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# THE INEQUALITY BETWEEN MASS AND ANGULAR MOMENTUM FOR AXIALLY SYMMETRIC BLACK HOLES*

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In this essay I first discuss the physical relevance of the inequality $m \geq \sqrt{|J|}$ for axially symmetric (nonstationary) black holes, where $m$ is the mass and $J$ the angular momentum of the space–time. Then, I present a proof of this inequality for the case of one spinning black hole. The proof involves a remarkable characterization of the extreme Kerr black hole as an absolute minimum of the total mass. Finally, I conjecture about the physical implications of this characterization for the nonlinear stability problem for black holes.

*Keywords*: Black holes; axial symmetry; angular momentum; mass; extreme Kerr.

## 1. Introduction

The Kerr metric is a solution to the vacuum Einstein equations which depends on two parameters, $m$ and $J$, the mass and the angular momentum of the space–time. It is well defined for any choice of the parameters; however, it describes a black hole only if the following — remarkable — inequality holds:

$$m \geq \sqrt{|J|}. \tag{1}$$

Roughly speaking, this inequality says that if an object is spinning too fast it cannot collapse to form a black hole.

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The inequality (1) has important consequences for a gravitational collapse, because the Kerr black hole is expected to play a unique role in such a process. To describe these consequences, let us first review what is known as the standard picture of the gravitational collapse. It consists mainly of the following two conjectures:

(i) Gravitational collapse results in a black hole (weak cosmic censorship). This conjecture points out the physical relevance of black hole solutions. Accordingly, it is assumed that solutions containing naked singularity [for example, the Kerr solution whose parameters do not satisfy (1)] would not generically occur.
(ii) The space–time settles down to a stationary final state, because only a finite amount of gravitational radiation can be emitted by an isolated system. It is also reasonable to assume that at some finite time all the matter fields have fallen into the black hole and hence the exterior region is a pure vacuum (for simplicity we discard electromagnetic fields in the exterior). Then, the black hole uniqueness theorem implies that the final state should be the Kerr black hole.

If the initial conditions for a collapse violate (1), then the extra angular momentum should be radiated away in gravitational waves. There exists, however, an important class of space–times in which angular momentum cannot be radiated by gravitational waves: axially symmetric space–times. For an axially symmetric space–time the angular momentum is a conserved quantity. Then, the angular momentum $J$ of the initial conditions must be equal to the final one, $J_0$. On the other hand, the mass of the initial conditions $m$ should be bigger than the final mass of the resulting Kerr black hole, $m_0$, because gravitational radiation carries positive energy. If we assume that the conjectures and (i) and (ii) hold, then the system will settle down to a final Kerr black hole, for which we have $m_0 \geq \sqrt{|J_0|}$. Then, we deduce that in this case the inequality (1) should be satisfied by the initial conditions. Also, if the initial conditions satisfy the equality $m = \sqrt{|J|}$, no mass can be radiated and hence we expect the system to be stationary. Since the only stationary black hole which satisfies this equality is extreme Kerr, in this case the system should be exactly extreme Kerr.

The argument presented above was essentially given in Ref. 1 and it is similar to the one used by Penrose[2] to obtain the inequality between mass and the area of the horizon on the initial data. As in the case of the Penrose inequality, a counterexample of (1) will imply that the standard picture of the gravitational collapse is not true. Conversely, a proof of (1) gives indirect evidence of its validity, since it is very hard to understand why this highly nontrivial inequality should hold unless (i) and (ii) can be thought of as providing the underlying physical reason behind it (see the discussion in Refs. 3 and 4).

The physical interpretation of (1) in the nonstationary case is the following. If we have a stationary vacuum black hole (i.e. Kerr) with mass $m_0$ and angular momentum $J_0$ and add to it axisymmetric gravitational waves, then the space–time will still contain a (nonstationary) black hole; these waves will only increase the mass and not the angular momentum of the space–time because they are axially symmetric. Since Kerr satisfies $m_0 \geq \sqrt{|J_0|}$, we get (1) for the resulting space–time. The difference $m - \sqrt{|J|}$, which can be calculated *a priori* on the initial conditions and by (1) is positive, provides an upper bound for the total amount of radiation $E$ emitted by the system:

$$E = m - m_0 \leq m - \sqrt{|J_0|} = m - \sqrt{|J|}. \tag{2}$$

This argument is the same as the one used by Hawking in Ref. 5.

## 2. The Variational Approach

The inequality (1) suggests a variational principle; namely, extreme Kerr realizes the minimum of the mass among all axially symmetric black holes with fixed angular momentum. However, it is important to note that for two related inequalities, i.e. the positive mass theorem and the Penrose inequality, a variational formulation was not successful. In the case of the positive mass theorem, only a local version was proved using a variational principle.[6]

The key difference in the present case it is axial symmetry. In that case it is possible to write the mass (in an appropriate gauge) as a positive definite integral on a spacelike hypersurface. The reason for this particular behavior of the mass is the following. In the presence of a symmetry, vacuum Einstein equations can be reduced *a la* Kaluza–Klein to a system on a three dimensional manifold, where it takes the form of three-dimensional Einstein equations coupled to a matter source. Since in three dimensions there is no radiation (the Weyl tensor is zero), this source represents the true gravitational degrees of freedom that have descended from four dimensions to appear as "matter" in three dimensions. Since all the energy is produced by these effective matter sources, one would expect that, as in other field theories, the total energy of the system can be expressed as a positive definite integral over them. This was in fact proved by Brill[7] in some restricted cases and then generalized in Refs. 8 and 9.

The mass integral essentially depends on two free functions: the norm and the twist potential of the axial Killing vector. This allows us as to make unconstrained variations of them and hence formulate a well-defined variational problem for the mass functional.

In a series of recent articles,[8,10–12] the inequality (1) has been proved for the case of one spinning black hole using this variational formulation. These results can be summarized as follows (we have suppressed some technical assumptions; for a precise formulation see Ref. 8):

**Theorem 1.** *Extreme Kerr realizes the unique absolute minimum of the mass functional among all axially symmetric spinning black holes with only one connected component and fixed angular momentum.*

While restricted to axially symmetric solutions, this theorem represents the first nonlinear and nonstationary result concerning the Kerr black hole.

Black holes are critical points of the mass at fixed angular momentum and horizon area.[13] This is, in essence, the first law of black hole mechanics. However, only extreme Kerr is in addition a minimum of the mass.

This characterization of extreme Kerr as a minimum of the mass implies a kind of stability of this solution. It is *a priori* not directly related to the nonlinear stability of extreme Kerr under the evolution of Einstein equations. But it suggests that the extreme Kerr black hole is stable in a more fundamental way than Schwarzschild or nonextreme Kerr among axially symmetric deformations.

This suggestion is supported by quantum effects. There are two quantum effects which can make the black hole unstable: Hawking radiation and particle production by superradiance. For the extreme Kerr black hole the temperature is zero and hence there is no Hawking radiation. The superradiance effect is related to the transfer of angular momentum from the black hole to the exterior (similar to the Penrose process). If we restrict ourselves to axially symmetric configurations, this transfer is not possible. In this sense, the extreme Kerr black hole is quantum-stable among axially symmetric configurations.

The nonlinear stability of black holes is a major open problem in general relativity. The first nontrivial vacuum model to be studied is represented by axisymmetric space–times. The results presented here reveal the following two relevant features of axial symmetry, which are likely to play an important role in this problem. First, the mass is a positive definite integral on the spacelike hypersurfaces. Since the mass is a conserved quantity, the norm defined by this integral controls the fields during the evolution. Second, the above considerations indicate that in the class of axisymmetric solutions the extreme Kerr black hole possesses hidden properties which are not present in the nonextreme case. Their analysis may significantly simplify the study of its nonlinear stability.

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