

NEAR-CONSTANT MEAN CURVATURE SOLUTIONS OF THE EINSTEIN CONSTRAINT EQUATIONS WITH NON-NEGATIVE YAMABE METRICS

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ABSTRACT. We show that sets of conformal data on closed manifolds with the metric in the positive or zero Yamabe class, and with the gradient of the mean curvature function sufficiently small, are mapped to solutions of the Einstein constraint equations. This result extends previous work which required the conformal metric to be in the negative Yamabe class, and required the mean curvature function to be nonzero.

1. INTRODUCTION

The set of smooth, constant mean curvature (CMC) solutions of the Einstein constraint equations is fairly well understood. For closed manifolds, there is a complete parameterization of these solutions in terms of conformal data [I]. For the asymptotically Euclidean and asymptotically hyperbolic cases, similar results hold [CBIY, AC].

Much less is known about non constant mean curvature solutions. The mathematical reason for this is that while the CMC condition effectively eliminates three of the four Einstein constraint equations from the analysis, in the non-CMC case one must handle the full, coupled system.

All of the non-CMC results to date [IM, CBIY, IP] require that the gradient of the mean curvature τ be sufficiently small; we call such solutions “near-CMC”. In the case of closed manifolds, these results also require that the metric be in the negative Yamabe class, and that the mean curvature function have no zeroes. While we have not yet managed to relax the small $|\nabla\tau|$ condition, in this paper we show that we can construct non CMC solutions on closed manifolds with the metric in the positive or zero Yamabe class, and with the mean curvature function allowed to have zeroes in the positive Yamabe case.

The procedure we use for proving our results here is the semi-decoupled sequence (constructive) method, which we have introduced in [IM]. The chief difference between our work here and [IM] is that while we can use a sequence of constant sub and super solutions for sets of conformal data with negative Yamabe class metrics and τ nowhere zero, for solutions with positive or zero Yamabe class metrics we require non-constant sub solutions. The focus in this paper is on how to obtain such sub solutions and how to control them. We discuss this issue in Section 3, after a brief introduction to the conformal method in Section 2. Also in Section 3 we show that the Lichnerowicz equations with negative Yamabe class metric and τ allowed to have zeroes (but not allowed to identically vanish) always admits solutions. In Section 4 we describe the semi-decoupling method for constructing near-CMC solutions of the constraints, and then state and prove our main theorems. We make some concluding remarks in Section 5. Note that, in this paper, we are not concerned with optimizing the regularity conditions on either the choice of conformal data or on the solutions of the constraints which we obtain. Presumably one could produce solutions with the same degree of roughness as is discussed in [M] and [CB].

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2. THE CONFORMAL METHOD AND THE LICHNEROWICZ EQUATION

The Einstein vacuum constraint equations require that a set of initial data (Σ, γ, K) consisting of a Riemannian metric γ and a symmetric tensor K specified on a three-dimensional manifold Σ^3 , satisfy the following

$$R_\gamma + K *_{\gamma} K - (\text{tr}_\gamma K)^2 = 0 \quad (2.1)$$

and

$$\nabla_\gamma K_b^a - \nabla_\gamma (\text{tr}_\gamma K) = 0 \quad (2.2)$$

where the covariant derivative ∇ , the scalar curvature R_γ , and all contractions and traces are calculated with respect to the metric γ .

The idea of the conformal method is that one may construct and parameterize solutions of the constraints (2.1)-(2.2) by splitting γ and K into a set of freely specified data and a set of determined data. The freely specified ‘‘conformal data’’ consists of a Riemannian metric λ , a symmetric tensor σ which is trace-free and divergence-free with respect to λ , and a function τ , all specified on the manifold Σ . The determined data consists of a vectorfield W and a positive definite scalar field ϕ . Using the conformal data to define the covariant derivative ∇_λ together with the corresponding Laplacian Δ_λ , scalar curvature R_λ , and conformal Killing operator

$$(LW)_\lambda^{ab} = \nabla_\lambda^a W_b + \nabla_\lambda^b W_a - \frac{2}{3} \lambda_{ab} \nabla_\lambda^c W^c,$$

and to define contractions, we write out the constraint equations as follows:

$$\Delta_\lambda \phi = \frac{1}{8} R_\lambda \phi - \frac{1}{8} (\sigma + LW)_\lambda^2 \phi^{-7} + \frac{1}{12} \tau^2 \phi^5 \quad (2.3)$$

$$\nabla_\lambda \cdot LW = \phi^6 \nabla_\lambda \tau. \quad (2.4)$$

If, for a given set of conformal data $(\Sigma, \lambda, \sigma, \tau)$, equations (2.3)-(2.4) can be solved for (ϕ, W) , then one readily verifies that the reconstituted data

$$\gamma = \phi^4 \lambda \quad (2.5)$$

$$K = \phi^{-10} (\sigma + LW) + \frac{2}{3} \phi^{-4} \tau \lambda \quad (2.6)$$

satisfy the constraint equations.

Since $\text{tr}_\gamma K = \tau$ and since K is the second fundamental form for the embedded Cauchy hypersurface Σ in a spacetime development evolved from the initial data (Σ, γ, K) , the function τ represents the mean curvature of the Cauchy surface. Specifying $\tau = \text{constant}$ results in constant mean curvature (CMC) data. This condition is important, since if we specify conformal data with constant mean curvature then the equations (2.3)-(2.4) decouple. Equation (2.4) becomes a homogeneous linear elliptic equation for W and we have $LW = 0$ in all (compact) cases. The determination of whether a particular set of CMC conformal data produces a solution of the constraints is thus determined entirely by the solubility of the non-linear elliptic ‘‘Lichnerowicz’’ equation (2.3), with $LW = 0$.

3. SOLVING THE LICHNEROWICZ EQUATION

In this section we discuss the solubility of the Lichnerowicz equation, independent of possible coupling to the other constraint equations. (We return to the system (2.4) in Section 4.) To

emphasize this, in (2.3) we replace the term $(\sigma + LW)^2$ which involves the product of tensor fields, by the simple function μ^2 . Thus we work with the Lichnerowicz equation in the form

$$\Delta_{\lambda} \phi = \frac{1}{8} R_{\lambda} \phi - \frac{1}{8} \mu^2 \phi^{-7} + \frac{1}{12} \tau^2 \phi^5. \quad (3.1)$$

Here μ and τ are arbitrary smooth functions, which may or may not have zeroes.

We may further simplify the analysis of the Lichnerowicz equation by re-scaling, making use of its conformal covariance which tells us that there is a solution to (3.1) for a given set of data (λ, μ, τ) if and only if there is a solution to (3.1) for the related set of data $(\theta^4 \lambda, \theta^{-6} \mu, \tau)$ (see [BI]). Combining this property with the Yamabe theorem [S] (see also [LP]), we find that to determine the solubility of the Lichnerowicz equation for general sets of conformal data, it is sufficient to study (3.1) for metrics having scalar curvature of either $+8$, 0 , or -8 .¹

The key tool we employ for proving the existence of solutions is the sub and super solution theorem. The most useful version for our work here (see [IM]; the same result is proven for rougher data in [M]) is as follows

Theorem 3.1. *Let (Σ^3, λ) be a closed Riemannian manifold with C^2 metric and let $f \in C^1(\Sigma \times \mathbb{R}_+)$. Assume that there exists $\phi_-, \phi_+ : \Sigma \rightarrow \mathbb{R}_+$ such that*

- (1) $\phi_{\pm} \in W^{2,p}(\Sigma)$,
- (2) $0 < \phi_-(x) \leq \phi_+(x)$ for all $x \in \Sigma$,
- (3) $\Delta_{\lambda} \phi_- \geq f(x, \phi_-)$, and
- (4) $\Delta_{\lambda} \phi_+ \leq f(x, \phi_+)$.

Then there exists $\phi : \Sigma \rightarrow \mathbb{R}_+$ such that

- (1) $\phi \in C^{2+\alpha}(\Sigma)$ for $\alpha \in (0, 1 - \frac{3}{p})$,
- (2) $\phi_-(x) \leq \phi(x) \leq \phi_+(x)$ for all $x \in \Sigma$, and
- (3) $\Delta_{\lambda} \phi = f(x, \phi)$.

We first focus on positive Yamabe metrics. We show the following

Proposition 3.1. *Let Σ^3 be a closed manifold, let λ be a smooth, positive Yamabe class metric on Σ , and let μ and τ be smooth functions on Σ with μ not identically zero. Then there exists a unique smooth ² solution ϕ to the Lichnerowicz equation (3.1).*

Proof. As noted above, it is sufficient to prove existence and uniqueness of solutions for which $\frac{R}{\lambda}$ is constant; thus we set $\frac{R}{\lambda} = +8$ and work with the equation

$$\Delta \phi = \phi - \frac{1}{8} \mu^2 \phi^{-1} + \frac{1}{12} \tau^2 \phi^5. \quad (3.2)$$

Step 1. (Sub solution for $\tau = 0$ case)

In Section 5 of [I] (see class $(\mathcal{Y}^+, \sigma \neq 0, \tau = 0)$) it is shown that there exists a smooth function ζ_- such that

$$\Delta_{\lambda} \zeta_- \geq \zeta_- - \frac{1}{8} \mu^2 \zeta_-^{-7}. \quad (3.3)$$

¹In fact, the full Yamabe Theorem is not needed; it suffices that each conformal class contain a metric with scalar curvature having definite sign; for a proof of this more elementary fact see [A].

²The proof in [I] produces a function $\zeta_- \in W^{4,p}(\Sigma)$ for $p > 3$; one readily bootstraps the argument to show that for smooth data, we obtain a smooth solution as well.

For completeness, we summarize the argument presented there: Let

$$A := \max \left\{ 1, \frac{1}{8} \max \mu^2 \right\}$$

and consider the linear PDE

$$-\Delta_{\lambda} \zeta_{-} + \zeta_{-} = \frac{1}{8} \mu^2 A^{-7}. \quad (3.4)$$

It follows from the nondegeneracy of the operator $(-\Delta_{\lambda} + 1)$ on compact Σ^3 and from the smoothness of λ and μ that there exists a unique, smooth solution ζ_{-} to (3.4). To show that this function satisfies the inequality (3.3) as well – and therefore is a sub solution for the Lichnerowicz equation with $\tau = 0$ – we first note that since $\frac{1}{8} \mu^2 A^{-7}$ is non-negative and is not identically zero, the maximum principle guarantees that $\zeta_{-} > 0$. Next, since the function $G(x, s) := \frac{1}{8} \mu^2 s^{-7}$ is monotonically non-increasing in s , and since by definition $A \geq 1$ and $A \geq \frac{1}{8} \mu^2$, we have $G(x, A) \leq G(x, 1)$ and also $G(x, A) \leq A$ for all $x \in \Sigma$. The latter inequality, together with the definition of ζ_{-} , guarantees that ζ_{-} satisfies the inequality

$$-\Delta_{\lambda} \zeta_{-} + \zeta_{-} \leq A \quad (3.5)$$

from which we infer (via the maximum principle)³ that $\zeta_{-} \leq A$.

Finally, using this last inequality together with the monotonicity (in s) of G to infer that $G(x, A) \leq G(x, \zeta_{-})$, and writing (3.4) as

$$\Delta_{\lambda} \zeta_{-} = \zeta_{-} - G(x, A) \quad (3.6)$$

we verify that indeed ζ_{-} satisfies the sub-solution inequality (3.3).

Step 2. (General sub solution)

The inequality (3.3) is not strict; in order to obtain a sub solution for the Lichnerowicz equation (3.2) with positive Yamabe metric and τ^2 not identically zero, it is useful to first replace ζ_{-} by a function ξ_{-} for which (3.3) is a strict inequality. This is easily done by setting

$$\xi_{-} := \zeta_{-} - \frac{1}{2} \min_{\Sigma} \zeta_{-}. \quad (3.7)$$

Here we use the continuity of ζ_{-} and the fact that $\zeta_{-} > 0$ to verify that $\hat{\zeta}_{-} := \min_{\Sigma} \zeta_{-}$ exists and is positive. Consequently we have that

$$\zeta_{-}(x) > \xi_{-}(x) \geq \frac{1}{2} \hat{\zeta}_{-} > 0, \quad x \in \Sigma. \quad (3.8)$$

Using the monotonicity of $G(x, \cdot)$ we have $G(x, \xi_{-}) \geq G(x, \zeta_{-})$ for all $x \in \Sigma$, from which it follows that

$$\begin{aligned} \Delta_{\lambda} \xi_{-} - \xi_{-} + \mu^2 \xi_{-}^{-7} &\geq \Delta_{\lambda} \zeta_{-} - \zeta_{-} + \mu^2 \zeta_{-}^{-7} \\ &\geq \frac{1}{2} \hat{\zeta}_{-} \\ &> 0. \end{aligned} \quad (3.9)$$

Let us now multiply ξ_{-} by a positive number $\beta \in (0, 1)$ (to be determined later); we obtain

$$\Delta_{\lambda}(\beta \xi_{-}) - (\beta \xi_{-}) + \frac{\beta^8}{8} \mu^2 (\beta \xi_{-})^{-7} \geq \frac{1}{2} \mu \hat{\zeta}_{-}. \quad (3.10)$$

Since $\beta \in (0, 1)$, one has $\frac{1}{8} \mu^2 \geq \frac{1}{8} \beta^8 \mu^2$; hence

$$\Delta_{\lambda}(\beta \xi_{-}) - (\beta \xi_{-}) + \frac{1}{8} \mu^2 (\beta \xi_{-})^{-7} \geq \frac{1}{2} \mu \hat{\zeta}_{-}. \quad (3.11)$$

³The version we use here appears as #3 in [1].

We wish to make a choice of the constant β so that if we set $\phi_- = \beta\xi_-$, then ϕ_- is a sub solution for (3.2). This is accomplished by choosing $\beta \in (0, 1)$ so that

$$\frac{1}{2}\beta\hat{\zeta}_- \geq \frac{1}{12}\tau^2(\beta\xi_-)^5. \quad (3.12)$$

One readily verifies that this is satisfied provided

$$\beta \leq \left[\frac{6\hat{\zeta}_-}{(\max_{\Sigma} \tau^2)(\max_{\Sigma} \xi_-)^5} \right]^{1/4}. \quad (3.13)$$

Making such a choice for β we have that $\phi_- = \beta\xi_-$ is a sub solution; i.e.,

$$\Delta_{\lambda} \phi_- \geq \phi_- - \frac{1}{8}\mu^2\phi_-^{-7} + \frac{1}{12}\tau^2\phi_-^5. \quad (3.14)$$

Step 3. (Super solution)

A constant ϕ_+ is a super solution if it satisfies the inequality

$$\phi_+ - \frac{1}{8}\mu^2\phi_+^{-7} + \frac{1}{12}\tau^2\phi_+^5 \geq 0. \quad (3.15)$$

Clearly if one chooses ϕ_+ to be $(\frac{1}{8}\max_{\Sigma} \mu^2)^{1/8}$, then the inequality above is satisfied. However, this choice does not guarantee that we have $\phi_+ \geq \phi_-$. To ensure this latter condition, we choose

$$\phi_+ = \max \left\{ 1, \frac{1}{8}\max_{\Sigma} \mu^2 \right\}. \quad (3.16)$$

Recalling (as determined in the previous step) that $\phi_- \leq \xi_- < \zeta_- \leq A$, we verify that $\phi_- \leq \phi_+$. One also readily verifies that this choice satisfies (3.15) for any μ and τ .

Step 4. (Existence of Solution)

Since ϕ_- and ϕ_+ together constitute a set of smooth sub and super solutions which satisfy the hypotheses of Theorem 3.1, it follows that (3.2) has a smooth solution which is pointwise bounded by ϕ_{\pm} .

Step 5. (Uniqueness of Solution)

The sub and super solution theorem can be used to guarantee that a solution exists, but tells us nothing about the uniqueness of that solution. To show that solutions of the Lichnerowicz equation are unique, we rely on the following lemma, proved in [I].

Lemma 3.1. *Let $f : \Sigma^3 \times \mathbb{R} \rightarrow \mathbb{R}$ be C^1 and satisfy*

$$\frac{\partial f}{\partial s}(x, s) \geq 0 \quad (3.17)$$

for all $x \in \Sigma$ and all $s \in I$, where I is some interval (possibly infinite) in \mathbb{R}_+ . If Ψ_i , $i = 1, 2$, are both solutions of

$$\Delta \Psi = f(x, \Psi(x)) \quad (3.18)$$

and if Ψ_i take values in I for all $x \in \Sigma$, then $\Psi_1(x) = \Psi_2(x)$ for all $x \in \Sigma$.

For (3.2) we have

$$f(x, s) = s - \frac{1}{8}\mu^2 s^{-7} + \frac{1}{12}\tau^2 s^5, \quad (3.19)$$

and therefore

$$\begin{aligned} \frac{\partial f}{\partial s}(x, s) &= 1 + \frac{7}{8}\mu^2 s^{-8} + \frac{5}{12}\tau^2 s^4 \\ &> 0. \end{aligned} \quad (3.20)$$

Uniqueness of solutions to (3.2) follows immediately. \square

The result for zero Yamabe class metrics follows from the work done to prove Proposition 3.1. In particular, we can show the following.

Proposition 3.2. *Let Σ^3 be a closed manifold; let λ be a smooth, zero Yamabe class metric on Σ ; and let μ and τ be smooth functions on Σ with τ nowhere zero and μ not identically zero. Then there exists a unique smooth solution ϕ to the Lichnerowicz equation (3.1).*

Proof. It is sufficient to prove existence and uniqueness of solutions when $\frac{R}{\lambda} = 0$; thus we work with the equation

$$\Delta\phi = -\frac{1}{8}\mu^2\phi^{-1} + \frac{1}{12}\tau^2\phi^5. \quad (3.21)$$

Step 1. (Sub solution)

For the given choice of data $\{\lambda, \mu, \tau\}$ with zero Yamabe class metric λ , we seek a function ψ_- which satisfies the inequality

$$\Delta\psi_- \geq -\frac{1}{8}\mu^2\psi_-^{-1} + \frac{1}{12}\tau^2\psi_-^5. \quad (3.22)$$

Let us consider the inequality

$$\Delta\psi_- \geq \psi_- - \frac{1}{8}\mu^2\psi_-^{-1} + \frac{1}{12}\tau^2\psi_-^5. \quad (3.23)$$

We have shown in the proof of Proposition 3.1 that there is a function ϕ_- which satisfies (3.23). The argument for the existence of ϕ_- does not depend on the Yamabe class of the metric; hence we may take $\psi_- = \phi_-$, since $\phi_- \geq 0$.

Step 2. (Super solution)

Any constant ψ_+ which satisfies the condition

$$\tau^2\psi_+ \geq \frac{3}{2}\mu^2\psi_+^{-7} \quad (3.24)$$

serves as a supersolution for (3.21). To ensure as well that $\psi_+ \geq \psi_-$, we choose

$$\psi_+ = \left(\max \left\{ 2, \left(\min_{\Sigma} \tau^2 \right)^{-1/12} \right\} \right) \left(\max \left\{ 1, \max_{\Sigma} \mu^2 \right\} \right) \quad (3.25)$$

(Note that we require τ to be nowhere vanishing, unlike for metrics in the positive and negative Yamabe classes.)

Step 3. (Existence of Solutions)

Since ψ_{\pm} constitute smooth sub and super solutions for (3.21), it follows that (3.21) has a smooth solution which is pointwise bounded by ψ_- and ϕ_+ .

Step 4. (Uniqueness)

For (3.21), the function $f(x, s)$ to which we apply Lemma 3 takes the form

$$f(x, s) = -\frac{1}{8}\mu(x)^2s^{-7} + \frac{1}{12}\tau(x)^2s^5. \quad (3.26)$$

Since $\tau^2 > 0$ we immediately see that $\partial f / \partial s > 0$; hence the solution to (3.21) obtained is indeed unique. \square

We use Propositions 3.1 and 3.2 as key results for proving that the conformal method maps certain sets of near-CMC conformal data to solutions of the constraint equations, as we show in the next section (see Theorems 4.1 and 4.3)

4. NEAR-CMC SOLUTIONS OF THE CONSTRAINT EQUATIONS

The semi-decoupled sequence method for obtaining near-CMC solutions of the coupled system (2.3)-(2.4), introduced in [IM], focusses on the sequence of equations

$$\Delta_n \phi = \frac{1}{8} R_n \phi - \frac{1}{8} (\sigma^{ab} + L_n W^{ab}) (\sigma_{ab} + L_n W_{ab}) \phi^{-7} + \frac{1}{12} \tau_n^2 \phi^5 \quad (4.1)$$

$$\nabla_a (L_n W)_b^a = \frac{2}{3} \phi_n^6 \nabla_b \tau. \quad (4.2)$$

Let us presume that we have made a specific choice of conformal data $\{\Sigma^3, \lambda, \sigma, \tau\}$ satisfying appropriate hypotheses; for convenience we have in these equations suppressed explicit reference to the conformal metric λ . The idea is to iteratively define a sequence $\{(\phi_n, W_n)\}$ satisfying (4.1)-(4.2) by first choosing ϕ_0 arbitrarily⁴, then solving (4.2) with $n = 1$ to obtain W_1 , then substituting W_1 into (4.1) with $n = 1$ and solving (4.1) for ϕ_1 , and thus proceeding to solve (4.2) and (4.1) alternately and iteratively so as to obtain the entire sequence $\{(\phi_n, W_n)\}$. Once the sequence is obtained, one proceeds to prove that it converges to a smooth limit (ϕ_∞, W_∞) which satisfies (2.3)-(2.4). One finally shows that, for a given choice of conformal data (and for any choice of ϕ_0), the solutions obtained are unique.

Our method for showing that this sequence exists involves obtaining a sequence of sub solutions ϕ_n^- and super solutions ϕ_n^+ , as discussed in the last section. Both to show that these sub and super solutions exist and are controlled, and also to prove convergence of the sequence, we seek uniform upper and lower bounds for the set of all sub and super solutions, which consequently uniformly bound the sequence ϕ_n itself; these in turn imply uniform estimates for each W_n . Once we find these uniform bounds ϕ_∞^- and ϕ_∞^+ , we have at our disposal the estimates

$$0 < \phi_\infty^- \leq \phi_n^- \leq \phi_n \leq \phi_n^+ \leq \phi_\infty^+ < \infty \quad (4.3)$$

which hold for all $n = 1, 2, 3, \dots$. In fact, in our construction, we may use $\phi_n^+ = \phi_\infty^+$ (i.e., ϕ_∞^+ is a super solution for (4.1) for all n), while we inductively show the existence of sub solutions $\{\phi_n^-\}$, which are uniformly bounded below by a positive constant ϕ_∞^- .

Unlike the Lichnerowicz equation (3.1), the coupled system (2.3)-(2.4) is *not* conformally covariant (see [IM]). Consequently, the clearest statement of our results here regarding the solvability of (2.3)-(2.4) for a given set of conformal data involve two steps: We first state and prove solvability for conformal data with constant positive curvature (Theorem 4.1) and then use that result to prove a corollary for data with any positive Yamabe class metric (Corollary 4.2). Similarly, we prove a solvability theorem for data including zero curvature metrics (Theorem 4.3) and then extend the results to data with any zero Yamabe class metric.

Theorem 4.1. *Let Σ^3 be a closed three-dimensional manifold, let λ be a smooth Riemannian metric on Σ which admits no conformal Killing fields and has constant positive scalar curvature $R = +8$, and let σ be a smooth symmetric 2-tensor on Σ which is trace-free and divergence-free (with respect to λ) and not identically zero. For every smooth function $\tau : \Sigma \rightarrow \mathbb{R}_+$ which satisfies the gradient conditions given by (4.13) and (4.18) and which also satisfies the gradient condition*

⁴We do require that this choice of ϕ_0 satisfy the inequality $\phi_\infty^- \leq \phi_0 \leq \phi_\infty^+$ as discussed below.

that the coefficient of $\left| \begin{smallmatrix} \phi & \\ \hline n & n-1 \end{smallmatrix} - \phi \right|$ in equation (4.36) is sufficiently small, the equations (2.3)-(2.4) with data $\{\Sigma^3, \lambda, \sigma, \tau\}$ admit a unique smooth solution (ϕ, W) . Consequently for every such set of data $\{\Sigma^3, \lambda, \sigma, \tau\}$, there exists a unique solution (Σ, γ, K) of the constraint equations (2.1)-(2.2), taking the form (2.5)-(2.6).

Proof. For conformal data of the sort hypothesized here, the semi-decoupled system (4.1)-(4.2) takes the form

$$\Delta \begin{smallmatrix} \phi \\ \hline n \end{smallmatrix} = \begin{smallmatrix} \phi \\ \hline n \end{smallmatrix} - \frac{1}{8} |\sigma + LW|_n^2 \begin{smallmatrix} \phi \\ \hline n \end{smallmatrix}^{-7} + \frac{1}{12} \tau^2 \begin{smallmatrix} \phi \\ \hline n \end{smallmatrix}^5 \quad (4.4)$$

$$\nabla \cdot LW \begin{smallmatrix} \\ \hline n \end{smallmatrix} = \frac{2}{3} \begin{smallmatrix} \phi \\ \hline n-1 \end{smallmatrix}^6 \nabla \tau. \quad (4.5)$$

Step 1. (Construction of the sequence)

We begin by choosing $\begin{smallmatrix} \phi \\ \hline 0 \end{smallmatrix}$ such that $\begin{smallmatrix} \phi \\ \hline 0 \end{smallmatrix}^- \leq \begin{smallmatrix} \phi \\ \hline 0 \end{smallmatrix} \leq \begin{smallmatrix} \phi \\ \hline 0 \end{smallmatrix}^+$, for some constants $\begin{smallmatrix} \phi \\ \hline 0 \end{smallmatrix}_\pm$ to be chosen later and which depend only on the choice of conformal data. As is evident below, the value of $\begin{smallmatrix} \phi \\ \hline 0 \end{smallmatrix}$ is irrelevant, provided it does satisfy the above inequality. Under our assumption that (Σ, λ) admits no conformal Killing vectorfields, the operator $\nabla \cdot L$ is elliptic, self-adjoint, and invertible. Thus by standard elliptic theory (See, for example, the appendix of Besse [B].), the equation (4.5) with $n = 1$ admits a unique solution $\begin{smallmatrix} W \\ \hline 1 \end{smallmatrix}$, which as a consequence of the smoothness of τ satisfies

$$\| \begin{smallmatrix} W \\ \hline 1 \end{smallmatrix} \|_{C^{k+2, \alpha}} \leq c \| \begin{smallmatrix} \phi \\ \hline 0 \end{smallmatrix}^6 \nabla \tau \|_{C^{k, \alpha}}, \quad k \geq 0, \quad \alpha \in (0, 1), \quad (4.6)$$

where $C^{l, \alpha}$ denotes the (l, α) Hölder norm of vector fields on Σ given by λ . Furthermore, as argued in [IM], it follows from (4.6) along with geometric considerations that there exists C_S , depending only on the Riemannian manifold (Σ, λ) , such that we have the pointwise estimate

$$|LW \begin{smallmatrix} \\ \hline 1 \end{smallmatrix}| \leq C_S (\max_{\Sigma} \begin{smallmatrix} \phi \\ \hline 0 \end{smallmatrix}^+)^6 (\max_{\Sigma} |\nabla \tau|); \quad (4.7)$$

similarly we find

$$|LW \begin{smallmatrix} \\ \hline n+1 \end{smallmatrix}| \leq C_S (\max_{\Sigma} \begin{smallmatrix} \phi \\ \hline n \end{smallmatrix}^+)^6 (\max_{\Sigma} |\nabla \tau|). \quad (4.8)$$

We now describe how to choose uniformly bounded sub and super solutions $\begin{smallmatrix} \phi \\ \hline n \end{smallmatrix}^-, \begin{smallmatrix} \phi \\ \hline n \end{smallmatrix}^+$ for (4.1). This allows us to inductively construct a sequence $\{(\begin{smallmatrix} \phi \\ \hline n \end{smallmatrix}, \begin{smallmatrix} W \\ \hline n \end{smallmatrix})\}$ satisfying (4.4)-(4.5). To this end we assume the existence of $\begin{smallmatrix} \phi \\ \hline n-1 \end{smallmatrix}$ such that $0 < \begin{smallmatrix} \phi \\ \hline n-1 \end{smallmatrix} \leq \begin{smallmatrix} \phi \\ \hline n \end{smallmatrix}^+$, for some constant $\begin{smallmatrix} \phi \\ \hline \infty \end{smallmatrix}^+$. By our inductive assumption, it follows that

$$|LW \begin{smallmatrix} \\ \hline n \end{smallmatrix}| \leq C_S (\max_{\Sigma} \begin{smallmatrix} \phi \\ \hline n \end{smallmatrix}^+)^6 (\max_{\Sigma} |\nabla \tau|). \quad (4.9)$$

We desire to choose $\begin{smallmatrix} \phi \\ \hline \infty \end{smallmatrix}^+$ so that it is a constant super solution for (4.4) for all $n \in \mathbb{N}$; it suffices that the estimate

$$0 \leq \begin{smallmatrix} \phi \\ \hline \infty \end{smallmatrix}^+ - \frac{1}{8} |\sigma + LW|_{\infty}^2 \begin{smallmatrix} \phi \\ \hline \infty \end{smallmatrix}^+{}^{-7} + \frac{1}{12} \tau^2 \begin{smallmatrix} \phi \\ \hline \infty \end{smallmatrix}^+{}^5 \quad (4.10)$$

holds on Σ . For this to hold it suffices that

$$\begin{smallmatrix} \phi \\ \hline \infty \end{smallmatrix}^+{}^8 + \frac{1}{12} \tau^2 \begin{smallmatrix} \phi \\ \hline \infty \end{smallmatrix}^+{}^{12} \geq \frac{1}{8} (|\sigma|^2 + |LW|_n^2), \quad (4.11)$$

which in turn holds provided

$$\begin{smallmatrix} \phi \\ \hline \infty \end{smallmatrix}^+{}^8 \geq \frac{1}{8} |\sigma|^2 \quad \text{and} \quad \frac{1}{12} \tau^2 \begin{smallmatrix} \phi \\ \hline \infty \end{smallmatrix}^+{}^{12} \geq \frac{1}{8} C_S^2 \begin{smallmatrix} \phi \\ \hline \infty \end{smallmatrix}^+{}^{12} \max_{\Sigma} |\nabla \tau|^2. \quad (4.12)$$

We now see that so long as we restrict τ so that

$$\frac{3}{2}C_S^2 \left(\frac{\max_{\Sigma} |\nabla \tau|}{\min_{\Sigma} \tau} \right)^2 < 1, \quad (4.13)$$

it suffices to choose ϕ_+ such that

$$\phi_+^8 \geq \frac{1}{8} \max_{\Sigma} |\sigma|^2. \quad (4.14)$$

We turn to the task of finding a sequence of sub solutions ϕ_- , which are defined to be a sequence of functions such that

$$\Delta \phi_- \geq \phi_- - \frac{1}{8} |\sigma + LW|^2 \phi_-^{-7} + \frac{1}{12} \tau^2 \phi_-^5, \quad (4.15)$$

where W satisfies (4.5). Here, we seek a positive constant ϕ_- , independent of n , such that $\phi_+ \geq \phi_- \geq \phi_- > 0$.

Following the method used in the proof of Proposition 3.1, we first study the solution ψ to

$$\Delta \psi - \psi = -\frac{1}{8} |\sigma + LW|^2 A^{-7}, \quad (4.16)$$

where $A = \max \{1, \frac{1}{8} \max_{\Sigma} |\sigma + LW|^2\}$. We estimate ψ using ξ , the (smooth) solution to

$$\Delta \xi - \xi = -\frac{1}{8} |\sigma|^2 A^{-7}, \quad (4.17)$$

where $A = \max \{1, \max_{\Sigma} |\sigma|^2\}$. By the maximum principle⁵ and the compactness of Σ , there exists a constant $\delta > 0$, depending on $(\Sigma, \lambda, \sigma)$, such that $\xi \geq \delta$.

We claim that one may choose, depending only on ϕ_+ (and hence on $\max_{\Sigma} |\sigma|^2$), a constant C_{τ} such that the condition

$$|\nabla \tau| \leq C_{\tau} \quad (4.18)$$

implies that $|\psi - \xi|$ is small enough to ensure that $\psi \geq \frac{1}{2} \delta$. The claim follows from examining the equation

$$(\Delta - 1)(\psi - \xi) = -\frac{1}{8} |\sigma + LW|^2 A^{-7} + \frac{1}{8} |\sigma|^2 A^{-7}. \quad (4.19)$$

We first assume that $A = 1$ and $A = 1$. In this case, we employ the estimate (4.9) to bound the right hand side of (4.19) by

$$\frac{1}{8} |\nabla \tau| \left(C_S^2 \phi_+^{12} \max_{\Sigma} |\nabla \tau| + C_S \phi_+^6 \max_{\Sigma} |\sigma| \right). \quad (4.20)$$

Thus by the maximum principle⁶, we have $|\psi - \xi|$ is small whenever $\nabla \tau$ is small.

In the case that $A > 1$, we can choose C_{τ} small so that (4.18) implies $A > 1$ for all n . Then the right side of (4.19) is equal to

$$8^6 \frac{|\sigma + LW|^2 - |\sigma|^2}{\left(\max_{\Sigma} |\sigma + LW|^2 \right)^7} + 8^6 |\sigma|^2 \left[\left(\max_{\Sigma} |\sigma + LW|^2 \right)^{-7} - \left(\max_{\Sigma} |\sigma|^2 \right)^{-7} \right]. \quad (4.21)$$

⁵See, for example, version 2 in [I].

⁶Version 3 in [IM].

Making use of (4.9) once again, we see that this quantity can be made small by controlling $|\nabla\tau|$. Thus an application of the maximum principle yields the claim in this case.

With the claim in hand, one easily verifies that $-\Delta\psi + \psi \leq \frac{A}{n}$ and hence by the maximum principle we have $\psi \leq \frac{A}{n}$. From this it follows that

$$\Delta\psi \geq \psi - \frac{1}{8}|\sigma + LW|^2\psi^{-7}. \quad (4.22)$$

Replacing ψ by $\psi - \frac{1}{4}\delta$, we see that the previous estimate holds with a strict inequality.

We now choose a constant $\beta \in (0, 1)$, independent of n , so that $\phi_- := \beta\psi$ is a sub solution for (4.1). One verifies, using an argument similar to that in the proof of Proposition 3.1, that for any choice of $\beta \in (0, 1)$ satisfying

$$\beta^5 \leq \frac{3\delta}{(\max_{\Sigma} \tau^2) \phi_+^5} \quad (4.23)$$

we have

$$\Delta(\beta\psi) \geq (\beta\psi) - \frac{1}{8}|\sigma + LW|^2(\beta\psi)^{-7} + \frac{1}{12}\tau^2(\beta\psi)^5. \quad (4.24)$$

Note that any such sub solution $\beta\psi$ is bounded below by $\phi_- := \frac{1}{5}\beta\delta > 0$, independently of n .

Finally, if necessary, we choose a larger ϕ_+ to ensure $\phi_- \leq \phi_+$. This allows construction of the sequence $\{(\phi, W)\}$ such that

$$0 < \phi_- \leq \phi \leq \phi_+ \quad (4.25)$$

and

$$|LW| \leq C_S \phi^6 \max_{\Sigma} |\sigma|^2. \quad (4.26)$$

Step 2. (Convergence of the sequence)

We now show that the sequence $\{(\phi, W)\}$ converges to a smooth limit (ϕ, W) . Standard elliptic estimates applied to

$$(\nabla \cdot L)(W - W) = \frac{2}{3}[\phi^6 - \phi^6]\nabla\tau \quad (4.27)$$

imply that $\{W\}$ is Cauchy in $W^{p,2}(\Sigma)$, with $p > 3$, provided $\{\phi\}$ is Cauchy in $C^0(\Sigma)$. In light of the Sobolev embedding $W^{p,2}(\Sigma^3) \subset C^0(\Sigma^3)$, the sequence $\{(\phi, W)\}$ converges to $(\phi, W) \in C^0(\Sigma^3) \times C^0(\Sigma^3)$ provided $\{\phi\}$ converges. Thus we turn our attention to this sequence.

We study this sequence by considering the quantity

$$\mathcal{J}(x, \phi, \phi, \phi) := \int_0^1 \frac{d}{dt} \left[\Delta \psi(t, x) - F(x, \phi(t, x), \phi(t, x)) \right] dt, \quad (4.28)$$

where $\psi(t, x) = t\phi(x) + (1-t)\phi(x)$ and where

$$F(x, \phi(t), \phi(t)) := \frac{1}{8}\psi - \frac{1}{8}|\sigma + LW|^2\psi^{-7} + \frac{1}{12}\tau^2\psi^5. \quad (4.29)$$

Here the vector field W satisfies

$$\nabla \cdot LW = \frac{2}{3}\psi^6 \nabla\tau \quad (4.30)$$

and we have suppressed dependence on the point $x \in \Sigma$. Computing the quantity \mathcal{J} via the Fundamental Theorem of Calculus and also by direct computation, we obtain

$$\Delta \begin{bmatrix} \phi \\ n+1 \end{bmatrix} - \mathcal{G} \begin{bmatrix} \phi \\ n+1 \end{bmatrix} = \mathcal{F} \begin{bmatrix} \phi \\ n \end{bmatrix} - \begin{bmatrix} \phi \\ n-1 \end{bmatrix}, \quad (4.31)$$

where

$$\begin{aligned} \mathcal{F} \begin{bmatrix} \phi \\ n \end{bmatrix} &= \int_0^1 D_2 F(\cdot, \phi(t), \begin{bmatrix} \phi \\ n+1 \end{bmatrix}(t)) dt \begin{bmatrix} \phi \\ n \end{bmatrix} - \begin{bmatrix} \phi \\ n-1 \end{bmatrix} \\ \mathcal{G} \begin{bmatrix} \phi \\ n+1 \end{bmatrix} &= \int_0^1 D_3 F(\cdot, \phi(t), \begin{bmatrix} \phi \\ n+1 \end{bmatrix}(t)) dt \begin{bmatrix} \phi \\ n+1 \end{bmatrix} - \begin{bmatrix} \phi \\ n \end{bmatrix}; \end{aligned} \quad (4.32)$$

here D_i is differentiation with respect to the i^{th} variable. One easily sees that \mathcal{G} satisfies

$$\mathcal{G} \begin{bmatrix} \phi \\ n+1 \end{bmatrix} - \begin{bmatrix} \phi \\ n \end{bmatrix} \geq \frac{1}{8} \begin{bmatrix} \phi \\ n+1 \end{bmatrix} - \begin{bmatrix} \phi \\ n \end{bmatrix}. \quad (4.33)$$

An estimate for \mathcal{F} can be obtained by observing that

$$\mathcal{F} \begin{bmatrix} \phi \\ n \end{bmatrix} - \begin{bmatrix} \phi \\ n-1 \end{bmatrix} = \frac{1}{4} \int_0^1 \left(\sigma^{ab} + LW \begin{bmatrix} \psi(t) \\ n \end{bmatrix}^{ab} \right) \left(L\omega(t) \begin{bmatrix} \phi - \phi \\ n \end{bmatrix} \right)_{ab} \begin{bmatrix} \phi - \phi \\ n+1 \end{bmatrix} \psi^{-7}(t) dt, \quad (4.34)$$

where the vector field $\omega \begin{bmatrix} \phi - \phi \\ n \end{bmatrix}$ is defined to be the solution to

$$\nabla \cdot L\omega \begin{bmatrix} \phi - \phi \\ n \end{bmatrix} = 4\psi^5 \nabla \tau \begin{bmatrix} \phi - \phi \\ n \end{bmatrix} - \begin{bmatrix} \phi - \phi \\ n-1 \end{bmatrix}. \quad (4.35)$$

Thus by standard elliptic estimates used above, we see that

$$\left| \mathcal{F} \begin{bmatrix} \phi \\ n \end{bmatrix} - \begin{bmatrix} \phi \\ n-1 \end{bmatrix} \right| \leq \frac{1}{4} \left(\max |\sigma| + \hat{C} \frac{4}{3} \phi_+^6 \max |\nabla \tau| \right) \left(8 \hat{C} \phi_+^5 \max |\nabla \tau| \right) \left(\phi_+^{-7} \right) \cdot \left| \begin{bmatrix} \phi - \phi \\ n \end{bmatrix} - \begin{bmatrix} \phi - \phi \\ n-1 \end{bmatrix} \right|. \quad (4.36)$$

Hence, provided $\max |\nabla \tau|$ is small, we obtain via the maximum principle that

$$\left| \begin{bmatrix} \phi \\ n+1 \end{bmatrix} - \begin{bmatrix} \phi \\ n \end{bmatrix} \right| \leq \Lambda \left| \begin{bmatrix} \phi \\ n \end{bmatrix} - \begin{bmatrix} \phi \\ n-1 \end{bmatrix} \right| \quad (4.37)$$

for some positive constant $\Lambda < 1$. It follows that $\{\phi\}$ is a Cauchy sequence in $C^0(\Sigma)$, and converges to $\phi \in C^0(\Sigma)$. Further, as a consequence of the argument described above, there is a C^0 limit W_∞ for the sequence $\{W_n\}$. We may then adapt an argument from [IM], together with the smoothness of the data, to verify that in fact ϕ and W_∞ are smooth.

Step 3. (Showing that the limit is the unique solution)

To see that (ϕ, W_∞) constitutes a solution to (2.3)-(2.4), it suffices to observe that for $p > 3$ as above, we have the estimate

$$\left\| \Delta \begin{bmatrix} \phi \\ n \end{bmatrix} - F(\cdot, \begin{bmatrix} \phi \\ n \end{bmatrix}, \begin{bmatrix} \phi \\ n \end{bmatrix}) \right\|_{W^{p,0}} \leq C_{\text{Sobolev}} \left\| F(\cdot, \begin{bmatrix} \phi \\ n-1 \end{bmatrix}, \begin{bmatrix} \phi \\ n \end{bmatrix}) - F(\cdot, \begin{bmatrix} \phi \\ n \end{bmatrix}, \begin{bmatrix} \phi \\ n \end{bmatrix}) \right\|_{C^0}. \quad (4.38)$$

The continuity of F implies that the right side tends to zero as $n \rightarrow \infty$; thus we see that ϕ is a weak solution to (2.3). The smoothness of ϕ implies that this weak solution is in fact a classical solution; from a similar argument we obtain that (2.4) is weakly (and therefore classically) satisfied.

To verify uniqueness, let (ϕ, W) and $(\hat{\phi}, \hat{W})$ be a pair solutions. Define $\Phi(t) = t\phi + (1-t)\hat{\phi}$ and consider the quantity

$$\mathcal{K}(x, \phi, \hat{\phi}) = \int_0^1 \frac{d}{dt} [\Delta \Phi - F(x, \Phi, \Phi)] dt. \quad (4.39)$$

The analysis used to show convergence of (ϕ, W) yields the estimate

$$\|\phi - \hat{\phi}\|_{C^0} \leq \Lambda \|\phi - \hat{\phi}\|_{C^0}, \quad (4.40)$$

where, as before, $\Lambda < 1$. Thus $\phi = \hat{\phi}$, from which it follows immediately that $W = \hat{W}$. \square

It is possible to use this theorem, in conjunction with the Yamabe theorem, to find a solution to the constraint equations in the conformal class of any Yamabe positive metric λ on Σ^3 . Given a Yamabe positive Riemannian manifold (Σ, λ) , by the Yamabe theorem we can find a smooth, positive function θ such that $(\Sigma, \theta^4 \lambda)$ has scalar curvature $R = 8$. Then by applying Theorem 4.1 to the system

$$\frac{\Delta}{\theta^4 \lambda} \phi = \frac{1}{8} R \phi - \frac{1}{8} (\sigma + LW)_{\theta^4 \lambda}^2 \phi^{-7} + \frac{1}{12} \tau^2 \phi^5 \quad (4.41)$$

$$\nabla_{\theta^4 \lambda} \cdot LW = \phi^6 \nabla_{\theta^4 \lambda} \tau, \quad (4.42)$$

we arrive at the following corollary.

Corollary 4.2. *Suppose that Σ^3 is a Riemannian manifold with smooth metric λ in the Yamabe positive class having no conformal Killing vector fields. For each smooth symmetric 2-tensor σ which is trace-free and divergence-free with respect to λ , and for each smooth function $\tau : \Sigma \rightarrow \mathbb{R}$ which is non zero and which satisfies the hypotheses of Theorem 4.1, there exist smooth positive functions ϕ and θ and a smooth vector field W such that the data*

$$\begin{aligned} \gamma_{ab} &= (\phi\theta)^4 \lambda_{ab} \\ K^{ab} &= \phi^{-10} \left(\theta^{-10} \sigma + LW \right)_{\theta^4 \lambda}^{ab} + \frac{1}{3} (\phi\theta)^{-4} \lambda^{ab} \tau, \end{aligned} \quad (4.43)$$

comprise a solution to the Einstein constraint equations.

We now turn our attention to the case of conformal data which includes metrics which lie in the zero Yamabe class (i.e., metrics which can be conformally transformed to a metric with zero scalar curvature). We start by proving a result for metrics with $R = 0$:

Theorem 4.3. *Let Σ^3 be a closed three-dimensional manifold, let λ be a smooth Riemannian metric on Σ which admits no conformal Killing fields and which has identically vanishing scalar curvature, and let σ be a smooth symmetric 2-tensor on Σ which is trace-free and divergence-free (with respect to λ) and not identically zero. For every smooth function $\tau : \Sigma \rightarrow \mathbb{R}_+$ which is nowhere zero and which satisfies the gradient conditions (4.49) and (4.18) and which also satisfies the gradient condition that the coefficient of $\left| \begin{smallmatrix} \phi - \phi \\ n & n-1 \end{smallmatrix} \right|$ in equation (4.36) is sufficiently small, the equations (2.3)-(2.4) with data $\{\Sigma^3, \lambda, \sigma, \tau\}$ admit a unique smooth solution (ϕ, W) . Consequently for every such set of data $\{\Sigma^3, \lambda, \sigma, \tau\}$, there exists a unique solution (Σ, γ, K) of the constraint equations (2.1)-(2.2), taking the form (2.5)-(2.6).*

Proof. The proof of Theorem 4.3 (Yamabe zero metrics) is very much like that of Theorem 4.1 (positive Yamabe class metrics). We sketch the steps here, emphasizing the differences from the proof of Theorem 4.1.

Step 1. (Construction of the sequence)

We make use of Proposition 3.2 to construct a sequence $\{(\phi_n, W_n)\}$ satisfying

$$\Delta \phi_n = -\frac{1}{8}|\sigma + LW_n|^2 \phi_n^{-7} + \frac{1}{12}\tau^2 \phi_n^5 \quad (4.44)$$

$$\nabla \cdot LW_n = \frac{2}{3} \phi_{n-1}^6 \nabla \tau, \quad (4.45)$$

and in doing so we obtain uniform upper and lower bounds for the corresponding sequence of sub and super solutions. As in the positive curvature case, we may choose a constant ϕ_+ which is a super solution for (4.44) for all n . It suffices that the constant ϕ_+ satisfy

$$\frac{\max_{\Sigma} |\sigma|^2}{3 \min_{\Sigma} \tau^2} \leq \phi_+^{12}, \quad (4.46)$$

provided the estimate

$$|LW_n|^2 \leq \frac{1}{3} \tau^2 \phi_+^{12} \quad (4.47)$$

also holds. In light of the elliptic estimate

$$|LW_n| \leq C_S \max_{\Sigma} \phi_{n-1}^6 \max_{\Sigma} \tau^2, \quad (4.48)$$

a constant ϕ_+ satisfying (4.46) is a super solution so long as we require that the conformal data satisfy the condition

$$\frac{\max_{\Sigma} |\nabla \tau|^2}{\min_{\Sigma} \tau^2} \leq \frac{1}{3C_S}. \quad (4.49)$$

We now show that the sequence of sub solutions $\{\phi_n\}$ provided by the proof of Proposition 3.2 is bounded below by a positive function ϕ_- . Recall from the proof that the sub solutions for equation (3.21) were in fact chosen to satisfy (3.23). Thus the sub solutions for (4.44) in fact satisfy (4.15) and hence, by the proof of Theorem 4.1, are indeed uniformly bounded below by a positive constant ϕ_- . With ϕ_- in hand, we may increase ϕ_+ if necessary to ensure that, for all n , the following holds

$$0 < \phi_- \leq \phi_n \leq \phi_n \leq \phi_+ \leq \phi_+ < \infty. \quad (4.50)$$

Recall that the argument constructing ϕ_- places conditions on the size of $\nabla \tau$.

Step 2. (Convergence of the sequence)

The argument for convergence of the sequence here (with $R = 0$ conformal data) is very similar to that given for convergence of the sequence in the proof of Theorem 4.1 ($R > 0$ conformal data). We define \mathcal{J} much as in (4.28) (with the $\frac{1}{8} \psi_{n+1}$ term subtracted from the quantity F in (4.29)).

Then to obtain a contraction map of the form (4.37) (with $\Lambda < 1$), we need to carry out the estimates for the operators \mathcal{F} and \mathcal{G} as in (4.32).

The estimate for \mathcal{F} is precisely the same as above; we obtain (4.36). For \mathcal{G} we easily calculate that

$$\mathcal{G}[\phi_{n+1} - \phi_n] \geq \frac{5}{12} \min_{\Sigma} \tau^2 \phi_- (\phi_{n+1} - \phi_n). \quad (4.51)$$

Combining these estimates, we readily determine that for sufficiently small $\max |\nabla \tau|$, we have (4.37) with $\Lambda < 1$. The convergence of the sequence $\{(\phi_n, W_n)\}$ then follows, as in the proof of Theorem 4.1.

Step 3. (Showing that the limit is the unique solution)

The argument that the limit of the sequence $\{(\phi_n, W_n)\}$ is a smooth solution, and that it is unique for the given set of conformal data, proceeds exactly as in the proof of Theorem 4.1.

□

Combining this result with the Yamabe theorem for metrics of the zero Yamabe class, we produce (analogous to Corollary 4.2 the following:

Corollary 4.4. *Suppose Σ^3 is a Riemannian manifold with smooth metric λ in the Yamabe zero class having no conformal Killing vector fields. For each smooth symmetric 2-tensor σ which is trace-free and divergence-free with respect to λ , and for each smooth function $\tau : \Sigma \rightarrow \mathbb{R}$ which is nowhere zero and which satisfies the hypotheses of Theorem 4.3, there exist smooth positive functions ϕ and θ and a smooth vector field W such that the data*

$$\begin{aligned} \gamma_{ab} &= (\phi\theta)^4 \lambda_{ab} \\ K^{ab} &= \phi^{-10} \left(\theta^{-10} \sigma + \frac{LW}{\theta^4 \lambda} \right)^{ab} + \frac{1}{3} (\phi\theta)^{-4} \lambda^{ab} \tau, \end{aligned} \tag{4.52}$$

is a solution to the Einstein constraint equations.

5. CONCLUSIONS

The results we present here, together with those of the earlier papers [IM], and [IO], provide a fairly complete picture of which sets of near-CMC conformal data on compact manifolds lead to solutions of the Einstein constraint equations and which do not. Similarly, the picture for near-CMC asymptotically Euclidean data [CBIY] and for near-CMC asymptotically hyperbolic data [IP] is fairly clear as well. Even for the case of near-CMC data on manifolds with boundary, the recent results of [HKN] point toward increasing clarity.

On the other hand, almost nothing is understood about conformal data which is neither CMC nor near-CMC. This is the direction which future research into the use of the conformal method for obtaining solutions of the Einstein constraint equations is bound to explore.

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