ON THE EXISTENCE AND UNIQUENESS OF THE SOLUTION OF A PARABOLIC OPTIMAL CONTROL PROBLEM WITH UNCERTAIN INPUTS

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Abstract. In this note, we consider the existence and uniqueness of the solution of a time-dependent optimal control problem constrained by a partial differential equation with uncertain inputs. Relying on the Lions' Lemma for deterministic problems, we furthermore characterize the optimal control of the stochastic problem.

Key words. PDE-constrained optimization, stochastic optimal control, Lions' lemma.

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1. Introduction.

This work is essentially concerned with the theoretical analysis of the existence and uniqueness of a parabolic stochastic optimal control problem (SOCP) considered in an earlier paper [1] by the authors. In [1], we propose efficient low-rank iterative solvers for solving the linear systems resulting from the stochastic Galerkin finite element method discretization of the SOCP. From a computational point of view, the work of Rosseel and Wells in [6] is directly related to the problem considered herein. However, the paper [6] treats only stationary SOCP and, as [1], does not deal with the theoretical issues of the existence and uniqueness of the solution of the considered problem. So, one goal of this paper is to close this apparent gap in the literature regarding the existence and uniqueness of SOCPs. Unlike other related literature on SOCPs (see e.g. [3] and the references therein), a special feature of the parabolic SOCP considered herein (as well as in [1, 5, 6]) is the presence of the standard deviation of the state variable as a risk measure in the cost functional.

In this work, we first establish the existence and uniqueness of the solution of the parabolic SOCP. Next, we rely mainly on the Lions' Lemma [4] to characterize the optimal control of the stochastic problem. However, we proceed first to Section 2 to provide the mathematical setting for the considered problem and recall Lions' Lemma. The main results are stated and proved in Section 3.

2. Problem statement.

In this note, we study the existence and uniqueness of the solution of a parabolic optimal control problem with stochastic inputs (SOCP). Before we proceed to state our problem, we first fix some notation that we will use in the sequel. To that end, let $\mathcal{D} \subset \mathbb{R}^d$ with $d \in \{1, 2, 3\}$ be a domain with Lipschitz boundary $\partial\mathcal{D}$. Moreover, for $T > 0$, we denote the time interval by $[0, T]$. We recall that by a random field $z : \mathcal{D} \times \Omega \to \mathbb{R}$, we mean that $z(x, \cdot)$ is a random variable defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for each $x \in \mathcal{D}$. Here, Ω is the set of outcomes, $\mathcal{F} \subset 2^\Omega$ is the σ-algebra of events, and $\mathbb{P} : \mathcal{F} \to [0, 1]$ is an appropriate probability measure. We consider only time-dependent random fields and assume

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that they are in the tensor product Hilbert space $L^2(0,T;L^2(\mathcal{D}))\otimes L^2(\Omega)$ which is endowed with the norm

$$||v||_{L^2(0,T;\mathcal{D})\otimes L^2(\Omega)} := \left(\int_\Omega ||v(\cdot,\cdot,\omega)||^2_{L^2(0,T;L^2(\mathcal{D}))}\, d\mathbb{P}(\omega)\right)^{\frac{1}{2}} < \infty,$$

where $L^2(\Omega) := L^2(\Omega,\mathcal{F},\mathbb{P})$. For any random variable z defined on $(\Omega,\mathcal{F},\mathbb{P})$, the standard deviation $\mathrm{std}(z)$ and the mean $\mathbb{E}(z)$ of z are given, respectively, by

$$\mathrm{std}(z) = \left[\int_\Omega (z-\mathbb{E}(z))^2\, d\mathbb{P}(\omega)\right]^{\frac{1}{2}} \text{ and } \mathbb{E}(z) = \int_\Omega z\, d\mathbb{P}(\omega) < \infty. \tag{2.1}$$

It is pertinent here to recall also that the variance $\mathrm{var}(z)$ of z is given by

$$\mathrm{var}(z) = (\mathrm{std}(z))^2 = \mathbb{E}(z^2) - (\mathbb{E}(z))^2. \tag{2.2}$$

In what follows, we write $\mathbb{P}$-a.s to mean $\mathbb{P}$-almost surely. Next, we set $\mathcal{X} := L^2(0,T;H^1_0(\mathcal{D}))\otimes L^2(\Omega)$ and let $\partial_t y \in L^2(0,T;H^{-1}(\mathcal{D}))\otimes L^2(\Omega)$. Also, let $\mathcal{U} := L^2(0,T;L^2(\mathcal{D}))\otimes L^2(\Omega)$ be the control space and set $\mathcal{Y} = \mathcal{U}$, where $\mathcal{Y}$ is the state space. Moreover, let $\mathcal{W} := L^2(0,T;L^2(\mathcal{D}))$. Note then that $\mathcal{W},\ \mathcal{X} \subset L^2(0,T;L^2(\mathcal{D}))\otimes L^2(\Omega) = \mathcal{Y}$. Finally, we set the Hilbert space $\mathcal{V} := L^2(\mathcal{D})\otimes L^2(\Omega)$ and let $\mathcal{V}'$ be the dual of $\mathcal{V}$.

In this work, we shall focus on distributed control problems, although we believe that our discussion generalizes to boundary control problems. More explicitly, we formulate our model problems as:

$$\min_{u\in\mathcal{U}_{ad}} \mathcal{J}(y,u) := \frac{1}{2}||y-y_d||^2_{\mathcal{Y}} + \frac{\alpha}{2}||\mathrm{std}(y)||^2_{\mathcal{W}} + \frac{\beta}{2}||u||^2_{\mathcal{U}} \tag{2.3}$$

subject, $\mathbb{P}$-a.s, to

$$\begin{cases} \dfrac{\partial y(t,x,\omega)}{\partial t} + A(x,\omega)y(t,x,\omega)) = u(t,x,\omega), \text{ in } (0,T]\times\mathcal{D}\times\Omega, \\ y(t,x,\omega) = 0, \text{ on } (0,T]\times\partial\mathcal{D}\times\Omega, \\ y(0,x,\omega) = 0, \text{ in } \mathcal{D}\times\Omega. \end{cases} \tag{2.4}$$

where $A : \mathcal{V} \to \mathcal{V}'$ is a linear operator that contains some random parameters. Moreover, $\mathcal{J}$ is a cost functional of tracking-type, which includes a risk penalization via the standard deviation. The functions y, u and y_d are, in general, real-valued functions representing, respectively, the state, the control and the prescribed target system response (or desired state). Without loss of generality, we assume that the state $y \in \mathcal{Y}$ and the control function $u \in \mathcal{U}$ are random fields while the desired state $y_d \in \mathcal{Y}$ is modeled deterministically. The constant $\beta > 0$ in (2.3) represents the parameter for the penalization of the action of the control u, whereas $\alpha \geq 0$ is the so-called *risk-aversion* parameter that penalizes the standard deviation $\mathrm{std}(y)$ of the state y. The objective functional $\mathcal{J}$ is a deterministic quantity with uncertain terms. The set $\mathcal{U}_{ad}$ is the so-called convex admissible set, and is given by

$$\mathcal{U}_{ad} := \{u \in \mathcal{U} : u(t,x,\omega) \geq 0 \ \mathbb{P}\text{-a.s in } [0,T]\times\mathcal{D}\times\Omega\}. \tag{2.5}$$

We shall need the following assumptions on A in the sequel.

Assumption 1.
(a) A is coercive: there exist constants $c > 0$ and θ such that, $\mathbb{P}$-a.s,

$$(Av, v) + \theta ||v||^2_{\mathcal{H}} \geq c||v||_{\mathcal{V}}, \ \ \forall v \in \mathcal{V},$$

where the space $\mathcal{H}$ is chosen such that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ and $\mathcal{V}$ is dense in $\mathcal{H}$.
(b) A is self-adjoint: $(Au, v) = (u, A^\star v)\,, \quad \forall u, v \in \mathcal{V}, \quad \mathbb{P}$-a.s.

A prominent example of the operator A is the diffusion operator considered, for instance, in [1, 6]:

$$A := -\nabla \cdot a(x, \omega)\nabla, \tag{2.6}$$

in which case we assume that the random field $a(x, \omega)$ is uniformly positive in $\mathcal{D} \times \Omega$. That is, there exist strictly positive constants $a_{\min}$ and $a_{\max}$, with $a_{\min} \leq a_{\max}$, such that

$$\mathbb{P}\left(\omega \in \Omega : a(x, \omega) \in [a_{\min}, a_{\max}], \forall x \in \mathcal{D}\right) = 1.$$

The weak formulation of the SOCP (2.3) and (2.4) above is given by

$$\begin{aligned} \min_{u \in \mathcal{U}_{ad}} \mathcal{J}(u) = \frac{1}{2}\mathbb{E}\int_0^T \int_{\mathcal{D}} (y(u) - y_d)^2 \ dxdt + \frac{\alpha}{2}\mathbb{E}\int_0^T \int_{\mathcal{D}} \left[y(u) - (\mathbb{E}y(u))\right]^2 \ dxdt \\ + \frac{\beta}{2}\mathbb{E}\int_0^T \int_{\mathcal{D}} u^2 \ dxdt \end{aligned} \tag{2.7}$$

subject, $\mathbb{P}$-a.s, to

$$\mathbb{E}\int_0^T \int_{\mathcal{D}} \partial_t y v \ dxdt + \mathcal{B}(y, v) = \mathbb{E}\int_0^T \int_{\mathcal{D}} uv \ dxdt, \quad v \in H_0^1(\mathcal{D}) \otimes L^2(\Omega), \tag{2.8}$$

where $\mathcal{B}$ is a bilinear form of the operator A defined on the tensor product space $H_0^1(\mathcal{D}) \otimes L^2(\Omega)$. In particular, if $A = -\nabla \cdot a(x, \omega)\nabla$, then

$$\mathcal{B}(y, v) := \mathbb{E}\int_0^T \int_{\mathcal{D}} a\nabla y \cdot \nabla v \ dxdt.$$

Next, we proceed to Section 3 to establish our existence and uniqueness results for the parabolic SOCP (2.3) – (2.4).

3. Existence and uniqueness results. In our subsequent discussion, we shall rely explicitly on the following definition.

Definition 3.1. *A function $\bar{u} \in \mathcal{U}_{ad}$ is called an optimal control and $\bar{y} = y(\bar{u})$ the associated optimal state corresponding to the the SOCP (2.3) and (2.4) if, $\mathbb{P}$-a.s, the following expression holds:*

$$\mathcal{J}(\bar{y}, \bar{u}) \leq \mathcal{J}(y(u), u), \qquad \forall u \in \mathcal{U}_{ad}.$$

We can now state the following result.

Theorem 3.2. *Let $\{\mathcal{H}_1, ||\cdot||\}$ and $\{\mathcal{H}_2, ||\cdot||\}$ be Hilbert spaces. Suppose that $\tilde{\mathcal{H}}_1 \subset \mathcal{H}_1$ is a non-empty, closed and convex set. Let $y_d \in \mathcal{H}_1$ and the constants*

$\gamma, \eta \geq 0$ be given. Furthermore, let $S : \mathcal{H}_1 \mapsto \mathcal{H}_2$ be a continuous linear operator. Then, the quadratic Hilbert space optimization problem

$$\min_{u \in \tilde{\mathcal{H}}_1} f(u) = \frac{1}{2}||Su - y_d||^2_{\mathcal{H}_2} + \frac{\gamma}{2}||std(Su)||^2_{\mathcal{H}_2} + \frac{\eta}{2}||u||^2_{\mathcal{H}_1}$$

admits, $\mathbb{P}$-a.s, an optimal solution $\bar{u} \in \tilde{\mathcal{H}}_1$. If $\eta > 0$, then $\bar{u}$ is uniquely determined.

Proof. Note first that the function $f(u) \geq 0$ is continuous and convex. The proof of Theorem 3.2 therefore follows analogously to that of [7, Theorem 2.14]. ☐

Suppose now that we set $\tilde{\mathcal{H}}_1 = \mathcal{U}_{ad}$, $\mathcal{H}_1 = \mathcal{U}$ and $\mathcal{H}_2 = \mathcal{Y}$. Observe then that, by [7, Theorems 3.12, 3.13], for any $u \in \mathcal{U}$, there exists a unique solution to the parabolic initial-boundary value problem (2.4). Now, let the mapping

$$G_{\mathcal{U}} : \mathcal{U} \mapsto \mathcal{X} \subset \mathcal{Y},\ u \mapsto y(u)$$

be the so-called control-to-state operator. Observe then that $G_{\mathcal{U}}$ is a continuous linear operator [7, pp. 50]. Moreover, let $E_{\mathcal{Y}} : \mathcal{X} \mapsto \mathcal{Y}$ denote the embedding operator that assigns to each $y \in \mathcal{X} \subset \mathcal{Y}$ the same function in $\mathcal{Y}$. Note that $E_{\mathcal{Y}}$ is also linear and continuous. Thus, the composition $u \mapsto y(u) \mapsto y(u)$ is a continuous linear operator $S := E_{\mathcal{Y}} G_{\mathcal{U}} : u \mapsto y(u)$. Hence, it turns out that if we substitute S into the cost functional $\mathcal{J}(y, u)$ in (2.3), then we eliminate the parabolic initial-boundary value problem to arrive at the following quadratic minimization problem in the Hilbert space $\mathcal{U}$:

$$\begin{aligned}\min_{u \in \mathcal{U}_{ad}} \mathcal{J}(u) = &\frac{1}{2}\mathbb{E}\int_0^T \int_{\mathcal{D}} (Su - y_d)^2 \, dxdt + \frac{\alpha}{2}\mathbb{E}\int_0^T \int_{\mathcal{D}} [Su - (\mathbb{E}Su)]^2 \, dxdt \\ &+ \frac{\beta}{2}\mathbb{E}\int_0^T \int_{\mathcal{D}} u^2 \, dxdt. \end{aligned} \tag{3.1}$$

Furthermore, note that $\mathcal{U}_{ad}$ as defined by (2.5) is indeed non-empty, closed and convex. Thus, we can infer from Theorem 3.2 that there exists an optimal control $\bar{u}$ for the problem (3.1). Moreover, by our initial assumption the regularization parameter $\beta > 0$. Hence, we have indeed established the following result.

THEOREM 3.3. *Under the assumptions of Theorem 3.2, the SOCP (2.3) – (2.4) has at least one optimal control $\bar{u} \in \mathcal{U}_{ad}$. If $\beta > 0$, then $\bar{u}$ is uniquely determined.*

Although Theorem 3.3 establishes the existence and uniqueness of a solution to the optimal control problem (2.3) – (2.4), it is not constructive in the sense that the theorem provides no indication as to how this solution can be obtained. In the following, we will address this issue and provide a characterization of the optimal control. To this end, a chief corner stone in our subsequent discussion in this contribution is the following result in the deterministic framework, which is often known as the Lions' lemma [4, p. 10].

THEOREM 3.4. *[4, Theorem 1.3] Suppose the cost functional $v \mapsto \mathcal{J}(v)$ is strictly convex and differentiable. Then, there exists a unique optimal control $\bar{u} \in \mathcal{U}_{ad}$ if and only if*

$$\mathcal{J}'(\bar{u}) \cdot (v - \bar{u}) \geq 0, \quad \forall v \in \mathcal{U}_{ad}, \tag{3.2}$$

where

$$\mathcal{J}'(\bar{u}) \cdot (w) := \lim_{h \to 0} \frac{\mathcal{J}(\bar{u} + hw) - \mathcal{J}(\bar{u})}{h}, \tag{3.3}$$

is the derivative of $\mathcal{J}$ with respect to u in the direction of w.

Note that it is very easy to check that the cost functional $\mathcal{J}$ in (2.3) is strictly convex. We can now prove the following characterization result for the optimal control.

THEOREM 3.5. *The SOCP given by (2.3) – (2.4) has a unique solution (y,u) if and only if there exists a co-state variable $\lambda \in \mathcal{Y}$ such that the triplet (y,u,λ) satisfies, $\mathbb{P}$-a.s , the following optimality system:*

$$\frac{\partial y(u)}{\partial t} + Ay(u) = u, \tag{3.4}$$

$$y(u)\mid_{t=0} = 0, \quad y(u) \in \mathcal{Y}. \tag{3.5}$$

$$-\frac{\partial \lambda(u)}{\partial t} + A^*\lambda(u) = (1+\alpha)y(u) - \alpha\mathbb{E}(y(u)) - y_d, \tag{3.6}$$

$$\lambda(u)\mid_{t=T} = 0, \quad \lambda(u) \in \mathcal{Y}. \tag{3.7}$$

$$\mathbb{E}\int_0^T \int_{\mathcal{D}} (\lambda(u) + \beta u)\cdot(v-u)\, dxdt \geq 0, \quad u, v \in \mathcal{U}_{ad}. \tag{3.8}$$

We note here that we have dropped the dependence on (t,x,ω) for notational convenience.

Proof. It suffices to show that the condition (3.2) in Theorem 3.4 holds. Now, using (2.2) and the fact that $y = y(u)$, note that (2.3) can be re-written as

$$\begin{aligned}\mathcal{J}(u) &:= \mathcal{J}_1(u) + \mathcal{J}_2(u) - \mathcal{J}_3(u) + \mathcal{J}_4(u)\\ &= \frac{1}{2}\mathbb{E}\int_0^T\int_{\mathcal{D}} (y(u)-y_d)^2\, dxdt + \frac{\alpha}{2}\mathbb{E}\int_0^T\int_{\mathcal{D}} y(u)^2\, dxdt\\ &\quad -\frac{\alpha}{2}\int_0^T\int_{\mathcal{D}} (\mathbb{E}y(u))^2\, dxdt + \frac{\beta}{2}\mathbb{E}\int_0^T\int_{\mathcal{D}} u^2\, dxdt.\end{aligned} \tag{3.9}$$

Using the definition (3.3), we find that

$$\begin{aligned}\mathcal{J}_1'(u)\cdot(v-u) &= \lim_{h\to 0}\frac{\mathbb{E}\int_0^T\int_{\mathcal{D}}[y(u+h(v-u))-y_d]^2\, dxdt - \mathbb{E}\int_0^T\int_{\mathcal{D}}[y(u)-y_d]^2\, dxdt}{2h}\\ &= \lim_{h\to 0}\frac{\mathbb{E}\int_0^T\int_{\mathcal{D}} y(u+h(v-u))^2 - y(u)^2\, dxdt}{2h}\\ &\quad - \lim_{h\to 0}\frac{\mathbb{E}\int_0^T\int_{\mathcal{D}} 2y_d(y(u+h(v-u))-y(u))\, dxdt}{2h}\\ &= \mathbb{E}\int_0^T\int_{\mathcal{D}} y(u)y'(u)\cdot(v-u)\, dxdt - \mathbb{E}\int_0^T\int_{\mathcal{D}} y_d y'(u)\cdot(v-u)\, dxdt\\ &= \mathbb{E}\int_0^T\int_{\mathcal{D}} (y(u)-y_d)y'(u)\cdot(v-u)\, dxdt,\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_4'(u)\cdot(v-u) &= \beta\cdot\lim_{h\to 0}\frac{\mathbb{E}\int_0^T\int_{\mathcal{D}}(u+h(v-u))^2\, dxdt - \mathbb{E}\int_0^T\int_{\mathcal{D}} u^2\, dxdt}{2h}\\ &= \beta\cdot\lim_{h\to 0}\frac{\mathbb{E}\int_0^T\int_{\mathcal{D}}(u^2+h^2(v-u)^2+2hu(v-u))\, dxdt - \mathbb{E}\int_0^T\int_{\mathcal{D}} u^2\, dxdt}{2h}\\ &= \beta\mathbb{E}\int_0^T\int_{\mathcal{D}} u\cdot(v-u)\, dxdt.\end{aligned}$$

One can easily perform similar calculations with the terms $\mathcal{J}_2$ and $\mathcal{J}_3$ in (3.9) to obtain (see e.g. [6]), respectively,

$$\mathcal{J}_2'(u)\cdot(v-u) = \alpha\mathbb{E}\int_0^T\int_{\mathcal{D}} y(u)y'(u)\cdot(v-u)\,dxdt,$$

and

$$\mathcal{J}_3'(u)\cdot(v-u) = \alpha\mathbb{E}\int_0^T\int_{\mathcal{D}} (\mathbb{E}y(u))y'(u)\cdot(v-u)\,dxdt.$$

Hence, we have

$$\begin{aligned}\mathcal{J}'(u)\cdot(v-u) &= [\mathcal{J}_1'(u)+\mathcal{J}_2'(u)-\mathcal{J}_3'(u)+\mathcal{J}_4'(u)]\cdot(v-u)\\ &= \mathbb{E}\int_0^T\int_{\mathcal{D}}(y(u)-y_d)y'(u)\cdot(v-u)\,dxdt\\ &\quad+\alpha\mathbb{E}\int_0^T\int_{\mathcal{D}} y(u)y'(u)\cdot(v-u)\,dxdt\\ &\quad-\alpha\mathbb{E}\int_0^T\int_{\mathcal{D}}(\mathbb{E}y(u))y'(u)\cdot(v-u)\,dxdt\\ &\quad+\beta\mathbb{E}\int_0^T\int_{\mathcal{D}} u\cdot(v-u)\,dxdt.\end{aligned}\tag{3.10}$$

Next, let $\mathcal{L} := \frac{\partial}{\partial t} + A$. Note then that Assumption 1, as given by (a) and (b) which are directly below equation (2.4), implies that $\mathcal{L}$ is invertible and, indeed, from (3.4) one then gets

$$\mathcal{L}y(u) = u \Longrightarrow y(u) = \mathcal{L}^{-1}u,\tag{3.11}$$

so that, using (3.11), the quantity $y'(u)\cdot(v-u)$ appearing in the first three terms of the expression (3.10) now yields

$$y'(u)\cdot(v-u) = \mathcal{L}^{-1}(v-u) = \mathcal{L}^{-1}(v) - \mathcal{L}^{-1}(u) = y(v)-y(u).$$

Thus, we have from (3.10) that

$$\mathcal{J}'(u)\cdot(v-u) = \Psi(\alpha) + \beta\mathbb{E}\int_0^T\int_{\mathcal{D}} u\cdot(v-u)\,dxdt,\tag{3.12}$$

where

$$\begin{aligned}\Psi(\alpha) &= (1+\alpha)\mathbb{E}\int_0^T\int_{\mathcal{D}} y(u)\cdot(y(v)-y(u))\,dxdt\\ &\quad-\alpha\mathbb{E}\int_0^T\int_{\mathcal{D}}\mathbb{E}(y(u))\cdot(y(v)-y(u))\,dxdt\\ &\quad-\mathbb{E}\int_0^T\int_{\mathcal{D}} y_d\cdot(y(v)-y(u))\,dxdt.\end{aligned}\tag{3.13}$$

Observe, once again, that Lions' lemma demands that the expression (3.12) be non-negative to ensure the existence and uniqueness of the solution to the SOCP (2.3) and (2.4). To that end, we next introduce the adjoint state $\lambda(v)$ by

$$-\frac{\partial \lambda(v)}{\partial t} + A^*\lambda(v) = (1+\alpha)y(v) - \alpha\mathbb{E}(y(v)) - y_d, \tag{3.14}$$

$$\lambda(v)\mid_{T=0} = 0, \quad \lambda(v) \in \mathcal{Y}. \tag{3.15}$$

Now, set $v = u$ in (3.14) and multiply both sides of the equation by $y(v) - y(u)$. Observe first that, by taking the expectation of the resulting expression and integrating it over $[0,T]$ and $\mathcal{D}$, these two operations essentially transform the right hand side of (3.14) to the expression $\Psi(\alpha)$ in (3.13). Moreover, the two terms on the left hand side of (3.14) now read

$$\mathbb{E}\int_0^T\int_{\mathcal{D}} -\frac{\partial \lambda(u)}{\partial t}\cdot(y(v)-y(u))\,dxdt = \mathbb{E}\int_0^T\int_{\mathcal{D}} \lambda(u)\cdot\left(\frac{\partial y(v)}{\partial t} - \frac{\partial y(u)}{\partial t}\right)\,dxdt, \tag{3.16}$$

$$\mathbb{E}\int_0^T\int_{\mathcal{D}} A^*\lambda(u)\cdot(y(v)-y(u))\,dxdt = \mathbb{E}\int_0^T\int_{\mathcal{D}} \lambda(u)\cdot(Ay(v)-Ay(u))\,dxdt. \tag{3.17}$$

To obtain (3.17), we have used the fact that the operator A is self-adjoint. Furthermore, we have used integration by parts, together with the conditions (3.5) and (3.7) to obtain (3.16). Thus, summing up (3.16) and (3.17), one gets

$$\begin{aligned}\mathbb{E}\int_0^T\int_{\mathcal{D}}\left(\lambda(u)\cdot\left(\frac{\partial}{\partial t}+A\right)(y(v)-y(u))\right)\,dxdt &= \mathbb{E}\int_0^T\int_{\mathcal{D}}\lambda(u)\cdot(v-u)\,dxdt\\ &= \Psi(\alpha),\end{aligned} \tag{3.18}$$

where we have explicitly used (3.4) in the first line of (3.18). Hence, it follows from (3.12), (3.13) and Theorem 3.4 that

$$\begin{aligned}\mathcal{J}'(u)\cdot(v-u) &= \Psi(\alpha) + \mathbb{E}\int_0^T\int_{\mathcal{D}}\beta u\cdot(v-u)\,dxdt,\\ &= \mathbb{E}\int_0^T\int_{\mathcal{D}}\lambda(u)\cdot(v-u)\,dxdt + \mathbb{E}\int_0^T\int_{\mathcal{D}}\beta u\cdot(v-u)\,dxdt,\\ &= \mathbb{E}\int_0^T\int_{\mathcal{D}}(\lambda(u)+\beta u)\cdot(v-u)\,dxdt \geq 0,\end{aligned} \tag{3.19}$$

yields the desired result, thereby completing the proof of the theorem. □

Remark 1. *Note that if we set $\mathcal{U}_{ad} = \mathcal{U}$, then (3.19) implies that*

$$\lambda(u) + \beta u = 0.$$

Remark 2. *It is pertinent to point out here that the papers [2, 3] prove the existence and uniqueness of the solution of SOCPs in the particular case of the steady-state diffusion equation constraint, with the operator A given by (2.6). However, ours is a generalization of their result to the parabolic case, with operator A satisfying the assumptions (a) and (b) in Section 2. Moreover, unlike this work, [2, 3] do not consider the inclusion of the standard deviation of the state variable (or any other risk measure) in the cost functional.*

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