mathbf{R}} = \sum_{j=1}^{k^\dagger} \mathbf{Y}_{\mathbf{R}}^{(j)}$ and also $\mathbf{Y}_{\mathbf{L}} = \sum_{j=1}^{q^\dagger} \mathbf{Y}_{\mathbf{L}}^{(j)}$. Next, arrange the interpolation points in diagonal matrices format (as for the linear and bilinear cases), i.e.,

$$\mathbf{M} = \text{blkdiag}\,[\mathbf{M}_1, \ \ \mathbf{M}_2, \ \ \cdots, \ \mathbf{M}_{q^\dagger}], \quad \mathbf{\Lambda} = \text{blkdiag}\,[\mathbf{\Lambda}_1, \ \ \mathbf{\Lambda}_2, \ \ \cdots, \ \mathbf{\Lambda}_{k^\dagger}]. \tag{20}$$

where $\mathbf{M}_j = \text{diag}\,[\mu_1^{(j)}, \ \ \mu_2^{(j)}, \ \ \mu_3^{(j)}]$ and $\mathbf{\Lambda}_i = \text{diag}\,[\lambda_1^{(i)}, \ \ \lambda_2^{(i)}, \ \ \lambda_3^{(i)}]$.

Additionally, introduce the following matrices (for $j \in \{1, 2, \ldots, q^\dagger\}$)

$$\mathbf{X}^{(j)} = \mathbf{e}_{j,q^\dagger} \otimes \mathbf{e}_{j,q^\dagger}^T \otimes \mathbf{e}_{1,3} \otimes \mathbf{e}_{3,3}^T \ \in \mathbb{R}^{q \times q}, \tag{21}$$

$$\mathbf{T}^{(j)} = \mathbf{I}_n \otimes \mathbf{e}_{3j-2,k} \ \in \mathbb{R}^{nk \times n}, \tag{22}$$

$$\mathbf{U}^{(j)} = \mathbf{e}_{3j-2,k} \otimes \mathbf{I}_k \ \in \mathbb{R}^{k^2 \times k}. \tag{23}$$

Lemma 3.2 *The generalized reachability matrix $\mathcal{R}$ defined in (10) satisfy the following generalized Sylvester equation:*

$$\mathbf{A}\,\mathcal{R} + \mathbf{Q}\big(\mathcal{R} \otimes \mathcal{R}\big)\mathbf{Y}_{\mathbf{R}} + \mathbf{N}\,\mathcal{R}\,\mathbf{Z}_{\mathbf{R}} + \mathbf{B}\,\mathbf{R} = \mathbf{E}\,\mathcal{R}\,\mathbf{\Lambda}. \tag{24}$$

Proof 3.1 Multiply equation (24) to the right with the unit vector $\mathbf{e}_{3j-2,k}$ $(1 \leqslant j \leqslant k^\dagger)$

$$\mathbf{A}\mathcal{R}_{3j-2} + \mathbf{B} = \lambda_{3j-2}\mathbf{E}\mathcal{R}_{3j-2} \quad \Leftrightarrow \quad \mathcal{R}_{3j-2} = (\lambda_{3j-2}\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} = \mathbf{\Phi}(\lambda_{3j-2})\mathbf{B}. \tag{25}$$

Thus the $(3j-2)^{\text{th}}$ column of the matrix which is the solution of (24) is indeed equal to the $(3j-2)^{\text{th}}$ column of the generalized controllability matrix $\mathcal{R}$. By multiplying the same equation on the right with the unit vector $\mathbf{e}_{3j-1,k}$ obtain

$$\mathbf{A}\mathcal{R}_{3j-1} + \mathbf{N}\mathcal{R}_{3j-2} = \lambda_{3j-1}\mathbf{E}\mathcal{R}_{3j-1} \quad \Leftrightarrow \quad \mathcal{R}_{3j-1} = (\lambda_{3j-1}\mathbf{E} - \mathbf{A})^{-1}\mathbf{N}\mathcal{R}_{3j-2}. \tag{26}$$

By substituting (25) into (26), we get that

$$\mathcal{R}_{3j-1} = (\lambda_{3j-1}\mathbf{E} - \mathbf{A})^{-1}\mathbf{N}(\lambda_{3j-2}\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} = \mathbf{\Phi}(\lambda_{3j-1})\mathbf{N}\mathbf{\Phi}(\lambda_{3j-2})\mathbf{B}.$$

By again multiplying equation (24) to the right, this time with the unit vector $\mathbf{e}_{3j,k}$, write

$$\begin{aligned} &\mathbf{A}\mathcal{R}_{3j} + \mathbf{Q}\big(\mathcal{R} \otimes \mathcal{R}\big)\big(\mathbf{e}_{3j-2,k} \otimes \mathbf{e}_{3j-2,k}\big) = \lambda_{3j}\mathbf{E}\mathcal{R}_{3j} \Leftrightarrow (\lambda_{3j}\mathbf{E} - \mathbf{A})\mathcal{R}_{3j} = \mathbf{Q}\big(\mathcal{R}_{3j-2} \otimes \mathcal{R}_{3j-2}\big) \\ &\Leftrightarrow \mathcal{R}_{3j} = (\lambda_{3j}\mathbf{E} - \mathbf{A})^{-1}\mathbf{Q}\big(\mathcal{R}_{3j-2} \otimes \mathcal{R}_{3j-2}\big) = \mathbf{\Phi}_{\lambda_{3j}}\mathbf{Q}\big(\mathcal{R}_{3j-2} \otimes \mathcal{R}_{3j-2}\big). \end{aligned} \tag{27}$$

By substituting (25) into (27), we get that

$$\mathcal{R}_{3j} = \mathbf{\Phi}(\lambda_{3j})\mathbf{Q}\big(\mathbf{\Phi}(\lambda_{3j-2})\mathbf{B} \otimes \mathbf{\Phi}(\lambda_{3j-2})\mathbf{B}\big).$$

By putting together all the results above, it follows that indeed, the generalized reachability matrix $\mathcal{R}$ construcetd in (10) satisfies equation (24).

Lemma 3.3 *The generalized observability matrix $\mathcal{O}$ defined in (11) satisfies the following generalized Sylvester equation:*

$$\mathcal{O}\,\mathbf{A} + \sum_{j=1}^{q^\dagger} \mathbf{X}^{(j)}\mathcal{O}\mathbf{Q}\big(\mathcal{R}_{3j-2} \otimes \mathbf{I}\big) + \mathbf{Z}_{\mathbf{L}}\,\mathcal{O}\,\mathbf{N} + \mathbf{L}\,\mathbf{C} = \mathbf{M}\,\mathcal{O}\,\mathbf{E}, \tag{28}$$

or equivalently

$$\mathcal{O}\,\mathbf{A} + \sum_{j=1}^{q^\dagger} \mathbf{X}^{(j)}\mathcal{O}\mathbf{Q}\big(\mathbf{I} \otimes \mathcal{R}\big)\mathbf{T}^{(j)} + \mathbf{Z}_{\mathbf{L}}\,\mathcal{O}\,\mathbf{N} + \mathbf{L}\,\mathbf{C} = \mathbf{M}\,\mathcal{O}\,\mathbf{E}. \tag{29}$$

Proof 3.2 Multiply equation (28) to the right with the row vector $\mathbf{e}_{3j-2,k}^T$ $(1 \leqslant j \leqslant k^\dagger)$

$$\mathcal{O}_{3j-2}^T\mathbf{A} + \mathbf{C} = \mu_{3j-2}\mathcal{O}_{3j-2}^T\mathbf{E} \quad \Leftrightarrow \quad \mathcal{O}_{3j-2}^T = \mathbf{C}(\mu_{3j-2}\mathbf{E} - \mathbf{A})^{-1} = \mathbf{C}\boldsymbol{\Phi}(\mu_{3j-2}). \tag{30}$$

Multiplying the same equation on the right with the row vector $\mathbf{e}_{3j-1,k}^T$ obtain:

$$\mathcal{O}_{3j-1}^T\mathbf{A} + \mathcal{O}_{3j-2}^T\mathbf{N} = \mu_{3j-1}\mathcal{O}_{3j-1}^T\mathbf{E} \quad \Leftrightarrow \quad \mathcal{O}_{3j-1}^T = \mathcal{O}_{3j-2}^T\mathbf{N}(\mu_{3j-1}\mathbf{E} - \mathbf{A})^{-1}. \tag{31}$$

By substituting (30) into (31), we get that

$$\mathcal{O}_{3j-1}^T = \mathbf{C}(\mu_{3j-2}\mathbf{E} - \mathbf{A})^{-1}\mathbf{N}(\mu_{3j-1}\mathbf{E} - \mathbf{A})^{-1} = \mathbf{C}\boldsymbol{\Phi}(\mu_{3j-2})\mathbf{N}\boldsymbol{\Phi}(\mu_{3j-1}).$$

Finally, multiplying equation (28) to the right, this time with the row vector $\mathbf{e}_{3j,k}^T$, write

$$\begin{aligned} &\mathcal{O}_{3j}^T\mathbf{A} + \mathcal{O}_{3j-2}^T\mathbf{Q}\big(\mathcal{R} \otimes \mathbf{I}\big) = \mu_{3j}\mathcal{O}_{3j}^T\mathbf{E} \quad \Leftrightarrow \quad \mathcal{O}_{3j}^T(\mu_{3j}\mathbf{E} - \mathbf{A}) = \mathcal{O}_{3j-2}^T\mathbf{Q}\big(\mathcal{R}_{3j-2} \otimes \mathbf{I}\big) \\ &\Leftrightarrow \mathcal{O}_{3j}^T = \mathcal{O}_{3j-2}^T\mathbf{Q}\big(\mathcal{R}_{3j-2} \otimes \mathbf{I}\big)(\mu_{3j}\mathbf{E} - \mathbf{A})^{-1} = \mathcal{O}_{3j-2}^T\mathbf{Q}\big(\mathcal{R}_{3j-2} \otimes \boldsymbol{\Phi}(\mu_{3j})\big). \end{aligned} \tag{32}$$

By substituting (30) into (32), we get that

$$\mathcal{O}_{3j} = \mathbf{C}\boldsymbol{\Phi}(\mu_{3j-2})\mathbf{Q}\big(\boldsymbol{\Phi}(\lambda_{3j-2})\mathbf{B} \otimes \boldsymbol{\Phi}(\mu_{3j})\big).$$

By putting together all the results above, it follows that indeed, the generalized observability matrix $\mathcal{O}$ construcetd in (11) satisfies equation (28). We can write for $j \in \{1, 2, \ldots, k^\dagger\}$

$$\mathbf{Q}\big(\mathcal{R}_{3j-2} \otimes \mathbf{I}\big) = \mathbf{Q}\big(\mathbf{I} \otimes \mathcal{R}_{3j-2}\big) = \mathbf{Q}\big(\mathbf{I} \otimes \mathcal{R}\mathbf{e}_{3j-2,k}\big) = \mathbf{Q}\big(\mathbf{I} \otimes \mathcal{R}\big)\big(\mathbf{I} \otimes \mathbf{e}_{3j-2,k}\big) = \mathbf{Q}\big(\mathbf{I} \otimes \mathcal{R}\big)\mathbf{T}^{(j)}.$$

Hence justify that the equation (28) can be rewritten as (29).

Proposition 3.1 *Moreover, equation (28) can be further simplified by replacing the* $\mathbf{Q}$ *matrix with* $\mathbf{Q}^{(-1)}$ *(introduced in Definition 2.2.19) as follows,*

$$\mathcal{O}\,\mathbf{A} + \mathbf{Y_L}\big(\mathcal{O} \otimes \mathcal{R}^T\big)\big(\mathbf{Q}^{(-1)}\big)^T + \mathbf{Z_L}\,\mathcal{O}\,\mathbf{N} + \mathbf{L}\,\mathbf{C} = \mathbf{M}\,\mathcal{O}\,\mathbf{E}. \tag{33}$$

Proof 3.3 First show that, for all $j \in \{1, 2, \ldots, q^\dagger\}$, we have that $\mathbf{X}^{(j)} \otimes \mathbf{e}_{3j-2,k}^T = \mathbf{Y}_\mathbf{L}^{(j)}$. Note that, using the original definition of $\mathbf{X}^{(j)}$ and $\mathbf{Y}_\mathbf{L}^{(j)}$ from (21) and (19) as well as the the result in (5), one can write the following

$$\begin{aligned} \mathbf{X}^{(j)} \otimes \mathbf{e}_{3j-2,k}^T &= \mathbf{e}_{j,k^\dagger} \otimes \mathbf{e}_{j,k^\dagger}^T \otimes \mathbf{e}_{1,3}^T \otimes \mathbf{e}_{3,3} \otimes \mathbf{e}_{3j-2,k}^T = \mathbf{e}_{j,k^\dagger} \otimes \mathbf{e}_{3j-2,k}^T \otimes \mathbf{e}_{3,3} \otimes \mathbf{e}_{3j-2,k}^T \\ &= \big(\mathbf{e}_{j,k^\dagger}\mathbf{e}_{3j-2,k}^T\big) \otimes \big(\mathbf{e}_{3,3} \otimes \mathbf{e}_{3j-2,k}^T\big) = \big(\mathbf{e}_{j,k^\dagger} \otimes \mathbf{e}_{3,3}\big)\big(\mathbf{e}_{3j-2,k}^T \otimes \mathbf{e}_{3j-2,k}^T\big) \\ &= \mathbf{e}_{3j,k}\big(\mathbf{e}_{3j-2,k}^T \otimes \mathbf{e}_{3j-2,k}^T\big), \\ \mathbf{Y}_\mathbf{L}^{(j)} &= \mathbf{e}_{3j-2,k}^T \otimes \mathbf{e}_{3j-2,k}^T \otimes \mathbf{e}_{3j,k} = \mathbf{e}_{3j,k}\big(\mathbf{e}_{3j-2,k}^T \otimes \mathbf{e}_{3j-2,k}^T\big). \end{aligned}$$

We apply the result in (7) for $\mathbf{v} = \mathcal{R}_{3j-2}$, $j \in \{1, 2, \ldots, k^\dagger\}$, i.e.,

$$\mathbf{Q}\big(\mathcal{R}_{3j-2} \otimes \mathbf{I}\big) = \big(\mathbf{I} \otimes \mathcal{R}_{3j-2}^T\big)\big(\mathbf{Q}^{(-1)}\big)^T.$$

By multiplying to the left with $\mathbf{X}^{(j)}\mathcal{O}$ (for $j \in \{1, 2\}$), it follows that

$$\begin{aligned} &\mathbf{X}^{(j)}\mathcal{O}\mathbf{Q}\big(\mathcal{R}_{3j-2} \otimes \mathbf{I}\big) = \mathbf{X}^{(j)}\mathcal{O}\big(\mathbf{I} \otimes \mathcal{R}_{3j-2}^T\big)\big(\mathbf{Q}^{(-1)}\big)^T = \mathbf{X}^{(j)}\big(\mathcal{O} \otimes \mathcal{R}_{3j-2}^T\big)\big(\mathbf{Q}^{(-1)}\big)^T \\ &= \mathbf{X}^{(j)}\Big[\big(\mathbf{I}_k\mathcal{O}\big) \otimes \big(\mathbf{e}_{3j-2,k}^T\mathcal{R}^T\big)\Big]\big(\mathbf{Q}^{(-1)}\big)^T = \mathbf{X}^{(j)}\big(\mathbf{I}_k \otimes \mathbf{e}_{3j-2,k}^T\big)\big(\mathcal{O} \otimes \mathcal{R}^T\big)\big(\mathbf{Q}^{(-1)}\big)^T \\ &= \underbrace{\big(\mathbf{X}^{(j)} \otimes \mathbf{e}_{3j-2,k}^T}_{\mathbf{Y}_\mathbf{L}^{(j)}}\big)\big(\mathcal{O} \otimes \mathcal{R}^T\big)\big(\mathbf{Q}^{(-1)}\big)^T = \mathbf{Y}_\mathbf{L}^{(j)}\big(\mathcal{O} \otimes \mathcal{R}^T\big)\big(\mathbf{Q}^{(-1)}\big)^T. \end{aligned} \tag{34}$$

By substituting (34) into (28), it follows that the equation which characterizes the observability matrix $\mathcal{O}$, can be rewritten as in (33).

Corollary 3.1 The Sylvester equation in (33) has unique solution if the interpolation points in (20)are chosen so that the Sylvester operator

$$\mathcal{L}_{\mathcal{O}} = \mathbf{A}^T \otimes \mathbf{I}_n - \mathbf{E}^T \otimes \mathbf{M} + \big(\mathbf{Q}^{(-1)} \otimes \mathbf{Y}_{\mathbf{L}}\big)\mathbf{J}\big(\mathrm{vec}(\mathcal{R}^T) \otimes \mathbf{I}_{n^2}\big) + \mathbf{N}^T \otimes \mathbf{Z}_{\mathbf{L}}$$

is invertible, i.e., has no zero eigenvalues. Here, $\mathbf{J} \in \mathbb{R}^{n^4 \times n^4}$ is a permutation matrix that allows to write the vectorization of the Kronecker product between any two matrices $\mathbf{U}, \mathbf{V} \in \mathbb{R}^n$ as follows

$$\mathrm{vec}(\mathbf{U} \otimes \mathbf{V}) = \mathbf{J}\big(\mathrm{vec}(\mathbf{V}) \otimes \mathbf{I}_{n^2}\big)\mathrm{vec}(\mathbf{U})$$

Here it is assumed that the reachability matrix $\mathcal{R}$ has been already computed.

3.2 The generalized Loewner pencil

Given the notations introduced in section 5.3.1, we introduce the appropriate generalizations of the Loewner matrices.

Definition 3.3 *Consider a quadratic-bilinear system* $\mathbf{\Sigma}_{QB}$*, and let* $\mathcal{R}$ *and* $\mathcal{O}$ *be the reachability and observability matrices associated with the multi-tuples (8) and defined by (10), (11) respectively. The* **Loewner matrix** $\mathbb{L}$*, and the* **shifted Loewner** *matrix* $\mathbb{L}_s$ *are defined as*

$$\mathbb{L} = -\mathcal{O}\,\mathbf{E}\,\mathcal{R}, \quad \mathbb{L}_s = -\mathcal{O}\,\mathbf{A}\,\mathcal{R}. \tag{35}$$

In addition we define the quantities

$$\mathbf{\Omega} = \mathcal{O}\,\mathbf{Q}\,(\mathcal{R} \otimes \mathcal{R}), \quad \mathbf{\Psi} = \mathcal{O}\,\mathbf{N}\,\mathcal{R}, \quad \mathbf{V} = \mathcal{O}\,\mathbf{B} \;\; \textit{and} \;\; \mathbf{W} = \mathbf{C}\,\mathcal{R}. \tag{36}$$

Note that $\mathbb{L}$ and $\mathbb{L}_s$ as defined above are indeed Loewner matrices, that is, they can be expressed as divided differences of appropriate transfer function values of the underlying bilinear system, as shown in the next example.

Example 3.2 Given the SISO quadratic-bilinear system $\mathbf{\Sigma}_{QB}$ characterized by the collection of matrices $(\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{Q}, \mathbf{N}, \mathbf{B})$, where $\mathbf{E}, \mathbf{A}, \mathbf{N} \in \mathbb{R}^{n \times n}, \mathbf{Q} \in \mathbb{R}^{n \times n^2}$ and $\mathbf{B}, \mathbf{C}^T \in \mathbb{R}^n$, consider 6 interpolation points $\{\omega_1, \omega_2, \ldots, \omega_6\}$. First partition this set into two disjoint sets corresponding to left and right points, as $\{\mu_1, \mu_2, \mu_3\} \cup \{\lambda_1, \lambda_2, \lambda_3\}$. Next consider the ordered tuples of right and left interpolation points, i.e.,

$$\boldsymbol{\lambda} = \{\ (\lambda_1),\ (\lambda_2, \lambda_1),\ (\lambda_3, \lambda_1, \lambda_1)\ \}, \quad \boldsymbol{\mu} = \{\ (\mu_1),\ (\mu_1, \mu_2),\ (\mu_1, \lambda_1, \mu_3)\ \}.$$

The associated *generalized observability and reachability* matrices are constructed

$$\mathcal{R} = \left[\ \mathbf{\Phi}(\lambda_1)\mathbf{B} \quad \mathbf{\Phi}(\lambda_2)\mathbf{N}\mathbf{\Phi}(\lambda_1)\mathbf{B} \quad \mathbf{\Phi}(\lambda_3)\mathbf{Q}\big(\mathbf{\Phi}(\lambda_1)\mathbf{B} \otimes \mathbf{\Phi}(\lambda_1)\mathbf{B}\big)\ \right],$$

$$\mathcal{O} = \begin{bmatrix} \mathbf{C}\mathbf{\Phi}(\mu_1) \\ \mathbf{C}\mathbf{\Phi}(\mu_1)\mathbf{N}\mathbf{\Phi}(\mu_2) \\ \mathbf{C}\mathbf{\Phi}(\mu_1)\mathbf{Q}\big(\mathbf{\Phi}(\lambda_1)\mathbf{B} \otimes \mathbf{\Phi}(\mu_3)\big) \end{bmatrix}.$$

Apart from the linear transfer function $\mathbf{H}(s) = \mathbf{C}\mathbf{\Phi}(s)\mathbf{B}$, consider the following *level 1* transfer functions, as

$$\begin{cases} \mathbf{H}_1^{\mathbf{N}}(s_1, s_2) = \mathbf{C}\mathbf{\Phi}(s_1)\mathbf{N}\mathbf{\Phi}(s_2)\mathbf{B}, \\ \mathbf{H}_1^{\mathbf{Q}}(s_1, s_2, s_3) = \mathbf{C}\mathbf{\Phi}(s_1)\mathbf{Q}\big(\mathbf{\Phi}(s_2)\mathbf{B} \otimes \mathbf{\Phi}(s_3)\big)\mathbf{B}. \end{cases} \tag{37}$$

Also mention the following *level 2* transfer functions

$$\begin{cases} \mathbf{H}_2^{\mathbf{N},\mathbf{N}}(s_1,s_2,s_3) = \mathbf{C}\mathbf{\Phi}(s_1)\mathbf{N}\mathbf{\Phi}(s_2)\mathbf{N}\mathbf{\Phi}(s_3)\mathbf{B}, \\ \mathbf{H}_2^{\mathbf{N},\mathbf{Q}}(s_1,s_2,s_3,s_4) = \mathbf{C}\mathbf{\Phi}(s_1)\mathbf{N}\mathbf{\Phi}(s_2)\mathbf{Q}\big(\mathbf{\Phi}(s_3)\mathbf{B}\otimes\mathbf{\Phi}(s_4)\big)\mathbf{B}, \\ \mathbf{H}_2^{\mathbf{Q},\mathbf{N}}(s_1,s_2,s_3,s_4) = \mathbf{C}\mathbf{\Phi}(s_1)\mathbf{Q}\big(\mathbf{\Phi}(s_2)\mathbf{B}\otimes\mathbf{\Phi}(s_3)\big)\mathbf{N}\mathbf{\Phi}(s_4)\mathbf{B}, \\ \mathbf{H}_2^{\mathbf{Q},\mathbf{Q}}(s_1,s_2,s_3,s_4,s_5) = \mathbf{C}\mathbf{\Phi}(s_1)\mathbf{Q}\big(\mathbf{\Phi}(s_2)\mathbf{B}\otimes\mathbf{\Phi}(s_3)\big)\mathbf{Q}\big(\mathbf{\Phi}(s_4)\mathbf{B}\otimes\mathbf{\Phi}(s_5)\big)\mathbf{B}. \end{cases} \tag{38}$$

Remark 3.2 The purely bilinear transfer functions were already used in chapter 4 in (**??**). To make the notation consistent, one can use the following conversion formula

$$\mathbf{H}_j^{\mathbf{N},\mathbf{N},\ldots,\mathbf{N}}(s_1,\ldots,s_{j+1}) = \mathbf{H}_{j+1}(s_1,\ldots,s_{j+1}).$$

Remark 3.3 *The recovered system matrices corresponding to* $\mathbf{E},\mathbf{A},\mathbf{B}$ *and* $\mathbf{C}$ *can be directly written using only samples coming from the 7 transfer functions mentioned above in (37) and in (38) (including the linear one* $\mathbf{H}(s)$*).*

Next write the Loewner matrix $\mathbb{L} = -\mathcal{O}\mathbf{E}\mathcal{R}$ as a divided difference matrix as follows

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{H}(\mu_1)-\mathbf{H}(\lambda_1)}{\mu_1-\lambda_1} & \frac{\mathbf{H}_1^{\mathbf{N}}(\mu_1,\lambda_1)-\mathbf{H}_1^{\mathbf{N}}(\lambda_2,\lambda_1)}{\mu_1-\lambda_2} & \frac{\mathbf{H}_1^{\mathbf{Q}}(\mu_1,\lambda_1,\lambda_1)-\mathbf{H}_1^{\mathbf{Q}}(\lambda_3,\lambda_1,\lambda_1)}{\mu_1-\lambda_3} \\ \frac{\mathbf{H}_1^{\mathbf{N}}(\mu_1,\mu_2)-\mathbf{H}_1^{\mathbf{N}}(\mu_1,\lambda_1)}{\mu_2-\lambda_1} & \frac{\mathbf{H}_2^{\mathbf{N},\mathbf{N}}(\mu_1,\mu_2,\lambda_1)-\mathbf{H}_2^{\mathbf{N},\mathbf{N}}(\mu_1,\lambda_2,\lambda_1)}{\mu_2-\lambda_2} & \frac{\mathbf{H}_2^{\mathbf{N},\mathbf{Q}}(\mu_1,\mu_2,\lambda_1,\lambda_1)-\mathbf{H}_2^{\mathbf{N},\mathbf{Q}}(\mu_1,\lambda_3,\lambda_1,\lambda_1)}{\mu_2-\lambda_3} \\ \frac{\mathbf{H}_1^{\mathbf{Q}}(\mu_1,\lambda_1,\mu_3)-\mathbf{H}_1^{\mathbf{Q}}(\mu_1,\lambda_1,\lambda_1)}{\mu_3-\lambda_1} & \frac{\mathbf{H}_2^{\mathbf{Q},\mathbf{N}}(\mu_1,\lambda_1,\mu_3,\lambda_1)-\mathbf{H}_2^{\mathbf{Q},\mathbf{N}}(\mu_1,\lambda_1,\lambda_2,\lambda_1)}{\mu_3-\lambda_2} & \frac{\mathbf{H}_2^{\mathbf{Q},\mathbf{Q}}(\mu_1,\lambda_1,\mu_3,\lambda_1,\lambda_1)-\mathbf{H}_2^{\mathbf{Q},\mathbf{Q}}(\mu_1,\lambda_1,\lambda_3,\lambda_1,\lambda_1)}{\mu_3-\lambda_3} \end{bmatrix}$$

Similarly, the shifted Loewner matrix is written as a divided difference matrix of samples coming from the same type of transfer functions.

$$\mathbb{L}_s = \begin{bmatrix} \frac{\mu_1\mathbf{H}(\mu_1)-\lambda_1\mathbf{H}(\lambda_1)}{\mu_1-\lambda_1} & \cdots & \frac{\mu_1\mathbf{H}_1^{\mathbf{Q}}(\mu_1,\lambda_1,\lambda_1)-\lambda_3\mathbf{H}_1^{\mathbf{Q}}(\lambda_3,\lambda_1,\lambda_1)}{\mu_1-\lambda_3} \\ \frac{\mu_2\mathbf{H}_1^{\mathbf{N}}(\mu_1,\mu_2)-\lambda_1\mathbf{H}_1^{\mathbf{N}}(\mu_1,\lambda_1)}{\mu_2-\lambda_1} & \ddots & \frac{\mu_2\mathbf{H}_2^{\mathbf{N},\mathbf{Q}}(\mu_1,\mu_2,\lambda_1,\lambda_1)-\lambda_3\mathbf{H}_2^{\mathbf{N},\mathbf{Q}}(\mu_1,\lambda_3,\lambda_1,\lambda_1)}{\mu_2-\lambda_3} \\ \frac{\mu_3\mathbf{H}_1^{\mathbf{Q}}(\mu_1,\lambda_1,\mu_3)-\lambda_1\mathbf{H}_1^{\mathbf{Q}}(\mu_1,\lambda_1,\lambda_1)}{\mu_3-\lambda_1} & \cdots & \frac{\mu_3\mathbf{H}_2^{\mathbf{Q},\mathbf{Q}}(\mu_1,\lambda_1,\mu_3,\lambda_1,\lambda_1)-\lambda_3\mathbf{H}_2^{\mathbf{Q},\mathbf{Q}}(\mu_1,\lambda_1,\lambda_3,\lambda_1,\lambda_1)}{\mu_3-\lambda_3} \end{bmatrix}.$$

Note that the $\mathbf{V}$ and $\mathbf{W}$ vectors can be written in terms of samples coming from *level* 0 and 1 transfer functions in the following way

$$\mathbf{V} = \begin{bmatrix} \mathbf{H}(\mu_1) \\ \mathbf{H}_1^{\mathbf{N}}(\mu_1,\mu_2) \\ \mathbf{H}_1^{\mathbf{Q}}(\mu_1,\lambda_1,\mu_3) \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} \mathbf{H}(\lambda_1) & \mathbf{H}_1^{\mathbf{N}}(\lambda_2,\lambda_1) & \mathbf{H}_1^{\mathbf{Q}}(\lambda_3,\lambda_1,\lambda_1) \end{bmatrix}.$$

Consider the following *level* 3 transfer functions

$$\begin{cases} \mathbf{H}_3^{\mathbf{N},\mathbf{N},\mathbf{N}}(s_1,s_2,s_3,s_4) = \mathbf{C}\mathbf{\Phi}(s_1)\mathbf{N}\mathbf{\Phi}(s_2)\mathbf{N}\mathbf{\Phi}(s_3)\mathbf{N}\mathbf{\Phi}(s_4)\mathbf{B}, \\ \mathbf{H}_2^{\mathbf{N},\mathbf{N},\mathbf{Q}}(s_1,s_2,s_3,s_4,s_5) = \mathbf{C}\mathbf{\Phi}(s_1)\mathbf{N}\mathbf{\Phi}(s_2)\mathbf{N}\mathbf{\Phi}(s_3)\mathbf{Q}\big(\mathbf{\Phi}(s_4)\mathbf{B}\otimes\mathbf{\Phi}(s_5)\big)\mathbf{B}, \\ \mathbf{H}_2^{\mathbf{Q},\mathbf{N},\mathbf{N}}(s_1,s_2,s_3,s_4,s_5) = \mathbf{C}\mathbf{\Phi}(s_1)\mathbf{Q}\big(\mathbf{\Phi}(s_2)\mathbf{B}\otimes\mathbf{\Phi}(s_3)\big)\mathbf{N}\mathbf{\Phi}(s_4)\mathbf{N}\mathbf{\Phi}(s_5)\mathbf{B}, \\ \mathbf{H}_2^{\mathbf{Q},\mathbf{N},\mathbf{Q}}(s_1,\ldots,s_6) = \mathbf{C}\mathbf{\Phi}(s_1)\mathbf{Q}\big(\mathbf{\Phi}(s_2)\mathbf{B}\otimes\mathbf{\Phi}(s_3)\big)\mathbf{N}\mathbf{\Phi}(s_4)\mathbf{Q}\big(\mathbf{\Phi}(s_5)\mathbf{B}\otimes\mathbf{\Phi}(s_6)\mathbf{B}\big). \end{cases} \tag{39}$$

Remark 3.4 The recovered $\mathbf{N}$ matrix, i.e., $\Xi = \mathcal{O}\mathbf{N}\mathcal{R}$, can be written solely in terms of samples coming from transfer functions of the first two levels and in terms of the four functions mentioned in (39). Hence write the matrix $\Xi \in \mathbb{R}^{3\times 3}$ as

$$\mathbf{\Xi} = \begin{bmatrix} \mathbf{H}_1^{\mathbf{N}}(\mu_1,\lambda_1) & \mathbf{H}_2^{\mathbf{N},\mathbf{N}}(\mu_1,\lambda_2,\lambda_1) & \mathbf{H}_2^{\mathbf{N},\mathbf{Q}}(\mu_1,\lambda_3,\lambda_2,\lambda_1) \\ \mathbf{H}_2^{\mathbf{N},\mathbf{N}}(\mu_1,\mu_2,\lambda_1) & \mathbf{H}_3^{\mathbf{N},\mathbf{N},\mathbf{N}}(\mu_1,\mu_2,\lambda_2,\lambda_1) & \mathbf{H}_3^{\mathbf{N},\mathbf{N},\mathbf{Q}}(\mu_1,\mu_2,\lambda_3,\lambda_2,\lambda_1) \\ \mathbf{H}_2^{\mathbf{Q},\mathbf{N}}(\mu_1,\lambda_1,\mu_3,\lambda_1) & \mathbf{H}_3^{\mathbf{Q},\mathbf{N},\mathbf{N}}(\mu_1,\lambda_1,\mu_3,\lambda_2,\lambda_1) & \mathbf{H}_3^{\mathbf{Q},\mathbf{N},\mathbf{Q}}(\mu_1,\lambda_1,\mu_2,\lambda_3,\lambda_2,\lambda_1) \end{bmatrix}$$

Consider the other four transfer functions that correspond to *level* 3, i.e.,

$$\begin{cases} \mathbf{H}_2^{\mathbf{N},\mathbf{Q},\mathbf{N}}(s_1,\dots,s_5) = \mathbf{C}\mathbf{\Phi}(s_1)\mathbf{N}\mathbf{\Phi}(s_2)\mathbf{Q}\big(\mathbf{\Phi}(s_3)\mathbf{B}\otimes\mathbf{\Phi}(s_4)\big)\mathbf{N}\mathbf{\Phi}(s_5)\mathbf{B}, \\ \mathbf{H}_2^{\mathbf{N},\mathbf{Q},\mathbf{Q}}(s_1,\dots,s_6) = \mathbf{C}\mathbf{\Phi}(s_1)\mathbf{N}\mathbf{\Phi}(s_2)\mathbf{Q}\big(\mathbf{\Phi}(s_3)\mathbf{B}\otimes\mathbf{\Phi}(s_4)\big)\mathbf{Q}\big(\mathbf{\Phi}(s_5)\mathbf{B}\otimes\mathbf{\Phi}(s_6)\big)\mathbf{B}, \\ \mathbf{H}_3^{\mathbf{Q},\mathbf{Q},\mathbf{N}}(s_1,\dots,s_6) = \mathbf{C}\mathbf{\Phi}(s_1)\mathbf{Q}\big(\mathbf{\Phi}(s_2)\mathbf{B}\otimes\mathbf{\Phi}(s_3)\big)\mathbf{Q}\big(\mathbf{\Phi}(s_4)\mathbf{B}\otimes\mathbf{\Phi}(s_5)\big)\mathbf{N}\mathbf{\Phi}(s_6)\mathbf{B}, \\ \mathbf{H}_2^{\mathbf{Q},\mathbf{Q},\mathbf{Q}}(s_1,\dots,s_7) = \mathbf{C}\mathbf{\Phi}(s_1)\mathbf{Q}\big(\mathbf{\Phi}(s_2)\mathbf{B}\otimes\mathbf{\Phi}(s_3)\big)\mathbf{Q}\big(\mathbf{\Phi}(s_4)\mathbf{B}\otimes\mathbf{\Phi}(s_5)\big)\mathbf{Q}\big(\mathbf{\Phi}(s_6)\mathbf{B}\otimes\mathbf{\Phi}(s_7)\big)\mathbf{B}. \end{cases} \tag{40}$$

Remark 3.5 The recovered $\mathbf{Q}$ matrix, i.e., $\mathbf{\Omega} = \mathcal{O}\mathbf{Q}\mathcal{R}$, can be written in terms of samples coming from transfer functions previously mentioned in (37), (38) and (39) and, on top of that, in terms of functions in (40).

Hence write the matrix $\mathbf{\Omega} \in \mathbb{R}^{3\times 9}$ as follows

$$\mathbf{\Omega} = \begin{bmatrix} \mathbf{H}_1^{\mathbf{Q}}(\mu_1,\lambda_1,\lambda_1) & \mathbf{H}_2^{\mathbf{Q},\mathbf{N}}(\mu_1,\lambda_1,\lambda_2,\lambda_1) & \mathbf{H}_2^{\mathbf{Q},\mathbf{Q}}(\mu_1,\lambda_1,\lambda_3,\lambda_1,\lambda_1) & \cdots \\ \mathbf{H}_2^{\mathbf{N},\mathbf{Q}}(\mu_1,\mu_2,\lambda_1,\lambda_1) & \mathbf{H}_3^{\mathbf{N},\mathbf{Q},\mathbf{N}}(\mu_1,\mu_2,\lambda_1,\lambda_2,\lambda_1) & \mathbf{H}_3^{\mathbf{N},\mathbf{Q},\mathbf{Q}}(\mu_1,\mu_2,\lambda_1,\lambda_3,\lambda_1,\lambda_1) & \ddots \\ \mathbf{H}_2^{\mathbf{Q},\mathbf{Q}}(\mu_1,\lambda_1,\mu_3,\lambda_1,\lambda_1) & \mathbf{H}_3^{\mathbf{Q},\mathbf{Q},\mathbf{N}}(\mu_1,\lambda_1,\mu_3,\lambda_1,\lambda_2,\lambda_1) & \mathbf{H}_3^{\mathbf{Q},\mathbf{Q},\mathbf{Q}}(\mu_1,\lambda_1,\mu_3,\lambda_3,\lambda_1,\lambda_3,\lambda_1,\lambda_1) & \cdots \end{bmatrix}$$

Remark 3.6 Some entries of the $\mathbf{\Omega}$ matrix can not be directly written in terms of samples of transfer functions defined in (4). Nevertheless, we can overcome this issue by altering the $\mathbf{Q}$ matrix. In this way we successfully keep the simplified format of transfer functions from (4). For example, we rewrite the entries of $\mathbf{\Omega}$ as follows (for brevity choose only two examples)

$$\begin{aligned} \mathbf{\Omega}(1,5) &= \mathcal{O}_1\mathbf{Q}\big(\mathcal{R}_2\otimes\mathcal{R}_2\big) = \mathbf{C}\mathbf{\Phi}(\mu_1)\mathbf{Q}\big(\mathbf{\Phi}(\lambda_2)\mathbf{N}\mathbf{\Phi}(\lambda_1)\mathbf{B}\otimes\mathbf{\Phi}(\lambda_2)\mathbf{N}\mathbf{\Phi}(\lambda_1)\mathbf{B}\big) \\ &= \mathbf{C}\mathbf{\Phi}(\mu_1)\mathbf{Q}\big(\mathbf{\Phi}(\lambda_2)\mathbf{N}\mathbf{\Phi}(\lambda_1)\mathbf{B}\otimes\mathbf{\Phi}(\lambda_2)\big)\mathbf{N}\mathbf{\Phi}(\lambda_1)\mathbf{B} \\ &= \mathbf{C}\mathbf{\Phi}(\mu_1)\underbrace{\mathbf{Q}\big(\mathbf{\Phi}(\lambda_2)\mathbf{N}\otimes\mathbf{I}\big)}_{\bar{\mathbf{Q}}}\big(\mathbf{\Phi}(\lambda_1)\mathbf{B}\otimes\mathbf{\Phi}(\lambda_2)\big)\mathbf{N}\mathbf{\Phi}(\lambda_1)\mathbf{B} = \mathbf{H}_2^{\bar{\mathbf{Q}},\mathbf{N}}(\mu_1,\lambda_1,\lambda_2,\lambda_1). \end{aligned}$$

$$\begin{aligned} \mathbf{\Omega}(1,9) &= \mathcal{O}_1\mathbf{Q}\big(\mathcal{R}_3\otimes\mathcal{R}_3\big) = \mathbf{C}\mathbf{\Phi}(\mu_1)\mathbf{Q}\big(\mathbf{\Phi}(\lambda_3)\mathbf{Q}\mathbf{\Phi}(\lambda_1)\mathbf{B}\otimes\mathbf{\Phi}(\lambda_1)\mathbf{B}\big)\otimes\big(\mathbf{\Phi}(\lambda_3)\mathbf{Q}\mathbf{\Phi}(\lambda_1)\mathbf{B}\otimes\mathbf{\Phi}(\lambda_1)\mathbf{B}\big) \\ &= \mathbf{C}\mathbf{\Phi}(\mu_1)\mathbf{Q}\big[\big(\mathbf{\Phi}(\lambda_3)\mathbf{Q}\mathbf{\Phi}(\lambda_1)\mathbf{B}\otimes\mathbf{\Phi}(\lambda_1)\mathbf{B}\big)\otimes\mathbf{\Phi}(\lambda_3)\big]\mathbf{Q}\big(\mathbf{\Phi}(\lambda_1)\mathbf{B}\otimes\mathbf{\Phi}(\lambda_1)\mathbf{B}\big) \\ &= \mathbf{C}\mathbf{\Phi}(\mu_1)\underbrace{\mathbf{Q}\big[\mathbf{\Phi}(\lambda_3)\mathbf{Q}\big(\mathbf{\Phi}(\lambda_1)\mathbf{B}\otimes\mathbf{I}\big)\otimes\mathbf{I}\big]}_{\tilde{\mathbf{Q}}}\big(\mathbf{\Phi}(\lambda_1)\mathbf{B}\otimes\mathbf{\Phi}(\lambda_3)\big)\mathbf{Q}\big(\mathbf{\Phi}(\lambda_1)\mathbf{B}\otimes\mathbf{\Phi}(\lambda_1)\mathbf{B}\big) \\ &= \mathbf{H}_2^{\tilde{\mathbf{Q}},\mathbf{Q}}(\mu_1,\lambda_1,\lambda_3,\lambda_1,\lambda_1). \end{aligned}$$

It readily follows that given the QB system $\mathbf{\Sigma}_{QB}$, a reduced QB system of order three, can be directly obtained without computation (matrix factorizations or solves). This reduced system matches $2k + k^2 + \frac{(k+1)k^2}{3} - \frac{(2k+3)k^2}{9} = 24$ (for $k = 3$) moments (of the original system) that can be directly written in the format defined in (4). Below, enumerate these values as follows (two corresponding to the linear transfer function on *level* 1, six corresponding to *level* 1, eight corresponding to *level* 2 and finally, eight corresponding to *level* 3), as

$$
\begin{aligned}
\text{two of } \mathbf{H}: &\quad \mathbf{H}(\mu_1),\ \mathbf{H}(\lambda_1),\\
\text{three of } \mathbf{H}_1^{\mathbf{N}}: &\quad \mathbf{H}_1^{\mathbf{N}}(\mu_1,\mu_2),\ \mathbf{H}_1^{\mathbf{N}}(\mu_1,\lambda_1),\ \mathbf{H}_1^{\mathbf{N}}(\lambda_2,\lambda_1),\\
\text{three of } \mathbf{H}_1^{\mathbf{Q}}: &\quad \mathbf{H}_1^{\mathbf{Q}}(\mu_1,\lambda_1,\mu_3),\ \mathbf{H}_1^{\mathbf{Q}}(\mu_1,\mu_2,\lambda_1),\ \mathbf{H}_1^{\mathbf{Q}}(\mu_1,\lambda_2,\lambda_1),\\
\text{two of } \mathbf{H}_2^{\mathbf{N},\mathbf{N}}: &\quad \mathbf{H}_2^{\mathbf{N},\mathbf{N}}(\mu_1,\mu_2,\lambda_1),\ \mathbf{H}_2^{\mathbf{N},\mathbf{N}}(\mu_1,\lambda_2,\lambda_1),\\
\text{two of } \mathbf{H}_2^{\mathbf{N},\mathbf{Q}}: &\quad \mathbf{H}_2^{\mathbf{N},\mathbf{Q}}(\mu_1,\lambda_3,\lambda_2,\lambda_1),\ \mathbf{H}_2^{\mathbf{N},\mathbf{Q}}(\mu_1,\mu_2,\lambda_1,\lambda_1),\\
\text{two of } \mathbf{H}_2^{\mathbf{Q},\mathbf{N}}: &\quad \mathbf{H}_2^{\mathbf{Q},\mathbf{N}}(\mu_1,\lambda_1,\mu_3,\lambda_1),\ \mathbf{H}_2^{\mathbf{Q},\mathbf{N}}(\mu_1,\lambda_1,\lambda_2,\lambda_1),\\
\text{two of } \mathbf{H}_2^{\mathbf{Q},\mathbf{Q}}: &\quad \mathbf{H}_2^{\mathbf{Q},\mathbf{Q}}(\mu_1,\lambda_1,\lambda_3,\lambda_1,\lambda_1),\ \mathbf{H}_2^{\mathbf{Q},\mathbf{Q}}(\mu_1,\lambda_1,\mu_3,\lambda_1,\lambda_1),\\
\text{eight of } \mathbf{H}_3^{\mathbf{w}}: &\quad \mathbf{H}_3^{\mathbf{N},\mathbf{N},\mathbf{N}}(\mu_1,\mu_2,\lambda_2,\lambda_1),\ \mathbf{H}_3^{\mathbf{N},\mathbf{N},\mathbf{Q}}(\mu_1,\mu_2,\lambda_3,\lambda_2,\lambda_1),\\
&\quad \ldots,\ \mathbf{H}_3^{\mathbf{Q},\mathbf{Q},\mathbf{Q}}(\mu_1,\lambda_1,\mu_3,\lambda_3,\lambda_1,\lambda_3,\lambda_1,\lambda_1),\ \mathbf{w}\in\Upsilon^3.
\end{aligned}
$$

as well as $\frac{(2k+3)k^2}{9} = 9$ (for $k = 3$) moments (of the original system) that can not be directly written in the format from (4). Hence the total number of moments matched using this procedure is $2k + k^2 + \frac{(k+1)k^2}{2} = 33$, for $k = 3$.

3.2.1 Properties of the Loewner pencil

We will prove that the quantities defined earlier satisfy various equations which generalize the ones for the linear and bilinear cases.

Proposition 3.2 *The Loewner and shifted Loewner matrices satify the following relations*

$$\mathbb{L}_s = \mathbb{L}\mathbf{\Lambda} + \mathbf{\Omega}\mathbf{Y}_{\mathbf{R}} + \mathbf{\Xi}\mathbf{Z}_{\mathbf{R}} + \mathbf{V}\mathbf{R}, \tag{41}$$

$$\mathbb{L}_s = \mathbf{M}\mathbb{L} + \sum_{j=1}^{k^\dagger} \mathbf{X}^{(j)}\mathbf{\Omega}\mathbf{U}^{(j)} + \mathbf{Z}^T\mathbf{\Xi} + \mathbf{L}\mathbf{W}. \tag{42}$$

Proof 3.4 By multiplying equation (24) with $\mathcal{O}$ to the left, it follows that we can write:

$$\underbrace{\mathcal{O}\mathbf{A}\mathcal{R}}_{-\mathbb{L}_s} + \underbrace{\mathcal{O}\mathbf{Q}\big(\mathcal{R}\otimes\mathcal{R}}_{\mathbf{\Omega}}\big)\mathbf{Y}_{\mathbf{R}} + \underbrace{\mathcal{O}\mathbf{N}\mathcal{R}}_{\mathbf{\Psi}}\mathbf{Z}_{\mathbf{R}} + \underbrace{\mathcal{O}\mathbf{B}}_{\mathbf{V}}\mathbf{R} = \underbrace{\mathcal{O}\mathbf{E}\mathcal{R}}_{-\mathbb{L}}\mathbf{\Lambda}.$$

Now by substituting the projected matrices defined in (35) and (36) onto the above equation, it directly follows that the relation (41) is verified. Moreover, by multiplying equation (28) with $\mathcal{R}$ to the right, it follows that we can write

$$\mathcal{O}\mathbf{A}\mathcal{R} + \sum_{j=1}^{k^\dagger} \mathbf{X}^{(j)}\mathcal{O}\mathbf{Q}\big(\mathcal{R}_{3j-2}\otimes\mathbf{I}\big)\mathcal{R} + \mathbf{Z}^T\mathcal{O}\mathbf{N}\mathcal{R} + \mathbf{L}\mathbf{C}\mathcal{R} = \mathbf{M}\mathcal{O}\mathbf{E}\mathcal{R}. \tag{43}$$

Using basic properties of the Kronecker product from Proposition 5.2.1, the following holds for $j \in \{1, 2\}$

$$\big(\mathcal{R}_{3j-2}\otimes\mathbf{I}\big)\mathcal{R} = \mathcal{R}_{3j-2}\otimes\mathcal{R} = (\mathcal{R}\mathbf{e}_{3j-2,k})\otimes(\mathcal{R}\mathbf{I}_k) = (\mathcal{R}\otimes\mathcal{R})(\mathbf{e}_{3j-2,k}\otimes\mathbf{I}_k) = (\mathcal{R}\otimes\mathcal{R})\mathbf{U}^{(j)}.$$

Now replacing this equality into (43), we write

$$\underbrace{\mathcal{O}\mathbf{A}\mathcal{R}}_{-\mathbb{L}_s}+\sum_{j=1}^{k^\dagger}\mathbf{X}^{(j)}\underbrace{\mathcal{O}\mathbf{Q}\big(\mathcal{R}\otimes\mathcal{R}\big)}_{\mathbf{\Omega}}\mathbf{U}^{(j)}+\mathbf{Z}^T\underbrace{\mathcal{O}\mathbf{N}\mathcal{R}}_{\mathbf{\Phi}}+\mathbf{L}\underbrace{\mathbf{C}\mathcal{R}}_{\mathbf{W}}=\mathbf{M}\underbrace{\mathcal{O}\mathbf{E}\mathcal{R}}_{-\mathbb{L}}.$$

Again, by substituting the projected matrices defined in (35) and (36) onto the above equation, it directly follows that the relation (42) is verified.

Proposition 3.3 *The Loewner matrix satisfies the following Sylvester equation*

$$\mathbf{M}\mathbb{L}-\mathbb{L}\mathbf{\Lambda}=(\mathbf{V}\mathbf{R}+\mathbf{\Xi}\mathbf{Z}++\mathbf{\Omega}\mathbf{Y})-(\mathbf{L}\mathbf{W}+\mathbf{Z}^T\Xi+\sum_{j=1}^{k^\dagger}\mathbf{X}^{(j)}\mathbf{\Omega}\mathbf{U}^{(j)}). \tag{44}$$

Proof 3.5 This result directly follows by subtracting equation (41) from equation (42).

Proposition 3.4 *The shifted Loewner matrix satisfies the following Sylvester equation:*

$$\mathbf{M}\mathbb{L}_s-\mathbb{L}_s\mathbf{\Lambda}=(\mathbf{M}\mathbf{V}\mathbf{R}+\mathbf{M}\mathbf{\Xi}\mathbf{Z}+\mathbf{M}\mathbf{\Omega}\mathbf{Y})-(\mathbf{L}\mathbf{W}\mathbf{\Lambda}+\mathbf{Z}^T\Xi\mathbf{\Lambda}+\sum_{j=1}^{k^\dagger}\mathbf{X}^{(j)}\mathbf{\Omega}\mathbf{U}^{(j)}\mathbf{\Lambda}) \tag{45}$$

Proof 3.6 This result directly follows by subtracting equation (42) multiplied with $\mathbf{\Lambda}$ to the right from equation (41) multiplied with $\mathbf{M}$ to the left.

3.3 Construction of reduced order models

As was already specified, the interpolation data for the quadratic-bilinear case is more complex than for the bilinear case, as higher order transfer function of purely quadratic as well as mixed quadratic-bilinear transfer functions functions need to betaken into account. However, the rest of the procedure remains more or less unchanged. As in the case of linear and bilinear systems, the following result holds

Lemma 3.4 *Assume that $k = q$, and let $(\mathbb{L}_s, \mathbb{L})$, be a regular pencil, such that none of the interpolation points λ_i, μ_j are its eigenvalues. Then it follows that the matrices*

$$\hat{\mathbf{E}}=-\mathbb{L},\quad \hat{\mathbf{A}}=-\mathbb{L}_s,\quad \hat{\mathbf{Q}}=\mathbf{\Omega},\quad \hat{\mathbf{N}}=\mathbf{\Psi},\quad \hat{\mathbf{B}}=\mathbf{V},\quad \hat{\mathbf{C}}=\mathbf{W},$$

form a realization of a QB system that interpolates the data.

In the case of redundant data, the pencil $(\mathbb{L}_s, \mathbb{L})$ is singular, and if the condition (**??**) is satisfied, we construct the projection matrices $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{k\times r}$ as in (**??**).

Theorem 3.1 *The sextuple* $(\hat{\mathbf{C}},\hat{\mathbf{E}},\hat{\mathbf{A}},\hat{\mathbf{Q}},\hat{\mathbf{N}},\hat{\mathbf{B}})$ *given by*

$$\hat{\mathbf{E}}=-\mathbf{Y}^*\mathbb{L}\mathbf{X},\ \hat{\mathbf{A}}=-\mathbf{Y}^*\mathbb{L}_s\mathbf{X},\ \hat{\mathbf{Q}}=\mathbf{Y}^*\mathbf{\Omega}\big(\mathbf{X}\otimes\mathbf{X}\big),\ \hat{\mathbf{N}}=\mathbf{Y}^*\mathbf{\Psi}\mathbf{X},\ \hat{\mathbf{B}}=\mathbf{Y}^*\mathbf{V},\ \hat{\mathbf{C}}=\mathbf{W}\mathbf{X},$$

is the realization of a reduced QB system that approximately interpolates the given data. If the truncated singular values are all zero, then the interpolation is exact.

Remark 3.7 Thus, as in the linear case, if we have more data than necessary, we can either consider $(\mathbf{W},-\mathbb{L},-\mathbb{L}_s,\ \mathbf{\Psi},\mathbf{V})$ as an exact but singular model of the data or

$$(\mathbf{W}\mathbf{X},-\mathbf{Y}^*\mathbb{L}\mathbf{X},-\mathbf{Y}^*\mathbb{L}_s\mathbf{X},\ \mathbf{Y}^*\mathbf{\Omega}\big(\mathbf{X}\otimes\mathbf{X}\big),\ \mathbf{Y}^*\mathbf{\Psi}\mathbf{X},\ \mathbf{Y}^*\mathbf{V}),$$

as an approximate (nonsingular) model of the data. The use of the *Drazin* or *Moore-Penrose* pseudo inverses holds as in the linear case (see [**?**]).

4 Numerical experiments

4.1 Burgers' equation

The first example we present in this section is the viscous Burgers' equation which was already studied in the context of model reduction for Carleman linearized large-scale bilinear systems in chapter 4, section 4.5.3.

We denote by $\mathbf{\Sigma}_0$ the original nonlinear system for which the state variable has dimesnion $n = 50$; furthermore, $\mathbf{\Sigma}_B$ denotes 2550^{th} order approximation of $\mathbf{\Sigma}_0$, obtained by means of the Carleman bilinearization, and $\mathbf{\Sigma}_{QB}$ denotes the quadratic bilinear form of $\mathbf{\Sigma}_0$ (no approximation involved) of order 50. The system will be reduced by means of the following four methods:

1. $\mathbf{\Sigma}_B$ is reduced using Loewner to obtain $\mathbf{\Sigma}_1$ of order 30.

2. $\mathbf{\Sigma}_{QB}$ is reduced using Loewner to obtain $\mathbf{\Sigma}_2$ of order 16.

3. $\mathbf{\Sigma}_0$ is reduced using standard POD method to obtain $\mathbf{\Sigma}_3$ of order 16.

4. $\mathbf{\Sigma}_0$ is reduced using discrete empirical interpolation method (or DEIM, see [16, 17]) to obtain $\mathbf{\Sigma}_4$ of order 16.

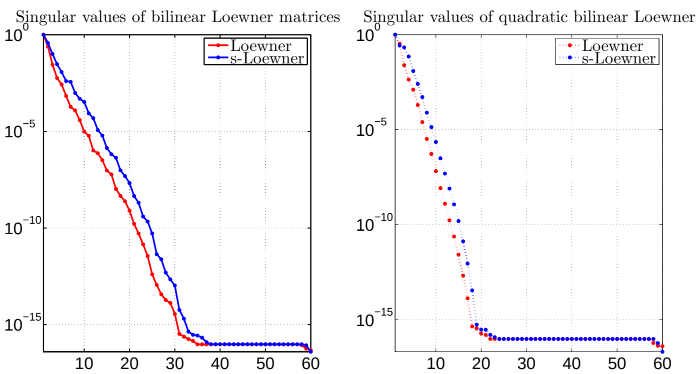

Figure 1: Singular values of the Loewner pencil; (a) bilinear; (b) quadratic-bilinear.

The first step is to collect samples from appropriately defined generalized transfer functions and plot the singular values of the ensuing Loewner pencil. As illustrated in Fig. 5.1, we notice that $\sigma_1 = 1$, $\sigma_{30} \approx 10^{-15}$, i.e., the 30^{th} singular value attains machine precision. We choose the reduced order $r_b = 30$ for the bilinear case. For the quadratic=bilinear case, a steeper drop in singular values is noticed. The order $r_q = 16$ was chosen instead.

In Fig. 5.2, the distribution of the poles corresponding to both $\mathbf{\Sigma}_B$ and $\mathbf{\Sigma}_{QB}$ is depicted. Note that the $\mathbf{A}$ matrix corresponding to both reduced systems is Hurwitz (since all the poles are in the left half of the complex plane).

Next, we compare the time-domain response of the original nonlinear system against the responses of the reduced models , when the input is $\mathbf{u}(t) = 0.2(\cos(2\pi t) + \cos(4\pi t))$. For the POD based approximation, collect snapshots of the true solution for the training input $\mathbf{u}_1(t) = 10\sin(4t)e^{-t/2}$, and then compute the projection by taking the most 16 dominant basis vectors.

In Fig. 5.3 we depict the respective outputs. By examining the plot, observe that all but one outputs seem to follow the same path; the one corresponding to the bilinear Loewner method deviates from the original output, i.e., the dotted red curve does not follow the black curve.

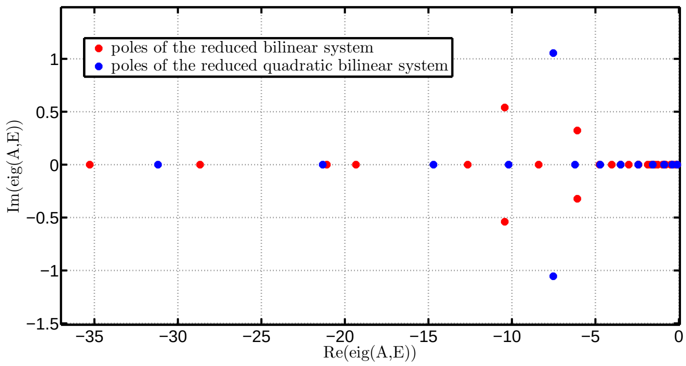

Figure 2: The poles of the reduced Loewner models.

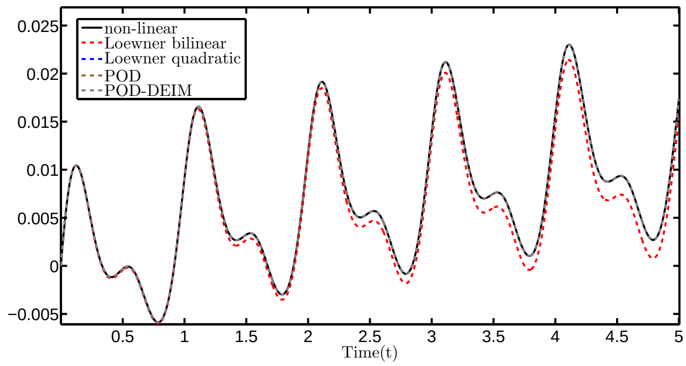

Figure 3: Time domain simulations – original vs. reduced systems

Finally, in Fig. 5.4, we depict the error between the response of $\mathbf{\Sigma}_0$ and the responses of all the reduced systems. We notice that the error when applying the quadratic-bilinear Loewner procedure is slightly lower than the error for the POD type methods.

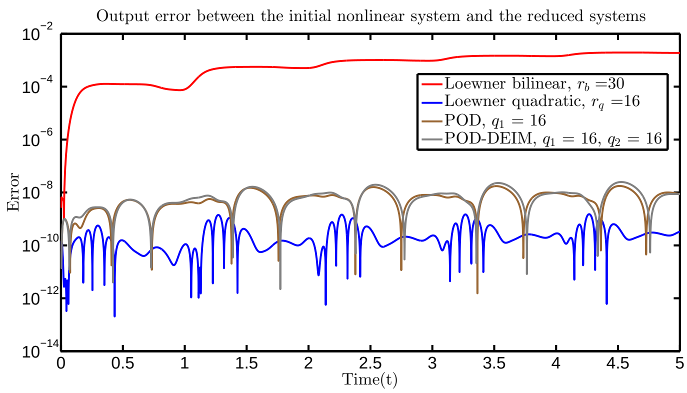

Figure 4: Time-domain approximation error between original and reduced systems

4.2 Nonlinear RLC network

The nonlinear transmission line circuit (for which the schematic is depicted in Fig. 5.5) is a very commonly used circuit for testing nonlinear model reduction techniques (see [6, 25, 21, 14]).

Consider all resistors and capacitors to be set to 1 and the diode to be characterized by the following nonlinear current/voltage dependency $i_D = g(v_D) = e^{40v_D} - 1$. The input is set to the

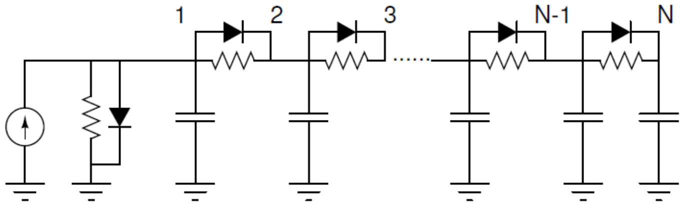

Figure 5: Circuit schematic

current source and the output is the voltage at node 1. By writing the corresponding equations, we construct a nonlinear system in state space representation that characterize the dynamics of the circuit, as

$$\mathbf{\Sigma}_N : \begin{cases} \dot{\mathbf{v}}(t) = \mathbf{f}(\mathbf{v}(t)) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{v}(t), \end{cases}$$

where in this context, identify

$$\mathbf{f}(\mathbf{v}_k) = \begin{cases} -g(\mathbf{v}_1) - g(\mathbf{v}_1 - \mathbf{v}_2), & \text{for } k = 1 \\ g(\mathbf{x}_{k-1} - \mathbf{x}_k) - g(\mathbf{v}_k - \mathbf{v}_{k+1}), & \text{for } 1 < k < N \\ g(\mathbf{v}_{N-1}) - g(\mathbf{v}_N), & \text{for } k = N \end{cases}, \quad \mathbf{B} = \mathbf{C}^T = \mathbf{e}_1.$$

As was pointed out in [23], a transformation to quadratic-bilinear form is easily obtained by introducing additional state variables $\mathbf{x}_i = e^{40\mathbf{v}_i}$ and $\mathbf{z}_i = e^{-40\mathbf{v}_i}$. It follows that the state variable of the new transformed system will have dimension $3N$, i.e., $\hat{\mathbf{v}} = [\mathbf{v}\ ;\ \mathbf{x}\ ;\ \mathbf{z}] \in \mathbb{R}^{3N}$.

An alternative to this is presented in [14], where the new state variables are introduced $\mathbf{x}_1 = \mathbf{v}_1$, $\mathbf{x}_k = \mathbf{v}_k - \mathbf{v}_{k+1}$, $\mathbf{z}_1 = e^{40\mathbf{v}_1} - 1$ and $\mathbf{z}_k = e^{40\mathbf{x}_k}$ for $k \in \{2, \ldots, N\}$. Hence it turns out that it is possible to construct an equivalent quadratic-bilinear system of a lower dimension ($2N$ to be precise) where, the new variable $\hat{\mathbf{v}} = [\mathbf{x}\ ;\ \mathbf{z}] \in \mathbb{R}^{2N}$ is defined. This alternative final system is quadratic-bilinear, and its dynamic is determined by the equations

$$\begin{aligned}
\dot{\mathbf{z}}_1 &= 40(\mathbf{z}_1 + 1)(\underbrace{-\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{z}_1 - \mathbf{z}_2 + \mathbf{u}}_{\dot{\mathbf{x}}_1}), \\
\dot{\mathbf{z}}_2 &= 40(\mathbf{z}_2 + 1)(\underbrace{-\mathbf{x}_1 - 2\mathbf{x}_2 + \mathbf{x}_3 - \mathbf{z}_1 - 2\mathbf{z}_2 + \mathbf{z}_3 + \mathbf{u}}_{\dot{\mathbf{x}}_2}), \\
&\vdots \\
\dot{\mathbf{z}}_k &= 40(\mathbf{z}_k + 1)(\underbrace{\mathbf{z}_{k-1} - 2\mathbf{z}_k + \mathbf{w}_{k+1} + \mathbf{z}_{k-1} - 2\mathbf{z}_k + \mathbf{z}_{k+1}}_{\dot{\mathbf{x}}_k}),\ 2 < k < N, \\
&\vdots \\
\dot{\mathbf{z}}_N &= 40(\mathbf{z}_N + 1)(\underbrace{\mathbf{x}_{N-1} - 2\mathbf{x}_N + \mathbf{z}_{N-1} - 2\mathbf{z}_N}_{\dot{\mathbf{x}}_N}).
\end{aligned}$$

Hence a quadratic-bilinear representation of the original non-linear system is computed (of order $n = 2N$). This will be considered as the original system in the following computations; it will be reduced to a much smaller dimension by means of the QB Loewner framework and of the TQB-IRKA method (as introduced in [12]).

First, proceed by collecting samples from generalized linear, quadratic, bilinear and quadratic-bilinear transfer functions up to *level* 3 (the procedure was previously described in Section 5.3).

In total consider 60 interpolations points that are logarithmically spaced inside $[10^{-3}, 10^3]j$. As illustrated in Fig. 5.6, the 18^{th} singular value (of the Loewner matrix) attains machine precision. We choose the reduced-order $k = 12$ for the Loewner reduced system as well as the one obtained via the TQB-IRKA procedure.

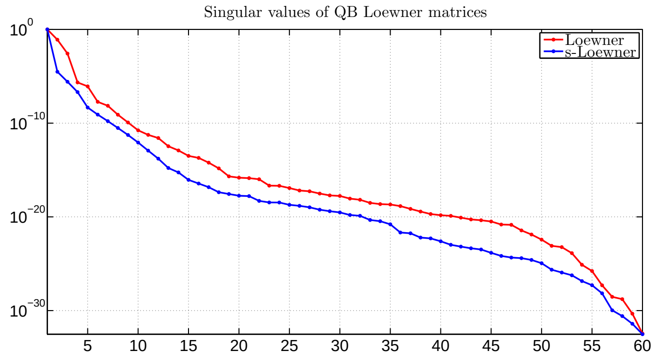

Figure 6: Singular value decay of the QB Loewner pencil

As it can be observed Fig. 5.7, the responses corresponding to both reduction methods seem to faithfully duplicate the original response. The control input was chosen as $\mathbf{u}(t) = (1 + \cos(\pi t))/2$.

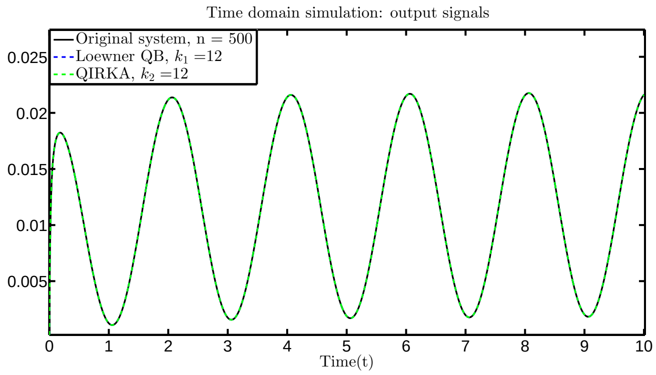

Figure 7: Time-domain simulation

When analyzing the magnitude of the relative error between the original response and the two responses of the reduced systems, notice that the method that we propose produces slightly better approximation than TQB-IRKA does (in Fig. 5.8).

4.3 Chafee-Infante equation

Next, consider the same example as in section 4.5.4, i.e., the one-dimensional Chafee-Infante equation with cubic nonlinearity.

By means of a finite difference scheme (with n equidistant points over the length), construct a semi-discretized quadratic-bilinear system of order $2n$. The output $\mathbf{y}(t)$ is chosen to be the response at the right boundary. Take $n = 500$ which will result in a 1000^{th} initial QB system.

Previously in section 4.5.4, the discretized quadratic-bilinear system was approximated by a purely bilinear system by means of the Carleman's linearization technique. Then, two reduction procedures were applied for decreasing the dimension of the aforementioned bilinear system.

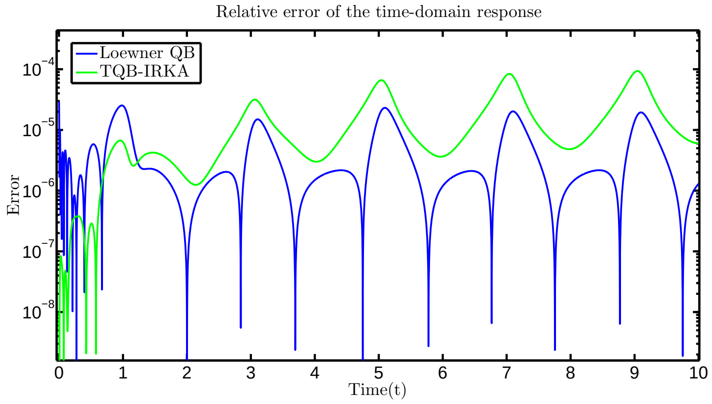

Figure 8: Relative error between the response of the original system and of the reduced ones

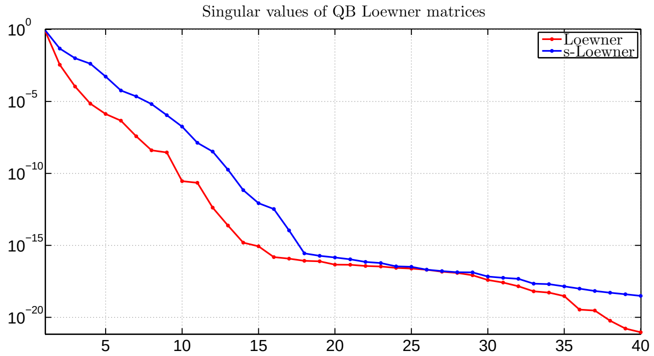

Figure 9: Singular value decay of the QB Loewner pencil

Now, we can skeep the (bi)linearization step and apply the MOR tools directly for the QB finite element model. As for the bilinear case, proceed by collecting samples from various generalized transfer functions. This time, the transfer functions are more diverse and are written in terms of both the $\mathbf{N}$ and the $\mathbf{Q}$ matrices.

Consider in total 40 interpolations points that are logarithmically spaced inside $[10^{-2}, 10^{2}]j$. As illustrated in Fig. 5.9, the 12^{th} singular value attains machine precision. We choose the reduced-order $k = 10$ for both reduced systems.

Again, both reduction methods seem to produce good approximations of the original transient response (as it can be observed in Fig. 5.10). In this case, the control input was chosen as in [12], i.e., as a decaying oscillatory exponential $\mathbf{u}(t) = (1 + \sin(\pi t))e^{-t/5}$.

By analyzing the absolute value of the offset between the original response and the two responses of the reduced systems, notice that the proposed method that again produces better approximation than the truncated quadratic-bilinear generalization of IRKA (in Fig. 5.11).

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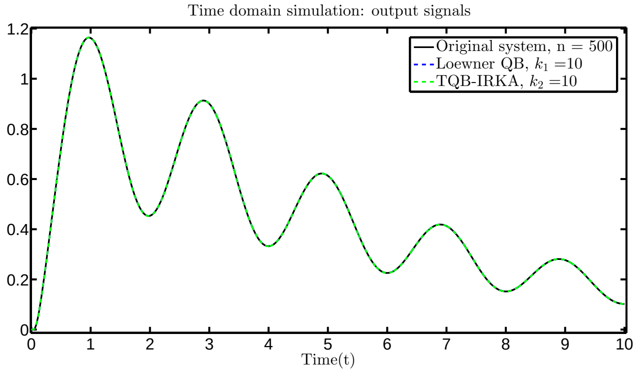

Figure 10: Time-domain simulation

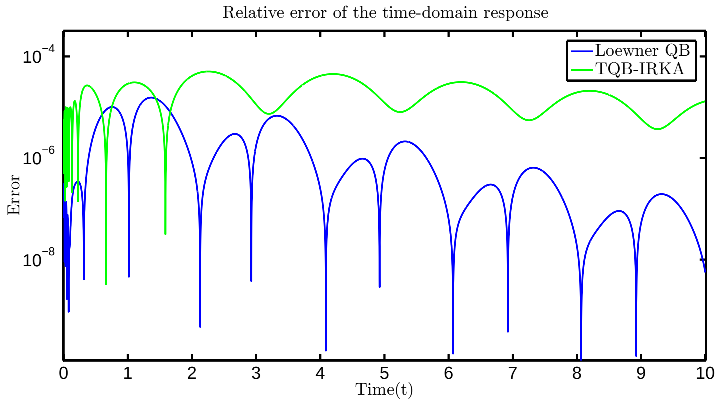

Figure 11: Relative error between the response of the original system and of the reduced ones

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5 Appendix

Proof of Proposition 2.3 We will prove the equality of the two $n \times n$ matrices by showing that each column is the same for both; this is done by multiplying the equality in (7) to the right with the unit vector $\mathbf{e}_k$, for $k \in \{1, 2, \ldots, n\}$. By using the third identity from Proposition 5.2.1, we have that

$$\mathbf{Q}^{(-1)}\big(\mathbf{I}_n \otimes \mathbf{w}\big)\mathbf{e}_k = \big(\mathbf{w}^T \otimes \mathbf{I}_n\big)\mathbf{Q}^T\mathbf{e}_k \Leftrightarrow \mathbf{Q}^{(-1)}\big(\mathbf{e}_k \otimes \mathbf{w}\big) = \big(\mathbf{w}^T \otimes \mathbf{I}_n\big)\mathbf{Q}^T\mathbf{e}_k \tag{46}$$

We further write

$$\mathbf{Q}^{(-1)}\big(\mathbf{e}_k \otimes \mathbf{w}\big) = \bar{\mathbf{Q}}_k^{tens}\mathbf{w} \tag{47}$$

where $\bar{\mathbf{Q}}_k^{tens}$ is the k^{th} frontal slice of tensor $\bar{\mathbf{Q}}$; the matrix $\mathbf{Q}^{(-1)}$ is the mode-1 matricization of the same tensor $\bar{\mathbf{Q}}$ (see Definition 2.2.19). Moreover, from the same definition, we have that $\mathbf{Q}^T\mathbf{e}_k = \text{vec}(\bar{\mathbf{Q}}_k^{tens})$. Combining this result with the first identity from Proposition 5.2.1, it follows

$$\big(\mathbf{w}^T \otimes \mathbf{I}_n\big)\mathbf{Q}^T\mathbf{e}_k = \big(\mathbf{w}^T \otimes \mathbf{I}_n\big)\text{vec}(\bar{\mathbf{Q}}_k^{tens}) = \text{vec}\big(\mathbf{I}_n\bar{\mathbf{Q}}_k^{tens}\mathbf{w}\big) = \bar{\mathbf{Q}}_k^{tens}\mathbf{w} \tag{48}$$

since $\bar{\mathbf{Q}}_k^{tens}\mathbf{w} \in \mathbb{R}^n$. From (46), (47) and (48) it follows that

$$\mathbf{Q}^{(-1)}\big(\mathbf{I}_n \otimes \mathbf{w}\big)\mathbf{e}_k = \big(\mathbf{w}^T \otimes \mathbf{I}_n\big)\mathbf{Q}^T\mathbf{e}_k, \; \forall \, k \in \{1, 2, \ldots, n\}$$

and hence (7) is proven.

Proof of Lemma 3.1 We project the quadratic-bilinear system $\boldsymbol{\Sigma}_{QB}$ with $\mathbf{X} = \mathcal{R} \in \mathbb{C}^{n\times k}$, and an arbitrary matrix $\mathbf{Y} \in \mathbb{C}^{n\times k}$ (so that $\mathbf{Y}^T\mathbf{X}$ is nonsingular). It readily follows that, for $i \in \{1, 2, \ldots, k\}$

$$\textbf{(a)}\ \ \hat{\boldsymbol{\Phi}}(\lambda_1^{(i)})\,\hat{\mathbf{B}} = \mathbf{e}_{3i-2} \quad \textbf{(b)}\ \ \hat{\boldsymbol{\Phi}}(\lambda_2^{(i)})\,\hat{\mathbf{N}}\,\mathbf{e}_{3i-2} = \mathbf{e}_{3i-1} \quad \text{and} \quad \textbf{(c)}\ \ \hat{\boldsymbol{\Phi}}(\lambda_3^{(i)})\,\hat{\mathbf{Q}}\,\big(\mathbf{e}_{3i-2}\otimes\mathbf{e}_{3i-2}\big) = \mathbf{e}_{3i}.$$

We make use of the following result:

$$\hat{\boldsymbol{\Phi}}(s)^{-1} = s\hat{\mathbf{E}} - \hat{\mathbf{A}} = \mathbf{Y}^T(s\mathbf{E}-\mathbf{A})\mathcal{R} = \mathbf{Y}^T\boldsymbol{\Phi}(s)^{-1}\mathcal{R}. \tag{49}$$

To prove **(a)**, we first notice that by multiplying $\mathcal{R}$ to the right with $\mathbf{e}_{3i-2}$ we can write

$$\mathcal{R}\mathbf{e}_{3i-2} = \boldsymbol{\Phi}(\lambda_1^{(i)})\mathbf{B} \Rightarrow \boldsymbol{\Phi}(\lambda_1^{(i)})^{-1}\mathcal{R}\mathbf{e}_{3i-2} = \mathbf{B} \Rightarrow \mathbf{Y}^T\boldsymbol{\Phi}(\lambda_1^{(i)})^{-1}\mathcal{R}\mathbf{e}_{3i-2} = \mathbf{Y}^T\mathbf{B}.$$

Using the notation $\hat{\mathbf{B}} = \mathbf{Y}^T\mathbf{B}$ and the result in (49), we write

$$\hat{\boldsymbol{\Phi}}(\lambda_1^{(i)})^{-1}\mathcal{R}\mathbf{e}_{3i-2} = \hat{\mathbf{B}} \Rightarrow \hat{\boldsymbol{\Phi}}(\lambda_1^{(i)})\hat{\mathbf{B}} = \mathbf{e}_{3i-2}.$$

To prove **(b)**, note that by multiplying $\mathcal{R}$ to the right with $\mathbf{e}_{3i-1}$, we can write

$$\mathcal{R}\mathbf{e}_{3i-1} = \boldsymbol{\Phi}(\lambda_2^{(i)})\mathbf{N}\boldsymbol{\Phi}(\lambda_1^{(i)})\mathbf{B} \Rightarrow \boldsymbol{\Phi}(\lambda_2^{(i)})^{-1}\mathcal{R}\mathbf{e}_{3i-1} = \mathbf{N}\mathcal{R}\mathbf{e}_{3i-2}. \tag{50}$$

By multiplying (50) with $\mathbf{Y}^T$ to the left and then using the notation $\hat{\mathbf{N}} = \mathbf{Y}^T\mathbf{N}\mathcal{R}$ and the result in (49), we have that

$$\mathbf{Y}^T\boldsymbol{\Phi}(\lambda_2^{(i)})^{-1}\mathcal{R}\mathbf{e}_{3i-1} = \mathbf{Y}^T\mathbf{N}\mathcal{R}\mathbf{e}_{3i-2} \Rightarrow \hat{\boldsymbol{\Phi}}(\lambda_2^{(i)})^{-1}\mathbf{e}_{3i-1} = \hat{\mathbf{N}}\mathbf{e}_{3i-2} \Rightarrow \hat{\boldsymbol{\Phi}}(\lambda_2^{(i)})\hat{\mathbf{N}}\mathbf{e}_{3i-2} = \mathbf{e}_{3i-1}.$$

To prove **(c)**, note that by multiplying $\mathcal{R}$ to the right with $\mathbf{e}_{3i}$, we can write

$$\mathcal{R}\mathbf{e}_{3i} = \boldsymbol{\Phi}(\lambda_3^{(i)})\mathbf{Q}\big(\boldsymbol{\Phi}(\lambda_1^{(i)})\mathbf{B}\otimes\boldsymbol{\Phi}(\lambda_1^{(i)})\big) \Rightarrow \boldsymbol{\Phi}(\lambda_3^{(i)})^{-1}\mathcal{R}\mathbf{e}_{3i} = \mathbf{Q}\big(\mathcal{R}\mathbf{e}_{3i-2}\otimes\mathcal{R}\mathbf{e}_{3i-2}\big). \tag{51}$$

By multiplying (51) with $\mathbf{Y}^T$ to the left and then using the notation $\hat{\mathbf{Q}} = \mathbf{Y}^T\mathbf{Q}\big(\mathcal{R}\otimes\mathcal{R}\big)$ as well as the result in (49), we have that

$$\begin{aligned}
&\mathbf{Y}^T\boldsymbol{\Phi}(\lambda_3^{(i)})^{-1}\mathcal{R}\mathbf{e}_{3i} = \mathbf{Y}^T\mathbf{Q}\big(\mathcal{R}\otimes\mathcal{R}\big)\big(\mathbf{e}_{3i-2}\otimes\mathbf{e}_{3i-2}\big)\\
&\Rightarrow \hat{\boldsymbol{\Phi}}(\lambda_3^{(i)})^{-1}\mathbf{e}_{3i} = \hat{\mathbf{Q}}\big(\mathbf{e}_{3i-2}\otimes\mathbf{e}_{3i-2}\big) \Rightarrow \hat{\boldsymbol{\Phi}}(\lambda_3^{(i)})\hat{\mathbf{Q}}\big(\mathbf{e}_{3i-2}\otimes\mathbf{e}_{3i-2}\big) = \mathbf{e}_{3i}.
\end{aligned}$$

The equalities in (a),(b) and (c) imply the right-hand conditions in (13). For instance, by multiplying the relation stated in (a) with $\hat{\mathbf{C}}$ to the left we obtain:

$$\hat{\mathbf{C}}\hat{\boldsymbol{\Phi}}(\lambda_1^{(i)})\hat{\mathbf{B}} = \hat{\mathbf{C}}\mathbf{e}_1 = \mathbf{C}\mathcal{R}\mathbf{e}_{3i-2} = \mathbf{C}\boldsymbol{\Phi}(\lambda_1^{(i)})\mathbf{B} \Rightarrow \mathbf{H}_0^{\epsilon}(\lambda_1^{(i)}) = \hat{\mathbf{H}}_0^{\epsilon}(\lambda_1^{(i)}).$$

Also, by multiplying the relation stated in (b) with $\hat{\mathbf{C}}$ to the left we obtain

$$\hat{\mathbf{C}}\hat{\boldsymbol{\Phi}}(\lambda_2^{(i)})\hat{\mathbf{N}}\mathbf{e}_{3i-2} = \hat{\mathbf{C}}\mathbf{e}_{3i-1} = \mathbf{C}\mathcal{R}\mathbf{e}_{3i-1} = \mathbf{C}\boldsymbol{\Phi}(\lambda_2^{(i)})\mathbf{N}\boldsymbol{\Phi}(\lambda_1^{(i)})\mathbf{B} \Rightarrow \mathbf{H}_1^{\mathbf{N}}(\lambda_2^{(i)},\lambda_1^{(i)}) = \hat{\mathbf{H}}_1^{\hat{\mathbf{N}}}(\lambda_2^{(i)},\lambda_1^{(i)}).$$

Finally, by multiplying the relation stated in (c) with $\hat{\mathbf{C}}$ to the left, we write

$$\hat{\mathbf{C}}\hat{\boldsymbol{\Phi}}(\lambda_3^{(i)})\hat{\mathbf{Q}}\big(\mathbf{e}_{3i-2}\otimes\mathbf{e}_{3i-2}\big) = \hat{\mathbf{C}}\mathbf{e}_{3i} = \mathbf{C}\mathcal{R}\mathbf{e}_{3i} = \mathbf{C}\boldsymbol{\Phi}(\lambda_2^{(i)})\mathbf{Q}\big(\boldsymbol{\Phi}(\lambda_1^{(i)})\mathbf{B}\otimes\lambda_1^{(i)})\mathbf{B}\big)$$

$$\Rightarrow \mathbf{H}_1^{\mathbf{Q}}(\lambda_3^{(i)},\lambda_1^{(i)},\lambda_1^{(i)}) = \hat{\mathbf{H}}_1^{\hat{\mathbf{Q}}}(\lambda_3^{(i)},\lambda_1^{(i)},\lambda_1^{(i)}).$$

Similarly, if $\mathbf{Y}^T = \mathcal{O}$ and $\mathbf{X}$ arbitrary, we have that the following holds for $j \in \{1,2,\ldots,k\}$, i.e.,

$$\textbf{(d)}\ \hat{\mathbf{C}}\,\hat{\mathbf{\Phi}}(\mu_1^{(j)}) = \mathbf{e}_{3j-2}^T \quad \textbf{(e)}\ \mathbf{e}_{3j-2}^T\hat{\mathbf{N}}\hat{\mathbf{\Phi}}(\mu_2^{(j)}) = \mathbf{e}_{3j-1}^T \quad \text{and} \quad \textbf{(f)}\ \mathbf{e}_{3j-2}^T\hat{\mathbf{Q}}\big(\hat{\mathbf{\Phi}}(\lambda_1^{(j)})\hat{\mathbf{B}}\otimes\mathbf{\Phi}(\mu_3^{(j)})\big) = \mathbf{e}_{3j}^T,$$

which imply the left-hand conditions (12). Finally, with $\mathbf{X} = \mathcal{R}$, $\mathbf{Y}^T = \mathcal{O}$, and combining **(a) - (f)**, interpolation conditions (14) and (15) follow.

For instance, by fixing $i, j \in \{1,2,\ldots,k\}$ and $\ell, h \in \{1,2,3\}$, we would like to show that the equalities in (14) hold. By choosing $\ell = 2$ and $h = 3$, it follows that we would need to show the following equality

$$\mathbf{H}_3^{\mathbf{N},\mathbf{N},\mathbf{Q}}(\mu_1^{(j)},\mu_2^{(j)},\lambda_3^{(i)},\lambda_1^{(i)},\lambda_1^{(i)}) = \hat{\mathbf{H}}_3^{\mathbf{N},\mathbf{N},\mathbf{Q}}(\mu_1^{(j)},\mu_2^{(j)},\lambda_3^{(i)},\lambda_1^{(i)},\lambda_1^{(i)}). \tag{52}$$

From (c) and (e), we write

$$\hat{\mathbf{C}}\hat{\mathbf{\Phi}}(\mu_1^{(j)})\hat{\mathbf{N}}\hat{\mathbf{\Phi}}(\mu_2^{(j)}) = \mathbf{e}_{3j-1}^T \quad \text{and} \quad \hat{\mathbf{\Phi}}(\lambda_3^{(i)})\,\hat{\mathbf{Q}}\left(\hat{\mathbf{\Phi}}(\lambda_1^{(i)})\hat{\mathbf{B}}\otimes\hat{\mathbf{\Phi}}(\lambda_1^{(i)})\hat{\mathbf{B}}\right) = \mathbf{e}_{3i}.$$

By multiplying these two relations, we hence write that

$$\begin{aligned}&\Big(\hat{\mathbf{C}}\hat{\mathbf{\Phi}}(\mu_1^{(j)})\hat{\mathbf{N}}\hat{\mathbf{\Phi}}(\mu_2^{(j)})\Big)\hat{\mathbf{N}}\Big(\hat{\mathbf{\Phi}}(\lambda_3^{(i)})\,\hat{\mathbf{Q}}\left(\hat{\mathbf{\Phi}}(\lambda_1^{(i)})\hat{\mathbf{B}}\otimes\hat{\mathbf{\Phi}}(\lambda_1^{(i)})\hat{\mathbf{B}}\right)\Big) = \mathbf{e}_{3j-1}^T\hat{\mathbf{N}}\mathbf{e}_{3i} = \mathbf{e}_{3j-1}^T\big(\mathcal{O}^T\mathbf{N}\mathcal{R}\big)\mathbf{e}_{3i}\\ &= \big(\mathbf{e}_{3j-1}^T\mathcal{O}^T\big)\mathbf{N}\big(\mathcal{R}\mathbf{e}_{3i}\big) = \mathbf{C}\mathbf{\Phi}(\mu_1^{(j)}\mathbf{N}\mathbf{\Phi}(\mu_2^{(j)}\mathbf{N}\Big(\mathbf{\Phi}(\lambda_3^{(i)})\,\mathbf{Q}\left(\mathbf{\Phi}(\lambda_1^{(i)})\mathbf{B}\otimes\mathbf{\Phi}(\lambda_1^{(i)})\mathbf{B}\right)\Big),\end{aligned}$$

which proves the equality in (52). In general, we show that the interpolation conditions in (14) hold

$$\begin{aligned}&\hat{\mathbf{H}}_{|\hat{\mathbf{w}}(\ell)|+|\hat{\mathbf{w}}(h)|+1}^{\hat{\mathbf{w}}(\ell)\circledcirc\hat{\mathbf{N}}\circledcirc\mathbf{w}\hat{(}h)}(\boldsymbol{\mu}^{(j)}(\ell)\circledcirc\boldsymbol{\lambda}^{(i)}(h)) = \mathbf{e}_{3j-3+\ell}^T\hat{\mathbf{N}}\mathbf{e}_{3i-3+h} = \mathbf{e}_{3j-3+\ell}^T\big(\mathcal{O}^T\mathbf{N}\mathcal{R}\big)\mathbf{e}_{3i-3+h}\\ &= \big(\mathbf{e}_{3j-3+\ell}^T\mathcal{O}^T\big)\mathbf{N}\big(\mathcal{R}\mathbf{e}_{3i-3+h}\big) = \mathbf{H}_{|w(\ell)|+|v(h)|+1}^{w(\ell)\circledcirc\mathbf{N}\circledcirc\mathbf{v}(h)}(\boldsymbol{\mu}^{(j)}(\ell)\circledcirc\boldsymbol{\lambda}^{(i)}(h)).\end{aligned}$$

Also, it can be shown that the interpolation conditions in (15) hold in general (by choosing $h_1, h_2 \in \{1,2,3\}$ so that $h_1 \vee h_2 = 1$)

$$\begin{aligned}&\hat{\mathbf{H}}_{|\hat{\mathbf{w}}(\ell)|+|\hat{\mathbf{w}}(h_1)|+|\hat{\mathbf{w}}(h_2)|+1}^{\hat{\mathbf{w}}(\ell)\circledcirc\mathbf{Q}\circledcirc\hat{\mathbf{w}}(h_1)\circledcirc\hat{\mathbf{w}}(h_2)}(\boldsymbol{\mu}^{(j)}(\ell)\circledcirc\boldsymbol{\lambda}^{(i)}(h_1)\circledcirc\boldsymbol{\lambda}^{(i)}(h_2)) = \mathbf{e}_{3j-3+\ell}^T\hat{\mathbf{Q}}\big(\mathbf{e}_{3i-3+h_1}\otimes\mathbf{e}_{3i-3+h_2}\big)\\ &= \mathbf{e}_{3j-3+\ell}^T\mathcal{O}^T\mathbf{Q}\big(\mathcal{R}\otimes\mathcal{R}\big)\big(\mathbf{e}_{3i-3+h_1}\otimes\mathbf{e}_{3i-3+h_2}\big) = \mathbf{e}_{3j-3+\ell}^T\mathcal{O}^T\mathbf{Q}\big(\mathcal{R}\mathbf{e}_{3i-3+h_1}\big)\otimes\big(\mathcal{R}\mathbf{e}_{3i-3+h_2}\big)\\ &= \mathbf{H}_{|\mathbf{w}(\ell)|+|\mathbf{w}(h_1)|+|\mathbf{w}(h_2)|+1}^{\mathbf{w}(\ell)\circledcirc\mathbf{Q}\circledcirc\mathbf{w}(h_1)\circledcirc\mathbf{w}(h_2)}(\boldsymbol{\mu}^{(j)}(\ell)\circledcirc\boldsymbol{\lambda}^{(i)}(h_1)\circledcirc\boldsymbol{\lambda}^{(i)}(h_2)).\end{aligned}$$