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Spontaneous particle creation in time-dependent overcritical fields

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Abstract
It is believed that in the presence of some strong electromagnetic fields, called *overcritical*, the (Dirac) vacuum becomes unstable and decays, leading to a spontaneous production of an electron–positron pair. However, most of the arguments are based on the analysis of static fields and are insufficient to explain this phenomenon completely. Therefore, we consider time-dependent overcritical fields and show, within the external field formulation, how spontaneous particle creation can be defined and measured in physical processes. We prove that the effect exists always when a *overcritical* field is switched on, but it becomes unstable and hence generically only approximate and non-unique when the field is switched on and off. In the latter case, it becomes unique and stable only in the adiabatic limit.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In a long debate [1–8] on whether and how spontaneous particle creation (of e^+e^- pairs) can be uniquely defined as an effect of the vacuum decay in the presence of overcritical (electromagnetic) fields either static or adiabatic fields have been considered, what does not really answer the question in a realistic physical situation. The main problem in time-dependent overcritical fields is to distinguish between two sources of particle creation: dynamical, due to the time-dependence, and spontaneous, due to the overcriticality of the external field. In [9], we have studied analytically and numerically various time-dependent overcritical fields and have shown when the effect can be uniquely defined and when it behaves in a stable way.

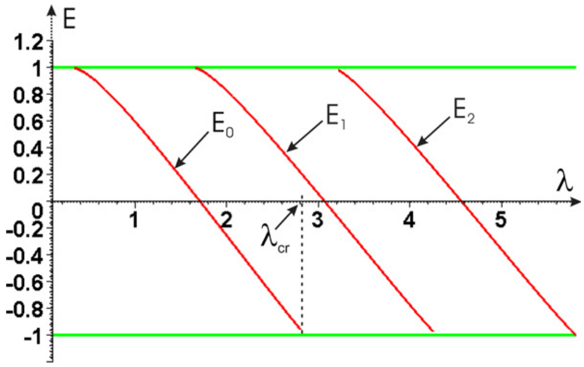

Figure 1. Spectrum of $H_\lambda = H_0 + \lambda V$ as a function of λ.

1.1. Classical Dirac equation

Electrons in an external time-dependent electromagnetic field $A_\mu(x)$ are described by the Dirac equation which can be written in the evolution form

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(t,\vec{x}) = H(t)\Psi(t,\vec{x}) \qquad \text{with} \qquad H(t) \equiv H_0 + V(t), \tag{1}$$

where $V(t) = eA_0 + e\alpha^i A_i$ is the time-dependent external potential and $H_0 = -\mathrm{i}\hbar c\alpha^i \partial_i + mc^2\beta$ is the free Hamiltonian[1]. Consider every $H(t)$ separately treating t as a parameter. For atomic-like (localized) potentials the spectrum $\sigma(H)$ has two continuous parts $(-\infty, -mc^2) \cup (mc^2, \infty)$ and a possible discrete part $\{E_n \in (-mc^2, mc^2)\}$. The corresponding wavefunctions $\psi_E(\vec{x})$, satisfying $H\psi_E = E\psi_E$, describe: electron scattering states for $E > mc^2$, bound states for $|E| < mc^2$ and positron scattering states for $E < -mc^2$.

1.2. Overcriticality on the classical level

Consider a one-parameter family of Hamiltonians $H_\lambda \equiv H_0 + \lambda V(\vec{x})$ having bound states $E_0(\lambda) < E_1(\lambda) < E_2(\lambda) < \cdots$. For a large class of negative potentials $V(\vec{x})$ (attractive for electrons) $E_n(\lambda)$ decrease continuously towards $-mc^2$ as λ increases and the lowest bound-state energy reaches $E_0(\lambda_{\mathrm{cr}}) = -mc^2$ (with a finite slope $\mathrm{d}E_0/d\lambda < 0$ [10]) for a finite $\lambda = \lambda_{\mathrm{cr}}$, called *critical* (figure 1). For $\lambda > \lambda_{\mathrm{cr}}$ the bound-state E_0 disappears from the spectrum. Such potentials V_λ are called *overcritical*. Further, next bound states $E_1, E_2, \ldots,$ disappear as λ grows.

1.3. Resonances

Bound states crossing the value $E = -mc^2$ are forbidden to get embedded in the negative continuum, so they disappear from the spectrum and turn into resonances, which can be traced as poles on the analytic continuation of the resolvent $R_\lambda \equiv (H_\lambda - E)^{-1}$ (figure 2).

As the Hamiltonian changes so that a bound state turns into a resonance, the dominant part of the bound-state wavefunction forms a wave packet localized spectrally in the continuum around the *resonance energy* $E_R \equiv \mathrm{Re}\, E$. The half-width of the packet is equal to $\Gamma \equiv \Im E$.

[1] For a large class of potentials the Hamiltonians $H(t)$ are essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^3)$ and can be uniquely extended to self-adjoint operators $\hat{H}(t)$ on $\mathcal{L}^2(\mathbb{R}^n)$. For brevity we identify H with $\hat{H}$.

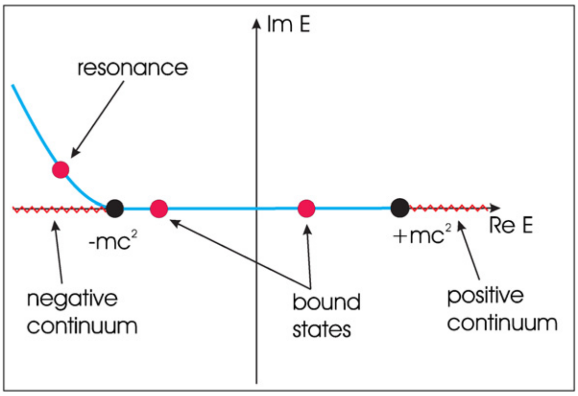

Figure 2. Position of the real or complex pole of the resolvent R_λ meaning a bound state or a resonance, respectively.

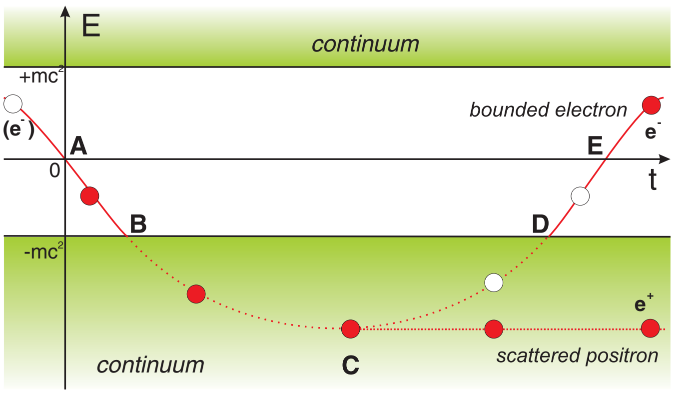

Figure 3. Pair production in an overcritical process.

During evolution generated by a static overcritical Hamiltonian the wave packet decays spatially (but stays localized spectrally).

1.4. What is expected by overcriticality?

Consider time-dependent Hamiltonians $H(t) = H_0 + \lambda(t)V$ with an overcritical period $\lambda(t) > \lambda_{cr}$ for $|t| < T$. When $\lambda(t)$ changes adiabatically the bound states (defined at every instant t) change slowly and according to the adiabatic theorem the wavefunction 'follows' them. However, there is no adiabatic theorem for resonances, thus during the overcritical period the wave packet may decay and stay trapped in the continuum forever.

To explain these processes in terms of particle creation and annihilation one needs a many-particle description, i.e. the second quantized Dirac theory. Formulated in a non-rigorous language it provides the following scenario [9] (see figure 3): (A) An empty particle bound state crosses the boundary between particle and antiparticle subspaces and turns into an occupied antiparticle bound state, what we call *weak overcriticality*. (B) The bound state turns into a resonance and the wavefunction forms a wave packet in the negative continuum,

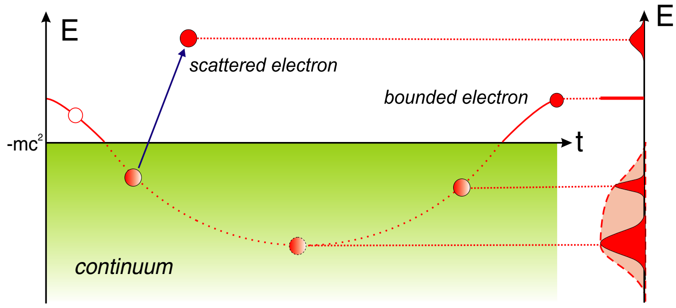

Figure 4. Spectrum of produced particles and antiparticles in result of a time-dependent overcritical field.

what we call *strong overcriticality*. (C) The wave packet decays. (D) An empty antiparticle bound state reappears while the whole wavefunction stays trapped in the negative continuum. (E) The empty antiparticle bound state turns into an occupied particle bound state. Finally, we obtain a pair: a bound particle and a scattered antiparticle.

In practice, except the adiabatic case, the wavefunction disperses during evolution over the whole spectrum what results in an additional *dynamical* pair creation. Essential is the question whether it is possible to separate the *spontaneous particle creation* occurring due to the overcriticality of the potential from the dynamical one.

2. QED in external fields

To describe physical processes one needs full QED, but it appears too difficult to be solved in our case, so we consider the external field approximation, i.e. treat the electromagnetic field as classical and quantize only the Dirac field. This approximation is believed to be accurate as long as the number of charged Dirac particles stays small and does not influence the electromagnetic field.

In the presence of overcritical fields the vacuum and particles change their properties qualitatively so that in order to treat them correctly the analysis must start from the first principles.

2.1. Fock space and the field operator

The construction of the theory begins with the algebra of fields $\hat{\Psi}(f)$ anti-linear in $f \in \mathcal{H}$ – the canonical anticommutation relations

$$\{\hat{\Psi}(f), \hat{\Psi}^*(g)\}_+ = (f, g)$$
$$\{\hat{\Psi}(f), \hat{\Psi}(g)\}_+ = \{\hat{\Psi}^*(f), \hat{\Psi}^*(g)\}_+ = 0,$$

$\forall f, g \in \mathcal{H}$ with (f, g) a scalar product in $\mathcal{H}$. We represent them as self-adjoint operators acting in the Fock space. The representation is unique up to the choice of a pair of projectors $P_\pm : \mathcal{H} \to \mathcal{H}_\pm$ which split the Hilbert space $\mathcal{H} \equiv \mathcal{H}_+ \bigoplus \mathcal{H}_-$ on the particle and antiparticle subspaces. Then, the Fock space is constructed as

$$\mathcal{F} = \bigoplus_{m,n=0}^{\infty} \mathcal{F}^{(n,m)} = \bigoplus_{m,n=0}^{\infty} \overbrace{\underbrace{\mathcal{H}_+ \otimes \cdots \otimes \mathcal{H}_+}_{\text{n times}} \otimes \underbrace{\mathcal{H}_- \otimes \cdots \otimes \mathcal{H}_-}_{\text{m times}}}^{\text{antisymmetric}}, \tag{2}$$

with $\mathcal{F}^{(0,0)}$ one-dimensional space with a unit vector Ω, a no-particle state called *vacuum*. Using the projectors $P_\pm$ we split

$$\hat{\Psi}(f) = \hat{b}(P_+ f) + \hat{d}^*(P_- f) \qquad \forall f \in \mathcal{H} \tag{3}$$

so that $\hat{b}(P_+ f)\Omega = \hat{d}(P_- f)\Omega = 0$. Then $\hat{b}^*(f)$ and $\hat{d}^*(g)$ create particles and antiparticles in states $f \in \mathcal{H}_+$ and $g \in \mathcal{H}_-$, respectively, and $\hat{b}(f)$ and $\hat{d}(g)$ annihilate.

The choice of the projectors $P_\pm$ is equivalent to the choice of the vacuum vector $\Omega \in \mathcal{F}$. Two representations based on $P_\pm$ and $P'_\pm$ are unitary equivalent if and only if the Hilbert–Schmidt norm

$$\|P_\pm - P'_\pm\|_{\text{H.S.}} < \infty. \tag{4}$$

So unitarily non-equivalent representations, giving different physical predictions, are possible. To exclude them one distinguishes $P_\pm$ as spectral projections on the positive and negative energy subspaces of the Hamiltonian H. Then the induced vacuum vector Ω is a ground state of the Hamiltonian $\hat{H}$ implemented in the Fock space $\mathcal{F}$.

2.2. Particle scattering in Fock space

For time-dependent Hamiltonians $H(t)$ the above construction must be repeated at every instant of time with time-dependent projectors $P_\pm(t)$ and leads to a family of Fock spaces $\mathcal{F}(t)$. To implement scattering processes it is sufficient to consider only the initial $\mathcal{F}_{\text{in}}$ and final $\mathcal{F}_{\text{out}}$. The classical (one-particle) unitary scattering operator $S : \mathcal{H} \to \mathcal{H}$ is implemented in the Fock space by a unitary $\hat{S} : \mathcal{F} \to \mathcal{F}$ such that

$$\hat{\Psi}_{\text{out}}(f) \equiv \hat{\Psi}_{\text{in}}(Sf) = \hat{S}\hat{\Psi}_{\text{in}}(f)\hat{S}^*. \tag{5}$$

Assume that $P_\pm(-\infty) = P_\pm(+\infty) \equiv P_\pm$. Then $\hat{S}$ exists when S satisfies the Shale–Stinespring criterion $\|S_{\pm\mp}\|_{\text{H.S.}} := \|P_\pm S P_\mp\|_{\text{H.S.}} < \infty$, which guarantees that the initial vacuum state $\Omega \in \mathcal{F}$ evolves to $\hat{S}\Omega \in \mathcal{F}$.

It is remarkable that the whole information about scattering in $\mathcal{F}$ is encoded in one-particle S. It means that every observable can be expressed via S, e.g. the expectation value of the 'particle number' operator in state $\hat{S}\Omega$ evolved from initial vacuum Ω reads

$$N_n = (\hat{S}\Omega, \hat{b}_n^*\hat{b}_n\hat{S}\Omega) = \sum_k |(S_{+-})_{nk}|^2 = \|(S^*)_{-+}\phi_n\|^2. \tag{6}$$

3. Structure of $\hat{S}$

For time-dependent processes with $H(t) \equiv H_0 + V(t)$ the scattering operator $\hat{S}$ has the form $\hat{S} = C_0 : \hat{S}_0\widetilde{S} :$ where C_0 is a normalization constant

$$C_0 \equiv [\det(1 + A^*A)]^{-1/2}, \tag{7}$$

$\widetilde{S}$ describes creation and annihilation of particle–antiparticle pairs and their scattering

$$\begin{aligned}\widetilde{S}^* =: &\exp\left(\sum_{k,l} A_{kl}\hat{b}_k^*\hat{d}_l^*\right)\exp\left(\sum_{k,l}(B_{kl} - \delta_{kl})\hat{b}_k^*\hat{b}_l\right)\\ &\times \exp\left(\sum_{k,l}(C_{kl} - \delta_{kl})\hat{d}_k^*\hat{d}_l\right)\exp\left(\sum_{k,l} D_{kl}\hat{b}_k\hat{d}_l\right) : \\ &k = n_+ + 1, 2, \ldots, \infty; \qquad l = n_- + 1, \ldots, \infty\end{aligned} \tag{8}$$

and the exceptional part $\hat{S}_0$ describes creation and annihilation of single particles

$$\hat{S}_0^* \equiv: \left(\hat{b}_1^* \mp \sum_{k=1}^{n_+} (S_{-+})_{k1} \hat{d}_k\right) \cdots \left(\hat{b}_{n_+}^* \mp \sum_{k=1}^{n_+} (S_{-+})_{kn_+} \hat{d}_k\right) \times \left(\hat{d}_1^* \mp \sum_{k=1}^{n_-} \overline{(S_{+-})_{k1}} \hat{b}_k\right) \cdots \left(\hat{d}_{n_-}^* \mp \sum_{k=1}^{n_-} \overline{(S_{+-})_{kn_-}} \hat{b}_k\right), \tag{9}$$

where

$$n_+ = \dim\ker S_{++}, \qquad n_- = \dim\ker S_{--}$$
$$A \equiv -S_{++}^{-1} S_{+-}, \qquad B \equiv \pm S_{++}^{-1}, \qquad C \equiv \pm\overline{(S_{--}^{-1})}, \qquad D \equiv \pm\left(S_{-+}S_{++}^{-1}\right)^T. \tag{10}$$

Acting on vacuum

$$\hat{S}\Omega = C_0 \hat{d}_{n_-}^* \cdots \hat{d}_1^* \hat{b}_{n_+}^* \cdots \hat{b}_1^* \exp\left(\sum_{k,l} A_{kl} \hat{b}_k^* \hat{d}_l^*\right) \Omega \tag{11}$$

$\hat{S}_0$ creates single particles and antiparticles $\hat{b}_i^*$ and $\hat{d}_i^*$ which correspond to the *special states* $\phi_i^\pm$, which are mapped by S from $\mathcal{H}_\pm$ to $\mathcal{H}_\mp$. It has been conjectured that the spontaneous particle creation is associated with the presence of the exceptional part $\hat{S}_0$ [1].

3.1. Potential switched on and off: $H(-\infty) = H(+\infty) = H_0$

Time-dependent processes $H_\lambda(t) \equiv H_0 + \lambda V(t)$ give $\hat{S}(\lambda)$. Since $P_\pm S(\lambda) P_\mp$ are analytic in λ, the special states exist only for discrete values of λ [9][theorem 16] and hence are unstable w.r.t. perturbations of λ!

We discuss two ways to handle this problem: either to defend the role of *special states* in the definition of spontaneous particle creation and consider the adiabatic limit which is free from the above instability, or to relax the condition of *special states* and define the spontaneous pair creation in a *weaker sense*.

4. Definition by the adiabatic limit

In the adiabatic limit the dynamical pair production tends to 0, while the spontaneous pair production survives the limit. Let us consider processes where the potential $V_{\lambda,\epsilon}(t,\vec{x}) \equiv \lambda\, e^{-\epsilon^2 t^2} \tilde{V}(\vec{x})$ varies arbitrarily slow in time ($\epsilon \to 0$) and vanishes as $t \to \pm\infty$. We can calculate the scattering operators $\hat{S}_{\lambda,\epsilon}$. In the adiabatic limit it is possible that $\lim_{\epsilon\to 0} \hat{S}_{\lambda,\epsilon} \neq \hat{S}_{\lambda,0}$ and the probability of particle creation: $r_\lambda = 1 - \lim_{\epsilon\to 0} |(\Omega, \hat{S}_{\lambda,\epsilon}\Omega)|^2$ has a jump at $\lambda = \lambda_{\text{cr}}$, because

- for $0 < \lambda < \lambda_{\text{cr}}$ (V_λ subcritical): $r_\lambda \leqslant \|P_- S_{\lambda,\epsilon} P_+\|_{\text{H.S.}} \xrightarrow{\epsilon\to 0} 0$,
- for $\lambda > \lambda_{\text{cr}}$ (V_λ overcritical): $r_\lambda \geqslant \|P_- S_{\lambda,\epsilon} P_+\| \xrightarrow{\epsilon\to 0} 1$.

This conjecture was posed and proved in the subcritical case by Nenciu [2]. The overcritical case has been recently proved by Pickl [3]. For the considered class of short-range potentials the resonance decays already at the edge of the negative continuum and leads to creation of an antiparticle with vanishing momentum.

4.1. Results: switch on and off processes

- In subcritical processes, when $E_0(t) > -mc^2$ exists for all t, no particles are created: $\hat{S}\Omega = \Omega$ [2].
- When, for some interval of time $t_1 < t < t_2$, the bound state turns into a resonance in the negative continuum Re $E_0(t) < -mc^2$, which we call *(strongly) overcritical*, there is exactly one pair[2] created spontaneously: $\hat{S}\Omega = \hat{b}^*(\chi)\hat{d}^*(\psi)\Omega$ [3].

We conclude that *only the strong overcriticality leads to physically observable effects*, as was conjectured by Greiner *et al* [11], but in contrast to Klaus and Scharf who argued for the *weak overcriticality* (Re $E_0(t) < 0$) [1].

5. Spontaneous pair creation in a 'weaker sense'

Instability of the special states of $\hat{S}(\lambda)$ means that

$$\hat{S}(\lambda)\Omega = C_0\hat{d}^*(\psi)\hat{b}^*(\chi)\exp\left(\sum_{k,l} A_{kl}\hat{b}_k^*\hat{d}_l^*\right)\Omega, \tag{12}$$

after perturbation $\lambda \to \lambda + \delta\lambda$ goes over into [12]

$$\hat{\tilde{S}}(\lambda)\Omega = C_0\exp\left(\sum_{k,l}\tilde{A}_{kl}\hat{b}_k^*\hat{d}_l^* + B\hat{d}^*(\tilde{\psi})\hat{b}^*(\tilde{\chi})\right)\Omega, \tag{13}$$

with $|B| < 1$, but by continuity $|B| \approx 1$. One can try to relax the condition of creation of a pair in the special states (with probability 1) to creation of a pair with probability ≈ 1 in the corresponding states. However, it is difficult to make this definition rigorous and unique.

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