

Conformal Anomaly for Non-Conformal Scalar Fields

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Abstract

We give a general definition of the conformal anomaly for theories that are not classically Weyl invariant and show that this definition yields a quantity that is both finite and local. As an example we study the conformal anomaly for a non-minimally coupled massless scalar and show that our definition coincides with results obtained using the heat kernel method.

1 Introduction

Conformal anomalies have been the focus of much study since the 1970s [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. One aspect, however, that has not received so much attention concerns the significance of conformal anomalies in theories that are *not* classically conformal (or Weyl) invariant. One main reason for the interest in this question is the fact that the most interesting cancellations of conformal anomalies occur in *non*-conformal supergravities for $N \geq 5$ [17] (whereas conformal supergravities stop at $N = 4$, and cancellations require very special matter couplings [9]). The full significance of these cancellations has not been fully understood so far; moreover there is a pending issue concerning the dependence of the a and c coefficients on the gauge choice for the external gravitational field for fields of spin $\geq \frac{3}{2}$ that remains to be resolved. In this paper we focus on the investigation of the conformal anomaly and its significance for the specific example of the non-conformal scalar field. For this we follow a recent recalculation of the anomaly for spin- $\frac{1}{2}$ fields [18], which is based on an evaluation of Feynman diagrams largely analogous to (though much more involved than) the textbook derivation of the axial anomaly in gauge theories. The spin-0 field coupled to a background gravitational field represents the simplest example in which to consider non-conformal deformations, and has the added advantage that there is a tunable parameter ξ , such that for one special value of ξ the theory becomes conformal.

For any theory, whether conformal or not, we adopt the following *general definition* of the conformal anomaly

$$\mathcal{A}(x) := \lim_{\varepsilon \rightarrow 0} \left[g^{(4)\mu\nu} \langle T_{\mu\nu}(x) \rangle - \langle g^{\mu\nu} T_{\mu\nu}(x) \rangle \right]. \quad (1)$$

where $\varepsilon \equiv \frac{1}{2}(d-4)$ is the regularization parameter in dimensional regularization, and the superscript on $g^{\mu\nu}$ indicates the dimension in which the trace is to be performed (we do not yet specify this dimension in the second term, because there is a choice which does not affect the physically significant part of the anomaly, as we will explain below). Here the second term removes the classical violation of conformal invariance, reflected in a non-vanishing trace of the classical stress-energy tensor – the difference between the quantum trace and the expectation value of the classical trace is what produces the conformal anomaly. We show in section 2 that this definition yields a result that is always finite and local for any theory, including theories that are not classically Weyl invariant, and hence to which one would not normally associate a Weyl anomaly. Furthermore we will see that in the non-conformal case the two terms on the r.h.s. of (1) by themselves do not produce a meaningful answer because they separately exhibit divergences and non-local terms which only cancel in the difference: this is the reason the difference must be taken *before* removing the regulator. In section 2 below we present a general argument why this is always true. Furthermore,

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in the case of a non-minimally coupled scalar we show that the coefficients of \mathbf{E}_4 (Euler number) and C^2 in the anomaly do not depend on the value of ξ , whereas the coefficient of $\square R$ does depend on ξ (but the a and c coefficients may start to depend on various couplings in higher loop orders in the presence of interactions, as there appears to be no analog of the Adler-Bardeen theorem for the conformal anomaly).

For the non-minimally coupled scalar, our conformal anomaly agrees with the heat kernel coefficient a_2 up to the coefficient of the scheme-dependent contribution $\square R$ [7], giving an interpretation for the heat kernel coefficient in the non-conformal case. Although our calculation therefore mostly recovers known results, our derivation differs from previous ones and exhibits several new features. One of these is that, for non-conformal theories, the anomaly as defined in (1) need *not* satisfy the Wess-Zumino (WZ) consistency condition

$$\frac{\delta(\sqrt{-g}\mathcal{A}(x))}{\delta\sigma(y)} = \frac{\delta(\sqrt{-g}\mathcal{A}(y))}{\delta\sigma(x)} \quad (2)$$

where $\sigma \equiv \log(-g)$ is the conformal factor. As a result on the r.h.s. of (1) for non-conformal theories there will appear in addition to the usual \mathbf{E}_4 , C^2 and $\square R$ anomalies (which do satisfy the WZ condition) extra terms proportional to R^2 (which does not satisfy the WZ condition). A further new result of this paper is the explicit structure of the pole terms which has not been given in the literature to the best of our knowledge, but which can alternatively be derived from heat kernel methods as we will show.

While the properties of a quantum field theory at criticality are well-understood because of Weyl symmetry, or conformal symmetry in the flat space limit, it is less understood how these properties are modified away from criticality. For example, it is known that there is a monotonically decreasing function that interpolates between a UV fixed point and an IR fixed point [19]. By studying the trace of the stress tensor away from criticality we hope to identify properties of the trace that may be preserved even away from the point, albeit, in this case, with a deformation that requires a non-trivial metric background.

2 Conformal anomaly for non-Weyl invariant theories

Before entering into the details of the spin zero case, we would like to present a general argument why (1) always produces a finite and local result, provided all divergences are local. As we said above, we wish to define the analogue of the conformal anomaly for theories that are not necessarily classically Weyl invariant. In the general case, the trace of the expectation value of the stress tensor (first term on the r.h.s. of (1)) will be both divergent and non-local. Even if we renormalise the theory in order to remove the divergence we will only be guaranteed a local expression when the theory is classically Weyl invariant, the expression thus being the anomaly. If we however take definition (1), which reduces to the standard anomaly for theories that are Weyl invariant, we can show that the expression will be both finite and local.

The expectation value of the operator $T_{\mu\nu}(x)$, in a theory regularised with dimensional regularisation, reads

$$\langle T_{\mu\nu}(x) \rangle = \frac{P_{\mu\nu}(x)}{\varepsilon} + F_{\mu\nu}(x) \quad (3)$$

where $P_{\mu\nu}$ and $F_{\mu\nu}$ are the pole and the finite part of the expansion of the expectation value. In general such expectation value is divergent (and requires renormalisation to yield a finite result). Here we show that, provided the pole $P_{\mu\nu}$ is local, the quantity (1) is finite and local. We stress that this is the case in any local quantum field theory where the calculation is performed to n -loops provided all the divergences from $(n-1)$ loops have been subtracted; in particular this is true for $n=1$, that is the case of interest here. For the conformal anomaly, as defined in equation (1), to be local at higher loop order we must therefore use the action renormalised up to the previous loop order. The first term on the r.h.s. of (1) is to be computed considering the four-dimensional trace (i.e. $g^{\mu\nu}g_{\mu\nu} = 4$) of (3) *after* computing the regularised expectation value of $T_{\mu\nu}$. For the second we take the classical trace *before* computing the expectation value. Here we have the choice of taking the trace in d or in four dimensions. For the d -dimensional trace we note the identity

$$g^{(d)\mu\nu}\langle T_{\mu\nu} \rangle = \langle g^{(d)\mu\nu}T_{\mu\nu} \rangle \quad (4)$$

whence the two operations of taking the trace and taking the expectation value commute with this choice. This is no longer the case if the trace is taken in four dimensions. If the trace inside the

bracket is taken in d dimensions we thus arrive at the alternative formula

$$\mathcal{A}(x) := \lim_{\varepsilon \rightarrow 0} \left[(g^{(4)\mu\nu} - g^{(d)\mu\nu}) \langle T_{\mu\nu}(x) \rangle \right]. \quad (5)$$

For the special example of the scalar field studied in this paper we will find that the difference $\langle (g^{(d)\mu\nu} - g^{(4)\mu\nu}) T_{\mu\nu}(x) \rangle$ is proportional to $\square R$, and can thus be absorbed into a local counterterm. Hence the choice of dimension in the second term on the r.h.s. of (1) has no intrinsic physical significance, at least for the case at hand. The above formula also shows why the WZ consistency condition fails if the second term on the r.h.s. of (5) does not vanish: there is then no functional differential operator $\delta/\delta\tilde{\sigma}(x)$ to reproduce the r.h.s. by acting on some regularized effective functional.

From these observations it follows that

$$g^{(4)\mu\nu} \langle T_{\mu\nu}(x) \rangle = \frac{g^{(4)\mu\nu} P_{\mu\nu}(x)}{\varepsilon} + g^{(4)\mu\nu} F_{\mu\nu}(x) = \frac{P(x, 4)}{\varepsilon} + F(x, 4) \quad (6)$$

$$\langle g^{(4-2\varepsilon)\mu\nu} T_{\mu\nu}(x) \rangle = \frac{g^{(4-2\varepsilon)\mu\nu} P_{\mu\nu}(x)}{\varepsilon} + g^{(4-2\varepsilon)\mu\nu} F_{\mu\nu}(x) = \frac{P(x, 4-2\varepsilon)}{\varepsilon} + F(x, 4-2\varepsilon) \quad (7)$$

making use of (4) and defining $P(x, d) := g^{(d)\mu\nu} P_{\mu\nu}(x)$, namely the second argument of P is the trace of g (similarly for F) or the dimension of spacetime. Expanding (7) in powers of ε yields for $\mathcal{A}(x)$ the expression

$$\mathcal{A}(x) = 2P'(x, 4) + O(\varepsilon) \quad (8)$$

where the $'$ indicate derivative with respect to second argument. This discussion also shows that the terms contributing to (8) are only those with an explicit factor of $g_{\mu\nu}(x)$, and not other tensors for which the difference between a contraction in different dimensions vanishes (e.g. $g^{\mu\nu} R_{\mu\nu} = R$ both for $D = 4$ and for $D = 4 - 2\varepsilon$). We will verify this explicitly in the case of the non-minimally coupled scalar.

3 Scalar field

We start with the action for a real scalar in d dimensions¹

$$S = -\frac{1}{2} \int d^d x \sqrt{-g} \phi (-\square + \xi R) \phi \quad (9)$$

with the associated stress-energy tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} g_{\mu\alpha} g_{\nu\beta} \frac{\delta S}{\delta g_{\alpha\beta}} \quad (10)$$

$$= \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi + \xi \phi^2 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) - \xi (\nabla_\mu \partial_\nu \phi^2 - g_{\mu\nu} \nabla^\alpha \partial_\alpha \phi^2). \quad (11)$$

By virtue of the equation of motion ($\square \phi = \xi R \phi$) this trace is covariantly conserved, $\nabla^\mu T_{\mu\nu} = 0$, for any ξ . In d dimensions, this action is furthermore conformally invariant if and only if $\xi = \xi_d$ with

$$\xi_d \equiv \frac{d-2}{4(d-1)}. \quad (12)$$

Accordingly, the trace of the stress-energy tensor vanishes on-shell for $\xi = \xi_d$, since

$$g^{\mu\nu} T_{\mu\nu} = 2(d-1)(\xi - \xi_d) (\partial_\alpha \phi \partial^\alpha \phi + \xi R \phi^2) = (d-1)(\xi - \xi_d) \square(\phi^2). \quad (13)$$

For the perturbative determination of the anomaly (1) we follow the same procedure as in [18] and expand the metric in the usual way

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \quad (14)$$

with ensuing expansions of the action and the stress-energy tensor in powers of $h_{\mu\nu}$,

$$S = S^{(0)} + S^{(1)} + S^{(2)} + \dots \quad (15)$$

$$T_{\mu\nu} = T_{\mu\nu}^{(0)} + T_{\mu\nu}^{(1)} + T_{\mu\nu}^{(2)} + \dots \quad (16)$$

¹Throughout this paper we use the mostly plus signature $\eta_{\mu\nu} = \text{diag}(- + + +)$.

where the superscript (n) corresponds to the collection of all term of $\mathcal{O}(h^n)$. The quantity to be computed is then

$$\langle T_{\mu\nu}(x) \rangle = \left\langle (T_{\mu\nu}^{(0)}(x) + T_{\mu\nu}^{(1)}(x) + \dots) e^{i(S^{(1)}+S^{(2)}+\dots)} \right\rangle_0 \quad (17)$$

where $\langle \dots \rangle_0$ refers to the free scalar expectation value (below we will often drop the subscript 0 when it is obvious what is meant). The evaluation of the Feynman diagrams resulting from this expansion is completely analogous to the calculation performed in [18] to which we refer for further technical details.

4 Computations at $\mathcal{O}(h)$

The computations at $\mathcal{O}(h)$ are straightforward, and are only included here for completeness. At first order in the metric perturbation the expectation value of the stress tensor is

$$\langle T_{\mu\nu}(x) \rangle = i \left\langle T_{\mu\nu}^{(0)}(x) S^{(1)} \right\rangle_0 + \mathcal{O}(h^2). \quad (18)$$

We write

$$i \left\langle T_{\mu\nu}^{(0)}(x) S^{(1)} \right\rangle_0 = -i \int d^d y \int \frac{d^d p}{(2\pi)^d} e^{-ip(x-y)} T_{\mu\nu\alpha\beta}(p) h^{\alpha\beta}(y), \quad (19)$$

where $T_{\mu\nu\alpha\beta}(p)$ is the two-point function of stress tensor,

$$\begin{array}{c} \mu\nu \\ \otimes \\ \begin{array}{c} \text{---} k-p \\ \text{---} k \\ \text{---} p \\ \text{---} \alpha\beta \end{array} \end{array} = T_{\mu\nu\alpha\beta}(p) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k-p)^2} V_{\mu\nu}(k-p, -k) V_{\alpha\beta}(k, p-k) \quad (20)$$

with

$$\begin{array}{c} \tau\sigma \\ \text{---} -k-\ell \\ \text{---} k \\ \text{---} \ell \end{array} = \begin{array}{c} \otimes \\ \text{---} k \\ \text{---} \ell \end{array} = V_{\tau\sigma}(k, \ell) \equiv \frac{1}{2} (k \cdot \ell) \eta_{\tau\sigma} - k_{(\tau} \ell_{\sigma)} + \xi \left((k+\ell)_{\tau} (k+\ell)_{\sigma} - \eta_{\tau\sigma} (k+\ell)^2 \right) \quad (21)$$

Conservation of the stress tensor at $\mathcal{O}(h)$ follows directly from

$$p^\mu T_{\mu\nu\alpha\beta}(p) = 0, \quad (22)$$

which itself is a consequence of the vanishing of tadpole integrals.

It is now straightforward to take the trace of the expectation value of the stress-energy tensor

$$g^{\mu\nu} \langle T_{\mu\nu}(x) \rangle = -i \eta^{\mu\nu} \int d^d y \int \frac{d^d p}{(2\pi)^d} e^{-ip(x-y)} T_{\mu\nu\alpha\beta}(p) h^{\alpha\beta}(y) + \mathcal{O}(h^2). \quad (23)$$

We are interested in the result *for arbitrary* ξ . After some calculation, and taking the four-dimensional trace of the regularised integral we obtain, up to higher powers of ε

$$\eta^{(4)\mu\nu} T_{\mu\nu\alpha\beta}(p) = -\frac{ip^2 (p_\alpha p_\beta - \eta_{\alpha\beta} p^2)}{(4\pi)^2} \left[\frac{(6\xi - 1)^2}{12} \left(\frac{1}{\varepsilon} + 2 - \gamma_E - \log \frac{p^2}{4\pi\mu^2} \right) - \frac{1}{15} \left(\frac{11}{12} - 5\xi \right) \right]. \quad (24)$$

For non-conformal values of $\xi \neq \frac{1}{6}$, this trace exhibits a pole as well as a non-local contribution $\propto \log(p^2/\mu^2)$ (μ^2 is the usual regularisation scale). To remove these terms we next evaluate the expectation value of the regularised on-shell trace of the energy-momentum tensor at order $\mathcal{O}(h)$, i.e.

$$\langle g^{(d)\mu\nu} T_{\mu\nu}(x) \rangle|_{\mathcal{O}(h)} = i \left\langle (d-1)(\xi - \xi_d) \square(\phi^2) S^{(1)} \right\rangle = -i \int d^d y \int \frac{d^d p}{(2\pi)^d} e^{-ip(x-y)} \tau_{\alpha\beta}(p) h^{\alpha\beta}(y); \quad (25)$$

where we kept the factor $(d-1)(\xi-\xi_d)$ inside the expectation value to indicate that it has to be expanded as $d=4-2\varepsilon$; $\tau_{\alpha\beta}(p)$ is given by the expression

$$\tau_{\alpha\beta}(p) \equiv (d-1)(\xi-\xi_d)p^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k-p)^2} V_{\alpha\beta}(p-k, k). \quad (26)$$

The evaluation of (26) is straightforward, and leads to

$$\tau_{\alpha\beta}(p) = -\frac{ip^2(p_\alpha p_\beta - p^2 \eta_{\alpha\beta})}{(4\pi)^2} \left[\frac{(6\xi-1)^2}{12} \left(\frac{1}{\varepsilon} + 2 - \gamma_E - \log \frac{p^2}{4\pi\mu^2} \right) - \frac{(6\xi-1)(3\xi-1)}{9} \right]. \quad (27)$$

We see that both the pole and the non-local term match precisely with (24) to produce a finite and local result. We also notice that the subtraction alters the coefficient of $\square R$, so in the limit $\varepsilon \rightarrow 0$ we end up with

$$\mathcal{A}_\xi = \lim_{\varepsilon \rightarrow 0} \left[g^{\mu\nu} \langle T_{\mu\nu}(x) \rangle_\xi - \langle g^{(4-2\varepsilon)\mu\nu} T_{\mu\nu}(x) \rangle_\xi \right] = \frac{1}{180(4\pi)^2} (1 - 10(1-6\xi)^2) \square R. \quad (28)$$

In removing the classical trace, we could also use dimensional regularisation by dimensional reduction whereby we treat the contractions over momenta (or derivatives in position space) as d -dimensional but traces as 4-dimensional. The result in this case reads

$$\mathcal{A}_\xi^{(BD)} = g^{\mu\nu} \langle T_{\mu\nu}(x) \rangle - \langle g^{(4)\mu\nu} T_{\mu\nu}(x) \rangle = \frac{1}{30(4\pi)^2} (1 - 5\xi) \square R. \quad (29)$$

The new coefficient matches with that given by Birrell and Davies [7, p. 179]. We see that the different prescription affects the coefficient of the $\square R$ contribution, which is a scheme-dependent contribution and can in any case be tuned to any desired value by choice of a suitable R^2 counterterm, whence this coefficient has no intrinsic significance. This is in marked contrast to the coefficients of the \mathbf{E}_4 and C^2 anomalies at $\mathcal{O}(h^2)$ which exhibit no such prescription dependence. This is the reason why the nice trick that allows the c -coefficient to be determined from the $\mathcal{O}(h)$ computation [2] (see also [18]) no longer works for non-conformal theories. Consequently for $\xi \neq \frac{1}{6}$ the determination of the a and c coefficients requires a calculation at $\mathcal{O}(h^2)$.

Let us also consider the tensor structure of the pole of $\langle T_{\mu\nu} \rangle$ at first order in h of the expectation value of the stress energy tensor as computed through (19). Given that the expression must be local, generally covariant and must have dimension $+4$, this restricts it to the form

$$\langle T_{\mu\nu} \rangle = \frac{1}{(4\pi)^2 \varepsilon} \left[a_1 g_{\mu\nu} \square R + a_2 \nabla_\mu \nabla_\nu R + a_3 \square R_{\mu\nu} \right] + \mathcal{O}(h^2). \quad (30)$$

Using the first order expansions for the Ricci tensor and Ricci scalar, we can match the expansion term by term and we get

$$a_1 = \frac{-3 + 40\xi - 120\xi^2}{120}, \quad a_2 = \frac{1 - 10\xi + 30\xi^2}{30}, \quad a_3 = -\frac{1}{60}. \quad (31)$$

As a check, we can trace over $\mu\nu$ indices, and indeed we obtain, to first order in h

$$g^{\mu\nu} \langle T_{\mu\nu} \rangle = \frac{1}{(4\pi)^2} \frac{4a_1 + a_2 + a_3}{\varepsilon} \square R = -\frac{(1-6\xi)^2}{12(4\pi)^2 \varepsilon} \square R \quad (32)$$

which matches with the pole of (24).

Furthermore, using the arguments of section 2, we can see that the anomaly, \mathcal{A}_ξ , should only depend on the coefficient a_1 in (30), namely

$$\mathcal{A}_\xi = \frac{1}{(4\pi)^2} 2a_1, \quad (33)$$

which indeed agrees with equation (28).

5 Computations at $\mathcal{O}(h^2)$

The computation is considerably more involved at second order in h , but works along similar lines to those in [18]; for this reason we here display only the salient results.² At second order we have

$$\langle T_{\mu\nu}(x) \rangle \Big|_{\mathcal{O}(h^2)} = i \langle T_{\mu\nu}^{(0)}(x) S^{(2)} \rangle_0 - \frac{1}{2} \langle T_{\mu\nu}^{(0)}(x) S^{(1)} S^{(1)} \rangle_0 + i \langle T_{\mu\nu}^{(1)}(x) S^{(1)} \rangle_0. \quad (34)$$

²Full details of the computation will be provided in the forthcoming thesis by one of the authors (L. Casarin).

We write

$$i \left\langle T_{\mu\nu}^{(0)}(x) S^{(2)} \right\rangle_0 = -i \int d^d y d^d z \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} e^{ip(x-y)} e^{iq(z-y)} h^{\alpha\beta}(y) h^{\rho\sigma}(z) T_{\mu\nu\alpha\beta\rho\sigma}^{[2]}(p, q), \quad (35)$$

$$-\frac{1}{2} \left\langle T_{\mu\nu}^{(0)}(x) S^{(1)} S^{(1)} \right\rangle_0 = -i \int d^d y d^d z \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} e^{ip(x-y)} e^{iq(z-y)} h^{\alpha\beta}(y) h^{\rho\sigma}(z) T_{\mu\nu\alpha\beta\rho\sigma}^{[3]}(p, q), \quad (36)$$

$$i \left\langle T_{\mu\nu}^{(1)}(x) S^{(1)} \right\rangle_0 = -i \int d^d y d^d z \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} e^{ip(x-y)} e^{iq(z-y)} h^{\alpha\beta}(y) h^{\rho\sigma}(z) T_{\mu\nu\alpha\beta\rho\sigma}^{[4]}(p+q, -q). \quad (37)$$

In the last integral, we have rewritten $h(x)$ as the inverse Fourier transform of its Fourier transform and shifted the integration variables as $(p, q) \rightarrow (p+q, -q)$ to make the exponential factors uniform. The functions above read

$$T_{\mu\nu\alpha\beta\rho\sigma}^{[2]}(p, q) = \mu\nu \begin{array}{c} \text{Diagram: A circle with a cross in the center. An incoming arrow from the top is labeled } k-p. \text{ An outgoing arrow from the bottom is labeled } k. \text{ An incoming wavy arrow from the right is labeled } p+q \text{ with index } \alpha\beta. \text{ An outgoing wavy arrow from the left is labeled } -q \text{ with index } \rho\sigma. \end{array} \\ = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k-p)^2} V_{\mu\nu}(k-p, -k) W_{\alpha\beta\rho\sigma}(k-p, -k, p+q, -q), \quad (38)$$

$$T_{\mu\nu\alpha\beta\rho\sigma}^{[3]}(p, q) = \mu\nu \begin{array}{c} \text{Diagram: A circle with a cross in the center. An incoming arrow from the top is labeled } k-p. \text{ An outgoing arrow from the bottom is labeled } k. \text{ An incoming wavy arrow from the right is labeled } p+q \text{ with index } \alpha\beta. \text{ An outgoing wavy arrow from the left is labeled } -q \text{ with index } \rho\sigma. \end{array} \\ = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k-p)^2 (k+q)^2} V_{\mu\nu}(k-p, -k) V_{\alpha\beta}(k+q, p-k) V_{\rho\sigma}(k+q, -k), \quad (39)$$

$$T_{\mu\nu\alpha\beta\rho\sigma}^{[4]}(p, q) = \rho\sigma \begin{array}{c} \text{Diagram: A circle with a cross in the center. An incoming wavy arrow from the left is labeled } q \text{ with index } \rho\sigma. \text{ An outgoing wavy arrow from the right is labeled } p \text{ with index } \alpha\beta. \text{ An incoming arrow from the top is labeled } k-p. \text{ An outgoing arrow from the bottom is labeled } k. \end{array} \\ = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k-p)^2} V_{\mu\nu; \rho\sigma}^{(1)}(k, p-k, q) V_{\alpha\beta}(k, p-k). \quad (40)$$

The vertex function $V_{\mu\nu}(k, \ell)$ was already defined in (21); the remaining ones are

$$\begin{array}{c} \text{Diagram: A vertex with four external lines. Top-left: wavy arrow labeled } p \text{ with index } \alpha\beta. \text{ Top-right: straight arrow labeled } k. \text{ Bottom-left: wavy arrow labeled } q \text{ with index } \rho\sigma. \text{ Bottom-right: straight arrow labeled } \ell. \end{array} = W_{\alpha\beta\rho\sigma}(k, \ell, p, q)|_{p+q+k+\ell=0} = W_{\alpha\beta\rho\sigma}^{(1)}(k, \ell) + \xi W_{\alpha\beta\rho\sigma}^{(2)}(p, q) \quad (41)$$

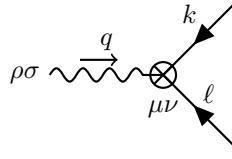
where

$$W_{\alpha\beta\rho\sigma}^{(1)}(k, \ell) = -\frac{1}{4} \eta_{\rho(\alpha\eta\beta)\sigma} k\ell + \frac{1}{8} \eta_{\alpha\beta} \eta_{\rho\sigma} k\ell - \frac{1}{4} \eta_{\alpha\beta} k_{(\rho} \ell_{\sigma)} - \frac{1}{4} \eta_{\rho\sigma} k_{(\alpha} \ell_{\beta)} + \frac{1}{2} k_{(\alpha\eta\beta)(\rho} \ell_{\sigma)} + \frac{1}{2} \ell_{(\alpha\eta\beta)(\rho} k_{\sigma)} \quad (42)$$

$$W_{\alpha\beta\rho\sigma}^{(2)}(p, q) = \frac{1}{4} \eta_{\alpha\beta} q_{\rho} q_{\sigma} + \frac{1}{4} \eta_{\rho\sigma} p_{\alpha} p_{\beta} - \frac{1}{4} \eta_{\alpha\beta} \eta_{\rho\sigma} q^2 - \frac{1}{4} \eta_{\alpha\beta} \eta_{\rho\sigma} p^2 + \frac{3}{4} \eta_{\rho(\alpha\eta\beta)\sigma} p q - \frac{1}{2} q_{(\alpha\eta\beta)(\rho} p_{\sigma)} \quad (43)$$

$$\begin{aligned}
& + \frac{1}{2}\eta_{\rho(\alpha}\eta_{\beta)\sigma}q^2 + \frac{1}{2}\eta_{\rho(\alpha}\eta_{\beta)\sigma}p^2 + \frac{1}{2}\eta_{\rho\sigma}q_{\alpha}q_{\beta} + \frac{1}{2}\eta_{\alpha\beta}p_{\rho}p_{\sigma} - q_{(\alpha}\eta_{\beta)(\rho}q_{\sigma)} - p_{(\alpha}\eta_{\beta)(\rho}p_{\sigma)} \\
& + \frac{1}{2}\eta_{\alpha\beta}p_{(\rho}q_{\sigma)} + \frac{1}{2}\eta_{\rho\sigma}p_{(\alpha}q_{\beta)} - \frac{1}{4}\eta_{\alpha\beta}\eta_{\rho\sigma}p_{\mu}q_{\nu} - p_{(\alpha}\eta_{\beta)(\rho}q_{\sigma)}
\end{aligned}$$

and



$$= V_{\mu\nu;\rho\sigma}^{(1)}(k, \ell, q) = V_{\mu\nu;\rho\sigma}^{(1);0}(k, \ell) + V_{\mu\nu;\rho\sigma}^{(1);1}(k + \ell, q) + V_{\mu\nu;\rho\sigma}^{(1);2}(q) \quad (44)$$

where

$$V_{\mu\nu;\rho\sigma}^{(1);0}(k, \ell) = \frac{1}{2}\eta_{\mu(\rho}\eta_{\sigma)\nu}k \cdot \ell - \frac{1}{2}\eta_{\mu\nu}k_{(\rho}\ell_{\sigma)} - \xi(\eta_{\mu(\rho}\eta_{\sigma)\nu}(k + \ell)^2 - \eta_{\mu\nu}(k + \ell)_{(\rho}(k + \ell)_{\sigma)}) \quad (45)$$

$$V_{\mu\nu;\rho\sigma}^{(1);1}(\ell, q) = -\xi \left[q_{(\mu}\eta_{\nu)(\rho}\ell_{\sigma)} - \eta_{\mu\nu}q_{(\rho}\ell_{\sigma)} - \frac{1}{2}\eta_{\mu(\rho}\eta_{\sigma)\nu}q \cdot \ell + \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}q \cdot \ell \right] \quad (46)$$

$$V_{\mu\nu;\rho\sigma}^{(1);2}(q) = -\xi \left[q_{(\rho}\eta_{\sigma)(\mu}q_{\nu)} - \frac{1}{2}q^2\eta_{\mu(\rho}\eta_{\sigma)\nu} - \frac{1}{2}\eta_{\rho\sigma}q_{\mu}q_{\nu} - \frac{1}{2}\eta_{\mu\nu}q_{\rho}q_{\sigma} + \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}q^2 \right] \quad (47)$$

At second order in h covariant conservation of the stress tensor requires

$$\begin{aligned}
\nabla^{\mu} \langle T_{\mu\nu}(x) \rangle &= \partial^{\mu} \langle T_{\mu\nu}(x) \rangle_{\mathcal{O}(h^2)} \\
& - h^{\mu\rho} \partial_{\rho} \langle T_{\mu\nu}(x) \rangle_{\mathcal{O}(h)} - \frac{1}{2}(2\partial_{\mu}h^{\mu\rho} - \partial^{\rho}h) \langle T_{\rho\nu}(x) \rangle_{\mathcal{O}(h)} - \frac{1}{2}\partial_{\nu}h^{\mu\rho} \langle T_{\mu\rho}(x) \rangle_{\mathcal{O}(h)}
\end{aligned} \quad (48)$$

as can be confirmed by a somewhat tedious calculation which is, however, completely analogous to the one performed in [18].

To determine the anomaly we recall the known result for the Weyl invariant case ($\xi = \frac{1}{6}$), which reads [4, 5]

$$\mathcal{A} = g^{(4)\mu\nu}(x) \langle T_{\mu\nu}(x) \rangle \Big|_{\xi=\frac{1}{6}} = \frac{1}{180(4\pi)^2} [\text{Riem}^2 - \text{Ric}^2 + \square R] \quad (49)$$

$$= \frac{1}{180(4\pi)^2} \left[-\frac{1}{2}\mathbf{E}_4 + \frac{3}{2}C^2 + \square R \right]. \quad (50)$$

We now perform the calculation for arbitrary ξ . In this case the computation is substantially more involved, and for this reason we had to make use of a Mathematica code, in particular we exploited the HEPMath package [22]. Schematically for the two contributions to (1) we find

$$\begin{aligned}
g^{(4)\mu\nu}(x) \langle T_{\mu\nu}(x) \rangle_{\xi} &= -\frac{(6\xi - 1)^2}{12(4\pi)^2} \square R + A + \mathcal{O}(\varepsilon) \\
\langle g^{\mu\nu}(x) T_{\mu\nu}(x) \rangle_{\xi} &= -\frac{(6\xi - 1)^2}{12(4\pi)^2} \square R + B + \mathcal{O}(\varepsilon)
\end{aligned} \quad (51)$$

for the regularized expressions. The poles correctly cancel with each other and vanish, as does B , when $\xi = 1/6$. However, for generic ξ the functions A and B are very complicated with about 15 000 terms each; most of these are non-local, involving expressions like $1/((pq)^2 - p^2q^2)^4$, $\log p^2$, $\log(p + q)^2$ in the momentum space integrals. All these terms come from the diagrams with three external legs, as well as the finite scalar loop integral J_{111} (see [18]). Remarkably in the difference $A - B$, all these unwanted terms cancel, leaving a much simpler expression that in momentum space contains less than 200 terms and combines correctly into the second order expressions required for the covariant expressions in the curvature tensor. The final result is

$$\begin{aligned}
\mathcal{A}_{\xi} &= g^{(4)\mu\nu}(x) \langle T_{\mu\nu}(x) \rangle_{\xi} - \langle g^{\mu\nu}(x) T_{\mu\nu}(x) \rangle_{\xi} \\
&= \frac{1}{180(4\pi)^2} \left[\text{Riem}^2 - \text{Ric}^2 + (1 - 10(1 - 6\xi)^2) \square R + \frac{5}{2}(1 - 6\xi)^2 R^2 \right]
\end{aligned} \quad (52)$$

notice that the coefficient in front of $\square R$ matches the result from $\mathcal{O}(h)$ in (28). This result matches with the one reported in [7] up to the coefficient of $\square R$, which matches the one we found in

(29). Following the discussion around equations (28) and (29), we can trace the difference to the subtraction of a different classical contribution, which is not made explicit in [7]. Observe also that, as we already anticipated, for $\xi \neq \frac{1}{6}$ there appears on the r.h.s. a contribution $\propto R^2$ which does not satisfy the WZ condition. The result shows that the anomaly proper – that is, the terms that cannot be removed by local counterterms and that satisfy the WZ condition – are indeed independent of ξ and thus universal. However it remains to be seen whether this conclusion also holds in a more general context.

As for $\mathcal{O}(h)$ we can now explicitly exhibit the structure of the pole of $\langle T_{\mu\nu} \rangle$ at order $\mathcal{O}(h^2)$. The pole of the expectation value of the stress-energy tensor is a local generally covariant expression with four derivatives acting on the metric. This constrains the expression to be of the form

$$\begin{aligned} \langle T_{\mu\nu} \rangle = \frac{1}{(4\pi)^2 \varepsilon} & \left[a_1 g_{\mu\nu} R^2 + a_2 R R_{\mu\nu} + a_3 g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} + a_4 R_{\mu}^{\alpha} R_{\alpha\nu} + a_5 R^{\alpha\beta} R_{\mu\alpha\beta\nu} \right. \\ & \left. + a_6 R_{\mu}^{\alpha\beta\gamma} R_{\nu\alpha\beta\gamma} + a_7 g_{\mu\nu} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + a_8 \nabla_{\mu} \nabla_{\nu} R + a_9 g_{\mu\nu} \square R + a_{10} \square R_{\mu\nu} \right], \end{aligned} \quad (53)$$

Any other term can be related to those written above via Bianchi identities and symmetry arguments. Writing out the $\mathcal{O}(h^2)$ expansions for all these contributions, and matching with the second order results of our computations we get

$$\begin{aligned} a_1 &= \frac{(1-6\xi)^2}{144}, & a_2 &= -\frac{(1-6\xi)^2}{36}, & a_3 &= -\frac{1}{360}, \\ a_4 &= \frac{1}{45}, & a_5 &= \frac{1}{90}, & a_6 &= -\frac{1}{90}, & a_7 &= \frac{1}{360}, \\ a_8 &= \frac{1-10\xi+30\xi^2}{30}, & a_9 &= -\frac{3-40\xi+120\xi^2}{120}, & a_{10} &= -\frac{1}{60}. \end{aligned} \quad (54)$$

The coefficients a_8, a_9, a_{10} match those computed at order $\mathcal{O}(h)$ (as they should), and therefore considering the trace we recover also (51). It is also noteworthy that, since $g^{(4)\mu\nu} \langle T_{\mu\nu} \rangle \sim \square R/\varepsilon$, it follows that

$$4a_1 + a_2 = 0 \qquad 4a_3 + a_4 - a_5 = 0 \qquad 4a_7 + a_6 = 0 \quad (55)$$

as they correspond to the coefficients of R^2 , Ric^2 and Riem^2 . We can see that the coefficients in (54) indeed respect this constraint, and this is a nontrivial consistency check of the result. Furthermore, from the general arguments of section 2, and more specifically exploiting formula (5), the anomaly is

$$\mathcal{A}_{\xi} = \frac{2}{(4\pi)^2} \left[a_1 R^2 + a_3 \text{Ric}^2 + a_7 \text{Riem}^2 + a_9 \square R \right], \quad (56)$$

which indeed agrees with expression (52) upon substituting (54).

Following the derivation of the conformal anomaly often done in the literature (see e.g. [7] for a complete exposition), we have independently confirmed the coefficients (54) by computing the pole of $\langle T_{\mu\nu} \rangle = -(2/\sqrt{-g})\delta\Gamma/\delta g^{\mu\nu}$ from the regularised effective action Γ computed with a heat kernel expansion. The heat kernel method yields the following explicit expression for the (regularised) effective action:

$$\Gamma[g] = -\frac{1}{2} \log \det(-\square + \xi R) = \frac{1}{(4\pi)^2 2\varepsilon} \int \sqrt{-g} a_2 + \mathcal{O}(\varepsilon^0) \quad (57)$$

where $a_2(x)$ reads (we are neglecting here a $\square R$ contribution, as it is a boundary term)

$$a_2(x) = \frac{1}{180} \text{Riem}^2 - \frac{1}{180} \text{Ric}^2 + \frac{1}{72} (1-6\xi)^2 R^2. \quad (58)$$

Explicit expressions for the variations $\delta(\sqrt{-g} \text{Riem}^2)$, $\delta(\sqrt{-g} \text{Ric}^2)$, $\delta(\sqrt{-g} R^2)$ that can be usefully employed for this calculation can be found in [20].

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