

Effective action of dilaton gravity as the classical double copy of Yang-Mills theory

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We compute the classical effective action of color charges moving along worldlines by integrating out the Yang-Mills gauge field to next-to-leading order in the coupling. An adapted version of the Bern-Carrasco-Johansson (BCJ) double-copy construction known from quantum scattering amplitudes is then applied to the Feynman integrands, yielding the prediction for the classical effective action of point masses in dilaton gravity. We check the validity of the result by independently constructing the effective action in dilaton gravity employing field redefinitions and gauge choices that greatly simplify the perturbative construction. Complete agreement is found at next-to-leading order. Finally, upon performing the post-Newtonian expansion of our result, we find agreement with the corresponding action of scalar-tensor theories known from the literature. Our results represent a proof of concept for the classical double-copy construction of the gravitational effective action and provides another application of a BCJ-like double copy beyond scattering amplitudes.

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I. INTRODUCTION

There is a growing body of evidence for a fascinating perturbative duality between Yang-Mills theory and quantum gravity known as the double-copy construction or color-kinematics duality due to Bern, Carrasco and Johansson (BCJ) [1–3]. It provides a concrete prescription for transforming scattering amplitudes in non-Abelian gauge theories into scattering amplitudes in gravitational theories upon replacing the non-Abelian color degrees of freedom by kinematical ones. In the simplest scenario pure Yang-Mills theory double copies to dilaton gravity coupled to an axion also termed as $\mathcal{N} = 0$ supergravity (the massless sector of bosonic strings). This relation was proven for tree-level amplitudes [2], where the double copy is equivalent to the Kawai-Lewellen-Tye relations of string theory [4], a precursor of the BCJ duality. It is also manifest in the Cachazo-He-Yuan formulation of gluon and graviton tree-level amplitudes [5]. While remaining

conjectural at the loop level, the double-copy procedure is extremely efficient in generating integrands for gravitational theories at high-perturbative orders in theories with and without supersymmetry [6–14] and as such has passed many nontrivial checks. The present record being at the four point five-loop level for maximal supergravity [15]. Moreover, an elaborate web of theories connected via color-kinematics duality exists that includes matter couplings and various numbers of supersymmetries [16–18]. All these results point towards the double copy being a generic property of gravity alluding at a hidden kinematical algebra which has resisted discovery so far.

In light of these findings the natural question arises, whether the double copy generalizes beyond the realm of scattering amplitudes. In particular, does it also play a role in classical general relativity? Here a number of encouraging results have been obtained: Schwarzschild, Kerr and Taub-NUT spacetimes were shown to be double copies of classical gauge-theory solutions [19,20], were further extended to certain classes of perturbative spacetimes [21], and a classical double-copy description of the gravitational radiation from accelerating black holes was developed in Ref. [22]. Following up the last direction, a very interesting setting is that of gravitational radiation produced by binary sources, not the least due to the spectacular observations of gravitational waves and the need for high precision predictions to generate waveform templates [23]. Pioneering work was done in [24–26]

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where perturbative classical solutions for binary color-charged point particles coupled via Yang-Mills theory [27] were shown to double copy to their counterparts in dilaton-gravity radiation (see also [28,29]). In these works certain color-kinematics replacement rules were employed which appear somewhat distinct to the color factor/numerator replacement rules familiar from scattering amplitudes. Also the question of a double-copy-respecting representation of the perturbative gauge-theory solutions remained open. This situation was clarified very recently in the work of Shen [30], which pushed the perturbative approach of solving the equations of motions via double copy to the next-to-leading order (NLO) in the coupling constant expansion. In this process a color-kinematics duality representation of the Yang-Mills-radiation solution was found, which parallels the color-kinematics duality rules of BCJ [1] and replaces the nonstandard double-copy rules of [24]. An alternative route for finding the graviton radiation of binaries via the double copy was taken in [31], where a tree scattering amplitude in gauge theory coupled to scalar fields was double copied to establish the leading order gravitational radiation emitted from the scattering of two black holes (modeled by two massive scalar fields).

In our present work, we generalize these approaches to the double copy of gravitationally interacting binaries by ascending from the level of equations of motion to the classical effective action. This approach makes direct contact to the post-Minkowskian (weak-field) and post-Newtonian (weak-field and slow-motion) expansions of the gravitational potential for which high-order results exist in the literature, namely at the fourth post-Newtonian order (four loop) for nonspinning bodies: using a canonical formalism of general relativity [32], a Fokker Lagrangian [33,34], and partial results within an effective field theory formalism [35]. The latter makes use of Feynman diagrammatic methods and sophisticated tools for effective field theories in the context of classical gravity [36]. The question how the classical gravitational potential may be extracted from the quantum scattering amplitude of massive scalars has in fact a long history starting in the 1970s [37–39]. Recent works have updated this by employing modern unitarity methods of quantum field theory [40–42]. Damour recently proposed an alternative approach for converting scattering angles and amplitudes to the effective one-body Hamiltonian [43,44]. Hence, in order to perform the construction of the interaction potential using the double copy, one could study the scattering amplitude of a massive scalar field coupled to either Yang-Mills or dilaton-gravity theory and relate to a potential using one of the mentioned works. Representing the gauge-theory result in a color-kinematics respecting fashion should then yield the effective gravitational potential via a suitably defined double copy. Here we follow a more direct approach, where the classical interaction potential is the direct outcome of a path integral. For this purpose, we

start out with color charges moving along a worldline as in [24–26,30]. This may be viewed as a classical limit of the massive scalar field.

Our paper is organized as follows: In Sec. II we give an overview of the structure of a theory of classical color-charged massive point particles moving along worldlines, interacting via a Yang-Mills theory, in the first-order formalism. In Sec. III we propose a double-copy prescription for the classical effective action. We compute the effective action of the theory introduced in Sec. II up to next-to-leading order in the gluon coupling. Along the way we introduce the notion of a trivalent representation for graphs involving worldlines and at the end of the section we perform the double copy. In Sec. IV we repeat the computation of the effective action in the weak-field/post-Minkowskian approximation for massive point particles interacting in dilaton-gravity theory and we compare the resulting expression to the double-copy result. In Sec. V we reexpand our result for the post-Minkowskian dilaton-gravity effective action in velocities, thus yielding the post-Newtonian approximation. After solving the Feynman integrals, we compare the resulting expression to known results in the literature. Our conclusions are presented in Sec. VI. Finally, our conventions, the Feynman rules and a discussion of self-interactions can be found in the appendices.

II. YANG-MILLS INTERACTION OF COLOR-CHARGED MASSIVE PARTICLES

We consider the worldline action of a massive, color-charged (non-Abelian) point particle coupled to the Yang-Mills gauge field A_μ^a ; see appendix A for notation and conventions. The action for the classical colored point charge (pc) is given by [24,25,27]

$$S_{\text{pc}} = - \int d\tau L_{\text{pc}} = - \int d\tau (m\sqrt{u^2} - \psi^\dagger i u^\mu D_\mu \psi) \quad (1)$$

$$= - \int d\tau (m\sqrt{u^2} - i\psi^\dagger \dot{\psi} - g u^\mu A_\mu^a c^a), \quad (2)$$

which is invariant under reparametrization of τ . Here $u^\mu = \dot{x}^\mu$ is the 4-velocity of the particle along the worldline $x^\mu(\tau)$ and $\psi(\tau)$ an associated fundamental vector carrying the color degrees of freedom of the particle—often called the color wave function. Moreover

$$c^a := \psi^\dagger T^a \psi \quad (3)$$

is the color charge carried by the particle. In this form, there is only a single gluon coupling to the scalar worldline. As the gravity counterpart also has higher graviton couplings to the worldline of a massive particle, this poses an immediate obstacle to a double-copy relation. Also a massive scalar field, for which a double copy of scattering

amplitudes exist [31], has linear *and quadratic* couplings to gluons. However, in a first-order formalism this situation is remedied. Defining canonical momenta

$$p_\mu = \frac{\partial L_{\text{pc}}}{\partial u^\mu} = m \frac{u_\mu}{\sqrt{u^2}} - g A_\mu^a c^a, \Rightarrow m^2 = (p + gAc)^2, \quad (4)$$

we find the Hamiltonian and the action in the first-order formalism

$$\begin{aligned} H_{\text{pc}} &= H_{\text{can}} + \lambda[(p + gAc)^2 - m^2], \\ H_{\text{can}} &= p_\mu u^\mu - L_{\text{pc}} \stackrel{(4)}{=} i\psi^\dagger \dot{\psi}, \end{aligned} \quad (5)$$

$$\begin{aligned} S_{\text{pc}} &= - \int d\tau (p_\mu u^\mu - H_{\text{pc}}) \\ &= - \int d\tau (p_\mu u^\mu - i\psi^\dagger \dot{\psi} - \lambda[p^2 + 2gp_\mu A_\mu^a c^a \\ &\quad + g^2 A_\mu^b c^b A_\mu^a c^a - m^2]). \end{aligned} \quad (6)$$

Here, H_{can} is the ‘‘canonical’’ Hamiltonian and H_{pc} is the ‘‘Dirac’’ Hamiltonian where the mass-shell constraint was added using a Lagrange multiplier $\lambda(\tau)$. Now we have both a single and a double gluon coupling to the worldline, similar to a massive scalar field. Note that $\lambda(\tau)d\tau$ is reparametrization invariant.

We now consider Yang-Mills gauge theory coupled to a set of two scalar particles in the worldline description. Notationally we separate the worldlines by placing a tilde on all variables associated to one of the worldlines. Therefore the full action for classical Yang-Mills (cYM) reads

$$S_{\text{cYM}} = S_{\text{YM}} + S_{\text{gf}} + S_{\text{pc}} + \tilde{S}_{\text{pc}}, \quad (7)$$

where a gauge fixing part S_{gf} was added. With this, the gluon propagator reads

$$\langle A_\mu^a(x) A_\nu^b(y) \rangle_0 = \frac{\hbar}{i} \eta_{\mu\nu} \delta_{ab} D(x-y), \quad (8)$$

where $D(x-y)$ is defined via

$$\square D(x-y) = -\delta(x-y). \quad (9)$$

III. EFFECTIVE ACTION OF COLORED MASSIVE PARTICLES AND ITS DOUBLE COPY

A. Principle of the double-copy construction

The double-copy scheme to be applied here is to first integrate out the gauge field A_μ^a in order to obtain the effective action $S_{\text{eff,YM}}$ for two colored, massive particles with Yang-Mills interactions, i.e.,

$$e^{\frac{i}{\hbar} S_{\text{eff,YM}}} = e^{\frac{i}{\hbar} S_{\text{pc,free}}} \mathcal{M}_{\text{YM}} = \text{const} \int \mathcal{D}A e^{\frac{i}{\hbar} S_{\text{cYM}}}. \quad (10)$$

The (divergent) normalization constant on the right-hand side is chosen such that $\mathcal{M}_{\text{YM}} = 1$ for $g \rightarrow 0$ and $S_{\text{pc,free}}$ is given by $S_{\text{pc}} + \tilde{S}_{\text{pc}}$ for $g \rightarrow 0$. The perturbative expansion in the coupling g of \mathcal{M}_{YM} can always be brought into the schematic form

$$\mathcal{M}_{\text{YM}} = \sum_{n=0}^{\infty} (2g)^{2n} \sum_{I \in \Gamma_n} \left(\frac{i}{\hbar}\right)^{x_I} \int \prod_{i_I} d\hat{\tau}_{i_I} \int d^{4l_I} x \frac{C_I N_I}{S_I D_I}, \quad (11)$$

where Γ_n is the set of N^{n-1} LO trivalent graphs with two external worldlines of the colored point particles, C_I denotes the color factor associated with graph I , i.e., functions of the $c(\tau)$'s, N_I the associated kinematic numerator factors and D_I the spacetime propagators appearing in the graph I . The x_I is defined as the number of vertices (bulk and worldline) minus the number of propagators in the trivalent diagram and l_I is the number of bulk vertices. Note that the integral measure along the worldline $d\tau$ always comes with a corresponding Lagrange multiplier $\lambda(\tau)$ in order to guarantee reparametrization invariance. We collect this into a new density notation via

$$d\hat{\tau}_i := d\tau_i \lambda(\tau_i). \quad (12)$$

The S_I finally, denote the symmetry factor of the graph I in the trivalent representation, i.e., the graph has to be drawn in such a way that it reproduces the color structure C_I . We will explain our notion of a trivalent representation more precisely in the next section.

The double-copy prescription that we are adopting here then amounts to representing the exponential of the effective action of two massive particles coupled to a weak-field (post-Minkowskian) expansion of dilaton gravity (dg) as

$$\mathcal{M}_{\text{dg}} = \sum_{n=0}^{\infty} (i\kappa)^{2n} \sum_{I \in \Gamma_n} \left(\frac{i}{\hbar}\right)^{x_I} \int \prod_{i_I} d\hat{\tau}_{i_I} \int d^{4l_I} x \frac{N_I N_I}{S_I D_I}, \quad (13)$$

where κ is the gravitational coupling constant. After that, the effective action in dilaton gravity follows from $e^{\frac{i}{\hbar} S_{\text{eff,dg}}} = e^{\frac{i}{\hbar} S_{\text{pc,free}}} \mathcal{M}_{\text{dg}}$. Only classical terms ($\hbar \rightarrow 0$) are retained in $S_{\text{eff,dg}}$ in our considerations. At higher order, C_I and N_I are required to fulfill the usual BCJ color-kinematics duality (Jacobi relations). We conclude this section by stressing that our double-copy prescription is proposed at the level of \mathcal{M} and not at the level of the effective action.

B. The trivalent representation of the effective action of colored particles to next-to-leading order

We now compute the effective action for two color-charged massive point particles moving on their worldlines to next-to-leading order, $\mathcal{O}(g^4)$. Furthermore, we will clarify what we mean by a trivalent representation of a diagram involving worldlines. We will use the short-hand notation

$$c_i := c(\tau_i), \quad p_i := p(\tau_i),$$

$$D_{ij} := D(x(\tau_i) - x(\tau_j)), \quad d\hat{\tau}_{1\dots n} := \prod_{i=1}^n d\tau_i \lambda(\tau_i). \quad (14)$$

Additionally, we will use the tilde notation for the ‘‘right’’ worldline in our diagrams.

At leading order we have the three graphs

$$\begin{array}{c} \tau_1 \\ \bullet \\ \text{---} \\ \bullet \\ \tilde{\tau}_2 \end{array} \quad \begin{array}{c} | \\ \text{---} \\ | \end{array} = 4g^2 \frac{i}{\hbar} \int d\hat{\tau}_{1\bar{2}} (c_1 \cdot \tilde{c}_2) (p_1 \cdot \tilde{p}_2) D_{1\bar{2}}, \quad (15)$$

$$\begin{array}{c} \tau_1 \\ \bullet \\ \text{---} \\ \bullet \\ \tau_2 \end{array} \quad \begin{array}{c} | \\ \text{---} \\ | \end{array} = 2g^2 \frac{i}{\hbar} \int d\hat{\tau}_{12} (c_1 \cdot c_2) (p_1 \cdot p_2) D_{12}, \quad (16)$$

$$\begin{array}{c} \tau_1 \\ \bullet \\ \text{---} \\ \bullet \\ | \end{array} \quad \begin{array}{c} | \\ \text{---} \\ | \end{array} \propto \int d\tau_1 D_{11} = 0 \quad \text{in dim. reg.} \quad (17)$$

Since the last term vanishes in dimensional regularization, we neglect any graph involving a bubble, as shown above, or any other scaleless integral. Note that we do not mention the mirrored counterparts to every graph with an uneven number of untilded and tilded variables. They can be trivially obtained by replacing untilded and tilded and it is understood that we add them to our final results.

At the next-to-leading order we encounter the graphs

$$\begin{array}{c} \tau_3 \\ \bullet \\ \text{---} \\ \bullet \\ \tau_1 \end{array} \quad \begin{array}{c} \tilde{\tau}_4 \\ \bullet \\ \text{---} \\ \bullet \\ \tilde{\tau}_2 \end{array} = 8g^4 \left(\frac{i}{\hbar}\right)^2 \int d\hat{\tau}_{1\bar{2}3\bar{4}} (c_1 \cdot \tilde{c}_2) (c_3 \cdot \tilde{c}_4) (p_1 \cdot \tilde{p}_2) (p_3 \cdot \tilde{p}_4) D_{1\bar{2}} D_{3\bar{4}}, \quad (18)$$

$$\begin{array}{c} \tau_1 \\ \bullet \\ \text{---} \\ \bullet \\ \tilde{\tau}_2 \end{array} \quad \begin{array}{c} \tilde{\tau}_4 \\ \bullet \\ \text{---} \\ \bullet \\ \tilde{\tau}_3 \end{array} = 4g^4 \frac{i}{\hbar} \int d\hat{\tau}_{1\bar{2}\bar{4}} (c_1 \cdot \tilde{c}_2) (c_1 \cdot \tilde{c}_4) (\tilde{p}_2 \cdot \tilde{p}_4) D_{1\bar{2}} D_{1\bar{4}} \\ = 4g^4 \frac{i}{\hbar} \int d\hat{\tau}_{1\bar{2}\bar{4}3} (c_1 \cdot \tilde{c}_2) (c_3 \cdot \tilde{c}_4) (\tilde{p}_2 \cdot \tilde{p}_4) D_{1\bar{2}} D_{3\bar{4}} \frac{\delta(\tau_3 - \tau_1)}{\lambda_3}, \quad (19)$$

where we used a delta function in the last step in order to introduce a dummy τ_3 integration. This can be understood as pulling apart the gluon-gluon-worldline vertex into two gluon-worldline vertices, i.e., arriving at our definition of a trivalent representation necessary for our proposed double-copy prescription (13).

The next graph is

$$\begin{array}{c} \tau_1 \\ \bullet \\ \text{---} \\ \bullet \\ \tilde{\tau}_2 \end{array} \quad \begin{array}{c} \tilde{\tau}_3 \\ \bullet \\ \text{---} \\ \bullet \\ \tilde{\tau}_2 \end{array} = -4g^4 \frac{i}{\hbar} \int d\hat{\tau}_{1\bar{2}\bar{3}} f^{abc} c_1^a \tilde{c}_2^b \tilde{c}_3^c V_{1\bar{2}\bar{3}}^{\mu\nu\rho} p_{1\mu} \tilde{p}_{2\nu} \tilde{p}_{3\rho} G_{1\bar{2}\bar{3}}, \quad (20)$$

where $V_{123}^{\mu_1\mu_2\mu_3}$ is the color independent part of the Yang-Mills three gluon vertex

$$V_{123}^{\mu_1\mu_2\mu_3} = \eta^{\mu_1\mu_2}(\partial_1^{\mu_3} - \partial_2^{\mu_3}) + \text{cyclic}, \quad (21)$$

and $G_{1\bar{2}\bar{3}} = \int d^4x D_{1x} D_{\bar{2}x} D_{\bar{3}x}$. We proceed in an analogous fashion with

$$\begin{aligned} \tau_1 \left| \begin{array}{c} \tilde{\tau}_3 \\ \tilde{\tau}_2 \end{array} \right. &= 8g^4 \frac{i}{\hbar} \int d\hat{\tau}_{1\bar{2}\bar{3}} (c_1 \cdot \tilde{c}_2) (\tilde{c}_2 \cdot \tilde{c}_3) (p_1 \cdot \tilde{p}_3) D_{1\bar{2}} D_{\bar{2}\bar{3}} \\ &= 8g^4 \frac{i}{\hbar} \int d\hat{\tau}_{1\bar{2}\bar{3}\bar{4}} (c_1 \cdot \tilde{c}_2) (\tilde{c}_4 \cdot \tilde{c}_3) (p_1 \cdot \tilde{p}_3) D_{1\bar{2}} D_{\bar{4}\bar{3}} \frac{\delta(\tilde{\tau}_4 - \tilde{\tau}_2)}{\tilde{\lambda}_4}, \end{aligned} \quad (22)$$

$$\tau_1 \left| \begin{array}{c} \tilde{\tau}_3 \\ \tilde{\tau}_4 \end{array} \right. = 8g^4 \left(\frac{i}{\hbar}\right)^2 \int d\hat{\tau}_{1\bar{2}\bar{3}\bar{4}} (c_1 \cdot \tilde{c}_2) (\tilde{c}_3 \cdot \tilde{c}_4) (p_1 \cdot \tilde{p}_2) (\tilde{p}_3 \cdot \tilde{p}_4) D_{1\bar{2}} D_{\bar{3}\bar{4}}, \quad (23)$$

$$\left. \begin{array}{c} \tau_3 \\ \tau_4 \\ \tau_1 \\ \tau_2 \end{array} \right| = 2g^4 \left(\frac{i}{\hbar}\right)^2 \int d\hat{\tau}_{1234} (c_1 \cdot c_2) (c_3 \cdot c_4) (p_1 \cdot p_2) (p_3 \cdot p_4) D_{12} D_{43}, \quad (24)$$

$$\begin{aligned} \left. \begin{array}{c} \tau_3 \\ \tau_1 \\ \tau_2 \end{array} \right| &= 4g^4 \frac{i}{\hbar} \int d\hat{\tau}_{123} (c_1 \cdot c_2) (c_1 \cdot c_3) (p_2 \cdot p_3) D_{12} D_{13} \\ &= 4g^4 \frac{i}{\hbar} \int d\hat{\tau}_{1234} (c_1 \cdot c_2) (c_4 \cdot c_3) (p_2 \cdot p_3) D_{12} D_{43} \frac{\delta(\tau_4 - \tau_1)}{\lambda_4}. \end{aligned} \quad (25)$$

We also have graphs that are not mirrored counterparts of above mentioned graphs, i.e.,

$$\left. \begin{array}{c} \tau_1 \\ \tau_2 \end{array} \right|, \quad \left. \begin{array}{c} \tau_2 \\ \tau_1 \end{array} \right|, \quad \left. \begin{array}{c} \tau_1 \\ \tau_2 \end{array} \right|, \quad (26)$$

which we only include implicitly due to their very similar analytic form as previously computed graphs. In addition we neglect graphs that represent quantum corrections, since they are suppressed in the classical effective action. In particular we neglect the following type of graphs:

$$\left. \begin{array}{c} \tau_1 \\ \tau_2 \end{array} \right| \left(\text{blob} \right), \quad (27)$$

where the grey blob denotes loop topologies.

C. The double copy of the gauge-theory effective action

In order to perform the double copy according to (11) we first need to investigate the independent color structures from the previous section. They are given by

$$\begin{aligned} \text{LO: } & (c \cdot \tilde{c}), (c \cdot c), (\tilde{c} \cdot \tilde{c}). \\ \text{NLO: } & (c \cdot \tilde{c})^2, (c \cdot \tilde{c})(\tilde{c} \cdot \tilde{c}), (\tilde{c} \cdot c)(c \cdot c), (c \cdot c)^2, (\tilde{c} \cdot \tilde{c})^2, \\ & f^{abc} c_a c_b c_c, f^{abc} c_a \tilde{c}_b \tilde{c}_c, f^{abc} \tilde{c}_a c_b c_c, f^{abc} \tilde{c}_a \tilde{c}_b \tilde{c}_c, \end{aligned} \quad (28)$$

yielding the backbone to write down the terms in (11). We start with the H -graph

$$H : \tau_1 \left| \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \right| \tau_2 = (2g)^2 \left(\frac{i}{\hbar} \right) \int d\hat{\tau}_{1\bar{2}} \frac{C_H N_H}{S_H D_H}, \quad (29)$$

where

$$\begin{aligned} C_H &= (c_1 \cdot \tilde{c}_2), & D_H^{-1} &= D_{1\bar{2}}, \\ S_H &= 1, & N_H &= (p_1 \cdot \tilde{p}_2). \end{aligned} \quad (30)$$

Next we have the single-worldline (self-interaction) D -graph

$$D : \tau_1 \left| \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array} \right| \tau_2 = (2g)^2 \left(\frac{i}{\hbar} \right) \int d\hat{\tau}_{1\bar{2}} \frac{C_D N_D}{S_D D_D}, \quad (31)$$

where

$$\begin{aligned} C_D &= (c_1 \cdot c_2), & D_D^{-1} &= D_{12}, \\ S_D &= 2, & N_D &= (p_1 \cdot p_2). \end{aligned} \quad (32)$$

Turning to next-to-leading order contributions, we have the V -graph

$$\begin{aligned} V : \tau_1 \left| \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array} \right| \tau_2 + \tau_1 \left| \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array} \right| \tau_3 + \tau_1 \left| \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array} \right| \tau_3 \\ = (2g)^4 \left(\frac{i}{\hbar} \right)^2 \int d\hat{\tau}_{1\bar{2}\bar{3}\bar{4}} \frac{C_V N_V}{S_V D_V}, \end{aligned} \quad (33)$$

where

$$\begin{aligned} D_V^{-1} &= D_{1\bar{3}} D_{2\bar{4}}, & C_V &= (c_1 \cdot \tilde{c}_3)(c_2 \cdot \tilde{c}_4), & S_V &= 2, \\ N_V &= (p_1 \cdot \tilde{p}_3)(p_2 \cdot \tilde{p}_4) + \frac{\hbar}{2i} \frac{\delta(\tau_1 - \tau_2)}{\lambda_2} (\tilde{p}_3 \cdot \tilde{p}_4) \\ &\quad + \frac{\hbar}{2i} \frac{\delta(\tilde{\tau}_3 - \tilde{\tau}_4)}{\tilde{\lambda}_4} (p_1 \cdot p_2), \end{aligned} \quad (34)$$

and the Y -graph

$$Y : \tau_1 \left| \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array} \right| \tau_2 = (2g)^4 \left(\frac{i}{\hbar} \right) \int d\hat{\tau}_{1\bar{2}\bar{3}} \frac{C_Y N_Y}{S_Y D_Y}, \quad (35)$$

where

$$\begin{aligned} C_Y &= f^{abc} c_1^a \tilde{c}_2^b \tilde{c}_3^c, & D_Y^{-1} &= G_{1\bar{2}\bar{3}}, & S_Y &= 2, \\ N_Y &= -\frac{1}{2} V_{1\bar{2}\bar{3}}^{\mu\nu\rho} p_{1\mu} \tilde{p}_{2\nu} \tilde{p}_{3\rho}. \end{aligned} \quad (36)$$

Next there is the C -graph

$$C : \tau_1 \left| \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array} \right| \tau_2 + \tau_1 \left| \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array} \right| \tau_3 = (2g)^4 \left(\frac{i}{\hbar} \right)^2 \int d\hat{\tau}_{1\bar{2}\bar{3}\bar{4}} \frac{C_C N_C}{S_C D_C}, \quad (37)$$

where

$$\begin{aligned} C_C &= (c_1 \cdot \tilde{c}_2)(\tilde{c}_3 \cdot \tilde{c}_4), & D_C^{-1} &= D_{1\bar{2}} D_{\bar{3}\bar{4}}, & S_C &= 2, \\ N_C &= (p_1 \cdot \tilde{p}_2)(\tilde{p}_3 \cdot \tilde{p}_4) + \frac{\hbar}{i} \frac{\delta(\tilde{\tau}_2 - \tilde{\tau}_4)}{\tilde{\lambda}_4} (p_1 \cdot \tilde{p}_3). \end{aligned} \quad (38)$$

Note that here we use the symmetry factor of the first topology of the C -graph. This is consistent with our double-copy prescription which states that the correct symmetry factor is the one attributed to the trivalent graph.

Next in line is the B -graph

$$B : \left| \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array} \right| + \left| \begin{array}{c} \bullet \\ \text{---} \\ \bullet \\ \text{---} \\ \bullet \end{array} \right| = (2g)^4 \left(\frac{i}{\hbar} \right)^2 \int d\hat{\tau}_{1\bar{2}\bar{3}\bar{4}} \frac{C_B N_B}{S_B D_B}, \quad (39)$$

where

$$C_B = (\tilde{c}_1 \cdot \tilde{c}_2)(\tilde{c}_4 \cdot \tilde{c}_3), \quad D_B^{-1} = D_{\tilde{1}\tilde{2}}D_{\tilde{3}\tilde{4}}, \quad S_B = 8, \\ N_B = (\tilde{p}_1 \cdot \tilde{p}_2)(\tilde{p}_3 \cdot \tilde{p}_4) + 2\frac{\hbar \delta(\tilde{\tau}_4 - \tilde{\tau}_1)}{\tilde{\lambda}_4}(\tilde{p}_2 \cdot \tilde{p}_3). \quad (40)$$

Again, the symmetry factor is given by the first topology of the B -graph.

The same constructions apply to the mirrored graphs and the graphs mentioned in (26). Performing the double copy as discussed in Sec. III A yields our prediction for the exponential of the effective action in dilaton gravity

$$\bar{\mathcal{M}}_{\text{dg}} = 1 - \kappa^2 \frac{i}{\hbar} \left(\int d\hat{\tau}_H \frac{N_H N_H}{S_H D_H} + \int d\hat{\tau}_D \frac{N_D N_D}{S_D D_D} \right) \\ + \kappa^4 \left(\frac{i}{\hbar} \right)^2 \left(\int d\hat{\tau}_V \frac{N_V N_V}{S_V D_V} + \int d\hat{\tau}_C \frac{N_C N_C}{S_C D_C} + \int d\hat{\tau}_B \frac{N_B N_B}{S_B D_B} \right) \\ + \kappa^4 \frac{i}{\hbar} \int d\hat{\tau}_Y \frac{N_Y N_Y}{S_Y D_Y} + (\text{mirrored}), \quad (41)$$

where we introduced the bar notation for $\bar{\mathcal{M}}_{\text{dg}}$ to point out that it is computed via our double-copy prescription. At this point it is necessary to comment on a subtle feature of the above result. An essential technicality in the previous construction was the notion of pulling apart the gluon-gluon-worldline vertex using a delta function, i.e., obtaining a trivalent representation. Doing so, our double-copy prescription introduces $\delta(0)$ terms in $\bar{\mathcal{M}}_{\text{dg}}$. This is a potential hazard but since such terms are of $\mathcal{O}(\hbar^0)$, they are quantum corrections to $S_{\text{eff,dg}}$ and we neglect them for the time being.

Another important property of $S_{\text{eff,dg}}$ is that all negative powers of \hbar exponentiate, which is *a priori* not obvious. We check this by taking the logarithm of our double-copy result, obtaining a perturbative expansion of the effective action,

$$\bar{S}_{\text{eff,dg}} = S_{\text{pc,free}} + \frac{\hbar}{i} \log \bar{\mathcal{M}}_{\text{dg}} = S_{\text{pc,free}} - \kappa^2 \int d\hat{\tau}_{1\tilde{2}}(p_1 \cdot \tilde{p}_2)^2 D_{1\tilde{2}} - \frac{\kappa^2}{2} \int d\hat{\tau}_{12}(p_1 \cdot p_2)^2 D_{12} \\ + \frac{\kappa^4}{2} \int d\hat{\tau}_{1\tilde{2}\tilde{3}} \left(\frac{1}{2} V_{1\tilde{2}\tilde{3}}^{\mu\nu\rho} p_{1\mu} \tilde{p}_{2\nu} \tilde{p}_{3\rho} \right)^2 G_{1\tilde{2}\tilde{3}} \\ + \frac{\kappa^4}{2} \int d\hat{\tau}_{1\tilde{2}\tilde{3}}(p_1 \cdot \tilde{p}_2)(p_1 \cdot \tilde{p}_3)(\tilde{p}_3 \cdot \tilde{p}_2) D_{1\tilde{2}} D_{1\tilde{3}} \\ + \kappa^4 \int d\hat{\tau}_{1\tilde{2}\tilde{3}}(p_1 \cdot \tilde{p}_3)(p_1 \cdot \tilde{p}_2)(\tilde{p}_3 \cdot \tilde{p}_2) D_{1\tilde{2}} D_{\tilde{2}\tilde{3}} \\ + \frac{\kappa^4}{2} \int d\hat{\tau}_{123}(p_1 \cdot p_2)(p_1 \cdot p_3)(p_2 \cdot p_3) D_{12} D_{13} \\ + (\text{mirrored}) + \mathcal{O}(\hbar). \quad (42)$$

In the following we will check whether a direct construction within dilaton gravity reproduces this prediction.

IV. INTERACTION OF POINT MASSES IN DILATON GRAVITY

In this section we are performing an analogous analysis as in the previous section for a system of two massive worldlines that are interacting in dilaton gravity.

A. Dilaton gravity

The action of dilaton gravity is given by¹

$$S_{\text{dg}} = -\frac{2}{\kappa^2} \int d^4x \sqrt{-g} [R - 2\partial_\mu \phi \partial^\mu \phi] \\ + (\text{GHY boundary term}) \quad (43)$$

$$= -\frac{2}{\kappa^2} \int d^4x \sqrt{-g} [g^{\mu\nu} (\Gamma^\rho{}_{\mu\lambda} \Gamma^\lambda{}_{\nu\rho} \\ - \Gamma^\rho{}_{\mu\nu} \Gamma^\lambda{}_{\rho\lambda}) - 2\partial_\mu \phi \partial^\mu \phi], \quad (44)$$

where $\kappa = m_{\text{Pl}}^{-1} = \sqrt{32\pi G}$ is the gravitational coupling (with Newton constant G and Planck mass m_{Pl}), ϕ is a real scalar field called the dilaton, R is the usual Ricci scalar, $\Gamma^\alpha{}_{\mu\nu} = (\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu})g^{\alpha\beta}/2$ is the Christoffel connection, $g_{\mu\nu}$ is the metric, and $g = \det(g_{\mu\nu})$. The worldline action of a point mass (pm) coupled to gravity and a dilaton is defined by

¹Note that the Gibbons-Hawking-York (GHY) boundary term [45,46] is nonzero also in dimensional regularization for asymptotically flat spacetimes, leading to (44).

$$S_{\text{pm}} = -m \int d\tau e^\phi \sqrt{g_{\mu\nu} u^\mu u^\nu}, \quad (45)$$

with the worldline $x^\mu(\tau)$, the 4-velocity $u^\mu = \dot{x}^\mu$ and $\phi, g_{\mu\nu}$ are evaluated at $x^\mu(\tau)$. We again pass to the first-order version

$$S_{\text{pm}} = - \int d\tau (p_\mu u^\mu - \lambda [e^{-2\phi} g^{\mu\nu} p_\mu p_\nu - m^2]). \quad (46)$$

The full action reads

$$S_{\text{dg}} + S_{\text{gf}} + S_{\text{pm}} + \tilde{S}_{\text{pm}}, \quad (47)$$

with a gauge fixing part S_{gf} specified below. Note that we do not need an axion field here, which will be important when adding spin to the point particles [26].

B. Weak-field expansion

The weak-field approximation is defined as the expansion of the full metric around a flat Minkowski background, i.e.,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x), \quad (48)$$

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - \kappa h^{\mu\nu}(x) + \kappa^2 h^{\mu\lambda}(x) h_\lambda{}^\nu(x) + \mathcal{O}(\kappa^3), \quad (49)$$

$$\begin{aligned} \sqrt{-g} &= 1 + \kappa h^\mu{}_\mu(x) - \frac{\kappa^2}{2} (h^{\mu\nu}(x) h_{\mu\nu}(x) - (h^\mu{}_\mu(x))^2) \\ &+ \mathcal{O}(\kappa^3), \end{aligned} \quad (50)$$

where κ is our weak-field (or post-Minkowskian) perturbation parameter and $h_{\mu\nu}$ is the graviton. In principle we are ready to start the computation of the effective action by integrating out the graviton and dilaton field. Nevertheless, at this point one encounters large expressions during intermediate steps, e.g., the three-graviton vertex will have around 170 terms [47]. However, it is known that one can achieve a high simplification by performing field redefinitions and adding the appropriate gauge fixings [48]. Here, we aim at a field redefinition that removes the coupling of the dilaton to the worldline and also simplifies the three-graviton vertex to the square of the Yang-Mills one. Our procedure is given by:

First, choosing the following gauge fixing terms to $\mathcal{O}(\kappa^2)$,

$$S_{\text{gf}} = \frac{1}{\kappa^2} \int d^4x \sqrt{-g} f^\mu f_\mu, \quad (51)$$

$$\begin{aligned} f^\mu &= \Gamma^\mu{}_{\nu\sigma} g^{\nu\sigma} + \frac{\kappa^2}{2} \left[-\frac{1}{4} (\partial_\kappa h^{\kappa\lambda}) h_\lambda{}^\mu - \frac{1}{4} (\partial^\mu h^{\kappa\lambda}) h_{\kappa\lambda} \right. \\ &+ (\partial^\kappa h^{\mu\lambda}) h_{\kappa\lambda} + \frac{3}{16} (\partial^\mu h^\kappa{}_\kappa) h^\lambda{}_\lambda - \frac{3}{8} (\partial^\kappa h^\mu{}_\kappa) h^\lambda{}_\lambda \\ &\left. - \frac{3}{8} (\partial^\lambda h^\kappa{}_\kappa) h^\mu{}_\lambda \right], \end{aligned} \quad (52)$$

then performing the field redefinitions

$$\begin{aligned} h_{\mu\nu} &\rightarrow h_{\mu\nu} - \eta_{\mu\nu} \left(\frac{1}{2} h^\mu{}_\mu + 2\phi \right) \\ &+ \kappa \left(-\frac{1}{2} h_{\mu\nu} h^\rho{}_\rho + \frac{1}{8} \eta_{\mu\nu} h^\rho{}_\rho h^\sigma{}_\sigma + \frac{1}{2} h_{\mu\rho} h^\rho{}_\nu - 2\phi h_{\mu\nu} \right. \\ &\left. + 2\phi^2 \eta_{\mu\nu} + \phi h_{\mu\nu} h^\rho{}_\rho \right), \end{aligned} \quad (53)$$

$$\phi \rightarrow \phi + \frac{1}{4} h^\mu{}_\mu, \quad (54)$$

and finally, adding the total derivative

$$\begin{aligned} 0 = S_{\text{TD}} &= \int d^4x \left[\partial_\mu ((\partial_\nu h^{\mu\kappa}) h^\nu{}_\kappa) - \partial_\mu (h^{\mu\nu} (\partial_\kappa h^\kappa{}_\nu)) \right. \\ &+ \kappa \left(\frac{1}{4} \partial_\mu (h^{\mu\nu} (\partial_\nu h^{\sigma\nu}) h_{\sigma\nu}) - \frac{1}{4} \partial_\mu ((\partial_\nu h^{\mu\lambda}) h_{\lambda\kappa} h^{\kappa\nu}) \right. \\ &\left. - \frac{1}{4} \partial_\mu (h^{\mu\nu} (\partial_\lambda h_{\nu\rho}) h^{\rho\lambda}) + \frac{1}{4} \partial_\mu (h^\mu{}_\nu h^{\nu\lambda} (\partial_\sigma h^\lambda{}_\sigma)) \right), \end{aligned} \quad (55)$$

to the action. This procedure decouples the point masses on the worldline from the dilaton,

$$\begin{aligned} S_{\text{pm}} &= - \int d\tau \left(p_\mu u^\mu - \lambda \left[(\eta^{\mu\nu} - \kappa h^{\mu\nu} + \frac{\kappa^2}{2} h^\mu{}_\rho h^{\rho\nu}) p_\mu p_\nu \right. \right. \\ &\left. \left. - m^2 \right] \right) + \mathcal{O}(\kappa^3), \end{aligned} \quad (56)$$

which means we only need to worry about the graviton-dependent part of the field action for the computation of the classical effective action $S_{\text{eff,dg}}$ at next-to-leading order. Furthermore we observe a considerable simplification in the three-graviton interaction, i.e., it is given by

$$S_{\text{dg}} + S_{\text{gf}} + S_{\text{TD}} \quad (57)$$

$$\begin{aligned} &= \int d^4x \left[\frac{1}{2} \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} + \frac{\kappa}{4} (h_{\mu\nu} \partial^\mu \partial^\nu h_{\rho\sigma} h^{\rho\sigma} \right. \\ &+ 2h_{\mu\nu} \partial^\sigma h^\mu{}_\rho \partial^\nu h^\rho{}_\sigma - h_{\mu\nu} \partial^\sigma h_\rho{}^\mu \partial^\rho h^\nu{}_\sigma - h_{\rho\sigma} \partial^\rho h_{\mu\nu} \partial^\sigma h^{\mu\nu} \\ &\left. - \partial_\rho \partial_\sigma h_{\mu\nu} h^{\rho\mu} h^{\sigma\nu} \right] + \mathcal{O}(\kappa^2, \phi) \end{aligned} \quad (58)$$

$$\begin{aligned} &= \int d^4x \left[\frac{1}{2} \partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu} + \frac{\kappa}{4 \cdot 3!} V_{123}^{\mu\alpha\gamma} V_{123}^{\nu\beta\delta} h_{1\mu\nu} h_{2\alpha\beta} h_{3\gamma\delta} \right] \\ &+ \mathcal{O}(\kappa^2, \phi), \end{aligned} \quad (59)$$

where $V_{123}^{\mu\alpha\gamma}$ is again the color independent part of the three gluon interaction defined by (21). We also introduced fiducial indices 1, 2, 3, on $h_{\mu\nu}$ to indicate on which $h_{\mu\nu}$

the partial derivatives in $V_{123}^{\mu_1\mu_2\mu_3}$ are applied. The propagator therefore reads

$$\langle h_{\mu\nu}(x)h_{\rho\sigma}(y) \rangle_0 = \frac{\hbar}{i} \eta_{\mu(\rho} \eta_{\sigma)\nu} D(x-y), \quad (60)$$

where the round brackets on the indices indicate a symmetrization of unit weight,

$$\eta_{\mu(\rho} \eta_{\sigma)\nu} = \frac{1}{2} (\eta_{\mu\rho} \eta_{\sigma\nu} + \eta_{\mu\sigma} \eta_{\rho\nu}). \quad (61)$$

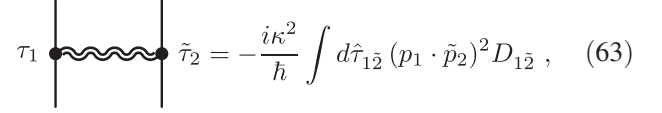
C. Effective action of gravitating particles to next-to-leading order

Similar to the Yang-Mills case in Sec. III B, we compute $S_{\text{eff,dg}}$ by integrating out the graviton and scalar fields,

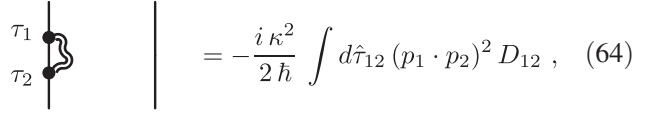
$$e^{\frac{i}{\hbar} S_{\text{eff,dg}}} = e^{\frac{i}{\hbar} S_{\text{pm,free}}} \mathcal{M}_{\text{dg}} = \text{const} \int \mathcal{D}h \mathcal{D}\phi e^{\frac{i}{\hbar} (S_{\text{dg}} + S_{\text{gf}} + S_{\text{pm}} + \tilde{S}_{\text{pm}})}. \quad (62)$$

Again the (divergent) normalization constant on the right-hand side is chosen such that $\mathcal{M}_{\text{dg}} = 1$ for $\kappa \rightarrow 0$ and $S_{\text{pm,free}}$ is given by $S_{\text{pm}} + \tilde{S}_{\text{pm}}$ for $\kappa \rightarrow 0$. We also neglect quantum corrections and keep all mirrored graphs implicit.

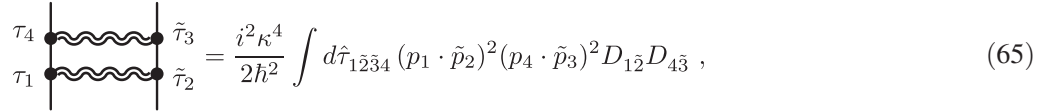
The relevant classical diagrams for \mathcal{M}_{dg} to order κ^4 read



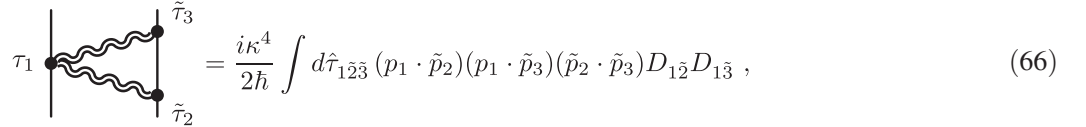
$$\tau_1 \left[\text{diagram} \right] \tau_2 = -\frac{i\kappa^2}{\hbar} \int d\hat{\tau}_{1\bar{2}} (p_1 \cdot \tilde{p}_2)^2 D_{1\bar{2}}, \quad (63)$$



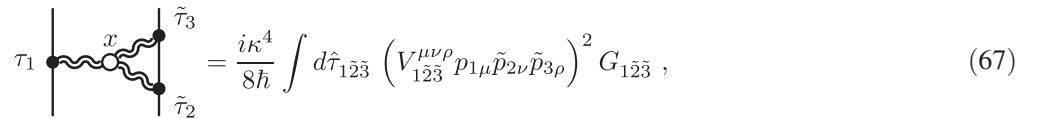
$$\tau_1 \left[\text{diagram} \right] \tau_2 = -\frac{i\kappa^2}{2\hbar} \int d\hat{\tau}_{12} (p_1 \cdot p_2)^2 D_{12}, \quad (64)$$



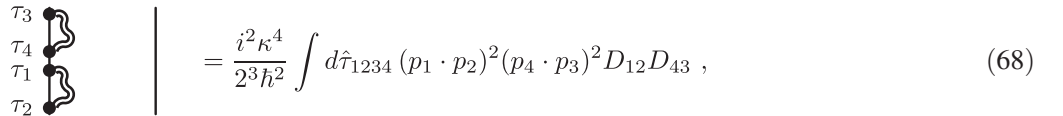
$$\tau_1 \left[\text{diagram} \right] \tau_2 = \frac{i^2 \kappa^4}{2\hbar^2} \int d\hat{\tau}_{1\bar{2}\bar{3}\bar{4}} (p_1 \cdot \tilde{p}_2)^2 (p_4 \cdot \tilde{p}_3)^2 D_{1\bar{2}} D_{4\bar{3}}, \quad (65)$$



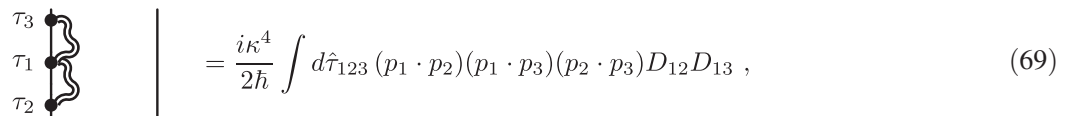
$$\tau_1 \left[\text{diagram} \right] \tau_2 = \frac{i\kappa^4}{2\hbar} \int d\hat{\tau}_{1\bar{2}\bar{3}} (p_1 \cdot \tilde{p}_2) (p_1 \cdot \tilde{p}_3) (\tilde{p}_2 \cdot \tilde{p}_3) D_{1\bar{2}} D_{1\bar{3}}, \quad (66)$$




$$\tau_1 \left[\text{diagram} \right] \tau_2 = \frac{i\kappa^4}{8\hbar} \int d\hat{\tau}_{1\bar{2}\bar{3}} \left(V_{1\bar{2}\bar{3}}^{\mu\nu\rho} p_{1\mu} \tilde{p}_{2\nu} \tilde{p}_{3\rho} \right)^2 G_{1\bar{2}\bar{3}}, \quad (67)$$



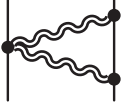
$$\tau_3 \left[\text{diagram} \right] \tau_2 = \frac{i^2 \kappa^4}{2^3 \hbar^2} \int d\hat{\tau}_{1234} (p_1 \cdot p_2)^2 (p_4 \cdot p_3)^2 D_{12} D_{43}, \quad (68)$$




$$\tau_3 \left[\text{diagram} \right] \tau_2 = \frac{i\kappa^4}{2\hbar} \int d\hat{\tau}_{123} (p_1 \cdot p_2) (p_1 \cdot p_3) (p_2 \cdot p_3) D_{12} D_{13}, \quad (69)$$



$$= \frac{\kappa^2}{4\pi} \int dt \frac{\lambda \tilde{\lambda} E^2 \tilde{E}^2}{2r} \left[2 - 4 \frac{\mathbf{p} \cdot \tilde{\mathbf{p}}}{E \tilde{E}} + \mathbf{v} \cdot \tilde{\mathbf{v}} - (\mathbf{n} \cdot \mathbf{v})(\mathbf{n} \cdot \tilde{\mathbf{v}}) \right] + \mathcal{O}(c^{-6}), \quad (78)$$



$$= \frac{\kappa^4}{(4\pi)^2} \int dt \frac{\lambda \tilde{\lambda}^2 E^2 \tilde{E}^4}{2r^2} + \mathcal{O}(c^{-6}), \quad (79)$$



$$= 0 + \mathcal{O}(c^{-6}), \quad (80)$$

where $\mathbf{n} = (\mathbf{x} - \tilde{\mathbf{x}})/r$, and we used

$$\int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{e^{i\mathbf{k} \cdot \mathbf{x}}}{(\mathbf{k}^2)^\alpha} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(d/2 - \alpha)}{\Gamma(\alpha)} \left(\frac{\mathbf{x}^2}{4} \right)^{\alpha - d/2}. \quad (81)$$

Moreover in (78) we anticipated that $\lambda = \text{const} + \mathcal{O}(c^{-2}) = E$, allowing us to drop time derivatives of λ , E . The effective action in the post-Newtonian approximation finally reads

$$S_{\text{eff,dg}} = \int dt \left[\mathbf{p} \cdot \mathbf{v} - E + \lambda(E^2 - m^2 - \mathbf{p}^2) + \tilde{\mathbf{p}} \cdot \tilde{\mathbf{v}} - \tilde{E} + \tilde{\lambda}(\tilde{E}^2 - \tilde{m}^2 - \tilde{\mathbf{p}}^2) + \frac{\kappa^2 \lambda \tilde{\lambda} E^2 \tilde{E}^2}{8\pi r} \left(2 - 4 \frac{\mathbf{p} \cdot \tilde{\mathbf{p}}}{E \tilde{E}} + \mathbf{v} \cdot \tilde{\mathbf{v}} - (\mathbf{n} \cdot \mathbf{v})(\mathbf{n} \cdot \tilde{\mathbf{v}}) \right) + \frac{\kappa^4 \lambda \tilde{\lambda}^2 E^2 \tilde{E}^4}{2(4\pi r)^2} + \frac{\kappa^4 \tilde{\lambda}^2 \tilde{E}^2 E^4}{2(4\pi r)^2} + \mathcal{O}(c^{-6}) \right]. \quad (82)$$

Next, we vary the effective action with respect to λ and E to arrive at

$$0 = E^2 - m^2 - \mathbf{p}^2 + \frac{\kappa^2 \tilde{\lambda} E^2 \tilde{E}^2}{8\pi r} \left[2 - 4 \frac{\mathbf{p} \cdot \tilde{\mathbf{p}}}{E \tilde{E}} + \mathbf{v} \cdot \tilde{\mathbf{v}} - (\mathbf{n} \cdot \mathbf{v})(\mathbf{n} \cdot \tilde{\mathbf{v}}) \right] + \frac{\kappa^4 \tilde{\lambda}^2 E^2 \tilde{E}^4 + 2\lambda \tilde{\lambda} E^4 \tilde{E}^2}{(4\pi)^2 2r^2} + \mathcal{O}(c^{-6}), \quad (83)$$

$$0 = -1 + 2\lambda E + \frac{\kappa^2 \lambda \tilde{\lambda} E \tilde{E}^2}{2\pi r} + \mathcal{O}(c^{-4}), \quad (84)$$

and similar for the tilded variables. Solving iteratively for λ , E , using also $\mathbf{p} = m\mathbf{v} + \mathcal{O}(c^{-3})$,

$$E = m + \frac{\mathbf{p}^2}{2m} - \frac{\kappa^2 m \tilde{m}}{16\pi r} - \frac{\mathbf{p}^4}{8m^3} + \frac{2\kappa^4 m \tilde{m} (2m + \tilde{m})}{(32\pi r)^2} \quad (85)$$

$$- \frac{\kappa^2 m \tilde{m}}{32\pi r} \left[\frac{\mathbf{p}^2}{m^2} + \frac{\tilde{\mathbf{p}}^2}{\tilde{m}^2} - \frac{3(\mathbf{p} \cdot \tilde{\mathbf{p}})}{m \tilde{m}} - \frac{(\mathbf{n} \cdot \mathbf{p})(\mathbf{n} \cdot \tilde{\mathbf{p}})}{m \tilde{m}} \right] + \mathcal{O}(c^{-6}), \quad (86)$$

$$\lambda = \frac{1}{2m} \left[1 - \frac{\mathbf{p}^2}{2m^2} - \frac{\kappa^2 \tilde{m}}{16\pi r} \right] + \mathcal{O}(c^{-4}), \quad (87)$$

and substituting these solutions into the action, we arrive at (recall $\kappa^2 = 32\pi G$)

$$S_{\text{eff,dg}} = \int dt [\mathbf{p} \cdot \mathbf{v} + \tilde{\mathbf{p}} \cdot \tilde{\mathbf{v}} - H], \quad (88)$$

$$H = m + \tilde{m} + \frac{\mathbf{p}^2}{2m} + \frac{\tilde{\mathbf{p}}^2}{2\tilde{m}} - \frac{2Gm\tilde{m}}{r} - \frac{\mathbf{p}^4}{8m^3} - \frac{\tilde{\mathbf{p}}^4}{8\tilde{m}^3} + \frac{2G^2 m \tilde{m} (m + \tilde{m})}{r^2} - \frac{Gm\tilde{m}}{r} \left[\frac{\mathbf{p}^2}{m^2} + \frac{\tilde{\mathbf{p}}^2}{\tilde{m}^2} - \frac{3\mathbf{p} \cdot \tilde{\mathbf{p}}}{m \tilde{m}} - \frac{(\mathbf{n} \cdot \mathbf{p})(\mathbf{n} \cdot \tilde{\mathbf{p}})}{m \tilde{m}} \right] + \mathcal{O}(c^{-6}). \quad (89)$$

Notice the extra factor of 2 in the Newtonian potential, which is due to the dilaton. Throughout this derivation, higher-order time derivatives (like accelerations) are removed by inserting the respective equations of motion. This corresponds to a variable redefinition at the considered order [50]. [In particular, this justifies the dropping of time derivatives of λ , E and the use of $\mathbf{p} = m\mathbf{v} + \mathcal{O}(c^{-3})$ above.] Further note that H is the Hamiltonian describing the motion of the binary system in dilaton gravity.

We may compare our result to scalar-tensor theory [49]. Scalar-tensor theories (in the so called Einstein frame) are based on the same field action as dilaton gravity (43). However, the source part (45) is more generic in scalar-tensor theories,

$$S_{\text{pm,st}} = -m \int d\tau \sqrt{g_{\mu\nu} u^\mu u^\nu} \times \left[1 + \alpha_0 \phi + \frac{1}{2} (\alpha_0^2 + \beta_0) \phi^2 + \mathcal{O}(\kappa^3) \right], \quad (90)$$

with the so-called sensitivities α_0 and β_0 . Comparing to (45) we find that $\alpha_0 = 1$ and $\beta_0 = 0$ for dilaton gravity. The Hamiltonian for scalar-tensor gravity, following via a

Legendre transform from the Lagrangian stated in Eq. (3.7) of Ref. [49], reads

$$\begin{aligned}
H = & m + \tilde{m} + \frac{\mathbf{p}^2}{2m} + \frac{\tilde{\mathbf{p}}^2}{2\tilde{m}} - \frac{\tilde{G}m\tilde{m}}{r} - \frac{\mathbf{p}^4}{8m^3} - \frac{\tilde{\mathbf{p}}^4}{8\tilde{m}^3} \\
& - \frac{\tilde{G}m\tilde{m}}{r} \left[\frac{3\mathbf{p}^2}{2m^2} + \frac{3\tilde{\mathbf{p}}^2}{2\tilde{m}^2} - \frac{7\mathbf{p} \cdot \tilde{\mathbf{p}}}{2m\tilde{m}} - \frac{(\mathbf{n} \cdot \mathbf{p})(\mathbf{n} \cdot \tilde{\mathbf{p}})}{2m\tilde{m}} \right] \\
& + \frac{2Gm\alpha_0\tilde{m}\tilde{\alpha}_0}{r} \left[\frac{\mathbf{p}}{m} - \frac{\tilde{\mathbf{p}}}{\tilde{m}} \right]^2 + \frac{\tilde{G}^2 m\tilde{m}}{2r^2} (m + \tilde{m}) \\
& + \frac{G^2 m\tilde{m}}{2r^2} (m\alpha_0^2\tilde{\beta}_0 + \tilde{m}\tilde{\alpha}_0^2\beta_0) + \mathcal{O}(c^{-6}), \quad (91)
\end{aligned}$$

where $\tilde{G} = G(1 + \alpha_0\tilde{\alpha}_0)$. This Hamiltonian agrees with ours for $\alpha_0 = 1$ and $\beta_0 = 0$, as expected. Note that for $\alpha_0 = 0 = \beta_0$ one obtains the result of general relativity.

In closing let us compare the complexities of establishing the result above with the more traditional approach of Ref. [49]. At this low order in perturbation theory it is hard to claim a clear superiority of the double-copy approach. We regard our work as a proof of concept, which we expect to simplify calculations at higher orders dramatically, as it is the case in scattering amplitudes. But already at the next-to-leading order established here, our approach allows one to obtain the post-Minkowskian Feynman integrals in a simple way, where in a more traditional approach the three-graviton vertex contains around 170 terms [47].

VI. CONCLUSIONS

In the present paper, we proposed an adaption of the BCJ double copy [1–3] to the classical effective action of binary systems and demonstrated its validity to next-to-leading order in the weak-field/post-Minkowskian approximation. This is another application for a BCJ-like double copy beyond (quantum) scattering amplitudes, next to classical solutions for the field equations [19–21] and weak-field approximations for the radiation from classical binary point-sources [24–26,28–30]. The former application of the double copy operates at the level of the gauge field and metric which are gauge-covariant quantities, unlike gauge-invariant scattering amplitudes. Similarly, in the present work, we formulated a double copy between effective actions, which are not *manifestly* gauge independent either: the actions depend on gauge-dependent canonical momenta, and on the dilaton-gravity side the positions are gauge dependent, too.² This provides growing evidence that the BCJ double copy can be adapted to gauge-covariant quantities, which, if true, would liberate it to a considerably larger realm of applications. Furthermore, our work is an

²The (conservative) effective actions encode the binding energy (energy levels in a quantum mechanical analog), which are gauge independent. However, the variables (like momenta) that are used when the double copy is performed are not gauge invariant.

adaptation of the BCJ double copy to classical gravity. This is an important research direction due to the foreseeable improvements of gravitational wave observations over the next decade(s), demanding ever more accurate predictions for the motion and radiation of compact astrophysical binaries.

An improvable part of our double-copy procedure is the split of worldline vertices with more than one gluon, using a delta distribution of the worldline parameter. This leads to singular terms involving $\delta(0)$ in the quantum corrections to the effective action, which we boldly drop in our classical consideration. While one cannot expect a consistent treatment of all quantum corrections in an approach based on classical worldlines, addressing this problem might still lead to a crucial improvement of our approach. An interesting solution is to integrate out the auxiliary field ψ along the worldline. Since its propagator is a step function, the delta distributions used to split worldline vertices can be produced by derivatives in the numerators acting on these worldline propagators. This will avoid the terms $\delta(0)$. However, after the double-copy step, this will produce an auxiliary field ψ propagating along the worldline in the dilaton gravity, too. But it appears possible to surgically remove this unwanted feature from the result. Alternatively, one can possibly give the auxiliary field an additional property that one desires of the theory, like a spin. It is interesting to note that step-function propagators along the worldline basically introduce a time ordering. This time ordering is also present in the Wilson loop, which can be used to calculate the potential between a quark-antiquark pair.

We may also compare our work to a direct calculation of radiation via equations of motion in [24–26,30], which is based on the same classical color charges (Yang-Mills) and point-masses (dilaton gravity) as our work. If we descend from the effective action to the equations of motion through a variation, then derivatives of the propagators may appear. These derivatives are not part of the numerators which take part in the double copy. Adapting the double copy at the level of equations of motion then faces the challenge of identifying numerators and (derivatives of) propagators. This problem was elegantly solved by Shen [30]: replacing the Yang-Mills numerators by a copy of the color factor should lead to the corresponding result in bi-adjoint scalar theory, thus providing a prescription to separate numerator and propagator structures. Here we avoided this complication by working at the action level. On the other hand, it is straightforward to calculate the (gauge independent) radiation emitted by a binary system when working with equations of motion [24–26,30].

While maybe less straightforward, extending our work to radiation from binaries will be an interesting future direction. Note that the calculation of classical binary interaction potentials presented here was an effective field theory approach similar to [36]. Following this line of research [51], the calculation of radiation aided by a double

copy (at the level of the action) could succeed. Past work also suggests that an extension of our work to spinning point particles [26] or a projection to pure general relativity [14,31] will be possible. But the most important future extension of our present paper will be to demonstrate the double-copy prescription at higher orders. At next-to-next-to-leading order in the effective action, the BCJ color-kinematics duality (Jacobi identity) will start to play a crucial role in identifying suitable color factors and numerators for the double copy. As for all work in the area of color-kinematics duality and double copy, it will be highly interesting to see if our proposal (or a modification thereof) survives at the next order. For the original BCJ case, the answer to this was so far always positive.

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APPENDIX A: CONVENTIONS AND FEYNMAN RULES

The signature of spacetime is -2 (“mostly minus”). We use units such that the speed of light $c = 1$ and use c^{-1} as a formal post-Newtonian counting parameter.

The Yang-Mills action is the standard one, $S_{\text{YM}} = -\frac{1}{4} \int d^4x (F_{\mu\nu}^a)^2$, with the color convention $\text{tr}(T^a T^b) = \delta^{ab}/2$. The field strength $F_{\mu\nu}^a$ is given by

$$[D_\mu, D_\nu] = -igF_{\mu\nu}^a T^a \Rightarrow F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{bca} A_\mu^b A_\nu^c, \quad (\text{A1})$$

where $D_\mu = \partial_\mu - igA_\mu^a T^a$, $[T^a, T^b] = if^{abc} T^c$ and the structure constants f^{abc} are totally antisymmetric here. Working in Feynman gauge we have the coordinate space Feynman rules of Yang-Mills theory,

$$\langle A_\mu^a(x) A_\nu^b(y) \rangle_0 = a \mu \sim b \nu = \frac{\hbar}{i} \eta_{\mu\nu} \delta_{ab} D(x-y), \quad (\text{A2})$$

$$\tau \bullet \sim a \mu = \frac{i}{\hbar} 2g \lambda(\tau) p^\mu(\tau) c^a(\tau), \quad (\text{A3})$$

$$\tau \bullet \begin{array}{l} \sim b \nu \\ \sim a \mu \end{array} = \frac{i}{\hbar} 2g^2 \lambda(\tau) c^a(\tau) c^b(\tau) \eta^{\mu\nu}, \quad (\text{A4})$$

$$\begin{array}{l} a \mu_1 \\ \bullet \\ b \mu_2 \end{array} \sim c \mu_3 = -\frac{i}{\hbar} g f^{abc} V_{123}^{\mu_1 \mu_2 \mu_3}, \quad (\text{A5})$$

where we used the abbreviation $V_{123}^{\mu_1 \mu_2 \mu_3}$ from (21).

Our definition for the Riemann tensor is $R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu{}_{\nu\beta} - \partial_\beta \Gamma^\mu{}_{\nu\alpha} + \Gamma^\mu{}_{\lambda\alpha} \Gamma^\lambda{}_{\nu\beta} - \Gamma^\mu{}_{\lambda\beta} \Gamma^\lambda{}_{\nu\alpha}$, from which the Ricci tensor $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ and Ricci scalar $R = R^\mu{}_\mu$ are obtained through contractions. The Feynman rules for dilaton gravity, for our choice of gauge and field redefinitions, follow straightforwardly from (59) and (56) as

$$\langle g_{\mu\nu}(x) g_{\alpha\beta}(y) \rangle_0 = \alpha\beta \sim \mu\nu = -\frac{\hbar}{i} P_{\mu\nu\alpha\beta} D(x-y), \quad (\text{A6})$$

$$\tau \bullet \sim \mu\nu = -\frac{i}{\hbar} \kappa \lambda(\tau) p^\mu(\tau) p^\nu(\tau), \quad (\text{A7})$$

$$\tau \bullet \begin{array}{l} \sim \alpha\beta \\ \sim \mu\nu \end{array} = \frac{i}{\hbar} \kappa^2 \lambda(\tau) p^{(\mu}(\tau) \eta^{\nu)(\alpha} p^{\beta)}(\tau), \quad (\text{A8})$$

$$\begin{array}{l} \mu_1 \nu_1 \\ \bullet \\ \mu_2 \nu_2 \end{array} \sim \mu_3 \nu_3 = \frac{i}{\hbar} \frac{\kappa}{4} V_{123}^{\alpha_1 \alpha_2 \alpha_3} V_{123}^{\beta_1 \beta_2 \beta_3} \prod_{i=1}^3 P_{\alpha_i \beta_i}{}^{\mu_i \nu_i}, \quad (\text{A9})$$

where $P_{\mu\nu\alpha\beta} = \frac{1}{2} (\eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\nu\alpha} \eta^{\mu\beta})$.

By comparing the Yang-Mills and dilaton-gravity Feynman rules, it is clear that the double copy of many diagrams is trivial, since the vertices already show the double-copy structure, except for the vertex (A8).

APPENDIX B: SELF-INTERACTIONS IN THE POST-NEWTONIAN APPROXIMATION

In this appendix, we wish to discuss the self-interactions, which vanish using the post-Newtonian expansion of the propagator (77). Using the gauge choice $\tau = t$, the simplest self-interaction at leading order reads

$$\left| \begin{array}{l} \bullet \\ \bullet \end{array} \right| = -\frac{i\kappa^2}{2\hbar} \int dt dt' [p(t) \cdot p(t')]^2 D(x(t) - x(t')) \quad (\text{B1})$$

$$\begin{aligned}
&= -\frac{i\kappa^2}{2\hbar} \int \frac{d^3\mathbf{k} dt dt'}{(2\pi)^3} [p(t) \cdot p(t')]^2 \frac{e^{i\mathbf{k} \cdot [\mathbf{x}(t) - \mathbf{x}(t')]}{\mathbf{k}^2} \\
&\times \left[1 + \frac{\partial_t \partial_{t'}}{\mathbf{k}^2} + \dots \right] \delta(t - t'). \tag{B2}
\end{aligned}$$

After performing one of the time integrations, the times are identified $t = t'$ and in the end $e^{i\mathbf{k} \cdot [\mathbf{x}(t) - \mathbf{x}(t')]} = 1$. The \mathbf{k} -integral then turns into a scaleless one and vanishes in dimensional regularization. Hence one gets a vanishing result at each post-Newtonian order. This result is physically sensible. Indeed, a crucial property of the propagator (in Minkowski signature) is that its support is

on the light-cone. Now, the worldlines of massive particles can not intersect the light cones emanating from them, so that these self-interaction should be zero. Similar arguments apply to the other self-interactions at next-to-leading order in the effective action



which all vanish in dimensional regularization.

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