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Abstract

The one-dimensional Hubbard model is an exceptional integrable spin chain which is apparently based on a deformation of the Yangian for the superalgebra $\mathfrak{gl}(2|2)$. Here we investigate the quantum deformation of the Hubbard model in the classical limit. This leads to a novel classical *r*-matrix of trigonometric kind. We derive the corresponding one-parameter family of Lie bialgebras as a deformation of the affine $\mathfrak{gl}(2|2)$ Kac–Moody superalgebra. In particular, we discuss the affine extension as well as discrete symmetries, and we scan for simpler limiting cases, such as the rational r-matrix for the undeformed Hubbard model.

1. Introduction and overview

The Hubbard model [1] is a model of spin-half electrons hopping around on a lattice of atoms (see [2] for an introduction). It has several useful features that make it attractive for the investigation of aspects of electron transport, in particular, superconductivity. An unrelated property of its one-dimensional incarnation is *integrability* which enabled Lieb and Wu to find the spectrum by means of Bethe equations [3]. Remarkably, the integrable structure is different from conventional spin chain models in several respects: the most striking distinction is, arguably, that the *R*-matrix, which was found by Shastry [4], is not of difference form[1]. This

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[1] Two similar cases have previously been discussed: these are based on the twisted affine superalgebras $\mathfrak{gl}(N|N)^{(2')}$ [5] and $\mathfrak{d}(2, 1; e^{2\pi i/3})^{(3)}$ [6]. Their Cartan–Killing forms are charged under the twisting automorphism which leads to unconventional quantum algebras. Another exceptional case involving the twisted affine superalgebra $\mathfrak{gl}(2|2)^{(2)}$ is discussed in section 6.4.

implies that the description of the integrable structure through standard *Yangian* or *quantum affine algebras* [7, 8] cannot apply to this case[2].

For a long time the question of the algebraic structure underlying the Hubbard chain was left at rest. Recent progress towards this goal came from a totally unexpected direction. It turned out that Shastry's *R*-matrix is equivalent [9] to a scattering matrix [10, 11] found in the context of the AdS/CFT correspondence [12] (see [13] for reviews of integrability in AdS/CFT). This matrix has a *centrally extended* $\mathfrak{psl}(2|2)$ supersymmetry by construction which includes the two (more or less) manifest $\mathfrak{sl}(2)$ symmetries of the Hubbard model [14]. Since then, there has been a lot of progress in the formulation of a quantum symmetry algebra for Shastry's *R*-matrix [15–18]. In particular, the construction for higher representations has advanced significantly [9, 19, 20]. Still, it is fair to say that a satisfactory quantization to a *quasi-triangular Hopf algebra* similar to a Yangian has not yet been achieved.

By quantum-deforming[3] the Hubbard chain, we hope to get further insights into the Hopf algebra underlying this special model. For conventional integrable spin chains based on Lie (super)algebra symmetries, the quantum deformation lifts the Yangian to a quantum affine algebra. This has some drawbacks, but also benefits. One the one hand, the deformation breaks the manifest Lie symmetry down to its Cartan subalgebra. On the other hand, one gains a more uniform and symmetric description of the algebra itself. It is then possible to return to the undeformed model and recover the Yangian as a particular limit. The limit is singular, and it obscures some of the symmetry of the quantum affine formulation. An increased internal symmetry will hopefully simplify the formulation of a quasi-triangular Hopf algebra for the (quantum-deformed) Hubbard chain. Another motivation to study the quantum-deformation is that some of the structures in the centrally extended $\mathfrak{psl}(2|2)$ algebra [10] for Shastry's *R*-matrix are reminiscent of quantum affine algebras.

The quantum-deformation of the Hubbard Hamiltonian along with its *R*-matrix was performed in [21]. It has an additional parameter q, and therefore yields a bigger class of models. It turned out that this class contains a multi-parameter family of deformations of the Hubbard model proposed earlier by Alcaraz and Bariev [22]. In fact, many of the variants of the Hubbard chain (see references in [9, 21]) are special cases of this model. The deformed and undeformed model and *R*-matrix have in common a rather complicated structure which obstructs direct attempts to set up a quasi-triangular Hopf algebra.

Fortunately, there is a limit, the *classical limit*, which makes the algebraic structure much more tractable. The classical framework consists of some Lie algebra $\mathfrak{g}$ along with an element r of the tensor product $\mathfrak{g} \otimes \mathfrak{g}$ serving as the *classical r-matrix*. For the quantum algebra $\mathfrak{g}$ is promoted to a deformation of its universal enveloping algebra $U_q(\mathfrak{g})$ which is substantially bigger than $\mathfrak{g}$ itself. For *r*-matrices with a spectral parameter, the Lie algebra $\mathfrak{g}$ is typically of affine Kac–Moody type, for which an efficient and uniform description exists. All in all, the manipulations in the classical limit can usually be performed very explicitly with pen and paper, much in contradistinction to the quantum case.

The classical limit of Shastry's *R*-matrix was derived in [23]. The underlying Lie algebra with the universal classical *r*-matrix was found in [24]. This algebra turned out to be a peculiar deformation of the loop algebra $\mathfrak{gl}(2|2)[u, u^{-1}]$. Note that the $\mathfrak{gl}(2|2)$ algebra is not simple; it contains central charges as well as derivations [16, 24, 25], and thus it escapes the classification of *r*-matrices in [6]. The algebra is curious because it is not a loop algebra of some deformed algebra, the deformation applies to the loop algebra structure itself, in particular to

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[2] The *R*-matrix must be invariant under the affine shift which enforces the difference form.
[3] The q-deformation lifts a rational to a trigonometric *R*-matrix, e.g. Heisenberg XXX to XXZ.
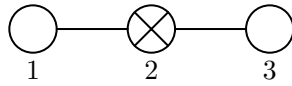
Figure 1. Distinguished Dynkin diagram for $\mathfrak{sl}(2|2)$.

the derivations and charges. Yet, surprisingly, the algebra admits a quasi-triangular bialgebra structure.

In this paper, we will derive the classical r-matrix for the quantum-deformed Hubbard chain. This is the trigonometric analogue of the rational r-matrix in [23, 24]. We expect that it will be of help in deriving the full quantum algebra framework for the (quantum-deformed) Hubbard model.

The paper is organized as follows. We start with a brief review of the quantum R-matrix in section 2. In section 3, we perform the classical limit and show that it leads to a quasi-triangular Lie bialgebra. Next we consider its affine extension in section 4 which provides some more structure to the algebra. The r-matrix and the algebra have several discrete symmetries and special points which are discussed in section 5. Section 6 is devoted to the enumeration of simpler limiting cases of the r-matrix and the algebra. Finally, in section 7, we conclude and give an outlook.

2. Quantum-deformed *S*-matrix

In [21] a quantum deformation of the centrally extended $\mathfrak{psl}(2|2)$ algebra was defined. Subsequently, the fundamental R-matrix for this algebra was derived. In this section, we will summarize the results of [21] important to this paper.

2.1. Serre–Chevalley presentation

We first define the quantum deformation of the extended $\mathfrak{psl}(2|2)$ algebra in the Serre–Chevalley presentation, cf [26, 27] for the case of conventional (affine) $\mathfrak{gl}(2|2)$. It has nine Serre–Chevalley generators H_j, E_j, F_j with $j = 1, 2, 3$. For the distinguished choice of Dynkin diagram of $\mathfrak{psl}(2|2)$, see figure 1, the generators E_2, F_2 are fermionic while the remaining seven are bosonic. The symmetric Cartan matrix A_{jk} reads[4]

$$A_{jk} = \begin{pmatrix} +2 & -1 & 0 \\ -1 & 0 & +1 \\ 0 & +1 & -2 \end{pmatrix}. \tag{2.1}$$

Algebra. The commutators with symmetrized Cartan elements H_j are determined by the Cartan matrix A_{jk}:

$$[H_j, H_k] = 0, \qquad [H_j, E_k] = +A_{jk}E_k, \qquad [H_j, F_k] = -A_{jk}F_k. \tag{2.2}$$

The commutators between E_j and F_k are non-trivial only for $j = k$:

$$[E_1, F_1] = \frac{q^{H_1} - q^{-H_1}}{q - q^{-1}}, \quad \{E_2, F_2\} = -\frac{q^{H_2} - q^{-H_2}}{q - q^{-1}}, \quad [E_3, F_3] = -\frac{q^{H_3} - q^{-H_3}}{q - q^{-1}}. \tag{2.3}$$

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[4] For superalgebras it is sometimes convenient to flip the signs of some rows/columns to make the matrix symmetric.

The Serre relations between alike generators E_j or F_j read

$$
\begin{aligned}
0 &= [E_1, E_3] = [F_1, F_3] = E_2 E_2 = F_2 F_2 \\
&= E_1 E_1 E_2 - (q + q^{-1}) E_1 E_2 E_1 + E_2 E_1 E_1 = E_3 E_3 E_2 - (q + q^{-1}) E_3 E_2 E_3 + E_2 E_3 E_3 \\
&= F_1 F_1 F_2 - (q + q^{-1}) F_1 F_2 F_1 + F_2 F_1 F_1 = F_3 F_3 F_2 - (q + q^{-1}) F_3 F_2 F_3 + F_2 F_3 F_3.
\end{aligned}
\tag{2.4}
$$

Central elements. What singles out $\mathfrak{psl}(2|2)$ from the other simple superalgebras is that it has three non-trivial central extensions [28]. Our algebra has two central elements C, D, and they are the key to the peculiar features discussed in this paper. The standard central element C in $\mathfrak{sl}(2|2)$ reads

$$
C = -\tfrac{1}{2} H_1 - H_2 - \tfrac{1}{2} H_3.
\tag{2.5}
$$

In addition, there are two exceptional central elements P, K which originate from dropping the two Serre relations $P = K = 0$ particular to superalgebras [26]

$$
\begin{aligned}
P &= E_1 E_2 E_3 E_2 + E_2 E_3 E_2 E_1 + E_3 E_2 E_1 E_2 + E_2 E_1 E_2 E_3 - (q + q^{-1}) E_2 E_1 E_3 E_2, \\
K &= F_1 F_2 F_3 F_2 + F_2 F_3 F_2 F_1 + F_3 F_2 F_1 F_2 + F_2 F_1 F_2 F_3 - (q + q^{-1}) F_2 F_1 F_3 F_2.
\end{aligned}
\tag{2.6}
$$

In order to get an interesting quantum algebra structure, the two extra central elements have to be constrained. We introduce a new central element D as well as two global constants g, α, and express P, K through them[5]:

$$
P = g\alpha(1 - q^{2C} q^{2D}), \qquad K = g\alpha^{-1}(q^{-2C} - q^{-2D}).
\tag{2.7}
$$

Coalgebra. The standard quantum-deformed coproduct applies to all *bosonic* generators E_j, F_j, H_j (i.e. all except E_2 and F_2):

$$
\begin{aligned}
\Delta(H_j) &= H_j \otimes 1 + 1 \otimes H_j, \\
\Delta(E_j) &= E_j \otimes 1 + q^{-H_j} \otimes E_j, \\
\Delta(F_j) &= F_j \otimes q^{H_j} + 1 \otimes F_j.
\end{aligned}
\tag{2.8}
$$

For the two fermionic generators E_2, F_2 an additional braiding with the generator D is introduced:

$$
\begin{aligned}
\Delta(E_2) &= E_2 \otimes 1 + q^{-H_2} q^{D} \otimes E_2, \\
\Delta(F_2) &= F_2 \otimes q^{H_2} + q^{-D} \otimes F_2, \\
\Delta(D) &= D \otimes 1 + 1 \otimes D.
\end{aligned}
\tag{2.9}
$$

For convenience we have stated the coproduct of the central charge D which actually follows from the other coproducts.

2.2. Fundamental representation

The above algebra has a family of four-dimensional fundamental representations. Its vector space $\mathbb{V}$ has two bosonic and two fermionic directions. We assume it to be spanned by the four states

$$
|\phi^1\rangle, |\phi^2\rangle \quad \text{and} \quad |\psi^1\rangle, |\psi^2\rangle.
\tag{2.10}
$$

The former two are bosonic and the latter two are fermionic.

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[5] Note the similarity between (2.6), (2.7) and (2.3).

Representation.   The fundamental action of the Chevalley–Serre generators is given by[6]

$$H_1|\phi^1\rangle = -|\phi^1\rangle, \quad H_2|\phi^1\rangle = -\left(C - \tfrac{1}{2}\right)|\phi^1\rangle, \quad E_1|\phi^1\rangle = q^{+1/2}|\phi^2\rangle, \quad F_2|\phi^1\rangle = c|\psi^2\rangle,$$
$$H_1|\phi^2\rangle = +|\phi^2\rangle, \quad H_2|\phi^2\rangle = -\left(C + \tfrac{1}{2}\right)|\phi^2\rangle, \quad E_2|\phi^2\rangle = a|\psi^1\rangle, \quad F_1|\phi^2\rangle = q^{-1/2}|\phi^1\rangle,$$
$$H_3|\psi^1\rangle = +|\psi^1\rangle, \quad H_2|\psi^1\rangle = -\left(C + \tfrac{1}{2}\right)|\psi^1\rangle, \quad E_3|\psi^1\rangle = q^{-1/2}|\psi^2\rangle, \quad F_2|\psi^1\rangle = d|\phi^2\rangle,$$
$$H_3|\psi^2\rangle = -|\psi^2\rangle, \quad H_2|\psi^2\rangle = -\left(C - \tfrac{1}{2}\right)|\psi^2\rangle, \quad E_2|\psi^2\rangle = b|\phi^1\rangle, \quad F_3|\psi^2\rangle = q^{+1/2}|\psi^1\rangle.$$

$$(2.11)$$

The representation parameters a, b, c, d must obey the constraint $(ad - qbc)(ad - q^{-1}bc) = 1$. They can be expressed in terms of new parameters $x^{\pm}, \gamma$ as follows[7]:

$$a = \gamma, \qquad\qquad b = \frac{g\alpha}{\gamma}\frac{1}{x^-}(x^- - q^{2C-1}x^+),$$
$$c = \frac{i\gamma}{\alpha}\frac{q^{-C+1/2}}{x^+}, \qquad d = \frac{ig}{\gamma}q^{C+1/2}(x^- - q^{-2C-1}x^+).$$

$$(2.12)$$

In terms of these parameters, the constraint implies the following quadratic relation between $x^{\pm}$:

$$\frac{x^+}{q} + \frac{q}{x^+} - qx^- - \frac{1}{qx^-} + ig(q - q^{-1})\left(\frac{x^+}{qx^-} - \frac{qx^-}{x^+}\right) = \frac{i}{g}.$$

$$(2.13)$$

Central charges.   The central charge eigenvalues D, C cannot be written unambiguously using $x^{\pm}$, but the combinations q^{2D}, q^{2C} are well defined:

$$q^{2D} = \frac{x^+}{qx^-}, \qquad q^{2C} = q\frac{(q - q^{-1})/x^+ - ig^{-1}}{(q - q^{-1})/x^- - ig^{-1}} = q^{-1}\frac{(q - q^{-1})x^+ + ig^{-1}}{(q - q^{-1})x^- + ig^{-1}}.$$

$$(2.14)$$

The latter two expressions are equivalent upon (2.13). Finally, the central charge eigenvalues P, K follow from (2.7):

$$P = g\alpha(1 - q^{2C}q^{2D}), \qquad K = g\alpha^{-1}(q^{-2C} - q^{-2D}).$$

$$(2.15)$$

The parameter γ adjusts the normalization of bosons w.r.t. fermions in the representation; it is unphysical, but there is a preferable choice.

Fundamental R-matrix.   The quantum fundamental R-matrix $\mathcal{R} : \mathbb{V} \otimes \mathbb{V} \to \mathbb{V} \otimes \mathbb{V}$ can be found by demanding that it satisfies the cocommutativity relation

$$\widetilde{\Delta}(J)\mathcal{R} = \mathcal{R}\Delta(J)$$

$$(2.16)$$

for all generators J of the algebra, where $\widetilde{\Delta}$ is the opposite coproduct. It turns out to be fully constrained by this relation up to one overall factor R_{12}^0. The result is lengthy, and it can be found in [21]; we refrain from reproducing it here.

3. Classical limit

The classical limit of quantum-deformed R-matrices typically consists in sending the deformation parameter q to unity, $q \to 1$. For the undeformed R-matrix [10], however, the classical limit involves a large coupling constant, $g \to \infty$, cf [23]. Furthermore, the

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[6] We have interchanged the states $|\psi^1\rangle$ and $|\psi^2\rangle$ as compared to [21].
[7] As compared to [21] we have rescaled γ by $1/\sqrt{g}$ for later convenience.

parameters for the fundamental representation have to scale in a particular fashion such that $x^\pm$ approach a common finite value, $x^\pm \to x$.

We find that a reasonable classical limit consists in setting

$$q = 1 + \frac{h}{2g} + \mathcal{O}(g^{-2}), \tag{3.1}$$

with the inverse coupling constant g^{-1} taking the role of the quantum parameter $\hbar$:

$$g \to \infty, \tag{3.2}$$

while h remains a finite deformation parameter even in the classical limit. For later purposes, we shall also introduce h' as the combination

$$h' = \sqrt{1 - h^2}. \tag{3.3}$$

3.1. Fundamental representation

For the parameters of the fundamental representation we assume the following classical limit[8]:

$$x^\pm = (h'x - ih)\left(1 \pm \frac{1}{2g}\frac{x(hx + ih')}{x^2 - 1} + \mathcal{O}(g^{-2})\right). \tag{3.4}$$

These obey constraint (2.13) up to the order given. The coefficients a, b, c, d in (2.12) then take the classical values

$$a = \gamma, \quad b = -\frac{\alpha(h'x - ih)(hx + ih')}{\gamma h'(x^2 - 1)}, \quad c = \frac{i\gamma}{\alpha(h'x - ih)}, \quad d = \frac{x(h'x - ih)}{\gamma h'(x^2 - 1)}. \tag{3.5}$$

One can see that $ad - bc = 1$ as desired for the classical limit $q \to 1$. The limit of the central charges D, C, P, K in (2.14), (2.15) then follows as

$$D = -\tfrac{1}{2}h^{-1}(z + 1)q, \qquad C = \tfrac{1}{2}h^{-1}(z - 1)q, \qquad P = \alpha q, \qquad K = -\alpha^{-1}zq, \tag{3.6}$$

where z and q (the quantum parameter q will not appear in the classical limit and we can use the letter for a different purpose) are defined by

$$z = \frac{ix}{(h'x - ih)(hx + ih')}, \qquad q = \frac{-(h'x - ih)(hx + ih')}{h'(x^2 - 1)}. \tag{3.7}$$

3.2. Fundamental r-matrix

We now take the classical limit (3.1), (3.2) and (3.4) on the fundamental R-matrix $\mathcal{R}$ found in [21]. In the strict classical limit, it reduces to the unity operator and the first non-trivial order equals the fundamental classical r-matrix r:

$$(-A_{12}D_{12})^{-1/2}\mathcal{R} = 1 \otimes 1 + \frac{h}{g}r + \mathcal{O}(g^{-2}). \tag{3.8}$$

Note that for definiteness we have multiplied the fundamental R-matrix by a combination of the coefficient functions A_{12} and D_{12} in [21]. This removes the undetermined overall coefficient function R_{12}^0, or it effectively fixes it to a convenient expression.

The resulting form of the fundamental classical r-matrix is given in table 1. It is determined by ten coefficient functions $A, B, C, D, E, F, G, H, K, L$. Their values in the classical limit are given in table 2. The matrix r inherits the classical Yang–Baxter equation $[[r, r]] = 0$ from its quantum counterpart [21], where

$$[[r, s]] := [r_{12}, s_{13}] + [r_{12}, s_{23}] + [r_{13}, s_{23}]. \tag{3.9}$$

---

[8] A simultaneous sign flip of h' and x changes nothing.

Table 1. The fundamental trigonometric r-matrix.

$r|\phi^1\phi^1\rangle = A_{12}|\phi^1\phi^1\rangle$

$r|\phi^1\phi^2\rangle = \frac{1}{2}(A_{12} + B_{12} + 1)|\phi^2\phi^1\rangle + \frac{1}{2}(A_{12} - B_{12})|\phi^1\phi^2\rangle - \frac{1}{2}C_{12}\varepsilon_{\alpha\beta}|\psi^\alpha\psi^\beta\rangle$

$r|\phi^2\phi^1\rangle = \frac{1}{2}(A_{12} - B_{12})|\phi^2\phi^1\rangle + \frac{1}{2}(A_{12} + B_{12} - 1)|\phi^1\phi^2\rangle + \frac{1}{2}C_{12}\varepsilon_{\alpha\beta}|\psi^\alpha\psi^\beta\rangle$

$r|\phi^2\phi^2\rangle = A_{12}|\phi^2\phi^2\rangle$

$r|\psi^1\psi^1\rangle = -D_{12}|\psi^1\psi^1\rangle$

$r|\psi^1\psi^2\rangle = -\frac{1}{2}(D_{12} + E_{12} + 1)|\psi^2\psi^1\rangle - \frac{1}{2}(D_{12} - E_{12})|\psi^1\psi^2\rangle + \frac{1}{2}F_{12}\varepsilon_{ab}|\phi^a\phi^b\rangle$

$r|\psi^2\psi^1\rangle = -\frac{1}{2}(D_{12} - E_{12})|\psi^2\psi^1\rangle - \frac{1}{2}(D_{12} + E_{12} - 1)|\psi^1\psi^2\rangle - \frac{1}{2}F_{12}\varepsilon_{ab}|\phi^a\phi^b\rangle$

$r|\psi^2\psi^2\rangle = -D_{12}|\psi^2\psi^2\rangle$

$r|\phi^a\psi^\beta\rangle = G_{12}|\phi^a\psi^\beta\rangle + H_{12}|\psi^\beta\phi^a\rangle$

$r|\psi^\alpha\phi^b\rangle = K_{12}|\phi^b\psi^\alpha\rangle + L_{12}|\psi^\alpha\phi^b\rangle$

Table 2. The coefficients for the fundamental r-matrix.

$$A_{12} = D_{12} = \frac{\frac{1}{4}z_1 + \frac{1}{4}z_2 + \frac{1}{4}z_1q_1q_2^{-1} + \frac{1}{4}z_2q_2q_1^{-1}}{z_1 - z_2}$$

$$\frac{1}{2}(A_{12} + B_{12} + 1) = \frac{1}{2}(D_{12} + E_{12} + 1) = \frac{z_1}{z_1 - z_2}$$

$$\frac{1}{2}(A_{12} + B_{12} - 1) = \frac{1}{2}(D_{12} + E_{12} - 1) = \frac{z_2}{z_1 - z_2}$$

$$\frac{1}{2}(A_{12} - B_{12}) = \frac{1}{2}(D_{12} - E_{12}) = \frac{-\frac{1}{4}z_1 - \frac{1}{4}z_2 + \frac{1}{4}z_1q_1q_2^{-1} + \frac{1}{4}z_2q_2q_1^{-1}}{z_1 - z_2}$$

$$C_{12} = \frac{z_1a_1c_2 - z_2a_2c_1}{z_1 - z_2}$$

$$F_{12} = \frac{z_1b_1d_2 - z_2b_2d_1}{z_1 - z_2}$$

$$G_{12} = -L_{12} = \frac{-\frac{1}{4}z_1q_1q_2^{-1} + \frac{1}{4}z_2q_2q_1^{-1}}{z_1 - z_2}$$

$$H_{12} = \frac{z_1a_1d_2 - z_2b_2c_1}{z_1 - z_2}$$

$$K_{12} = \frac{-z_1b_1c_2 + z_2a_2d_1}{z_1 - z_2}$$

Taking a closer look at the coefficients we find four identities among them: two linear ones

$$A - D = -B + E = G + L \tag{3.10}$$

and two quadratic identities

$$\frac{1}{4}(A + B - 1)(A + B + 1) = \frac{1}{4}(3A - B)(3D - E) + 4GL = CF + HK. \tag{3.11}$$

Note that we cannot in general claim that $A = D$ as suggested by table 2 because it follows only from our above choice of prefactor R^0_{12} in (3.8). In other words, unlike the above four constraints, the latter one is not invariant under the shift proportional to the identity matrix

$$\delta(A, B, C, D, E, F, G, H, K, L) \sim (+1, -1, 0, -1, +1, 0, +1, 0, 0, +1), \tag{3.12}$$

which corresponds to changing the overall scattering phase. Altogether this reduces the ten coefficient functions to merely six independent ones. This equals the number of free parameters: $x_1, x_2, \gamma_1, \gamma_2, h$ and the freedom to shift by the identity matrix (3.12).

3.3. Lie bialgebra

In the following, we shall derive a framework for the above r-matrix in terms of a Lie bialgebra.

Fundamental representation and r-matrix.   First we would like to turn the fundamental r-matrix into a universal one to make it applicable to arbitrary representations. This is achieved by converting the operations in r, e.g. $|\phi^1\phi^2\rangle \mapsto |\phi^2\phi^1\rangle$, to representations of symmetry generators, e.g. $-R^{22} \otimes R^{11}$, acting individually on the two sites.

   The operators R, L, Q, S, A, B are meant to mimic the fundamental representation of $\mathfrak{gl}(2|2)$: the two sets of $\mathfrak{sl}(2)$ generators $R^{ab} = R^{ba}$ and $L^{\alpha\beta} = L^{\beta\alpha}$ act canonically on the two pairs of states $|\phi^a\rangle$, $|\psi^\alpha\rangle$. The remaining operators are set up in analogy to [24] to be able to reproduce the coefficients in table 2. The action of the supercharges $Q^{\alpha b}$ and $S^{\alpha b}$ is specified through the parameters a, b, c, d. Finally, the action of the derivation B and the central charge A involves q. Altogether the action reads

$$
\begin{aligned}
&R^{ab}|\phi^c\rangle = \tfrac{1}{2}\varepsilon^{bc}|\phi^a\rangle + \tfrac{1}{2}\varepsilon^{ac}|\phi^b\rangle, &\quad &L^{\alpha\beta}|\psi^\gamma\rangle = \tfrac{1}{2}\varepsilon^{\beta\gamma}|\psi^\alpha\rangle - \tfrac{1}{2}\varepsilon^{\alpha\gamma}|\psi^\beta\rangle, \\
&Q^{\alpha b}|\phi^c\rangle = a\varepsilon^{bc}|\psi^\alpha\rangle, &\quad &Q^{\alpha b}|\psi^\gamma\rangle = -b\varepsilon^{\alpha\gamma}|\phi^b\rangle, \\
&S^{\alpha b}|\phi^c\rangle = -c\varepsilon^{bc}|\psi^\alpha\rangle, &\quad &S^{\alpha b}|\psi^\gamma\rangle = d\varepsilon^{\alpha\gamma}|\phi^b\rangle, \\
&A|\phi^a\rangle = \tfrac{1}{2}q|\phi^a\rangle, &\quad &A|\psi^\alpha\rangle = \tfrac{1}{2}q|\psi^\alpha\rangle, \\
&B|\phi^a\rangle = -\tfrac{1}{2}q^{-1}|\phi^a\rangle, &\quad &B|\psi^\alpha\rangle = +\tfrac{1}{2}q^{-1}|\psi^\alpha\rangle.
\end{aligned}
\tag{3.13}
$$

The symbol $\varepsilon^{\cdot\cdot}$ is the antisymmetric 2×2 matrix with $\varepsilon^{12} = +1$. One can make contact with the fundamental representation of the quantum algebra in (2.11) by means of the following identification with the Chevalley–Serre generators:

$$
\begin{aligned}
&H_1 = +2R^{12}, &\quad &E_1 = -R^{22}, \quad F_1 = +R^{11}, \\
&H_2 = -h^{-1}(z-1)A - \tfrac{1}{2}H_1 - \tfrac{1}{2}H_3, &\quad &E_2 = +Q^{11}, \quad F_2 = -S^{22}, \\
&H_3 = -2L^{12}, &\quad &E_3 = -L^{22}, \quad F_3 = +L^{11}.
\end{aligned}
\tag{3.14}
$$

We are then led to the following form for the classical r-matrix from which the various coefficients in table 2 are easily reproduced

$$
\begin{aligned}
r = &+\frac{\tfrac{1}{2}z_1 + \tfrac{1}{2}z_2}{z_1 - z_2} 2R^{12} \otimes R^{12} - \frac{z_1}{z_1 - z_2} R^{22} \otimes R^{11} - \frac{z_2}{z_1 - z_2} R^{11} \otimes R^{22} \\
&-\frac{\tfrac{1}{2}z_1 + \tfrac{1}{2}z_2}{z_1 - z_2} 2L^{12} \otimes L^{12} + \frac{z_1}{z_1 - z_2} L^{22} \otimes L^{11} + \frac{z_2}{z_1 - z_2} L^{11} \otimes L^{22} \\
&-\frac{z_1}{z_1 - z_2} \varepsilon_{\alpha\gamma} \varepsilon_{bd} Q^{\alpha b} \otimes S^{\gamma d} + \frac{z_2}{z_1 - z_2} \varepsilon_{\alpha\gamma} \varepsilon_{bd} S^{\alpha b} \otimes Q^{\gamma d} \\
&-\frac{z_1}{z_1 - z_2} A \otimes B - \frac{z_2}{z_1 - z_2} B \otimes A.
\end{aligned}
\tag{3.15}
$$

Lie brackets.   Next we consider the commutators of the operators in (3.13) to the end that they become the brackets of a Lie algebra and (3.13) define the fundamental representation. From the way the indices are contracted in (3.13), it is evident that R and L form two $\mathfrak{sl}(2)$ algebras and that the generators Q and S transform in fundamental representations under these:

$$
\begin{aligned}
&[R^{ab}, R^{cd}] = \varepsilon^{bc}R^{ad} + \varepsilon^{ad}R^{bc}, &\quad &[L^{\alpha\beta}, L^{\gamma\delta}] = \varepsilon^{\beta\gamma}L^{\alpha\delta} + \varepsilon^{\alpha\delta}L^{\beta\gamma}, \\
&[R^{ab}, Q^{\gamma d}] = \tfrac{1}{2}\varepsilon^{bd}Q^{\gamma a} + \tfrac{1}{2}\varepsilon^{ad}Q^{\gamma b}, &\quad &[L^{\alpha\beta}, Q^{\gamma d}] = \tfrac{1}{2}\varepsilon^{\beta\gamma}Q^{\alpha d} + \tfrac{1}{2}\varepsilon^{\alpha\gamma}Q^{\beta d}, \\
&[R^{ab}, S^{\gamma d}] = \tfrac{1}{2}\varepsilon^{bd}S^{\gamma a} + \tfrac{1}{2}\varepsilon^{ad}S^{\gamma b}, &\quad &[L^{\alpha\beta}, S^{\gamma d}] = \tfrac{1}{2}\varepsilon^{\beta\gamma}S^{\alpha d} + \tfrac{1}{2}\varepsilon^{\alpha\gamma}S^{\beta d}.
\end{aligned}
\tag{3.16}
$$

8

The action of Q, S, A, B depends on the parameters a, b, c, d, q but their commutators can be written using only z defined in (3.7):

$$
\begin{aligned}
\{Q^{\alpha b}, Q^{\gamma d}\} &= 2\alpha \varepsilon^{\alpha\gamma} \varepsilon^{bd} A, \\
\{Q^{\alpha b}, S^{\gamma d}\} &= -\varepsilon^{\alpha\gamma} R^{bd} + \varepsilon^{bd} L^{\alpha\gamma} - \varepsilon^{\alpha\gamma} \varepsilon^{bd} h^{-1}(z-1) A, \\
\{S^{\alpha b}, S^{\gamma d}\} &= -2\alpha^{-1} \varepsilon^{\alpha\gamma} \varepsilon^{bd} z A, \\
[B, Q^{\alpha b}] &= h^{-1}(z-1) Q^{\alpha b} + 2\alpha S^{\alpha b}, \\
[B, S^{\alpha b}] &= 2\alpha^{-1} z Q^{\alpha b} - h^{-1}(z-1) S^{\alpha b}.
\end{aligned}
\tag{3.17}
$$

Although the above generators and their relations are reminiscent of $\mathfrak{gl}(2|2)$, they cannot form a Lie algebra as they stand. The point is that the above commutators depend on the representation parameter z, whereas the structure constants must be universal to the Lie algebra as a whole. The way out is to consider instead the loop algebra of $\mathfrak{gl}(2|2)[z, z^{-1}]$ in the way proposed in [24]. The variable z can be interpreted as the formal loop variable and the above action as an evaluation representation. Then the above commutation relations define Lie brackets on the loop space $\mathfrak{gl}(2|2)[z, z^{-1}]$. The algebra is however not $\mathfrak{gl}(2|2)[z, z^{-1}]$ because the above relations are not homogeneous in z. It is rather a non-trivial deformation of $\mathfrak{gl}(2|2)[z, z^{-1}]$.

We observe that several of the coefficients appearing in (3.17) coincide. It turns out to be useful to combine these coefficients as well as a, b, c, d into 2×2 matrices W and T, respectively:

$$
T = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}, \qquad
W = \begin{pmatrix} +h^{-1}(z-1) & 2\alpha \\ 2\alpha^{-1}z & -h^{-1}(z-1) \end{pmatrix}.
\tag{3.18}
$$

We note that $\det T = 1$ and $\operatorname{Tr} W = 0$. Introducing a constant matrix M, we can write the relations required to derive (3.17) in the compact form

$$
TM = qWT, \qquad M = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}.
\tag{3.19}
$$

The above commutation relations (3.17) then read

$$
\begin{aligned}
\{Q^{\alpha b}, Q^{\gamma d}\} &= \varepsilon^{\alpha\gamma} \varepsilon^{bd} W_{12}(z) A, \\
\{Q^{\alpha b}, S^{\gamma d}\} &= -\varepsilon^{\alpha\gamma} R^{bd} + \varepsilon^{bd} L^{\alpha\gamma} - \varepsilon^{\alpha\gamma} \varepsilon^{bd} W_{11}(z) A, \\
\{S^{\alpha b}, S^{\gamma d}\} &= -\varepsilon^{\alpha\gamma} \varepsilon^{bd} W_{21}(z) A, \\
[B, Q^{\alpha b}] &= W_{11}(z) Q^{\alpha b} + W_{12}(z) S^{\alpha b}, \\
[B, S^{\alpha b}] &= W_{21}(z) Q^{\alpha b} + W_{22}(z) S^{\alpha b},
\end{aligned}
\tag{3.20}
$$

suggesting that $Q^{\alpha b}$ and $S^{\alpha b}$ form a two-component vector on which these matrices can act. Note that the combinations for the brackets of supercharges are naturally associated with the symmetric matrix

$$
-WN = \begin{pmatrix} +W_{12} & -W_{11} \\ +W_{22} & -W_{21} \end{pmatrix} \quad \text{with} \quad N = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}.
\tag{3.21}
$$

Universal r-matrix. The combinations of z_1 and z_2 appearing in (3.15) are common for trigonometric classical *r*-matrices. We can split all of them into terms proportional to $z_1/(z_1 - z_2)$ and $z_2/(z_1 - z_2)$

$$
r_{12} = \frac{z_1}{z_1 - z_2} s_{12} + \frac{z_2}{z_1 - z_2} s_{21} = s_{12} + \frac{z_2}{z_1 - z_2} t_{12}.
\tag{3.22}
$$

Here, s and t are the following tensor products of generators:

$$s_{12} = \mathrm{R}^{12} \otimes \mathrm{R}^{12} - \mathrm{R}^{22} \otimes \mathrm{R}^{11} - \mathrm{L}^{12} \otimes \mathrm{L}^{12} + \mathrm{L}^{22} \otimes \mathrm{L}^{11} - \varepsilon_{\alpha\gamma}\varepsilon_{bd}\mathrm{Q}^{\alpha b} \otimes \mathrm{S}^{\gamma d} - \mathrm{A} \otimes \mathrm{B},$$

$$s_{21} = \mathrm{R}^{12} \otimes \mathrm{R}^{12} - \mathrm{R}^{11} \otimes \mathrm{R}^{22} - \mathrm{L}^{12} \otimes \mathrm{L}^{12} + \mathrm{L}^{11} \otimes \mathrm{L}^{22} + \varepsilon_{\alpha\gamma}\varepsilon_{bd}\mathrm{S}^{\gamma d} \otimes \mathrm{Q}^{\alpha b} - \mathrm{B} \otimes \mathrm{A},$$

$$t_{12} = s_{12} + s_{21}$$

$$= -\varepsilon_{ac}\varepsilon_{bd}\mathrm{R}^{ab} \otimes \mathrm{R}^{cd} + \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}\mathrm{L}^{\alpha\beta} \otimes \mathrm{L}^{\gamma\delta} - \varepsilon_{\alpha\gamma}\varepsilon_{bd}\mathrm{Q}^{\alpha b} \otimes \mathrm{S}^{\gamma d}$$

$$+ \varepsilon_{\alpha\gamma}\varepsilon_{bd}\mathrm{S}^{\alpha b} \otimes \mathrm{Q}^{\gamma d} - \mathrm{A} \otimes \mathrm{B} - \mathrm{B} \otimes \mathrm{A}. \tag{3.23}$$

The term t is (graded) symmetric and r is (graded) anti-symmetric

$$t_{12} = t_{21}, \qquad r_{12} + r_{21} = 0. \tag{3.24}$$

We can now consider the classical Yang–Baxter equation $[[r, r]] = 0$. The form of r coincides with the conventional trigonometric r-matrix for the superalgebra $\mathfrak{gl}(2|2)$ [6] for which the CYBE holds indeed. Therefore, the only violations could arise from the deformations in (3.20). We calculate the terms in $[[r, r]]$ which consist of one factor of A and two supercharges. These turn out to vanish if the following three equations hold for $F_1(z) = zW_{12}(z)$, $F_2(z) = W_{11}(z)$ and $F_3(z) = z^{-1}W_{21}(z)$:

$$\frac{F_k(z_1)}{(z_1 - z_2)(z_1 - z_3)} - \frac{F_k(z_2)}{(z_1 - z_2)(z_2 - z_3)} + \frac{F_k(z_3)}{(z_1 - z_3)(z_2 - z_3)} = 0. \tag{3.25}$$

Setting $z_3 = 0$ the equation reduces to

$$\frac{F_k(z_1) - F_k(0)}{z_1} = \frac{F_k(z_2) - F_k(0)}{z_2}. \tag{3.26}$$

This implies that $F_k(z)$ must be a polynomial of degree 1 which is indeed a solution of the above equation and which is also true for all matrix elements in (3.18). Therefore, the CYBE is fulfilled, and the r-matrix enhances the loop algebra to a triangular Lie bialgebra. Note that the above three conditions also guarantee that the algebra has a positive, a negative and a Cartan subalgebra, see (3.33) for more details.

3.4. Loop level form

In order to define the loop algebra more rigorously, we shall provide an alternative presentation in terms of the generators at definite levels of the loop algebra

$$\mathrm{J}_n \simeq z^n \mathrm{J}. \tag{3.27}$$

This description of the loop algebra is instructive, and it has in fact a slightly different bialgebra structure. Nevertheless, in the remainder of this paper we shall mostly employ the functional description introduced above.

Lie brackets. Based on the above operators $\mathrm{J} \in \langle \mathrm{R}, \mathrm{L}, \mathrm{Q}, \mathrm{S}, \mathrm{A}, \mathrm{B} \rangle = \mathfrak{gl}(2|2)$ we define an algebra spanned by J_n for $n \in \mathbb{Z}$. The vector space of the algebra is the one of $\mathfrak{gl}(2|2)[z, z^{-1}]$, but the Lie brackets are deformed. The brackets involving the two sets of $\mathfrak{sl}(2)$ generators R and L are precisely as in $\mathfrak{gl}(2|2)[z, z^{-1}]$:

$$\left[\mathrm{R}_m^{ab}, \mathrm{R}_n^{cd}\right] = \varepsilon^{bc}\mathrm{R}_{m+n}^{ad} + \varepsilon^{ad}\mathrm{R}_{m+n}^{bc}, \qquad \left[\mathrm{L}_m^{\alpha\beta}, \mathrm{L}_n^{\gamma\delta}\right] = \varepsilon^{\beta\gamma}\mathrm{L}_{m+n}^{\alpha\delta} + \varepsilon^{\alpha\delta}\mathrm{L}_{m+n}^{\beta\gamma},$$

$$\left[\mathrm{R}_m^{ab}, \mathrm{Q}_n^{\gamma d}\right] = \tfrac{1}{2}\varepsilon^{bd}\mathrm{Q}_{m+n}^{\gamma a} + \tfrac{1}{2}\varepsilon^{ad}\mathrm{Q}_{m+n}^{\gamma b}, \qquad \left[\mathrm{L}_m^{\alpha\beta}, \mathrm{Q}_n^{\gamma d}\right] = \tfrac{1}{2}\varepsilon^{\beta\gamma}\mathrm{Q}_{m+n}^{\alpha d} + \tfrac{1}{2}\varepsilon^{\alpha\gamma}\mathrm{Q}_{m+n}^{\beta d}, \tag{3.28}$$

$$\left[\mathrm{R}_m^{ab}, \mathrm{S}_n^{\gamma d}\right] = \tfrac{1}{2}\varepsilon^{bd}\mathrm{S}_{m+n}^{\gamma a} + \tfrac{1}{2}\varepsilon^{ad}\mathrm{S}_{m+n}^{\gamma b}, \qquad \left[\mathrm{L}_m^{\alpha\beta}, \mathrm{S}_n^{\gamma d}\right] = \tfrac{1}{2}\varepsilon^{\beta\gamma}\mathrm{S}_{m+n}^{\alpha d} + \tfrac{1}{2}\varepsilon^{\alpha\gamma}\mathrm{S}_{m+n}^{\beta d}.$$

Only the brackets between supercharges Q, S and the derivation B are modified. They follow from the above commutators for the fundamental representation (3.17) where the variable z is interpreted as a shift by one level:

$$
\begin{aligned}
&\{Q_m^{\alpha b}, Q_n^{\gamma d}\} = 2\alpha\varepsilon^{\alpha\gamma}\varepsilon^{bd}A_{m+n},\\
&\{Q_m^{\alpha b}, S_n^{\gamma d}\} = -\varepsilon^{\alpha\gamma}R_{m+n}^{bd} + \varepsilon^{bd}L_{m+n}^{\alpha\gamma} - \varepsilon^{\alpha\gamma}\varepsilon^{bd}h^{-1}(A_{m+n+1} - A_{m+n}),\\
&\{S_m^{\alpha b}, S_n^{\gamma d}\} = -2\alpha^{-1}\varepsilon^{\alpha\gamma}\varepsilon^{bd}A_{m+n+1},\\
&[B_m, Q_n^{\alpha b}] = h^{-1}(Q_{m+n+1}^{\alpha b} - Q_{m+n}^{\alpha b}) + 2\alpha S_{m+n}^{\alpha b},\\
&[B_m, S_n^{\alpha b}] = 2\alpha^{-1}Q_{m+n+1}^{\alpha b} - h^{-1}(S_{m+n+1}^{\alpha b} - S_{m+n}^{\alpha b}).
\end{aligned}
\tag{3.29}
$$

The remaining unspecified Lie brackets are trivial. Altogether the Jacobi identities are satisfied as can be confirmed explicitly. The algebra has a family of four-dimensional evaluation representations with $J_n \simeq z^n J$ and the action of J specified in (3.13).

Universal r-matrix.   The functional *r*-matrix in (3.22) can be cast into the loop level form. To that end one expands the above function of z's into a geometric series[9]

$$
\frac{z_2}{z_1 - z_2} = \sum_{k=1}^{\infty}\left(\frac{z_2}{z_1}\right)^k.
\tag{3.30}
$$

The resulting *r*-matrix then reads explicitly

$$
\begin{aligned}
r = {}& R_0^{12} \otimes R_0^{12} - R_0^{22} \otimes R_0^{11} - L_0^{12} \otimes L_0^{12} + L_0^{22} \otimes L_0^{11} - \varepsilon_{\alpha\gamma}\varepsilon_{bd}Q_0^{\alpha b} \otimes S_0^{\gamma d} - A_0 \otimes B_0 \\
&+ \sum_{k=1}^{\infty}\Big[-\varepsilon_{ac}\varepsilon_{bd}R_{-k}^{ab} \otimes R_{+k}^{cd} + \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}L_{-k}^{\alpha\beta} \otimes L_{+k}^{\gamma\delta} - \varepsilon_{\alpha\gamma}\varepsilon_{bd}Q_{-k}^{\alpha b} \otimes S_{+k}^{\gamma d} \\
&+ \varepsilon_{\alpha\gamma}\varepsilon_{bd}S_{-k}^{\alpha b} \otimes Q_{+k}^{\gamma d} - A_{-k} \otimes B_{+k} - B_{-k} \otimes A_{+k}\Big].
\end{aligned}
\tag{3.31}
$$

This *r*-matrix defines a quasi-triangular Lie bialgebra: first of all the symmetric part of *r* equals

$$
r_{12} + r_{21} = \sum_{k=-\infty}^{\infty}\left(\frac{z_2}{z_1}\right)^k t,
\tag{3.32}
$$

which is an invertible quadratic invariant of the algebra. It is straightforward to convince oneself of this fact. Note that this is slightly different than in the functional form where $r_{12} + r_{21} = 0$, cf (3.24). Consequently, the resulting algebra is merely *quasi*-triangular.

Furthermore, the classical Yang–Baxter equation $[[r, r]] = 0$, see (3.9), holds. This is not as easily seen; in particular, it does not follow right away from the discussion at the end of section 3.3 because the expansion into loop levels introduces slight modifications, cf (3.24) versus (3.32). The CYBE eventually follows from the triangular decomposition of $\mathfrak{gl}(2|2)[z, z^{-1}]$ into a positive, negative and Cartan subalgebra $\mathfrak{g}^+ \oplus \mathfrak{g}^- \oplus \mathfrak{g}^0$ with

$$
\begin{aligned}
\mathfrak{g}^+ &= \langle R_0^{11}, L_0^{11}, S_0, B_0\rangle \oplus z\mathfrak{gl}(2|2)[z],\\
\mathfrak{g}^0 &= \langle R_0^{12}, L_0^{12}\rangle,\\
\mathfrak{g}^- &= \langle R_0^{22}, L_0^{22}, Q_0, A_0\rangle \oplus z^{-1}\mathfrak{gl}(2|2)[z^{-1}].
\end{aligned}
\tag{3.33}
$$

The crucial observation which ensures quasi-triangularity is that the *r*-matrix (3.31) belongs to the following subspace (more precisely its compactification):

$$
r \in (\mathfrak{g}^- \otimes \mathfrak{g}^+) \oplus (\mathfrak{g}^0 \otimes \mathfrak{g}^0).
\tag{3.34}
$$

[9] This formula represents an analytic continuation of the series, see below for additional distributional contributions.

This *r*-matrix takes the form of the classical double of $\mathfrak{g}^+ \oplus \mathfrak{g}^0$ divided by the centre generated by a combination of $\mathfrak{g}^0$ and its dual. Alternatively, one can say that the decomposition $(\mathfrak{g}^+ \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^-, \mathfrak{g}^+ \oplus \mathfrak{g}^0, \mathfrak{g}^- \oplus \mathfrak{g}^0)$ is a Manin triple up to the double appearance of the Cartan subalgebra $\mathfrak{g}^0$.

Distributions on the complex plane. To convert between the above two pictures for loop algebras, one conventionally uses the geometric series

$$g(z) := \sum_{n=0}^{\infty} z^n, \qquad g(z) = \frac{1}{1-z} \quad \text{for} \quad |z| < 1. \tag{3.35}$$

It is convenient to continue the function $g(z)$ analytically to all $z \neq 1$, but some care is required because it actually introduces inconsistencies. Consider the contour integral of $z^k g(z)$ for a circle of radius r around the origin. One would like to obtain the following result for the geometric series (i.e. when performing the integral prior to the infinite sum)

$$\frac{1}{2\pi i} \oint_r z^k g(z)\, dz = \delta_{k<0}. \tag{3.36}$$

When analytically continuing the series $g(z)$ to all $z \neq 1$ one obtains a different result

$$\frac{1}{2\pi i} \oint_r \frac{z^k\, dz}{1-z} = \begin{cases} +\delta_{k<0} & \text{for } r < 1, \\ -\delta_{k\geqslant 0} & \text{for } r > 1. \end{cases} \tag{3.37}$$

The difference between the two integrals equals -1 for $r > 1$ irrespectively of the value of k. Such a term can be thought of as to originate from a distributional term $\delta_{a,b}(z)$ which is supported on a curve between a and b.[10] The distribution is defined such that for each (directed) crossing of the contour through the supporting curve, the distribution contributes the value of the integrand at $z = 0$. Now the distributional result of the geometric series reads

$$g(z) = \frac{1}{1-z} + 2\pi i \delta_{0,\infty}(z-1), \tag{3.38}$$

and the extra term w.r.t. (3.35) is what reduces a triangular algebra to a *quasi*-triangular one. Now $g(z)$ has a cut on the positive real axis extending from $z = 1$ to $z = \infty$. Each crossing of the cut from the lower towards the upper half-plane contributes the value of the integrand at $z = 1$:

$$\frac{1}{2\pi i} \oint_r z^k 2\pi i \delta_{0,\infty}(z-1)\, dz = \begin{cases} 0 & \text{for } r < 1, \\ 1 & \text{for } r > 1. \end{cases} \tag{3.39}$$

The quadratic invariant requires a geometric series over both positive and negative powers. For such series (3.32), the analytic contribution vanishes exactly, while a distributional contribution remains:

$$\sum_{n=-\infty}^{\infty} z^n = g(z) + g(1/z) - 1 = 2\pi i \delta_{0,\infty}(z-1) + 2\pi i \delta_{0,\infty}(1/z-1) = 2\pi i \delta_{-1,\infty}(z-1). \tag{3.40}$$

We have made use of proper transformation rules for this distribution which are analogous to those for delta functions. Here the resulting branch cut extends from $z = 0$ to $z = \infty$, and for

---

[10] The distributions become somewhat more familiar, see e.g. [29], when the variables and integrations are restricted to the unit circle, $|z| = 1$.

each crossing it contributes the value of the integrand at $z = 1$. In the remainder of the paper, we will only make reference to this type of distribution, written in the form

$$\sum_{n=-\infty}^{\infty} \left(\frac{z_1}{z_2}\right)^n = 2\pi i \delta_{-1,\infty}(z_1/z_2 - 1) =: 2\pi i z_1 \delta(z_1 - z_2). \tag{3.41}$$

The latter is a convenient abbreviation of the former distribution. Here the cut extends from $z_1 = 0$ to $z_1 = \infty$ or alternatively from $z_2 = \infty$ to $z_2 = 0$.

4. Affine extension

A loop algebra can be extended by one derivation D and one central charge C to an affine (Kac–Moody) Lie algebra. Here we show that our deformed loop algebra also admits such an affine extension.

4.1. Example

We shall use the example of $\mathfrak{sl}(2)$ to illustrate the construction of the affine extension. The derivation D is defined as the following derivative w.r.t. z:

$$D = \frac{z d}{dz}. \tag{4.1}$$

Put differently, D generates a scaling transformation of z. Alternatively, we can define D through its action on the loop variable z and the base generators R^{ab}:

$$[D, z] = z, \qquad [D, R^{ab}] = 0. \tag{4.2}$$

The central charge appears in the brackets as follows:

$$[f(z)R^{ab}, g(z)R^{cd}] = f(z)g(z)(\varepsilon^{bc}R^{ad} + \varepsilon^{ad}R^{bc})$$
$$- \frac{1}{2}\left(\varepsilon^{ac}\varepsilon^{bd} + \varepsilon^{ad}\varepsilon^{bc}\right)\frac{1}{2\pi i}\oint (f(z)\,dg(z))C. \tag{4.3}$$

Here the contour integral winds once around $z = 0$ (or $z = \infty$). Finally, we can write the quadratic invariant using the delta distribution in (3.41)

$$\hat{t} = -2\pi i z_1 \delta(z_1 - z_2)\varepsilon_{ac}\varepsilon_{bd}R^{ab} \otimes R^{cd} - D \otimes C - C \otimes D. \tag{4.4}$$

The above construction can be generalized straightforwardly to any loop algebra, but for our deformed loop algebra some more work is needed because of the non-homogeneous structure of the loop levels in (3.17).

4.2. Derivation

Now we have to generalize the brackets with the affine derivation to all generators of our loop algebra. First of all, it acts on the loop parameter z as a scaling transformation

$$[D, z] = z \quad \text{or} \quad D \simeq \frac{z d}{dz}. \tag{4.5}$$

The derivations of the two sets of $\mathfrak{sl}(2)$ generators R and L take the standard form

$$[D, R^{ab}] = 0, \qquad [D, L^{\alpha\beta}] = 0. \tag{4.6}$$

For the remaining generators Q, S, A, B, we can gain inspiration from the fundamental representation in (3.13). As compared to the fundamental representation of the undeformed $\mathfrak{gl}(2|2)$, the representations of Q and S (as a two-vector) are rotated by the SL(2) matrix T

in (3.18) [30]. Furthermore the action of A and B is scaled by q w.r.t. the undeformed $\mathfrak{gl}(2|2)$. The parameters a, b, c, d, q depend on x which is related to z through (3.7). The derivation D transforms z according to (4.1); hence, it modifies the matrix T. The brackets of the derivation with the supercharges must reflect this transformation in order to find a suitable representation of D. We are thus led to the following combinations:

$$z \frac{\mathrm{d}T}{\mathrm{d}z} T^{-1} = U + f(z)W, \qquad \frac{z}{q} \frac{\mathrm{d}q}{\mathrm{d}z} = V, \tag{4.7}$$

with[11]

$$U = \frac{1}{z + z^{-1} - 2 + 4h^2} \begin{pmatrix} -h^2 & +h\alpha \\ -h\alpha^{-1} & +h^2 \end{pmatrix}, \qquad V = -\frac{z - 1 + 2h^2}{z + z^{-1} - 2 + 4h^2}. \tag{4.8}$$

The precise functional form of γ influences the undetermined function $f(z)$. For $f(z) = 0$, we get a reasonably simple final expression corresponding to the choice

$$\gamma \simeq \frac{h'x - \mathrm{i}h}{h'\sqrt{x^2 - 1}}. \tag{4.9}$$

The matrix U now appears as the derivation of the two-vector of the bare supercharges Q and S. Altogether the derivations are specified by

$$\begin{aligned}
[\mathrm{D}, \mathrm{Q}^{\alpha b}] &= U_{11}(z)\mathrm{Q}^{\alpha b} + U_{12}(z)\mathrm{S}^{\alpha b}, \\
[\mathrm{D}, \mathrm{S}^{\alpha b}] &= U_{21}(z)\mathrm{Q}^{\alpha b} + U_{22}(z)\mathrm{S}^{\alpha b}, \\
[\mathrm{D}, \mathrm{A}] &= +V(z)\mathrm{A}, \\
[\mathrm{D}, \mathrm{B}] &= -V(z)\mathrm{B}.
\end{aligned} \tag{4.10}$$

Note that the ambiguity in (4.7) corresponds to shifting D by $f(z)$B, cf (3.20); nothing is lost by making a specific choice as the above. The Jacobi identities require

$$z \frac{\mathrm{d}W}{\mathrm{d}z} = [U, W] - VW, \tag{4.11}$$

which follows by combining (3.19) with (4.7). As an aside, we note that the derivation D can be extended to a Virasoro algebra $\mathrm{D}_n = z^n \mathrm{D}$ with a new central charge c, but we will not make use of it here.

4.3. Central charge

The central charge appears in the brackets of the two sets of $\mathfrak{sl}(2)$ generators in the standard fashion:

$$\begin{aligned}
[f(z)\mathrm{R}^{ab}, g(z)\mathrm{R}^{cd}] &= f(z)g(z)(\varepsilon^{bc}\mathrm{R}^{ad} - \varepsilon^{ad}\mathrm{R}^{bc}) \\
&\quad - \frac{1}{2}(\varepsilon^{ac}\varepsilon^{bd} + \varepsilon^{ad}\varepsilon^{bc}) \frac{1}{2\pi\mathrm{i}} \oint (f(z)\,\mathrm{d}g(z))\mathrm{C}, \\
[f(z)\mathrm{L}^{\alpha\beta}, g(z)\mathrm{L}^{\gamma\delta}] &= f(z)g(z)(\varepsilon^{\beta\gamma}\mathrm{L}^{\alpha\delta} - \varepsilon^{\alpha\delta}\mathrm{L}^{\beta\gamma}) \\
&\quad + \frac{1}{2}(\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta} + \varepsilon^{\alpha\delta}\varepsilon^{\beta\gamma}) \frac{1}{2\pi\mathrm{i}} \oint (f(z)\,\mathrm{d}g(z))\mathrm{C}.
\end{aligned} \tag{4.12}$$

For the remaining generators Q, S, A, B, the brackets leading to the central charge have to be adjusted to the deformations in (4.10). There are several ways to derive a central charge for

---

[11] The conversion to levels of the loop algebra along the lines of section 3.4 is somewhat problematic due to the presence of poles in $U(z)$ and $V(z)$ at $z \neq 0, \infty$. This issue deserves further investigations.

the above loop algebra. A very convenient method consists in demanding invariance of the quadratic invariant, cf (3.41):

$$\hat{t} = 2\pi i z_1 \delta(z_1 - z_2)t - C \otimes D - D \otimes C \tag{4.13}$$

where t is given in (3.23). The invariance under the loop generators requires a balancing of two types of terms. The contributions from brackets with D must cancel the contribution from brackets proportional to the central charge. One can easily figure out the central charge contributions complementary to (4.10):

$$\{f(z)Q^{\alpha b}, g(z)Q^{\gamma d}\} = \varepsilon^{\alpha\gamma}\varepsilon^{bd} f(z)g(z)W_{12}(z)A + \varepsilon^{\alpha\gamma}\varepsilon^{bd} \frac{1}{2\pi i} \oint \left(f(z)g(z)U_{12}(z)\frac{dz}{z} \right) C,$$

$$\{f(z)Q^{\alpha b}, g(z)S^{\gamma d}\} = f(z)g(z)(-\varepsilon^{\alpha\gamma}R^{bd} + \varepsilon^{bd}L^{\alpha\gamma} - \varepsilon^{\alpha\gamma}\varepsilon^{bd}W_{11}(z)A)$$
$$+ \varepsilon^{\alpha\gamma}\varepsilon^{bd} \frac{1}{2\pi i} \oint \left(f(z)\,dg(z) - f(z)g(z)U_{11}(z)\frac{dz}{z} \right) C,$$

$$\{f(z)S^{\alpha b}, g(z)S^{\gamma d}\} = -\varepsilon^{\alpha\gamma}\varepsilon^{bd} f(z)g(z)W_{21}(z)A + \varepsilon^{\alpha\gamma}\varepsilon^{bd} \frac{1}{2\pi i} \oint \left(-f(z)g(z)U_{21}(z)\frac{dz}{z} \right) C,$$

$$[f(z)A, g(z)B] = -\frac{1}{2\pi i} \oint \left(f(z)\,dg(z) - V(z)f(z)g(z)\frac{dz}{z} \right) C. \tag{4.14}$$

Note that the value of the above integrals depends on the choice of contours. Conventionally one assumes that the functions $f(z)$ and $g(z)$ are holomorphic except at $z = 0$ and $z = \infty$. In that case, the contour can take any path that winds once around $z = 0$ and $z = \infty$. Here the functions $W(z)$ and $V(z)$ in (4.8) introduce two extra poles $z_\pm^*$. These could be used to define two additional central charges. It appears that they behave much like $\delta(z - z_\pm^*)A$ and therefore there may be no need to enlarge the algebra further. The issues of how to put the contours and how to define the affine central charge(s) need further investigations.

4.4. Affine r-matrix

The affine extension of the r-matrix in (3.22) reads

$$\hat{r} = r - C \otimes D. \tag{4.15}$$

In the presence of C, the additional term is needed to fulfil the CYBE. Note that one is free to add an antisymmetric term proportional to $C \otimes D - D \otimes C$ to the above $\hat{r}$ [31].

A curious feature of the r-matrix is that it is not invariant under the affine derivation D. This is because the coefficients U, V in (4.8) of action (4.10) depend on z. Effectively this implies that the r-matrix is not a function of z_2/z_1 alone, but it depends separately on z_1 and z_2. Correspondingly, the cobracket of D becomes non-trivial.

A possible benefit of the affine extension is that it may add further constraints on the r-matrix. Without the extension it is possible to add to r terms of the form

$$z_1^m z_2^n A \otimes A \tag{4.16}$$

because A is a central element and hence it cannot be seen within the CYBE. One could also view the deformation as a deformation of $z^m B$ by $z^n A$ which affects only the coalgebra but not the algebra. In the presence of the affine extensions, such deformations may no longer be possible because A is no longer in the centre; it has non-trivial brackets with D and B. Thus, it would be interesting to derive constraints on the permissible deformations of r by $A \otimes A$.

4.5. Fundamental representation

Let us reconsider the fundamental evaluation representation (3.13). The regular loop generators act on a four-dimensional space spanned by $|\phi^a\rangle$ and $|\psi^\alpha\rangle$. Conversely, the derivation D acts as a scaling transformation (4.1) for the parameter z which is related to the representation parameter x through (3.7). Consequently we must promote the states to fields $|\phi^a, x\rangle$ and $|\psi^\alpha, x\rangle$ so that D can act on them. The representation of the affine algebra is therefore infinite dimensional, and it naturally models a field on a one-dimensional mass shell. Effectively, the affine derivation corresponds to a (Lorentz) boost of the mass shell. In this case, the cobrackets for D are non-trivial and therefore Lorentz symmetry must be considered as deformed.

In the picture of fields $|\phi^a, x\rangle$ and $|\psi^\alpha, x\rangle$, one can get a clearer understanding of the role of the generator z^kB (note that x is related to z): equation (3.13) suggests us that it induces an x-dependent (i.e. gauge) transformation for the normalization of bosons w.r.t. fermions. The role of γ (which can now depend on x) is related. It serves as a (functional) parameter of the representation, and it fixes a particular normalization for it.

This evaluation-type representation clearly has vanishing central charge C $\simeq$ 0. However, there surely exist representations with non-vanishing central charge, such as highest-weight representations. In the physical context, these may correspond to vertex operators. It would be interesting to investigate charged representations of this algebra.

5. Discrete symmetries

Before we continue with particular limiting cases of the classical r-matrix, we shall discuss some of its discrete symmetries. These will help us understand the limits better and also relate some cases to others.

5.1. Conjugation

The map (3.7) between x and z is quadratic and thus 2:1. The underlying reason for this property is that for unitary superalgebras there are four conjugate fundamental representations. In the present algebra, however, there is just a one-parameter family of fundamental representations parametrized through x. For each value of z there are two values of x corresponding to a pair of representations and its conjugate. That this is possible in the first place is a special property of $\mathfrak{sl}(2|2)$. The representation of each $\mathfrak{sl}(2)$ subalgebra is fundamental; as such it is self-conjugate under transposition and conjugation by an antisymmetric 2×2 matrix ε, i.e. for a traceless 2×2 matrix E, one has

$$E' = -\varepsilon E^{\mathrm{T}} \varepsilon^{-1} = E. \tag{5.1}$$

However, the representation of the remaining generators is not self-conjugate under the combined map for a 4×4 supermatrix E written in 2×2 blocks

$$E' = - \left(\begin{array}{c|c} \varepsilon & 0 \\ \hline 0 & \varepsilon \end{array} \right) E^{\mathrm{ST}} \left(\begin{array}{c|c} \varepsilon^{-1} & 0 \\ \hline 0 & \varepsilon^{-1} \end{array} \right). \tag{5.2}$$

The representation E is parametrized through x and γ. The two values of x corresponding to conjugate representations are related by inversion [21]

$$x' = \frac{1}{x}, \qquad z' = z. \tag{5.3}$$

The parameters (3.5), (3.7) and (3.18) for the fundamental representation map according to

$$T' = T \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}, \qquad q' = -q, \qquad \gamma' = \frac{\alpha}{\gamma} \frac{(h'x - ih)(hx + ih')}{h'(x^2 - 1)}. \tag{5.4}$$

Note that the matrix multiplying T corresponds to the supertranspose operation which is $\mathbb{Z}_4$ periodic. So for each value of x there are two representations which differ in sign for the odd generators corresponding to a total of four fundamentals in superalgebras. See also section 5.5 for further comments.

This transformation involves only representations and thus it can be applied to each of the two sites of the fundamental r-matrix individually. Under such a crossing transformation of x_1, γ_1 the coefficients in table 2 permute as follows:

$$\begin{aligned}
A_{\bar{1}2} &= -\tfrac{1}{2}(A_{12} - B_{12}), & D_{\bar{1}2} &= -\tfrac{1}{2}(D_{12} - E_{12}), \\
\tfrac{1}{2}(A_{\bar{1}2} - B_{\bar{1}2}) &= -A_{12}, & \tfrac{1}{2}(D_{\bar{1}2} - E_{\bar{1}2}) &= -D_{12}, \\
G_{\bar{1}2} &= L_{12}, & L_{\bar{1}2} &= G_{12}, \\
H_{\bar{1}2} &= -F_{12}, & F_{\bar{1}2} &= +H_{12}, \\
K_{\bar{1}2} &= -C_{12}, & C_{\bar{1}2} &= +K_{12}.
\end{aligned} \tag{5.5}$$

The combinations $\tfrac{1}{2}(A_{12} + B_{12} \pm 1)$ and $\tfrac{1}{2}(D_{12} + E_{12} \pm 1)$ remain invariant. This transformation is compensated by the map (5.2) on the first site of the fundamental r-matrix in table 1. The transformation for x_2, γ_2 is the same as above except that C, F, H, K transform differently

$$H_{1\bar{2}} = +C_{12}, \qquad C_{1\bar{2}} = -H_{12}, \qquad K_{1\bar{2}} = +F_{12}, \qquad F_{1\bar{2}} = -K_{12}. \tag{5.6}$$

Under the combined transformation of both sites, the coefficients are invariant up to the following permutations:

$$C_{\bar{1}\bar{2}} = F_{12}, \qquad F_{\bar{1}\bar{2}} = C_{12}, \qquad H_{\bar{1}\bar{2}} = K_{12}, \qquad K_{\bar{1}\bar{2}} = H_{12}. \tag{5.7}$$

The above transposition map has two fixed points which will be of importance later

$$x^*_\pm = \pm 1, \qquad z^*_\pm = (ih \pm h')^2, \qquad q^* = \infty. \tag{5.8}$$

These two points will be called *self-dual*.

5.2. Inversion

Another useful discrete map is the inversion of z. It implies the following transformations of the related parameters

$$x' = i\frac{hx + ih'}{h'x - ih}, \qquad z' = \frac{1}{z}. \tag{5.9}$$

The parameters of the fundamental representation transform according to

$$T' = RT, \qquad q' = zq, \qquad \gamma' = \frac{\gamma}{h'x - ih}, \qquad R = \begin{pmatrix} 0 & i\alpha \\ i\alpha^{-1} & 0 \end{pmatrix}. \tag{5.10}$$

These rules suggest that the map

$$\begin{pmatrix} Q'^{\alpha b} \\ S'^{\alpha b} \end{pmatrix} = R \begin{pmatrix} Q^{\alpha b} \\ S^{\alpha b} \end{pmatrix}, \qquad A' = zA, \qquad B' = z^{-1}B, \tag{5.11}$$

together with $z' = 1/z$ is an algebra automorphism. Indeed one can confirm that the algebra in sections 3 and 4 is invariant under the map. It does not, however, respect the decomposition in (3.33) underlying the r-matrix which is therefore not invariant. In particular, the subalgebras

$\mathfrak{g}^+$ and $\mathfrak{g}^-$ in (3.33) are interchanged except for the elements R^{11}, R^{22}, L^{11}, L^{22}. To achieve a proper transformation we have to interchange them using the map $z' = 1/z$ with

$$
\begin{aligned}
&R'^{ab} = \varepsilon_{ac}\varepsilon_{bd}R^{cd}, &
\begin{pmatrix} Q'^{\alpha b} \\ S'^{\alpha b} \end{pmatrix} = \varepsilon_{\alpha\gamma}\varepsilon_{bd}R\begin{pmatrix} Q^{\gamma d} \\ S^{\gamma d} \end{pmatrix}, &
&A' = zA, \\
&L'^{\alpha\beta} = \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}L^{\gamma\delta}, & &
&B' = z^{-1}B.
\end{aligned}
\tag{5.12}
$$

Under the inversion, all the coefficients $A_{12}, \ldots, L_{12}$ for the fundamental r-matrix in table 2 flip sign. This yields an overall sign in the r-matrix except for the elements $\frac{1}{2}(A_{12} + B_{12} \pm 1)$ and $\frac{1}{2}(D_{12} + E_{12} \pm 1)$ which are permuted. The permutation is compensated by the transformation in (5.12).

5.3. Statistics flip

The superalgebras of the kind $\mathfrak{psl}(n|n)$ have an exceptional automorphism [32]. It interchanges the two $\mathfrak{sl}(n)$ factors and thus flips the two gradings in certain representations. It is responsible for the existence of the two types of strange superalgebras.

The Lie brackets are invariant under the exchange of the two $\mathfrak{sl}(2)$ subalgebras:

$$
R'^{ab} = L^{ab}, \qquad L'^{\alpha\beta} = R^{\alpha\beta}.
\tag{5.13}
$$

At the level of the fundamental representation, the exchange is compensated by the map

$$
|\phi^a\rangle' = |\psi^a\rangle, \qquad |\psi^\alpha\rangle' = |\phi^\alpha\rangle.
\tag{5.14}
$$

Under this map, the fundamental r-matrix in table 1 flips sign provided that the coefficients transform according to

$$
\begin{aligned}
&A'_{12} = D_{12}, & &B'_{12} = E_{12}, & &G'_{12} = -L_{12}, & &C'_{12} = F_{12}, & &H'_{12} = K_{12}, \\
&D'_{12} = A_{12}, & &E'_{12} = B_{12}, & &L'_{12} = -G_{12}, & &F'_{12} = C_{12}, & &K'_{12} = H_{12}.
\end{aligned}
\tag{5.15}
$$

Note that the elements C, F, H, K in table 1 receive an extra sign due to the change of statistics of the states when acting with the bifermionic contributions (3.31). For the coefficients in table 2, this transformation is realized by mapping the parameter γ according to

$$
\gamma' = \frac{\alpha(h'x - ih)(hx + ih')x}{h'\gamma(x^2 - 1)}.
\tag{5.16}
$$

The transformation of the coefficients for the fundamental representation in (3.5) and (3.18) reads

$$
T' = RT\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad q' = q, \qquad R = \frac{i}{h'}\begin{pmatrix} +h & +\alpha z^{-1} \\ -\alpha^{-1}z & -h \end{pmatrix}.
\tag{5.17}
$$

The off-diagonal matrix multiplying T corresponds to the action (5.14). The map implies the following transformation for the remaining generators:

$$
\begin{pmatrix} Q'^{\alpha b} \\ S'^{\alpha b} \end{pmatrix} = R\begin{pmatrix} Q^{\alpha b} \\ S^{\alpha b} \end{pmatrix}, \qquad A' = A, \qquad B' = -B.
\tag{5.18}
$$

This transformation respects the algebra in sections 3 and 4 and the decomposition (3.33) while it flips the sign of the r-matrix in (3.22).

5.4. Duality

Further scrutiny suggests that there is a relationship between r-matrices with global parameters h and h' interchanged. The quadratic relation $h^2 + h'^2 = 1$ implies various sign ambiguities in the map which we can lift by choosing a different parameter

$$
h = \tfrac{1}{2}(k + k^{-1}), \qquad h' = -\tfrac{i}{2}(k - k^{-1}).
\tag{5.19}
$$

The interchange corresponds to the map $k' = ik$. The coefficients of the fundamental r-matrix in table 2 turn out to be invariant under the transformation

$$k' = ik, \qquad z' = -z, \qquad x' = x, \qquad \alpha' = -i\frac{k + k^{-1}}{k - k^{-1}}\alpha. \qquad (5.20)$$

The remaining parameters of the fundamental representation transform according to

$$T' = RT, \qquad q' = i\frac{k - k^{-1}}{k + k^{-1}}q, \qquad \gamma' = \gamma, \qquad R = \begin{pmatrix} 1 & 0 \\ \alpha^{-1}h^{-1}z & 1 \end{pmatrix}. \qquad (5.21)$$

Again the algebra in sections 3 and 4 is invariant if one imposes the following map for the generators:

$$\begin{pmatrix} Q'^{\alpha b} \\ S'^{\alpha b} \end{pmatrix} = R \begin{pmatrix} Q^{\alpha b} \\ S^{\alpha b} \end{pmatrix}, \qquad A' = i\frac{k - k^{-1}}{k + k^{-1}}A, \qquad B' = -i\frac{k + k^{-1}}{k - k^{-1}}B. \qquad (5.22)$$

Also the decomposition (3.33) is respected, and consequently the r-matrix is invariant. In particular, the coefficients in table 2 transform trivially.

5.5. Reparametrization

Here we introduce a change of variables which helps us to make some features of the algebra discussed above somewhat more transparent. This will be instructive to some extent, but in the remainder of the paper we shall nevertheless stick to the old variables.

Reparametrization. We have seen in section 5.1 that for each value of z there are four fundamental representations. They are distinguished by different values of x and γ. For instance, for each z the map (3.7) permits two values for x, and for each x there is a pair of representations distinguished by different signs for γ.

In fact one can introduce a new parameter y to distinguish all four fundamental representations corresponding to a particular value z:

$$x = -\frac{y^2 - 1}{y^2 + 1}, \quad \gamma = \frac{y^2 + k^2}{2ky}\eta, \quad z = -k^2\frac{y^4 - 1}{y^4 - k^4}, \quad q = \frac{-1}{k^2(k - k^{-1})}\frac{y^4 - k^4}{y^2}. \qquad (5.23)$$

At the same time we shall use the parameter k introduced in section 5.4 instead of h:

$$h = \tfrac{1}{2}(k + k^{-1}), \qquad h' = -\tfrac{i}{2}(k - k^{-1}), \qquad \alpha = \tfrac{1}{2}(k - k^{-1})\kappa. \qquad (5.24)$$

Altogether the following parametrization yields a slightly more transparent picture. This can be observed for the coefficients a, b, c, d of the matrix T which now take a very symmetric form:

$$a = \frac{\eta(y^2 + k^2)}{2ky}, \quad b = \frac{-\kappa(y^2 - k^2)}{2k\eta y}, \quad c = \frac{-\eta(y^2 + 1)}{\kappa(k - k^{-1})y}, \quad d = \frac{y^2 - 1}{\eta(k - k^{-1})y}. \qquad (5.25)$$

Notably, all the coefficients in table 2 now factor completely into terms $y^2 \pm 1$, $y^2 \pm k^2$ and $y_1^2 \pm y_2^2$.

Special points. Investigating the above expressions it becomes clear that the y-plane has several special points. The points $y^\circ_{+1,2,3,4} = \pm 1, \pm i$ map to $z = 0$ while $y^\circ_{-1,2,3,4} = \pm k, \pm ik$ map to $z = \infty$. Finally, the two points $y^*_\pm = 0, \infty$ map to the self-dual points $z^*_\pm = -k^{\mp 2}$ or $x^*_\pm = \pm 1$ in (5.8). The configurations of special points are displayed in figure 2.
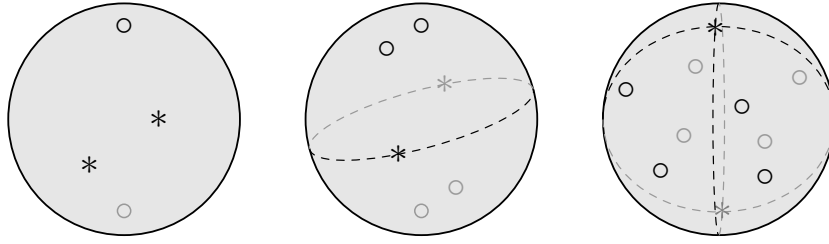
Figure 2. The compactified complex plane for z, x or y, respectively. The points corresponding to $z_\pm^\circ = 0, \infty$ are marked by $\circ$. The self-dual points corresponding to $z_\pm^*$ are marked by $*$. The spheres are divided into one, two or four regions which are identified by a twist.

Eigenbasis. A curious feature of the matrix U in (4.8) is that the z-dependence is in the prefactor only. Hence, the eigenvectors are constants and we can use them as a new basis for Q and S.

A matrix to perform the similarity transformation to the eigenvectors is given by

$$R = \begin{pmatrix} -2/(k - k^{-1}) & \kappa k \\ -\kappa^{-1}/(k - k^{-1}) & \frac{1}{2} k^{-1} \end{pmatrix}. \tag{5.26}$$

The resulting matrix $\tilde{T}$ containing the coefficients $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ then simply reads

$$\tilde{T} = RT = \begin{pmatrix} \eta y & \kappa \eta^{-1} y \\ -\frac{1}{2} \kappa^{-1} \eta y^{-1} & \frac{1}{2} \eta^{-1} y^{-1} \end{pmatrix}. \tag{5.27}$$

The transformation curiously removes the diagonal terms in the matrix $\tilde{W}$:

$$\tilde{W} = RWR^{-1} = \frac{1}{k + k^{-1}} \begin{pmatrix} 0 & -4\kappa(1 + k^2 z) \\ \kappa^{-1}(1 + k^{-2} z) & 0 \end{pmatrix}, \tag{5.28}$$

whereas by construction the matrix $\tilde{U}$ is diagonal:

$$\tilde{U} = RUR^{-1} = \frac{(k^4 - 1)z}{4(k^2 + z)(1 + k^2 z)} \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.29}$$

The only reason not to perform this similarity transformation once and for all is that it obscures the linear combinations of $\tilde{Q}$ and $\tilde{S}$ which appear in the contribution (3.23) to the r-matrix and in the triangular decomposition (3.33). We will thus stick to the original basis of Q and S.

Embedding. The above reparametrization has led to rational expressions for the parameters a, b, c, d, q of the fundamental representation[12]. We can use them to go one step further, and embed our algebra into the standard algebra $\mathfrak{gl}(2|2)[y, y^{-1}]$ (with $\bar{W} = M$, $\bar{U} = \bar{V} = 0$) in analogy to the transformation in [24]:

$$\begin{array}{lll} \mathrm{R}^{ab} = \bar{\mathrm{R}}^{ab}, & \begin{pmatrix} \mathrm{Q}^{\alpha b} \\ \mathrm{S}^{\alpha b} \end{pmatrix} = T(y) \begin{pmatrix} \bar{\mathrm{Q}}^{\alpha b} \\ \bar{\mathrm{S}}^{\alpha b} \end{pmatrix}, & \mathrm{A} = q(y)\bar{\mathrm{A}}, \\ \mathrm{L}^{\alpha\beta} = \bar{\mathrm{L}}^{\alpha\beta}, & & \mathrm{B} = q(y)^{-1}\bar{\mathrm{B}}, \end{array} \tag{5.30}$$

with $\eta = \sqrt{\kappa}$. Note that one must allow for pole singularities at the special points y°, y^*. In this sense, one has to require that the Riemann surface underlying the ambient algebra is a sphere with punctures at all of these points, see figure 2, not just at $y = 0, \infty$ as for conventional loop algebras.

---

[12] In fact, also x (but not z itself) is permissible because $T(x)$, $q(x)$ are rational.

The reduction to our subalgebra is done by twisting with the $\mathbb{Z}_4$-periodic automorphism of $\mathfrak{gl}(2|2)$[13]:

$$y \to iy, \quad \bar{Q} \to i\bar{S}, \quad \bar{S} \to i\bar{Q}, \quad \bar{A} \to -\bar{A}, \quad \bar{B} \to -\bar{B}. \tag{5.31}$$

Furthermore, singularities at the fixed points $y^* = 0, \infty$ are restricted to be at most double poles while there can be poles of arbitrary order at the points y°.

As above, this redefinition changes the form of the r-matrix (3.22) and the triangular decomposition (3.33), and we shall refrain from making use of it subsequently. It is nevertheless interesting because it shifts the deformation from the algebra to the r-matrix, i.e. the conventional affine $\mathfrak{gl}(2|2)$ algebra apparently admits a non-standard r-matrix.

Discrete transformations. The discrete transformations discussed above also simplify. Essentially they map the various special points y° and y^* into each other. The conjugation symmetry discussed in section 5.1 translates between the four conjugate fundamental representations for each value of z. This is achieved through

$$y' = iy, \qquad \eta' = \frac{i\kappa}{\eta}. \tag{5.32}$$

The inversion symmetry discussed in section 5.2 is invoked by

$$y \to \frac{k}{y}, \qquad \eta' = i\eta. \tag{5.33}$$

The statistics flip symmetry in section 5.3 requires to change η according to

$$\eta' = \frac{\kappa}{\eta} \frac{y^2 - 1}{y^2 + 1} \frac{y^2 - k^2}{y^2 + k^2}. \tag{5.34}$$

Finally, there is the duality discussed in section 5.4 which relates four different values of k:

$$k' = ik, \qquad \eta' = i\frac{y^2 + k^2}{y^2 - k^2}\eta, \qquad \kappa' = -\kappa. \tag{5.35}$$

A similar transformation does not change anything in the original parametrization

$$k' = \frac{1}{k}, \qquad y' = \frac{1}{y}, \qquad \kappa' = -\kappa. \tag{5.36}$$

Note that the point $k = \sqrt{i}$ is self-dual under a combination of the above two duality maps. This map thus becomes an additional symmetry of the $k = \sqrt{i}$ system

$$y' = \frac{1}{y}, \qquad \eta' = -i\frac{y^2 + i}{y^2 - i}\eta, \qquad x' = -x, \qquad z' = -z. \tag{5.37}$$

It might be worth investigating if the self-dual point $k = \sqrt{i}$ has further interesting properties. The other self-dual point $k = 1$ is discussed in the following section.

6. Limits

The r-matrix presented in tables 1 and 2 has a couple of interesting limits which themselves lead to quasi-triangular Lie algebras. We shall call the r-matrix of section 3 the 'full trigonometric r-matrix'. The limits will modify the attributes of the name accordingly.

---

[13] Although the automorphism is $\mathbb{Z}_4$-periodic, it merely corresponds to a $\mathbb{Z}_2$-periodic *outer* automorphism of the $\mathfrak{gl}(2|2)$ algebra, see the discussion below (6.34).

6.1. Full rational case

The trigonometric r-matrix can be reduced to the rational r-matrix [23] obtained in the context of the AdS/CFT duality. To that end, one takes the limit

$$h = \epsilon \to 0, \qquad x \sim \epsilon^0, \qquad z = 1 + i\epsilon u + \mathcal{O}(\epsilon^2). \tag{6.1}$$

All of the following results are in full agreement with [24] where the structure and the underlying quasi-triangular Lie bialgebra were obtained.

Fundamental r-matrix.   The parameters of the fundamental representation (3.5) and (3.7) become[14]

$$a = \gamma, \quad b = \frac{-i\alpha x}{\gamma(x^2 - 1)}, \quad c = \frac{i\gamma}{\alpha x}, \quad d = \frac{x^2}{\gamma(x^2 - 1)}, \quad u = x + \frac{1}{x}, \quad q = \frac{-ix}{x^2 - 1}. \tag{6.2}$$

In this limit, the r-matrix diverges like ϵ^{-1} and needs to be renormalized:

$$\tilde{r} = i\epsilon r. \tag{6.3}$$

Most importantly, the divergence reduces the structure of the r-matrix in table 1 because the constant terms in the combinations $\frac{1}{2}(A + B \pm 1)$ and $\frac{1}{2}(D + E \pm 1)$ drop out. It can then be written in a manifestly $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ invariant fashion known for *rational r*-matrices:

$$\tilde{r}|\phi^a \phi^b\rangle = \tfrac{1}{2}(\tilde{A}_{12} + \tilde{B}_{12})|\phi^b \phi^a\rangle + \tfrac{1}{2}(\tilde{A}_{12} - \tilde{B}_{12})|\phi^a \phi^b\rangle + \tfrac{1}{2}\tilde{C}_{12}\varepsilon^{ab}\varepsilon_{\gamma\delta}|\psi^\gamma \psi^\delta\rangle,$$
$$\tilde{r}|\psi^\alpha \psi^\beta\rangle = -\tfrac{1}{2}(\tilde{D}_{12} + \tilde{E}_{12})|\psi^\beta \psi^\alpha\rangle - \tfrac{1}{2}(\tilde{D}_{12} - \tilde{E}_{12})|\psi^\alpha \psi^\beta\rangle - \tfrac{1}{2}\tilde{F}_{12}\varepsilon^{\alpha\beta}\varepsilon_{cd}|\phi^c \phi^d\rangle,$$
$$\tilde{r}|\phi^a \psi^\beta\rangle = \tilde{G}_{12}|\phi^a \psi^\beta\rangle + \tilde{H}_{12}|\psi^\beta \phi^a\rangle,$$
$$\tilde{r}|\psi^\alpha \phi^b\rangle = \tilde{K}_{12}|\phi^b \psi^\alpha\rangle + \tilde{L}_{12}|\psi^\alpha \phi^b\rangle. \tag{6.4}$$

The coefficient functions $\tilde{A}, \ldots, \tilde{L}$ are essentially the same as $A, \ldots, L$ in table 2, but the z-dependence reduces according to the limit

$$\frac{z_1}{z_1 - z_2} \sim \frac{\tfrac{1}{2}z_1 + \tfrac{1}{2}z_2}{z_1 - z_2} \sim \frac{z_2}{z_1 - z_2} \to \frac{1}{i\epsilon}\frac{1}{u_1 - u_2}, \tag{6.5}$$

where the factor of $1/i\epsilon$ is absorbed into the definition of the r-matrix $\tilde{r}$. The coefficients obey the same linear relations (3.10) as in the trigonometric case, but the constant shift disappears from the quadratic relations (3.11)

$$\tfrac{1}{4}(\tilde{A} + \tilde{B})(\tilde{D} + \tilde{E}) = \tfrac{1}{4}(3\tilde{A} - \tilde{B})(3\tilde{D} - \tilde{E}) + 4\tilde{G}\tilde{L} = \tilde{C}\tilde{F} + \tilde{H}\tilde{K}. \tag{6.6}$$

Algebra and universal r-matrix.   The loop algebra derived in section 3.3 remains essentially the same. One difference is that we shall use u as the formal loop variable instead of z. Thus, the brackets in (3.20) now read

$$\{Q^{\alpha b}, Q^{\gamma d}\} = \varepsilon^{\alpha\gamma}\varepsilon^{bd}W_{12}(u)A,$$
$$\{Q^{\alpha b}, S^{\gamma d}\} = -\varepsilon^{\alpha\gamma}R^{bd} + \varepsilon^{bd}L^{\alpha\gamma} - \varepsilon^{\alpha\gamma}\varepsilon^{bd}W_{11}(u)A,$$
$$\{S^{\alpha b}, S^{\gamma d}\} = -\varepsilon^{\alpha\gamma}\varepsilon^{bd}W_{21}(u)A, \tag{6.7}$$
$$[B, Q^{\alpha b}] = W_{11}(u)Q^{\alpha b} + W_{12}(u)S^{\alpha b},$$
$$[B, S^{\alpha b}] = W_{21}(u)Q^{\alpha b} + W_{22}(u)S^{\alpha b}.$$

---

[14] For convenience, one might absorb several factors of i into the definition of q, A, B, α.

Furthermore, the matrix W in (3.18) reduces to[15]

$$W = \begin{pmatrix} +iu & 2\alpha \\ 2\alpha^{-1} & -iu \end{pmatrix}. \tag{6.8}$$

The limit of the universal trigonometric r-matrix in (3.22) reads

$$\tilde{r} = \frac{1}{u_1 - u_2} \tilde{t}. \tag{6.9}$$

When expanded into loop levels using a geometric series (cf. section 3.4) one finds the analogue of (3.31):

$$\tilde{r} = + \sum_{k=0}^{\infty} \left[-\varepsilon_{ac}\varepsilon_{bd} R_{-1-k}^{ab} \otimes R_{+k}^{cd} + \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} L_{-1-k}^{\alpha\beta} \otimes L_{+k}^{\gamma\delta} - \varepsilon_{\alpha\gamma}\varepsilon_{bd} Q_{-1-k}^{\alpha b} \otimes S_{+k}^{\gamma d} \right.$$
$$\left. + \varepsilon_{\alpha\gamma}\varepsilon_{bd} S_{-1-k}^{\alpha b} \otimes Q_{+k}^{\gamma d} - A_{-1-k} \otimes B_{+k} - B_{-1-k} \otimes A_{+k} \right]. \tag{6.10}$$

Affine extension. The loop variable z is replaced by u according to (6.1):

$$z = 1 + i\epsilon u + \mathcal{O}(h^2). \tag{6.11}$$

After a rescaling

$$\tilde{D} = i\epsilon D, \tag{6.12}$$

the affine derivation (4.5) transforms into a derivative w.r.t. u:

$$[\tilde{D}, u] = 1 \quad \text{or} \quad \tilde{D} \simeq \frac{d}{du}. \tag{6.13}$$

The structure of the affine derivations remains the same as in (4.10):

$$\begin{aligned} [\tilde{D}, Q^{\alpha b}] &= \tilde{U}_{11}(u) Q^{\alpha b} + \tilde{U}_{12}(u) S^{\alpha b}, \\ [\tilde{D}, S^{\alpha b}] &= \tilde{U}_{21}(u) Q^{\alpha b} + \tilde{U}_{22}(u) S^{\alpha b}, \\ [\tilde{D}, A] &= +\tilde{V}(u) A, \\ [\tilde{D}, B] &= -\tilde{V}(u) B, \end{aligned} \tag{6.14}$$

whereas for the central charges in (4.14) one has to replace dz/z by du:

$$\begin{aligned} \{f(u)Q^{\alpha b}, g(u)Q^{\gamma d}\} &= \varepsilon^{\alpha\gamma}\varepsilon^{bd} f(u)g(u) W_{12}(u) A \\ &\quad + \varepsilon^{\alpha\gamma}\varepsilon^{bd} \frac{1}{2\pi i} \oint (f(u)g(u)U_{12}(u)\, du) C, \\ \{f(u)Q^{\alpha b}, g(u)S^{\gamma d}\} &= f(u)g(u)(-\varepsilon^{\alpha\gamma} R^{bd} + \varepsilon^{bd} L^{\alpha\gamma} - \varepsilon^{\alpha\gamma}\varepsilon^{bd} W_{11}(u) A) \\ &\quad + \varepsilon^{\alpha\gamma}\varepsilon^{bd} \frac{1}{2\pi i} \oint (f(u)\, dg(u) - f(u)g(u)U_{11}(u)\, du) C, \\ \{f(u)S^{\alpha b}, g(u)S^{\gamma d}\} &= -\varepsilon^{\alpha\gamma}\varepsilon^{bd} f(u)g(u) W_{21}(u) A \\ &\quad + \varepsilon^{\alpha\gamma}\varepsilon^{bd} \frac{1}{2\pi i} \oint (-f(u)g(u)U_{21}(u)\, du) C, \\ [f(u)A, g(u)B] &= -\frac{1}{2\pi i} \oint (f(u)\, dg(u) - V(u)f(u)g(u)\, du) C. \end{aligned} \tag{6.15}$$

The parameters have to be rescaled w.r.t. (4.8); now they read, see also [33],

$$\tilde{U} = i\epsilon U = \frac{1}{u^2 - 4} \begin{pmatrix} 0 & -i\alpha \\ +i\alpha^{-1} & 0 \end{pmatrix}, \qquad \tilde{V} = i\epsilon V = -\frac{u}{u^2 - 4}. \tag{6.16}$$

---

[15] The generators B, A are shifted by the one loop level w.r.t. the corresponding ones in [24], i.e. W differs by a factor of u.

Note that the non-vanishing of the above parameters leads to the non-invariance of the r-matrix under $\tilde{D}$ and thus to a non-trivial cobracket (see section 4). When $\tilde{D}$ is interpreted as a two-dimensional (Lorentz) boost, the corresponding (Lorentz) symmetry would be deformed along the lines discussed in [33, 34].

It appears that the exponentiated affine derivation $\exp(\frac{i}{2}g^{-1}\tilde{D})$ (note that exponentiated generators naturally appear in quantum algebras, see section 2) plays an important role in the quantization of the algebra. The generator induces a finite shift of u by an amount which frequently occurs in the quantum R-matrix, e.g.

$$x^{\pm}(u) = x\left(u \pm \frac{i}{2}g^{-1}\right) = \exp\left(\pm\frac{i}{2}g^{-1}\tilde{D}\right)x(u). \tag{6.17}$$

It would be interesting to pursue the role of the affine derivation further.

6.2. Conventional rational case

The simplest limit of the fundamental r-matrix is obtained when the two parameters x_k approach each other at a generic point x_0:

$$x = x_0(1 + \epsilon u). \tag{6.18}$$

The r-matrix diverges in the limit $\epsilon \to 0$ and one obtains a rational r-matrix $\tilde{r}$ (6.4):

$$\tilde{r} = \frac{-hh'\epsilon\left(x_0^2 - 1\right)}{(hx_0 + ih')(h'x_0 - ih)}r. \tag{6.19}$$

The new coefficient functions all have the same simple singularity at $u_1 = u_2$:

$$\tilde{A}_{12} = \tilde{B}_{12} = \tilde{D}_{12} = \tilde{E}_{12} = \frac{\gamma_2}{\gamma_1}\tilde{H}_{12} = \frac{\gamma_1}{\gamma_2}\tilde{K}_{12} = \frac{1}{u_1 - u_2}, \quad \tilde{C}_{12} = \tilde{F}_{12} = \tilde{G}_{12} = \tilde{L}_{12} = 0. \tag{6.20}$$

This r-matrix is the fundamental representation of the classical rational r-matrix for the conventional affine $\mathfrak{gl}(2|2)$ algebra.

Interestingly, the parameter h has dropped out completely from the r-matrix $\tilde{r}$ and from the associated affine bialgebra. However, this does not mean that the limit is the same for all h and for all x_0. In particular one can see that the prefactor in (6.19) is singular at certain points x_0, namely $x_0 = -ih'/h$, $x_0 = ih/h'$ and $x_0 = \pm 1$. The first pair of points corresponds to $z_0 = \infty$ and the second pair to the self-dual points $z_0 = z_{\pm}^* = (ih \pm h')^2$ discussed around (5.8). In the following, we shall discuss the limits at these points.

6.3. Conventional trigonometric case

Let us next discuss the point $z_0 = \infty$. The point $z_0 = 0$ is analogous according to the discussion in section 5.2, and there is no need to discuss it separately. Similarly, we can safely restrict to one of the two corresponding points $x = ih/h'$ and $x = -ih'/h$, cf section 5.1. Here we take the limit

$$x = \frac{ih}{h'}\left(1 + \frac{\epsilon}{\tilde{z}} + \mathcal{O}(\epsilon^2)\right) \qquad z = \epsilon^{-1}\tilde{z}. \tag{6.21}$$

At the same time, the parameter α should scale like $\alpha \sim \epsilon^{-1}$. In this case, the r-matrix remains finite in the limit $\epsilon \to 0$. Thus, the trigonometric structure in table 1 applies, and its

coefficients in table 2 reduce to

$$A_{12} = B_{12} = D_{12} = E_{12} = \frac{\frac{1}{2}\tilde{z}_1 + \frac{1}{2}\tilde{z}_2}{\tilde{z}_1 - \tilde{z}_2}, \quad C_{12} = F_{12} = 0, \tag{6.22}$$

$$G_{12} = -L_{12} = +\frac{1}{4}, \quad H_{12} = \frac{\gamma_1}{\gamma_2}\frac{\tilde{z}_1}{\tilde{z}_1 - \tilde{z}_2}, \quad K_{12} = \frac{\gamma_2}{\gamma_1}\frac{\tilde{z}_2}{\tilde{z}_1 - \tilde{z}_2}. \tag{6.22}$$

These coefficients are precisely the coefficients of the conventional trigonometric r-matrix for $\mathfrak{gl}(2|2)$. The underlying algebraic structure is thus the standard affine $\mathfrak{gl}(2|2)$ algebra with the trigonometric r-matrix.

6.4. Twisted rational case

The self-dual points $z_\pm^*$ lead to a more elaborate limit. According to section 5.2 the two limits are equivalent and we choose to investigate

$$z_0 = (\mathrm{i}h + h')^2, \qquad x_0 = +1. \tag{6.23}$$

The limit is defined by

$$x = 1 + \epsilon\frac{h' - \mathrm{i}h}{h'}y, \qquad z = z_0\left(1 + \frac{\mathrm{i}h\epsilon^2}{h'}u\right), \qquad u = y^2, \qquad \alpha = \frac{\mathrm{i}\epsilon}{h' - \mathrm{i}h}\tilde{\alpha}. \tag{6.24}$$

Here, the r-matrix diverges quadratically

$$\tilde{r} = \frac{\mathrm{i}h\epsilon^2}{h'}r \tag{6.25}$$

with $\tilde{r}$ a rational r-matrix of the form (6.4). The coefficients of this fundamental r-matrix read

$$
\begin{aligned}
\tilde{A}_{12} &= \tilde{D}_{12} = \frac{1}{4y_1y_2}\frac{y_1 + y_2}{y_1 - y_2}, \\
\tfrac{1}{2}(\tilde{A}_{12} + \tilde{B}_{12}) &= \tfrac{1}{2}(\tilde{D}_{12} + \tilde{E}_{12}) = \frac{1}{y_1^2 - y_2^2}, \\
\tfrac{1}{2}(\tilde{A}_{12} - \tilde{B}_{12}) &= \tfrac{1}{2}(\tilde{D}_{12} - \tilde{E}_{12}) = \frac{1}{4y_1y_2}\frac{y_1 - y_2}{y_1 + y_2}, \\
\frac{\tilde{\alpha}}{2\gamma_1\gamma_2}\tilde{C}_{12} &= -\frac{2\gamma_1\gamma_2 y_1 y_2}{\tilde{\alpha}}\tilde{F}_{12} = \frac{1}{2}\frac{1}{y_1 + y_2}, \\
\tilde{G}_{12} &= -\tilde{L}_{12} = \frac{1}{4y_1y_2}, \\
\frac{\gamma_2 y_2}{\gamma_1}\tilde{H}_{12} &= \frac{\gamma_1 y_1}{\gamma_2}\tilde{K}_{12} = \frac{1}{2}\frac{1}{y_1 - y_2}.
\end{aligned}
\tag{6.26}
$$

These coefficients along with the rational r-matrix structure in (6.4) agree with equations (4.2) and (4.10) in [35] when setting $y = 1/2p_-$, $\gamma = \sqrt{\tilde{\alpha}p_-}$. This case therefore provides the classical s-matrix for strings in $\mathrm{AdS}_5 \times S^5$ in the near flat space limit [36].

In order to understand the algebra underlying this r-matrix, we consider the coefficients (3.5) and (3.7) for the fundamental representation first. It turns out that a and b are finite while c and d diverge. In combination with a and b one can nevertheless find finite combinations $\tilde{c}$ and $\tilde{d}$:

$$a = \gamma, \quad b = \frac{\tilde{\alpha}}{2y\gamma}, \quad \tilde{c} = c - \frac{a}{\tilde{\alpha}\epsilon} = -\frac{\gamma y}{\tilde{\alpha}}, \quad \tilde{d} = d - \frac{b}{\tilde{\alpha}\epsilon} = \frac{1}{2\gamma}, \quad \tilde{q} = \frac{\mathrm{i}\epsilon}{x_0}q = \frac{1}{2y}. \tag{6.27}$$

In the matrix notation (3.18), this corresponds to a multiplication by a matrix R:

$$\tilde{T} = RT, \qquad R = \begin{pmatrix} 1 & 0 \\ \tilde{\alpha}^{-1}\epsilon^{-1} & 1 \end{pmatrix}. \tag{6.28}$$

This implies that we should consider the following redefined generators:

$$\begin{pmatrix} \tilde{Q}^{\alpha b} \\ \tilde{S}^{\alpha b} \end{pmatrix} = R \begin{pmatrix} Q^{\alpha b} \\ S^{\alpha b} \end{pmatrix}, \qquad \tilde{A} = \frac{i\epsilon}{x_0}A, \qquad \tilde{B} = -\frac{ix_0}{\epsilon}B. \tag{6.29}$$

They have a well-defined algebra in the limit $\epsilon \to 0$, cf (3.20) and (3.18) with the new matrix

$$\tilde{W} = \frac{x_0}{i\epsilon}RWR^{-1} = \begin{pmatrix} 0 & 2\tilde{\alpha} \\ 2\tilde{\alpha}^{-1}u & 0 \end{pmatrix}. \tag{6.30}$$

Next we consider the limit of the affine extension of the algebra. The affine derivation must be rescaled

$$\tilde{D} = \frac{ih\epsilon^2}{h'}D \simeq \frac{d}{du}. \tag{6.31}$$

The action on the generators is defined by (6.14) with coefficients $\tilde{U}$, $\tilde{V}$ (4.8) limiting to

$$\tilde{U} = \frac{ih\epsilon^2}{h'}RUR^{-1} = \frac{1}{4u}\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}, \qquad \tilde{V} = \frac{ih\epsilon^2}{h'}V = -\frac{1}{2u}. \tag{6.32}$$

The above algebra is in fact a twisted affine algebra. This can be observed if we write the Lie brackets in terms of the generators

$$\bar{Q} = u^{+1/4}\tilde{Q}, \qquad \bar{S} = u^{-1/4}\tilde{S}, \qquad \bar{A} = u^{+1/2}\tilde{A}, \qquad \bar{B} = u^{-1/2}\tilde{B}. \tag{6.33}$$

Now the above algebra is defined by the parameters

$$\bar{W} = \begin{pmatrix} 0 & 2\tilde{\alpha} \\ 2\tilde{\alpha}^{-1} & 0 \end{pmatrix}, \qquad \bar{U} = \bar{V} = 0. \tag{6.34}$$

That is, the loop levels of the generators add up simply, and the affine extension acts canonically. The automorphism defining the above twist has a period of 4. It corresponds to an *outer* $\mathbb{Z}_2$-automorphism of $\mathfrak{gl}(2|2)$ which acts non-trivially on one of the two $\mathfrak{sl}(2)$ subalgebra. Note that for the simple superalgebra $\mathfrak{psl}(2|2)$ the corresponding automorphism is inner [32], so the non-triviality of the twist is only due to the central charge A and the derivation B.

6.5. Twisted trigonometric case

We have exhausted all the special values of z for generic values of the global parameter h. As h varies also the self-dual points (5.8)

$$z_{\pm}^* = \left(ih \pm h'\right)^2 \tag{6.35}$$

move around in the complex plane while the special values $z = 0$ and $z = \infty$ remain fixed. For particular values of h, namely $h = 0, \pm 1, \infty$, some of the special values coincide giving rise to further limits of interest. We have seen in section 5.4 that the points $h = \pm 1$ are equivalent to $h = 0$; consequently, there is no need to discuss them separately.

Let us first consider the case $h = \infty$. There both self-dual points approach the other two special values, $z_+^* = 0$, $z_-^* = \infty$. Now we take the limit

$$h = \epsilon^{-1}, \qquad z \sim \epsilon^0, \qquad \epsilon \to 0. \tag{6.36}$$

Here the parameters (3.5) and (3.7) of the fundamental representation (3.13) read

$$a = \gamma, \quad b = \frac{\alpha}{2\gamma y}, \quad c = -\frac{\gamma y}{\alpha}, \quad d = \frac{1}{2\gamma}, \quad x = 1 - \frac{\epsilon}{y}, \quad z = y^2, \quad q = \frac{1}{2y}, \tag{6.37}$$

and the fundamental classical r-matrix in tables 1 and 2 takes the same form using these simplified parameters a, b, c, d, q. In particular, we find

$$
\begin{aligned}
A_{12} = D_{12} &= \frac{1}{4}\frac{y_1 + y_2}{y_1 - y_2}, \\
\frac{1}{2}(A_{12} + B_{12} + 1) = \frac{1}{2}(D_{12} + E_{12} + 1) &= \frac{y_1^2}{y_1^2 - y_2^2}, \\
\frac{1}{2}(A_{12} + B_{12} - 1) = \frac{1}{2}(D_{12} + E_{12} - 1) &= \frac{y_2^2}{y_1^2 - y_2^2}, \\
\frac{1}{2}(A_{12} - B_{12}) = \frac{1}{2}(D_{12} - E_{12}) &= -\frac{1}{4}\frac{y_1 - y_2}{y_1 + y_2}, \\
\frac{\alpha}{2\gamma_1\gamma_2 y_1 y_2}C_{12} = -\frac{2\gamma_1\gamma_2}{\alpha}F_{12} &= -\frac{1}{2}\frac{1}{y_1 + y_2}, \\
G_{12} = L_{12} &= 0, \\
\frac{\gamma_2}{y_1\gamma_1}H_{12} = \frac{\gamma_1}{y_2\gamma_2}K_{12} &= \frac{1}{2}\frac{1}{y_1 - y_2}.
\end{aligned}
\tag{6.38}
$$

These coefficients are reminiscent of those for the twisted rational r-matrix in (6.26). In fact, the representation parameters in (6.37) agree precisely with those in (6.27). Effectively, it means that the two algebras are equivalent (up to the affine extensions). The parameters therefore read

$$
W = \begin{pmatrix} 0 & 2\alpha \\ 2\alpha^{-1}z & 0 \end{pmatrix}, \qquad U = \frac{1}{4}\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}, \qquad V = -\frac{1}{2}. \tag{6.39}
$$

As explained in section 6.4, they describe a $\mathbb{Z}_2$-twisted affine $\mathfrak{gl}(2|2)$ algebra. A quantum R-matrix for this algebra was derived in [37]. Therefore, one would expect that its classical limit is related to the trigonometric r-matrix described above.

It is curious to observe that the coefficients in (6.38) match almost exactly with those found for the scattering matrix derived in (4.9)–(4.11) in [38] including the functional form of its prefactor when equating $y_k = \exp(\theta_k)$. Nevertheless, there is a crucial difference. The s-matrix in [38] is based on the rational structure (6.4) with manifest $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ symmetry, while the coefficients are intimately associated with the trigonometric structure in table 1 with broken $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Effectively K_2 in [38] compares to $\frac{1}{2}(A_{12} + B_{12})$ rather than $\frac{1}{2}(A_{12} + B_{12} \pm 1)$.

6.6. Special trigonometric case at $h = \infty$

The value $h = \infty$ considered above is subtle, and the result depends on the details of taking the limit $h \to \infty$. Previously we have assumed that z remains finite, but there is also the option of scaling $z \to 0$ or $z \to \infty$ in correlation with $h \to \infty$. Moreover, the result generally depends on how fast z converges in comparison to h. A suitable limit with $z \to 0$ (according to section 5.2 this is equivalent to $z \to \infty$) turns out to be

$$
h = \epsilon^{-1}, \qquad z = -\tfrac{1}{4}\epsilon^2\tilde{z}, \qquad x \sim \epsilon^0, \qquad \alpha = \tfrac{1}{2}\epsilon\tilde{\alpha}. \tag{6.40}
$$

This limit is distinguished from the previous one by the fact that one of the self-dual points (5.8) remains finite while the other approaches infinity:

$$
\tilde{z}^*_- = 1, \qquad \tilde{z}^*_+ \sim 16\epsilon^{-4}. \tag{6.41}
$$

The parameters a, b, c, d (3.5) for the supercharges in the fundamental representation remain finite:

$$a = \gamma, \qquad b = -\frac{\tilde{\alpha}}{2\gamma}\frac{x-1}{x+1}, \qquad c = \frac{2\gamma}{\tilde{\alpha}}\frac{1}{x-1}, \qquad d = \frac{1}{\gamma}\frac{x}{x+1}, \qquad (6.42)$$

while the parameters q, z are singular and must be renormalized

$$\tilde{q} = \epsilon q = -\frac{x-1}{x+1}, \qquad \tilde{z} = -4\epsilon^{-2}z = -\frac{4x}{(x-1)^2}. \qquad (6.43)$$

Using these parameters, the fundamental r-matrix takes the same form as in table 1.

For the algebra, we have to rescale some generators

$$\tilde{A} = \epsilon A, \qquad \tilde{B} = \epsilon^{-1}B. \qquad (6.44)$$

The parameters U, V, W in (3.18) and (4.8) for the Lie brackets (3.20), (4.10) and (4.14) read in this case

$$\tilde{W} = \epsilon^{-1}W = \begin{pmatrix} -1 & +\tilde{\alpha} \\ -\tilde{\alpha}^{-1}\tilde{z} & +1 \end{pmatrix}, \quad U = \frac{1}{4}\frac{\tilde{z}}{\tilde{z}-1}\begin{pmatrix} -1 & 0 \\ -2\tilde{\alpha}^{-1} & +1 \end{pmatrix}, \quad V = -\frac{1}{2}\frac{\tilde{z}}{\tilde{z}-1}. \quad (6.45)$$

6.7. Special trigonometric case at $h = 0$

The limit $h \to 0$ was discussed already in section 6.1; it yields the full rational r-matrix [23, 24]. In this limit, it was assumed that x remains finite whereas $z \to 1$. Likewise one can demand that z remains finite and arbitrary while $x \to 0$ or $x \to \infty$; this turns out to yield an inequivalent limit. Let us consider the case of large x

$$h = \epsilon, \qquad z \sim \epsilon^0, \qquad x = -i\epsilon^{-1}\frac{z-1}{z} + \mathcal{O}(\epsilon^0). \qquad (6.46)$$

Then the parameters of the fundamental representation (3.5) and (3.7) read

$$a = \gamma, \qquad b = c = 0, \qquad d = \frac{1}{\gamma}, \qquad \tilde{q} = \epsilon^{-1}q = \frac{1}{z-1}. \qquad (6.47)$$

This is almost the fundamental representation of the standard $\mathfrak{gl}(2|2)$, but the central charge $\tilde{q}$ behaves differently. Consequently, the r-matrix coefficients in table 2 take a slightly non-standard form. The case of $x \to 0$ leads to the conjugate fundamental representation.

Next, let us consider the algebra. In this case, we should rescale the generators A and B according to

$$\tilde{B} = \epsilon B, \qquad \tilde{A} = \epsilon^{-1}A \qquad (6.48)$$

in order to make their action finite. The algebra now takes the standard form with the parameters (3.18) and (4.8):

$$\tilde{W} = \epsilon W = (z-1)\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad U = 0, \qquad V = -\frac{z}{z-1}. \qquad (6.49)$$

All the off-diagonal elements of the matrices are absent as in the conventional affine $\mathfrak{gl}(2|2)$. Only the central charge A appears with a non-trivial dependence on the loop variable z.

In fact, we can formally make all the algebra relations like those for affine $\mathfrak{gl}(2|2)$ by redefining the loop levels of $\tilde{A}$ and $\tilde{B}$:

$$\bar{B} = (z-1)^{-1}\tilde{B}, \qquad \bar{A} = (z-1)\tilde{A}. \qquad (6.50)$$

This leads to the standard affine algebra with parameters:

$$\bar{W} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \bar{U} = \bar{V} = 0. \qquad (6.51)$$

The simplification is however at the cost of changing the universal r-matrix in (3.15) because the transformation does not respect the decomposition (3.33).

6.8. Special rational case

There is even a combination of the two different limits at $h \to 0$. Here h should approach 0 faster than z approaches 1. For example we can define the limit

$$h = \epsilon^2, \qquad z = 1 + i\epsilon u + \mathcal{O}(\epsilon^2), \qquad x = \frac{u}{\epsilon} + \mathcal{O}(\epsilon^0). \tag{6.52}$$

The parameters of the fundamental representation reduce to

$$a = \gamma, \qquad b = c = 0, \qquad d = \frac{1}{\gamma}, \qquad \tilde{q} = \frac{1}{i\epsilon}q = -\frac{1}{u}. \tag{6.53}$$

The r-matrix diverges and becomes of rational type (6.4):

$$\tilde{r} = i\epsilon r. \tag{6.54}$$

The coefficients are almost those of the conventional rational r-matrix, but there are a few important modifications:

$$\frac{1}{2}(\tilde{A}_{12} + \tilde{B}_{12}) = \frac{1}{2}(\tilde{D}_{12} + \tilde{E}_{12}) = \frac{\gamma_2}{\gamma_1}\tilde{H}_{12} = \frac{\gamma_1}{\gamma_2}\tilde{K}_{12} = \frac{1}{u_1 - u_2}, \qquad \tilde{C}_{12} = \tilde{F}_{12} = 0,$$

$$\frac{1}{2}(\tilde{A}_{12} - \tilde{B}_{12}) = \frac{1}{2}(\tilde{D}_{12} - \tilde{E}_{12}) = \frac{u_1 - u_2}{4u_1 u_2}, \qquad \tilde{G}_{12} = -\tilde{L}_{12} = \frac{u_1 + u_2}{4u_1 u_2}. \tag{6.55}$$

The algebra is specified by the following parameters:

$$\tilde{W} = i\epsilon W = u \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}, \qquad \tilde{U} = i\epsilon U = 0, \qquad \tilde{V} = i\epsilon V = -\frac{1}{u}. \tag{6.56}$$

This case may be viewed as the rational analogue of the special trigonometric case at $h = 0$ in section 6.7.

6.9. Summary

In this section, we have found more than a handful special limits of the r-matrix. What makes these limits special and how can we be sure that we have not missed an interesting case? To answer the question we should consider special points in the z-plane. The affine algebra specializes the two points $z_\pm^\circ = 0, \infty$. Furthermore, there are two points $z_\pm^*$ which lead to certain self-duality properties of representations, see (5.8). In total, there are four special points:

$$z_\pm^\circ = 0, \infty, \qquad z_\pm^* = (ih \pm h')^2. \tag{6.57}$$

Above we have constructed limits by zooming into the neighbourhood of certain points while potentially taking a simultaneous limit for h. There is however a different point of view which makes the various limits more transparent. By zooming into the neighbourhood of one point, we effectively shift all other special points to the point at infinity. Hence, the various limits correspond to grouping the special points in different ways.

What is the role of the parameter h in the limits? Zooming into a neighbourhood can be achieved by Möbius transformations of the z-plane with coefficients depending on the limiting procedure. The transformation maps the special points to different positions, but there exists one conformal cross-ratio which remains invariant. Its value $s = (ih + h')^4$ is a function of h. Alternatively one can consider $h = h(s)$ to be a function of the cross-ratio s. This allows us to view the four special points as independent, and $h = h(z_\pm^\circ, z_\pm^*)$ as a function of their distribution modulo Möbius transformations.

To understand the various limits, we should group the four special points in all possible ways. Up to trivial permutations there are nine choices corresponding to the full trigonometric
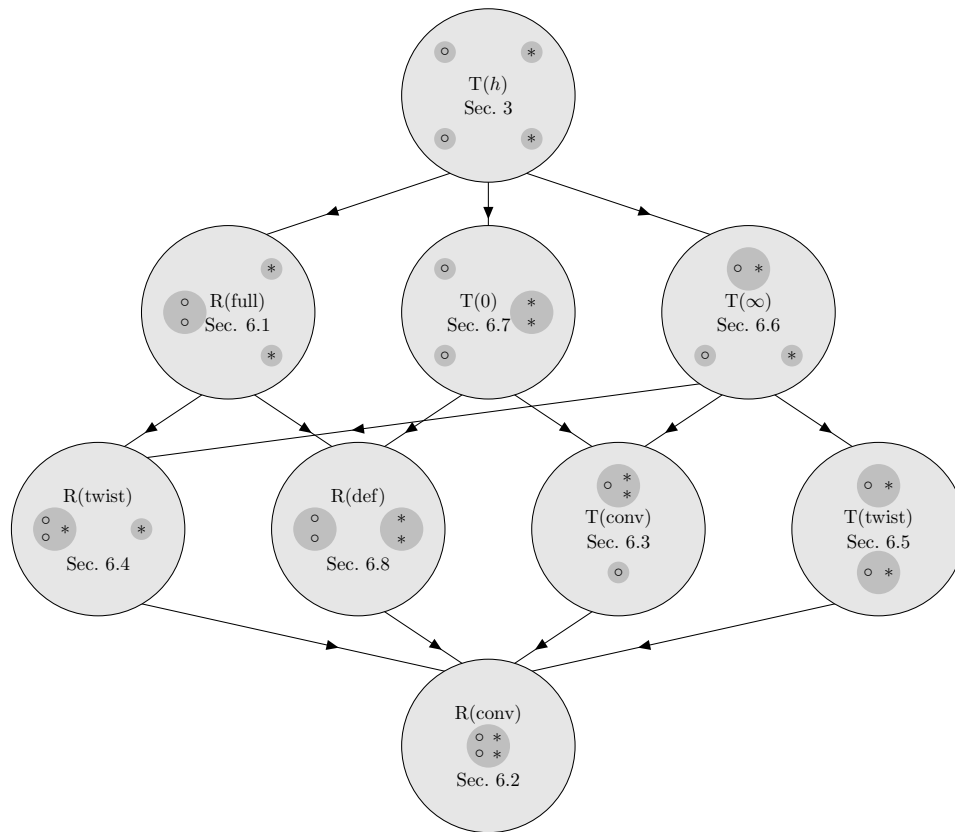
Figure 3. Analytic structure and limits of various *r*-matrices. T(*h*): full trigonometric (section 3); T(0): special trigonometric at $h = 0$ (section 6.7); T(∞): special trigonometric at $h = \infty$ (section 6.6); T(twist): twisted trigonometric (section 6.5); T(conv): conventional trigonometric (section 6.3); R(full): full rational (section 6.1); R(twist): twisted rational (section 6.4); R(def): special rational (section 6.8); R(conv): conventional rational (section 6.2). Special points $z_\pm^\circ = 0, \infty$ and $z_\pm^*$ are marked by $\circ$ and $*$, respectively. A circle is drawn around coincident special points. Two cases are connected by an arrow if the second is a particular limit of the first.

case with parameter *h* and its eight limiting cases considered above, see figure 3. Note that the trigonometric cases have two distinct points $z_\pm^\circ$ while the rational cases have identical points $z_+^\circ = z_-^\circ$. Two cases are linked by a limiting procedure if the special points of the first can be combined to the special points of the second.

7. Conclusions and outlook

Classical *r*-matrices for Lie algebras were classified in [39]. Three main classes, distinguished by the distribution of poles in the complex plane, were identified: rational, trigonometric and elliptic. The classification is analogous for simple Lie superalgebras [6]. In the case of the (non-simple) Lie superalgebra $\mathfrak{gl}(2|2)$, an exceptional *r*-matrix was identified in [24]. This *r*-matrix is of rational type, but it is not of difference form. Its quantization leads to Shastry's *R*-matrix for the Hubbard model [4] or equivalently [9] to the *S*-matrix for the AdS/CFT integrable system [10]. Hence, this *r*-matrix is responsible for the exceptional integrable structure in these models at the classical level.

In this paper, we have developed and investigated the trigonometric generalization of the exceptional r-matrix for $\mathfrak{gl}(2|2)$. The corresponding fundamental quantum R-matrix was derived in [21], and it defines the integrable structure of the Alcaraz–Bariev model [22] (type B). As for the rational case, the underlying Lie algebra is a deformation of the loop algebra $\mathfrak{gl}(2|2)[z, z^{-1}]$. The deformation is special in the sense that the Lie brackets are not homogeneous in the level of the loop algebra. Nevertheless, the algebra admits solutions to the classical Yang–Baxter equation. Analogously, it admits a decomposition into positive and negative subalgebras. An interesting feature of the algebra is that it has one modulus h whose value has a significant impact on the algebra. One may wonder whether there are other similar cases of deformed loop algebras or if $\mathfrak{gl}(2|2)$ is truly exceptional in this regard. In other words, which is the precise (co)homological property of $\mathfrak{gl}(2|2)$ or its loop algebra giving rise to the deformation?

The deformed loop algebra also admits the extension by a derivation and a central charge to an affine Lie algebra. This algebra is not of Kac–Moody type, but its structure is similar in many respects. The affine derivation serves as a scaling of the loop variable z (or a shift in u in the rational case). In a physical scattering context, it can be viewed as a boost operator in analogy to Lorentz boosts in two spacetime dimensions. Also we must extend the notion of particles to fields, because the particle momentum does not commute with boosts. Interestingly, the boost has non-trivial cobrackets; hence, the symmetry should be viewed as deformed or non-commutative [33, 34]. Non-invariance of the r-matrix also explains the violation of the difference form for the r-matrix. Finally, the extension of a symmetry often leads to additional restrictions. Here it would be interesting to see if, e.g., the overall prefactor of the r-matrix can be constrained by the affine extension.

Subsequently, we have investigated discrete transformations and special points of the r-matrix. Transformations include conjugation, inversion of the loop variable, a flip of statistics and a duality for the global parameter. Conjugation maps different representations into each other. In particular, the family of fundamental representations is self-conjugate, and thus conjugation extends to a crossing symmetry of the r-matrix, cf [40]. Inversion symmetry of the r-matrix can be viewed as a scattering unitarity condition. The statistics flip interchanges bosons and fermions in the fundamental representation. At the level of the algebra, it permutes the two $\mathfrak{sl}(2)$ subalgebras. Last but not least, the duality map relates algebras/r-matrices with different moduli h. An important insight gained from the discrete transformations is that next to the special points $z = z^\circ_\pm = 0, \infty$, which exist for any trigonometric r-matrix, there are two self-dual points $z = z^*_\pm$ whose value depends on h.

Finally, several r-matrices with simpler structures were recovered as limiting cases. For example, our trigonometric r-matrix reduces to the exceptional rational r-matrix of [23, 24] in a particular limit. The latter can be reduced further to the conventional rational $\mathfrak{gl}(2|2)$ r-matrix as well as to two other intermediate cases. In total there is the one-parameter family of exceptional trigonometric r-matrices and eight singular cases, see figure 3. The trigonometric family has the most sophisticated structure while the conventional rational r-matrix is the plainest. All intermediate cases can be obtained from the former and be reduced to the latter. These include some special cases with the $\mathfrak{gl}(2|2)$ structure discovered earlier in various contexts. They can be of trigonometric or of rational type, conventional or deformed, and untwisted or $\mathbb{Z}_2$-twisted. In terms of algebra, all cases follow from the one discussed in this paper. Its structure can be simplified through limits and algebraic contractions down to the plain $\mathfrak{gl}(2|2)$ affine Kac–Moody algebra. It would be interesting to find out whether the trigonometric structure is itself a limiting case of some exceptional elliptic r-matrix (note that both $\mathfrak{psl}(2|2)$ and $\mathfrak{osp}(4|2)$ admit elliptic r-matrices [6]).
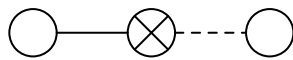
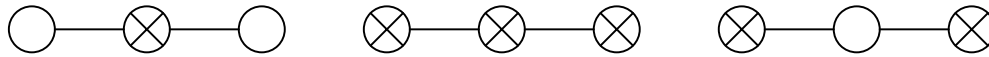**Figure 4.** Dynkin diagram for $\mathfrak{d}(2, 1; 0)$.

Figure 5. All Dynkin diagrams for $\mathfrak{sl}(2|2)$.

With a good part of the classical framework established, several open questions concerning the exceptional trigonometric r-matrix remain. For instance, we would like to promote the Lie bialgebra to a quantum affine Hopf algebra (cf. [41]). Are there any obstacles due to the non-standard structure of the affine algebra? So far only the fundamental quantum R-matrix has been established. However, there is little doubt that R-matrices for higher representations can indeed be constructed as in the rational case [9, 19, 20]. This would be very suggestive of a universal R-matrix[16].

Developing the quantum affine algebra would establish, as a by-product, the Yangian for the undeformed Hubbard model or for integrable scattering in AdS/CFT. One complication in the formulation might reside in the existence of the tower of derivations z^nB for which Drinfeld's first presentation [7] using Chevalley–Serre generators is not ideally suited. Instead, Drinfeld's second realization [42] along the lines of [17] may prove to be more helpful.

Also the choice of the Dynkin diagram may play a role. For instance, the Bethe equations [3, 43] cannot be formulated (easily) for the distinguished diagram in figure 1 (leftmost in figure 5), but it appears to prefer a structure reminiscent of the exceptional superalgebra $\mathfrak{d}(2, 1; \alpha)$ with the singular parameter $\alpha = 0$ in figure 4. The latter has a non-symmetrizable Cartan matrix, cf [44]. It would be interesting to derive the r-matrices for the various other Dynkin diagrams (see figure 5) and to understand how to transform between them, see also [45].

Acknowledgments

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