Hyperbolic Weyl groups and
the four normed division algebras

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Abstract: We study the Weyl groups of hyperbolic Kac–Moody algebras of 'over-extended' type and ranks 3, 4, 6 and 10, which are intimately linked with the four normed division algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively. A crucial role is played by integral lattices of the division algebras and associated discrete matrix groups. Our findings can be summarized by saying that the even subgroups, W^+, of the Kac–Moody Weyl groups, W, are isomorphic to generalized modular groups over $\mathbb{K}$ for the simply laced algebras, and to certain finite extensions thereof for the non-simply laced algebras. This hints at an extended theory of modular forms and functions.

"The mathematical universe is inhabited not only by important species but
also by interesting individuals" – C. L. Siegel

1   Introduction

In [20] Feingold and Frenkel gained significant new insight into the structure of a particularly interesting rank 3 hyperbolic Kac–Moody algebra which they called $\mathcal{F}$, along with some connections to the theory of Siegel modular forms of genus 2. The first vital step in their work was the discovery that the Weyl group of that hyperbolic algebra is $W(\mathcal{F}) \cong PGL_2(\mathbb{Z})$, the projective group of (2×2) integral matrices with determinant ± 1, isomorphic to the hyperbolic triangle group $T(2, 3, \infty)$. They showed that the root system of that algebra could be realized as the set of (2×2) symmetric integral matrices X with $\det(\mathtt{X}) \geq -1$, and that the action of $M \in W(\mathcal{F})$ on X is given by $M\mathtt{X}M^T$. In notation used commonly by physicists today, the hyperbolic algebra studied in [20] is designated as A_1^{++} since it is obtained from the finite dimensional Lie algebra sl_2 of type A_1 by a process of double extension. The first step of the extension gives the affine algebra $A_1^{(1)}$, and the second step, often referred to as over-extension, gives the hyperbolic algebra. In the realization of the root lattice via symmetric matrices X, the real roots consist of the integral points X with $\det(\mathtt{X}) = -1$ on a single-sheeted hyperboloid, and the imaginary roots consist of the integral points X on the light-cone $\det(\mathtt{X}) = 0$ and on the two-sheeted hyperboloids $\det(\mathtt{X}) > 0$. In [20] it was also mentioned that these results could be extended to two other (dual) rank 4 hyperbolic algebras whose Weyl groups were both the Klein–Fricke group Ψ_1^* containing as an index 4 subgroup the Picard group $\Psi_1 = PSL_2(\mathbb{Z}(i))$ whose entries are from the Gaussian integers. The results expected to hold for these rank 4 hyperbolic algebras included connections to the theory of Hermitian modular forms, but that line of research was not pursued in later work. A paper by Kac, Moody and Wakimoto [25] generalized the structural results of [20] to the hyperbolic algebra $E_{10} = E_8^{++}$, but until now there has not been any new insight into the structure of the Weyl groups of hyperbolic algebras which are usually just given as Coxeter groups defined by generators and relations.

In the present work, we take up this line of development again, and present a coherent picture for many higher rank hyperbolic Kac–Moody algebras which is based on the relation to generalized modular groups associated with lattices and subrings of the four normed division algebras. More specifically, we shall show that the Weyl groups of all hyperbolic algebras of ranks 4, 6 and 10 which can be obtained by the process of double extension described above, admit realizations in terms of generalized modular groups over the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$, respectively. For $\mathbb{K} = \mathbb{C}$, the present work on these hyperbolic Weyl groups is thus a very natural extension of [20], showing in particular that the rank 4 hyperbolics A_2^{++}, C_2^{++} and G_2^{++} are naturally connected with certain subrings in the normed division algebra $\mathbb{C}$. Analogous results are obtained for all the 'over-extended' rank 6 hyperbolics $A_4^{++}, B_4^{++}, C_4^{++}, D_4^{++}$ and F_4^{++}, whose (even) Weyl groups can be described in terms of quaternionic modular groups. Finally, $\mathbb{K} = \mathbb{O}$, the largest and non-associative division algebra of octonions, is associated with rank 10 hyperbolics, and in particular, the maximally extended hyperbolic Kac–Moody algebra E_{10}. The other two hyperbolic over-extended algebras B_8^{++} and D_8^{++} can also be

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$\mathbb{K}$	Root System $\mathfrak{g}$	$W(\mathfrak{g})$	$W^+(\mathfrak{g}^{++})$
$\mathbb{R}$	A_1	$2 \equiv \mathbb{Z}_2$	$PSL_2(\mathbb{Z})$
$\mathbb{C}$	A_2	$\mathbb{Z}_3 \rtimes 2$	$PSL_2(\mathtt{E})$
$\mathbb{C}$	$B_2 \equiv C_2$	$\mathbb{Z}_4 \rtimes 2$	$PSL_2(\mathtt{G}) \rtimes 2$
$\mathbb{C}$	G_2	$\mathbb{Z}_6 \rtimes 2$	$PSL_2(\mathtt{E}) \rtimes 2$
$\mathbb{H}$	A_4	$\mathfrak{S}_5$	$PSL_2^{(0)}(\mathtt{I})$
$\mathbb{H}$	B_4	$2^4 \rtimes \mathfrak{S}_4$	$PSL_2^{(0)}(\mathtt{H}) \rtimes 2$
$\mathbb{H}$	C_4	$2^4 \rtimes \mathfrak{S}_4$	$\widetilde{PSL}_2^{(0)}(\mathtt{H}) \rtimes 2$
$\mathbb{H}$	D_4	$2^3 \rtimes \mathfrak{S}_4$	$PSL_2^{(0)}(\mathtt{H})$
$\mathbb{H}$	F_4	$2^5 \rtimes (\mathfrak{S}_3 \times \mathfrak{S}_3)$	$PSL_2(\mathtt{H}) \rtimes 2$
$\mathbb{O}$	D_8	$2^7 \rtimes \mathfrak{S}_8$	$PSL_2^{(0)}(\mathtt{O})$
$\mathbb{O}$	B_8	$2^8 \rtimes \mathfrak{S}_8$	$PSL_2^{(0)}(\mathtt{O}) \rtimes 2$
$\mathbb{O}$	E_8	$2 \,.\, O_8^+(2) \,.\, 2$	$PSL_2(\mathtt{O})$

Table 1: *Root systems, number systems and Weyl groups within the normed division algebras. The groups in the right column are defined in detail in the relevant sections.*

described using octonions. In this paper we present the rich structure which we found in the complex and quaternionic cases, as well as partial (and intriguing) results for the octonionic case. As explained in more detail in section 3, a new feature for the division algebras beyond $\mathbb{R}$ is that simple reflections involve complex conjugation of all entries of $\mathtt{X}$. For that reason only the *even* part of a given Weyl group will act by matrix conjugation of $\mathtt{X}$ and is therefore the main focus of our study. Throughout the paper we will denote by W^+ the even part of a Weyl group W. In the case of $\mathcal{F}$ one has $W^+(\mathcal{F}) \cong PSL_2(\mathbb{Z})$, the modular group, and for the other division algebras we discover a number of apparently new modular groups.

In table 1 we summarize our findings for the (even) Weyl groups of the finite and hyperbolic Kac–Moody algebras studied in relation to the various division algebras. By $\mathfrak{g}^{++}$ we mean the over-extension of the finite dimensional Lie algebra $\mathfrak{g}$ to a hyperbolic Kac–Moody algebra, where the first extension $\mathfrak{g}^+$ is an untwisted affine algebra. In the appendix we discuss other cases where the first extension is twisted affine, so a different notation is required. In the table, we use the standard group theory notation $C = A \,.\, B$ to mean a group C which contains group A as a normal subgroup, with quotient group C/A isomorphic to B. Such a group C is called an extension of A by B. If B has order $|B|$, A is said to be of index $|B|$ in $A \,.\, B$. It can happen that the extension is a semi-direct product, so that B is a subgroup of C which acts on A via conjugation as automorphisms, and in this case the product is denoted by $A \rtimes B$. By $\mathfrak{S}_n$ we denote the symmetric group on n letters.

The various rings appearing in the table are as follows. For $\mathbb{K} = \mathbb{C}$ these are the Gaussian integers $\mathtt{G} \equiv \mathbb{Z}(i)$ and the Eisenstein integers $\mathtt{E}$; for the quaternions

$\mathbb{H}$ we use the terminology of [8], referring to the maximal order of quaternions having all coefficients in $\mathbb{Z}$ or all in $\mathbb{Z}+\frac{1}{2}$ as *Hurwitz* integers $\mathtt{H}$. To understand $W^+(A_4^{++})$ we will need the icosian quaternions $\mathtt{I}$ [7, 33]. The octonionic integers are called the *octavians* $\mathtt{O}$ [8]. The generalized modular groups that we find are all discrete as matrix groups.

Our results are complete for the cases $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{H}$. For all of these we can reformulate the even Weyl groups as new kinds of modular groups, most of which have not yet appeared in the literature so far. We note that A. Krieg [28, 29] has developed a theory of modular forms over the quaternions, however the modular groups defined there do not appear to coincide with the ones found here. Nevertheless, we will make extensive use of some results of [28] in our analysis, in particular Theorem 2.2 on p. 16.

The most interesting Weyl group, however, is the one of the maximally extended hyperbolic Kac–Moody algebra $E_{10} \equiv E_8^{++}$, where our results are still incomplete, and which provided the chief motivation for the present work. In this case, the even Weyl group is given by

$$W^+(E_{10}) \cong PSL_2(\mathtt{O}) \, . \tag{1.1}$$

Because the (2×2) matrices over the octonions obviously do not form a group, the 'modular group' over the octavians $\mathtt{O}$ appearing on the r.h.s. of this equation can so far only be defined recursively, by nested sequences of matrix conjugations with the generating (octonionic) matrices introduced in section 3 of this paper. This is in analogy with the generators and relations description of the E_{10} Weyl group as products of fundamental reflections (and also in analogy with the description of the continuous Lorentz group $SO(1, 9; \mathbb{R})$ via octonionic (2×2) matrices in [39, 32]). It remains an outstanding problem to find a more manageable realization of this modular group directly in terms of (2×2) matrices with octavian entries and the $G_2(2)$ automorphism group of the octavian integers. In the final section we present some results in this direction which we believe are new, and which may be of use in future investigations. The case E_{10} is also the most important because it has recently been shown that all simply laced hyperbolic Kac–Moody algebras can be embedded into E_{10} [40]. Because the associated Weyl groups are then subgroups of $W(E_{10})$, all these Weyl subgroups should admit an octonionic realization. Conversely, the structure and explicit realizations of modular groups found here for the quaternionic case should help in understanding the group $PSL_2(\mathtt{O})$, because all rank 6 algebras occur as subalgebras inside E_{10} (and A_4^{++} and D_4^{++} as regular subalgebras, in particular). The remarks above also apply to the Weyl groups of B_8^{++} and D_8^{++}.

Ref. [20] also highlighted a possible link with the theory of *Siegel modular forms*. In the present context, one would thus start from the Jordan algebra $H_2(\mathbb{K}_{\mathbb{C}})$ over the complexified division algebra $\mathbb{K}_{\mathbb{C}}$, and consider a generalized Siegel upper half plane for $\mathcal{Z} \in H_2(\mathbb{K}_{\mathbb{C}})$. This can be done not only for the quaternions [29], but also for $\mathbb{K} = \mathbb{O}$, where the outlines of a corresponding theory of 'Siegel modular forms' have been described and developed in [16]. However, it remains to be seen what role $PSL_2(\mathtt{O})$ as defined here has to play

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in this context, and whether results in this direction can substantially advance our understanding of the hyperbolic algebra E_{10}.

We believe that our results are also interesting in the light of recent developments in the study of black hole microstates for particular classes of black holes. It was found already in [14] that the degeneracy formula of dyonic quarter BPS black holes of $\mathcal{N} = 4$ is controlled by the denominator formula of a specific Borcherds completion of a hyperbolic subalgebra of $\mathcal{F}$, whose Weyl group is index 6 in $W(\mathcal{F})$. In more recent work [5] it was argued that the Weyl group has a more immediate interpretation in this setup as realizing the so-called attractor flow of a given dyonic solution to an 'immortal dyon' by crossing walls of marginal stability in moduli space [3, 38, 10]. The correspondence to the Weyl group is such that a point X inside the forward light-cone corresponds to dyonic solutions with the given moduli and charges. As the light-cone is tessellated by the action of the Weyl group one can move such a point to a standard fundamental domain by a finite number of Weyl reflections and the endpoint of this motion is the immortal dyon. The walls between the different cells crossed during the motion in moduli space are the walls of marginal stability. It would be interesting to study the degeneracy and the attractor flow of less supersymmetric solutions and their relation to the higher rank hyperbolic algebras and modular groups discussed in this work. The possible relevance of modular forms is also mentioned in recent work on E_{10} unification and quantum gravity [4, 12]. Weyl groups of the so-called hidden symmetry groups also appear in the context of U-duality groups [22, 17, 34, 19].[1]

The organization of the paper is as follows. In section 2.1 we fix the common notation for the four division algebras and describe the general structure of the hyperbolic root systems. Section 3 contains the central general results about the realization of the simple reflections, the even hyperbolic Weyl group and the affine and finite Weyl groups for all division algebras. The commutative division algebras $\mathbb{R}$ and $\mathbb{C}$ and the associated hyperbolic Weyl groups are treated in section 4, the algebras admitting a quaternionic realization in section 5 and the case E_8^{++} which requires the octonions in section 6. In appendix A we also present two cases involving complex numbers which are not of over-extended type but involve a twisted affine extension.

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[1] We stress, however, that the Weyl groups discussed here must not be confused with the arithmetic duality groups conjectured to be symmetries of (compactified) M theory. For instance, $W(E_{10})$ can be realized as an arithmetic subgroup of $O(1, 9)$, whereas the hypothetical arithmetic duality group $E_{10}(\mathbb{Z})$ is infinitely larger. Independently of its possible physical significance, a proper definition of this object at the very least would presuppose properly understanding the continuous group E_{10}, a goal still beyond reach.

2 Division algebras and hyperbolic root lattices

We wish to describe the Weyl groups of the various hyperbolic Kac–Moody algebras as matrix groups which are to be interpreted as modular groups. In order to accomplish this, we will describe in this section a lattice (integral linear combinations of explicit elements) in the Lorentzian space of Hermitian (2×2) matrices over one of the normed division algebras $\mathbb{K}$. This lattice will be shown to be isometric to the root lattice of the hyperbolic algebra. In the following section we will show how the Weyl group acts as on this lattice by matrix conjugation, leading to interesting matrix groups in the associative cases.

2.1 Subrings and integers in division algebras

To begin we review some facts about division algebras, their subrings and orders in certain algebraic extensions of the rational division algebras.

Let $\mathbb{K}$ be one of the four normed division algebras (over $\mathbb{R}$): $\mathbb{R}$ (the real numbers), $\mathbb{C}$ (the complex numbers), $\mathbb{H}$ (the Hamilton quaternions) or $\mathbb{O}$ (the Cayley octonions). $\mathbb{K}$ will also be a topological metric space with respect to the norm topology. Each algebra $\mathbb{K}$ has an involution, sending $a \in \mathbb{K}$ to $\bar{a}$, generalizing the complex conjugation in $\mathbb{C}$, such that the norm of $a \in \mathbb{K}$, $N(a) = a\bar{a}$, is a non-negative real number, and satisfies the composition law $N(ab) = N(a)N(b)$ for all $a, b \in \mathbb{K}$. For $a \in \mathbb{R}$, we have $a = \bar{a}$, and for $a \in \mathbb{C}$, $\bar{a}$ is the usual complex conjugation in $\mathbb{C}$. One also defines the real part $a + \bar{a} = 2Re(a) \in \mathbb{R}$ (also sometimes called trace), and the imaginary part of a is defined by $a - \bar{a} = 2Im(a)$. The real-valued symmetric bilinear form on $\mathbb{K}$

$$(a, b) = N(a + b) - N(a) - N(b) = a\bar{b} + b\bar{a} \tag{2.1}$$

is positive definite, giving $\mathbb{K}$ the structure of a real Euclidean space.

For each choice of $\mathbb{K}$ there is a standard basis such that the structure constants are all in $\{-1, 0, 1\}$, so if one takes only rational linear combinations of these basis vectors, one gets four rational normed division algebras $\mathbb{K}_\mathbb{Q}$ (over $\mathbb{Q}$). If $\mathbb{F}$ is any subfield of $\mathbb{R}$ one has the normed division algebra $\mathbb{K}_\mathbb{F}$ (over $\mathbb{F}$), consisting of all $\mathbb{F}$-linear combinations of the standard basis elements of $\mathbb{K}$, such that $\mathbb{K} = \mathbb{K}_\mathbb{F} \otimes_\mathbb{F} \mathbb{R}$. We will actually only be using a few specific choices for $\mathbb{F}$, namely, $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{5})$.

The consideration of $\mathbb{K}_\mathbb{F}$ is necessary to describe the various finite root systems which can have angles with cosines involving the listed square roots. The matrices acting on the root lattices will have entries from certain *orders* $\mathcal{O}$ within $\mathbb{K}_\mathbb{F}$. We briefly review this concept from algebraic number theory (see e.g. [36, 8]). The subfield $\mathbb{F}$ of $\mathbb{R}$, assumed to be a finite extension of $\mathbb{Q}$, determines a ring $\mathbb{S}$ of algebraic integers in $\mathbb{F}$, which for our cases are simply $\mathbb{Z}$, $\mathbb{Z}(\sqrt{2})$, $\mathbb{Z}(\sqrt{3})$ and $\mathbb{Z}((1 + \sqrt{5})/2)$.[2] Note that, as a subring of a field, $\mathbb{S}$ is always commutative. An order $\mathcal{O}$ in $\mathbb{K}_\mathbb{F}$ over $\mathbb{S}$ is a subring of $\mathbb{K}_\mathbb{F}$ (non-associative

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[2]These examples follow the general pattern of algebraic integers in $\mathbb{Q}(\sqrt{D})$ which are those elements that satisfy a monic minimal polynomial with coefficients in $\mathbb{Z}$, that is, $\mathbb{Z}(\sqrt{D})$ or $\mathbb{Z}((1 + \sqrt{D})/2)$, depending on $D(\mathrm{mod}\ 4)$.

for $\mathbb{K} = \mathbb{O}$) containing 1 and which is *finitely* generated as an $\mathbb{S}$-module (under addition). Furthermore it has to contain a basis of $\mathbb{K}_\mathbb{F}$ over $\mathbb{F}$, such that $\mathcal{O} \otimes_\mathbb{F} \mathbb{F} = \mathbb{K}_\mathbb{F}$. An order is called a *maximal order* if it is an order not properly contained in another order. Orders in rings are a generalization of the integers in the rationals, and occupy an important role in number theory.

We stress that it is important for an order $\mathcal{O}$ to be finitely generated and that this implies that elements $x \in \mathcal{O}$ satisfy a monic polynomial equation

$$x^n + c_{n-1}x^{n-1} + \cdots + c_0 = 0 \qquad \text{with } c_{n-1}, \ldots, c_0 \in \mathbb{Z}, \qquad (2.2)$$

which also can be used for defining orders for the non-associative octonions [8]. The polynomial (2.2) arises as follows. Let $B = \{x_1, \cdots, x_m\}$ be a finite set of generators for $\mathcal{O}$ over $\mathbb{S}$, so that

$$\mathcal{O} = \left\{ \sum_{i=1}^m s_i x_i \;\middle|\; s_i \in \mathbb{S} \right\}. \qquad (2.3)$$

Left multiplication by $x \in \mathcal{O}$, L_x, can be represented by a matrix $[l_{ij}]$ such that

$$x x_j = \sum_{i=1}^m l_{ij} x_i \quad \text{and all } l_{ij} \in \mathbb{S} \qquad (2.4)$$

since $\mathcal{O}$ is a subring and an $\mathbb{S}$-module. The characteristic polynomial $\det(\lambda I - L_x)$ of the linear transformation L_x is a monic polynomial of degree m with coefficients in the commutative ring $\mathbb{S}$ and which is satisfied by L_x. Since $\mathbb{K}$ is a division algebra, the polynomial is also satisfied by x, and so x satifies a monic polynomial equation with coefficients in $\mathbb{S}$. Multiplying this polynomial by the conjugate polynomial, where $\sqrt{D}$ has been replaced by $-\sqrt{D}$ in the coefficients (where $D = 2, 3, 5$), will give a polynomial of degree $2m$ for x with coefficients in $\mathbb{Z}$ as claimed.[3] It can happen that the minimal polynomial satisfied by x is of lower degree but for our purposes it is sufficient to know that some such polynomial exists. Since in all cases we consider in this paper there is finite group of units $\mathcal{E}_\mathcal{O}$ we know that the ring spanned by them under addition and multiplication will form an order in the corresponding $\mathbb{K}_\mathbb{F}$, which we interpret as integers.

Although one usually thinks of integers as being 'discrete' in some topology, note that orders need not be discrete in the norm topology. For example, the order $\mathcal{O} = \mathbb{Z}(\sqrt{2}) = \mathbb{S}$ is a dense subring of $\mathbb{F} = \mathbb{Q}(\sqrt{2})$ where $\mathbb{K} = \mathbb{R}$ (and thus also dense in $\mathbb{R}$). Although we will use in some cases such dense subrings (cf. in particular section 5.2), the Weyl groups we define will be discrete groups (in the sense that there exists a fixed $\epsilon > 0$ such that the ϵ-neighborhoods of different matrices do not intersect). As we will explain in section 3.4, this discreteness can be traced back to the discreteness of the hyperbolic root lattices.

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[3]This can also be stated as follows: Since $\mathbb{S}$ is the ring of algebraic integers over $\mathbb{F}$ the norm from $\mathbb{F}$ to $\mathbb{Q}$ sends $\mathbb{S}$ to $\mathbb{Z}$. Applying the norm to the equation satisfied by x gives another equation satisfied by x, but with coefficients in $\mathbb{Z}$.

The discreteness is also important with regard to the properly discontinuous action of the matrix group on a suitable 'upper half plane' and the existence of fundamental domains with non-empty interior.[4]

For the division algebra $\mathbb{K} = \mathbb{R}$ we take $\mathbb{F} = \mathbb{Q}$, $\mathbb{S} = \mathbb{Z}$ and $\mathcal{O} = \mathbb{Z}$. For $\mathbb{K} = \mathbb{C}$ we use different choices of $\mathbb{F}$ and order $\mathcal{O}$. Either $\mathbb{F} = \mathbb{Q}$, $\mathbb{S} = \mathbb{Z}$, such that $\mathbb{K}_{\mathbb{F}} = \mathbb{Q}(i)$ is the 'standard' rational form of $\mathbb{C}$, and the order is the ring of Gaussian integers $\mathcal{O} = \mathtt{G} \equiv \mathbb{Z}(i)$; or we choose $\mathbb{F} = \mathbb{Q}(\sqrt{3})$ so that $\mathbb{K}_{\mathbb{F}} = \mathbb{F}(i)$ is an $\mathbb{F}$-form of $\mathbb{C}$ containing the order of Eisenstein integers $\mathcal{O} = \mathtt{E} \equiv \mathbb{Z}(\sqrt[3]{1})$, where $\sqrt[3]{1} = e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2}$. Each of these is a maximal order in its algebra $\mathbb{K}_{\mathbb{F}}$. Matters get a little more complicated for $\mathbb{K} = \mathbb{H}$, as is to be expected in order to get the different rank 6 hyperbolic Weyl groups. In one case we choose $\mathbb{F} = \mathbb{Q}$, $\mathbb{S} = \mathbb{Z}$ and the (non-maximal) order is the ring of Lipschitz integers $\mathcal{O} = \mathtt{L}$ which are generated over $\mathbb{Z}$ by the standard basis elements $\{1, i, j, k\}$. The same rational form contains the maximal order of Hurwitz integers $\mathcal{O} = \mathtt{H}$ which is generated over $\mathbb{Z}$ by $\{i, j, k, \frac{1+i+j+k}{2}\}$, or by the elements given below in (5.40). In another case we add to $\mathtt{H}$ the octahedral units (see section 5.2), producing an order $\mathtt{R}$ which is generated over $\mathbb{Z}(\sqrt{2})$ by $\{a, b, ab, ba\}$ where $a = \frac{1+i}{\sqrt{2}}$ and $b = \frac{1+j}{\sqrt{2}}$. In yet another case we will need to make use of icosian units, and appropriate choices of field $\mathbb{F} = \mathbb{Q}(\sqrt{5})$, algebraic integers $\mathbb{S}$ and order $\mathcal{O}$, which are described in detail in section 5. For the octonions $\mathbb{K} = \mathbb{O}$, and to get the Weyl group of E_{10} we will need the octavians which are both discrete and a maximal order in the (non-associative) ring of rational octonions.

Finally, recall that a unit $\varepsilon \in \mathcal{O}$ is an element having a multiplicative inverse $\nu \in \mathcal{O}$ in the order, but since $1 = N(\varepsilon\nu) = N(\varepsilon)N(\nu)$ and for all the orders we use, $N(\varepsilon), N(\nu) \in \mathbb{Z}$, we find that $N(\varepsilon) = \bar{\varepsilon}\varepsilon = \varepsilon\bar{\varepsilon} = 1$. We will use the symbol $\mathcal{E}_{\mathcal{O}}$ to denote the finite group (unless $\mathbb{K} = \mathbb{O}$, where the product is not associative) of units in $\mathcal{O}$.

2.2   Root lattices

The Kac–Moody algebras we are concerned with are hyperbolic extensions of finite-dimensional algebras associated with discrete subsets of the normed division algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and, therefore, are of ranks 3, 4, 6, and 10, respectively. In the main text we consider algebras $\mathfrak{g}^{++}$ of over-extended type [21, 11] which arise from simple finite-dimensional Lie algebras $\mathfrak{g}$ by constructing the non-twisted affine extension $\mathfrak{g}^{+}$ (also often denoted by $\mathfrak{g}^{(1)}$), and connecting the hyperbolic node of the Dynkin diagram by a single line to the affine node. Our methods also apply to hyperbolic algebras which are extensions of twisted affine algebras, and we discuss examples in the appendix. But to make the required formulas more uniform we restrict ourselves to the non-twisted cases in the main text.

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[4]This can also be traced back to the fact that the matrices making up the Weyl group, though formally taking entries in some dense order, always are constructed in such a way that an underlying discrete order governs the structure, see table 1, consistent with the group multiplication laws of the matrices.

We begin with (the Jordan algebra of) all Hermitian (2×2) matrices over $\mathbb{K}$, $\mathtt{X} = \mathtt{X}^\dagger$,

$$H_2(\mathbb{K}) = \left\{ \mathtt{X} = \left[\begin{array}{cc} x^+ & z \\ \bar{z} & x^- \end{array} \right] \; \middle| \; x^+,\, x^- \in \mathbb{R},\; z \in \mathbb{K} \right\} \qquad (2.5)$$

where $\mathtt{X}^\dagger = \overline{\mathtt{X}}^T$ is the conjugate transpose of $\mathtt{X}$. $H_2(\mathbb{K})$ is equipped with a quadratic form

$$||\mathtt{X}||^2 = -2 \det(\mathtt{X}) = -2\big(x^+ x^- - z\bar{z} \big) \qquad (2.6)$$

and a corresponding symmetric bilinear form $(\mathtt{X}, \mathtt{Y}) := \frac{1}{2}(||\mathtt{X}+\mathtt{Y}||^2 - ||\mathtt{X}||^2 - ||\mathtt{Y}||^2)$ which is Lorentzian. The subspace of matrices $\mathtt{X} \in H_2(\mathbb{K})$ with $x^+ = x^- = 0$ is isomorphic to $\mathbb{K}$, and the restriction of the bilinear form to this subspace agrees with the positive definite form on $\mathbb{K}$, making the isomorphism an isometry. It is in that Euclidean subspace that we will find root systems of finite type, and extend them to hyperbolic root systems in $H_2(\mathbb{K})$.

Suppose that we have been able to find a set of simple roots $a_i \in \mathbb{K}$, $i = 1, \ldots, \ell$, where $\ell = \dim_{\mathbb{R}}(\mathbb{K})$, with Cartan matrix

$$C = [C_{ij}] = \left[\frac{2(a_i, a_j)}{(a_j, a_j)} \right] . \qquad (2.7)$$

of finite type (the bilinear product is defined in (2.1)). Let $Q = \sum_i \mathbb{Z} a_i \subset \mathbb{K}$ be the finite type root lattice additively generated by those (finite) simple roots. If necessary, we will denote the Lie algebra type by a subscript on Q. We also find the highest root θ of the finite type root system, and normalize the lengths of the simple roots a_i so that $\theta\bar{\theta} = 1$ always. In some twisted affine cases described in the appendix, we may instead choose θ to be the highest short root, but we still assume that $\theta\bar{\theta} = 1$.

In the Jordan algebra $H_2(\mathbb{K})$, depending on the choice of Q, we define the lattice

$$\Lambda = \Lambda(Q) := \left\{ \mathtt{X} = \left[\begin{array}{cc} x^+ & z \\ \bar{z} & x^- \end{array} \right] \in H_2(\mathbb{K}) \; \middle| \; x^+,\, x^- \in \mathbb{Z},\; z \in Q \right\} . \qquad (2.8)$$

The restriction of the bilinear form to Λ makes it a Lorentzian lattice, which we will identify as the root lattice of a hyperbolic Kac–Moody Lie algebra.

Our first step is to identify the hyperbolic simple roots in the lattice Λ as follows:

$$\alpha_{-1} = \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right], \quad \alpha_0 = \left[\begin{array}{cc} -1 & -\theta \\ -\bar{\theta} & 0 \end{array} \right], \quad \alpha_i = \left[\begin{array}{cc} 0 & a_i \\ \bar{a}_i & 0 \end{array} \right], \; 1 \le i \le \ell. \; (2.9)$$

It follows immediately that

$$(\alpha_{-1}, \alpha_{-1}) = 2 \;, \quad (\alpha_{-1}, \alpha_0) = -1, \qquad (2.10)$$

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and

$$\begin{aligned}
(\alpha_0, \alpha_0) &= 2\theta\overline{\theta} = 2 \\
(\alpha_i, \alpha_j) &= a_i\bar{a}_j + a_j\bar{a}_i = (a_i, a_j) \quad \text{for } 1 \le i, j \le \ell. \tag{2.11}
\end{aligned}$$

We will give specific choices for the a_i so that the matrix C in (2.7) is the Cartan matrix of a finite type root system, consistent with the embedding of the corresponding Euclidean root lattice Q into Λ. We will see that choosing θ to be the highest root of the finite root system (or the highest short root) makes $\{\alpha_0, \alpha_1, \cdots, \alpha_\ell\}$ the simple roots of an untwisted (or a twisted) affine root system. Furthermore we will see that including α_{-1} to get all the simple roots in (2.9) gives the simple roots of a hyperbolic root system with Cartan matrix

$$[A_{IJ}] := \left[\frac{2(\alpha_I, \alpha_J)}{(\alpha_J, \alpha_J)}\right] \qquad \text{for } I, J = -1, 0, 1, \cdots, \ell. \tag{2.12}$$

The unit norm condition on θ is necessary to obtain the single line between the hyperbolic and affine node. For simply laced algebras all the a_i can also be chosen as units whereas this is no longer true for cases where the Dynkin diagram has arrows.

The hyperbolic Kac–Moody algebras (of over-extended type) of rank 3,4,6 and 10, respectively, which we obtain by this construction are as follows: $\mathcal{F} \equiv A_1^{++}$ (for $\mathbb{K} = \mathbb{R}$), A_2^{++}, C_2^{++} and G_2^{++} (for $\mathbb{K} = \mathbb{C}$), $A_4^{++}, B_4^{++}, C_4^{++}, D_4^{++}$ and F_4^{++} (for $\mathbb{K} = \mathbb{H}$), and $E_{10} \equiv E_8^{++}, D_8^{++}$ and B_8^{++} (for $\mathbb{K} = \mathbb{O}$).[5] In all cases, real roots are characterized by $\det \mathtt{X} < 0$ and imaginary roots by $\det \mathtt{X} \ge 0$, with null roots obeying $\det \mathtt{X} = 0$.

3 Weyl groups

In this section we study abstractly the Weyl groups of the hyperbolic Kac–Moody algebras identified in the preceding section. As we define the root lattice using the Jordan algebra $H_2(\mathbb{K})$ we aim to find a description of the Weyl group acting on this space. Except where explicitly stated otherwise, the results of this section apply to all four division algebras, including $\mathbb{K} = \mathbb{O}$. Specific features of the four individual division algebras will be studied separately in the following sections.

3.1 The simple reflections

The Weyl group associated with the hyperbolic Cartan matrix (2.12) is the Coxeter group generated by the simple reflections

$$w_I(\mathtt{X}) = \mathtt{X} - \frac{2(\mathtt{X}, \alpha_I)}{(\alpha_I, \alpha_I)} \, \alpha_I \, , \quad I = -1, 0, 1, \cdots, \ell. \tag{3.1}$$

---

[5]The over-extensions of the other finite simple rank 8 root systems C_8^{++} and A_8^{++} do not give rise to hyperbolic reflection groups.

These generators are known to satisfy the Coxeter relations, which give a complete presentation for the Weyl group. Recall [23] that a Coxeter group is defined by the presentation

$$\left\{ R_I \mid R_I^2 = \mathbb{1}, (R_I R_J)^{m_{IJ}} = \mathbb{1} \text{ for } I \neq J \right\}. \tag{3.2}$$

For Weyl groups only special values may occur for the entries of the matrix $M = [m_{IJ}]$, namely $m_{IJ} \in \{2, 3, 4, 6, \infty\}$. More specifically, the relations (3.2) say that $|w_I| = 2$, that is, each generator has order 2, and they give the orders of the products of pairs of distinct generators, determined by the entries of the Cartan matrix as follows:

$$|w_I w_J| = m_{IJ}, \qquad I \neq J \in \{-1, 0, 1, \cdots, \ell\}, \tag{3.3}$$

where

$$m_{IJ} = \begin{cases} 2 & \text{if } A_{IJ} A_{JI} = 0 \\ 3 & \text{if } A_{IJ} A_{JI} = 1 \\ 4 & \text{if } A_{IJ} A_{JI} = 2 \\ 6 & \text{if } A_{IJ} A_{JI} = 3 \\ \infty & \text{if } A_{IJ} A_{JI} \geq 4. \end{cases} \tag{3.4}$$

For other values of m_{IJ}, the Coxeter group defined by (3.2) cannot be the Weyl group of a Kac–Moody algebra. Nevertheless, Coxeter groups with other values of m_{IJ} besides $2, 3, 4, 6, \infty$ may be of interest in the present context, as such groups may occur as *subgroups* of Weyl groups. A prominent example of such a non-crystallographic Coxeter group is the group H_4, which is a subgroup of the Weyl group of E_8 [18, 33, 27].

Of central importance for our analysis is that the action of the simple hyperbolic Weyl reflections w_I can be rewritten as a matrix action on $\mathtt{X}$.

Theorem 1. *Denote by $\overline{\mathtt{X}} = (\mathtt{X}^\dagger)^T$ the matrix $\mathtt{X}$ with each entry conjugated but the matrix not transposed, let*

$$\varepsilon_i = a_i / \sqrt{N(a_i)} \quad \text{for } i = 1, \cdots, \ell \tag{3.5}$$

be the unit norm versions of the simple roots $a_i \in Q$, and let $\theta \in Q$ be the highest root of the finite root system with simple roots a_i, always normalized so that $\theta\overline{\theta} = 1$. (For simply laced finite root systems we have $\varepsilon_i = a_i$.) Define the (2×2) matrices M_I, $I = -1, 0, 1, \cdots, \ell$, by

$$M_{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad M_0 = \begin{bmatrix} -\theta & 1 \\ 0 & \theta \end{bmatrix}, \quad M_i = \begin{bmatrix} \varepsilon_i & 0 \\ 0 & -\bar{\varepsilon}_i \end{bmatrix}, \tag{3.6}$$

$i = 1, \cdots, \ell$. *Then the simple reflections (3.1) can be written as*

$$w_I(\mathtt{X}) = M_I \overline{\mathtt{X}} M_I^\dagger \quad, \quad I = -1, 0, 1, \ldots, \ell. \tag{3.7}$$

Proof: In the non-associative octonionic case one needs to check that the expression on the right side of (3.7) is well-defined for the matrices in (3.6) without placing parentheses. Since each matrix M_I involves only one non-real octonion, this follows immediately from the alternativity property of $\mathbb{O}$. (See the section on octonions below for more details.)

Now we will check directly that the two formulas (3.1) and (3.7) for w_I agree. First, note that

$$(\mathbf{X}, \alpha_{-1}) = x^+ - x^-, \quad (\mathbf{X}, \alpha_0) = -z\overline{\theta} - \theta\overline{z} + x^-, \quad (\mathbf{X}, \alpha_i) = z\overline{a}_i + a_i\overline{z} \quad (3.8)$$

for $1 \leq i \leq \ell$, so

$$
\begin{aligned}
w_{-1}(\mathbf{X}) &= \mathbf{X} - \frac{2(\mathbf{X}, \alpha_{-1})}{(\alpha_{-1}, \alpha_{-1})}\, \alpha_{-1} = \mathbf{X} - (x^+ - x^-)\alpha_{-1} = \begin{bmatrix} x^- & z \\ \overline{z} & x^+ \end{bmatrix}, \\
w_0(\mathbf{X}) &= \mathbf{X} - \frac{2(\mathbf{X}, \alpha_0)}{(\alpha_0, \alpha_0)}\, \alpha_0 = \mathbf{X} - (-z\overline{\theta} - \theta\overline{z} + x^-)\alpha_0 \\
&= \begin{bmatrix} (x^+ - z\overline{\theta} - \theta\overline{z} + x^-) & (z - z\overline{\theta}\theta - \theta\overline{z}\theta + x^-\theta) \\ (\overline{z} - \overline{\theta}z\overline{\theta} - \overline{\theta}\theta\overline{z} + \overline{\theta}x^-) & x^- \end{bmatrix}, \\
w_i(\mathbf{X}) &= \mathbf{X} - \frac{2(\mathbf{X}, \alpha_i)}{(\alpha_i, \alpha_i)}\, \alpha_i = \mathbf{X} - \frac{z\overline{a}_i + a_i\overline{z}}{N(a_i)}\alpha_i \\
&= \begin{bmatrix} x^+ & -\varepsilon_i\overline{z}\varepsilon_i \\ -\overline{\varepsilon}_i z\overline{\varepsilon}_i & x^- \end{bmatrix}. \quad (3.9)
\end{aligned}
$$

where we used the definition (3.5) in the last equation. Compare these with

$$
\begin{aligned}
M_{-1}\overline{\mathbf{X}}M_{-1}^\dagger &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x^+ & \overline{z} \\ z & x^- \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} x^- & z \\ \overline{z} & x^+ \end{bmatrix}, \\
M_0\overline{\mathbf{X}}M_0^\dagger &= \begin{bmatrix} -\theta & 1 \\ 0 & \overline{\theta} \end{bmatrix} \begin{bmatrix} x^+ & \overline{z} \\ z & x^- \end{bmatrix} \begin{bmatrix} -\overline{\theta} & 0 \\ 1 & \theta \end{bmatrix} \\
&= \begin{bmatrix} (\theta\overline{\theta}x^+ - z\overline{\theta} - \theta\overline{z} + x^-) & (-\theta\overline{z}\theta + x^-\theta) \\ (-\overline{\theta}z\overline{\theta} + \overline{\theta}x^-) & \overline{\theta}\theta x^- \end{bmatrix}, \\
M_i\overline{\mathbf{X}}M_i^\dagger &= \begin{bmatrix} \varepsilon_i & 0 \\ 0 & -\overline{\varepsilon}_i \end{bmatrix} \begin{bmatrix} x^+ & \overline{z} \\ z & x^- \end{bmatrix} \begin{bmatrix} \overline{\varepsilon}_i & 0 \\ 0 & -\varepsilon_i \end{bmatrix} \\
&= \begin{bmatrix} \varepsilon_i\overline{\varepsilon}_i x^+ & -\varepsilon_i\overline{z}\varepsilon_i \\ -\overline{\varepsilon}_i z\overline{\varepsilon}_i & \overline{\varepsilon}_i\varepsilon_i x^- \end{bmatrix}. \quad (3.10)
\end{aligned}
$$

Hence $w_{-1}(\mathbf{X}) = M_{-1}\overline{\mathbf{X}}M_{-1}^\dagger$, $w_0(\mathbf{X}) = M_0\overline{\mathbf{X}}M_0^\dagger$, since $\theta\overline{\theta} = 1$, and $w_i(\mathbf{X}) = M_i\overline{\mathbf{X}}M_i^\dagger$ since $\varepsilon_i\overline{\varepsilon}_i = 1$. The explicit expressions (3.8) and (3.10) are manifestly well-defined for octonions without placing parentheses. $\square$

Our aim in this paper is to study the group generated by the simple reflections (3.7) as an extension of a matrix group. This extension may always be realized in the associative cases ($\mathbb{K} \neq \mathbb{O}$) as an extension of a matrix group by a small finite group. Since the full Weyl group W is a semidirect product of the even part W^+ with $\mathbb{Z}_2 = \langle w_{-1}\rangle$, we will be satisfied to understand just the even Weyl group W^+ as an extension of a matrix group.

3.2 Even part of the Weyl group

The formula (3.7) for the simple reflections involves complex conjugation of $\mathtt{X}$. Therefore all *even* elements $s \in W^+ \subset W$ can be represented *without complex conjugation of* $\mathtt{X}$, and it turns out to be simpler to study the even Weyl group W^+ in many cases.

The even Weyl group W^+ is an index 2 normal subgroup of W and consists of those elements which can be expressed as the product of an even number of simple reflections. It is generated by the following list of $\ell + 1$ double reflections:

$$s_0 = w_{-1} w_0 \,, \quad s_i = w_{-1} w_i \quad (i = 1, \ldots, \ell) \,. \tag{3.11}$$

Of course, this is not a unique set of generating elements. From the Coxeter relations (3.2)–(3.4) satisfied by the simple reflections, w_I, these even elements satisfy the relations

$$s_0^3 = \mathbb{1} \,, \quad s_i^2 = \mathbb{1} \quad \text{for } i \neq 0 \tag{3.12}$$

and

$$(s_i^{-1} s_j)^{m_{ij}} = \mathbb{1} \ \text{ for } i \neq j \ \text{ and } \ i, j = 0, 1, \ldots, \ell, \tag{3.13}$$

where m_{ij} is given as before in (3.4).

In analogy with theorem 1 we have

Theorem 2. *Define the matrices*

$$S_0 = \begin{bmatrix} 0 & \theta \\ -\bar{\theta} & 1 \end{bmatrix} , \quad S_i = \begin{bmatrix} 0 & -\varepsilon_i \\ \bar{\varepsilon}_i & 0 \end{bmatrix} . \tag{3.14}$$

Then the generating double reflections (3.11) acting on $\mathtt{X} \in \Lambda$ can be written for all $\mathbb{K}$ as

$$s_I(\mathtt{X}) = S_I \mathtt{X} S_I^\dagger \quad , \ I = 0, 1, \ldots, \ell \,. \tag{3.15}$$

Proof: Follows by direct computation as in the proof of theorem 1. $\square$

An important corollary is

Corollary 1. *For the associative division algebras $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, all elements $s \in W^+$ can be realized by a matrix action according to*

$$s(\mathtt{X}) = S \mathtt{X} S^\dagger , \tag{3.16}$$

where $S = S_{i_1} \cdots S_{i_n}$ if $s = s_{i_1} \cdots s_{i_n} \in W^+$ in terms of the generating elements (3.11).

Proof: The iterated action of two even Weyl transformations is given by the associative product of matrices (for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$)

$$(s_1 s_2)(\mathtt{X}) = S_1 (S_2 \mathtt{X} S_2^\dagger) S_1^\dagger = (S_1 S_2) \mathtt{X} (S_1 S_2)^\dagger \tag{3.17}$$

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and has an obvious extension to arbitrary words in the even Weyl group by associativity. $\square$

For $\mathbb{K} = \mathbb{O}$, the formula (3.16) no longer holds (even though we have for any octonionic matrices $(S_1 S_2)^\dagger = S_2^\dagger S_1^\dagger$). This can be seen most easily in the continuous case by a dimension count as will be discussed in section 6 where we collect our results specific to the octonionic case.

Let us remark that in the commutative case (3.15) follows immediately by acting with two successive simple Weyl reflections, say w_J and w_I; the effect on $\mathtt{X}$ is

$$w_I(w_J(\mathtt{X})) = M_I \overline{(M_J \bar{\mathtt{X}} M_J^\dagger)} M_I^\dagger = S_{IJ} \mathtt{X} S_{IJ}^\dagger \,, \tag{3.18}$$

where $S_{IJ} \equiv M_I \overline{M}_J$. In this notation the matrices (3.14) are $S_i \equiv S_{-1\,i}$ for $i = 0, 1, \ldots, \ell$. However, in the non-commutative cases one has to be more careful because quaternionic and octonionic conjugation also reverses the order of factors inside a product, such that, in general

$$\overline{M_J \bar{\mathtt{X}} M_J^\dagger} \neq \overline{M}_J \mathtt{X} M_J^T \tag{3.19}$$

and M_I and $\overline{M}_J$ do not obviously combine into a matrix S_{IJ} which acts by conjugation as in (3.16). Therefore it is crucial that theorem 2 applies to *all* division algebras.

In the associative cases we obtain from corollary 1 that W^+ is isomorphic to a matrix group generated by (3.14). All elements $s \in W^+$ act by invertible matrices and we therefore obtain subgroups of $GL_2(\mathbb{K})$. Furthermore, they all act by matrix conjugation and therefore a matrix and its negative have the same action. Other scalar matrices $\varepsilon \mathbb{1} \in GL_2(\mathbb{K})$, for ε a central unit in $\mathbb{K}$, would act trivially, so we should be finding W^+ isomorphic to a subgroup of $PGL_2(\mathbb{K})$. It is important to note that for general (2×2) matrices over $\mathbb{K}$ the determinant is not well-defined unless $\mathbb{K}$ is commutative. In the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, the matrices M_I satisfy $\det M_I = -1$, so products of two such fundamental Weyl reflections have determinant $+1$ and hence W^+ is a subgroup of $PSL_2(\mathbb{K})$ for commutative $\mathbb{K}$. With a suitable definition of $PSL_2(\mathbb{K})$ this statement is also true for non-commutative $\mathbb{K}$. The necessary refinements required for $PSL_2(\mathbb{H})$ and $PSL_2(\mathbb{O})$ will be presented in the relevant sections. For all $\mathbb{K}$ the P in PSL means that only the quotient by $\{\mathbb{1}, -\mathbb{1}\}$ has been taken. We will argue that the even Weyl groups constitute interesting discrete 'modular' subgroups of $PSL_2(\mathbb{K})$.

3.3 Finite and affine Weyl subgroups

By specialization, the construction given above also yields matrix representations of the finite and the affine Weyl subgroups contained in the respective hyperbolic algebras. We first note that the action of the *finite* Weyl group $W_{\text{fin}} \equiv W(\mathfrak{g})$ on the root lattice of the finite subalgebra $\mathfrak{g}$ is obtained as a special case of (3.7) by setting $x^\pm = 0$ and restricting indices to $I \equiv i = 1, \cdots, \ell$;

as follows immediately from the last formula in (3.10), the simple reflections are thus realized on any lattice vector z via

$$w_i(z) = -\varepsilon_i \bar{z} \varepsilon_i \qquad \text{(for } z \in Q \subset \mathbb{K}\text{).} \qquad (3.20)$$

The *same* transformation on z is obtained by matrix conjugation (3.16) with the even element S_i with $i = 1, \ldots, \ell$, i.e. *without* complex conjugation of X. Similarly, it follows from (3.11) that

$$s_i s_j = w_i w_j \quad \text{for } i, j = 1, \ldots, \ell\,. \qquad (3.21)$$

so that, in terms of the matrix representation (3.14) for $\mathbb{K} \neq \mathbb{O}$ we obtain

$$S_i S_j = \begin{bmatrix} -\varepsilon_i \bar{\varepsilon}_j & 0 \\ 0 & -\bar{\varepsilon}_i \varepsilon_j \end{bmatrix} \qquad (3.22)$$

whence the even part $W^+(\mathfrak{g})$ of the finite Weyl group acts by diagonal matrices. Defining $u_{ij} = \varepsilon_i \bar{\varepsilon}_j$ and $v_{ij} = \bar{\varepsilon}_i \varepsilon_j$ we deduce $s_i(s_j(z)) = u_{ij} z \bar{v}_{ij} = u_{ij} z v_{ji}$ for $z \in Q$. To summarize: the even and odd parts of the finite Weyl group, respectively, act by purely diagonal or purely off-diagonal matrices for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. [6] Because of the extra w_{-1} contained in the definition of S_i the action of the odd part is only correct on the subspace with $x^\pm = 0$; on the full $H_2(\mathbb{K})$ the odd parts acts with an additional interchange of x^+ and x^-.

The affine subalgebra $\mathfrak{g}^+$ is characterized by all roots with $\det \mathtt{X} \leq 0$ which are of the form

$$\mathtt{X} = \begin{bmatrix} m & z \\ \bar{z} & 0 \end{bmatrix} \qquad \text{for } z \in Q \text{ and } m \in \mathbb{Z}\,. \qquad (3.23)$$

Its associated affine Weyl group W_{aff} is well known to be isomorphic to a semidirect product of the finite Weyl group and an abelian group of (affine) translations $\mathcal{T}$, such that $W_{\text{aff}} = \mathcal{T} \rtimes W_{\text{fin}}$. The latter is generated by elements of the form $w t_\theta w^{-1} = t_{w(\theta)}$ where $w \in W_{\text{fin}}$ and the relevant affine translation is

$$t_\theta = w_0 w_\theta = \begin{bmatrix} 1 & \theta \\ 0 & 1 \end{bmatrix}\,. \qquad (3.24)$$

Here, w_θ is the reflection about the highest root θ and this is the correct expression for the affine translation for all the algebras we consider in this paper. It is straightforward to check that this matrix indeed generates translations since

$$t_\theta \begin{bmatrix} m & z \\ \bar{z} & 0 \end{bmatrix} t_\theta^\dagger = \begin{bmatrix} m + \theta \bar{z} + z \bar{\theta} & z \\ \bar{z} & 0 \end{bmatrix} \qquad (3.25)$$

---

[6] For $\mathbb{K} = \mathbb{H}$ the same action can be alternatively written in terms of pairs $[l, r]$ and $*[l, r]$ of unit quaternions, cf. [8] p. 42. The relation to our notation is as follows:

$$[l, r] \longleftrightarrow S = \begin{bmatrix} \bar{l} & 0 \\ 0 & \bar{r} \end{bmatrix}\,, \qquad *[l, r] \longleftrightarrow S = \begin{bmatrix} 0 & \bar{l} \\ \bar{r} & 0 \end{bmatrix}\,.$$

This statement also holds for the octonionic case. We note that the interesting S-type transformations in the Weyl groups are then solely due to the hyperbolic extension.

If $\mathbb{K}$ is associative, and once the finite Weyl group has been identified in terms of diagonal and off-diagonal matrices, the full even hyperbolic Weyl group is obtained by adjoining the affine Weyl transformation (3.24) to the set of both diagonal and off-diagonal matrices as generating set.

3.4 Lattice symmetries and Weyl groups

In the associative cases, there are general necessary constraints on the structure of the matrices

$$S = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] , \tag{3.26}$$

which represent transformations of the even Weyl group acting on the hyperbolic root lattice Λ of (2.8). These follow from the fact that the transformed $\mathtt{X}' = S\mathtt{X}S^{\dagger}$ again has to lie in the root lattice Λ and should have the same norm as $\mathtt{X}$. Working out the matrix product one finds the following set of conditions for the transformed $\mathtt{X}'$ to lie in the root lattice:

$$\begin{aligned} a\bar{a},\, b\bar{b},\, c\bar{c},\, d\bar{d} &\in& \mathbb{Z}\,, \\ c\bar{a},\, d\bar{b} &\in& Q\,, \\ aa_i\bar{b} + b\bar{a}_i\bar{a},\, ca_i\bar{d} + d\bar{a}_i\bar{c} &\in& \mathbb{Z} \quad \text{for } i = 1,\ldots,\ell\,, \\ aa_i\bar{d} + b\bar{a}_i\bar{c} &\in& Q \quad \text{for } i = 1,\ldots,\ell\,. \end{aligned} \tag{3.27}$$

Here, Q is the finite root lattice with simple basis vectors a_i. It is the collection of the conditions (3.27) that will turn the matrices S into a *discrete subgroup* of the group of (2×2) matrices over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (even if the relevant rings inside $\mathbb{K}$ to which the matrix entries belong are not discrete). The norm preservation requirement will lead to additional determinant-type constraints. The conditions (3.27) are only necessary but not sufficient since they also allow for solutions which correspond to lattice symmetries of Q which are not elements of the Weyl group. This happens when the finite Dynkin diagram admits outer automorphisms, and we will be concerned with finding manageable conditions which eliminate these.

4 Commutative cases

We first discuss the commutative cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ where one has the usual definition of the determinant.

4.1 $\mathbb{K} = \mathbb{R}$, type A_1

For this case we recover the results of [20] for the rank 3 hyperbolic algebra $\mathcal{F} = A_1^{++}$, where $W(\mathcal{F}) = PGL_2(\mathbb{Z})$. The root system of type A_1 is shown in
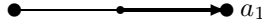
Figure 1: *The A_1 root system with simple root labeled. The root lattice is the lattice of (rational) integers $\mathbb{Z}$.*
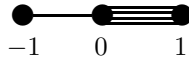

Figure 2: *Dynkin diagram of $\mathcal{F} \equiv A_1^{++}$ with numbering of nodes.*

fig. 1, and the Dynkin diagram of $\mathcal{F}$ is shown in fig. 2. A simplification here is that we do not have to worry about conjugation; it is for this reason that an isomorphism with a matrix group exists for the full Weyl group W, rather than only its even subgroup W^+. We have the simple root $a_1 = 1$, which is identical to the highest root $\theta = 1$, so that the simple roots of the hyperbolic algebra are represented by the three matrices

$$\alpha_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad \alpha_0 = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} \quad , \quad \alpha_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (4.1)$$

The corresponding matrices M_I representing the simple Weyl group generators are

$$M_{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad M_0 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \quad , \quad M_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (4.2)$$

The even part of the Weyl group is thus generated by the following two matrices (conjugation can be omitted)

$$S := M_{-1}M_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} , \quad T := M_0 M_1 = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \cong \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad (4.3)$$

implying that $W^+(\mathcal{F}) \cong PSL_2(\mathbb{Z})$. The full Weyl group in this case can be obtained by adjoining the matrix M_{-1} of determinant -1 from which one recovers $W(\mathcal{F}) \cong PGL_2(\mathbb{Z})$ [20].

4.2 $\mathbb{K} = \mathbb{C}$, simple reflections

For $\mathbb{K} = \mathbb{C}$ there are different choices of simple finite root systems which we discuss separately in the following sections, corresponding to the root lattices $A_2, B_2 \cong C_2, G_2$. Here, we collect some common features of all cases. However,

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the overextension of the $B_2 \cong C_2$ root system leads to the hyperbolic algebra C_2^{++} if θ is the highest root.

In all cases in this subsection, the matrices giving rise to the simple reflections are

$$M_{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad M_0 = \begin{bmatrix} -\theta & 1 \\ 0 & \theta \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} \varepsilon_2 & 0 \\ 0 & -\bar{\varepsilon}_2 \end{bmatrix}. \tag{4.4}$$

where we used the fact that for $\mathbb{K} = \mathbb{C}$ it is always possible to choose $\varepsilon_1 = 1$ (as for $\mathbb{K} = \mathbb{R}$). The even Weyl group W^+ is thus generated by the elements

$$S_0 = M_{-1}\overline{M}_0 = \begin{bmatrix} 0 & \theta \\ -\bar{\theta} & 1 \end{bmatrix}, \quad S_1 = M_{-1}\overline{M}_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$S_2 = M_{-1}\overline{M}_2 = \begin{bmatrix} 0 & -\varepsilon_2 \\ \bar{\varepsilon}_2 & 0 \end{bmatrix} \tag{4.5}$$

and we note that $S_2^2 = -\mathbb{1}$ which acts as the identity on X in agreement with the Coxeter relation $s_2^2 = \mathbb{1}$. We repeat that we will always normalise $\theta\bar{\theta} = 1$. In the above form the main difference between the three algebras is encoded in θ and ε_2. The three algebras are then distinguished simply by the multiplicative order of these numbers, which will be specified below for each case.

We note that the even parts of the finite Weyl groups $W^+(A_2)$, $W^+(B_2) \cong W^+(C_2)$ and $W^+(G_2)$ are cyclic groups of orders 3, 4, and 6 respectively. This follows also since their generating elements are rotations in the plane and so one obtains finite subgroups of the abelian group $SO(2)$. From this point of view the non-abelian nature of the full finite Weyl group arises because of a single reflection realized as complex conjugation, which shows that the full finite Weyl group is a dihedral group.

In order to determine the even Weyl group W^+ for the hyperbolic algebras A_2^{++} and G_2^{++} the following result will be useful.

Proposition 1. *Let $\mathcal{O}$ be a discrete Euclidean ring in $\mathbb{C}$, that is, a discrete additive group, closed under multiplication, satisfying the Euclidean algorithm. Furthermore, assume that $\mathcal{O} = \{m + n\theta^2 \in \mathbb{C} \mid m, n \in \mathbb{Z}\}$ with $\theta \in \mathcal{O}$ a unit, and that all units in $\mathcal{O}$ have norm 1. Then $SL_2(\mathcal{O})$, the group of (2×2) matrices with entries from $\mathcal{O}$ and with determinant 1, is generated by*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \varepsilon & 0 \\ 0 & \bar{\varepsilon} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.6}$$

where $\varepsilon \in \mathcal{E}_{\mathcal{O}}$ runs through all units of $\mathcal{O}$.

Proof: We prove the proposition by using arguments from [28]. Let Δ denote the group generated by the matrices in (4.6). These matrices belong to $SL_2(\mathcal{O})$

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and we want to show that $\Delta = SL_2(\mathcal{O})$. First, we claim that Δ contains all 'translation' matrices

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \quad \text{for } a \in \mathcal{O}. \tag{4.7}$$

To see this, note that

$$\begin{pmatrix} \theta & 0 \\ 0 & \overline{\theta} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{\theta} & 0 \\ 0 & \theta \end{pmatrix} = \begin{pmatrix} 1 & \theta^2 \\ 0 & 1 \end{pmatrix}. \tag{4.8}$$

We used that the unit $\theta \in \mathcal{O}$ satisfies $\theta\overline{\theta} = 1$. Since the translation matrices form an abelian group with addition of the upper right corner entry, and 1 and θ^2 are an integral basis of $\mathcal{O}$, for any $m, n \in \mathbb{Z}$, we have

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^m \begin{pmatrix} 1 & \theta^2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & m + n\theta^2 \\ 0 & 1 \end{pmatrix}, \tag{4.9}$$

so all matrices (4.7) are in Δ. We also get that all matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \tag{4.10}$$

are in Δ. To complete the proof, choose any matrix $A \in SL_2(\mathcal{O})$ and consider the set of norms

$$\mathcal{N} = \left\{ N(b_{ij}) \mid [b_{ij}] = B = UAV \text{ for some } U, V \in \Delta \right\} \backslash \{0\}. \tag{4.11}$$

Because the ring $\mathcal{O}$ is discrete, and the norm is positive definite, any set of non-zero norms has a least element. Then $\mathcal{N}$ contains a non-zero minimum element $N(b)$, and suppose

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{4.12}$$

is a matrix for which this occurs. Multiplying B on the left and/or right by the third generating matrix in (4.6) (the rotation matrix), we can move any of its entries into the upper left corner, so we may assume that the minimum occurs with $N(b) = N(b_{11})$. Now, using the Euclidean algorithm, we may write

$$\begin{aligned} b_{12} &= q_1 b_{11} + r_1 \quad \text{with } 0 \le N(r_1) < N(b_{11}) \\ b_{21} &= q_2 b_{11} + r_2 \quad \text{with } 0 \le N(r_2) < N(b_{11}) \end{aligned} \tag{4.13}$$

for $q_1, q_2, r_1, r_2 \in \mathcal{O}$. Then we have

$$B \begin{pmatrix} 1 & -q_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b_{11} & r_1 \\ b_{21} & b_{22} - q_1 b_{21} \end{pmatrix} \tag{4.14}$$

is a matrix of the form UAV for $U, V \in \Delta$, so the norms of its entries cannot be less than the minimal value $N(b_{11})$. This forces $r_1 = 0$, so $b_{12} = q_1 b_{11}$. Similarly, we have

$$\begin{pmatrix} 1 & 0 \\ -q_2 & 1 \end{pmatrix} B = \begin{pmatrix} b_{11} & b_{12} \\ r_2 & b_{22} - q_2 b_{12} \end{pmatrix} \qquad (4.15)$$

is a matrix of the form UAV for $U, V \in \Delta$, so the norms of its entries cannot be less than the minimal value $N(b_{11})$. This forces $r_2 = 0$, so $b_{21} = q_2 b_{11}$. Finally, we see that

$$\begin{pmatrix} 1 & 0 \\ -q_2 & 1 \end{pmatrix} B \begin{pmatrix} 1 & -q_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b_{11} & 0 \\ 0 & b'_{22} \end{pmatrix} = B', \qquad (4.16)$$

where $b'_{22} = b_{22} - q_1 q_2 b_{11}$, is a diagonal matrix of the form UAV for $U, V \in \Delta$, with determinant $b_{11} b'_{22} = 1$. Therefore, b_{11} must be a unit in $\mathcal{O}$ and $b'_{22} = b_{11}^{-1} = \overline{b_{11}}$ so that $B' \in \Delta$. All operations transforming A into $B' \in \Delta$ were performed using matrices from Δ, so we conclude that $A \in \Delta$, completing the proof. $\square$

We remark that this proposition does not apply to the Gaussian integers since all Gaussian units square to real numbers and one therefore cannot generate the whole ring from 1 and θ^2 for any unit θ.

4.3 $\mathbb{K} = \mathbb{C}$, type A_2

The first choice of integers we consider is the case of type A_2, which is simply laced. The simple roots can therefore be chosen as units $a_i = \varepsilon_i$ and we take them to be

$$\varepsilon_1 = a_1 = 1 \ , \ \varepsilon_2 = a_2 = \frac{-1 + i\sqrt{3}}{2} \ , \ \theta = -\bar{\varepsilon}_2 = \frac{1 + i\sqrt{3}}{2} \ ,$$

where also the highest root θ has been given. The A_2 root lattice is spanned by integral linear combinations of the simple roots, and they form the order of 'Eisenstein integers' E. The A_2 root system is depicted in fig. 3.

The hyperbolic algebra A_2^{++} has the Dynkin diagram shown in fig. 4. The following choice of simple roots in Λ provides us with that diagram:

$$\alpha_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \qquad \alpha_0 = \begin{bmatrix} -1 & -\theta \\ -\bar{\theta} & 0 \end{bmatrix} ,$$

$$\alpha_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \qquad \alpha_2 = \begin{bmatrix} 0 & -\bar{\theta} \\ -\theta & 0 \end{bmatrix} , \qquad (4.17)$$

where we used $\varepsilon_2 = -\bar{\theta}$. All roots have equal length in this case, so the hyperbolic extension A_2^{++} is also simply laced. We have

Proposition 2. *The even part of the Weyl group in this case is*

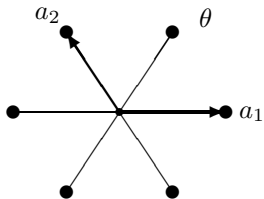$$W^+(A_2^{++}) \cong PSL_2(\mathrm{E}) \qquad (4.18)$$

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Figure 3: *The A_2 root system with simple roots labeled. The root lattice is the ring of Eisenstein integers.*
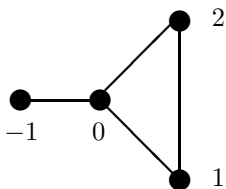

Figure 4: *Dynkin diagram of A_2^{++} with numbering of nodes.*

where $PSL_2(\mathtt{E})$ denotes the 'Eisenstein modular subgroup' of $PSL_2(\mathbb{C})$ obtained by restricting all entries to be Eisenstein integers.

Proof: The statement (4.18) is an immediate corollary of prop. 1, given that $\mathtt{E}$ satisfies the Euclidean algorithm, and all generating matrices in (4.6) can be obtained within the even part of the Weyl group of A_2^{++}. This is true by inspection of the matrices (4.5): the rotation matrix is just S_1^{-1}, and all diagonal matrices are obtained from powers of

$$B := S_1 S_2 = \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}. \tag{4.19}$$

Finally the translation matrix is obtained from

$$(S_2 S_0)(S_1 S_2)^{-1} = \begin{bmatrix} -\bar{\theta}^2 & \bar{\theta} \\ 0 & -\theta^2 \end{bmatrix} \cdot \begin{bmatrix} \bar{\theta} & 0 \\ 0 & \theta \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{4.20}$$

where we used $\theta^3 = -1$. Thus S_0, S_1 and S_2 generate a group isomorphic to $SL_2(\mathtt{E})$. Since in the action, the normal subgroup $\{\mathbb{1}, -\mathbb{1}\}$ acts trivially, the action of the even Weyl group on $\mathtt{X}$ is that of the quotient $PSL_2(\mathtt{E})$. $\square$

4.4 $\mathbb{K} = \mathbb{C}$, type C_2

The root system of type C_2 is shown in fig. 5. It is the same as the system of type B_2 but the over-extension constructed from this root system is C_2^{++} with
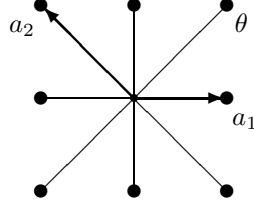
Figure 5: *The root system of type C_2, with simple roots labeled and indicated by arrows. The lattice they generate is a scaled version of the ring of Gaussian integers. There are both long and short roots, and the same system gives type B_2.*

Dynkin diagram shown in fig. 6. The dual diagram corresponds to the extension of the twisted affine Lie algebra $D_2^{(2)}$ (not B_2^{++}!) and will be discussed in the appendix.

The C_2 root system is not simply laced, having simple roots whose squared lengths are in the ratio 2 to 1:

$$\varepsilon_1 = a_1\sqrt{2} = 1 \ , \ \varepsilon_2 = a_2 = \frac{-1+i}{\sqrt{2}} \ , \ \theta = -\bar{\varepsilon}_2 = \frac{1+i}{\sqrt{2}} \ . \qquad (4.21)$$

We obtain the hyperbolic C_2^{++} Dynkin diagram since our simple roots satisfy

$$(\alpha_{-1}, \alpha_{-1}) = (\alpha_0, \alpha_0) = (\alpha_2, \alpha_2) = 2 \ , \ (\alpha_1, \alpha_1) = 1, \qquad (4.22)$$

$$(\alpha_{-1}, \alpha_0) = -1 \ , \ (\alpha_0, \alpha_1) = -1 \ , \ (\alpha_1, \alpha_2) = -1 \qquad (4.23)$$

and all other inner products are zero. From (4.21) we see that θ is a primitive eighth root of unity with $\theta^2 = -\bar{\theta}^2 = i$. The hyperbolic simple roots of C_2^{++} from (4.21) are:

$$\alpha_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \alpha_0 = \begin{bmatrix} -1 & \frac{-1-i}{\sqrt{2}} \\ \frac{-1+i}{\sqrt{2}} & 0 \end{bmatrix},$$

$$\alpha_1 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \qquad \alpha_2 = \begin{bmatrix} 0 & \frac{-1+i}{\sqrt{2}} \\ \frac{-1-i}{\sqrt{2}} & 0 \end{bmatrix}. \qquad (4.24)$$

The long root α_2 has a_2 as a unit whereas a_1, entering the short root α_1, is not a unit.

To determine $W^+(C_2^{++})$ it proves convenient to bring the matrices $\{S_0, S_i\}$ to another form by means of a similarity transformation

$$\tilde{S} = USU^{-1} \ , \quad U = \begin{bmatrix} \theta^{1/2} & 0 \\ 0 & \theta^{-1/2} \end{bmatrix} \qquad (4.25)$$

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which gives

$$\tilde{S}_0 = \begin{bmatrix} 0 & \theta^2 \\ -\bar{\theta}^2 & 1 \end{bmatrix}, \quad \tilde{S}_1 = \begin{bmatrix} 0 & -\theta \\ \bar{\theta} & 0 \end{bmatrix}, \quad \tilde{S}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \qquad (4.26)$$

Here, we have used $\theta = -\bar{\varepsilon}_2$. From these we can build the matrices

$$\begin{aligned}
A &= \tilde{S}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \\
B &= \tilde{S}_1\tilde{S}_2 = \begin{bmatrix} \theta & 0 \\ 0 & \bar{\theta} \end{bmatrix}, \\
C &= \tilde{S}_1\tilde{S}_2\tilde{S}_1 = \begin{bmatrix} 0 & -\theta^2 \\ \bar{\theta}^2 & 0 \end{bmatrix}, \\
D &= \tilde{S}_1\tilde{S}_0\tilde{S}_1\tilde{S}_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \\
E &= \tilde{S}_1\tilde{S}_2\tilde{S}_1\tilde{S}_0 = \begin{bmatrix} 1 & -\theta^2 \\ 0 & 1 \end{bmatrix}. \qquad (4.27)
\end{aligned}$$

The group generated by these matrices is isomorphic to the even part of the hyperbolic Weyl group. Hence, these matrices contain inversions and rotations (generated by A, B and C), and translations along some lattice directions (generated by D and E). The similarity transformation (4.25) is useful for explicitly exhibiting the correct lattice translations along two independent basis vectors 1 and θ^2 of the chosen integers via the matrices D and E, respectively. We note that for all θ the relation $C \cdot A = B^2$ is valid, showing that the group generated by A, B, C, D and E is an index 2 extension of the group generated by A, C, D and E.

With this we can easily recover the link with the so-called Klein–Fricke group which was first noticed in [20]. Because $\theta^2 = i$ the matrices C and E of (4.27) become

$$C = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}. \qquad (4.28)$$

Together with A and D, these matrices generate the Picard group $PSL_2(\mathtt{G})$ (where $\mathtt{G} \equiv \mathbb{Z}(i)$ are the Gaussian integers), see e.g. [30]. We thus recover the result of [20] (where, however, the explicit form of the embedding was not given).

Proposition 3. *The even Weyl group* $W^+(C_2^{++})$ *is an index 2 extension of* $PSL_2(\mathtt{G})$, *that is,*

$$W^+(C_2^{++}) \cong PSL_2(\mathtt{G}) \rtimes 2 \equiv PSL_2(\mathbb{Z}(i)) \rtimes 2. \qquad (4.29)$$

Semi-directness follows since conjugation of $PSL_2(\mathtt{G})$ by the matrix B is an automorphism.
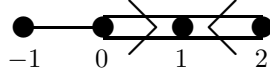
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Figure 6: C_2^{++} *Dynkin diagram with numbering of nodes.*

4.5 $\mathbb{K} = \mathbb{C}$, type G_2

The root system of type G_2 is depicted in fig. 7. It is not simply laced, having simple roots whose squared lengths are in the ratio 3 to 1.

$$\varepsilon_1 = a_1 = 1 \ , \ \varepsilon_2 = \sqrt{3}a_2 = \frac{-\sqrt{3}+i}{2} \ , \ \theta = \frac{1+\sqrt{3}i}{2} \ . \tag{4.30}$$

The G_2 root system is thus the superposition of two A_2 root systems which are scaled by a factor of $\sqrt{3}$ and rotated by 30° degrees relative to each other. Note that in this case $\theta \neq -\bar{\varepsilon}_2$; rather, we have $\varepsilon_2^2 = \bar{\theta}$, and that θ is identical to the highest root of A_2.

The Dynkin diagram of the hyperbolic algebra G_2^{++} is shown in fig. 8. We will get this diagram if our simple roots satisfy

$$(\alpha_{-1}, \alpha_{-1}) = (\alpha_0, \alpha_0) = (\alpha_1, \alpha_1) = 2 \ , \ (\alpha_2, \alpha_2) = \frac{2}{3}, \tag{4.31}$$

$$(\alpha_{-1}, \alpha_0) = -1 \ , \ (\alpha_0, \alpha_1) = -1 \ , \ (\alpha_1, \alpha_2) = -1 \tag{4.32}$$

and all others zero. The hyperbolic simple roots from (4.30) of G_2^{++} are:

$$\alpha_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \alpha_0 = \begin{bmatrix} -1 & \frac{-1-\sqrt{3}i}{2} \\ \frac{-1+\sqrt{3}i}{2} & 0 \end{bmatrix},$$

$$\alpha_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \alpha_2 = \begin{bmatrix} 0 & \frac{1+\sqrt{3}i}{2\sqrt{3}} \\ \frac{1-\sqrt{3}i}{2\sqrt{3}} & 0 \end{bmatrix}. \tag{4.33}$$

We label α_2 as a short root, so a_2 is not a unit.

The finite Weyl group $W(A_2)$ is a subgroup of index 2 in $W(G_2)$. The same is true for their hyperbolic extensions: we have

Proposition 4. *The even Weyl group $W^+(G_2^{++})$ is an index 2 extension of $W(A_2^{++})$, that is,*

$$W^+(G_2^{++}) = W^+(A_2^{++}) \rtimes 2 = PSL_2(\mathbb{E}) \rtimes 2 \ . \tag{4.34}$$

Proof: Because $\theta \neq -\bar{\varepsilon}_2$ we must proceed slightly differently than before. First we notice that

$$S_1 S_2 = \begin{bmatrix} -\bar{\varepsilon}_2 & 0 \\ 0 & -\varepsilon_2 \end{bmatrix} \quad \Rightarrow \quad (S_1 S_2)^2 = \begin{bmatrix} \theta & 0 \\ 0 & \bar{\theta} \end{bmatrix} \tag{4.35}$$
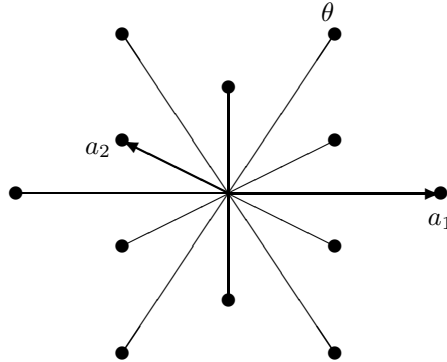
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Figure 7: *The root system of type G_2. The long and short roots are labeled and indicated by arrows.*
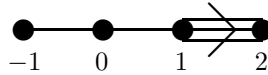

Figure 8: G_2^{++} *Dynkin diagram with numbering of nodes.*

Since $\theta^6 = 1$ (as for A_2), the set of matrices S_0, S_1 and $(S_1 S_2)^2 S_1$ coincides with the set (4.5) for A_2, hence these matrices generate the group $PSL_2(\mathtt{E})$, again by prop. 1. To get the full even Weyl group we must adjoin the matrix S_2 obeying $S_2^2 = -\mathbb{1}$, generating a $\mathbb{Z}_2$. The semi-directness of the product will follow from the action $S_2 S S_2^{-1} \cong S_2 S S_2$ of the extending matrix S_2 on $S \in PSL_2(\mathtt{E})$, if the resulting matrix is in $PSL_2(\mathtt{E})$ so that this action gives an automorphism of $PSL_2(\mathtt{E})$. By expanding the product, commutativity and the fact that $\varepsilon_2^2 \in \mathtt{E}$ the result follows.$\square$

5 Quaternions $\mathbb{K} = \mathbb{H}$

Within the four-dimensional quaternion algebra one can find the root systems of types A_4, B_4, C_4, D_4 and F_4. The associated hyperbolic rank 6 Kac–Moody algebras are now $A_4^{++}, B_4^{++}, C_4^{++}, D_4^{++}$, and F_4^{++}, and we will give explicit descriptions of their even Weyl groups in terms of matrix groups below. The quaternionic case is more subtle than the commutative cases because the criterion for selecting the matrix group to which the even Weyl group belongs cannot be so easily done via a determinant. Before turning to the issue of how to define determinants and matrix groups with quaternionic entries we first discuss different types of quaternionic integers and numbers required for exhibiting the root systems.

5.1 Hurwitz and Lipschitz integers

The standard basis for $\mathbb{H}$, $\{1, i, j, k\}$, has the famous products

$$i^2 = j^2 = k^2 = -1 , \quad ij = -ji = k , \quad jk = -kj = i , \quad ki = -ik = j, \quad (5.1)$$

so an obvious subring of integers is formed by the *Lipschitz integers*

$$\mathtt{L} = \{n_0 + n_1 i + n_2 j + n_3 k \,|\, n_0, n_1, n_2, n_3 \in \mathbb{Z}\} . \quad (5.2)$$

They form an order (a subring of the rational quaternions, $\mathbb{H}_\mathbb{Q}$, finitely generated as a $\mathbb{Z}$-module, containing a $\mathbb{Q}$-basis of $\mathbb{H}_\mathbb{Q}$) but are not a maximal order in $\mathbb{H}_\mathbb{Q}$ since they are contained in the ring of *Hurwitz integers*

$$\mathtt{H} = \left\{n_0 + n_1 i + n_2 j + n_3 k \,\middle|\, n_0, n_1, n_2, n_3 \in \mathbb{Z} \text{ or } n_0, n_1, n_2, n_3 \in \mathbb{Z} + \frac{1}{2}\right\} , \quad (5.3)$$

which constitute a maximal order in $\mathbb{H}_\mathbb{Q}$. We note that these two rings, $\mathtt{L}$ and $\mathtt{H}$, are generated by the Lipschitz and Hurwitz units given below in (5.9) and (5.10), respectively, and as $\mathbb{Z}$-modules, they are discrete lattices. Also note that the non-commutative ring $\mathtt{H}$ of Hurwitz integers satisfies the division with small remainder property required for the Euclidean algorithm (cf. prop. 1), whereas the Lipschitz integers $\mathtt{L}$ do not [8].

For the determination of some of the even Weyl groups we rely on the following definition and lemma.

Define $\mathtt{C}$ to be the two-sided ideal in the ring $\mathtt{H}$ generated by the commutators

$$[a, b] = ab - ba \qquad \text{for all } a, b \in \mathtt{H} . \quad (5.4)$$

To understand this we use the integral basis for $\mathtt{H}$ used in [28],

$$e_0 = \frac{1}{2}(1 + i + j + k), \qquad i, \qquad j, \qquad k, \quad (5.5)$$

which has the nice property that an integral combination $m_0 e_0 + m_1 i + m_2 j + m_3 k$ is in $\mathtt{L}$ when $m_0 \in 2\mathbb{Z}$. It is easy to check the commutators

$$
\begin{array}{lll}
[e_0, i] = j - k, & [e_0, j] = k - i, & [e_0, k] = i - j, \\
[i, j] = 2k, & [j, k] = 2i, & [k, i] = 2j,
\end{array}
\quad (5.6)
$$

which are all purely imaginary Lipschitz integers. $[\mathtt{H}, \mathtt{H}]$ is all integral linear combinations of the commutators above, but that is not even a subring of $\mathtt{H}$, as we can see, for example, from the fact that $(2i)(j - i) - 2k = 2$ is not a commutator. The ideal $\mathtt{C} = \mathtt{H}[\mathtt{H}, \mathtt{H}]\mathtt{H}$ consists of finite sums of the form $a[b, c]d$ for any $a, b, c, d \in \mathtt{H}$, and it is enough to compute these for a and d from the above integral basis and $[b, c]$ from (5.6). The only products we need to know are

$$
\begin{array}{lll}
e_0 i = -e_0 + i + j, & e_0 j = -e_0 + j + k, & e_0 k = -e_0 + i + k, \\
i e_0 = -e_0 + i + k, & j e_0 = -e_0 + i + j, & k e_0 = -e_0 + j + k, \\
& e_0 e_0 = -e_0 + i + j + k.
\end{array}
\quad (5.7)
$$

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Using these, it is clear that $a[b, c]d$ is in L, and consists of all integral linear combinations of the elements

$$\pm 1 \pm i, \quad \pm 1 \pm j, \quad \pm 1 \pm k, \quad \pm i \pm j, \quad \pm i \pm k, \quad \pm j \pm k, \tag{5.8}$$

which is clearly an index 2 integral lattice in L. Since L is an index 2 sublattice in H, we have that H/C is a ring of four elements, and we can take as coset representatives $\{0, -e_0, (-e_0)^2 = e_0 - 1, (-e_0)^3 = 1\}$ which form the field $\mathbb{F}_4$ of order 4 whose nonzero elements form the cyclic group of order 3. Also note that there are no units in C since its nonzero elements have minimal length 2.

Lemma 1. *Let $a_1, \ldots, a_n \in \mathsf{H}$ be any n Hurwitz numbers. Then the product $a_1 \cdots a_n$ is commutative modulo C.*

Proof: Since the quotient ring H/C is a field, where the product is commutative, the projection of the product $a_1 \cdots a_n$ is equal to the projection of $a_{\sigma(1)} \cdots a_{\sigma(n)}$ for any permutation σ of $\{1, \ldots, n\}$. $\square$

5.2 Quaternionic units and rings

For the root lattices of A_4, B_4, C_4 and F_4 (but not D_4) we also need quaternionic units which are neither Lipschitz nor Hurwitz numbers. These other units parametrize finite subgroups of $SU(2)$, and are, respectively, related to the octahedral (for B_4, C_4 and F_4) and icosahedral (for A_4) groups, as explained e.g. in [8]. Using diagonal and off-diagonal (2×2) matrices of such units (or alternatively *pairs* of units [8], cf. footnote 6 in section 3.3) we can then reconstruct the Weyl groups of all the finite simple rank 4 algebras, as we shall explain below. Besides the eight Lipschitz units

$$\mathcal{E}_{\mathsf{L}} = \{\pm 1, \pm i, \pm j, \pm k\} \tag{5.9}$$

which form the quaternionic group, often denoted by Q_8, we have the 24 Hurwitz units

$$\mathcal{E}_{\mathsf{H}} = \left\{ \pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k) \right\} \tag{5.10}$$

which form a subgroup in the unit quaternions. According to [7], p. 55, $\mathcal{E}_{\mathsf{H}} \cong 2 \cdot \mathfrak{A}_4$ is an index two extension of the alternating group on four letters. It can also be seen that $\mathcal{E}_{\mathsf{H}} \cong \mathcal{E}_{\mathsf{L}} \rtimes \mathbb{Z}_3$ is a semidirect product of the group of Lipschitz units with a cyclic group of order 3 related to triality and explained below in section 5.4. Any quaternion $z = n_0 + n_1 i + n_2 j + n_3 k$ is a root of the real polynomial

$$p_z(t) = (t - z)(t - \bar{z}) = t^2 - 2n_0 t + (n_0^2 + n_1^2 + n_2^2 + n_3^2) \tag{5.11}$$

whose coefficients will be integers if z is in L or H, and in that case, unless $z \in \mathbb{Z}$, this is the minimal polynomial satisfied by $z \in \mathsf{H}$.

We will also need the following set of 24 *octahedral* units

$$\left\{ \frac{\pm 1 \pm i}{\sqrt{2}}, \frac{\pm 1 \pm j}{\sqrt{2}}, \frac{\pm 1 \pm k}{\sqrt{2}}, \frac{\pm i \pm j}{\sqrt{2}}, \frac{\pm i \pm k}{\sqrt{2}}, \frac{\pm j \pm k}{\sqrt{2}} \right\} \tag{5.12}$$

which do not form a group by themselves, but the product of any two of them is a Hurwitz unit, and the product of any Hurwitz unit and an octahedral unit is an octahedral unit, so their union forms a group of order 48 which we call the octahedral subgroup, $\mathcal{E}_{\texttt{R}}$. We can write it as the disjoint union of two cosets of its normal subgroup, $\mathcal{E}_{\texttt{H}}$,

$$\mathcal{E}_{\texttt{R}} \equiv \mathcal{E}_{\texttt{H}} \cup i_{\text{O}} \cdot \mathcal{E}_{\texttt{H}} \qquad \text{where} \quad i_{\text{O}} \equiv \frac{j-k}{\sqrt{2}}. \tag{5.13}$$

These are, in fact, all the units in the ring, $\texttt{R}$, generated by integral linear combinations of units in $\mathcal{E}_{\texttt{R}}$. Contrary to the Lipschitz and Hurwitz numbers, the ring $\texttt{R}$ is not discrete but *dense* in $\mathbb{H}$ (using the usual topology). Nevertheless $\texttt{R}$ is an order of $\mathbb{H}_{\mathbb{F}}$ (with $\mathbb{F} = \mathbb{Q}(\sqrt{2})$) as defined in section 2.1, since it is finitely generated by its 48 units over $\mathbb{Z}$. It is not hard to show that it is generated over $\mathbb{S} = \mathbb{Z}[\sqrt{2}]$ just by the four elements, $\{a, b, ab, ba\}$ where $a = \frac{1+i}{\sqrt{2}}$ and $b = \frac{1+j}{\sqrt{2}}$. Note also that the integral span of the coset of 'purely octahedral numbers' $i_{\text{O}} \cdot \texttt{H}$ constitutes a lattice in $\mathbb{H}$ which can be regarded as a 'rotated' version of the lattice $\texttt{H}$ (but which, unlike $\texttt{H}$, is not closed under multiplication).[7]

Finally, we have 96 *icosian* units [7], p. 207, defined in terms of

$$\tau = \frac{1}{2}(1 + \sqrt{5}) \quad \text{and} \quad \sigma = \frac{1}{2}(1 - \sqrt{5}), \tag{5.14}$$

to be all elements obtained from the following eight basic elements

$$\frac{1}{2}\left(\pm i \pm \sigma j \pm \tau k \right) \tag{5.15}$$

by the 12 *even* permutations of the Lipschitz units $(1, i, j, k)$. Together with the 24 Hurwitz units they form the icosian subgroup $\mathcal{E}_{\texttt{I}}$, which are the 120 units in the ring $\texttt{I}$ defined to be their integral span. According to [7], the structure of the icosian group $\mathcal{E}_{\texttt{I}} \cong 2 \cdot \mathfrak{A}_5$ is an index two extension of the alternating group on five letters. $\texttt{I}$ is an order of $\mathbb{H}_{\mathbb{F}}$ where $\mathbb{F} = \mathbb{Q}(\sqrt{5})$. As in the order $\texttt{R}$, the ring $\texttt{I}$ is dense in $\mathbb{H}$, but it is finitely generated over the integers by its 120 units. In fact, it is generated over $\mathbb{S} = \mathbb{Z}(\tau)$ by the four elements

$$i, \quad j, \quad \omega = \frac{1}{2}(-1 + i + j + k), \quad i_{\texttt{I}} = \frac{1}{2}(i + \sigma j + \tau k). \tag{5.16}$$

---

[7]In fact, we will only encounter matrices S whose entries belong either to $\texttt{H}$ (and have minimal polynomial of degree two) or to $i_{\text{O}}\texttt{H}$ (and have minimal polynomial of degree four). This will be distributed over the matrix in such a way that this structure is preserved under multiplication.

It can be checked straight-forwardly that all their products can be expressed as linear combinations of those four elements with coefficients from $\mathbb{S}$, for example:

$$i^2 = -1 = -(\sigma + 1)i - (\sigma + 2)j + 2\omega + 2\sigma i_{\mathrm{I}}, \tag{5.17}$$

$$ji_{\mathrm{I}} = -2\sigma i - (2\sigma + 1)j + \sigma\omega + (2\sigma + 1)i_{\mathrm{I}}, \qquad \omega i_{\mathrm{I}} = \omega - i - k$$

$$i_{\mathrm{I}}\omega = -i - (\sigma + 1)j + \omega + \sigma i_{\mathrm{I}}, \qquad\qquad\qquad i_{\mathrm{I}}j = -i - \sigma j + \sigma\omega + i_{\mathrm{I}}\,.$$

These can be used to compute the commutators of pairs of generators, e.g.

$$[i, j] = 2k, \qquad\qquad [i, i_{\mathrm{I}}] = -\tau j + \sigma k, \tag{5.18}$$

If we try to imitate what we did for the Hurwitz integers, and let $\mathtt{C_I} = \mathtt{I}[\mathtt{I}, \mathtt{I}]\mathtt{I}$ be the ideal of $\mathtt{I}$ generated by the commutators $[\mathtt{I}, \mathtt{I}]$, we find that $\mathtt{C_I} = \mathtt{I}$ is the whole icosian ring. This follows since the two elements in (5.18) have norms 2 and 3 respecticely, so that the difference of their norms is 1 and hence $1 \in \mathtt{C_I}$. This means that the quotient ring, $\mathtt{I}/\mathtt{C_I}$ is trivial, not a field as was $\mathtt{H}/\mathtt{C}$, and we do not have an analog of lemma 1 for $\mathtt{I}$.

Even though most rank four root systems contain units from one of the dense rings $\mathtt{R}$ or $\mathtt{I}$ we will explain how one can nevertheless construct *discrete matrix groups* from them which are related to the even Weyl groups.

5.3   Quaternionic determinants

Useful introductory references for quaternionic determinants are [15, 1]. When $\mathbb{K} = \mathbb{H}$, the matrices in (3.16) are (2×2) matrices over the quaternions, that is, matrices of the form

$$S = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \qquad \text{with } a, b, c, d \in \mathbb{H}\,. \tag{5.19}$$

To describe the hyperbolic Weyl groups, we will only need to take these entries from the rings introduced above, but the following observations apply to all quaternionic matrices. If one tries to define the determinant of the quaternionic matrix S to be $ad - bc$, the non-commutativity of $\mathbb{H}$ spoils the usual multiplication law for determinants (see [1]). A quantity which does have good properties for $\mathbb{K} = \mathbb{H}$ is obtained by observing that the determinant is well-defined for *Hermitian* quaternionic matrices (for which a and d are real, and $b = \bar{c}$). Given any quaternionic matrix S, we therefore associate with it the Hermitian matrix $SS^\dagger$ which has a well-defined determinant given by $\det(SS^\dagger) = \det(S^\dagger S)$.[8] The multiplication law is then obeyed provided that

$$\det\left((S_1 S_2) \cdot (S_2^\dagger S_1^\dagger)\right) = \det\left(S_1 S_1^\dagger\right) \det\left(S_2 S_2^\dagger\right). \tag{5.20}$$

This relation (which is obviously satisfied for $\mathbb{K} = \mathbb{R}, \mathbb{C}$) can be verified for $\mathbb{K} = \mathbb{H}$ by direct calculation (however, it will fail for $\mathbb{K} = \mathbb{O}$ because of non-associativity). Any quaternionic matrix S satisfying $\det(SS^\dagger) \neq 0$ has left and

---

[8]These expressions also make sense for $\mathbb{K} = \mathbb{O}$ but in general are no longer equal.

right inverses given by

$$\left(S^\dagger S\right)^{-1} S^\dagger = S^\dagger \left(S S^\dagger\right)^{-1}. \tag{5.21}$$

Here, $(SS^\dagger)^{-1}$ denotes the well-defined inverse of a Hermitian matrix. That the two expressions coincide can be shown by explicit computation.[9] Hence, the condition $\det\left(SS^\dagger\right) = 1$ for (2×2) matrices with quaternionic entries indeed defines a continuous group which we denote by

$$SL_2(\mathbb{H}) := \left\{ S = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \ \middle| \ a, b, c, d \in \mathbb{H}, \ \det\left(SS^\dagger\right) = 1 \right\}. \tag{5.22}$$

Writing $\mathbb{1}$ for the (2×2) identity matrix, the subgroup $\{\pm\mathbb{1}\}$ is normal and we define $PSL_2(\mathbb{H}) = SL_2(\mathbb{H})/\{\pm\mathbb{1}\}$ to be the quotient group. We write elements in the quotient as matrices S but we identify S with $-S$.[10] This group has real dimension $15 = 4 \cdot 4 - 1$, and is known to be isomorphic to the Lorentz group in six dimensions [39, 2], i.e.

$$PSL_2(\mathbb{H}) \cong PSO(1, 5; \mathbb{R}). \tag{5.23}$$

We will soon see that the even Weyl groups of all rank six Kac–Moody algebras under consideration are discrete subgroups of $PSL_2(\mathbb{H})$.

The easiest of these discrete groups to describe is $PSL_2(\mathtt{H})$, which is obtained from $PSL_2(\mathbb{H})$ by restricting all matrix entries to be Hurwitz integers. We will also need its subgroup

$$PSL_2^{(0)}(\mathtt{H}) := \left\{ S = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right] \in PSL_2(\mathtt{H}) \ \middle| \ ad - bc \equiv 1 (\mathrm{mod}\, \mathtt{C}) \right\} \tag{5.24}$$

which can be understood in the following way. The composition of the group homomorphism

$$PSL_2(\mathtt{H}) \to PSL_2(\mathtt{H}/\mathtt{C}) \tag{5.25}$$

with the usual determinant of a (2×2) matrix over the field $\mathtt{H}/\mathtt{C} \cong \mathbb{F}_4$, yields a group homomorphism

$$\mathrm{Det} : PSL_2(\mathtt{H}) \to \mathbb{F}_4^* \tag{5.26}$$

onto $\mathbb{F}_4^* \cong \mathbb{Z}_3$ the cyclic group of order 3.[11] Since this composition is a group homomorphism, we have $\mathrm{Det}\,(S_1 S_2) = \mathrm{Det}\,S_1 \cdot \mathrm{Det}\,S_2$, and its kernel,

$$PSL_2^{(0)}(\mathtt{H}) = \left\{ S \in PSL_2(\mathtt{H}) \ \middle| \ \mathrm{Det}\,S = 1 \right\}, \tag{5.27}$$

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[9]For $\mathbb{K} = \mathbb{O}$, the two expressions in (5.21) are generally different because of non-associativity.

[10]In fact, if $S \in SL_2(\mathbb{H})$ then so is εS for any unit ε. Because of non-commutativity, only $\varepsilon = \pm 1$ are in the center and quotiented out.

[11]This 'determinant' is different from the Dieudonné determinant [1].

is a normal subgroup of index 3 in $PSL_2(\mathtt{H})$. Since $\mathtt{L} = \mathtt{C} \cup (1 + \mathtt{C})$ is the coset decomposition of $\mathtt{L}$, the condition $ad - bc \equiv 1(\mathrm{mod}\,\mathtt{C})$ says that $ad - bc \in (1 + \mathtt{C}) \subset \mathtt{L}$ and Lemma 1 says that the order of the products in the expression $ad - bc$ does not matter. Since $ad - bc \notin \mathtt{C}$, the condition for $S \in PSL_2^{(0)}(\mathtt{H})$ is just $ad - bc \in \mathtt{L}$. We have proved the following.

Lemma 2. $PSL_2^{(0)}(\mathtt{H})$ *is index 3 in* $PSL_2(\mathtt{H})$, *and*

$$PSL_2(\mathtt{H})/PSL_2^{(0)}(\mathtt{H}) = \mathfrak{A}_3 \tag{5.28}$$

where $\mathfrak{A}_3 \cong \mathbb{Z}_3$ *is the alternating group on three letters.*

The modular group $PSL_2^{(0)}(\mathtt{H})$ contains the modular group $PSL_2(\mathtt{L})$, but is strictly larger, so that we arrive at the following chain of subgroup relations

$$PSL_2(\mathtt{L}) \subset PSL_2^{(0)}(\mathtt{H}) \subset PSL_2(\mathtt{H}). \tag{5.29}$$

The map Det extends to the ring of all (2×2) Hurwitz matrices, first reducing entries modulo $\mathtt{C}$, and then taking the usual determinant. It is still a multiplicative map so that the invertible matrices have non-zero Det. But it is possible for a 'non-invertible' matrix in that ring to also have a non-zero Det. For example, the diagonal matrix $\mathrm{diag}(1, 3)$ does not have an inverse over the Hurwitz integers, but since $2 \in \mathtt{C}$, it reduces to the identity matrix modulo $\mathtt{C}$, whose usual determinant is $1 + \mathtt{C} \in \mathbb{F}_4^*$.

Proposition 5. *The modular group* $PSL_2^{(0)}(\mathtt{H})$ *is generated by the matrices*

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \tag{5.30}$$

where $a, b \in \mathcal{E}_{\mathtt{H}}$ *are Hurwitz units and* $ab \equiv 1(\mathrm{mod}\,\mathtt{C})$.

Proof: This claim can be proved by adapting Theorem 2.2 on page 16 of [28], and arguments very similar to those of proposition 1. According to Krieg's theorem, the set of all uni-modular (= invertible) (2×2) Hurwitz quaternionic matrices, here denoted by $SL_2(\mathtt{H}) \subset SL_2(\mathbb{H})$ in accordance with definition (5.22) (but designated as $GL_2(\mathtt{H})$ in [28]) is generated by taking products of the matrices in (5.30) without any restriction on the product ab, so that a, b run through all Hurwitz units. This is also true for the quotient $PSL_2(\mathtt{H})$. So we can write any element $S \in PSL_2(\mathtt{H})$ as

$$S = G_1 \cdots G_n \tag{5.31}$$

where each G_i, $i = 1, \dots, n$, is one of the three types of generating matrices in (5.30), but with no restriction on the product ab for the third type in accordance with Krieg's theorem. We will show that we can rewrite S in the form

$$S = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} G_1' \cdots G_{n'}', \tag{5.32}$$

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where $c \in \mathtt{H}$ and all G'_i matrices are from (5.30) with the third type satisfying the $ab \equiv 1(\mathrm{mod}\ \mathtt{C})$ condition. In order to arrive at this new presentation we show explicitly how to convert the presentation (5.31) to one of the form (5.32), and how the element c represents $\mathrm{Det}(S)$.

Examine the matrices in the expression $G_1 \cdots G_n$ from the right to the left. We don't need to change a matrix of the first or second kind from (5.30), nor do we change one of the third kind if it satisfies the $ab \equiv 1(\mathrm{mod}\ \mathtt{C})$ condition. Let i_0 be the largest index for which $G_{i_0} = \mathrm{diag}(a_{i_0}, b_{i_0})$ with $a_{i_0} b_{i_0} \not\equiv 1(\mathrm{mod}\ \mathtt{C})$. Then we can factorize it as

$$\begin{bmatrix} a_{i_0} & 0 \\ 0 & b_{i_0} \end{bmatrix} = \begin{bmatrix} \bar{\varepsilon} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon a_{i_0} & 0 \\ 0 & b_{i_0} \end{bmatrix} \tag{5.33}$$

where the unit $\varepsilon \in \mathcal{E}_{\mathtt{H}} \setminus \mathcal{E}_{\mathtt{L}}$ is chosen such that $\varepsilon a_{i_0} b_{i_0} = 1 \in \mathtt{L}$. We will show below how the new matrix $\mathrm{diag}(\bar{\varepsilon}, 1)$ can be moved to the left by passing through the other generators G_i with $i < i_0$, and leaving only acceptable generators on its right.

We first note that from (5.30) (with the restriction on the product) we can generate all translation matrices

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \quad \text{for } a \in \mathtt{H}. \tag{5.34}$$

For $a \in \{\pm i, \pm j, \pm k\}$ we have

$$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -a & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \tag{5.35}$$

and $a \equiv 1 \ (\mathrm{mod}\ \mathtt{C})$ so each matrix on the left side is from (5.30). Products of these provide all translation matrices (5.34) with $a \in \mathtt{L}$. To extend this to all $a \in \mathtt{H}$ it is sufficient to find one translation matrix with $a \in \mathtt{H} \backslash \mathtt{L}$. The computation in (4.8) where $\theta = \frac{1}{2}(1 - i - j - k)$ satisfies $\theta^2 = -\bar{\theta} = \frac{1}{2}(-1 - i - j - k)$, yields this, so that we indeed obtain all matrices (5.34) from (5.30). From this and

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{\varepsilon} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \bar{\varepsilon} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \varepsilon \\ 0 & 1 \end{bmatrix} \tag{5.36}$$

we see that we can pass the matrix $\mathrm{diag}(\bar{\varepsilon}, 1)$ across a matrix G_i of the first kind in (5.30) by replacing G_i by a translation matrix which is a product of the allowed generators.

Similarly we have

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{\varepsilon} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \bar{\varepsilon} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon & 0 \\ 0 & \bar{\varepsilon} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{5.37}$$

so that we can also pass through the second type of generating matrices by replacing it by a product of allowed matrices.

Finally, for any units $a, b \in \mathtt{H}$, we can also write

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \bar{\varepsilon} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a\bar{\varepsilon}b & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{b} & 0 \\ 0 & b \end{bmatrix} , \qquad (5.38)$$

so the matrix on the far right is an allowed generator, but the matrix being moved to the left has been changed to another of the form $\mathrm{diag}(c, 1)$ with $c = a\bar{\varepsilon}b \in \mathtt{H}$ a unit. Using (5.36)–(5.38), for some unit $c \in \mathtt{H}$, we finally arrive at the expression

$$S = \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix} G'_1 \cdots G'_{n'} \qquad (5.39)$$

with all G'_i belonging to the list of generating matrices (5.30) and those of the third kind have $ab \equiv 1 \pmod{\mathtt{C}}$, so all $\mathrm{Det}(G'_i) = 1 \in \mathbb{F}_4^*$. Then $\mathrm{Det}(S) = \mathrm{Det}(\mathrm{diag}(c, 1))\mathrm{Det}(G'_1) \cdots \mathrm{Det}(G'_{n'}) = c + \mathtt{C} \in \mathtt{H}/\mathtt{C} = \mathbb{F}_4$. If we assume that $S \in PSL_2^{(0)}(\mathtt{H})$, so $\mathrm{Det}(S) = 1$, then we must have $c \equiv 1 \pmod{\mathtt{C}}$ so that $\mathrm{diag}(c, 1)$ is an allowed generator from (5.30) and the proof is complete. $\square$

We close this section by counting the number of diagonal matrices $\mathrm{diag}(a, b)$ such that $a, b \in \mathtt{H}$ are units and $ab \in \mathtt{L}$. There are $64 = 8 \times 8$ such matrices when both $a, b \in \mathtt{L}$. If $a \in \mathtt{H} \backslash \mathtt{L}$ is one of the 16 pure Hurwitz units there are 8 choices for b such that $ab \in \mathtt{L}$, which gives $128 = 16 \times 8$ such matrices. Descending to the quotient by $\{ 1\!\!1, -1\!\!1 \}$ we are left with $96 = \frac{1}{2} \times 192$ diagonal matrices in $PSL_2^{(0)}(\mathtt{H})$. The full $PSL_2(\mathtt{H})$ has $288 = \frac{1}{2} \times 24 \times 24$ diagonal matrices. This gives another way to see lemma 2.

5.4   Even Weyl group $W^+(D_4^{++})$

We first study the Weyl group of the rank 6 hyperbolic algebra D_4^{++}. Among the Weyl groups associated to the rank 6 algebras, this is the 'easiest' to understand because it is the only one whose root system can be expressed solely in terms of Hurwitz numbers. As we will see, the Weyl groups in all the other cases, except for A_4^{++}, are finite extensions of this one.

In the lattice $\mathtt{H}$ equipped with the inner product (2.1), we can recognize that $\mathcal{E}_\mathtt{H}$ forms the D_4 root system, and among these we choose as simple roots

$$\varepsilon_1 = a_1 = 1 , \quad \varepsilon_2 = a_2 = \frac{1}{2}\left(-1 + i - j - k \right) , \qquad (5.40)$$

$$\varepsilon_3 = a_3 = \frac{1}{2}\left(-1 - i + j - k \right) , \quad \varepsilon_4 = a_4 = \frac{1}{2}\left(-1 - i - j + k \right) .$$

We can choose $a_i = \varepsilon_i$ for all i because D_4 is simply laced. This system of simple roots exhibits $\mathfrak{S}_3$ 'triality' symmetry as outer automorphisms, that is, it is symmetric under cyclic permutation of the three imaginary units (i, j, k), and under the exchange of any two imaginary units, e.g. $(j \leftrightarrow k)$. The former is concretely realized by the map $z \mapsto \theta z \bar{\theta}$, whereas the latter corresponds to $z \mapsto i_\mathtt{O} \bar{z} \bar{i}_\mathtt{O}$. Here, θ is the highest D_4 root

$$\theta = 2a_1 + a_2 + a_3 + a_4 = \frac{1}{2}\left(1 - i - j - k \right) = j\varepsilon_2 = k\varepsilon_3 = i\varepsilon_4 \qquad (5.41)$$
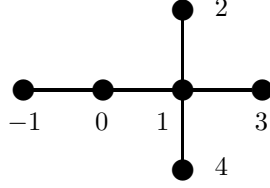
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Figure 9: *Dynkin diagram of D_4^{++} with numbering of nodes.*

and i_O is the specific octahedral unit of order four defined in (5.13). The 16 units in $\mathcal{E}_H \setminus \mathcal{E}_L$ are of order three or six, *viz.*

$$\varepsilon_2^3 = \varepsilon_3^3 = \varepsilon_4^3 = 1, \quad \theta^6 = 1. \tag{5.42}$$

With these choices we build the simple roots (2.9) of the hyperbolic algebra D_4^{++} in its root lattice $\Lambda(Q)$ where $Q = H$ is the D_4 root lattice. The corresponding Dynkin diagram of D_4^{++}, the hyperbolic over-extension of the D_4 Dynkin diagram, is depicted in fig. 9. Recall that the Weyl group of D_4 is $W(D_4) = 2^3 \rtimes \mathfrak{S}_4$.

Our central result is

Proposition 6. $W^+(D_4^{++}) \cong PSL_2^{(0)}(H)$.

Proof: Substituting (5.40) into the expressions of the generating matrices S_i, $0 \le i \le 4$, given in (3.14), one sees that all these matrices have $\det(S_i S_i^\dagger) = 1$ and $\mathrm{Det}(S_i) = 1$ so they belong to $PSL_2^{(0)}(H) \subset PSL_2(H)$ as defined in (5.24). Therefore, we get $W^+(D_4^{++}) \subset PSL_2^{(0)}(H)$. To prove the converse we show that all the generating matrices of proposition 5 are in $W^+(D_4^{++})$. To do this we compute

$$S_i S_1^{-1} = \begin{bmatrix} \varepsilon_i & 0 \\ 0 & \bar{\varepsilon}_i \end{bmatrix} \qquad \text{for } i = 2, 3, 4,$$

$$S_2 S_3 = \begin{bmatrix} i & 0 \\ 0 & j \end{bmatrix}, \quad S_3 S_4 = \begin{bmatrix} j & 0 \\ 0 & k \end{bmatrix}, \quad S_4 S_2 = \begin{bmatrix} k & 0 \\ 0 & i \end{bmatrix}. \tag{5.43}$$

It is straightforward to see that by further multiplication and permutation we can obtain from these matrices and S_1 any matrix of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \tag{5.44}$$

where (a, b) is any pair of Hurwitz units whose product is a Lipschitz unit, i.e. $N(a) = N(b) = 1$ and $ab \equiv 1 \pmod{C}$. We note that up to $(a, b) \cong (-a, -b)$ there are 96 pairs with this property so that there are 96 matrices of each type in (5.44) in $PSL_2(H)$ corresponding to the even and odd part of the finite Weyl group $W(D_4)$ of order 192.

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Given the matrices (5.44) we can also reconstruct the translation matrix from S_0 (using $\theta^3 = -1$) and

$$\begin{bmatrix} 0 & \theta^2 \\ \theta^{-2} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \theta \\ -\overline{\theta} & 1 \end{bmatrix} \cdot \begin{bmatrix} -\overline{\theta} & 0 \\ 0 & \theta \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}. \tag{5.45}$$

The translation matrix is the inverse of this, giving the last of the required generators (5.30). $\square$

The appearance of the group $PSL_2^{(0)}(\mathbb{H})$ for D_4^{++} (rather than $PSL_2(\mathbb{H})$) may be viewed as a manifestation of the triality symmetry of the D_4 Dynkin diagram. Since the even cyclic permutation of (i, j, k) is realized by conjugation by the Hurwitz unit θ, the associated diagonal matrix $\mathrm{diag}(\theta, \theta) \in PSL_2(\mathbb{H})$ solves the necessary conditions (3.27), but itself is not part of the Weyl group, and therefore has to be removed from $PSL_2(\mathbb{H})$. This is what the $\mathrm{Det}\,S = 1$ condition achieves, and since $\theta^3 = -1$ we find that the subgroup $PSL_2^{(0)}(\mathbb{H})$ is index three in $PSL_2(\mathbb{H})$, in accordance with lemma 2. In this way, the group of (even) *outer* automorphism of D_4 can also be realized by matrix conjugation.

For later reference we denote the alternative generating set of the group $PSL_2^{(0)}(\mathbb{H})$ furnished by the even D_4^{++} Weyl group by

$$g_i = S_i^{D_4} \qquad \text{for } i = 0, \dots, 4. \tag{5.46}$$

An alternative argument leading to the statement of the proposition can be based on the fact that *every* Hermitian matrix $\mathtt{X} \in \Lambda(D_4)$ obeying $\det \mathtt{X} = -1$ is a real root,[12] and that the (full) Weyl group acts transitively on the set of real roots. In the case of D_4^{++} they form a single Weyl orbit, since the Dynkin diagram has only single lines. This implies that we can generate *all* real roots from the hyperbolic simple root α_{-1} by acting with the full Weyl group, *viz.*

$$\Delta^{\mathrm{re}} = \left\{ \mathtt{X} \in \Lambda(D_4) \,|\, \det \mathtt{X} = -1 \right\} = W \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{5.47}$$

In particular, considering all even Weyl images of the form $s(\alpha_{-1}) = S\alpha_{-1}S^\dagger$, and using the fact that Weyl transformations preserve the norm, we get

$$-1 = \det(\alpha_{-1}) = \det(S\alpha_{-1}S^\dagger) = -\det(SS^\dagger) \tag{5.48}$$

where the third equality can be verified by direct computation for the quaternions. This is the same condition as (5.22). Conversely, to any $S \in PSL_2(\mathbb{H})$ satisfying (5.22) we can associate the real root $S\alpha_{-1}S^\dagger$, whence S becomes associated to some symmetry of $\Lambda(D_4)$. This seems to give all of $PSL_2(\mathbb{H})$ but the reasoning does not distinguish between inner and outer transformations. As argued above, taking the outer (diagram) automorphisms into account is the same as demanding the extra condition (5.24).

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[12]It is here that the hyperbolicity of the Kac–Moody algebra is essential; this statement is no longer true if the indefinite Kac–Moody algebra is not hyperbolic.
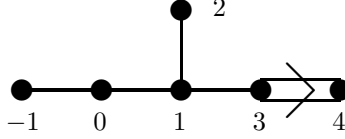
Figure 10: *Dynkin diagram of B_4^{++} with numbering of nodes.*

5.5 Even Weyl group $W^+(B_4^{++})$

The B_4 root lattice is isomorphic to the hypercubic lattice of the Lipschitz integers (in the same way that B_2 was associated with the cubic lattice of Gaussian integers). The simple roots can be chosen as follows

$$
\begin{aligned}
\varepsilon_1 &= a_1 = 1 \ , \quad \varepsilon_2 = a_2 = \frac{1}{2}\big(-1 + i - j - k\big) \ , \\
\varepsilon_3 &= a_3 = \frac{1}{2}\big(-1 - i + j - k\big) \ , \quad \varepsilon_4 = \sqrt{2}a_4 = \frac{-j + k}{\sqrt{2}} \ .
\end{aligned}
\tag{5.49}
$$

The highest (long) root is

$$
\theta = a_2 + 2a_1 + 2a_3 + 2a_4 = \frac{1}{2}\big(1 - i - j - k\big) .
\tag{5.50}
$$

Hence, a_4 is the short simple root, while the other (long) roots, including the highest root θ, are normalized to unity. These simple roots were chosen to be close to the D_4 simple roots: a_1, a_2 and a_3 agree and only a_4 is changed; the highest roots agree. This somewhat obscures the fact that the B_4 lattice is a scaled and rotated version of the lattice of Lipschitz numbers but makes the determination of the Weyl group simpler. The finite Weyl group is $W(B_4) = 2^4 \rtimes \mathfrak{S}_4$, and is concretely realized via permutations and reflections of the four basis elements $1, i, j, k$. Observe also that the above set of simple roots is not invariant under cyclic (triality) transformations of the three imaginary units, as was to be expected.

The hyperbolic extension B_4^{++} can be obtained by following the procedure described before and leads to the Dynkin diagram of fig. 10. Turning to its even hyperbolic Weyl group $W^+(B_4^{++})$ we notice that among the generators $S_i \in PSL_2(\mathbb{H})$ (for $i = 0, \ldots, 4$) the first four are identical to those of $W^+(D_4^{++})$. By explicit computation one finds furthermore that

$$
S_4 S_3 S_4 = \begin{bmatrix} 0 & \frac{1}{2}(1 + i + j - k) \\ \frac{1}{2}(-1 + i + j - k) & 0 \end{bmatrix} = g_4 ,
\tag{5.51}
$$

where g_4 was defined in (5.46) in the alternative generating set of $PSL_2^{(0)}(\mathbb{H})$. This shows that $W^+(B_4^{++})$ has a subgroup isomorphic to $PSL_2^{(0)}(\mathbb{H})$ generated

by S_0, S_1, S_2, S_3 and $S_4 S_3 S_4$. Adjoining to these the element S_4 which satisfies $S_4^2 = -\mathbb{1} \cong \mathbb{1}$ gives an extension in which $PSL_2^{(0)}(\mathtt{H})$ is a normal subgroup of index 2, and we arrive at

Proposition 7. $W^+(B_4^{++}) \cong PSL_2^{(0)}(\mathtt{H}) \rtimes 2$.

The semi-directness follows from expanding the action $S_4 S S_4^{-1}$ for $S \in PSL_2^{(0)}(\mathtt{H})$ and using the properties of ε_4. As the index 2 extension is realized via the matrix S_4 which contains $1/\sqrt{2}$ in all entries we see that $W^+(B_4^{++})$ consists of all matrices (5.19) in $PSL_2(\mathbb{H})$ such that either all a, b, c, d are Hurwitz numbers, or all are pure octahedral numbers $\in i_O \cdot \mathtt{H}$ (it is easy to see that this property is preserved under matrix multiplication because $i_O \mathtt{H} \cdot \mathtt{H} = i_O \mathtt{H}$ and $i_O \mathtt{H} \cdot i_O \mathtt{H} = \mathtt{H}$). This is the 'trick' by which the discreteness of the matrix group is achieved despite the fact that $\mathtt{R}$ is dense in $\mathbb{H}$.

To conclude the analysis of B_4^{++} we relate our results, restricted to the finite Weyl group $W^+(B_4)$, to the classification of [8] (see their table 4.2) where $W^+(B_4)$ is denoted by $\pm\frac{1}{6}[O \times O]$. It is not hard to check that all the generating elements given in [8] can be obtained from products of matrices of the type (3.22) when the B_4 units (5.49) are plugged in.

5.6 Even Weyl group $W^+(C_4^{++})$

For C_4 the direction of the arrow in the finite Dynkin diagram is reversed compared to B_4. A convenient choice of simple roots requires again quaternions which are not Hurwitz. In order to facilitate the comparison with D_4 we choose the first three simple roots of C_4 to coincide with the simple roots a_3, a_1, a_4 of D_4 up to a re-scaling required to maintain the unit normalization of the highest root θ. Thus

$$\varepsilon_1 = \sqrt{2}a_1 = 1 \ , \quad \varepsilon_2 = \sqrt{2}a_2 = \frac{1}{2}\big(-1 + i - j - k\big)$$

$$\varepsilon_3 = \sqrt{2}a_3 = \frac{1}{2}\big(-1 - i + j - k\big) \ , \quad \varepsilon_4 = a_4 = \frac{-j+k}{\sqrt{2}} \qquad (5.52)$$

with the (long) highest root

$$\theta = 2a_1 + 2a_2 + 2a_3 + a_4 = \frac{-j-k}{\sqrt{2}} \ , \qquad (5.53)$$

For the determination of the Weyl group we again choose a convenient basis. The hyperbolic extension C_4^{++} has Dynkin diagram given in fig. 11. Although the B_4 and C_4 Weyl groups are isomorphic (cf. table 1), the Weyl groups of C_4^{++} and B_4^{++} are nevertheless different. This difference in our basis is reflected only in the difference between the highest roots; for C_4 we have an octahedral unit of order four whereas for B_4 the highest root was a Hurwitz number of order six.
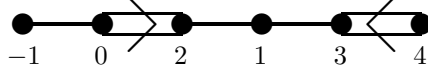
Figure 11: *Dynkin diagram of C_4^{++} with numbering of nodes.*

Proceeding as for $W^+(B_4^{++})$ we first observe that all S_i for $i = 1, \ldots, 4$ are identical to those of B_4^{++} (since the finite Weyl groups are isomorphic), and hence again

$$S_4 S_3 S_4 = \begin{bmatrix} 0 & \frac{1}{2}(1 + i + j - k) \\ \frac{1}{2}(-1 + i + j - k) & 0 \end{bmatrix} = g_4 \qquad (5.54)$$

in terms of (5.46). Therefore we obtain all diagonal and off-diagonal matrices (5.44) reflecting the fact that the finite Weyl groups of B_4 and C_4 are isomorphic. Using the C_4 highest root (5.53) we now define

$$\tilde{g}_i = \begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix} g_i \begin{bmatrix} \bar{\theta} & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{for} \; i = 1, 2, 3, 4, \qquad (5.55)$$

which is a unitary transformation of all the generators of $PSL_2^{(0)}(\mathtt{H})$ defined in (5.46) except for g_0. The matrices $\tilde{g}_i$ ($i = 1, 2, 3, 4$) also belong to $W^+(C_4^{++})$ since $\theta a \bar{\theta} b \equiv ab \, (\mathrm{mod} \, \mathtt{C})$. Furthermore, since

$$\begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{\theta} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \theta \\ 0 & 1 \end{bmatrix}. \qquad (5.56)$$

we also have the unitarily transformed translation matrix. From this we conclude that the group generated by S_0, S_1, S_2, and S_3 and $S_4 S_3 S_4$ consists of matrices of the type

$$\begin{bmatrix} \theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \bar{\theta} & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{for} \; \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL_2^{(0)}(\mathtt{H}). \qquad (5.57)$$

We denote this transformed (or 'twisted') $PSL_2^{(0)}(\mathtt{H})$ by $\widetilde{PSL}_2^{(0)}(\mathtt{H})$. In order to describe the full $W^+(C_4^{++})$ we still need to adjoin the matrix S_4 which still satisfies $S_4^2 = -\mathbb{1} \cong \mathbb{1}$ so that we find

Proposition 8. $W^+(C_4^{++}) \cong \widetilde{PSL}_2^{(0)}(\mathtt{H}) \rtimes 2$.

The semi-directness of the product can be verified by expanding the action of S_4 on $\widetilde{PSL}_2^{(0)}(\mathtt{H})$ which can be seen to give an automorphism of $\widetilde{PSL}_2^{(0)}(\mathtt{H})$.
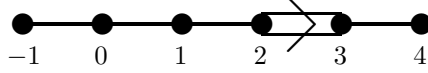
Figure 12: *Dynkin diagram of F_4^{++} with numbering of nodes.*

Although $\widetilde{PSL}_2^{(0)}(\mathtt{H}) \cong PSL_2^{(0)}(\mathtt{H})$, it must be stressed that the Weyl groups of C_4^{++} and B_4^{++} are nevertheless different, since for C_4^{++} the extension S_4 matrix is not unitarily transformed, so that the group extension is different.[13] The $PSL_2(\mathbb{H})$ matrices (5.19) belonging to $W^+(C_4^{++})$ can be described by saying that either $a, d \in \mathtt{H}$ and $b, c \in i_{\mathrm{O}}\cdot\mathtt{H}$, or conversely $a, d \in i_{\mathrm{O}}\cdot\mathtt{H}$ and $b, c \in \mathtt{H}$. Again, one can easily verify that this structure is preserved under multiplication.

Since the finite Weyl group satisfies $W(C_4) \cong W(B_4)$, the relation to the chiral group $\pm\frac{1}{6}[O \times O]$ of the classification in [8] holds analogously to the last section.

5.7 Even Weyl group $W^+(F_4^{++})$

The root system of F_4 consists of two copies of the D_4 system, one of which is rescaled. There are two short simple roots and two long simple roots. We choose the simple roots as follows

$$
\begin{aligned}
\varepsilon_1 &= a_1 = 1 \ , \quad \varepsilon_2 = a_2 = \frac{1}{2}\big(-1 + i - j - k\big) \ , \\
\varepsilon_3 &= \sqrt{2}a_3 = \frac{-i + j}{\sqrt{2}} \ , \quad \varepsilon_4 = \sqrt{2}a_4 = \frac{-j + k}{\sqrt{2}} \ ;
\end{aligned}
\tag{5.58}
$$

the highest (long) root is

$$
\theta = 2a_1 + 3a_2 + 4a_3 + 2a_4 = \frac{1}{2}\big(1 - i - j - k\big) .
\tag{5.59}
$$

The hyperbolic extension F_4^{++} has the Dynkin diagram shown in fig. 12.

The even Weyl group $W^+(F_4^{++})$ can be related to $PSL_2^{(0)}(\mathtt{H})$ in the following way: The generators g_0, g_1 and g_2 are identical to S_0, S_1 and S_2 constructed from the F_4 simple roots (5.58). Now consider

$$
S_3 S_2 S_3 = \left[\begin{array}{c|c} 0 & \frac{1}{2}(-1 - i + j - k) \\ \hline \frac{1}{2}(1 - i + j - k) & 0 \end{array} \right] \equiv g_3
\tag{5.60}
$$

---

[13]This is similar to the affine Weyl groups $W^+(B_4^+)$ and $W^+(C_4^+)$ although both are semi-direct products of isomorphic finite groups with abelian groups of the same rank. However, the automorphism involved in the definition of the semi-direct product is different.

and

$$S_4 S_3 S_2 S_3 S_4 = \begin{bmatrix} 0 & \frac{1}{2}(-1-i-j+k) \\ \frac{1}{2}(1-i-j+k) & 0 \end{bmatrix} \equiv g_4 , \qquad (5.61)$$

in terms of (5.46), so that we find all $PSL_2^{(0)}(\mathtt{H})$ generators in $W^+(F_4^{++})$. Adjoining to $PSL_2^{(0)}(\mathtt{H})$ the generators S_3 and S_4 one obtains all of $W^+(F_4^{++})$. Since S_3 and S_4 together generate the symmetric group on three letters we arrive at

Proposition 9. $W^+(F_4^{++}) \cong PSL_2^{(0)}(\mathtt{H}) \rtimes \mathfrak{S}_3 \cong PSL_2(\mathtt{H}) \rtimes 2$.

The last equality follows because the (diagonal) entries of $S_3 S_4$ are pure Hurwitz units, and both of order three, generating the cyclic group $\mathbb{Z}_3$ which extends $PSL_2^{(0)}(\mathtt{H})$ to $PSL_2(\mathtt{H})$. The semi-directness of the product can be verified explicitly by using the properties of ε_3 and ε_4.

The finite Weyl group $W^+(F_4)$ is now the chiral group $\pm\frac{1}{2}[O \times O]$ in the classification of [8]. Again it is not hard to see that the units (5.58) give rise to the generating elements given there.

5.8 Even Weyl group $W^+(A_4^{++})$

We discuss the algebra A_4^{++} last, because, somewhat surprisingly, it is the most cumbersome to describe in the present framework among the rank 6 algebras. In order to embed its root system into the quaternions one needs to make use of the *icosians* $\mathtt{I}$ introduced at the end of section 5.2. The algebra A_4^{++} is simply laced, and we choose the following four units

$$\varepsilon_1 = a_1 = 1 , \qquad\qquad \varepsilon_2 = a_2 = \frac{1}{2}(-1 + i - j - k) ,$$

$$\varepsilon_3 = a_3 = -i , \qquad\qquad \varepsilon_4 = a_4 = \frac{1}{2}(i + \sigma j + \tau k) , \qquad (5.62)$$

which have inner products corresponding to the A_4 Cartan matrix. The highest root is

$$\theta = a_1 + a_2 + a_3 + a_4 = \frac{1}{2}(1 - \tau j - \sigma k) . \qquad (5.63)$$

The Dynkin diagram of A_4^{++} is displayed in fig. 13.

As a first step we identify the finite Weyl group $W(A_4)$ of order 120 in terms of diagonal and off-diagonal matrices. The diagonal matrices can be generated from $\mathrm{diag}\,(\varepsilon_i, \bar{\varepsilon}_i)$ for $i = 2, 3, 4$ and, modulo $-\mathbb{1}$, one obtains all 60 even elements of $W(A_4)$ in this way. This shows that the root system of A_4 is not multiplicatively closed. Closer inspection reveals that the diagonal and off-diagonal matrices are of the form

$$\begin{bmatrix} a & 0 \\ 0 & a^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & a \\ -a^* & 0 \end{bmatrix} , \qquad (5.64)$$
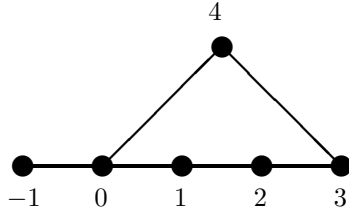
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Figure 13: *Dynkin diagram of A_4^{++} with numbering of nodes.*

where

$$a^* = a_0' - a_1'i - a_3'j - a_2'k \qquad \text{for } a = a_0 + a_1i + a_2j + a_3k \qquad (5.65)$$

with $(p + \sqrt{5}q)' := p - \sqrt{5}q$ for $p, q \in \mathbb{Q}$. For icosians the conjugation in $\sqrt{5}$ amounts to interchanging τ and σ everywhere. The operation (5.65) is an involutive automorphism of the ring $\mathtt{I}$ of icosians and satisfies in particular $(ab)^* = a^*b^*$. The automorphism is outer due to the 'conjugation' in $\sqrt{5}$; however, the exchange of j and k together with the complex conjugation alone is an *inner automorphism* of the quaternion algebra $\mathbb{H}$, and is explicitly realized by conjugation with the octahedral unit i_O from (5.13).

Since there are 120 icosian units there are 60 matrices of either type in (5.64) after quotienting out by $\{\mathbb{1}, -\mathbb{1}\}$ in agreement with the order of the Weyl group $W(A_4) \cong \mathfrak{S}_5$. Moreover, the A_4 root lattice is defined by the condition

$$Q_{A_4} = \{z \in \mathtt{I} \,|\, z^* = \bar{z}\} \,. \qquad (5.66)$$

Because conjugation reverses the order of factors but the $*$-automorphism does not, we see from (5.66) again that Q_{A_4} is not multiplicatively closed and therefore does not form a subring of $\mathtt{I}$. Q_{A_4} is closed under complex conjugation.[14] The action of the even finite Weyl group on $z \in Q_{A_4}$ is computed to be $z \mapsto az\bar{a}^*$, which manifestly preserves the root lattice.

The full $W^+(A_4^{++})$ is obtained by adjoining to (5.64) the matrix

$$\begin{bmatrix} 1 & \theta \\ 0 & 1 \end{bmatrix} \qquad (5.67)$$

and the result is

Proposition 10. $W^+(A_4^{++}) \cong PSL_2^{(0)}(\mathtt{I})$.

Here, $PSL_2^{(0)}(\mathtt{I})$ is defined as the discrete subgroup of $PSL_2(\mathtt{I})$ satisfying the conditions (3.27) with the four units a_i from (5.62); the (0)-superscript indicates that additionally the $\mathbb{Z}_2$ outer automorphism of the A_4 Dynkin diagram has to

---

[14]The fixed point set of $*$ on the set of icosian units is $\{1\}$.

be removed. This outer automorphism acts concretely as $z \mapsto -z$ on $z \in Q_{A_4}$ and the corresponding matrix is $\mathrm{diag}(1, -1)$ which has to be quotiented out. The explicit result of solving the conditions (3.27) and taking the quotient for diagonal and off-diagonal matrices leaves the ones listed in (5.64). For the general matrices S it leaves in particular the matrix (5.67).

The finite Weyl group $W^+(A_4)$ in terms of the classification of [8] is $+\frac{1}{60}[I \times \bar{I}]$ and again we see that there the generators given there can be reproduced from the icosian units (5.62).

One might hope to give a more natural description of $PSL_2^{(0)}(\mathtt{I})$ as the kernel of a group homomorphism analogous to Det used to define $PSL_2^{(0)}(\mathtt{H})$, but since the ideal $\mathtt{C_I}$ generated by the commutators in $\mathtt{I}$ is not proper, this idea fails.

5.9 Icosians and E_8

Using icosians it is also possible to give an embedding of the E_8 root system into $\mathbb{H}$ at the cost of introducing a new inner product. This inner product is defined by

$$(a, b)_\tau := x, \quad \text{if } (a, b) = a\bar{b} + b\bar{a} = x + \tau y \tag{5.68}$$

for a, b, x, y being rational quaternions. The E_8 simple roots are then given by (see e.g. [33])

$$a_1 = \frac{1}{2}(-\sigma - \tau i - k), \qquad a_2 = \frac{1}{2}(-\sigma i - \tau j + k),$$

$$a_3 = \frac{1}{2}(+i - \sigma j - \tau k), \qquad a_4 = \frac{1}{2}(-\tau i + j + \tau^2 k),$$

$$a_5 = \frac{1}{2}(+\tau i + j - \tau^2 k), \qquad a_6 = \frac{1}{2}(+i - \tau^2 j + \tau k),$$

$$a_7 = \frac{1}{2}(+1 - \tau^2 i - \tau k), \qquad a_8 = \frac{1}{2}(-i - \sigma j + \tau k). \tag{5.69}$$

Note that $\tau^2 = \tau + 1$. Here, a_4, a_5, a_6 and a_7 belong to $\tau\mathtt{I}$ constituting the second half of the 240 E_8 roots. The subspace spanned by integer linear combinations of the basis vectors (5.69) is dense in $\mathbb{H}$ (as one would expect, since we are projecting an 8-dimensional lattice onto a 4-dimensional hyperplane in such a way that distincts points remain distinct), and the same is, of course, true for the ring generated by the above elements in $\mathbb{H}$. In the following section we will present and discuss an *octonionic* realization of the E_8 root system, and we will see that the non-associativity leads to drastic changes in the matrix realization of the Weyl group. For this reason, one might have hoped to be able to avoid problems with non-associativity by working with the above quaternionic realization, but unfortunately the difficulties remain, mainly due to the modified inner product (5.68). Namely, the projection orthogonal to τ, which enters (5.68), invalidates our theorem 1, and we therefore cannot use the above representation to find a matrix representation of the E_8 Weyl group. For this reason we do not pursue this quaternionic realization of E_8 further in this paper.

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6 Octonions $\mathbb{K} = \mathbb{O}$

In this section we turn to the non-associative octonions and their relation to hyperbolic algebras of maximal rank 10. We restrict our attention mostly to the case $E_8^{++} \equiv E_{10}$ and exhibit in this case the kind of new complications arising due to the non-associativity of $\mathbb{O}$. For the other two hyperbolic over-extended algebras B_8^{++} and D_8^{++} we content ourselves with briefly presenting the octonionic realizations of their root lattices and stating their relation to that of E_{10}.

6.1 Octavians and the E_8 lattice

We use the octonionic multiplication conventions of Coxeter [9]. Hence, the seven imaginary units e_i appearing in the expansion of an octonion z via ($e_0 = 1$)

$$z = n_0 + \sum_{i=1}^{7} n_i e_i \tag{6.1}$$

multiply (in quaternionic subalgebras) according to

$$e_i e_{i+1} e_{i+3} = -1 \,, \tag{6.2}$$

where the indices are to be taken modulo seven and the relation is totally anti-symmetric. The multiplication table for the octonions can be neatly summarised by using the Fano plane, as shown in fig. 14. The following notation for the structure constants of the imaginary units can also be used

$$e_i e_j = -\delta_{ij} + f_{ijk} e_k \tag{6.3}$$

where we adopt the usual summation convention from now on. The structure constants f_{ijk} are totally antisymmetric, and satisfy (see e.g. [13] where many further identites involving these structure constants can be found)

$$f_{ijk} f^{kmn} = 2\delta_{ij}^{mn} - \frac{1}{6} \epsilon_{ij}{}^{mnrst} f_{rst} \,, \tag{6.4}$$

with $\delta_{ij}^{mn} := \frac{1}{2}\delta_i^m \delta_j^n - \frac{1}{2}\delta_i^n \delta_j^m$.

We introduce a few concepts which are useful for studying the E_8 and E_8^{++} Weyl groups. The *associator* of octonions is defined by

$$[a, b, c] = a(bc) - (ab)c \tag{6.5}$$

and is totally antisymmetric in a, b, c by virtue of the alternativity of $\mathbb{O}$. Put differently, any subalgebra generated by two elements is associative. In components it reads

$$[a, b, c] = [a, b, c]_i e_i \quad \text{with} \quad [a, b, c]_i = -\frac{1}{3}\epsilon_i{}^{jklrst} a_j b_k c_l f_{rst} \,. \tag{6.6}$$
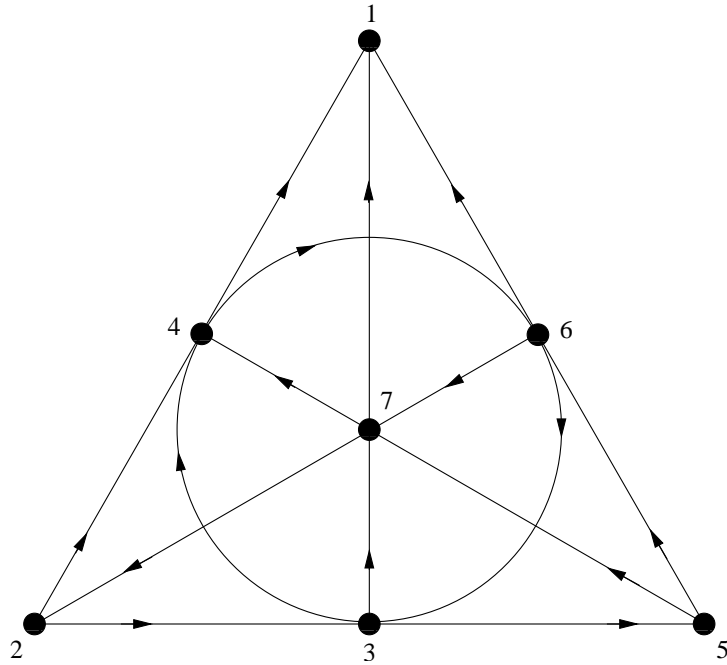
42

Figure 14: *The Fano plane illustrating associative triples and octonionic multiplication.*

The Jacobi identity gets modified by an associator term to

$$[[a,b],c] + [[b,c],a] + [[c,a],b] + 6\,[a,b,c] = 0\,. \tag{6.7}$$

The *Moufang laws* can be written as

$$x(yz) = (xyx)(x^{-1}z)\,, \qquad (yz)x = (yx^{-1})(xzx)\,. \tag{6.8}$$

They imply for example that $a(xy)a = (ax)(ya)$. Similarly one can transform axa^{-1} to show that this is an automorphism whenever $a \in \mathbb{O}$ satisfies $a^3 = \pm 1$. Another consequence of alternativity is that the following expressions are well-defined without brackets

$$axa = (ax)a = a(xa)\,, \qquad axa^{-1} = (ax)a^{-1} = a(xa^{-1})\,. \tag{6.9}$$

The first one implies the second one since $a^{-1} = \bar{a}/N(a)$. These relations are the basic reason why our formulas (3.7) with the generating matrices (3.6) also make sense for octonions.

Coxeter (following Bruck) has established that there is a maximal order of integers within $\mathbb{O}$ [9]. We briefly recall the construction from [8]. One starts from the sets of all Hurwitz numbers for all associative triples $(1, e_i, e_{i+1}, e_{i+3})$ of fig. 14 and their complements, arriving at the 'Kirmse numbers'. This is not a
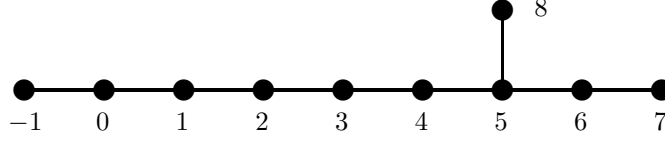
Figure 15: *Dynkin diagram of $E_{10} \equiv E_8^{++}$ with numbering of nodes.*

maximal order but will be so after exchanging 1 with any of the seven imaginary units; for definiteness we choose this unit as e_3. In addition, the lattice of integers is a scaled copy of the E_8 root lattice. The following octonionic units correspond to the simple roots of E_8 (labeled as in fig. 15)

$$
\begin{aligned}
a_1 &= e_3 &,&\quad a_2 = \frac{1}{2}\big(-e_1 - e_2 - e_3 + e_4\big) \\
a_3 &= e_1 &,&\quad a_4 = \frac{1}{2}\big(-1 - e_1 - e_4 + e_5\big) \\
a_5 &= 1 &,&\quad a_6 = \frac{1}{2}\big(-1 - e_5 - e_6 - e_7\big) \\
a_7 &= e_6 &,&\quad a_8 = \frac{1}{2}\big(-1 + e_2 + e_4 + e_7\big).
\end{aligned}
\tag{6.10}
$$

Their integer span gives all octonionic integers and, following [8], we call these integers *octavians* and denote them by $\mathbb{O}$. The highest E_8 root is represented by the unit

$$
\begin{aligned}
\theta &= 2a_1 + 3a_2 + 4a_3 + 5a_4 + 6a_5 + 4a_6 + 2a_7 + 3a_8 \\
&= \frac{1}{2}\big(e_3 + e_4 + e_5 - e_7\big).
\end{aligned}
\tag{6.11}
$$

There is a total of 240 unit octavians corresponding to the 240 roots of E_8. These definitions imply that no octavian contains only a quaternionic triple or its complement in its expansion.

6.2 D_8 and B_8 root systems

Before studying the E_8 and E_{10} Weyl groups we give the octonionic realizations of the root lattices of D_8 and B_8 which can be extended to the hyperbolic over-extensions in the usual manner.
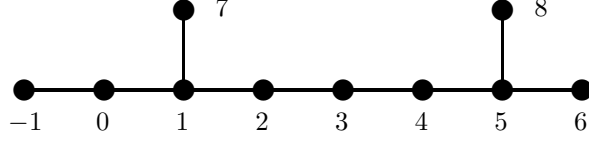
44

Figure 16: *Dynkin diagram of $DE_{10} \equiv D_8^{++}$ with numbering of nodes.*

For D_8 the simple roots are (as labeled in fig. 16)

$$D_8: \qquad a_1 = e_3 \quad , \quad a_2 = \frac{1}{2}\big(-e_1 - e_2 - e_3 + e_4\big)$$

$$a_3 = e_1 \quad , \quad a_4 = \frac{1}{2}\big(-1 - e_1 - e_4 + e_5\big)$$

$$a_5 = 1 \quad , \quad a_6 = \frac{1}{2}\big(-1 - e_5 - e_6 - e_7\big)$$

$$a_7 = \frac{1}{2}\big(e_2 - e_3 + e_6 - e_7\big) \quad , \quad a_8 = \frac{1}{2}\big(-1 + e_2 + e_4 + e_7\big). \quad (6.12)$$

Compared to the E_8 simple roots of (6.10) the only difference is in the simple root a_7 which is a specific linear combination describing an embedding of D_8 into E_8 [37, 26]. The highest root of D_8 is given by

$$\begin{aligned}
\theta^D &= 2a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + a_6 + a_7 + a_8 \\
&= \frac{1}{2}\big(e_3 + e_4 + e_5 - e_7\big).
\end{aligned} \qquad (6.13)$$

which is identical to the highest E_8 root. This will make it easy to relate the two hyperbolic Weyl groups.

For B_8 the simple roots are (as labeled in fig. 17)

$$B_8: \qquad a_1 = e_3 \quad , \quad a_2 = \frac{1}{2}\big(-e_1 - e_2 - e_3 + e_4\big)$$

$$a_3 = e_1 \quad , \quad a_4 = \frac{1}{2}\big(-1 - e_1 - e_4 + e_5\big)$$

$$a_5 = 1 \quad , \quad a_6 = \frac{1}{2}\big(-1 - e_5 - e_6 - e_7\big) \qquad (6.14)$$

$$a_7 = \frac{1}{2}\big(e_2 - e_3 + e_6 - e_7\big) \quad , \quad a_8 = \frac{1}{4}\big(e_2 + e_4 + e_5 + e_6 + 2e_7\big).$$

Compared to the D_8 simple roots of (6.12) the only difference is in the simple root a_8, which is no longer an octavian, in agreement with the fact that B_8 is not a subalgebra of E_8. A specific linear combination of the B_8 simple root gives the D_8 expressions, describing an embedding of D_8 into B_8. The highest
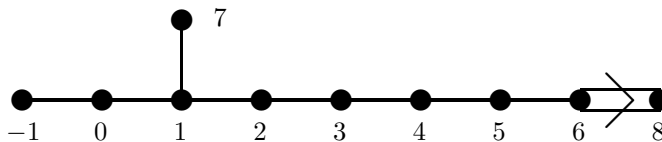
45

Figure 17: *Dynkin diagram of $BE_{10} \equiv B_8^{++}$ with numbering of nodes.*

root of B_8 is given by

$$
\begin{aligned}
\theta^B &= 2a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5 + 2a_6 + a_7 + 2a_8 \\
&= \frac{1}{2}\left(e_3 + e_4 + e_5 - e_7\right).
\end{aligned} \tag{6.15}
$$

which is identical to the highest E_8 and D_8 roots.

6.3 Weyl group $W(E_8)$

It is known that the Weyl group of E_8 is of order $696\,729\,600$ and has the structure [6]

$$
W(E_8) = 2 \cdot O_8^+(2) \cdot 2 \,. \tag{6.16}
$$

Since $W(E_8)$ naturally is a discrete subgroup of the continuous $O(8;\mathbb{R})$ of real dimension 28, we first record some facts about $O(8;\mathbb{R})$. The space of unit octonions is isomorphic to the seven sphere S^7 and hence of dimension 7. At the level of Lie algebras one has [39, 2]

$$
\mathfrak{so}(8;\mathbb{R}) \cong \mathrm{Im}(\mathbb{O}) \oplus G_2 \oplus \mathrm{Im}(\mathbb{O}) \tag{6.17}
$$

as a direct sum of vector spaces but not as Lie algebras. Here, $\mathrm{Im}(\mathbb{O})$ denotes the purely imaginary octonions which constitute the tangent space to S^7 at the octonion unit 1. The exceptional 14-dimensional Lie algebra G_2 is isomorphic to the algebra of derivations of the octonion algebra; the associated Lie group, which we also denote by G_2, is the group of automorphism of the octonionic multiplication table. The formula (6.17) suggests a representation of an arbitrary finite $SO(8;\mathbb{R})$ transformation as constructed from two unit octonions (a_L, a_R) and a G_2 automorphism γ. The following formula was proposed in [35]

$$
z \mapsto (a_L \gamma(z)) a_R \tag{6.18}
$$

where

$$
\gamma(z) = \bar{g}_1 \bar{g}_2 (g_1 (g_2 z \bar{g}_2) \bar{g}_1) g_2 g_1 \tag{6.19}
$$

is also expressed in terms of unit octonions g_1 and g_2. This formula reproduces the correct infinitesimal $\mathfrak{so}(8,\mathbb{R})$ transformations when all unit octonions are

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close to the identity and expressed as $a_L = \exp(\alpha_L)$ etc. for some $\alpha_L \in \mathrm{Im}(\mathbb{O})$. However, it is not true that any finite transformation is of this form as we will show below, see also [31]. It is known that any $SO(8;\mathbb{R})$ transformation can be written as seven times iterated (left) multiplication of z by unit octonions [8].

Turning to the finite E_8 Weyl group we observe that a statement similar to (6.17) is true for the integral octavians O. For this we first need some facts about the discrete automorphism group of O.

The group $Aut(\mathsf{O})$ is a finite group of order $12\,096$, usually denoted by $G_2(2)$ [8]. This group is not simple but has as simple part $U_3(3)$ of order $6\,048$. One obtains an explicit description of a generating set of $U_3(3)$ in the following way. As above, the transformation $z \mapsto az\bar{a}$ (for $a, z \in \mathsf{O}$) gives an automorphism iff $a^3 \in \mathbb{R}$, i.e. $a^3 = \pm 1$. Such $a \in \mathsf{O}$ were called *Brandt transformers* in [41]. Among the simple roots of E_8 given in (6.10) this is true for the simple roots a_4, a_5, a_6, a_8 which constitute a D_4 tree in the E_8 diagram. The automorphisms of O generated by $z \mapsto a_i z \bar{a}_i$ (for $i = 4, 5, 6, 8$) generate in fact all of $U_3(3)$. In order to describe the index two extension needed for the full $G_2(2)$ it is instructive to consider the following four 'diagonal' automorphisms of $z = \sum_{i=0}^{7} n_i e_i$ (where $e_0 = 1$)

$$
\begin{aligned}
(n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7) &\mapsto (n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7)\,, \\
(n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7) &\mapsto (n_0, -n_1, -n_2, n_3, n_4, -n_5, n_6, -n_7)\,, \\
(n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7) &\mapsto (n_0, n_1, -n_2, n_3, -n_4, -n_5, -n_6, n_7)\,, \\
(n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7) &\mapsto (n_0, -n_1, n_2, n_3, -n_4, n_5, -n_6, -n_7)\,,
\end{aligned}
$$

corresponding to choosing the three lines passing through unit e_3 as associative triples for the Dickson doubling process.[15] This suggests that one can obtain another automorphism by choosing a line not passing through e_3, for example

$$
(n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7) \mapsto (n_0, -n_1, -n_2, -n_3, n_4, n_5, -n_6, n_7)\,.
$$

This automorphism is a $\mathbb{Z}_2$ and one can check that it combines with $U_3(3)$ above to give all automorphisms of O. In this way one obtains eight diagonal automorphisms: one is the identity, while the other seven correspond to treating any of the seven lines of the Fano plane as the quaternion subalgebra entering the Dickson doubling process.[16]

Combining $Aut(\mathsf{O})$ with two sets of octavian units (of order 240), one finds that there is a total of

$$
240 \times |Aut(\mathsf{O})| \times 240 = 696\,729\,600 = |W(E_8)| \tag{6.20}
$$

elements, coinciding with the order of the E_8 Weyl group. This is the promised discrete (and finite) version of (6.17). It is again tempting to associate the

---

[15]Dickson doubled pairs (a, b) of quaternions manifestly have the automorphism $(a, b) \mapsto (a, -b)$.

[16]The Fano plane itself has the automorphism group $PSL(2, 7)$ of order 168 and there are several copies contained in $Aut(\mathsf{O})$. None is generated as a subgroup of the permutation group acting on the seven imaginary units.

action of this group on $z \in \mathbb{O}$ with the expression $z \mapsto (a_L \gamma(z)) a_R$, similar to (6.18), in particular since (6.17) is invariant under $(a_L, a_R) \leftrightarrow (-a_L, -a_R)$ thereby reducing the tentative number of transformation described in this way to the order of the even Weyl group $W^+(E_8)$, consistent with the fact that the odd Weyl transformations will involve an additional conjugation of z. However, the map (6.18) has a kernel in the discrete case in the sense that there are non-trivial triples (a_L, γ, a_R) which act trivially on $z \in \mathbb{O}$: Consider a γ given by a Brandt transformer $a \in \mathbb{O}$ with $a \neq 1$. Choosing $a_L = \bar{a}$ and $a_R = a$ leads to $z \mapsto z$ by alternativity of $\mathbb{O}$. There are 56 Brandt transformers among the 240 units of $\mathbb{O}$ and one can check that the kernel of the action (6.18) comes solely from situations of the type just described. Since we do not have the correct formula for any $w \in W(E_8)$ expressed in terms of unit octavians and a $G_2(2)$ transformation (also expressed in terms of unit octavians) we cannot give a closed description of $W(E_8)$ using the set (6.20). This will also impede our giving a fully explicit description of $W(E_8^{++})$.

We close this section by explaining why the formula for finite G_2 automorphisms given in (6.19) is not correct for arbitrary unit octonions g_1 and g_2. Using the Moufang identities one can show that $z \mapsto g_1(g_2 z \bar{g}_2) \bar{g}_1$ is an automorphism iff $g_1^2 g_2^3 g_1 \in \mathbb{R}$, so in particular whenever g_1 and g_2 are Brandt transformers. Conjugating with $\bar{g}_1 \bar{g}_2$ to obtain (6.19) leads to the necessary and sufficient criterion

$$\{\{\bar{g}_1, \bar{g}_2\}, g_2\} \in \mathbb{R} \tag{6.21}$$

for (6.19) to be an automorphism. Here, $\{x, y\} = xyx^{-1}y^{-1}$ is the (group) commutator which is well-defined by virtue of alternativity of $\mathbb{O}$. The criterion (6.21) is clearly not satisfied for arbitrary octavians; one example is given by $g_1 = a_2$ and $g_2 = a_1$ in terms of the E_8 simple roots. We note as a curiosity that $W^+(E_8)$ has a subgroup of type H_4 which is non-crystallographic and plays a role in the theory of quasi-crystals [33].

6.4 $W^+(E_{10}) \equiv W^+(E_8^{++})$

As already remarked after equation (3.16) the formula $S \mathbb{X} S^\dagger$ for general even Weyl transformations of the even hyperbolic Weyl group acting on Hermitian (2×2) matrices ceases to be valid in the octonionic case. The even Weyl transformations form a discrete subgroup of $SO(9, 1; \mathbb{R})$ and similar to (6.17) it is known that at the level of Lie algebras [39, 2]

$$\mathfrak{so}(1, 9; \mathbb{R}) \cong L_2'(\mathbb{O}) \oplus \text{Im}(\mathbb{O}) \oplus G_2 \ (\cong \mathfrak{sl}_2(\mathbb{O})), \tag{6.22}$$

again as a sum vector spaces. Here, $L_2'(\mathbb{O})$ denotes the 24-dimensional vector space of all octonionic traceless (2×2) matrices. Combining this with $\text{Im}(\mathbb{O})$ one obtains the 31-dimensional vector space of all octonionic (2×2) matrices with vanishing real part of the trace. This is analogous to (5.23) in the quaternionic case; the additional complication of non-associativity is that one also requires an 'intertwining' G_2 automorphism of $\mathbb{O}$.

Similar to the suggestive, but incorrect, formula (6.18) one could envisage the action of $PSO(1, 9; \mathbb{R})$ on $H_2(\mathbb{O})$ to be given by

$$\mathtt{X} \mapsto S\gamma(\mathtt{X})S^\dagger \,. \tag{6.23}$$

However, this 'definition' is ambiguous since it requires a prescription for how to place the parentheses in case there are more than two independent octonionic entries involved. Furthermore, we expect that in analogy with the finite E_8 case discussed above that (6.23) is not general enough to describe all transformations. Presumably both shortcomings of (6.23) are related and can be resolved together. We stress nevertheless that theorem 2 is still applicable and any even Weyl transformation can be described by an iterated action of (2×2) matrices.

Though we cannot offer a complete resolution to this problem we make a comment on the issue of placing parentheses. In [32] the approach was taken that S has to be such that there is no ambiguity when placing parentheses in the matrix expressions. This leads to very restrictive conditions on S, allowing essentially only one octonionic entry, which is the situation covered by theorem 1. However, we are here interested in the case with more than one independent octonionic entry. In order to make sense of (6.23), one must presumably define the triple product by putting parentheses *inside the matrix elements in such a way that the resulting matrix is again Hermitian* (simply bracketing whole matrices will not do). This is for example required to reduce (6.23) to (6.18) for the embedding of $W(E_8)$ in $W(E_{10})$ for the cases when (6.18) is correct.

Defining the Weyl group of E_{10} by iterated action of the basic matrices given in theorem 1 with the simple roots (6.10) (for which there arise no ambiguities due to non-associativity) we arrive at

$$W^+(E_8^{++}) \cong PSL_2(\mathtt{O}) \,, \tag{6.24}$$

where the group $PSL_2(\mathtt{O})$ on the r.h.s. is *defined* by the iterated action. Since, unlike the D_4 Dynkin diagram, the E_8 diagram has no outer automorphism no additional quotients of $PSL_2(\mathtt{O})$ are necessary for describing the Weyl group. It is an outstanding problem to find a better and more 'intrinsic' definition of the group $PSL_2(\mathtt{O})$, and to explore its implications for an associated theory of modular forms. We point out that $PSL_2(\mathtt{O})$ has a rich structure of subgroups, of types $PSL_2^{(0)}(\mathtt{H})$, $PSL_2(\mathtt{E})$, $W(D_9)$, and others, which remains to be exploited. We hope that the information on these subgroups we have obtained in the preceding sections will help to find a better description of (6.24).

6.5 $W^+(D_8^{++})$ and $W^+(B_8^{++})$

We close by giving the relations between the Weyl groups $W^+(D_8^{++})$ and $W^+(B_8^{++})$, and $W^+(D_8^{++})$ and $W^+(E_8^{++})$, respectively.

One can check that

$$S_8^D = S_8^B \circ S_6^B \circ S_8^B \quad \text{and} \quad (S_8^D)^2 = \mathbb{1} \tag{6.25}$$

by using either the abstract relations or iterated matrix action of octonionic matrices on X; the ∘ is meant to indicate that the product is to be understood as an iterated action. All other generators are identical and since S_8^B is acting by a conjugation automorphism this shows that

Proposition 11. $W^+(B_8^{++}) \cong W^+(D_8^{++}) \rtimes 2$.

Turning to $W^+(D_8^{++})$ one finds that S_7^D is the only generator which is not common to $W^+(D_8^{++})$ and $W^+(E_8^{++})$ since all other simple roots and the highest roots are identical. A calculation reveals that S_7^D can be expressed as the result of a conjugation action of the $W^+(E_8^{++})$ generators as

$$
\begin{aligned}
S_7^D &= S_7 S_6 S_5 S_8 S_4 S_3 S_5 S_4 S_6 S_7 S_5 S_8 S_6 S_5 S_4 S_3 S_2 \\
&\quad \circ S_3 S_4 S_5 S_6 S_8 S_5 S_7 S_6 S_4 S_5 S_3 S_4 S_8 S_5 S_6 S_7 \,,
\end{aligned} \tag{6.26}
$$

where we omitted a superscript on S_i^E and the ∘ symbols on the r.h.s. in order not to clutter notation. The Weyl transformation (6.26) maps the simple root α_7^E to α_7^D. This shows that $W^+(D_8^{++})$ is a subgroup of $W^+(E_{10})$ but one can check that it is not normal in $W^+(E_{10})$ since conjugation by S_7^E does not preserve the subgroup. For the finite Weyl groups $W(D_8)$ is index 135 in $W(E_8)$ and since the embedding (6.26) of $W^+(D_8^{++})$ in $W^+(E_{10})$ does not involve the affine or hyperbolic reflection the same is true for the hyperbolic Weyl groups.

Proposition 12. $W^+(D_8^{++})$ *is an index* 135 *subgroup of* $W^+(E_8^{++}) \cong PSL_2(\mathbb{O})$.

Without a direct definition of $PSL_2(\mathbb{O})$ we cannot give a more detailed octonionic description of $W^+(D_8^{++})$ that would be analogous to (5.24). There is no direct relation between $W^+(B_8^{++})$ and $W^+(E_8^{++})$.

A Examples with twisted affine algebras

In this appendix we show that our techniques are also suited for treating hyperbolic algebras which do not arise as over-extensions but whose Dynkin diagrams instead involve subdiagrams of twisted affine type. For simplicity we will exemplify this for two cases constructed over the complex numbers with twisted affine algebras $D_2^{(2)}$ and $D_4^{(3)}$ in the terminology of [24], both of which have rank three. The hyperbolic node is attached to the 'twisted affine' node with the single node for the cases we consider and we therefore denote the associated hyperbolic algebras by $D_2^{(2)+}$ and $D_4^{(3)+}$. However, we anticipate that there are also other, more general hyperbolic cases which fit into the picture we develop in this paper.

A.1 $\mathbb{K} = \mathbb{C}$, type $D_2^{(2)+}$

The hyperbolic Kac–Moody algebra $D_2^{(2)+}$ can be constructed from the root system of type B_2. Although the root system is identical to that of C_2 already
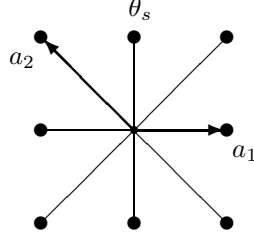
Figure 18: *The root system of type B_2, with simple roots labeled and indicated by arrows. The lattice they generate is the ring of Gaussian integers. There are both long and short roots and we indicate the highest short root θ_s.*
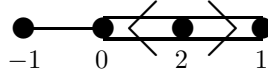

Figure 19: *$D_2^{(2)+}$ Dynkin diagram with numbering of nodes.*

depicted in fig. 5 we draw it again in fig. 18 where we highlight the short highest root which enters the twisted affine extension $D_2^{(2)}$.

The root system is not simply laced, having simple roots whose squared lengths are in the ratio 2 to 1. We choose the following units and simple roots for B_2 in the complex plane

$$\varepsilon_1 = a_1 = 1 \ , \ \varepsilon_2 = \frac{a_2}{\sqrt{2}} = \frac{-1+i}{\sqrt{2}} \ , \ \theta_s = i \,. \tag{A.1}$$

These formulas are identical to those of (4.21) except that the whole lattice has been rescaled by a factor $\sqrt{2}$.

The hyperbolic simple roots of $D_2^{(2)+}$ now take the form

$$\alpha_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \alpha_0 = \begin{bmatrix} -1 & -\theta_s \\ -\bar{\theta}_s & 0 \end{bmatrix} = \begin{bmatrix} -1 & -i \\ i & 0 \end{bmatrix},$$

$$\alpha_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \alpha_2 = \begin{bmatrix} 0 & -1+i \\ -1-i & 0 \end{bmatrix}. \tag{A.2}$$

The Dynkin diagram of the hyperbolic algebra $D_2^{(2)+}$ is shown in fig. 19. The non-zero inner products between the simple roots are

$$(\alpha_{-1}, \alpha_{-1}) = (\alpha_0, \alpha_0) = (\alpha_1, \alpha_1) = 2 \ , \ (\alpha_2, \alpha_2) = 4 \,,$$
$$(\alpha_{-1}, \alpha_0) = -1 \ , \ (\alpha_0, \alpha_2) = -2 \ , \ (\alpha_1, \alpha_2) = -2 \tag{A.3}$$

and these give rise to the $D_2^{(2)+}$ diagram.

Turning to the Weyl group of $D_2^{(2)^+}$, we realize that the Weyl group is isomorphic to that of C_2^{++} since the Coxeter group is insensitive to the direction of the arrows

Proposition 13. $W^+(D_2^{(2)+}) \cong W^+(B_2^{++}) \cong PSL_2(\mathbb{G}) \rtimes 2.$
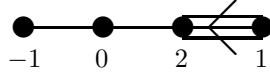
51

Figure 20: $D_4^{(3)+}$ *Dynkin diagram with numbering of nodes.*
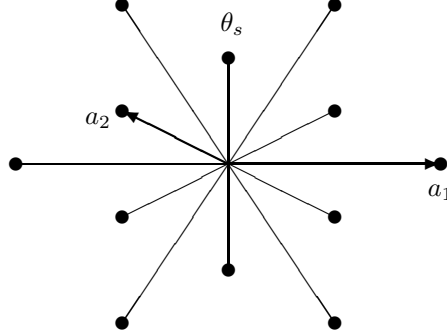

Figure 21: *The root system of type G_2 with highest short root indicated.*

A.2 $\mathbb{K} = \mathbb{C}$, type $D_4^{(3)+}$

The final hyperbolic algebra we consider is the hyperbolic extension of the twisted affine algebra of type $D_4^{(3)}$ (the only one twisted with an order three automorphism). We denote the hyperbolic extension by $D_4^{(3)+}$ and its diagram is shown in fig. 20. Its construction employs the G_2 root system, shown again in fig. 21, and we make the following choices of units and simple roots in the complex plane.

$$\varepsilon_1 = \frac{a_1}{\sqrt{3}} = 1 \ , \ \varepsilon_2 = a_2 = \frac{-\sqrt{3}+i}{2} \ , \ \theta_s = i \,. \tag{A.4}$$

Again, these formulas are those of (4.30) except for a rescaling by a factor $\sqrt{3}$ and the use of the highest short root θ_s instead of the highest (long) root θ.

The associated hyperbolic simple roots of $D_4^{(3)+}$ are then

$$\alpha_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \alpha_0 = \begin{bmatrix} -1 & -\theta_s \\ -\bar{\theta}_s & 0 \end{bmatrix} = \begin{bmatrix} -1 & -i \\ i & 0 \end{bmatrix},$$

$$\alpha_1 = \begin{bmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{bmatrix}, \qquad \alpha_2 = \begin{bmatrix} 0 & \frac{-\sqrt{3}+i}{2} \\ \frac{-\sqrt{3}-i}{2} & 0 \end{bmatrix}. \tag{A.5}$$

The non-zero inner products between these simple roots are

$$(\alpha_{-1}, \alpha_{-1}) = (\alpha_0, \alpha_0) = (\alpha_2, \alpha_2) = 2 \ , \ (\alpha_1, \alpha_1) = 6 \,,$$
$$(\alpha_{-1}, \alpha_0) = -1 \ , \ (\alpha_0, \alpha_2) = -1 \ , \ (\alpha_1, \alpha_2) = -3 \,. \tag{A.6}$$

The Weyl group is again easy to determine in this case since it is isomorphic to that of the standard G_2^{++} case

Proposition 14. $W^+(D_4^{(3)+}) \cong W^+(G_2^{++}) \cong PSL_2(\texttt{E}) \rtimes 2$.

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