A_∞ Algebras from Slightly Broken Higher Spin Symmetries

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Abstract

We define a class of A_∞-algebras that are obtained by deformations of higher spin symmetries. While higher spin symmetries of a free CFT form an associative algebra, the slightly broken higher spin symmetries give rise to a minimal A_∞-algebra extending the associative one. These A_∞-algebras are related to non-commutative deformation quantization much as the unbroken higher spin symmetries result from the conventional deformation quantization. In the case of three dimensions there is an additional parameter that the A_∞-structure depends on, which is to be related to the Chern-Simons level. The deformations corresponding to the bosonic and fermionic matter lead to the same A_∞-algebra, thus manifesting the three-dimensional bosonization conjecture. In all other cases we consider, the A_∞-deformation is determined by a generalized free field in one dimension lower.

Contents

1 Introduction

Strong homotopy algebras (SHA), which are also dubbed A_∞ and L_∞ for the cases generalizing associative and Lie algebras, are general enough structures that abstract and formalize what it means to be algebraically consistent in a broad sense. No wonder that many of physical problems can be cast into the framework of SHA, like string field theory [1–5] and the BV-BRST theory of gauge systems [6–9]. Even some of the problems that are seemingly unrelated to any SHA admit natural solutions by translating them to the SHA setup, e.g. the deformation quantization of Poisson manifolds [10]. In the present paper, we use the language of SHA in order to describe the slightly broken higher spin symmetries [11–18] that govern certain nontrivial conformal field theories (CFT) at least in the large-N limit.

It is well known that free CFT's have vast symmetries that extend the conformal algebra to infinite-dimensional associative algebras, called *higher spin algebras* in this context. In particular, the correlation functions can be computed as invariants of higher spin symmetries [19–22]. The structure of interacting CFT's, however, is much more interesting to study. If a given CFT admits a weakly-coupled limit, which is not necessarily free in terms of the fundamental constituents, then one can think of such a CFT as enjoying a *slightly broken higher spin symmetry*, the term coined in [11]. The examples of main interest include the critical vector model, the Gross-Neveu model and, more generally, the Chern-Simons matter theories in the large-N limit. The last model has recently been conjectured to exhibit a number of interesting dualities [11, 23–27] and our expectation is that the dualities can be explained by higher spin symmetries. The purpose of the paper is to define what a slightly broken higher spin symmetry means in mathematical terms.

One obstacle is that higher spin algebras are typically rigid, that is, admit no deformations. Indeed, in $d > 2$ the free CFT's are isolated points and do not form continuous families.[1] The opposite conclusion is also true: higher spin symmetries are the symmetries of free CFT's in $d > 2$ [32–35]. Therefore, a slightly broken higher spin symmetry is not about the usual deformation of associative algebras. Our proposal is that such deformations fall into the class of A_∞-algebras we are going to study.

[1]There is a one-parameter family of algebras in $4d$ [28–30], the free parameter λ being helicity. For non-(half)integer values of the parameter these algebras do not have any natural spacetime and CFT interpretation and for $|\lambda| > 1$ there is no local stress-tensor as well. Also, there is a one-parameter family of algebras relevant for higher spin theories in AdS_3, but the dual CFT's are minimal models [31] with W symmetry and higher-spin algebras do not seem to work in $2d$ CFT's the way they do $d > 2$.

On the other hand, the A_∞-algebras that describe slightly broken higher spin symmetries are still related to associative algebras and deformations thereof. The class of A_∞-algebras we construct may be of some interest by itself, being closely related to the so-called *non-commutative deformation quantization* [36, 37]. In a few words, suppose that we have an associative (in general non-commutative) algebra and, furthermore, that the product can be deformed as

$$a * b = ab + \phi(a,b)\hbar + \cdots \,, \tag{1.1}$$

$\hbar$ being a formal deformation parameter. Here ϕ is a Hochschild 2-cocycle. Classifying and constructing such deformations in the case of algebras of smooth functions on Poisson manifolds are the standard problems of deformation quantization that have been solved by Kontsevich [10] using a string-inspired construction of SHA. The deformation problem we are lead to is to promote the formal parameter $\hbar$ to an element of the algebra itself. This clearly has no sense in the realm of usual associative algebras. The idea is to go to the category of A_∞-algebras where it is legitimate to replace $\phi(a,b)\hbar$ with a tri-linear map $m_3(a,b;u)$, with the 'deformation parameter' u being now an element of the same algebra. The correspondence principle requires $m_3(a,b;\hbar) = \phi(a,b)\hbar$. This is only the starting point and all the higher structure maps $m_n(a,b;u,\ldots,w)$ are to be constructed. Taken together, the m's solve the Maurer–Cartan equation and this amounts to defining an A_∞-algebra. One of our results is that the A_∞-structure maps m_n can all be expressed through ϕ and the other coefficients of the expansion (1.1).

It is also important to realize the reason for such A_∞-structures to originate from higher spin algebras, given the fact that the latter algebras admit no relevant deformations. What is relevant for the slightly broken higher spin symmetries is not the higher spin algebras themselves, but certain simple extensions thereof. In the simplest cases these are $\mathbb{Z}_2$-extensions by the inversion map. We prove that as associative algebras these extensions admit at least a one-parameter family of deformations. It is this deformation that is plugged in into the general construction of A_∞-algebras just described.

The physical interpretation of the abstract discussion above is that higher spin symmetries originate from higher spin currents $J_s = \phi\partial\cdots\partial\phi + \ldots$, which are bi-linear in fields. When higher spin symmetry is slightly broken by interactions, the higher spin currents are no longer conserved. Nevertheless, the non-conservation, i.e., the breaking of the higher spin symmetry, has a very specific form of double trace operators built out of higher spin currents themselves. In a sense, higher spin currents are responsible for their own non-conservation.

Therefore, the higher spin algebra is deformed, the parameter of deformation being an infinite multiplet of higher spin currents suppressed by the factor $1/N$. A remarkable fact is that the multiplet of higher spin currents is isomorphic to the higher spin algebra itself (up to $\mathbb{Z}_2$-reflection by the inversion map). Therefore, in order to describe the slightly broken higher spin symmetry we should be looking for a deformation of the higher spin algebra which is controlled by another element of the algebra.

Of special interest is the case of three dimensions. Here the structure of the higher spin symmetry breaking is richer than in higher dimensions due to the presence of an additional parameter related to the level k of the Chern–Simons matter theories. The structure of correlation functions is also more complicated with certain parity-odd structures contributing to it [11, 14, 38]. More importantly, the theories with bosonic and fermionic matter seem to describe the same physics and this has lead to the conjecture of the three-dimensional bosonization and related ones [11, 23–27]. The first observation here is that the higher spin algebras of free boson and free fermion CFT's are isomorphic, meaning that they have to lead to the same A_∞-algebra. Secondly, the deformation is characterized by the second Hochschild cohomology and turns out to be two-dimensional, while it is one-dimensional in $d > 3$. The additional parameter is to be associated with the t'Hooft coupling $\lambda = N/k$. This provides a good evidence for the conjecture to the leading order in $1/N$. See also the comments at the very end.

The rest of the paper is organized as follows. We begin in Section 2 by recalling several definitions of A_∞-algebras, of which we prefer the Gerstenhaber bracket. In Section 3, we define and construct a class of A_∞-algebras that can be thought of as non-commutative deformation quantization of associative algebras. Various definitions and examples of higher spin algebras are recalled in Section 4. In Section 5, we prove that certain simple extensions of higher spin algebras admit deformations. Some explicit oscillator realizations of these deformations are discussed in Section 6. Conclusions are in Section 7.

2 A_∞ Algebras

There are several equivalent definitions of A_∞-algebras: (i) via Stasheff's relations [39]; (ii) via a nilpotent coderivation on the tensor coalgebra of the suspended graded algebra; (iii) via the Gerstenhaber bracket. While we prefer the last one and use it throughout the paper, let us briefly discuss the other two. The initial data is a graded vector space $V = \bigoplus_k V^k$.

Stasheff's relations. A_∞-structure is defined via multilinear maps $m_n : T^nV \to V$ of degree $2-n$ that obey the Stasheff relations [40]

$$\sum_{a+b+c=n} (-)^{a+bc}\, m_{a+b+1}(1^{\otimes a} \otimes m_b \otimes 1^{\otimes c}) = 0\,. \tag{2.1}$$

It is difficult to explain the origin of signs and gradings. Moreover, additional signs will be generated due to the Koszul rule when the actual arguments are plugged in.[2]

Coderivations. The Stasheff relations can be extracted from nilpotency of a coderivation. To this end, one takes the reduced tensor algebra $\bar{T}V$ by dropping the zeroth component:

$$\bar{T}V = \bigoplus_{k>0} T^kV\,. \tag{2.2}$$

There is a natural coproduct $\Delta : \bar{T}V \to \bar{T}V \otimes \bar{T}V$ defined by

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n-1} (v_1 \otimes \cdots \otimes v_i) \bigotimes (v_{i+1} \otimes \cdots \otimes v_n)\,. \tag{2.3}$$

It is not hard to see that each degree-one map $Q : \bar{T}V \to V$ uniquely extends to a coderivation $Q : \bar{T}V \to \bar{T}V$ (denoted by the same symbol), i.e., $\Delta Q = (1 \otimes Q + Q \otimes 1)\Delta$. Each degree-one coderivation Q squaring to zero defines an A_∞-structure on V and the Stasheff relations, together with the right signs,[3] can be extracted by expanding the equality $Q^2 = 0$ in homogeneous components.

Gerstenhaber bracket. Our preferable choice is to describe A_∞-structure via the Gerstenhaber bracket $[\![\bullet,\bullet]\!]$, which is defined on the space $Hom(TV, V)$ of maps (cochains) from the tensor algebra of V to V itself. By definition,

$$[\![f, g]\!] = f \circ g - (-1)^{|f||g|} g \circ f\,, \tag{2.4}$$

[2]The Koszul rule is that a sign accompanies whenever two graded objects has to be permuted:

$$(f \otimes g)(u \otimes v) = (-)^{|g||u|} f(u) \otimes g(v)\,,$$

where $|x|$ denotes the degree of an object x (an element of the vector space or a map).

[3]To get exactly these signs one needs to replace V in the definition above with its suspension $V[1]$, i.e., $V[1]^k = V^{k+1}$. See also a comment below on how to avoid dealing with suspensions.

where the non-associative $\circ$-product is defined by

$$
\begin{aligned}
&(f \circ g)(a_1 \otimes a_2 \otimes \cdots \otimes a_{m+n-1}) = \\
&\quad = \sum_{i=0}^{n-1} (-1)^{|g| \sum_{j=1}^{i} |a_j|} f(a_1 \otimes \cdots \otimes a_i \otimes g(a_{i+1} \otimes \cdots \otimes a_{i+m}) \otimes \cdots \otimes a_{m+n-1}) \,.
\end{aligned} \tag{2.5}
$$

The Gerstenhaber bracket is graded skew-symmetric and obeys the Jacobi identity:

$$
[\![f, g]\!] = -(-1)^{|f||g|} [\![g, f]\!] \,, \qquad [\![[\![f, g]\!], h]\!] = [\![f, [\![g, h]\!]]\!] - (-1)^{|f||g|} [\![g, [\![f, h]\!]]\!] \,. \tag{2.6}
$$

Note: it is common in the literature to define A_∞-algebras via maps on the suspension $V[1]$ of the corresponding graded space V. Then m_n has degree $2-n$, as above. We prefer to prepare the experimental setup in such a way that V is already suspended. This prevents appearance of many sign factors and all m_n have now degree one. For example, an associative algebra A is understood as a graded algebra with the only nonzero component leaving at degree zero. Then the suspended space $A[1]$ is nonzero at degree -1. When treated as an A_∞-algebra, we assume A to have only the A_{-1} component, so that multiplication is a degree-one map m_2 taking $A_{-1} \otimes A_{-1}$ to A_{-1}.

Given a graded space V and a sum $m = m_1 + m_2 + \cdots$ of degree-one maps $m_n : T^n V \to V$, the A_∞-structure is defined simply as a solution to the Maurer–Cartan equation:

$$
[\![m, m]\!] = 0 \,. \tag{2.7}
$$

Upon expansion $m = m_1 + m_2 + \cdots$ the first few relations have a simple interpretation: m_1 is a differential, $m_1 m_1 = 0$; m_2 is a product differentiated by m_1 by the rule $-m_1 m_2(a, b) = m_2(m_1(a), b) + (-)^{|a|} m_2(a, m_1(b))$. However, m_2 is not associative in general, the associativity is true up to a coboundary controlled by m_3:

$$
\begin{aligned}
m_2(m_2(a,b),c) + &(-)^{|a|} m_2(a, m_2(b,c)) + m_1 m_3(a,b,c) + m_3(m_1(a),b,c) + \\
&+ (-)^{|a|} m_3(a, m_1(b), c) + (-)^{|a|+|b|} m_3(a, b, m_1(c)) = 0 \,.
\end{aligned}
$$

There are certain special cases of A_∞-algebras that deserve their own names. *Minimal* A_∞-algebras do not have the lowest map m_1, i.e., differential. Such algebras arise naturally when passing to the cohomology $H(m_1)$ of m_1 and dragging the A_∞-structures there, the resulting algebras are called *minimal models*, see e.g. [4]. *Differential graded algebras* (DGA) have only m_1 and m_2, i.e., a differential and a bi-linear product that respect the Leibniz rule.

Note that for a genuine A_∞-structure to arise it is necessary that V has more than one graded component due to the degree requirement. The only possibility with just one nontrivial component is $V = V_{-1}$, then m_2 is an associative product on V.

3 A_∞ from Deformations of Associative Algebras

In this section, we discuss what kind of A_∞-structures are related to higher spin symmetries. From the A_∞ point of view the only important property of higher spin algebras (HSA) to be abstracted is that they are associative algebras. There are certain special properties of HSA that allows one to describe the corresponding A_∞-algebras in more detail and there are tools to explicitly construct them, which will be discussed in Sections 5, 6. Throughout this section, we let A denote any associative algebra.

Given an associative algebra A, it is clear that due to the restrictions imposed by the grading, there cannot be any interesting A_∞-structure on it; the only possibility is to deform A itself as an associative algebra. We define the A_∞-structure perturbatively and the first step is to extend A by any its bimodule M; in so doing, A and M are prescribed the degrees -1 and 0, respectively. At the lowest order the A_∞-structure is simply equivalent to the definitions above: there is only m_2 that is defined for various pairs $A_{-1}\otimes A_{-1}$ (the A product), $A_{-1}\otimes A_0$ (the left action of A on M), $A_0\otimes A_{-1}$ (the right action of A on M). All these conditions are summarize by the Stasheff identity:

$$m_2(m_2(a,b),c)+(-)^{|a|}m_2(a,m_2(b,c))=0 \qquad\Longleftrightarrow\qquad [\![m_2,m_2]\!]=0\,. \tag{3.1}$$

Denoting elements of A_{-1} by $a,b,\ldots$ and elements of the bimodule A_0 by $u,v,\ldots$ we have [4]

$$m_2(a,b)=ab\,,\qquad m_2(a,u)=au\,,\qquad m_2(u,a)=-ua\,,\qquad m_2(u,v)=0\,. \tag{3.2}$$

Now one tries to deform this rather trivial A_∞-structure and the first deformations $m^{(1)}$ can be described in terms of the Hochschild cohomology of A. Introducing the Hochschild differential $\delta=[\![m_2,\bullet]\!]$, one can identify the nontrivial first-order deformations $m^{(1)}$ with the nontrivial δ-cocycles,

$$\delta m^{(1)}=0 \qquad\Longleftrightarrow\qquad [\![m_2,m^{(1)}]\!]=0\,. \tag{3.3}$$

In other words, the space of infinitesimal deformations is identified with the δ-cohomology in degree 1, while the second δ-cohomology group is responsible for possible obstructions to deformation.

The first order deformation should have the form $m^{(1)}=m_3(\bullet,\bullet,\bullet)$ with arguments from A_{-1} and A_0. Various homogeneous components of $\delta m_3=0$ are collected in Appendix A,

[4] The left/right action is denoted by multiplication, au and ua.

while the first and the last ones are:

$$-am_3(b,c,u)+m_3(ab,c,u)-m_3(a,bc,u)+m_3(a,b,cu)=0\,, \tag{3.4}$$

$$\ldots=0\,, \tag{3.5}$$

$$m_3(u,a,b)v+um_3(a,b,v)+m_3(ua,b,v)+m_3(u,ab,v)-m_3(u,a,bv)=0\,. \tag{3.6}$$

For any A there is at least one natural bimodule, that is, A itself. Let us take A_0 to be A, in which case the deformation can be described in more detail. If A admits a deformation as an associative algebra, then the second Hochschild cohomology group is nonzero, $HH^2(\mathrm{A},\mathrm{A})\neq 0$. Given an element $[\phi]\in HH^2(\mathrm{A},\mathrm{A})$ represented by the cocycle ϕ, the standard deformation of the associative structure reads

$$a*b=ab+\phi(a,b)\hbar+\mathcal{O}(\hbar^2)\,, \tag{3.7}$$

where the deformation parameter $\hbar$ can live in the base field or even in the center of A. If the deformation is unobstructed we can construct a one-parameter family of algebras $\mathrm{A}_\hbar$ that starts at A for $\hbar=0$. When $A_0\sim\mathrm{A}$ the A_∞-algebra we are trying to construct upgrades the deformation parameter $\hbar$ to an element of A_0. The observation is that for $A_0\sim\mathrm{A}$ one can always put

$$m_3(a,b,u)=\phi(a,b)u\,, \qquad m_3(a,u,v)=\phi(a,u)v\,, \tag{3.8a}$$

$$m_3(a,u,b)=0\,, \qquad m_3(u,a,v)=-u\phi(a,v)\,, \tag{3.8b}$$

$$m_3(u,a,b)=0\,, \qquad m_3(u,v,a)=0\,. \tag{3.8c}$$

Here the 'deformation parameter' $u\in\mathrm{A}$ was placed on the right in $m_3(a,b,u)$. It is also possible to place it on the left

$$m_3(a,b,u)=0\,, \qquad m_3(a,u,v)=0\,, \tag{3.9a}$$

$$m_3(a,u,b)=0\,, \qquad m_3(u,a,v)=\phi(u,a)v\,, \tag{3.9b}$$

$$m_3(u,a,b)=u\phi(a,b)\,, \qquad m_3(u,v,a)=-u\phi(v,a)\,. \tag{3.9c}$$

For u in the base field (or more generally in the center of A) the left $u\phi(a,b)$ and the right $\phi(a,b)u$ deformations are clearly equivalent. This property extends to the A_∞-structure, namely, the left and right deformations differ from each other by a trivial deformation $m_3=\delta g$, where $g(a,u)=\phi(a,u)$.

The A_∞-algebra we are constructing extends the deformation parameter $\hbar$ to an element of A_0, which may be the algebra itself (or its bimodule). This is referred to as deformation with noncommutative base. If such A_∞-algebra can be constructed, it admits a truncation where A_0 is replaced by the center $Z(\mathrm{A})$, or just by $\hbar$, that is closely related to the one-parameter family of algebras $\mathrm{A}_\hbar$.

3.1 Explicit Construction

The central statement of the present paper is that the A_∞-structure, discussed in the previous section, is fully determined by the deformation of the underlying associative algebra. Assuming that the deformed product

$$a * b = ab + \sum_{k>0} \phi_k(a,b)\hbar^k \tag{3.10}$$

is known, we give an explicit formula for all m_n. The defining relation for the A_∞-structure, i.e., the Maurer–Cartan equation

$$[\![m,m]\!] = 0 \qquad\qquad \Longleftrightarrow \qquad\qquad \delta m_n + \sum_{i+j=n+2} m_i \circ m_j = 0\,, \tag{3.11}$$

is satisfied as a consequence of the associativity of the deformed product

$$a * (b * c) - (a * b) * c = 0 \qquad\qquad \Longleftrightarrow \qquad\qquad \delta\phi_n + \sum_{i+j=n-1} \phi_i \circ \phi_j = 0\,. \tag{3.12}$$

Here $\delta = [\![m_2, \bullet]\!]$ is the Hochschild differential associated to the undeformed product (3.2). We have three equivalent forms of the solution for m_n: recursive, in terms of binary trees, and through generating equations. Let us discuss them in order.

In general, there are two types of ambiguities in the definition of m_n. (i) As usual in deformation quantization, one can redefine the deformed product $*$ via a linear change of variables $a \to D(a) = a + \sum_k D_k(a)\hbar^k$. Then, the new product is given by $D(D^{-1}(a) * D^{-1}(b))$. (ii) One can perform various redefinitions at the level of A_∞-structure, which is done by exponentiating the infinitesimal gauge transformation

$$\dot{m}(t) = [\![m(t), \xi]\!]\,, \qquad\qquad m(0) = m\,, \tag{3.13}$$

for some cochain ξ of degree zero. The A_∞ gauge transformations are more general than redefinitions of the associative product. We have observed that the A_∞-transformations allow one to cast the first order deformation into the right form (with all, or all but one, A_0-factors staying on the right):

$$m_3(a,b,u) = f_3(a,b)u\,, \qquad m_3(a,u,v) = f_3(a,u)v\,, \qquad m_3(u,a,v) = -f_3(u,a)v\,, \tag{3.14}$$

and all other orderings of a, b, u, v in m_3 give zero result. Here $f_3(a,b) = \phi_1(a,b)$ is determined by the first-order deformation in (3.10). The full solution can be sought for in the similar form:

$$m_n(a,b,u,\dots,v) = +f_n(a,b,u,\dots)v\,, \tag{3.15}$$

$$m_n(a,u,\dots,v,w) = +f_n(a,u,\dots,v)w\,, \tag{3.16}$$

$$m_n(u,a,\dots,v,w) = -f_n(u,a,\dots,v)w\,. \tag{3.17}$$

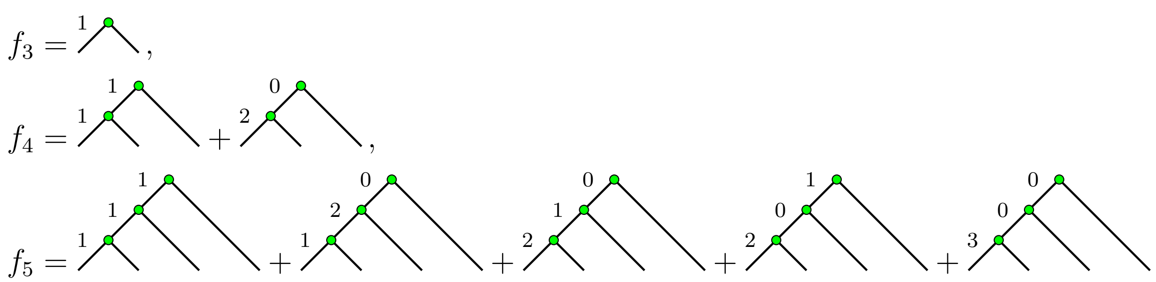
Therefore, the problem is reduced to defining one function f_n of $(n-1)$ arguments per each set of structure maps m_n with only three orderings being nontrivial. It is not hard to see that the equation for m_4, $\delta m_4 + m_3 \circ m_3 = 0$, is solved by

$$f_4(a,b,u) = \phi_2(a,b)u + \phi_1(\phi_1(a,b),u)\,. \tag{3.18}$$

At the next order we have to solve $\delta m_5 + m_3 \circ m_4 + m_4 \circ m_3 = 0$, which is satisfied by

$$\begin{aligned} f_5(a,b,u,v) = \phi_1(\phi_1(\phi_1(a,b),u),v) + \phi_2(\phi_1(a,b),u)v + \phi_1(\phi_2(a,b),u)v+\\ + \phi_1(\phi_2(a,b)u,v) + \phi_3(a,b)uv\,. \end{aligned} \tag{3.19}$$

The following graphical representation can be useful. We consider planar binary trees with vertices labelled by $1, 2, \dots$. A vertex with label k corresponds to ϕ_k and the two incoming edges correspond to the arguments. Functions f_3, f_4 and f_5 can then be depicted as



Solution, recursive formula. In order to write down a recursive formula for f_n let us introduce some further notation. It is clear that any f_n can be decomposed according to the number of the multiplicative arguments on the right:

$$f_n(a,b,u,\dots,v,w) = f_{n,0}(a,b,u,\dots,v,w) + f_{n,1}(a,b,u,\dots,v)w + f_{n,2}(a,b,u,\dots)vw + \dots\,.$$

There is an associated filtration, where the leftover $r_{n,k}$ contains all the terms in the decomposition with at least k multiplicative arguments on the right:

$$
\begin{aligned}
f_n(a,b,\ldots,v,w) &\equiv r_{n,0}(a,b,\ldots,v,w) && (3.20a)\\
&= f_{n,0}(a,b,\ldots,v,w) + r_{n,1}(a,b,\ldots,v)w && (3.20b)\\
&= f_{n,0}(a,b,\ldots,v,w) + f_{n,1}(a,b,\ldots,v)w + r_{n,2}(a,b,\ldots)vw\,, \quad \text{etc.} && (3.20c)
\end{aligned}
$$

Our claim is that all f_n are obtained by means of the following recursive relations:[5]

$$
\begin{aligned}
f_{n,0} &= \phi_1(r_{n-1,0},\bullet)\,, && (3.21a)\\
f_{n,1} &= \phi_2(r_{n-2,0},\bullet) + \phi_1(r_{n-1,1},\bullet)\,, && (3.21b)\\
f_{n,2} &= \phi_3(r_{n-2,0},\bullet) + \phi_2(r_{n-2,1},\bullet) + \phi_1(r_{n-3,0},\bullet)\,, && (3.21c)\\
&\ldots && (3.21d)\\
f_{n,k} &= \sum_{i=0}^{i=k} \phi_{k-i+1}(r_{n-k+i-1,i},\bullet)\,. && (3.21e)
\end{aligned}
$$

The formulae above together with the initial condition $f_3 = \phi_1$ allows one to reconstruct the A_∞-structure, m_n, in terms of the bi-linear maps ϕ_k defining the $*$-product (including the initial product $\phi_0(a,b) = ab$).

While f_n's are, in general, quite complicated functions with nested ϕ_k, there are some general properties that are easy to see. (a) The first and the last terms in f_n are of the form

$$
f_n(a,b,u,\ldots,v,w) = \phi_1(\phi_1(\ldots(\phi_1(a,b),u),\ldots,v),w) + \cdots + \phi_{n-2}(a,b)u\ldots vw\,. \quad (3.22)
$$

The presence of the last term is obvious as for $u,\ldots,v,w$ in the base field the deformation should reduce to the deformed product[6]

$$
f_n(a,b,\hbar,\ldots,\hbar,\hbar) = \phi_{n-2}(a,b)\hbar^{n-2}\,. \quad (3.23)
$$

(b) The graphs that show up in the decomposition of f_n are all left-aligned, i.e., are the simplest ones with all edges emerging from just one branch on the left. Such graphs can be parameterized by a sequence of numbers listing the indices of the vertices when read from left to right, e.g. $(2,0)$ and $(1,1)$ for f_4. Such simple form is the consequence of a particular A_∞ gauge we chose. By making an A_∞ gauge transformation one can arrive at various other forms. In particular, there exists the right-aligned form, which is obtained by reflection of

[5]It is useful to define $f_2 = r_{2,0}$ as identity map.

[6]We should assume here that the deformation of the product is properly normalized, $\phi_k(a,1) = 0$.

the graphs. (c) All graphs contributing to f_n have the total weight $n-2$, where the weight is the sum over the indices of the vertices in a graph. (d) Not all possible left-aligned graphs with a correct weight contribute to the expansion of f_n. All admissible graphs enter with multiplicity one.

Solution, explicit formula. Instead of the recursive definition given above it is possible to describe the set of trees that contribute to f_n in a more direct way. This is easier to do in terms of the sequences of natural numbers

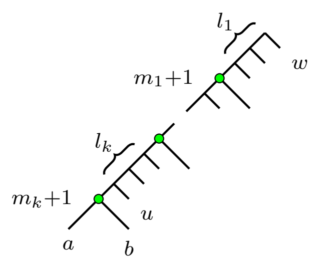
$$(m_k, l_k, \ldots, m_1, l_1) \tag{3.24}$$

that correspond to the trees encoded by the weights

$$m_k+1, \overbrace{0,\ldots,0}^{l_k}, m_{k-1}+1, 0, \ldots, 0, m_2+1, \overbrace{0,\ldots,0}^{l_2} m_1+1, \overbrace{0,\ldots,0}^{l_1}, \tag{3.25}$$

or, pictorially,

$$f_n(a,b,u,\ldots,w) \ni \tag{3.26}$$



Here the edges corresponding to the multiplicative arguments on the right are drawn a bit shorter and some of the arguments are indicated.

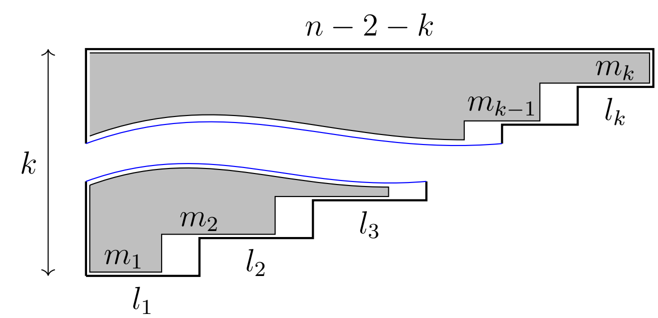
Equivalently, every such sequence corresponds to the expression

$$\phi_{m_1+1}\left(\ldots\phi_{m_{k-1}+1}(\phi_{m_k+1}(\mathbb{1},\mathbb{1})\mathbb{1}^{l_k},\mathbb{1})\mathbb{1}^{l_{k-1}},\ldots,\mathbb{1}^{l_2},\mathbb{1}\right)\mathbb{1}^{l_1}(a_0\otimes\ldots\otimes a_{n-2}), \tag{3.27}$$

where $\mathbb{1}$ is the identity map. In this notation l_i is the number of the multiplicative arguments on the right at level i and m_i stands for the insertion of ϕ_{m_i+1}. Now we need to specify which of the sequences or trees are admissible. They satisfy

$$\begin{aligned}
&l_1 \in [0, n-2-k]\,, && m_1 \in [0, l_1]\,,\\
&l_2 \in [0, n-2-k-l_1]\,, && m_2 \in [0, l_1+l_2-m_1]\,,\\
&\ldots\\
&l_k \in [0, n-2-k-l_1-\cdots-l_{k-1}]\,, && m_k \in [0, l_1+...+l_k-m_1-\cdots-m_{k-1}]\,.
\end{aligned}$$

Equivalently, all the terms (trees) contributing to f_n can be enumerated via pairs of Young diagrams. One should write down all possible Young diagrams with the first row of length $n-2-k$ for all $k = 1, \ldots, n-2$ and with k rows. Given such a diagram, one should write down all possible subdiagrams such that the first row is of the same length $n-2-k$. Any such pair of Young diagrams gives a sequence of l_i and m_i that are admissible. Some of l_i and m_i can be zero, provided that the Young diagram is a proper one (the length of the rows is nondecreasing upwards). Pictorially, such pairs look as follows:



(3.28)

For example, the pair of empty diagrams $(\bullet, \bullet)$ means $k = n-2$, $l_{1,\ldots,k} = m_{1,\ldots,k} = 0$ and corresponds to

$$\phi_1(\ldots\phi_1(\phi_1(a_0, a_1), a_2), \ldots, a_{n-2})\,. \tag{3.29}$$

The one-row Young diagram of length $n-3$ implies that $k = 1$, $m_1 = l_1 = n-3$ and corresponds to

$$\phi_{n-2}(a_0, a_1)a_2 \ldots a_{n-2}\,. \tag{3.30}$$

In this language the expansions for f_4, f_5 and f_6 can be written as

$$\begin{aligned}
f_4 &= (\bullet,\bullet) \oplus (\ydiagram{1},\ydiagram{1})\ ,\\
f_5 &= f_4 \oplus (\ydiagram{2},\ydiagram{2}) \oplus (\ydiagram{1,1},\ydiagram{1,1}) \oplus (\ydiagram{1,1},\ydiagram{1})\ ,\\
f_6 &= f_5 \oplus (\ydiagram{3},\ydiagram{3}) \oplus (\ydiagram{2,1},\ydiagram{2,1}) \oplus (\ydiagram{2,1},\ydiagram{2}) \oplus (\ydiagram{2,2},\ydiagram{2,2}) \oplus (\ydiagram{2,2},\ydiagram{2,1}) \oplus (\ydiagram{2,2},\ydiagram{2}) \oplus \left(\ydiagram{1,1,1},\ydiagram{1,1,1}\right) \oplus \left(\ydiagram{1,1,1},\ydiagram{1,1}\right) \oplus \left(\ydiagram{1,1,1},\ydiagram{1}\right).
\end{aligned}$$

Solution, generating equation. A combinatorial proof that the two forms above do solve our problem is sketched in Appendix B. Nevertheless, it is desirable to get all m's in a way that makes their existence obvious. To this end, we recall the construction of braces, which were first introduced in [41] (see also [42, 43]). A k-brace is a multi-linear map that assigns to $k+1$ Hochschild cochains $f, g_1, \dots g_k$ a new cochain $f\{g_1, \dots, g_k\}$ defined by the rule

$$f\{g_1,\dots,g_k\}(a_1,\dots) = \sum \pm f(a_1,\dots,g_1(\dots),\dots,g_2(\dots),\dots,g_k(\dots),\dots)\,. \tag{3.31}$$

Here the cochains g_i are inserted as arguments into the cochain f and the sum is over all unshuffles (i.e., the order of g_i is preserved) with the natural signs (whenever g_i has to jump over a_i an obvious sign is generated). For $k=1$ we get the Gerstenhaber $\circ$-product (2.5), that is, $f\{g\} = f\circ g$. As was shown in [42], any A_∞-structure m on V can be lifted to an A_∞-structure M on the space of Hochschild cochains $Hom(TV,V)$ by setting

$$M(g_1) = [\![m, g_1]\!]\,, \qquad M(g_1,\dots,g_k) = m\{g_1,\dots,g_k\}\,, \qquad k>1\,. \tag{3.32}$$

Using the properties of the braces, one can find [42, 43]

$$[\![M, M]\!](g_1,\dots) = [\![m, m]\!]\{g_1,\dots\} = 0\,. \tag{3.33}$$

In other words, M satisfies the Maurer–Cartan equation whenever m does so. Expanding the former structure in homogeneous components, $M = M_1 + M_2 + \cdots$, one gets an infinite sequence of relations

$$[\![M_1, M_1]\!] = 0\,, \qquad [\![M_1, M_2]\!] = 0\,, \qquad \dots\,.$$

As is seen the first term M_1 defines a differential $D = [\![m, \bullet]\!]$ on the space $Hom(TV,V)$. The second relation takes then the form

$$DM_2(g_1,g_2) + M_2(Dg_1,g_2) + (-1)^{|g_1|}M_2(g_1,Dg_2) = 0\,. \tag{3.34}$$

In particular, this means that M_2 maps any pair of D-cocycles g_1 and g_2 to a D-cocycle $M_2(g_1,g_2)$.

Suppose now that we are given a two-parameter family $m = m(\hbar, s)$ of A_∞-structures on V. Then, differentiating the defining condition $[\![m,m]\!]=0$ by the parameters, one readily concludes that the partial derivatives $\partial_\hbar m$ and $\partial_s m$ are D-cocycles for all $\hbar$ and s, i.e.,

$$D\partial_\hbar m = [\![m, \partial_\hbar m]\!] = 0\,, \qquad D\partial_s m = [\![m, \partial_s m]\!] = 0\,. \tag{3.35}$$

Applying to them the above operation M_2 yields one more family of D-cocycles

$$M_2(\partial_\hbar m, \partial_s m) = m\{\partial_\hbar m, \partial_s m\}\,.$$

We can increase the number of parameters entering m by considering the flow in the space of the Hochschild cochains

$$\partial_t m = m\,\{\partial_\hbar m, \partial_s m\} \tag{3.36}$$

with respect to the 'time' t. Solutions to this equation form a three-parameter family of the Hochschild cochains $m(t, \hbar, u)$. A simple observation is that the flow (3.36) can be consistently restricted to the surface $[\![m, m]\!] = 0$ identified with the set of Maurer–Cartan elements. Indeed, denoting $L = [\![m, m]\!]$, we find

$$\partial_t L = 2[\![m, m\,\{\partial_\hbar m, \partial_s m\}]\!] = -[\![\partial_\hbar L, \partial_s m]\!] + [\![\partial_\hbar m, \partial_s L]\!]\,. \tag{3.37}$$

Hence, choosing initial data $m(0, \hbar, s)$ for the solutions to Eq. (3.36) on the surface $L = 0$, we will get three-parameter families $m(t, \hbar, s)$ of the Maurer–Cartan elements. Let us take

$$m(0, \hbar, s) = \mu(\hbar) + s\partial\,. \tag{3.38}$$

Here the parameter s is prescribed the degree 2; ∂ is the degree -1 differential on $A_{-1} \oplus A_0$ that maps A_0 to A_{-1} as identity isomorphism and maps A_{-1} to 0, which is essentially a formal way to retract an element from the bimodule and reinterpret it as an element of the algebra again; $\mu(\hbar)$ is the algebra plus bimodule structure with respect to the full deformed product (3.10):

$$\mu(\hbar)(a, b) = a * b\,, \qquad \mu(\hbar)(a, u) = a * u\,, \qquad \mu(\hbar)(u, a) = -u * a\,.$$

The Maurer–Cartan equation for (3.38) is equivalent to the relations

$$[\![\mu(\hbar), \mu(\hbar)]\!] = 0\,, \qquad [\![\mu(\hbar), \partial]\!] = 0\,, \qquad [\![\partial, \partial]\!] = 0\,, \tag{3.39}$$

which are obviously satisfied. Notice that both (3.38) and the r.h.s. of (3.36) are of degree 1; hence, so is the solution $m(t, \hbar, s)$ to Eq. (3.36) with the initial condition (3.38).

Now, all the m_n can be generated systematically by solving (3.36) order by order in t, $m = m_2 + tm_3 + t^2 m_4 + \ldots$, and setting $\hbar = s = 0$ at the end. For example, at the first order we find

$$m_3 = \mu\{\mu', \partial\} \qquad \longrightarrow \qquad \begin{cases} \mu\{\mu', \partial\}(a, b, u) = +\mu(\mu'(a, b), \partial(u)) \overset{\hbar=0}{=} +\phi_1(a, b)u\,, \\ \mu\{\mu', \partial\}(a, u, v) = +\mu(\mu'(a, u), \partial(v)) \overset{\hbar=0}{=} +\phi_1(a, u)v\,, \\ \mu\{\mu', \partial\}(u, a, v) = -\mu(\mu'(u, a), \partial(v)) \overset{\hbar=0}{=} -\phi_1(u, a)v\,, \end{cases} \tag{3.40}$$

where on the right we evaluated the map on the left for various triplets of arguments. At the second order we obtain the relation

$$2m_4 = m_3\{\partial_\hbar \mu, \partial\} + \mu\{\partial_\hbar m_3, \partial\}\,, \tag{3.41}$$

and hence

$$m_4(a,b,u,v) = \mu(\mu'(\mu'(a,b),\partial(u)),\partial(v)) + \tfrac{1}{2}\mu(\mu(\mu''(a,b),\partial(u)),\partial(v)) = \tag{3.42}$$

$$\stackrel{\hbar=0}{=} \phi_1(\phi_1(a,b),u)v + \phi_2(a,b)uv\,, \tag{3.43}$$

which is in agreement with (3.18).

To summarize, given a deformation of an associative algebra, we can explicitly construct an A_∞-algebra that can be thought of as a noncommutative deformation of this algebra, where the deformation parameter is promoted to an element of the algebra itself.[7] Remarkably, the A_∞-structure is determined by the deformed product up to an A_∞ gauge transformation. While the construction above is quite general, in the sequel we focus upon the case of higher spin algebras and explain why and how these algebras can be deformed.

4 Higher Spin Algebras

In the first approximation, higher spin algebras (HSA) are just (infinite-dimensional) associative algebras that arise in the study of higher symmetries of linear conformally invariant equations or of higher spin extensions of gravity. Very often the same algebras show up in other contexts under different names. For instance, one of the simplest examples is just the Weyl algebra A_n. Rich examples of HSA are provided by various free conformal fields theories, being free they possess infinite-dimensional algebras of symmetries. Below we give a number of (almost) equivalent definitions and examples of HSA. The most important definitions for our subsequent discussion are due to free CFT's and universal enveloping algebras.

[7] Let us mention another quite general approach to the deformation problem above. It is based on the construction of an appropriate resolution for the initial algebra. The approach is applicable to associative [44, 45] as well as to A_∞/L_∞-algebra deformations [46, 47]. The choice of a resolution, however, is rather ambiguous and suitable resolutions may happen to be quite cumbersome. The advantage of the present approach is that it does not require any structure beyond the deformation of the underlying associative algebra and, in this sense, it is more universal.

4.1 Various Definitions and Constructions

1. Higher symmetries of linear equations. Given some linear equation $L\phi = 0$, where $\phi \equiv \phi(x)$ is a set of fields and $L = L(x, \partial)$ is a differential operator, it is useful to study its symmetries and the algebra they form. A differential operator $S = S(x, \partial)$ is called a symmetry if it maps solutions to solutions, i.e., $LS\phi = 0$ for any ϕ obeying $L\phi = 0$. In practice, the latter implies that L can be pushed through S, i.e., $LS = B_S L$ for some operator B_S. The operators of the form CL are called trivial symmetries. These should be quotiented out as they act trivially on-shell. It is also important that the product $S_1 S_2$ of two symmetries is a symmetry, as a consequence of the linearity of the equations. Therefore, the algebra of symmetries – the algebra of all symmetries modulo trivial ones – is associative.

A canonical example [48, 49] is a free scalar field $\phi(x)$ in d-dimensional flat space and $L = \Box$. The equation $\Box\phi = 0$ is well known to be conformally invariant, with conformal symmetries acting as[8]

$$\delta_\xi \phi(x) = \xi^a \partial_a \phi(x) + \frac{d-2}{2d}(\partial_a \xi^a)\phi(x)\,, \qquad \partial^a \xi^b + \partial^b \xi^a = \frac{2}{d}\eta^{ab}\partial_m \xi^m\,, \tag{4.1}$$

where $\xi^a(x)$ is a conformal Killing vector. These symmetries form the conformal algebra $so(d,2)$ with respect to the commutator $[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{[\xi_1,\xi_2]}$. As is pointed out above, the product $\delta_{\xi_1} \cdots \delta_{\xi_n}$ is a symmetry too and is represented by a higher-order differential operator. All such operators are related to the conformal Killing tensors

$$\delta_v \phi = v^{a_1 \ldots a_{k-1}} \partial_{a_1} \cdots \partial_{a_{k-1}} \phi + \text{more}\,, \qquad \partial^{a_1} v^{a_2 \ldots a_k} + \text{permutations} - \text{traces} = 0\,. \tag{4.2}$$

It can be shown that the products of conformal symmetries generate the full symmetry algebra [48, 49]. Higher powers of Laplacian, $L = \Box^k$, are also conformally-invariant operators and lead to interesting symmetry algebras [50, 51]. The symmetries of the free Dirac equation $\partial\!\!\!/\psi = 0$ are also known [52] as well as for many other differential operators.

The examples just given lead to infinite-dimensional associative algebras that contain the conformal algebra $so(d,2)$ as a (Lie) subalgebra under commutators. Possible generalizations are to consider other (not necessarily conformally invariant) differential operators, e.g. massive Klein-Gordon equation.

[8] $a, b, c, \ldots = 0, \ldots, d-1$ are the indices of the Lorentz algebra $so(d-1,1)$.

2. Higher spin currents and charges. Given a free field obeying $\Box$-type equations, e.g. $\Box\phi = 0$, one can construct an infinite number of conserved tensors [53]

$$j_{a_1\ldots a_s} = \phi\partial_{a_1}\ldots\partial_{a_s}\phi + \text{more terms}\,, \qquad\qquad \partial^m j_{ma_2\ldots a_s} = 0\,. \tag{4.3}$$

Due to the conformal invariance of $\Box\phi = 0$ the conserved tensors can be made traceless and are thereby quasi-primary operators of the free boson CFT. Contracting them with conformal Killing tensors, one obtains conserved currents and the corresponding charges:

$$j_m(v) = j_{ma_2\ldots a_s} v^{a_2\ldots a_s}\,, \qquad\qquad Q_v = \int d^{d-1}x\, j_0\,. \tag{4.4}$$

The Noether theorem relates these currents and charges with the symmetries described in item 1 above, i.e., Definition 2 is more or less equivalent to Definition 1. Such conserved tensors and symmetries associated to them have been known since the 60's, see e.g. [53, 54] and references therein. It was also shown that they do not survive when interactions are switched on. In the realm of CFT the opposite statement is also true: the existence of conserved higher rank tensors implies the theory is free. [32–35]. The extensions of the usual Poincare symmetry are constrained by Coleman–Mandula theorem [55].

3. Quotients of universal enveloping algebras. A more direct description of HSA associated with linear conformally-invariant equations is via universal enveloping algebra $U(so(d,2))$, as the last paragraph of item 1 suggests: juxtaposing conformal transformations generates the associative symmetry algebra. Therefore, we may collect the generators P^a, K^a, L^{ab}, D associated with the conformal algebra into T^{AB} of $so(d,2)$.[9] Then, any polynomial

$$f(T^{AB}) = f(P^a, K^a, L^{ab}, D) \tag{4.5}$$

generates a symmetry transformation. However, there are some relations meaning that not all the polynomials are independent and generate nontrivial transformations. For example, for the free scalar field we have $P_aP^a \sim 0$. Also, if the fundamental field corresponds to an irreducible representation of the conformal algebra, the Casimir operators should acquire fixed numerical values. As a result, the symmetry algebra is isomorphic to the quotient of the universal enveloping algebra $U(so(d,2))$ by a two-sided ideal (annihilator) $\mathcal{J}$:

$$hs \sim U(so(d,2))/\mathcal{J}\,. \tag{4.6}$$

[9] $A, B, C, ... = 0, ..., d+1$ are the indices of the conformal algebra $so(d,2)$ and $\eta^{AB} = (-+\cdots+-)$. Then, $L_{ab} = T_{ab}$, $D = -T_{d,d+1}$, $P_a = M_{a,d+1} - M_{a,d}$, $K_a = M_{a,d+1} + M_{a,d}$.

A concrete definition of $\mathcal{J}$ depends on the free CFT (irreducible representation) we consider, but, on general grounds, we expect all Casimir operators $\boldsymbol{C}_{2i}$ to have some fixed values C_{2i}. In most of the cases $\mathcal{J}$ can be generated by a few elements of $U(so(d,2))$.

In the case of the smallest unitary representation, e.g. the free conformal scalar field, the annihilator $\mathcal{J}$ is also known as Joseph ideal [56]. Possible generalizations here is to consider more general ideals in $U(g)$ for any (not necessarily conformal) Lie algebra g, see e.g. [28, 57, 58]. A useful for our studies example is provided by the HSA of the generalized free field CFT.[10]

4. Quantization of coadjoint orbits. There is also a relation [49, 59, 60] between HSA and deformation quantization [10, 61]. The fundamental field of any free CFT corresponds to some irreducible representation of the conformal algebra. This representation, it its turn, is associated to a certain coadjoint orbit (usually to a minimal nilpotent one). Not surprisingly that a given HSA can be identified with the quantized algebra of functions on this coadjoint orbit. Possible generalizations here is to consider deformation quantization in full generality, i.e., for general symplectic or Poisson manifolds.

4.2 Examples

Let us discuss a few simple examples of HSA that will be important later. We mostly employ the universal enveloping realization of HSA. The conformal or anti-de Sitter algebra generators T_{AB} obey

$$[T_{AB}, T_{CD}] = T_{AD}\eta_{BC} - T_{BD}\eta_{AC} - T_{AC}\eta_{BD} + T_{BC}\eta_{AD}\,. \tag{4.7}$$

and by the Poincaré–Birkhoff–Witt theorem, the decomposition of the universal enveloping algebra $U(so(d,2))$ is given by symmetrized tensor products of the adjoint representation[11]

$$U(so(d,2)) = \bullet \oplus \ydiagram{1,1} \oplus \left[\ydiagram{2,2} \oplus \ydiagram{2} \oplus \ydiagram{1,1,1,1} \oplus \bullet\right] \oplus \left[\ydiagram{3,3} \oplus \ydiagram{2,2,1,1} \oplus \cdots\right] \oplus \cdots\,. \tag{4.8}$$

[10]Note that Definitions 1 and 2 do not apply here, while the Definitions 3 and 4 can still be used, see below.

[11]The language of Young diagrams is useful here. For example, the fundamental and the adjoint representations are depicted by $\ydiagram{1}$ and $\ydiagram{1,1}$, respectively. The trivial representation is denoted by $\bullet$.

Here the first singlet $\bullet$ is the unit of $U(so(d,2))$, $\yng(1,1) \sim T^{AB}$ and the second $\bullet$ is the quadratic Casimir operator

$$\boldsymbol{C}_2 = -\frac{1}{2} T_{AB} T^{AB} \,. \tag{4.9}$$

In what follows we describe some ideals of $U(so(d,2))$ and the corresponding quotients that yield the HSA of interest.

Free Boson HSA. This is the simplest HSA and the generators of the ideal can be guessed from the symmetries of $\Box\phi = 0$. Since the solution space is an irreducible representation, the values of the Casimir operators are fixed. Decoupling of null states implies $P_a P^a = 0$ and $K^a K_a = 0$. Finally, all anti-symmetric combinations of the conformal symmetry generators, e.g. $L_{[ab}P_{c]}$ and $L_{[ab}L_{cd]}$, should vanish. All in all, the two-sided (Joseph) ideal is generated by [49][12]

$$\mathcal{J} = \yng(1,1,1,1) \oplus \yng(2) \oplus (\boldsymbol{C}_2 - C_2) \,, \qquad\qquad C_2 = -\frac{1}{4}(d^2-4) \,. \tag{4.10}$$

The $so(d,2)$ decomposition of the quotient algebra contains traceless tensors described by rectangular, two-row, Young diagrams:

$$hs_{F.B.} = \bullet \oplus \yng(1,1) \oplus \yng(2,2) \oplus \yng(3,3) \oplus \cdots \,. \tag{4.11}$$

More explicitly, the generators of the Joseph ideal read:

$$\mathcal{J}^{ABCD} = T^{[AB}T^{CD]} \,, \tag{4.12a}$$

$$\mathcal{J}^{AB} = T^{A}{}_{C} T^{BC} + T^{B}{}_{C} T^{AC} - (d-2)\eta^{AB} \,, \tag{4.12b}$$

$$\mathcal{J} = -\frac{1}{2} T_{AB} T^{AB} + \frac{1}{4}(d^2-4) \,. \tag{4.12c}$$

Free Boson and Free Fermion HSA in Three Dimensions. This is an even simpler example since all of the Joseph ideal relations can be resolved thanks to the isomorphism $so(3,2) \sim sp(4)$.[13] It turns out that the free boson and free fermion fields – as representations of $sp(4)$ – are equivalent to even and odd states in the Fock space of the $2d$ harmonic oscillator:

$$P^{a_1}...P^{a_k}|\phi\rangle \sim a^\dagger_{\alpha_1}...a^\dagger_{\alpha_{2k}}|0\rangle \,, \tag{4.13a}$$

$$P^{a_1}...P^{a_k}|\psi\rangle_\delta \sim a^\dagger_{\alpha_1}...a^\dagger_{\alpha_{2k}} a^\dagger_\delta|0\rangle \,. \tag{4.13b}$$

[12] The full two-sided ideal is obtained by taking the generators and multiplying them by $U(so(d,2))$.

[13] Some important facts contained already in [62]. Everything we discuss below can be found in [63–65].

Here a^α and $a^\dagger_\beta$ are the standard creation/annihilation operators satisfying

$$[a^\alpha, a^\dagger_\beta] = \delta^\alpha_\beta\,, \qquad a^\alpha|0\rangle = 0\,, \tag{4.14}$$

and $\alpha, \beta = 1, 2$ are the spinor indices from the $so(1,2)$ point of view. The spinor-vector dictionary is through the σ-matrices, e.g. $P_m = \sigma^{\alpha\beta}_m a^\dagger_\alpha a^\dagger_\beta$. The $sp(4)$ generators are realized by the ten bilinears in a^α and $a^\dagger_\alpha$:

$$K^{\alpha\beta} = a^\alpha a^\beta\,, \qquad \frac{1}{2}D\delta^\alpha_\beta + L^\alpha_\beta = \frac{1}{2}\{a^\alpha, a^\dagger_\beta\}\,, \qquad P_{\alpha\beta} = a^\dagger_\alpha a^\dagger_\beta\,. \tag{4.15}$$

This is the standard oscillator realization of $sp(4)$. The algebra of all operators $O(a, a^\dagger)$ in a^α, $a^\dagger_\beta$ is the Weyl algebra A_2.[14] The HSA, as an algebra that maps on-shell states (4.13) to the on-shell states, is the even subalgebra of the Weyl algebra A_2, i.e., $O(a, a^\dagger) = O(-a, -a^\dagger)$.

The most important feature of the $3d$ case is that the HSA of the free boson and of the free fermion are the same, the even subalgebra of A_2. This is not so in $d > 3$ for an obvious reason that the higher spin currents built out of the free fermion do not match those of the free boson, see e.g. [66].

Generalized Free Field HSA. A generalized free (scalar) field, i.e., a conformal scalar operator $O_\Delta(x)$ of some weight Δ such that all correlators are computed via free Wick contractions,[15] is a useful approximation in many situations. The corresponding HSA, denoted by hs_Δ, is defined to be the quotient $hs_\Delta = U(so(d,2))/\mathcal{J}_\Delta$ with respect to the ideal generated by

$$\mathcal{J}_\Delta = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline\end{array} \oplus \left(\boldsymbol{C}_2 - C_2(\Delta)\right)\,, \qquad C_2(\Delta) = \Delta(d - \Delta)\,, \tag{4.16}$$

or, in components,

$$\mathcal{J}^{ABCD} = T^{[AB}T^{CD]}\,, \qquad \mathcal{J} = -\frac{1}{2}T_{AB}T^{AB} - C_2(\Delta)\,. \tag{4.17}$$

The interpretation of the ideal is obvious. That $\mathcal{J}^{ABCD}$ must vanish is a manifestation of the lowest state $|\Delta\rangle$ being scalar, which implies that the descendants $P^a \ldots P^c|\Delta\rangle$ are symmetric

[14] The subscript indicates the number of canonical pairs, two in our case.

[15] For generic Δ, generalized free field does not have a local stress-tensor and does not have (local) higher spin currents. Also, there are no equations to be imposed. Therefore, the definitions (1) and (2) are not applicable. Nevertheless, the algebra can be defined via definition (3) (and also via (4)) as we do here. A good consistency check is that it reduces to the already known HSA at the expected values of Δ.

tensors and the combinations of the generators with more than two anti-symmetrized indices vanish. The $so(d,2)$-decomposition contains more tensors than that of the free boson HSA, namely,

$$hs_\Delta = \bullet \oplus \yng(1,1) \oplus \yng(2,2) \oplus \yng(2) \oplus \yng(3,3) \oplus \yng(3,1) \oplus \cdots . \tag{4.18}$$

The additional components are due to the absence of the $\yng(2)$ generator.[16]

Clearly, the HSA of generalized free field $O_\Delta(x)$ form a one-parameter family of algebras because Δ is a free parameter. At the critical values $\Delta_k = d/2 - k$, $k = 1, 2, \ldots$, the algebra is not simple and acquires a two-sided ideal. The resulting quotient algebra is the symmetry algebra of the free scalar field $\Box^k \phi = 0$ [50, 51]. The one-parameter family of HSA corresponding to generalized free fields will be important for the discussion below since it underlies the deformation of the other HSA.

4.3 Higher Spin Currents Equal Higher Spin Algebra

As it was already mentioned, the higher spin symmetry of free CFT's is manifested by an infinite number of higher-spin currents J_s, which are quasi-primary operators from the CFT point of view. Schematically, they are

$$J_{a_1\ldots a_s} = \phi \partial_{a_1} \ldots \partial_{a_s} \phi + \text{more}\,, \qquad\qquad \partial^c J_{ca_2\ldots a_s} = 0\,. \tag{4.19}$$

The stress-tensor, which is responsible for the $so(d,2)$-part of the HSA is the $s=2$ member of the family. By construction, the free field is a fundamental representation of this HSA.[17] The infinite multiplet J of higher spin currents J_s is the representation that is next to the fundamental one.[18] The lowest lying OPE's can be written as

$$\phi\phi = \mathbb{1} + J\,, \qquad\qquad JJ = \mathbb{1} + J + O_2\,, \tag{4.20}$$

[16]It may seem that one can pick several elements of $U(so(d,2))$ in random and declare them to generate an ideal, but in doing so one may discover that the ideal coincides with the full $U(so(d,2))$. In particular, it is impossible to add the $\yng(2)$ component to the generating set for generic Δ without trivializing the quotient.

[17]Representations (modules) of HSA are quite easy to describe, see e.g. [67]. Roughly speaking, the free field is a vector space V and HSA is $gl(V)$ for this V. Other representations are just tensor products $V \otimes \cdots \otimes V$ projected onto any irreducible representation of the permutation group (the permutation group commutes with the $gl(V)$-action on $T(V)$).

[18]One should be careful about tensor product vs. associativity issues and imply either the Lie subalgebra of a HSA (via commutators) or the tensor product of HSA that naturally acts on the tensor product of its representations.

where $\mathbb{1}$ is the identity operator and O_2 is a multiplet of double-trace operators, which is given by the quartic tensor product of the free field itself.

Regarding the free field as a vector space V and HSA as $gl(V)$, the higher spin currents belong to $V \otimes V$, which is very close to $gl(V) \sim V \otimes V^*$. This heuristic reasoning can be made more precise.[19] If $|\phi\rangle$ is the free field vacuum, then

$$K^a|\phi\rangle = 0\,, \qquad L^{ab}|\phi\rangle = 0\,, \qquad D|\phi\rangle = \tfrac{d-2}{2}|\phi\rangle \tag{4.21}$$

and the descendants correspond to $P^a \dots P^c|\phi\rangle$. Higher spin currents are the quasi-primary states in the tensor product

$$J \sim \phi \times \phi \sim P^a \dots P^c|\phi\rangle \otimes P^b \dots P^d|\phi\rangle\,, \tag{4.22}$$

while the HSA can be viewed as the span of operators of the form

$$P^a \dots P^c|\phi\rangle \otimes \langle\phi|K^b \dots K^d\,. \tag{4.23}$$

Clearly, the two spaces are formally isomorphic and the map between them is the conjugation $\langle\phi| = |\phi\rangle^\dagger$, which is defined via the inversion map $\mathtt{I}$.[20] Therefore, the higher spin currents together with descendants, as a module of the conformal (and also a HSA-module), can be viewed as the same HSA where the right action is twisted by $\mathtt{I}$. Therefore, $J\mathtt{I}$ is formally isomorphic to HSA.

In interacting CFT's with slightly broken higher spin symmetry higher spin currents are no longer conserved, but their non-conservation has a very specific form of

$$\partial \cdot J = \frac{1}{N}[JJ]\,, \tag{4.24}$$

where $[JJ]$ is a specific (set of) double-trace operators, whose form may also depend on the coupling constants, see [11–14, 71] for some explicit formulas. The non-conservation is supposed to be a small effect, which is controlled by $1/N$ for large-N. Therefore, J itself is a deformation parameter and we will take advantage of the fact that $J\mathtt{I}$ is isomorphic to the HSA originating from J in order to apply the construction of A_∞-algebras from Section 3.

[19] See [68] for subtleties that may arise in some formal manipulations. That the tensor product decomposes into (all) higher spin currents was shown, for $d = 3$, in [69] (the currents, as representations of $so(d, 2)$, viewed as anti-de Sitter algebra, are the same as massless fields in AdS_{d+1}, which is the interpretation adopted in [69]). See [53] for the result in any d. See also [70] that elaborates on the relation between this construction and $U(so(d, 2))$, showing, in particular, that the shadow of J_0 can also be treated by the same tools.

[20] Note that at the level of the Lie algebra we have $K^a = IP^aI$, $L^{ab} = IL^{ab}I$, $P^a = IK^aI$ and $-D = IDI$. We see that $P^a + K^a$ and L^{ab} are stable and form $so(d-1, 2)$ subalgebra of the conformal algebra $so(d, 2)$. We can also define $-K^a = IP^aI$, $L^{ab} = IL^{ab}I$, $-P^a = IK^aI$ and $-D = IDI$. Then, it is $P^a - K^a$ and L^{ab} that are stable and form $so(d, 1)$.

5 Deformations of Higher Spin Algebras

As it was already mentioned, typical HSA admit no deformations as associative algebras, which means that $HH^2(hs, hs) = \emptyset$.[21] Nevertheless, certain simple extensions of HSA do admit deformations and it is these deformations that are also responsible for the A_∞-structure that originate from any given HSA. The deformations turn out to exist due to the fact that certain subalgebras of HSA are related to the generalized free field whose weight is a free parameter.

In view of the way higher spin symmetry gets broken, the deformation parameter is the multiplet of higher spin currents, J. Also, $J\mathtt{I}$ – J twisted by the inversion map – is isomorphic to the HSA. In order to treat both the HSA hs and J on an equal footing, we take a bigger algebra, namely, HSA extended by $\mathtt{I}$, which we call the double and denote $D(hs)$. This just the simplest example of the smash product $B \rtimes \Gamma$, where B is an algebra and Γ is a finite group of automorphisms of B. Its elements have the form $a + b\mathtt{I}$, $a, b \in hs$ and the product law reads

$$(a + b\mathtt{I})(a' + b'\mathtt{I}) = (aa' + b\mathtt{I}(b')) + (ab' + b\mathtt{I}(a'))\mathtt{I}\,, \tag{5.1}$$

where $\mathtt{I}(a)$ is the action of the inversion on the algebra elements, which can be obtained by extending $\mathtt{I}(P^a) = \mathtt{I}P^a\mathtt{I} = K^a$, etc. to series in P^a, K^a, L^{ab}, D, i.e., to hs, and we used $\mathtt{I}^2 = 1$.

An important observation is that $D(hs)$ belongs to a one-parameter family of algebras (while hs usually does not). Then, we can apply the general construction of A_∞ from Section 3. Finally, we can take the truncation of the A_∞ that reduces $D(hs)$ to hs in the sector of A_{-1} and to $hs \cdot I$ – the bimodule of higher spin currents – in the sector of A_0. This gives precisely the A_∞-algebra we are looking for.[22]

The reason for $D(hs)$ to admit a deformation is quite simple. The $\mathtt{I}$-stable subalgebra, $\mathtt{I}(a) = a$, $a \in hs$, of hs, turns out to be the HSA of the generalized free field in $d-1$ dimension and, therefore, admits a deformation. Extending hs to $D(hs)$ allows one to uplift this deformation to $D(hs)$. Let us discuss this in more detail.

[21] What we discuss below applies also to the examples where they do admit such deformations.

[22] It is worth stressing that even if a given hs happens to belong to a one-parameter family of algebras, it will not lead to the A_∞-algebra we need, for the deformation parameter has to be in $hs \cdot \mathtt{I}$ rather than hs. Restriction of the A_∞ we need to the Lorentz subalgebra (see below) leads to this type of A_∞.

Higher Spin Lorentz Subalgebra. The most convenient definition of HSA at the moment is via universal enveloping algebra $U(so(d,2))$. Suppose we are given some hs as $hs = U(so(d,2))/\mathcal{J}$ for some $\mathcal{J}$. We also assume that hs corresponds to some free on-shell field. The $so(d,2)$-generators T^{AB} can be split into the AdS-Lorentz generators $\mathtt{L}^{\mathcal{AB}}$ and AdS-translations $\mathtt{P}^{\mathcal{A}}$.[23] The AdS-Lorentz subalgebra $L(hs)$ of hs is defined as the enveloping algebra of the $so(d,1)$ subalgebra generated by $\mathtt{L}^{\mathcal{AB}}$. This is the stability algebra of the inversion map.[24]

The Lorentz subalgebra $L(hs)$ can be understood as a HSA itself ($so(d,1)$ is viewed here as the Euclidian conformal algebra in $d-1$ dimensions): it has more or less the same properties, but the Casimir value corresponds to an off-shell conformal field in $(d-1)$ dimensions.

For example, the ideal that is responsible for the free boson HSA, when T^{AB} is decomposed into $\mathtt{L}^{\mathcal{AB}}$ and $\mathtt{P}^{\mathcal{A}}$, reads:

$$\mathcal{J}^{\mathcal{ABCD}} = \mathtt{L}^{[\mathcal{AB}}\mathtt{L}^{\mathcal{CD}]}\,, \tag{5.2a}$$

$$\mathcal{J}^{\mathcal{ABC}5} = \{\mathtt{L}^{[\mathcal{AB}}, \mathtt{P}^{\mathcal{C}]}\}\,, \tag{5.2b}$$

$$\mathcal{J}^{\mathcal{AB}} = \mathtt{L}^{\mathcal{A}}{}_{\mathcal{C}}\,\mathtt{L}^{\mathcal{BC}} + \mathtt{L}^{\mathcal{B}}{}_{\mathcal{C}}\,\mathtt{L}^{\mathcal{AC}} - \mathtt{P}^{\mathcal{A}}\mathtt{P}^{\mathcal{B}} - \mathtt{P}^{\mathcal{B}}\mathtt{P}^{\mathcal{A}} - (d-2)\eta^{\mathcal{AB}}\,, \tag{5.2c}$$

$$\mathcal{J}^{\mathcal{A}5} = \{\mathtt{L}^{\mathcal{A}}{}_{\mathcal{C}}\,, \mathtt{P}^{\mathcal{C}}\}\,, \tag{5.2d}$$

$$\mathcal{J}^{55} = 2\mathtt{P}_{\mathcal{A}}\mathtt{P}^{\mathcal{A}} + (d-2)\,, \tag{5.2e}$$

$$\mathcal{J} = -\frac{1}{2}\mathtt{L}_{\mathcal{AB}}\mathtt{L}^{\mathcal{AB}} + \mathtt{P}_{\mathcal{A}}\mathtt{P}^{\mathcal{A}} + \frac{1}{4}(d^2-4)\,, \tag{5.2f}$$

from which it follows

$$\mathcal{J}^{\mathcal{ABCD}} = \mathtt{L}^{[\mathcal{AB}}\mathtt{L}^{\mathcal{CD}]}\,, \qquad\qquad \mathcal{J} = -\frac{1}{2}\mathtt{L}_{\mathcal{AB}}\mathtt{L}^{\mathcal{AB}} + \frac{d}{4}(d-2)\,. \tag{5.3}$$

This is exactly the ideal that defines the HSA of the generalized free field, but in one dimension lower, cf. (4.17). The conformal weight of this fictitious generalized free field in $(d-1)$ dimensions is $(d-2)/2$ or $d/2$.[25]

[23] For example, $\mathtt{L}^{\mathcal{AB}} = T^{\mathcal{AB}}$ and $\mathtt{P}^{\mathcal{A}} = T^{\mathcal{A}5}$ where 5 is the extra dimension of an $so(d,2)$ vector as compared to an $so(d,1)$ one $\eta^{55} = -1$. Here $\mathcal{A}, \mathcal{B}, \ldots = 0, \ldots, d$ are the indices of the AdS Lorentz algebra $so(d,1)$.

[24] Another reason for the relevance of the AdS-Lorentz interpretation is that $\mathtt{I}\mathtt{L}^{\mathcal{AB}}\mathtt{I} = \mathtt{L}^{\mathcal{AB}}$ and $\mathtt{I}\mathtt{P}^{\mathcal{A}}\mathtt{I} = -\mathtt{P}^{\mathcal{A}}$ if we define $\mathtt{I}P^a\mathtt{I} = -K^a$ etc. Such automorphism of the AdS-algebra is used in the study of higher spin fields in AdS, see e.g. [72]. If we define $\mathtt{I}P^a\mathtt{I} = K^a$ etc., then the stability algebra is $so(d-1,2)$, which can be interpreted as the conformal algebra in $(d-1)$-dimension. Such definition is more physical since it is the $so(d-1,2)$ subalgebra that would admit supersymmetric extensions once $so(d,2)$ does for lower d. Nevertheless, below we mostly use the $so(d,1)$-interpretation.

[25] The value of the Casimir operator is $\Delta(\Delta-(d-1))$. Notice that both the roots are above the unitarity bound $(d-1)/2-1$.

Thanks to the fact that the weight of this fictitious generalized free field is generic the Lorentz subalgebra belongs to a one-parameter family of algebras. Therefore, the Lorentz subalgebra can be deformed.

To sum up, the Lorentz subalgebra $L(hs)$ of a HSA hs belongs to a one-parameter family of deformations. In particular, the second Hochschild cohomology group is not empty, $HH^2(L(hs), L(hs)) \neq 0$.

Deformation of the Double. That the $\mathtt{I}$-stable subalgebra $L(hs)$ admits a one-parameter family of deformations is a strong indication that the $\mathtt{I}$-extended algebra $D(hs)$ also does. However, there does not seem to be any general theorem that would allow one to directly construct such a deformation. The following three justifications are helpful. (1) In the case of the smash-product of Weyl algebra by a finite group of symplectic reflections (which is the case that many HSA can be reduced to) it can be shown that such deformations do exist and it is even possible to explicitly construct them, see [44, 45]. (2) For many algebras there is a duality [73] between Hochschild homology and cohomology[26] and we can explicitly construct the cycle that the sought for Hochschild two-cocycle is dual to (see below). (3) At least for the algebras we are interested in this paper there is a simple oscillator realization and in Section 6 we construct the deformed double $D_\hbar(hs)$ explicitly.

Dual Cycle. Cochains act in a natural way on the chains, the latter form a module over the former [74]. As different from cocycles, cycles are usually easier to find. Then, if the algebra falls into the class of algebras for which Hochschild cohomology $HH^\bullet(A)$ is dual to the homology $HH_\bullet(A)$, one can compute the dimension of various $HH^\bullet(A)$ from those of $HH_\bullet(A)$. Another usage of nontrivial cycles is to test whether a given cocycle is nontrivial since the chain differential is dual to the cochain differential with respect to the natural pairing. We will construct the dual cycle for $D(hs)$, which implies that there is a dual cocycle.

Note first, that the HSA hs of some free field[27] is determined by a certain two-sided ideal $\mathcal{J}$ of $U(so(d,2))$. For the free fields obeying the equation $\Box\phi = 0$ the ideal contains the generator described by the Young diagram $\square\square$. Taken together with the fixed value of the Casimir operator this means that the AdS-momentum squares to a constant:

$$\mathtt{P}_{\mathcal{A}}\mathtt{P}^{\mathcal{A}} = M^2\,. \tag{5.4}$$

[26] We do not list the quite technical assumptions of the theorem in [73].
[27] Here we avoid generalized free fields at generic value of the conformal weight.

For example, for the HSA of the free boson CFT we find [70] (5.2e)[28]

$$\mathtt{P}_{\mathcal{A}}\mathtt{P}^{\mathcal{A}} = -\frac{(d-2)}{2}\,. \tag{5.5}$$

Now, consider the two-chain[29]

$$\gamma = 1 \otimes \mathtt{P}_{\mathcal{A}} \otimes \mathtt{P}^{\mathcal{A}}\,. \tag{5.6}$$

It is a nontrivial cycle of hs with values in the representation that is twisted by $\mathtt{I}$:

$$\partial\gamma = \mathtt{P}_{\mathcal{A}} \otimes \mathtt{P}^{\mathcal{A}} - 1 \otimes \mathtt{P}_{\mathcal{A}}\mathtt{P}^{\mathcal{A}} + \mathtt{I}(\mathtt{P}^{\mathcal{A}}) \otimes \mathtt{P}_{\mathcal{A}} = 0\,. \tag{5.7}$$

Here we used (5.4) and the fact that the complex is normalized, i.e., $M^2 \sim 0$ when it appears in any of the factors except the first one. In this case it is easy to uplift the cycle from the normalized complex to the original one. Indeed,

$$\gamma' = 1 \otimes \mathtt{P}_{\mathcal{A}} \otimes \mathtt{P}^{\mathcal{A}} + M^2(1 \otimes 1 \otimes 1) \tag{5.8}$$

is closed as it is. Therefore, γ' represents a class in $HH_2(hs, hs^I)$. This cycle can also be uplifted to the cycle of the full double $D(hs)$

$$\gamma' = \mathtt{I} \otimes \mathtt{P}_{\mathcal{A}} \otimes \mathtt{P}^{\mathcal{A}} + M^2(1 \otimes 1 \otimes 1)\,, \tag{5.9}$$

representing thus an element of $HH_2(D(hs), D(hs))$. It should be dual to a nontrivial cocycle ϕ that is an element of $HH^2(D(hs), D(hs))$. This is the cocycle we need to deform $D(hs)$. It is a nontrivial solution to the equation

$$a\phi(b,c) - \phi(ab,c) + \phi(a,bc) - \phi(a,b)I(c) = 0\,, \qquad\qquad a,b,c \in hs\,. \tag{5.10}$$

The deformation of $D(hs)$ it induces can be written as

$$(a + a'I) * (b + b'I) = (ab + aI(b')) + (ab' + a'I(b))I + \hbar\phi(a + a'I, b + b'I)I + \mathcal{O}(\hbar^2)\,.$$

[28] This is the AdS-base rewriting of $P^aP_a = 0$, $K^aK_a = 0$, and $\boldsymbol{C}_2 - \frac{1}{4}(d^2-4) = 0$.

[29] The Hochschild differential acts as (note the twist by $\mathtt{I}$)

$$\partial(c_0 \otimes c_1 \otimes \cdots \otimes c_k) = c_0c_1 \otimes c_2 \cdots \otimes c_k - c_0 \otimes c_1c_2 \otimes \cdots \otimes c_k + \cdots + (-)^k I(c_k)c_0 \otimes c_1 \cdots \otimes c_{k-1}\,.$$

Also, note that $c_{k>0}$'s are assumed to take values in hs/K, where $K \subset hs$ is the base field. In practice this means that $K \sim 0$ for all the factors except for the first one. Such complex is called *normalized* and it is known to have the same homology $HH_\bullet(hs/K, hs^{\mathtt{I}}) \sim HH_\bullet(hs, hs^{\mathtt{I}})$.

Let us also consider the special case of three dimensions. Firstly, we can replace (5.9) with an equivalent two-cycle

$$\gamma' = \mathtt{L}_{\mathcal{AB}} \otimes \mathtt{P}^{\mathcal{A}} \otimes \mathtt{P}^{\mathcal{B}} + \frac{1}{4} 1 \otimes \mathtt{L}_{\mathcal{AB}} \otimes \mathtt{L}^{\mathcal{AB}} - \frac{1}{2} C_L 1 \otimes 1 \otimes 1 \,, \qquad \partial\gamma' = 0 \,, \tag{5.11}$$

where $C_L = -\frac{1}{2}\mathtt{L}_{\mathcal{AB}}\mathtt{L}^{\mathcal{AB}}$ is the value of the Casimir operator of the AdS-Lorentz subalgebra, see e.g. (5.3). Secondly, in the $sl(2,\mathbb{C})$ spinorial language generators T^{AB} of $so(3,2)$ decompose into $\mathtt{P}_{\alpha\dot\alpha}$, and $\mathtt{L}_{\alpha\beta}$, $\mathtt{L}_{\dot\alpha\dot\beta}$,[30] the latter being (anti)-selfdual components of $\mathtt{L}_{\mathcal{AB}}$. Then, (5.11) reduces to the two independent cycles

$$\gamma' = \mathtt{L}^{\alpha\beta} \otimes \mathtt{P}_{\alpha\dot\gamma} \otimes \mathtt{P}_{\beta}{}^{\dot\gamma} + \frac{1}{2} 1 \otimes \mathtt{L}_{\alpha\beta} \otimes \mathtt{L}^{\alpha\beta} - c_L 1 \otimes 1 \otimes 1 \,, \qquad \partial\gamma' = 0 \,, \tag{5.12}$$

where $c_L = -\frac{1}{2}\mathtt{L}_{\alpha\beta}\mathtt{L}^{\alpha\beta} = -3/4$ and the second cycle is obtained by complex conjugation. These two cycles imply that there are two independent deformations of the free boson HSA and free fermion HSA (which is the same) in three dimensions. Below, we provide a completion of the $*$-product for the examples of interest.

Constructing A_∞-algebra. The complete algorithm for constructing the A_∞ description of the slightly broken higher spin symmetry for any given HSA hs is as follows. Firstly, one takes the double $D(hs)$ by adding the inversion $\mathtt{I}$ and constructs the one-parameter family of associative algebras $D_\hbar(hs)$. Expanding the product $a *_\hbar b$ in powers of $\hbar$ yields then the bi-linear maps $\phi_k(\bullet,\bullet)$. Next, the A_∞-algebra is constructed by building up the structure maps m_n following the general method of Section 3. Lastly, it is easy to see that all elements from A_{-1} can be restricted to hs, while all elements from A_0 can be restricted to $hs\mathtt{I}$ to be interpreted as $J\mathtt{I}$ for the multiplet of higher spin currents J.

6 Explicit Oscillator Realizations

We have proved the existence of certain deformations of higher spin algebras (HSA), but in practice one may also need an efficient way to perform computations with the deformed algebras. All reasonable HSA admit oscillator realizations and, after briefly reviewing these realizations, we modify them as to construct the deformed HSA.

[30] $\alpha, \beta, \ldots = 1,2$ and $\dot\alpha, \dot\beta, \ldots = 1,2$ are the indices of the fundamental representation of $sl(2,\mathbb{C})$ and its conjugate. The dictionary between the vectorial and spinorial languages is via σ-matrices, e.g. $\mathtt{P}_{\mathcal{A}} = \sigma^{\alpha\dot\alpha}_{\mathcal{A}}\mathtt{P}_{\alpha\dot\alpha}$.

6.1 Toy Model: Weyl Algebra A_1

The simplest example, which nevertheless underlies all the other deformations, is the smallest Weyl algebra A_1, i.e., one-dimensional harmonic oscillator. The Weyl algebra A_1 is defined in our notation as[31]

$$[y_\alpha, y_\beta] = 2i\epsilon_{\alpha\beta}\,, \qquad\qquad \alpha,\beta = 1,2\,. \tag{6.1}$$

Let us define the automorphism $\mathtt{I}$ as the reflection $\mathtt{I}(y_\alpha) = -y_\alpha$. Therefore, the $\mathtt{I}$-stable subalgebra – the 'Lorentz' subalgebra – is simply the subalgebra A_1^e of even functions in y's, $f(y) = f(-y)$. It is well known that the Weyl algebra does not admit any deformation as an associative algebra, but the 'Lorentz' subalgebra does belong to a one-parameter family of algebras. Indeed, $sp(2) \sim sl(2)$ is a subalgebra of the Weyl algebra, which is realized by the three generators $t_{\alpha\beta} = t_{\beta\alpha}$:

$$t_{\alpha\beta} = -\tfrac{i}{4}\{y_\alpha, y_\beta\}\,, \qquad\qquad [t_{\alpha\beta}, t_{\gamma\delta}] = \epsilon_{\alpha\delta} t_{\beta\gamma} + \text{three more}\,. \tag{6.2}$$

The $\mathtt{I}$-stable subalgebra A_1^e coincides with the enveloping algebra of $t_{\alpha\beta}$ and it is not hard to see that it is the quotient of $U(sl_2)$ by the two-sided ideal generated by $\boldsymbol{C}_2 - (-\frac{3}{4})$, where $\boldsymbol{C}_2 = -\frac{1}{2} t_{\alpha\beta} t^{\alpha\beta}$ is the Casimir operator; the constant $-\frac{3}{4}$ is value of $\boldsymbol{C}_2$ in the oscillator realization. This algebra belongs to a one-parameter family of algebras,[32] called $hs(\lambda)$ that are obtained in the same way except that the value of the Casimir is kept to be a free parameter:

$$hs(\lambda) = U(sl_2)/\mathcal{J}\,, \qquad\qquad \mathcal{J} = U(sl_2)[\boldsymbol{C}_2 + (\lambda^2 - 1)]\,. \tag{6.3}$$

$hs(\lambda)$ is nothing but a noncommutative (fuzzy) sphere, whose radius is controlled by λ. Therefore, we have $A_1^e \sim hs(\lambda^*)$, where $\lambda^* = 1/2$. According to our general result, the Weyl algebra A_1 extended by the automorphism $\mathtt{I}$ should admit a one-parameter family of deformations. The double $D(A_1)$ of A_1 is defined by

$$[y_\alpha, y_\beta] = 2i\epsilon_{\alpha\beta}\,, \qquad \{y_\alpha, k\} = 0\,, \qquad k^2 = 1\,. \tag{6.4}$$

Indeed, the latter commutation relations and the algebra they generate are a particular case of the so-called deformed oscillator algebra $Aq(\nu)$,[33] which is defined by the following

[31] Here, $\epsilon_{\alpha\beta}$ is the invariant $sp(2)$-tensor, the anti-symmetric tensor with $\epsilon_{12} = -\epsilon_{21} = 1$.

[32] This algebras were defined in [75] and dubbed gl_λ since they reduce to gl_N for certain values of λ and can be thought of as algebras interpolating between gl_N and gl_{N+1}.

[33] Defined implicitly in [76] and explicitly in e.g. [77–79], see also [80].

relations on its generators:

$$[q_\alpha, q_\beta] = 2i\epsilon_{\alpha\beta}(1+\nu K)\,, \qquad \{q_\alpha, K\} = 0 \qquad K^2 = 1\,. \tag{6.5}$$

It is clear that the double of the Weyl algebra $D(A_1)$ is isomorphic to $Aq(0)$. Another description of the deformed oscillator algebra is

$$Aq(\nu) = U(osp(1|2))/\mathcal{J}\,, \qquad \mathcal{J} = U(osp(1|2))[C_2 + \frac{1}{4}(1-\nu^2)]\,. \tag{6.6}$$

This algebra is nothing but a noncommutative super-sphere $S^{2|2}$ whose radius is controlled by ν. The structure constants of $hs(\lambda)$ and of the deformed oscillators are available in the literature in several forms [30, 81–84]. Therefore, the components $\phi_k(\bullet,\bullet)$ of the deformed HSA-algebra product are known, which can be used to explicitly write down the A_∞-structure. Notice that the classical limit of the deformed algebra is just a two-dimensional symplectic space with coordinates y_α and k that anti-commutes to it.

One may wonder to which extent the deformation described above is unique. For the Weyl algebra it is well known that $HH^2(A, A^*)$ is one-dimensional. In the same time, the $\mathtt{I}$-map identifies the dual module A^* with the $\mathtt{I}$-twisted one. For the double $D(A_1)$ the cohomology is known to be one-dimensional and the deformation is unique.

6.2 Deformations of the Free Boson Algebra

The simplest example of a HSA is the symmetry algebra of the free boson CFT [49]. The case of three dimensions is somewhat special and is discussed in the next section. The A_∞-algebra originating from this HSA should be responsible for the breaking of higher spin symmetries in the large-N critical vector model in d dimensions.[34]

There exists a quasi-conformal realization of this HSA by the minimal number of oscillators where the Joseph ideal is completely resolved [86]. This realization is non-linear and for simplicity let us stick to another, linear, form [72], in which the Joseph ideal is partially resolved. Such realization appears naturally from the manifestly conformally-invariant description of the free conformal scalar field in the ambient space [87]. One begins with the

[34] Due to the unitarity constraints the unitary cases are confined to $2 < d < 4$ and $4 < d < 6$ [85]. It would be interesting to extend the A_∞-algebra to fractional dimensions d.

embedding of the HSA into the Weyl algebra A_{d+2}:[35]

$$[Y^A_\alpha, Y^B_\beta] = 2i\eta^{AB}\epsilon_{\alpha\beta}\,. \tag{6.7}$$

The bilinears in Y form $sp(2(d+2))$, which contains a Howe dual pair $so(d,2)\oplus sp(2)$ of algebras such that the $so(d,2)$ generators T^{AB} commute with the $sp(2)$ generators $t_{\alpha\beta}$:

$$T^{AB} = +\frac{i}{4}\epsilon^{\alpha\beta}\{Y^A_\alpha, Y^B_\beta\}\,, \qquad\qquad t_{\alpha\beta} = -\frac{i}{4}\{Y^A_\alpha, Y_{A\beta}\}\,. \tag{6.8}$$

We consider the enveloping algebra of T^{AB}, i.e., functions $f(Y)\equiv f(T)$, which can also be defined as the centralizer of $sp(2)$, $[t_{\alpha\beta}, f(Y)] = 0$. By construction, a part of the Joseph ideal vanishes identically since one cannot have more than two anti-symmetrized indices of $so(d,2)$:

$$T^{[AB}T^{CD]} \sim \begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix} \sim 0\,. \tag{6.9}$$

The resulting algebra is not simple and its $so(d,2)$ decomposition contains traceful tensors with the symmetry of two-row rectangular Young diagrams:

$$f(T) \sim \bullet \oplus \begin{smallmatrix}\boxtimes\\\boxtimes\end{smallmatrix} \oplus \begin{smallmatrix}\boxtimes\boxtimes\\\boxtimes\boxtimes\end{smallmatrix} \oplus \begin{smallmatrix}\boxtimes\boxtimes\boxtimes\\\boxtimes\boxtimes\boxtimes\end{smallmatrix} \oplus \cdots\,. \tag{6.10}$$

The HSA is defined as a quotient of this algebra by the ideal generated by traces:

$$f\in hs_{F.B.}: \qquad\qquad [t_{\alpha\beta}, f] = 0\,, \qquad\qquad f \sim f + t_{\alpha\beta}\star g^{\alpha\beta}\,, \tag{6.11}$$

where $g^{\alpha\beta}$ transforms as an $sp(2)$-tensor. Note that the $sp(2)$-generators $t_{\alpha\beta}$ are exactly the contractions of Y's, i.e., traces. The resulting spectrum is (4.11), as expected.

The automorphism $\mathtt{I}$ that corresponds to the inversion map in the CFT base and to the flip of the AdS-translations in the AdS base is realized as $\mathtt{I}(y^{\mathcal{A}}_\alpha, y_\alpha) = (y^{\mathcal{A}}_\alpha, -y_\alpha)$, i.e., it flips the sign of the A_1 subalgebra generators. As we already explained, the Lorentz subalgebra of the HSA belongs to a one-parameter family of algebras. Since the $\mathtt{I}$-map does not affect $y^{\mathcal{A}}_\alpha$, the whole construction is very similar to the A_1 toy model. The double $D(hs_{F.B.})$ is easy to construct:

$$[y^{\mathcal{A}}_\alpha, y^{\mathcal{B}}_\beta] = +2i\epsilon_{\alpha\beta}\eta^{\mathcal{AB}}\,, \qquad\qquad [y_\alpha, y_\beta] = -2i\epsilon_{\alpha\beta}\,, \qquad\qquad \{y_\alpha, k\} = 0\,. \tag{6.12}$$

[35]Here $A,B,\ldots = 0,\ldots,d+1$ are indices of $so(d,2)$. We will also split them as $A = \{\mathcal{A},5\}$, etc., where $\mathcal{A},\mathcal{B},\ldots = 0,\ldots,d$ are the indices of the AdS-Lorentz algebra $so(d,1)$ and 5 is an extra dimension. $L^{\mathcal{AB}} = T^{\mathcal{AB}}$, $P^{\mathcal{A}} = T^{\mathcal{A}5}$, $\eta^{55} = -1$, so that $[P^{\mathcal{A}}, P^{\mathcal{B}}] = L^{\mathcal{AB}}$.

The deformed double is then obtained with the help of the deformed oscillators,

$$[y^{\mathcal{A}}_\alpha, y^{\mathcal{B}}_\beta] = +2i\epsilon_{\alpha\beta}\eta^{\mathcal{AB}}\,, \qquad [q_\alpha, q_\beta] = -2i\epsilon_{\alpha\beta}(1+\nu k)\,, \qquad \{q_\alpha, k\} = 0\,, \tag{6.13}$$

and is defined following (6.11) as

$$D_\nu(hs) \ni f(y^{\mathcal{A}}_\alpha, q_\alpha, k): \qquad [f, t_{\alpha\beta}] = 0\,, \qquad f \sim f + t_{\alpha\beta} \star g^{\alpha\beta}(y,q,k)\,, \tag{6.14}$$

where the new $sp(2)$ generators are

$$t_{\alpha\beta} = -\frac{i}{4}\{y^{\mathcal{A}}_\alpha, y_{\mathcal{A}\beta}\} + \tau_{\alpha\beta}\,, \qquad \tau_{\alpha\beta} = \frac{i}{4}\{q_\alpha, q_\beta\}\,. \tag{6.15}$$

At this point, there is no need in the deformed oscillators themselves, it is sufficient to know that the deformation of the algebra in y_α, k is given by the quotient of $U(osp(1|2))$, the fuzzy super-sphere.

The first few levels of the deformed double are easy to explore. Following the general logic, one can define the Lorentz and translation generators

$$\mathtt{P}^{\mathcal{A}} = +\frac{i}{4}\{y^{\mathcal{A}}_\alpha, q_\beta\}\epsilon^{\alpha\beta}\,, \qquad \mathtt{L}^{\mathcal{AB}} = +\frac{i}{4}\{y^{\mathcal{A}}_\alpha, y^{\mathcal{B}}_\beta\}\epsilon^{\alpha\beta} \tag{6.16}$$

that commute with $sp(2)$:

$$[t_{\alpha\beta}, \mathtt{P}_{\mathcal{A}}] = 0\,, \qquad [t_{\alpha\beta}, \mathtt{L}_{\mathcal{AB}}] = 0\,, \qquad [t_{\alpha\beta}, k] = 0\,. \tag{6.17}$$

The relations of the $so(d,2)$ algebra get modified at one place

$$[\mathtt{P}^{\mathcal{A}}, \mathtt{P}^{\mathcal{B}}] = (1+\nu k)\mathtt{L}^{\mathcal{AB}}\,, \quad [\mathtt{L}^{\mathcal{AB}}, \mathtt{L}^{\mathcal{CD}}] = \mathtt{L}^{\mathcal{AD}}\eta^{\mathcal{BC}} + \ldots\,, \quad [\mathtt{L}^{\mathcal{AB}}, \mathtt{P}^{\mathcal{C}}] = \mathtt{P}^{\mathcal{A}}\eta^{\mathcal{BC}} - \mathtt{P}^{\mathcal{B}}\eta^{\mathcal{AC}}\,,$$

which is the first nontrivial component of the Hochschild cocycle. The term $\nu k \mathtt{L}^{\mathcal{AB}}$ in $[\mathtt{P}^{\mathcal{A}}, \mathtt{P}^{\mathcal{B}}]$ is dual to the cycle (5.11).

6.3 Three Dimensions

The case of three dimensions is special due to the fact that the HSA of the free boson CFT is the same as the HSA of the free fermion CFT. The unique HSA is the even subalgebra A_2^e of the Weyl algebra A_2:[36]

$$hs \ni f(Y): \qquad f(Y) = f(-Y)\,, \qquad [Y^A, Y^B] = 2iC^{AB}\,. \tag{6.18}$$

[36] $A, B, ... = 1, ..., 4$ are the $sp(4)$ vector indices, $sp(4) \sim so(3,2)$. The AdS-Lorentz algebra is $sl(2,\mathbb{C}) \sim so(3,1)$ and it is convenient to use the indices $\alpha, \beta, \ldots = 1,2$ and $\dot\alpha, \dot\beta, \ldots = 1,2$ of the fundamental of $sl(2,\mathbb{C})$ and its conjugate.

In the AdS-base the quartet Y^A can be split into the commuting $y_\alpha, \bar{y}_{\dot\alpha}$ in terms of which

$$\mathtt{L}_{\alpha\beta} = -\frac{i}{4}\{y_\alpha, y_\beta\}\,, \qquad \mathtt{P}_{\alpha\dot\alpha} = -\frac{i}{4}\{y_\alpha, \bar{y}_{\dot\alpha}\}\,, \qquad \bar{\mathtt{L}}_{\dot\alpha\dot\beta} = -\frac{i}{4}\{\bar{y}_{\dot\alpha}, \bar{y}_{\dot\beta}\}\,. \tag{6.19}$$

In the conformal base we have (4.15). The $\mathtt{I}$-map acts either as $\mathtt{I}(y_\alpha, \bar{y}_{\dot\alpha}) = (-y_\alpha, \bar{y}_{\dot\alpha})$ or as $\mathtt{I}(y_\alpha, \bar{y}_{\dot\alpha}) = (y_\alpha, -\bar{y}_{\dot\alpha})$. In the conformal base it corresponds to $\mathtt{I}(a^\alpha, a^\dagger_\beta) = (a^\dagger_\alpha, a^\beta)$ or $\mathtt{I}(a^\alpha, a^\dagger_\beta) = (-a^\dagger_\alpha, -a^\beta)$. That there are two different realizations of the $\mathtt{I}$-map is in accordance with the existence of two independent cocycles, which was already deduced in Section 5 from the dual cycles. The double of this algebra is just two copies of the one for A_1:[37]

$$\{y_\alpha, k\} = 0\,, \qquad [\bar{y}_{\dot\alpha}, k] = 0\,, \qquad \{\bar{y}_{\dot\alpha}, \bar{k}\} = 0\,, \qquad [y_\alpha, \bar{k}] = 0\,. \tag{6.20}$$

The exact deformation of the double is given by two copies of the deformed oscillators:

$$[q_\alpha, q_\beta] = 2i\epsilon_{\alpha\beta}(1 + \mu k)\,, \qquad \{q_\alpha, k\} = 0\,, \qquad [\bar{q}_{\dot\alpha}, k] = 0\,, \tag{6.21}$$

$$[\bar{q}_{\dot\alpha}, \bar{q}_{\dot\beta}] = 2i\epsilon_{\dot\alpha\dot\beta}(1 + \bar{\mu}\bar{k})\,, \qquad \{\bar{q}_{\dot\alpha}, \bar{k}\} = 0\,, \qquad [q_\alpha, \bar{k}] = 0\,. \tag{6.22}$$

The reality conditions $q^\dagger_\alpha = \bar{q}_{\dot\alpha}$ imply that $\mu = \nu e^{i\theta}$, $\bar{\mu} = \nu e^{-i\theta}$ for real ν. Therefore, the deformed double of the higher spin algebra leads to two copies of the noncommutative super-sphere $S^{2|2} \times S^{2|2}$ that have the same (absolute) value of radii.

The deformed Lorentz and translation generators are given by the same formulae

$$\mathtt{L}_{\alpha\beta} = -\frac{i}{4}\{q_\alpha, q_\beta\}\,, \qquad \mathtt{P}_{\alpha\dot\alpha} = -\frac{i}{4}\{q_\alpha, \bar{q}_{\dot\alpha}\}\,, \qquad \bar{\mathtt{L}}_{\dot\alpha\dot\beta} = -\frac{i}{4}\{\bar{q}_{\dot\alpha}, \bar{q}_{\dot\beta}\}\,. \tag{6.23}$$

The algebra they form is consistent with the structure of the dual cycles (5.12)

$$[\mathtt{P}_{\alpha\dot\alpha}, \mathtt{P}_{\beta\dot\beta}] = (1 + \mu k)\epsilon_{\alpha\beta}\bar{\mathtt{L}}_{\dot\alpha\dot\beta} + (1 + \bar{\mu}\bar{k})\epsilon_{\dot\alpha\dot\beta}\mathtt{L}_{\alpha\beta}\,. \tag{6.24}$$

The deformation that is isomorphic to the one obtained by setting $d = 3$ in the free boson case corresponds to $\theta = 0$ and projection by $(1 + k\bar{k})/2$. Now, the question of uniqueness of the deformation described above is of physical significance since the resulting A_∞-algebra is supposed to describe the slightly broken higher spin symmetry realized in the Chern-Simons matter theories. The double D is a particular case of the smash-product algebras. The Hochschild cohomology of such algebras is known [89] and in our case the second Hochschild cohomology group is two-dimensional [45, 90]. Therefore, we exhaust all possible deformations.

[37] The same algebra appeared in [88] as $\mathcal{N} = 2$ super-HSA.

7 Concluding Remarks

One of the main results of the paper is the explicit construction of a class of A_∞-algebras that can be viewed as noncommutative deformation quantization of a given associative algebra A. We show that if A admits a deformation as an associative algebra, then we can replace the formal deformation parameter $\hbar$ by an element of A itself by going to A_∞ setting. It turns out that the structure maps m_n of the A_∞-algebra are completely determined (up to a natural equivalence) by the deformation of A.

One straightforward application of the above construction is to describe the slightly broken higher spin symmetry that is present in a class of CFT's at least in the large-N limit. The associative algebra here is any higher spin algebra which is associated with the infinite dimensional symmetries of the free field theory limit. The deformation parameter is the multiplet of higher spin currents. It can be shown that up to an inversion map this multiplet is isomorphic to the algebra itself. The inversion map plays an important role since the algebra A that admits a deformation is not the higher spin algebra itself, but its $\mathbb{Z}_2$-extension by the inversion. It can also be shown that the reason for the deformation to exist is (generically) a generalized free field in one dimension lower whose dimension turns out to be the deformation parameter $\hbar$.

A more general point of view on the deformation we have faced is that a higher spin algebra is just a particular case of quantization of the algebra of functions on a Poisson manifold M. In the higher spin case the manifold is symplectic and coincides with the closure of the nilpotent coadjoint orbit of $so(d,2)$, see [59]. A Poisson manifold may have some discrete symmetries, e.g. the inversion map that we used. Once we restrict ourselves to the subalgebra of functions that are invariant under these symmetries we can find new deformations. Geometrically, one can view the algebra of invariant functions as the algebra of functions on the quotient space of M by the action of the symmetry group G. In case the action is not free, the quotient space M/G has the structure of an orbifold rather than a smooth Poisson manifold. In [37], it was shown that, in addition to the usual Poisson structures, the orbifold M/G admits non-commutative Poisson structures associated with the fixed-points of the G-action. Applying the deformation quantization technique to these new non-commutative Poisson structures gives rise to new deformations. This is the case of higher spin algebras: in general we have a two-parameter family of algebras $\mathrm{A}_{\hbar_1,\hbar_2}$, where the first deformation parameter $\hbar_1$ (that was implicit in the paper) comes from the usual deformation quantization and $\hbar_2$ is due to the action of $\mathbb{Z}_2$-automorphisms having the origin in the space of y's as the only fixed-point.

The case of three dimensions is special and there are two additional deformations $A_{\hbar,\mu=\nu e^{i\theta},\bar{\mu}=\nu e^{-i\theta}}$, while $\hbar$ leads to the Weyl–Moyal product. The microscopical description of these $3d$ CFT's with slightly broken higher spin symmetry is via the Chern-Simons matter theories with the two parameters N and k (in the simplest situation). So far the deformation parameters θ and ν are just phenomenological. At least in the large-N limit it is possible [24, 91] to relate them to the microscopical parameters $\theta = \frac{\pi}{2}\frac{N}{k}$, $\nu \sim \tilde{N}^{-1}$, $\tilde{N} = 2N\frac{\sin \pi\lambda}{\pi\lambda}$. It is remarkable that the higher spin symmetry breaking in these theories is fully described by a (two copies) rather simple associative algebra of fuzzy super-sphere $S^{2|2}$. In particular, the correlation functions of the single-trace operators should be expressed simply in terms of the invariants of this algebra, similarly to the $\nu = \bar{\nu} = 0$ case.

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A First Order Deformation

The first-order deformation of an A_{-1}-bimodule A_0, regarded as an A_∞-algebra, is a collection of six tri-linear maps $m_3(\bullet,\bullet,\bullet)$ that have to obey:

$$-am_3(b,c,u) + m_3(ab,c,u) - m_3(a,bc,u) + m_3(a,b,cu) = 0\,, \tag{A.1a}$$

$$m_3(a,b,u)c - am_3(b,u,c) + m_3(ab,u,c) - m_3(a,bu,c) - m_3(a,b,uc) = 0\,, \tag{A.1b}$$

$$m_3(a,u,b)c - am_3(u,b,c) + m_3(au,b,c) + m_3(a,ub,c) - m_3(a,u,bc) = 0\,, \tag{A.1c}$$

$$m_3(u,a,b)c - m_3(ua,b,c) + m_3(u,ab,c) - m_3(u,a,bc) = 0\,, \tag{A.1d}$$

and

$$m_3(a,b,u)v - am_3(b,u,v) + m_3(ab,u,v) - m_3(a,bu,v) = 0\,, \tag{A.1e}$$

$$m_3(u,v,a)b - um_3(v,a,b) + m_3(u,va,b) + m_3(u,v,ab) = 0\,, \tag{A.1f}$$

$$m_3(a,u,b)v - am_3(u,b,v) + m_3(au,b,v) + m_3(a,ub,v) - m_3(a,u,bv) = 0\,, \tag{A.1g}$$

$$-m_3(u,a,v)b - um_3(a,v,b) - m_3(ua,v,b) + m_3(u,av,b) + m_3(u,a,vb) = 0\,, \tag{A.1h}$$

$$-m_3(a,u,v)b - am_3(u,v,b) + m_3(au,v,b) + m_3(a,u,vb) = 0\,, \tag{A.1i}$$

$$m_3(u,a,b)v - um_3(a,b,v) - m_3(ua,b,v) + m_3(u,ab,v) - m_3(u,a,bv) = 0\,, \tag{A.1j}$$

where a,b,c are elements of A_{-1} and $u,v,w \in A_0$. It is easy to see that (3.8) and (3.8) are solutions. These two solutions are equivalent via an A_∞ change of variables, which at this order is $m_3 \to m_3 + \delta f$, for $f(a,u) = \phi_1(a,u)$.

More generally, the first equation (A.1a) seems to be the most important one. Its solution correspond to the second Hochschild cohomology group $HH^2(A,\mathcal{M})$, where $\mathcal{M}$ is $\mathrm{Hom}(M,M)$ endowed with the natural bimodule structure (in our case $M \sim A_0$). If $A_{-1} \sim A_1$ is the polynomial Weyl algebra on two generators, then $HH^3(A_1,N) = 0$ for any bimodule N as the enveloping algebra A_1^e admits a projective resolution of length 2. This means that the deformations are unobstructed. The same is true for the matrix algebras $\mathrm{Mat}_n(A_1)$ acting on the bimodule $\mathrm{Mat}_n(A_1)$ (the algebras A_1 and $\mathrm{Mat}_n(A_1)$ are Morita equivalent).

B Sketch of the Proof

We need to check that m_n defined in Section 3 do solve the Maurer-Cartan equation

$$\delta m_n + \sum_{i+j=n+2} m_i \circ m_j = 0\,. \tag{B.1}$$

Due to the specific form of m_n (with arguments from A_{-1} on the left) there are fewer equations to be checked. Firstly, one can restrict oneself to the sector with three A_{-1} factors and $n-2$ factors in A_0, i.e., the arguments in (B.1) are permutations of $abcuv\dots w$. Secondly, the nontrivial equations can be parameterized by the position of c:

$$E_k(a,b,\dots,u,c,\overbrace{v,\dots,w}^{k}) = \delta m_n + \sum_{i+j=n+2} m_i \circ m_j\Big|_{a,b,\dots,u,c,v,\dots,w} = 0\,. \tag{B.2}$$

The differential δ is very simple for most of k's, $k = 1, \dots, n-3$:

$$\delta m_n(a, b, \dots, u, c, v, \dots, w) = -m_n(a, b, \dots, uc, v, \dots, w) + m_n(a, b, \dots, u, cv, \dots, w) \quad \text{(B.3)}$$

and contains four terms for the maximal $k = n-2$

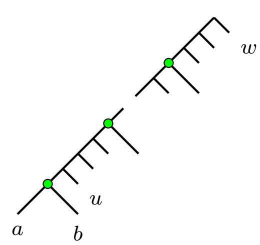
$$\begin{aligned} \delta m_n(a, b, c, v, \dots, w) = -a m_n(b, c, v, \dots, w) + m_n(ab, c, v, \dots, w) + \\ - m_n(a, bc, v, \dots, w) + m_n(a, b, cv, \dots, w)\,. \end{aligned} \quad \text{(B.4)}$$

The rationale for the recursive formula given in the main text is that the differential (B.3) annihilates those components of m_n that have too many multiplicative arguments on the right. Therefore, one can start at $k = 1$, to which only m_3 and m_{n-1} contribute:

$$\begin{aligned} E_1(a, b, \dots, u, c, w) = &- m_n(a, b, \dots, uc, w) + m_n(a, b, \dots, u, cv) + \\ &- m_{n-1}(a, b, \dots, m_3(u, c, w)) + m_3(m_{n-1}(a, b, \dots, u), c, v) = 0\,. \end{aligned}$$

This equation determines the part of f_n that has no multiplicative arguments at all. Using $m_{n-1} = f_{n-1}(a, b, \dots)u$ and explicit form of m_3, one observes that $f_n = \phi_1(f_{n-1}(a, b, \dots), u)$, i.e., $m_n = f_n(a, b, \dots, u)w$, up to the terms with more direct factors. Next, one should proceed to $k = 2$ and fix the part of f_n that has one direct factor. At each order one will get the equations that are supposed to be true for $m_{3,\dots,k+2}$. The trick here is that E_k contains Gerstenhaber products of $m_3, m_{n-1}, \dots, m_{k+2}, m_{n-k}$ and the lowest f_{n-k} always enters with the same arguments, i.e., can be treated as a single variable. Eventually, E_k can be reduced to equations for $m_3, \dots, m_{k+2}$ irrespective of n. For example, $E_1 = 0$ is solved by $f_n = \phi_1(f_{n-1}, \bullet)$ irrespective of what f_{n-1} is, but the same time this fixes the lowest component of f_{n-1} itself, and so on.

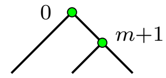
A non-recursive proof is based on manipulations with the trees. Let us recall that f_n is a sum over all terms that are depicted by trees (with one branch)[38]

$f_n(a, b, u, \dots, w) \ni$ 

(B.5)

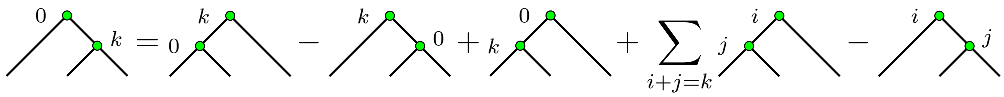
[38]Recall that the (green) dots correspond to some ϕ_{m+1}, while the simple vertices are mapped into insertions of multiplicative arguments on the right.

To deal with more complicated trees we introduce an order. A tree is called ordered if it does not contain vertices of the form



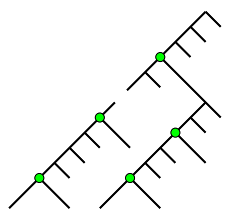
(B.6)

i.e., in the actual expression any $\phi_{m+1}(\bullet, \bullet)$ does not have any factors on the left, e.g. $a\phi_{m+1}(b, c)$, where a can be any expression possibly containing several factors and other ϕ's. The bad vertices can be ordered via



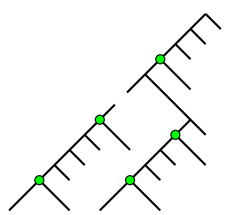
(B.7)

Equation E_k contains several terms, those coming from δm_n are already ordered (except for E_{n-2}). Also, only good vertices arise when m_i is inserted into m_j as an (right) argument of some ϕ_k:



(B.8)

The only source of bad vertices is when m_i in inserted into an argument of m_j that corresponds to a multiplicative argument (simple vertex):



(B.9)

These terms need to be reordered and will eventually generate (with the opposite sign) all the trees with one branch (B.5) or two branches (B.8) that are already present. Therefore, the Maurer–Cartan equation is indeed satisfied. It would be interesting to find an appropriate configuration space where the proof would reduce to the Stokes theorem.

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