A_∞ algebras from slightly broken higher spin symmetries

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ABSTRACT: We define a class of A_∞-algebras that are obtained by deformations of higher spin symmetries. While higher spin symmetries of a free CFT form an associative algebra, the slightly broken higher spin symmetries give rise to a minimal A_∞-algebra extending the associative one. These A_∞-algebras are related to non-commutative deformation quantization much as the unbroken higher spin symmetries result from the conventional deformation quantization. In the case of three dimensions there is an additional parameter that the A_∞-structure depends on, which is to be related to the Chern-Simons level. The deformations corresponding to the bosonic and fermionic matter lead to the same A_∞-algebra, thus manifesting the three-dimensional bosonization conjecture. In all other cases we consider, the A_∞-deformation is determined by a generalized free field in one dimension lower.

Contents

1 Introduction

Strong homotopy algebras (SHA), which are also dubbed A_∞ and L_∞ for the cases generalizing associative and Lie algebras, are general enough structures that abstract and formalize what it means to be algebraically consistent in a broad sense. No wonder that many of physical problems can be cast into the framework of SHA, like string field theory [1–5] and the BV-BRST theory of gauge systems [6–9]. Even some problems that are seemingly unrelated to any SHA admit natural solutions by translating them to the SHA setup, e.g. the deformation quantization of Poisson manifolds [10]. In the present paper, we use the language of SHA in order to describe the slightly broken higher spin symmetries [11–19] that govern certain nontrivial conformal field theories (CFT) at least in the large-N limit.

It is well known that free CFT's have vast symmetries — *higher spin symmetries* — that extend the conformal Lie algebra to infinite-dimensional associative algebras, called

higher spin algebras in this context. Higher spin symmetries are related to higher spin currents, i.e., conserved tensors that are bi-linear in the free fields, e.g. $J_s = \phi \partial^s \phi + \ldots$ for the free scalar field ϕ, with the stress-tensor being just the $s = 2$ member of the multiplet. On the other hand, higher spin symmetries generated by abstract conserved tensors J_s are powerful enough as to fix all the correlation functions [20–23] and these turn out to be necessarily given by a free CFT. Therefore, unbroken higher spin symmetries are in one-to-one correspondence with free CFT's. Still it is interesting that the correlators of J's can be directly computed as invariants of higher spin symmetries [24–28].

The structure of interacting CFT's is much more complicated. If a given CFT admits a weakly-coupled limit, which is not necessarily free in terms of the fundamental constituents, then one can think of such a CFT as enjoying a *slightly broken higher spin symmetry*, the term coined in [11]. In particular, the higher spin currents J_s are not conserved anymore, but their conservation is broken in a very specific way. The examples of main interest include the critical vector model, the Gross-Neveu model and, more generally, the Chern-Simons matter theories in the large-N limit. The last class of models has recently been conjectured to exhibit a number of interesting dualities [11, 29–33], in particular the three-dimensional bosonization duality. Our expectation is that the dualities can be explained by the slightly broken higher spin symmetries [11] and that this symmetry makes the models exactly soluble, at least in the large-N limit. The purpose of the paper is (i) to define what a slightly broken higher spin symmetry means in mathematical terms since it is not strictly speaking a symmetry in the conventional sense; (ii) to provide an explicit construction; (iii) to explore the simplest consequences including applications to the bosonization duality.

To begin with we would like to stress that higher spin algebras are typically rigid, that is, admit no deformations. Indeed, in $d > 2$ the free CFT's are isolated points and do not form continuous families.[1] The opposite conclusion is also true: higher spin symmetries are the symmetries of free CFT's in $d > 2$ [20–23]. Therefore, a slightly broken higher spin symmetry is not about deformation of higher spin algebras as associative (or Lie) algebras. Due to the specific way that higher spin currents J_s fail to be conserved, our proposal is that such deformations fall into the class of A_∞-algebras we construct.

The A_∞-algebras that describe slightly broken higher spin symmetries are still related to associative algebras and deformations thereof. The class of relevant A_∞-algebras may be of some interest by itself, being closely related to the so-called *noncommutative deformation quantization* [39, 40]. In a few words, suppose that we have an associative (in general non-commutative) algebra and, furthermore, that the product can be deformed as

$$a * b = ab + \phi(a,b)\hbar + \cdots , \tag{1.1}$$

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[1]There is a one-parameter family of algebras in $4d$ [34–37], the free parameter λ being helicity of a free $4d$ conformal field. For non-(half)integer values of the parameter these algebras do not have any natural spacetime and CFT interpretation and for $|\lambda| > 1$ there is no local stress-tensor as well. Therefore, only $|\lambda| = 0, \frac{1}{2}, 1$ correspond to free CFT's. Also, there is a one-parameter family of algebras relevant for higher spin theories in AdS_3 whose dual CFT's are W minimal models [38]. Lastly, $\mathcal{N} = 4$ SYM forms a continuous family of CFT's approaching the free limit at $g = 0$. However, for $g \neq 0$ the higher spin currents are not conserved and, therefore, SYM does not enjoy higher spin symmetry for $g \neq 0$.

$\hbar$ being a formal deformation parameter. Therefore, we have a one-parameter family of algebras $A_\hbar$. Here ϕ is a Hochschild 2-cocycle. Construction and classification of such deformations in the case of algebras of smooth functions on Poisson manifolds is the standard problems of deformation quantization that have been solved by Kontsevich [10] using a string-inspired construction. The deformation problem we are lead to is to promote the formal parameter $\hbar$ to an element of the algebra itself. This clearly has no sense in the realm of usual associative algebras. The idea is to go to the category of A_∞-algebras where it is legitimate to replace $\phi(a,b)\hbar$ with a tri-linear map $m_3(a,b;u)$, with the 'deformation parameter' u being now an element of the same algebra. The correspondence principle requires $m_3(a,b;\hbar) = \phi(a,b)\hbar$. This is only the starting point and all the higher structure maps $m_n(a,b;u,\ldots,w)$ are to be constructed. Taken together, the m's obey the Maurer-Cartan equation and this amounts to defining an A_∞-algebra. One of our results is that the A_∞-structure maps m_n can all be expressed through ϕ and the other coefficients in expansion (1.1). To summarize, given a one-parameter family $A_\hbar$ of associative algebras we can explicitly construct an A_∞-algebra that can be viewed as a non-commutative deformation quantization of $A_\hbar$ at $\hbar = 0$:

$$
\begin{array}{ccc}
\text{one-parameter family of} & \xrightarrow{\ \text{Section 3}\ } & \text{strong homotopy algebra} \\
\text{associative algebras } A_\hbar & & \text{extending } A_0
\end{array}
$$

The last step is to establish a relation between the physical realization of the slightly broken higher spin symmetry in vector models and the abstract construction above. When the higher spin symmetry is slightly broken by interactions, the higher spin currents J_s are no longer conserved. Nevertheless, the non-conservation of J_s has a very specific form of double trace operators built out of higher spin currents themselves

$$\partial \cdot J = g\,[JJ]\,, \tag{1.2}$$

where g is a small parameter of order $1/N$. In a sense, higher spin currents are responsible for their own non-conservation. A remarkable fact is that the multiplet of higher spin currents is isomorphic to the higher spin algebra itself up to a $\mathbb{Z}_2$-twist generated by the inversion map (section 4.3). Having in mind this correspondence we should be looking for a deformation of the higher spin algebra which is controlled by another element of the algebra (up to the inversion map). Thus, the $\mathbb{Z}_2$-extension of a given higher spin algebra by the inversion map is a useful object to incorporate both the algebra and the higher spin currents. We prove that the $\mathbb{Z}_2$-extended associative algebras are proved to admit at least a one-parameter family of deformations, which we call *deformed higher spin algebras*. It is this deformation that is plugged into the general construction of A_∞-algebras just described. The algorithm for constructing the strong homotopy algebra description of the slightly broken higher spin symmetry starting from any higher spin algebra hs looks as follows:

$$
\begin{array}{ccccccc}
hs & \xrightarrow{\ \text{extension}\ } & hs \rtimes \mathbb{Z}_2 & \xrightarrow{\ \text{deformation}\ } & \text{deformed higher} & \longrightarrow & \text{strong homotopy algebra} \\
\text{(rigid)} & & \text{(soft)} & & \text{spin algebra} & &
\end{array}
$$

In words, hs is rigid, but its $\mathbb{Z}_2$-extension, which is needed to incorporate the higher spin currents, is soft and can be deformed into an at least one-parameter family of associative algebras. Lastly, one can use our construction of the strong homotopy algebras out of a one-parameter family of associative algebras (non-commutative deformation quantization).

We also study particular examples of this construction. As is mentioned, higher spin algebras contain the conformal algebra $so(d,2)$ as a Lie subalgebra and have the full information about the spectrum of higher spin currents including the correlation functions. Therefore, they crucially depend on dimension d and on a type of a free CFT they originate from. Of special interest is the case of three dimensions. Here the structure of the higher spin symmetry breaking is richer than in higher dimensions. Microscopically, this happens due to the presence of an additional parameter related to the level k of the Chern-Simons matter theories. In $d \neq 3$ the vector models have a single parameter N.[2] The structure of correlation functions is also more complicated with certain parity-odd structures contributing to it [11, 14, 41]. More importantly, the Chern-Simons matter theories with bosonic and fermionic matter seem to describe the same physics and this has lead to the conjecture of the three-dimensional bosonization and related ones [11, 29–33].
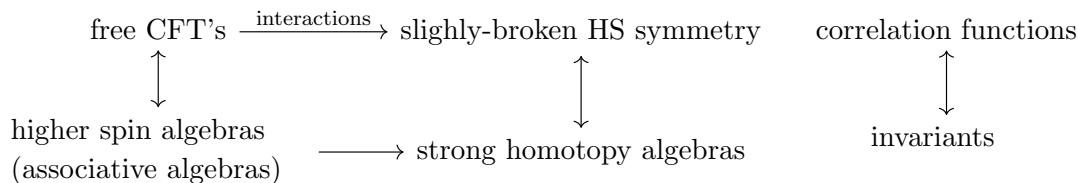
Concerning the three-dimensional bosonization duality, the first observation is that the higher spin algebras of $3d$ free boson and $3d$ free fermion CFT's are isomorphic, which is not the case when $d > 3$. This implies that they have to lead to the same A_∞-algebra. Secondly, the deformation that leads to A_∞-algebra is characterized by the second Hochschild cohomology and it turns out to be two-dimensional in $d = 3$, while it is one-dimensional in $d > 3$. The additional parameter is to be associated with the t'Hooft coupling $\lambda = N/k$. This provides a good evidence for the conjecture to the leading order in $1/N$. The correlation functions should be given by the invariants we discuss at the very end. The appearance of the additional parameter is a feature of the A_∞-algebra, i.e., of the slightly broken higher spin symmetry, and is not seen in the free limit governed by the higher spin algebra.

To summarize, the unbroken higher spin symmetry is powerful enough[3] in $d \geq 3$ as to fix all correlation functions. This property is expected to extend to the more interesting case of the slightly broken higher spin symmetry that underlies a number of nontrivial CFT's at least in the large-N limit. While the former is governed by associative higher spin algebras, the latter leads to the A_∞-algebras we propose in the paper. These A_∞-algebras still are fully controlled by *deformed higher spin algebras*. The invariants of these algebras should give the correlation functions, much as they do for the usual higher spin algebras. The relation between some of the structures on the physics and mathematical

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[2]Note that the breaking of higher spin symmetry in $\mathcal{N} = 4$ SYM is different from the one in vector models: the non-conservation equation does not have the form (1.2). It is the special form of (1.2) that makes higher spin currents close onto themselves at least in the large-N limit. While it would also be interesting to study the breaking of higher spin symmetries in models of matrix type, like SYM and ABJ, we restrict ourselves to models of vector type and use the term *slightly broken higher spin symmetry* for vector models only, as introduced in [11].

[3]The restriction to $d > 2$ is important, see e.g. [20], since this result does not hold in $d = 2$ and higher spin algebras do not seem to work in $2d$ CFT's the way they do in $d > 2$.

sides is illustrated by the following diagrams:

$$\text{free CFT's} \xrightarrow{\text{interactions}} \text{slighly-broken HS symmetry} \qquad \text{correlation functions}$$

$$\updownarrow \qquad\qquad\qquad\qquad \updownarrow \qquad\qquad\qquad\qquad \updownarrow$$

$$\begin{array}{l} \text{higher spin algebras} \\ \text{(associative algebras)} \end{array} \quad \longrightarrow \quad \text{strong homotopy algebras} \qquad \text{invariants}$$

The rest of the paper is organized as follows. We begin in section 2 with the definition of A_∞-algebras in terms of the Gerstenhaber bracket. In section 3, we define and construct a class of A_∞-algebras that can be thought of as non-commutative deformation quantization of associative algebras. Various definitions and examples of higher spin algebras are recalled in section 4. In section 5 we define the A_∞-algebra of the slightly broken higher spin symmetry. Some explicit oscillator realizations of these deformations are discussed in section 6. Conclusions are in section 7. Several appendices are devoted to more technical aspects, in particular in appendix B we prove that certain simple extensions of higher spin algebras admit deformations.

2 A_∞ algebras

There are several equivalent definitions of A_∞-algebras: (i) via Stasheff's relations [42]; (ii) via a nilpotent coderivation on the tensor coalgebra of the suspended graded algebra and (iii) via the Gerstenhaber bracket. Throughout the paper we will exclusively use the last one.

Let V be a $\mathbb{Z}$-graded vector space $V = \bigoplus_k V^k$. Consider the space $Hom(TV, V)$ of all maps from the tensor algebra $TV = \bigoplus_n T^n V$ of V to the space V itself. The element of $Hom(T^n V, V)$, called n-cochains, are multilinear functions $f(a_1, a_2, \ldots, a_n)$ on V. The $\mathbb{Z}$-grading on V induces that on $Hom(TV, V)$; by definition,

$$|f| = |f(a_1, a_2, \ldots, a_n)| - \sum_{k=1}^{n} |a_k|.$$

The $\circ$-product of an n-cochain f and an m-cochain g is a natural operation that nests one map into the other with the usual Koszul signs

$$(f \circ g)(a_1 \otimes a_2 \otimes \cdots \otimes a_{m+n-1}) =$$
$$= \sum_{i=0}^{n-1} (-1)^{|g| \sum_{j=1}^{i} |a_j|} f(a_1 \otimes \cdots \otimes a_i \otimes g(a_{i+1} \otimes \cdots \otimes a_{i+m}) \otimes \cdots \otimes a_{m+n-1}). \tag{2.1}$$

It should be noted that the $\circ$-product is non-associative. Nevertheless, the following bracket, called *Gerstenhaber bracket*,

$$[\![f, g]\!] = f \circ g - (-1)^{|f||g|} g \circ f, \tag{2.2}$$

is graded skew-symmetric and obeys the Jacobi identity:

$$[\![f, g]\!] = -(-1)^{|f||g|}[\![g, f]\!], \qquad [\![[\![f, g]\!], h]\!] = [\![f, [\![g, h]\!]]\!] - (-1)^{|f||g|}[\![g, [\![f, h]\!]]\!]. \tag{2.3}$$

Given a $\mathbb{Z}$-graded space V and a sum $m = m_1 + m_2 + \cdots$ of degree-one maps $m_n : T^n V \to V$, the A_∞-structure is defined simply as a solution to the Maurer-Cartan equation:

$$\llbracket m, m \rrbracket = 0 \,. \tag{2.4}$$

Upon expansion $m = m_1 + m_2 + \cdots$ the first few relations have a simple interpretation: m_1 is a differential, $m_1 m_1 = 0$; m_2 is a bi-linear product differentiated by m_1 by the graded Leibniz rule

$$-m_1 m_2(a, b) = m_2(m_1(a), b) + (-)^{|a|} m_2(a, m_1(b)) \,. \tag{2.5}$$

However, m_2 is not associative in general, associativity is true up to a coboundary controlled by m_3:

$$m_2(m_2(a, b), c) + (-)^{|a|} m_2(a, m_2(b, c)) + m_1 m_3(a, b, c) + m_3(m_1(a), b, c) +$$
$$+ (-)^{|a|} m_3(a, m_1(b), c) + (-)^{|a| + |b|} m_3(a, b, m_1(c)) = 0 \,.$$

NB: it is common in the literature to define A_∞-algebras via maps on the suspension $V[1]$ of the corresponding graded space V. Then m_n has degree $2 - n$. We prefer to prepare the 'experimental setup' in such a way that V is already suspended. This prevents appearance of many sign factors and all m_n have now degree one. For example, an associative algebra A is understood as a graded algebra with the only nonzero component leaving in degree -1, so that multiplication is a degree-one map m_2 taking $A_{-1} \otimes A_{-1}$ to A_{-1}. As a consequence of such a degree assignment the associativity condition has the right form

$$\llbracket m_2, m_2 \rrbracket(a, b, c) = 2 m_2(m_2(a, b), c) - 2 m_2(a, m_2(b, c)) = 0 \,.$$

Certain A_∞-algebras deserve their own names. *Minimal A_∞-algebras* do not have the lowest map m_1, i.e., differential. Such algebras arise naturally when passing to the cohomology $H(m_1)$ of m_1 and dragging the A_∞-structures there, the resulting algebras are called *minimal models*, see e.g. [4]. *Differential graded algebras* (DGA) have only m_1 and m_2, i.e., a differential and a bi-linear product that respect the Leibniz rule.

Note that for a genuine A_∞-structure to arise it is necessary that V has more than one graded component due to the degree requirement. The only possibility with just one nontrivial component is $V = V_{-1}$, then m_2 is just an associative product on V.

3 A_∞ from deformations of associative algebras

In this section, we construct an A_∞-algebra out of a one-parameter family of associative algebras. Even though the construction is inspired by the study of the slightly broken higher spin symmetry, it is quite general and may be of independent interest as a new way to build a large class of A_∞-algebras. There are certain special properties of HSA that allows one to describe the corresponding A_∞-algebras in more detail and there are tools to explicitly construct them, which will be discussed in sections 6 and appendix B. Throughout this section, we let A denote any associative algebra.

Given an associative algebra A, it is clear that due to the restrictions imposed by the grading, there cannot be any interesting A_∞-structure on it; the only possibility is to deform A itself as an associative algebra. We define the A_∞-structure perturbatively and the first step is to extend A by any its bimodule M; in so doing, A and M are prescribed the degrees -1 and 0, respectively. At the lowest order the A_∞-structure is simply equivalent to the definitions above: there is only m_2 that is defined for various pairs $A_{-1} \otimes A_{-1}$ (the A product), $A_{-1} \otimes A_0$ (the left action of A on M), $A_0 \otimes A_{-1}$ (the right action of A on M). All these conditions are summarize by the Stasheff identity:

$$m_2(m_2(a, b), c) + (-)^{|a|} m_2(a, m_2(b, c)) = 0 \qquad \Longleftrightarrow \qquad [\![m_2, m_2]\!] = 0. \qquad (3.1)$$

Denoting elements of A_{-1} by $a, b, \dots$ and elements of A_0 by $u, v, \dots$ we have[4]

$$m_2(a, b) = ab, \qquad m_2(a, u) = au, \qquad m_2(u, a) = -ua, \qquad m_2(u, v) = 0. \qquad (3.2)$$

Now one tries to deform this rather trivial A_∞-structure and the first-order deformations $m^{(1)}$ can be described in terms of the Hochschild cohomology of A. Introducing the Hochschild differential $\delta = [\![m_2, \bullet]\!]$, one can identify the nontrivial first-order deformations $m^{(1)}$ with the nontrivial δ-cocycles,

$$\delta m^{(1)} = 0 \qquad \Longleftrightarrow \qquad [\![m_2, m^{(1)}]\!] = 0. \qquad (3.3)$$

In other words, the space of infinitesimal deformations is identified with the δ-cohomology in degree 1, while the second δ-cohomology group is responsible for possible obstructions to deformation.

The first-order deformation should have the form $m^{(1)} = m_3(\bullet, \bullet, \bullet)$ with arguments from A_{-1} and A_0. Various homogeneous components of $\delta m_3 = 0$ are collected in appendix A, while the first and the last ones are:

$$-a m_3(b, c, u) + m_3(ab, c, u) - m_3(a, bc, u) + m_3(a, b, cu) = 0, \qquad (3.4)$$

$$\dots = 0, \qquad (3.5)$$

$$m_3(u, a, b)v + u m_3(a, b, v) + m_3(ua, b, v) + m_3(u, ab, v) - m_3(u, a, bv) = 0. \qquad (3.6)$$

For any associative algebra A there is at least one natural bimodule, that is, A itself. Let us take A_0 to be A, in which case the deformation can be described in more detail. If A admits a deformation as an associative algebra, then the second Hochschild cohomology group is nonzero, $HH^2(A, A) \neq 0$. Given an element $[\phi] \in HH^2(A, A)$ represented by a cocycle ϕ, the standard deformation of the associative structure reads

$$a * b = ab + \phi(a, b)\hbar + \mathcal{O}(\hbar^2), \qquad (3.7)$$

where the deformation parameter $\hbar$ can live in the base field or even in the center of A. If the deformation is unobstructed, we can construct a one-parameter family of algebras $A_\hbar$ that starts at A for $\hbar = 0$. When $A_0 \sim A$ the A_∞-algebra we are trying to construct

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[4]The left/right action is denoted by multiplication, au and ua.

upgrades the deformation parameter $\hbar$ to an element of A_0. The observation is that for $A_0 \sim A$ one can always put

$$m_3(a,b,u) = \phi(a,b)u\,, \qquad\qquad m_3(a,u,v) = \phi(a,u)v\,, \qquad (3.8\text{a})$$
$$m_3(a,u,b) = 0\,, \qquad\qquad m_3(u,a,v) = -\phi(u,a)v\,, \qquad (3.8\text{b})$$
$$m_3(u,a,b) = 0\,, \qquad\qquad m_3(u,v,a) = 0\,. \qquad (3.8\text{c})$$

Here the 'deformation parameter' $u \in A$ was placed on the right in $m_3(a,b,u)$. It is also possible to place it on the left

$$m_3(a,b,u) = 0\,, \qquad\qquad m_3(a,u,v) = 0\,, \qquad (3.9\text{a})$$
$$m_3(a,u,b) = 0\,, \qquad\qquad m_3(u,a,v) = u\phi(a,v)\,, \qquad (3.9\text{b})$$
$$m_3(u,a,b) = u\phi(a,b)\,, \qquad\qquad m_3(u,v,a) = -u\phi(v,a)\,. \qquad (3.9\text{c})$$

For u in the base field (or more generally in the center of A) the left $u\phi(a,b)$ and the right $\phi(a,b)u$ deformations are clearly equivalent. This property extends to the A_∞-structure, namely, the left and right deformations differ from each other by a trivial deformation $m_3 = \delta g$, where $g(a,u) = \phi(a,u)$.

The A_∞-algebra we are constructing extends the deformation parameter $\hbar$ to an element of A_0, which may be the algebra itself (or its bimodule). This is usually referred to as deformation with noncommutative base. If such an A_∞-algebra can be constructed, it admits a truncation where A_0 is replaced by the center $Z(A)$, or just by $\hbar$, that is closely related to the one-parameter family of algebras $A_\hbar$.

3.1 Explicit construction

The central statement of the present paper is that the A_∞-structure of the previous section, is fully determined by the deformation of the underlying associative algebra. Assuming that the deformed product

$$a * b = ab + \sum_{k>0} \phi_k(a,b)\hbar^k \qquad (3.10)$$

is known, we give an explicit formula for all m_n. The defining relation for the A_∞-structure, i.e., the Maurer-Cartan equation

$$[\![m,m]\!] = 0 \qquad\qquad \Longleftrightarrow \qquad\qquad \delta m_n + \sum_{i+j=n+2} m_i \circ m_j = 0\,, \qquad (3.11)$$

is satisfied as a consequence of the associativity of the deformed product

$$a * (b * c) - (a * b) * c = 0 \qquad\qquad \Longleftrightarrow \qquad\qquad \delta\phi_n + \sum_{i+j=n-1} \phi_i \circ \phi_j = 0\,. \qquad (3.12)$$

Here $\delta = [\![m_2, \bullet]\!]$ is the Hochschild differential associated to the undeformed product (3.2). We have three equivalent forms of m_n: recursive, in terms of binary trees, and through generating equations. Let us discuss them in order.

In general, there are two types of ambiguities in the definition of m_n. (i) As usual in deformation quantization, one can redefine the deformed product $*$ via a linear change of variables $a \to D(a) = a + \sum_k D_k(a)\hbar^k$. Then, the new product is given by $D(D^{-1}(a) * D^{-1}(b))$. (ii) One can perform various redefinitions at the level of A_∞-structure, which is done by exponentiating the infinitesimal gauge transformation

$$\dot{m}(t) = [\![m(t), \xi]\!] , \qquad\qquad m(0) = m , \qquad\qquad (3.13)$$

for some cochain ξ of degree zero. The A_∞ gauge transformations are more general than redefinitions of the associative product. We have observed that the A_∞-transformations allow one to cast the first-order deformation into the right form (with all, or all but one, A_0-factors staying on the right):

$$m_3(a, b, u) = f_3(a, b)u , \quad m_3(a, u, v) = f_3(a, u)v , \quad m_3(u, a, v) = -f_3(u, a)v , \quad (3.14)$$

and all other orderings of a, b, u, v in m_3 give zero result. Here $f_3(a, b) = \phi_1(a, b)$ is determined by the first-order deformation in (3.10). The full solution can be sought for in a similar form:

$$m_n(a, b, u, \ldots, v) = +f_n(a, b, u, \ldots)v , \qquad\qquad (3.15)$$
$$m_n(a, u, \ldots, v, w) = +f_n(a, u, \ldots, v)w , \qquad\qquad (3.16)$$
$$m_n(u, a, \ldots, v, w) = -f_n(u, a, \ldots, v)w . \qquad\qquad (3.17)$$

Therefore, the problem is reduced to defining one function f_n of $(n-1)$ arguments per each set of structure maps m_n with only three orderings being nontrivial. It is not hard to see that the equation for m_4, $\delta m_4 + m_3 \circ m_3 = 0$, is solved by

$$f_4(a, b, u) = \phi_2(a, b)u + \phi_1(\phi_1(a, b), u) . \qquad\qquad (3.18)$$

At the next order we have to solve $\delta m_5 + m_3 \circ m_4 + m_4 \circ m_3 = 0$, which is satisfied by

$$f_5(a, b, u, v) = \phi_1(\phi_1(\phi_1(a, b), u), v) + \phi_2(\phi_1(a, b), u)v + \phi_1(\phi_2(a, b), u)v + \\ + \phi_1(\phi_2(a, b)u, v) + \phi_3(a, b)uv . \qquad\qquad (3.19)$$

The following graphical representation can be useful. We consider planar binary trees with vertices labelled by $0, 1, 2, \ldots$. A vertex with label k corresponds to ϕ_k and the two incoming edges correspond to the arguments. Functions f_3, f_4 and f_5 can then be depicted as

Solution, recursive formula. In order to write down a recursive formula for f_n let us introduce some further notation. It is clear that any f_n can be decomposed according to the number of the multiplicative arguments on the right:

$$f_n(a,b,u,\ldots,v,w) = f_{n,0}(a,b,u,\ldots,v,w) + f_{n,1}(a,b,u,\ldots,v)w + f_{n,2}(a,b,u,\ldots)vw + \ldots.$$

There is an associated filtration, where the leftover $r_{n,k}$ contains all the terms in the decomposition with at least k multiplicative arguments on the right:

$$f_n(a,b,\ldots,v,w) \equiv r_{n,0}(a,b,\ldots,v,w) \tag{3.20a}$$

$$= f_{n,0}(a,b,\ldots,v,w) + r_{n,1}(a,b,\ldots,v)w \tag{3.20b}$$

$$= f_{n,0}(a,b,\ldots,v,w) + f_{n,1}(a,b,\ldots,v)w + r_{n,2}(a,b,\ldots)vw, \text{ etc.} \tag{3.20c}$$

Our claim is that all f_n are obtained by means of the following recursive relations:[5]

$$f_{n,0} = \phi_1(r_{n-1,0}, \bullet), \tag{3.21a}$$

$$f_{n,1} = \phi_2(r_{n-2,0}, \bullet) + \phi_1(r_{n-1,1}, \bullet), \tag{3.21b}$$

$$f_{n,2} = \phi_3(r_{n-2,0}, \bullet) + \phi_2(r_{n-2,1}, \bullet) + \phi_1(r_{n-3,0}, \bullet), \tag{3.21c}$$

$$\cdots \tag{3.21d}$$

$$f_{n,k} = \sum_{i=0}^{i=k} \phi_{k-i+1}(r_{n-k+i-1,i}, \bullet). \tag{3.21e}$$

The formulae above together with the initial condition $f_3 = \phi_1$ allow one to reconstruct the A_∞-structure, m_n, in terms of the bi-linear maps ϕ_k defining the $*$-product (including the initial product $\phi_0(a,b) = ab$).

While f_n's are, in general, quite complicated functions with nested ϕ_k, there are some general properties that are easy to see. (a) The first and the last terms in f_n are of the form

$$f_n(a,b,u,\ldots,v,w) = \phi_1(\phi_1(\ldots(\phi_1(a,b),u),\ldots,v),w) + \cdots + \phi_{n-2}(a,b)u\ldots vw. \tag{3.22}$$

The presence of the last term is obvious as for $u,\ldots,v,w$ in the base field the deformation should reduce to the deformed product[6]

$$f_n(a,b,\hbar,\ldots,\hbar,\hbar) = \phi_{n-2}(a,b)\hbar^{n-2}. \tag{3.23}$$

(b) The graphs that show up in the decomposition of f_n are all left-aligned, i.e., are the simplest ones with all edges emerging from just one branch on the left. Such graphs can be parameterized by a sequence of numbers listing the indices of the vertices when read from left to right, e.g. $(2,0)$ and $(1,1)$ for f_4. Such a simple form is the consequence of a particular A_∞ gauge we chose. By performing an A_∞ gauge transformations one can arrive at various other forms. In particular, there exists the right-aligned form, which is obtained by reflection of the graphs. (c) All graphs contributing to f_n have the total weight $n-2$, where the weight is the sum over the indices of the vertices in a graph. (d) Not all possible left-aligned graphs with a correct weight contribute to the expansion of f_n. All admissible graphs enter with multiplicity one.

---

[5]It is useful to define $f_2 = r_{2,0}$ as the identity map.

[6]We should assume here that the deformation of the product is properly normalized, $\phi_k(a,1) = 0$.

Solution, explicit formula. Instead of the recursive definition given above it is possible to describe the set of trees that contribute to f_n in a more direct way. This is easier to do in terms of the sequences of natural numbers

$$(m_k, l_k, \ldots, m_1, l_1) \tag{3.24}$$

that correspond to the trees encoded by the weights

$$m_k + 1, \overbrace{0, \ldots, 0}^{l_k}, m_{k-1} + 1, 0, \ldots, 0, m_2 + 1, \overbrace{0, \ldots, 0}^{l_2} m_1 + 1, \overbrace{0, \ldots, 0}^{l_1}, \tag{3.25}$$

or, pictorially,

$$f_n(a, b, u, \ldots, w) \ni \qquad \tag{3.26}$$

Here the edges corresponding to the multiplicative arguments on the right are drawn a bit shorter. Some of the arguments are displayed.

Equivalently, every such sequence corresponds to the expression

$$\phi_{m_1+1}\left(\ldots \phi_{m_{k-1}+1}(\phi_{m_k+1}(\mathbb{1}, \mathbb{1})\mathbb{1}^{l_k}, \mathbb{1})\mathbb{1}^{l_{k-1}}, \ldots, \mathbb{1}^{l_2}, \mathbb{1}\right)\mathbb{1}^{l_1}(a_0 \otimes \ldots \otimes a_{n-2}), \tag{3.27}$$

where $\mathbb{1}$ is the identity map. In this notation l_i is the number of the multiplicative arguments on the right at level i and m_i stands for the insertion of ϕ_{m_i+1}. Now we need to specify which of the sequences or trees are admissible. They satisfy

$$
\begin{aligned}
&l_1 \in [0, n-2-k], && m_1 \in [0, l_1], \\
&l_2 \in [0, n-2-k-l_1], && m_2 \in [0, l_1 + l_2 - m_1], \\
&\quad \ldots && \\
&l_k \in [0, n-2-k-l_1-\cdots-l_{k-1}], && m_k \in [0, l_1+\ldots+l_k-m_1-\cdots-m_{k-1}].
\end{aligned}
$$

Equivalently, all the terms (trees) contributing to f_n can be enumerated via pairs of Young diagrams. One should write down all possible Young diagrams with the first row of length $n-2-k$ for all $k = 1, \ldots, n-2$ and with k rows. Given such a diagram, one should write down all possible subdiagrams such that the first row is of the same length $n-2-k$. Any such pair of Young diagrams gives a sequence of l_i and m_i that are admissible. Some of l_i and m_i can be zero, provided that the Young diagram is a proper one (the length of the

rows is nondecreasing upwards). Pictorially, such pairs look as follows:

$$(3.28)$$

For example, the pair of empty diagrams $(\bullet, \bullet)$ means $k = n - 2$, $l_{1,\ldots,k} = m_{1,\ldots,k} = 0$ and corresponds to

$$\phi_1(\ldots \phi_1(\phi_1(a_0, a_1), a_2), \ldots, a_{n-2}).$$ (3.29)

The one-row Young diagram of length $n - 3$ implies that $k = 1$, $m_1 = l_1 = n - 3$ and corresponds to

$$\phi_{n-2}(a_0, a_1) a_2 \ldots a_{n-2}.$$ (3.30)

In this language the expansions for f_4, f_5 and f_6 can be written as

$$f_4 = (\bullet, \bullet) \oplus (\square, \square),$$
$$f_5 = f_4 \oplus (\square\square, \square\square) \oplus \left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\right),$$
$$f_6 = f_5 \oplus (\square\square\square, \square\square\square) \oplus (\square\square\square, \square\square\square) \oplus (\square\square\square, \square\square) \oplus (\square\square, \square\square) \oplus (\square\square, \square\square\square) \oplus (\square\square, \square\square)$$
$$\oplus \left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}, \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}, \begin{smallmatrix}\square\\\square\end{smallmatrix}\right) \oplus \left(\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}, \square\right).$$

Solution, generating equation. A combinatorial proof that the two forms above do solve our problem is sketched in appendix C. Nevertheless, it is desirable to get all m's in a way that makes their existence obvious. To this end, we recall the construction of braces, which were first introduced in [43] (see also [44, 45]). A k-brace is a multi-linear map that assigns to any set of $k + 1$ Hochschild cochains $f, g_1, \ldots g_k$ a new cochain $f\{g_1, \ldots, g_k\}$ defined by the rule

$$f\{g_1, \ldots, g_k\}(a_1, \ldots) = \sum \pm f(a_1, \ldots, g_1(\ldots), \ldots, g_2(\ldots), \ldots, g_k(\ldots), \ldots).$$ (3.31)

Here the cochains g_i are inserted as arguments into the cochain f and the sum is over all unshuffles (i.e., the order of g_i is preserved) with natural signs (whenever g_i has to jump over a_j an obvious sign $(-)^{|g_i||a_j|}$ is generated). For $k = 1$ we get the Gerstenhaber $\circ$-product (2.1), that is, $f\{g\} = f \circ g$.

As was shown in [44], any A_∞-structure m on V can be lifted to an A_∞-structure $M = M_1 + M_2 + \cdots$ on the space of Hochschild cochains $Hom(TV, V)$ by setting

$$M_1(g_1) = [\![m, g_1]\!], \qquad M_k(g_1, \ldots, g_k) = m\{g_1, \ldots, g_k\}, \qquad k > 1.$$ (3.32)

Using the properties of the braces, one can find [44, 45]

$$[\![M, M]\!](g_1, \ldots) = [\![m, m]\!]\{g_1, \ldots\} = 0\,. \tag{3.33}$$

In other words, M satisfies the Maurer-Cartan equation whenever m does so. Expanding the former structure in homogeneous components, M, one gets an infinite sequence of relations

$$[\![M_1, M_1]\!] = 0\,, \qquad [\![M_1, M_2]\!] = 0\,, \qquad \ldots\,.$$

As is seen the first term M_1 defines a differential $D = [\![m, \bullet]\!]$ on the space $Hom(TV, V)$. The second relation takes then the form

$$DM_2(g_1, g_2) + M_2(Dg_1, g_2) + (-1)^{|g_1|} M_2(g_1, Dg_2) = 0\,. \tag{3.34}$$

In particular, this means that M_2 maps any pair of D-cocycles g_1 and g_2 to a D-cocycle $M_2(g_1, g_2)$.

Suppose now that we are given a two-parameter family $m = m(\hbar, s)$ of A_∞-structures on V. Then, differentiating the defining condition $[\![m, m]\!] = 0$ by the parameters, one readily concludes that the partial derivatives $\partial_\hbar m$ and $\partial_s m$ are D-cocycles for all $\hbar$ and s. Indeed,

$$D\partial_\hbar m = [\![m, \partial_\hbar m]\!] = \frac{1}{2}\partial_\hbar[\![m, m]\!] = 0\,, \qquad D\partial_s m = [\![m, \partial_s m]\!] = \frac{1}{2}\partial_s[\![m, m]\!] = 0\,. \tag{3.35}$$

Applying to them M_2 yields then one more family of D-cocycles

$$M_2(\partial_\hbar m, \partial_s m) = m\{\partial_\hbar m, \partial_s m\}\,.$$

We can increase the number of parameters entering m by considering the flow in the space of cochains

$$\partial_t m = m\{\partial_\hbar m, \partial_s m\} \tag{3.36}$$

with respect to the 'time' t. Solutions to this equation form a three-parameter family of the cochains $m(t, \hbar, s)$. A simple observation is that the flow (3.36) can be consistently restricted to the surface $[\![m, m]\!] = 0$ identified with the set of Maurer-Cartan elements. Indeed, denoting $L = [\![m, m]\!]$, we find

$$\partial_t L = 2[\![m, m\{\partial_\hbar m, \partial_s m\}]\!] = -[\![\partial_\hbar L, \partial_s m]\!] + [\![\partial_\hbar m, \partial_s L]\!]\,. \tag{3.37}$$

Hence, choosing initial data $m(0, \hbar, s)$ for the solutions to eq. (3.36) on the surface $L = 0$, we will get three-parameter families $m(t, \hbar, s)$ of the Maurer-Cartan elements. Let us take

$$m(0, \hbar, s) = \mu(\hbar) + s\partial\,. \tag{3.38}$$

Here the parameter s is prescribed the degree 2; ∂ is the degree -1 differential on $A_{-1} \oplus A_0$ that maps A_0 to A_{-1} as identity isomorphism and maps A_{-1} to 0, which is essentially a formal way to retract an element from the bimodule and reinterpret it as an element

of the algebra again; $\mu(\hbar)$ is the bimodule structure with respect to the full deformed product (3.10):

$$\mu(\hbar)(a,b) = a * b \,, \qquad \mu(\hbar)(a,u) = a * u \,, \qquad \mu(\hbar)(u,a) = -u * a \,.$$

The Maurer-Cartan equation for (3.38) is equivalent to the relations

$$[\![\mu(\hbar), \mu(\hbar)]\!] = 0 \,, \qquad [\![\mu(\hbar), \partial]\!] = 0 \,, \qquad [\![\partial, \partial]\!] = 0 \,, \tag{3.39}$$

which are obviously satisfied. Notice that both (3.38) and the r.h.s. of (3.36) are of degree 1; hence, so is the solution $m(t, \hbar, s)$ to eq. (3.36) with the initial condition (3.38).

Now, all m_n can be generated systematically by solving (3.36) order by order in t, $m = m_2 + t m_3 + t^2 m_4 + \ldots$, and setting $\hbar = s = 0$ at the end. For example, at the first-order we find

$$m_3 = \mu\{\mu', \partial\} \quad \longrightarrow \quad \begin{cases} \mu\{\mu', \partial\}(a,b,u) = +\mu(\mu'(a,b), \partial(u)) \overset{\hbar=0}{=} +\phi_1(a,b)u \,, \\ \mu\{\mu', \partial\}(a,u,v) = -\mu(\mu'(a,u), \partial(v)) \overset{\hbar=0}{=} +\phi_1(a,u)v \,, \\ \mu\{\mu', \partial\}(u,a,v) = -\mu(\mu'(u,a), \partial(v)) \overset{\hbar=0}{=} -\phi_1(u,a)v \,, \end{cases} \tag{3.40}$$

where on the right we evaluated the map on the left for various triplets of arguments. At the second order we obtain the relation

$$2 m_4 = m_3 \{\partial_\hbar \mu, \partial\} + \mu\{\partial_\hbar m_3, \partial\} \,, \tag{3.41}$$

and hence

$$m_4(a,b,u,v) = \mu(\mu'(\mu'(a,b), \partial(u)), \partial(v)) + \tfrac{1}{2}\mu(\mu(\mu''(a,b), \partial(u)), \partial(v)) = \tag{3.42}$$

$$\overset{\hbar=0}{=} \phi_1(\phi_1(a,b), u)v + \phi_2(a,b)uv \,, \tag{3.43}$$

which is in agreement with (3.18).

To summarize, given a deformation of an associative algebra, we can explicitly construct an A_∞-algebra that can be thought of as a noncommutative deformation of this algebra, where the deformation parameter is promoted to an element of the algebra itself.[7] Remarkably, the A_∞-structure is determined by the deformed product up to an A_∞ gauge transformation. While the construction above is quite general, in the sequel we focus upon the case of higher spin algebras and explain why and how these algebras can be deformed.

---

[7]Let us mention another quite general approach to the deformation problem above. It is based on the construction of an appropriate resolution for the initial algebra. The approach is applicable to associative [46, 47] as well as to A_∞/L_∞-algebra deformations [48, 49]. The choice of a resolution, however, is rather ambiguous and suitable resolutions may happen to be quite cumbersome. The advantage of the present approach is that it does not require any structure beyond the deformation of the underlying associative algebra and, in this sense, it is more universal.

4 Higher spin algebras

In the first approximation, higher spin algebras (HSA) are just (infinite-dimensional) associative algebras that arise in the study of higher symmetries of linear conformally invariant equations or of higher spin extensions of gravity. Very often the same algebras show up in other contexts under different names. For instance, one of the simplest examples is just the Weyl algebra A_n. Many examples of HSA are provided by various free conformal fields theories, being free they possess infinite-dimensional algebras of symmetries. Below we give a number of (almost) equivalent definitions and examples of HSA. The most important definitions for our subsequent discussion are due to free CFT's and universal enveloping algebras.

4.1 Various definitions and constructions

1. Higher symmetries of linear equations. Given a linear equation $L\phi = 0$, where $\phi \equiv \phi(x)$ is a set of fields and $L = L(x, \partial)$ is a differential operator, it is useful to study its symmetries and the algebra they form. A differential operator $S = S(x, \partial)$ is called a symmetry if it maps solutions to solutions, i.e., $LS\phi = 0$ for any ϕ obeying $L\phi = 0$. In practice, this implies that L can be pushed through S, i.e., $LS = B_S L$ for some operator B_S. The operators of the form CL are called trivial symmetries. These should be quotiented out as they act trivially on-shell. It is also important that the product $S_1 S_2$ of two symmetries is a symmetry, as a consequence of linearity. Therefore, the algebra of symmetries — the algebra of all symmetries modulo trivial ones — is associative.

A canonical example [50, 51] is a free scalar field $\phi(x)$ in d-dimensional flat space and $L = \Box$. The equation $\Box\phi = 0$ is well known to be conformally invariant, with conformal symmetries acting as[8]

$$\delta_\xi \phi(x) = \xi^a \partial_a \phi(x) + \frac{d-2}{2d}(\partial_a \xi^a)\phi(x)\,, \qquad \partial^a \xi^b + \partial^b \xi^a = \frac{2}{d}\eta^{ab}\partial_m \xi^m\,, \qquad (4.1)$$

where $\xi^a(x)$ is a conformal Killing vector. These symmetries form the conformal algebra $so(d,2)$ with respect to the commutator $[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{[\xi_1, \xi_2]}$. As is pointed out above, the product $\delta_{\xi_1} \cdots \delta_{\xi_n}$ is a symmetry too and is represented by a higher-order differential operator. All such operators are related to the conformal Killing tensors

$$\delta_v \phi = v^{a_1 \dots a_{k-1}} \partial_{a_1} \cdots \partial_{a_{k-1}} \phi + \text{more}\,, \quad \partial^{a_1} v^{a_2 \dots a_k} + \text{permutations} - \text{traces} = 0\,. \quad (4.2)$$

It can be shown that the products of conformal symmetries generate the full symmetry algebra [50, 51]. Higher powers of Laplacian, $L = \Box^k$, are also conformally-invariant operators with interesting symmetry algebras [52, 53]. The symmetries of the free Dirac equation $\slashed{\partial}\psi = 0$ [54] and of many other relativistic wave equations are also known [55].

The examples just given lead to infinite-dimensional associative algebras that contain the conformal algebra $so(d,2)$ as a (Lie) subalgebra under commutators. A possible generalization is to consider other (not necessarily conformally invariant) differential operators, e.g. massive Klein-Gordon equation.

---

[8]$a, b, c, \dots = 0, \dots, d-1$ are the indices of the Lorentz algebra $so(d-1,1)$.

To summarize, Definition 1: the higher spin algebras are defined to be the (associative) symmetry algebras of linear conformally-invariant equations.

2. Higher spin currents and charges. Given a free field obeying $\Box$-type equations, e.g. $\Box\phi = 0$, one can construct an infinite number of conserved tensors [56, 57]

$$j_{a_1...a_s} = \phi\partial_{a_1}\ldots\partial_{a_s}\phi + \text{more terms}, \qquad \partial^m j_{ma_2...a_s} = 0. \qquad (4.3)$$

Due to the conformal invariance the conserved tensors can be made traceless and are thereby quasi-primary operators of the free boson CFT. Contracting them with conformal Killing tensors, one obtains conserved currents and the corresponding charges:

$$j_m(v) = j_{ma_2...a_s}v^{a_2...a_s}, \qquad Q_v = \int d^{d-1}x\, j_0. \qquad (4.4)$$

Definition 2 identifies higher spin algebras with the symmetries generated by the Noether charges associated to the higher spin currents. Via the Noether theorem Definition 2 is more or less equivalent to Definition 1. Such conserved tensors and symmetries associated to them have been known since the 60's, see e.g. [56, 58] and references therein. It was also shown that they do not survive when interactions are switched on. For CFT's the opposite statement is also true: the existence of conserved higher rank tensors implies that the theory is free in disguise [20–23]. The extensions of the Poincaré symmetry are constrained by the Coleman-Mandula theorem [59].

3. Quotients of universal enveloping algebras. A more direct description of HSA associated with linear conformally-invariant equations is via universal enveloping algebra $U(so(d,2))$, as the last paragraph of item 1 suggests: juxtaposing conformal transformations generates the associative symmetry algebra. Therefore, we may collect the generators P^a, K^a, L^{ab}, D associated with the conformal algebra into T^{AB} of $so(d,2)$.[9] Then, any polynomial

$$f(T^{AB}) = f(P^a, K^a, L^{ab}, D) \qquad (4.5)$$

generates a symmetry transformation. However, there are some relations meaning that not all the polynomials are independent and generate nontrivial transformations. For example, for the free scalar field we obviously have $P_a P^a \sim 0$. The fundamental field corresponds to an irreducible representation of the conformal algebra and hence the Casimir operators have fixed numerical values. As a result, the symmetry algebra is isomorphic to the quotient of the universal enveloping algebra $U(so(d,2))$ by a two-sided ideal (annihilator) $\mathcal{J}$:

$$hs \sim U(so(d,2))/\mathcal{J}. \qquad (4.6)$$

A concrete definition of $\mathcal{J}$ depends on a free CFT (irreducible representation) we consider, but, on general grounds, we expect all Casimir operators $\boldsymbol{C}_{2i}$ to have some fixed values C_{2i}. In the cases we are aware of $\mathcal{J}$ is generated by a few elements of $U(so(d,2))$.

---

[9]$A, B, C, \ldots = 0, \ldots, d+1$ are the indices of the conformal algebra $so(d,2)$ and $\eta^{AB} = (-+\cdots+-)$. Then, $L_{ab} = T_{ab}$, $D = -T_{d,d+1}$, $P_a = M_{a,d+1} - M_{a,d}$, $K_a = M_{a,d+1} + M_{a,d}$.

In the case of the smallest unitary representation, e.g. the free conformal scalar field, the annihilator $\mathcal{J}$ is also known as the Joseph ideal [60]. Possible generalizations here is to consider more general ideals in $U(g)$ for any (not necessarily conformal) Lie algebra g, see e.g. [35, 61, 62]. A useful for our studies example is provided by the HSA of the generalized free field CFT.[10]

We stress that higher spin algebras depend on dimension d and on the spectrum of the free CFT in a crucial way. Given a higher spin algebra one can read off the dimension of the spacetime, the spectrum of higher spin currents and the fundamental free field they are generated by. We conclude by Definition 3: the higher spin algebras are defined as various (simple) quotients of $U(so(d,2))$.

4. Quantization of coadjoint orbits.   There is also a relation [51, 63, 64] between HSA and deformation quantization [10, 65]. The fundamental field of any free CFT corresponds to some irreducible representation of the conformal algebra. This representation, in its turn, is associated to a certain coadjoint orbit (usually to a minimal nilpotent one). Not surprisingly that a given HSA can be identified with the quantized algebra of functions on this coadjoint orbit. Possible generalizations here is to consider deformation quantization in full generality, i.e., for general symplectic or Poisson manifolds.

4.2   Examples

Let us discuss a few simple examples of HSA that will be important later. We mostly employ the universal enveloping realization of HSA. The conformal or anti-de Sitter algebra generators T_{AB} obey

$$[T_{AB}, T_{CD}] = T_{AD}\eta_{BC} - T_{BD}\eta_{AC} - T_{AC}\eta_{BD} + T_{BC}\eta_{AD}, \qquad (4.7)$$

and by the Poincaré-Birkhoff-Witt theorem, the decomposition of the universal enveloping algebra $U(so(d,2))$ is given by symmetrized tensor products of the adjoint representation[11]

$$U(so(d,2)) = \bullet \oplus \square \oplus \left[\square\square \oplus \square\square \oplus \square \oplus \bullet \right] \oplus \left[\square\square\square \oplus \square\square \oplus \cdots \right] \oplus \cdots. \qquad (4.8)$$

Here the first singlet $\bullet$ is the unit of $U(so(d,2))$, $\square \sim T^{AB}$ and the second $\bullet$ is the quadratic Casimir operator

$$C_2 = -\frac{1}{2} T_{AB} T^{AB}. \qquad (4.9)$$

In what follows we describe some ideals of $U(so(d,2))$ and the corresponding quotients that yield the HSA of interest.

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[10]Note that Definitions 1 and 2 do not apply here, while the Definitions 3 and 4 can still be used, see below.

[11]The language of Young diagrams is useful here. For example, the fundamental and the adjoint representations are depicted by $\square$ and $\square$, respectively. The trivial representation is denoted by $\bullet$.

Free boson HSA. This is the simplest HSA and the generators of the ideal can be guessed from the symmetries of $\Box\phi = 0$. Since the solution space is an irreducible representation, the values of the Casimir operators are fixed. Decoupling of null states implies $P_a P^a = 0$ and $K^a K_a = 0$. Finally, all anti-symmetric combinations of the conformal symmetry generators, e.g. $L_{[ab}P_{c]}$ and $L_{[ab}L_{cd]}$, should vanish. All in all, the two-sided (Joseph) ideal is generated by [51][12]

$$\mathcal{J} = \begin{array}{c}\square\\\square\end{array} \oplus \,\square\square\, \oplus (\boldsymbol{C}_2 - C_2)\,, \qquad\qquad C_2 = -\frac{1}{4}(d^2 - 4)\,. \tag{4.10}$$

The $so(d,2)$ decomposition of the quotient algebra contains traceless tensors described by rectangular, two-row, Young diagrams:

$$hs_{F.B.} = \bullet \oplus \square\!\square \oplus \square\square\square\square \oplus \square\square\square\square\square\square \oplus \cdots\,. \tag{4.11}$$

More explicitly, the generators of the Joseph ideal read:

$$\mathcal{J}^{ABCD} = T^{[AB}T^{CD]}\,, \tag{4.12a}$$

$$\mathcal{J}^{AB} = T^A{}_C\,T^{BC} + T^B{}_C\,T^{AC} - (d-2)\eta^{AB}\,, \tag{4.12b}$$

$$\mathcal{J} = -\frac{1}{2}T_{AB}T^{AB} + \frac{1}{4}(d^2 - 4)\,. \tag{4.12c}$$

Free boson and free fermion HSA in three dimensions. This is an even simpler example since all of the Joseph ideal relations can be resolved thanks to the isomorphism $so(3,2) \sim sp(4)$.[13] It turns out that the free boson and free fermion fields — as representations of $sp(4)$ — are equivalent to even and odd states in the Fock space of the 2d harmonic oscillator:

$$P^{a_1}\ldots P^{a_k}|\phi\rangle \sim a^\dagger_{\alpha_1}\ldots a^\dagger_{\alpha_{2k}}|0\rangle\,, \tag{4.13a}$$

$$P^{a_1}\ldots P^{a_k}|\psi\rangle_\delta \sim a^\dagger_{\alpha_1}\ldots a^\dagger_{\alpha_{2k}}a^\dagger_\delta|0\rangle\,. \tag{4.13b}$$

Here a^α and $a^\dagger_\beta$ are the standard creation/annihilation operators satisfying

$$[a^\alpha, a^\dagger_\beta] = \delta^\alpha_\beta\,, \qquad a^\alpha|0\rangle = 0\,, \tag{4.14}$$

and $\alpha, \beta = 1, 2$ are the spinor indices from the $so(1,2)$ point of view. The spinor-vector dictionary is through the σ-matrices, e.g. $P_m = \sigma_m^{\alpha\beta}a^\dagger_\alpha a^\dagger_\beta$. The $sp(4)$ generators are realized by the ten bilinears in a^α and $a^\dagger_\alpha$:

$$K^{\alpha\beta} = a^\alpha a^\beta\,, \qquad \frac{1}{2}D\delta^\alpha_\beta + L^\alpha_\beta = \frac{1}{2}\{a^\alpha, a^\dagger_\beta\}\,, \qquad P_{\alpha\beta} = a^\dagger_\alpha a^\dagger_\beta\,. \tag{4.15}$$

This is the standard oscillator realization of $sp(4)$. The algebra of all ordered polynomials $O(a, a^\dagger)$ in a^α, $a^\dagger_\beta$ is the Weyl algebra A_2.[14] The HSA, as an algebra that maps on-shell

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[12]The full two-sided ideal is obtained by taking the generators and multiplying them by $U(so(d,2))$.

[13]Some important facts are contained already in [66]. Everything we discuss below can be found in [67–69].

[14]The subscript indicates the number of canonical pairs, two in our case.

states (4.13) to the on-shell states, is the even subalgebra of the Weyl algebra A_2, i.e., $O(a, a^\dagger) = O(-a, -a^\dagger)$.

The most important feature of the 3d case is that the HSA's of free boson and free fermion fields are equivalent and isomorphic to the even subalgebra of A_2. This is not true when $d > 3$ for an obvious reason that the higher spin currents built out of the free fermion do not match those of the free boson, see e.g. [70].

Generalized free field HSA. A generalized free (scalar) field, i.e., a conformal scalar operator $O_\Delta(x)$ of some weight Δ such that all correlators are computed via the free Wick contractions,[15] is a useful approximation in many situations. The corresponding HSA, denoted by hs_Δ, is defined to be the quotient $hs_\Delta = U(so(d, 2))/\mathcal{J}_\Delta$ with respect to the ideal generated by

$$\mathcal{J}_\Delta = \begin{array}{|c|} \hline \\ \hline \\ \hline \\ \hline \end{array} \oplus (\boldsymbol{C}_2 - C_2(\Delta)), \qquad C_2(\Delta) = \Delta(d - \Delta), \qquad (4.16)$$

or, in components,

$$\mathcal{J}^{ABCD} = T^{[AB}T^{CD]}, \qquad \mathcal{J} = -\frac{1}{2}T_{AB}T^{AB} - C_2(\Delta). \qquad (4.17)$$

The interpretation of the ideal is obvious. That $\mathcal{J}^{ABCD}$ must vanish is manifestation of the lowest state $|\Delta\rangle$ being scalar, which implies that the descendants $P^a \ldots P^c |\Delta\rangle$ are symmetric tensors and the combinations of the generators with more than two anti-symmetrized indices vanish. The $so(d, 2)$-decomposition contains more tensors than that of the free boson HSA, namely,

$$hs_\Delta = \bullet \oplus \begin{array}{|c|}\hline\\\hline\end{array} \oplus \begin{array}{|c|c|}\hline&\\\hline\end{array} \oplus \begin{array}{|c|c|}\hline&\\\hline\end{array} \oplus \begin{array}{|c|c|c|}\hline&&\\\hline\end{array} \oplus \begin{array}{|c|c|}\hline&\\\hline&\\\hline\end{array} \oplus \cdots. \qquad (4.18)$$

The additional components are due to the absence of the $\begin{array}{|c|c|}\hline&\\\hline\end{array}$ generator, cf. (4.10) and (4.16).[16]

Clearly, the HSA of generalized free field $O_\Delta(x)$ form a one-parameter family of algebras because Δ is a free parameter. At the critical values $\Delta_k = d/2 - k$, $k = 1, 2, \ldots$, the algebra is not simple and acquires a two-sided ideal. The resulting quotient algebra is the symmetry algebra of the free scalar field $\Box^k \phi = 0$ [52, 53]. The one-parameter family of HSA corresponding to generalized free fields will be important for the discussion in appendix B since it underlies the deformation of the other HSA.

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[15]For generic Δ, the generalized free field does not have a local stress-tensor and does not have (local) higher spin currents. Also, there are no equations to be imposed. Therefore, the definitions (1) and (2) are not applicable. Nevertheless, the algebra can be defined via definition (3) (and also via (4)) as we do here. A good consistency check is that it reduces to the already known HSA at the expected values of Δ.

[16]It may seem that one can pick several elements of $U(so(d, 2))$ in random and declare them to generate an ideal, but in doing so one may discover that the ideal coincides with the full $U(so(d, 2))$. In particular, it is impossible to add the $\begin{array}{|c|c|}\hline&\\\hline\end{array}$ component to the generating set for generic Δ without trivializing the quotient.

4.3 Higher spin currents equal higher spin algebra

As it was already mentioned, the higher spin symmetry of free CFT's is manifested by an infinite number of higher-spin currents J_s, which are quasi-primary operators from the CFT point of view. Schematically, e.g. in the free scalar CFT, they are

$$J_{a_1...a_s} = \phi \partial_{a_1} \ldots \partial_{a_s} \phi + \text{more}\,, \qquad\qquad \partial^c J_{ca_2...a_s} = 0\,. \qquad (4.19)$$

The stress-tensor, which is responsible for the $so(d, 2)$-part of the HSA is the $s = 2$ member of the family. By construction, the free field is a fundamental representation of this HSA.[17] The infinite multiplet J of higher spin currents J_s is the representation that is next to the fundamental one.[18] The lowest lying OPE's can be written as

$$\phi\phi = \mathbb{1} + J\,, \qquad\qquad JJ = \mathbb{1} + J + O_2\,, \qquad (4.20)$$

where $\mathbb{1}$ is the identity operator and O_2 is a multiplet of double-trace operators, which is given by the quartic tensor product of the free field itself.

Regarding the free field as a vector space V and HSA as $gl(V)$, the higher spin currents belong to $V \otimes V$, which is very close to $gl(V) \sim V \otimes V^*$. This heuristic reasoning can be made more precise.[19] If $|\phi\rangle$ is the free field vacuum, then

$$K^a|\phi\rangle = 0\,, \qquad L^{ab}|\phi\rangle = 0\,, \qquad D|\phi\rangle = \tfrac{d-2}{2}|\phi\rangle \qquad (4.21)$$

and the descendants correspond to $P^a \ldots P^c|\phi\rangle$. Higher spin currents are the quasi-primary states in the tensor product

$$J \sim \phi \times \phi \sim P^a \ldots P^c|\phi\rangle \otimes P^b \ldots P^d|\phi\rangle\,, \qquad (4.22)$$

while the HSA can be viewed as the span of operators of the form

$$P^a \ldots P^c|\phi\rangle \otimes \langle\phi|K^b \ldots K^d\,. \qquad (4.23)$$

Clearly, the two spaces are formally isomorphic and the map between them is the conjugation $\langle\phi| = |\phi\rangle^\dagger$, which is defined via the inversion map $\mathtt{I}$.[20] Therefore, the higher spin

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[17]Representations (modules) of HSA are quite easy to describe, see e.g. [71]. Roughly speaking, the free field is a vector space V and HSA is $gl(V)$ for this V. Other representations are just tensor products $V \otimes \cdots \otimes V$ projected onto any irreducible representation of the permutation group (the permutation group commutes with the $gl(V)$-action on $T(V)$).

[18]One should be careful about tensor product vs. associativity issues and imply either the Lie subalgebra of a HSA (via commutators) or the tensor product of HSA that naturally acts on the tensor product of its representations.

[19]See [72] for subtleties that may arise in some formal manipulations. That the tensor product decomposes into (all) higher spin currents was shown, for $d = 3$, in [73] (the currents, as representations of $so(d, 2)$, viewed as anti-de Sitter algebra, are the same as massless fields in AdS_{d+1}, which is the interpretation adopted in [73]). See [56] for the result in any d. See also [74] that elaborates on the relation between this construction and $U(so(d, 2))$, showing, in particular, that the shadow of J_0 can also be treated by the same tools.

[20]Note that at the level of the Lie algebra we have $K^a = IP^aI$, $L^{ab} = IL^{ab}I$, $P^a = IK^aI$ and $-D = IDI$. We see that $P^a + K^a$ and L^{ab} are stable and form $so(d-1, 2)$ subalgebra of the conformal algebra $so(d, 2)$. We can also define $-K^a = IP^aI$, $L^{ab} = IL^{ab}I$, $-P^a = IK^aI$ and $-D = IDI$. Then, it is $P^a - K^a$ and L^{ab} that are stable and form $so(d, 1)$.

currents together with their descendants, as a module of the conformal algebra (as well as a HSA-module), can be viewed as the same HSA where the right action is twisted by I. That is, JI is formally isomorphic to HSA.

5 Slightly broken higher spin symmetry

Now we are ready to explain the problem of the slightly broken higher spin symmetry and our proposal in greater detail. In interacting CFT's with slightly broken higher spin symmetry higher spin currents are no longer conserved, but their non-conservation has a very specific form of

$$\partial \cdot J = \frac{1}{N}[JJ] , \qquad (5.1)$$

where $[JJ]$ is a specific (set of) double-trace operators, whose form may also depend on the coupling constants (e.g. it depends on $\lambda = N/k$ for Chern-Simons matter theories), see [11–14, 75] for some explicit formulas. It is worth mentioning at this point, that the non-conservation equations for the two theories related by the bosonization duality can be directly mapped into each other [14], which again supports the statement that (5.1) and its consequences should explain the dualities.[21]

At the free level a given CFT, including the correlation functions, is fully controlled by the corresponding higher spin algebra (HSA), say hs. The multiplet of higher spin currents J form a representation of hs. Moreover, JI is formally isomorphic to hs (as representation). The slightly broken higher spin symmetry is a deformation of hs by J (to be precise by J/N to introduce a small parameter). Since JI $\sim hs$, we can interpret the sought for deformation as a non-commutative deformation of hs described in section 3. Then, the HSA belongs to A_{-1} and the non-commutative deformation parameter to A_0, i.e., $A_{-1} \sim hs$ and $A_0 \sim J$. The only subtlety is that one should take into account the inversion map I. From the A_∞ point of view the action of a HSA on the module should be defined as

$$m_2(a,u) = au , \qquad m_2(u,a) = -u\mathtt{I}(a) \quad \text{for} \quad a \in A_{-1} , u \in A_0 . \qquad (5.2)$$

We show below that this problem can be reduced to the one already solved in section 3. The procedure is three step. Firstly, we extend hs by adding the automorphism I and call the resulting algebra the double $D(hs)$. Secondly, we deform $D(hs)$ as an associative algebra. This deformation can be used to construct an appropriate A_∞-algebra with the help of section 3. Lastly, we can truncate the algebra in such a way that (5.2) is true.

As it was already mentioned, typical HSA's admit no deformations as associative algebras, which means that $HH^2(hs, hs) = 0$.[22] Nevertheless, certain simple extensions of

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[21]We also note that there are cases where the triple-trace terms $N^{-2}[JJJ]$ are possible. For example, this corresponds to the sextic coupling $\lambda_6(\phi^2)^3$ in the action. The fate of such terms is yet unclear to us. First of all, λ_6 is not an independent parameter since the conformal point corresponds to $\beta_{\lambda_6} = 0$ [76]. Secondly, the triple-trace terms are suppressed by an additional N^{-1} and, for example, play no role for the anomalous dimensions and three-point functions studied in [14]. The same time, such triple terms may be (already) accounted for by the higher structure maps of the A_∞-algebra.

[22]What we discuss below applies also to the examples where they do admit such deformations.

HSA's do admit deformations and it is these deformations that are also responsible for the A_∞-structure.

In order to treat both the HSA hs and J on an equal footing, we take a bigger algebra — the double $D(hs)$ — HSA extended by $\mathtt{I}$. This is just the simplest example of the smash product $B \rtimes \Gamma$, where B is an algebra and Γ is a finite group of its automorphisms of B. In our case $\Gamma = \mathbb{Z}_2$. Elements of $D(hs)$ have the form $a = a' + a''\mathtt{I}$, $a', a'' \in hs$ and the product law reads

$$(a' + a''\mathtt{I})(b' + b''\mathtt{I}) = (a'b' + a''\mathtt{I}(b'')) + (a'b'' + a''\mathtt{I}(b'))\mathtt{I}, \tag{5.3}$$

where $\mathtt{I}(a)$ is the action of the inversion on the algebra elements, which can be obtained by extending $\mathtt{I}(P^a) = \mathtt{I}P^a\mathtt{I} = K^a$, etc. to polynomials in P^a, K^a, L^{ab}, D and we used $\mathtt{I}^2 = 1$. Now, the usual adjoint action and the twisted action (5.2) are just different projections of the adjoint action in $D(hs)$.

An important observation is that $D(hs)$ belongs to a one-parameter family of algebras (while hs usually does not). We discuss various arguments in favor of this statement in appendix B and explicit examples in section 6. For now it is sufficient to assume that we have already constructed a one-parameter family of associative algebras $D_\hbar(hs)$ that deforms the double $D(hs)$. As a result one gets the deformed product (3.10):

$$a * b = ab + \sum_{k>0} \phi_k(a,b)\hbar^k \qquad\qquad a, b \in D(hs). \tag{5.4}$$

We also assume that such deformation does not originate from a deformation of hs itself in those exceptional cases when the latter exists.[23] This means that the first-order deformation has to obey

$$a\phi_1(b,c) - \phi_1(ab,c) + \phi_1(a,bc) - \phi_1(a,b)I(c) = 0, \qquad a, b, c \in hs. \tag{5.5}$$

The same equation without $\mathtt{I}$ determines the first-order deformations of hs. A nontrivial solution to (5.5) is a Hochschild cocycle in the representation twisted by $\mathtt{I}$. Then, the first-order deformation of $D(hs)$ induced by ϕ_1 can be written as

$$(a + a'I) * (b + b'I) = (ab + aI(b')) + (ab' + a'I(b))I + \hbar\phi_1(a + a'I, b + b'I)I + \mathcal{O}(\hbar^2).$$

This illustrates the relation between the Hochschild cohomology in the representation twisted by $\mathtt{I}$, (5.5), and the Hochschild cohomology of the double $D(hs)$.

The A_∞-algebra is constructed by building up the structure maps m_n following the general method of section 3. Since $\mathtt{I}^2 = 1$ the Taylor coefficients ϕ_k have a specific dependence on $\mathtt{I}$: $\phi_{2n+1}(a,b) = \varphi_{2n+1}(a,b)\mathtt{I}$ and $\phi_{2n}(a,b) = \varphi_{2n}(a,b)$ where φ_k do not depend on $\mathtt{I}$. This property makes it obvious that we can restrict A_{-1} to hs, while all elements from A_0 can be restricted to $hs\mathtt{I}$ to be interpreted as $J\mathtt{I}$ for the multiplet of higher spin

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[23]It is worth stressing that even if a given hs happens to belong to a one-parameter family of algebras, it will not lead to the A_∞-algebra we need, for the m_2-map has to be (5.2) to incorporate higher spin currents.

currents J. Assuming that a, b and u, v take values in hs, we can write for the first structure maps m_n

$$m_2(a, b) = ab, \qquad m_2(a, u) = au, \qquad m_2(u, a) = -u\mathtt{I}(a)$$
$$m_3(a, b, u) = \varphi_1(a, b)\mathtt{I}(u), \quad m_3(a, u, v) = \varphi_1(a, u)\mathtt{I}(v), \quad m_3(u, a, v) = -\varphi_1(u, \mathtt{I}(a))v,$$
$$m_4(a, b, u, v) = \varphi_2(a, b)u\mathtt{I}(v) + \varphi_1(\varphi_1(a, b), \mathtt{I}(u))\mathtt{I}(v), \qquad \dots$$

The bilinear structure maps m_2 are equivalent to having a higher spin algebra and its module in the adjoint representation twisted by $\mathtt{I}$, i.e., higher spin currents. One can check that these structure maps do obey the first defining relations of the A_∞-algebra. The general formula for m_n is in section 3. Further investigation requires explicit form of ϕ_k and some examples are provided in section 6.

6 Explicit oscillator realizations

In practice we may need an efficient way to perform computations with the deformed algebras. Many interesting HSA's admit oscillator realizations and, after briefly reviewing these realizations, we modify them as to construct the deformed HSA's. Note that the structure maps of the A_∞-algebra are expressed in terms of the deformed product and as such contain no new information compared to the one in the deformed HSA. Therefore, the problem of the slightly broken higher spin symmetry is reduced to a much simpler problem of constructing a deformed HSA. This brings up the question: how big is the space of all deformations? Quantitatively its size is defined by the second Hochschild cohomology group (if there are no obstructions), whose dimension equals the number of phenomenological parameters entering the correlation functions.

6.1 Toy model: Weyl algebra A_1

The simplest example, which nevertheless underlies all the other deformations, is the smallest Weyl algebra A_1, i.e., a one-dimensional harmonic oscillator. The Weyl algebra A_1 is defined in our notation as[24]

$$[y_\alpha, y_\beta] = 2i\epsilon_{\alpha\beta}, \qquad\qquad \alpha, \beta = 1, 2. \qquad (6.1)$$

Let us define the automorphism $\mathtt{I}$ as the reflection $\mathtt{I}(y_\alpha) = -y_\alpha$. Therefore, the $\mathtt{I}$-stable subalgebra — the 'Lorentz' subalgebra — is simply the subalgebra A_1^e of even polynomials in y's, $f(y) = f(-y)$. It is well known that the Weyl algebra does not admit any deformation as an associative algebra, but the 'Lorentz' subalgebra does belong to a one-parameter family of algebras. Indeed, $sp(2) \sim sl(2)$ is a subalgebra of the Weyl algebra, which is realized by the three generators $t_{\alpha\beta} = t_{\beta\alpha}$:

$$t_{\alpha\beta} = -\tfrac{i}{4}\{y_\alpha, y_\beta\}, \qquad\qquad [t_{\alpha\beta}, t_{\gamma\delta}] = \epsilon_{\alpha\delta}t_{\beta\gamma} + \text{three more}. \qquad (6.2)$$

The $\mathtt{I}$-stable subalgebra A_1^e coincides with the enveloping algebra of $t_{\alpha\beta}$ and it is not hard to see that this is the quotient of $U(sl_2)$ by the two-sided ideal generated by $\boldsymbol{C}_2 - \left(-\tfrac{3}{4}\right)$,

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[24]Here, $\epsilon_{\alpha\beta}$ is the invariant $sp(2)$-tensor, the anti-symmetric tensor with $\epsilon_{12} = -\epsilon_{21} = 1$.

where $\boldsymbol{C}_2 = -\frac{1}{2}t_{\alpha\beta}t^{\alpha\beta}$ is the Casimir operator; the constant $-\frac{3}{4}$ is the value of $\boldsymbol{C}_2$ in the oscillator realization. This algebra belongs to a one-parameter family of algebras,[25] called $hs(\lambda)$ that are obtained in the same way except that the eigen value of the Casimir operator is kept to be a free parameter:

$$hs(\lambda) = U(sl_2)/\mathcal{J}\,, \qquad\qquad \mathcal{J} = U(sl_2)[\boldsymbol{C}_2 + (\lambda^2 - 1)]\,. \qquad (6.3)$$

$hs(\lambda)$ is nothing but a noncommutative (fuzzy) sphere, whose radius is controlled by λ. Therefore, we have $A_1^e \sim hs(\lambda^*)$, where $\lambda^* = 1/2$. According to our general claim, the Weyl algebra A_1 extended by the automorphism I should admit a one-parameter family of deformations. The double $D(A_1)$ is defined by

$$[y_\alpha, y_\beta] = 2i\epsilon_{\alpha\beta}\,, \qquad\qquad \{y_\alpha, k\} = 0\,, \qquad\qquad k^2 = 1\,. \qquad (6.4)$$

Indeed, the algebra generated by the y's and k is a particular case of the so-called deformed oscillator algebra $Aq(\nu)$,[26] which is defined by the following relations on its generators:

$$[q_\alpha, q_\beta] = 2i\epsilon_{\alpha\beta}(1 + \nu K)\,, \qquad\qquad \{q_\alpha, K\} = 0 \qquad\qquad K^2 = 1\,. \qquad (6.5)$$

It is clear that the double of the Weyl algebra $D(A_1)$ is isomorphic to $Aq(0)$. Another description of the deformed oscillator algebra is

$$Aq(\nu) = U(osp(1|2))/\mathcal{J}\,, \qquad\qquad \mathcal{J} = U(osp(1|2))\left[C_2 + \frac{1}{4}(1 - \nu^2)\right]\,. \qquad (6.6)$$

This algebra is nothing but a noncommutative super-sphere $S^{2|2}$ whose radius is controlled by ν. The structure constants of $hs(\lambda)$ and of the deformed oscillators are available in the literature in several forms [37, 83–86]. Therefore, the components $\phi_k(\bullet, \bullet)$ of the deformed HSA product are known and can be used to explicitly write down the A_∞-structure. Notice that the classical limit of the deformed algebra is just a two-dimensional symplectic space endowed with a symplectic reflection k.

One may wonder to which extent the deformation described above is unique. For the Weyl algebra it is well known that $HH^2(A, A^*)$ is one-dimensional. At the same time, the I-map identifies the dual module A^* with the I-twisted one. For the double $D(A_1)$ the cohomology is known to be one-dimensional and the deformation is unique.

6.2 Deformations of the free boson algebra

The simplest example of a HSA is the symmetry algebra of the free boson CFT [51]. The case of three dimensions is somewhat special and is discussed in the next section. The A_∞-algebra originating from this HSA should be responsible for the breaking of higher spin symmetries in the large-N critical vector model in d dimensions.[27]

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[25]These algebras were defined in [77] and dubbed gl_λ since they reduce to gl_N for certain values of λ and can be thought of as algebras interpolating between gl_N and gl_{N+1}.

[26]Defined implicitly in [78] and explicitly in e.g. [79–81], see also [82].

[27]Due to the unitarity constraints the unitary cases are confined to $2 < d < 4$ and $4 < d < 6$ [87]. It would be interesting to extend the A_∞-algebra to fractional dimensions d.

There exists a quasi-conformal realization of this HSA by the minimal number of oscillators where the Joseph ideal is completely resolved [88]. This realization is non-linear and for simplicity let us stick to another, linear, form [89], in which the Joseph ideal is partially resolved. Such a realization appears naturally in the manifestly conformally-invariant description of the free conformal scalar field in the ambient space [90]. One begins with the embedding of the HSA into the Weyl algebra A_{d+2}:[28]

$$[Y_\alpha^A, Y_\beta^B] = 2i\eta^{AB}\epsilon_{\alpha\beta}. \tag{6.7}$$

The bilinears in Y form $sp(2(d+2))$, which contains a Howe dual pair $so(d,2) \oplus sp(2)$ of algebras such that the $so(d,2)$ generators T^{AB} commute with the $sp(2)$ generators $t_{\alpha\beta}$:

$$T^{AB} = +\frac{i}{4}\epsilon^{\alpha\beta}\{Y_\alpha^A, Y_\beta^B\}, \qquad t_{\alpha\beta} = -\frac{i}{4}\{Y_\alpha^A, Y_{A\beta}\}. \tag{6.8}$$

We consider the enveloping algebra of T^{AB}, i.e., polynomials $f(Y) \equiv f(T)$, which can also be defined as the centralizer of $sp(2)$, $[t_{\alpha\beta}, f(Y)] = 0$. By construction, a part of the Joseph ideal vanishes identically since one cannot have more than two anti-symmetrized indices of $so(d,2)$:

$$T^{[AB}T^{CD]} \sim \boxed{} \sim 0. \tag{6.9}$$

The resulting algebra is not simple and its $so(d,2)$ decomposition contains traceful tensors with the symmetry of two-row rectangular Young diagrams:

$$f(T) \sim \bullet \oplus \boxed{} \oplus \boxed{} \oplus \boxed{} \oplus \cdots. \tag{6.10}$$

The HSA is defined as a quotient of this algebra by the ideal generated by traces:

$$f \in hs_{F.B.}: \qquad [t_{\alpha\beta}, f] = 0, \qquad f \sim f + t_{\alpha\beta} \star g^{\alpha\beta}, \tag{6.11}$$

where $g^{\alpha\beta}$ transforms as an $sp(2)$-tensor. Note that the $sp(2)$-generators $t_{\alpha\beta}$ are exactly the contractions of Y's, that is, traces. The resulting spectrum is (4.11), as expected.

The automorphism $\mathtt{I}$ that corresponds to the inversion map in the CFT base and to the flip of the AdS-translations in the AdS base is realized as $\mathtt{I}(y_\alpha^\mathcal{A}, y_\alpha) = (y_\alpha^\mathcal{A}, -y_\alpha)$, i.e., it flips the sign of the A_1 subalgebra generators. Since the $\mathtt{I}$-map does not affect $y_\alpha^\mathcal{A}$, the whole construction is very similar to the A_1 toy model. The double $D(hs_{F.B.})$ is easy to construct:

$$[y_\alpha^\mathcal{A}, y_\beta^\mathcal{B}] = +2i\epsilon_{\alpha\beta}\eta^{\mathcal{AB}}, \qquad [y_\alpha, y_\beta] = -2i\epsilon_{\alpha\beta}, \qquad \{y_\alpha, k\} = 0. \tag{6.12}$$

The deformed double is then obtained with the help of the deformed oscillators,[29]

$$[y_\alpha^\mathcal{A}, y_\beta^\mathcal{B}] = +2i\epsilon_{\alpha\beta}\eta^{\mathcal{AB}}, \qquad [q_\alpha, q_\beta] = -2i\epsilon_{\alpha\beta}(1 + \nu k), \qquad \{q_\alpha, k\} = 0, \tag{6.13}$$

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[28]Here $A, B, \ldots = 0, \ldots, d+1$ are indices of $so(d,2)$. We will also split them as $A = \{\mathcal{A}, 5\}$, etc., where $\mathcal{A}, \mathcal{B}, \ldots = 0, \ldots, d$ are the indices of the AdS-Lorentz algebra $so(d,1)$ and 5 is an extra dimension. $L^{\mathcal{AB}} = T^{\mathcal{AB}}$, $P^\mathcal{A} = T^{\mathcal{A}5}$, $\eta^{55} = -1$, so that $[P^\mathcal{A}, P^\mathcal{B}] = L^{\mathcal{AB}}$.

[29]Note that the inversion map can also be realized as $\mathtt{I}(y_\alpha^\mathcal{A}, y_\alpha) = (-y_\alpha^\mathcal{A}, y_\alpha)$, but this realization does not admit the deformation we are looking for. It is important to note that while the double $D(hs)$ can always be deformed, a particular (oscillator) realization of $D(hs)$ may not admit any straightforward deformation. Indeed, in the present case hs is realized as a subquotient of the Weyl algebra A_{d+2} and we are to deform $D(A_{d+2})$ first, which may or may not be possible depending on how $D(A_{d+2})$ is realized.

and is defined following (6.11) as

$$D_\nu(hs) \ni f(y_\alpha^{\mathcal{A}}, q_\alpha, k) : \qquad [f, t_{\alpha\beta}] = 0, \qquad f \sim f + t_{\alpha\beta} \star g^{\alpha\beta}(y, q, k), \qquad (6.14)$$

where the new $sp(2)$ generators are

$$t_{\alpha\beta} = -\frac{i}{4}\{y_\alpha^{\mathcal{A}}, y_{\mathcal{A}\beta}\} + \tau_{\alpha\beta}, \qquad\qquad \tau_{\alpha\beta} = \frac{i}{4}\{q_\alpha, q_\beta\}. \qquad (6.15)$$

At this point, there is no need in the deformed oscillators themselves, it is sufficient to know that the deformation of the algebra in y_α and k is given by the quotient of $U(osp(1|2))$, the fuzzy super-sphere.

The first few levels of the deformed double are easy to explore. Following the general logic, one can define the Lorentz and translation generators

$$\mathsf{P}^{\mathcal{A}} = +\frac{i}{4}\{y_\alpha^{\mathcal{A}}, q_\beta\}\epsilon^{\alpha\beta}, \qquad\qquad \mathsf{L}^{\mathcal{AB}} = +\frac{i}{4}\{y_\alpha^{\mathcal{A}}, y_\beta^{\mathcal{B}}\}\epsilon^{\alpha\beta} \qquad (6.16)$$

that commute with $sp(2)$:

$$[t_{\alpha\beta}, \mathsf{P}_{\mathcal{A}}] = 0, \qquad\qquad [t_{\alpha\beta}, \mathsf{L}_{\mathcal{AB}}] = 0, \qquad\qquad [t_{\alpha\beta}, k] = 0. \qquad (6.17)$$

The relations of the $so(d, 2)$ algebra get modified at one place

$$[\mathsf{P}^{\mathcal{A}}, \mathsf{P}^{\mathcal{B}}] = (1 + \nu k)\mathsf{L}^{\mathcal{AB}}, \quad [\mathsf{L}^{\mathcal{AB}}, \mathsf{L}^{\mathcal{CD}}] = \mathsf{L}^{\mathcal{AD}}\eta^{\mathcal{BC}} + \dots, \quad [\mathsf{L}^{\mathcal{AB}}, \mathsf{P}^{\mathcal{C}}] = \mathsf{P}^{\mathcal{A}}\eta^{\mathcal{BC}} - \mathsf{P}^{\mathcal{B}}\eta^{\mathcal{AC}},$$

which is the first nontrivial component of the Hochschild cocycle.

6.3 Three dimensions

The case of three dimensions is special due to the fact that the HSA of the free boson CFT is the same as the HSA of the free fermion CFT. A unique HSA is the even subalgebra A_2^e of the Weyl algebra A_2:[30]

$$hs \ni f(Y) : \qquad f(Y) = f(-Y), \qquad [Y^A, Y^B] = 2iC^{AB}. \qquad (6.18)$$

In the AdS-base the quartet Y^A can be split into the commuting $y_\alpha, \bar{y}_{\dot\alpha}$ in terms of which

$$\mathsf{L}_{\alpha\beta} = -\frac{i}{4}\{y_\alpha, y_\beta\}, \qquad \mathsf{P}_{\alpha\dot\alpha} = -\frac{i}{4}\{y_\alpha, \bar{y}_{\dot\alpha}\}, \qquad \bar{\mathsf{L}}_{\dot\alpha\dot\beta} = -\frac{i}{4}\{\bar{y}_{\dot\alpha}, \bar{y}_{\dot\beta}\}. \qquad (6.19)$$

In the conformal base we have (4.15). The I-map acts either as $\mathtt{I}(y_\alpha, \bar{y}_{\dot\alpha}) = (-y_\alpha, \bar{y}_{\dot\alpha})$ or as $\mathtt{I}(y_\alpha, \bar{y}_{\dot\alpha}) = (y_\alpha, -\bar{y}_{\dot\alpha})$. In the conformal base it corresponds to $\mathtt{I}(a^\alpha, a_\beta^\dagger) = (a_\alpha^\dagger, a^\beta)$ or $\mathtt{I}(a^\alpha, a_\beta^\dagger) = (-a_\alpha^\dagger, -a^\beta)$. That there are two different realizations of the I-map is in accordance with the existence of two independent cocycles, which was already deduced in

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[30]$A, B, \dots = 1, \dots, 4$ are the $sp(4)$ vector indices, $sp(4) \sim so(3, 2)$. The AdS-Lorentz algebra is $sl(2, \mathbb{C}) \sim so(3, 1)$ and it is convenient to use the indices $\alpha, \beta, \dots = 1, 2$ and $\dot\alpha, \dot\beta, \dots = 1, 2$ of the fundamental of $sl(2, \mathbb{C})$ and its conjugate.

appendix B from the dual cycles. The double of this algebra is just the two copies of the one for A_1:[31]

$$\{y_\alpha, k\} = 0\,, \qquad [\bar{y}_{\dot\alpha}, k] = 0\,, \qquad \{\bar{y}_{\dot\alpha}, \bar{k}\} = 0\,, \qquad [y_\alpha, \bar{k}] = 0\,. \qquad (6.20)$$

The exact deformation of the double is given by the two pairs of deformed oscillators:

$$[q_\alpha, q_\beta] = 2i\epsilon_{\alpha\beta}(1 + \mu k)\,, \qquad \{q_\alpha, k\} = 0\,, \qquad [\bar{q}_{\dot\alpha}, k] = 0\,, \qquad (6.21)$$

$$[\bar{q}_{\dot\alpha}, \bar{q}_{\dot\beta}] = 2i\epsilon_{\dot\alpha\dot\beta}(1 + \bar{\mu}\bar{k})\,, \qquad \{\bar{q}_{\dot\alpha}, \bar{k}\} = 0\,, \qquad [q_\alpha, \bar{k}] = 0\,. \qquad (6.22)$$

The reality conditions $q_\alpha^\dagger = \bar{q}_{\dot\alpha}$ imply that $\mu = \nu e^{i\theta}$, $\bar{\mu} = \nu e^{-i\theta}$ for real ν. Geometrically, the deformed double of the HSA corresponds to the direct product of two noncommutative super-spheres $S^{2|2} \times S^{2|2}$ that have the same (absolute) value of radii.

The deformed Lorentz and translation generators are given by the same formulae

$$\mathsf{L}_{\alpha\beta} = -\frac{i}{4}\{q_\alpha, q_\beta\}\,, \qquad \mathsf{P}_{\alpha\dot\alpha} = -\frac{i}{4}\{q_\alpha, \bar{q}_{\dot\alpha}\}\,, \qquad \bar{\mathsf{L}}_{\dot\alpha\dot\beta} = -\frac{i}{4}\{\bar{q}_{\dot\alpha}, \bar{q}_{\dot\beta}\}\,. \qquad (6.23)$$

The first place where the commutators deform is

$$[\mathsf{P}_{\alpha\dot\alpha}, \mathsf{P}_{\beta\dot\beta}] = (1 + \mu k)\epsilon_{\alpha\beta}\bar{\mathsf{L}}_{\dot\alpha\dot\beta} + (1 + \bar{\mu}\bar{k})\epsilon_{\dot\alpha\dot\beta}\mathsf{L}_{\alpha\beta}\,. \qquad (6.24)$$

This is consistent with the structure of the dual cycles (B.10). The deformation that is isomorphic to the one obtained by setting $d = 3$ in the free boson case corresponds to $\theta = 0$ and projection by $(1 + k\bar{k})/2$. Now, the question of uniqueness of the deformation described above is of physical significance since the resulting A_∞-algebra is supposed to describe the slightly broken higher spin symmetry realized in the Chern-Simons matter theories. The double D is a particular case of the smash-product algebras. The Hochschild cohomology of such algebras is known [92] and in our case the second Hochschild cohomology group is two-dimensional [47, 93]. Therefore, we exhaust all possible deformations. The relevance of these statements is also discussed below.

7 Concluding remarks

The slightly broken higher spin symmetry is expected to fix all correlation functions in Chern-Simons matter theories, at least in the large-N limit, which would also explain and prove the three-dimensional bosonization duality. The problem is that it is not a symmetry in any usual sense. In the free limit each CFT leads to a well-defined infinite-dimensional associative algebra, a higher spin algebra (HSA). HSA determines the correlation functions of higher spin currents. When interactions are turned on, the conservation of the higher spin currents is broken by the double-trace operators built of the currents themselves. The fact that higher spin currents, as a representation, are formally isomorphic to the HSA (up to the $\mathbb{Z}_2$-automorphism given by the inversion map $\mathtt{I}$) makes it clear that slightly broken higher spin symmetry is a deformation of a given HSA by an element of HSA itself (up to $\mathtt{I}$).

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[31]The same algebra appeared in [91] as $\mathcal{N} = 2$ super-HSA.

While there is no place for such a deformation in the realm of associative algebras, this can be achieved by going to the A_∞-setting, which is the main proposal of the paper.

One of the main results of the paper is the explicit construction of a class of A_∞-algebras that can be viewed as noncommutative deformation quantization of a given associative algebra A. We show that if A admits a deformation as an associative algebra, then we can replace the formal deformation parameter $\hbar$ by an element of A itself by going to the A_∞ setting. It turns out that the structure maps m_n of the A_∞-algebra are completely determined (up to a natural equivalence) by the deformation of A. Therefore, this new class of A_∞-algebras is completely determined by deformations of associative algebras.

Combining these two findings, the problem of the slightly broken higher spin symmetry gets reduced to a much simpler problem of deforming the $\mathbb{Z}_2$-extension of a given HSA. This deformation may depend on several parameters, the number being given by the size of the Hochschild cohomology. We argue that there is an at least one-parameter deformation, the parameter being $1/N$.

The case of three dimensions is special. The deformation is found to involve the pair of free parameters μ and $\bar\mu$, while it has only one deformation parameter for the free boson HSA in $d > 3$. Taking the reality conditions into account, one can put $\mu = \nu e^{i\theta}, \bar\mu = \nu e^{-i\theta}$. The microscopical description of these $3d$ CFT's with slightly broken higher spin symmetry is via the Chern-Simons matter theories with the two parameters N and k (in the simplest situation). So far the deformation parameters θ and ν are just phenomenological. At least in the large-N limit it is possible [30, 94] to relate them to the microscopical parameters $\theta = \frac{\pi}{2}\frac{N}{k}, \nu \sim \tilde{N}^{-1}, \tilde{N} = 2N\frac{\sin \pi\lambda}{\pi\lambda}$. It is remarkable that the higher spin symmetry breaking in these theories is fully described by a (two copies) rather simple associative algebra of fuzzy super-sphere $S^{2|2}$. For the free case the correlation functions of higher spin currents are given by a unique invariant of the HSA — the trace [24–26, 28]. Since the deformed HSA does also admit a trace, it is natural to conjecture that the correlation functions of the single-trace operators in the Chern-Simons matter theories are expressible in terms of the same type of invariants:

$$\langle J_1 \ldots J_n \rangle \qquad \longleftrightarrow \qquad \mathrm{Tr}(C_1 * \ldots * C_n) + \text{permutations}. \qquad (7.1)$$

Here the trace is the invariant trace of the deformed higher spin algebra and C_i are 'wavefunctions' similar to those of [24–26, 28]. The expression is manifestly invariant under the infinite-dimensional deformed symmetries. The dependence on θ enters implicitly via Tr, $*$ and C_i. This is a smooth deformation of the free CFT correlation functions.

As a side remark, let us point out a relation between the deformation of the $\mathbb{Z}_2$-extended higher spin algebras (HSA) and deformation quantization. HSA result from quantization of the algebra of functions on a Poisson manifold M which is the closure of a nilpotent coadjoint orbit of $so(d,2)$, see [63]. A Poisson manifold may have some discrete symmetries G, e.g. $G = \mathbb{Z}_2$ related to the inversion map. Given G there are two complementary algebras: functions on the Poisson orbifold M/G and the smashed product of functions on M with G (we called it a double, $D(hs)$). These two algebras can have new deformations compared to the usual deformation quantization, see e.g. [40]. The deformed HSA's we constructed are examples of this situation. The algebras depend on (at

least) two deformation parameters, where the first deformation parameter (it was implicit in the paper) comes from the usual deformation quantization and the second one is due to the $\mathbb{Z}_2$-automorphism. These remarks make deformed HSA's be a part of deformation quantization.

Lastly, it is worth mentioning that the same A_∞-algebras that we constructed in the paper allows one to solve [95] the problem of Formal Higher Spin Gravities, which can indirectly, via AdS/CFT, explain the relevance of these A_∞-algebras.

Acknowledgments

A First order deformation

The first-order deformation of an A_{-1}-bimodule A_0, regarded as an A_∞-algebra, is a collection of six tri-linear maps $m_3(\bullet, \bullet, \bullet)$ obeying the equations

$$-am_3(b, c, u) + m_3(ab, c, u) - m_3(a, bc, u) + m_3(a, b, cu) = 0, \quad \text{(A.1a)}$$

$$m_3(a, b, u)c - am_3(b, u, c) + m_3(ab, u, c) - m_3(a, bu, c) - m_3(a, b, uc) = 0, \quad \text{(A.1b)}$$

$$m_3(a, u, b)c - am_3(u, b, c) + m_3(au, b, c) + m_3(a, ub, c) - m_3(a, u, bc) = 0, \quad \text{(A.1c)}$$

$$m_3(u, a, b)c - m_3(ua, b, c) + m_3(u, ab, c) - m_3(u, a, bc) = 0, \quad \text{(A.1d)}$$

and

$$m_3(a, b, u)v - am_3(b, u, v) + m_3(ab, u, v) - m_3(a, bu, v) = 0, \quad \text{(A.1e)}$$

$$m_3(u, v, a)b - um_3(v, a, b) + m_3(u, va, b) + m_3(u, v, ab) = 0, \quad \text{(A.1f)}$$

$$m_3(a, u, b)v - am_3(u, b, v) + m_3(au, b, v) + m_3(a, ub, v) - m_3(a, u, bv) = 0, \quad \text{(A.1g)}$$

$$-m_3(u, a, v)b - um_3(a, v, b) - m_3(ua, v, b) + m_3(u, av, b) + m_3(u, a, vb) = 0, \quad \text{(A.1h)}$$

$$-m_3(a, u, v)b - am_3(u, v, b) + m_3(au, v, b) + m_3(a, u, vb) = 0, \quad \text{(A.1i)}$$

$$m_3(u, a, b)v - um_3(a, b, v) - m_3(ua, b, v) + m_3(u, ab, v) - m_3(u, a, bv) = 0, \quad \text{(A.1j)}$$

where a, b, c are elements of A_{-1} and $u, v, w \in A_0$. It is easy to see that (3.8) and (3.8) are solutions. These two solutions are equivalent via an A_∞ change of variables, which at this order is $m_3 \to m_3 + \delta f$, for $f(a, u) = \phi_1(a, u)$.

More generally, the first equation (A.1a) seems to be the most important one. Its nontrivial solutions correspond to the second Hochschild cohomology group $HH^2(A, \mathcal{M})$, where $\mathcal{M}$ is $\text{Hom}(M, M)$ endowed with the natural bimodule structure (in our case $M \sim A_0$). If $A_{-1} \sim A_1$ is the polynomial Weyl algebra on two generators, then $HH^3(A_1, N) = 0$

for any bimodule N as the enveloping algebra A_1^e admits a projective resolution of length 2. This means that the deformations are unobstructed. The same holds true for the matrix algebras $\text{Mat}_n(A_1)$ acting on the bimodule $\text{Mat}_n(A_1)$ (the algebras A_1 and $\text{Mat}_n(A_1)$ are Morita equivalent).

B   Deformations of higher spin algebras

In order to construct the A_∞-algebra description of the slightly broken higher spin symmetry we need to construct a deformation of the double $D(hs)$ of a given HSA. It is hard to prove that such deformation always exist due to the flexibility of what HSA means. We adopt Definition 3 via quotients of $U(so(d,2))$. We show that the I-stable subalgebra, $\text{I}(a) = a$, $a \in hs$, of hs, turns out to be the HSA of the generalized free field in $d-1$ dimension and its weight is generic. Therefore, the subalgebra admits a deformation. It turns out that this deformation can be uplifted to $D(hs)$.

Higher spin Lorentz subalgebra.   The most convenient definition of HSA at the moment is via the universal enveloping algebra $U(so(d,2))$. Suppose we are given some hs as $hs = U(so(d,2))/\mathcal{J}$ for some $\mathcal{J}$. Let us also assume that hs corresponds to some free on-shell field. The $so(d,2)$-generators T^{AB} can be split into the AdS-Lorentz generators $\text{L}^{\mathcal{AB}}$ and AdS-translations $\text{P}^{\mathcal{A}}$.[32] The AdS-Lorentz subalgebra $L(hs)$ of hs is defined as the enveloping algebra of the $so(d,1)$ subalgebra generated by $\text{L}^{\mathcal{AB}}$. This is the stability algebra of the inversion map.[33]

The Lorentz subalgebra $L(hs)$ can be understood as a HSA itself ($so(d,1)$ is viewed here as the Euclidian conformal algebra in $d-1$ dimensions): it has more or less the same properties, but the Casimir value corresponds to an off-shell conformal field in $(d-1)$ dimensions.

For example, the ideal that is responsible for the free boson HSA, when T^{AB} is decomposed into $\text{L}^{\mathcal{AB}}$ and $\text{P}^{\mathcal{A}}$, reads:

$$\mathcal{J}^{\mathcal{ABCD}} = \text{L}^{[\mathcal{AB}}\text{L}^{\mathcal{CD}]} \,, \tag{B.1a}$$

$$\mathcal{J}^{\mathcal{ABC}5} = \{\text{L}^{[\mathcal{AB}}, \text{P}^{\mathcal{C}]}\} \,, \tag{B.1b}$$

$$\mathcal{J}^{\mathcal{AB}} = \text{L}^{\mathcal{A}}{}_{\mathcal{C}}\,\text{L}^{\mathcal{BC}} + \text{L}^{\mathcal{B}}{}_{\mathcal{C}}\,\text{L}^{\mathcal{AC}} - \text{P}^{\mathcal{A}}\text{P}^{\mathcal{B}} - \text{P}^{\mathcal{B}}\text{P}^{\mathcal{A}} - (d-2)\eta^{\mathcal{AB}} \,, \tag{B.1c}$$

$$\mathcal{J}^{\mathcal{A}5} = \{\text{L}^{\mathcal{A}}{}_{\mathcal{C}}, \text{P}^{\mathcal{C}}\} \,, \tag{B.1d}$$

$$\mathcal{J}^{55} = 2\text{P}_{\mathcal{A}}\text{P}^{\mathcal{A}} + (d-2) \,, \tag{B.1e}$$

$$\mathcal{J} = -\frac{1}{2}\text{L}_{\mathcal{AB}}\text{L}^{\mathcal{AB}} + \text{P}_{\mathcal{A}}\text{P}^{\mathcal{A}} + \frac{1}{4}(d^2 - 4) \,, \tag{B.1f}$$

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[32]For example, $\text{L}^{\mathcal{AB}} = T^{\mathcal{AB}}$ and $\text{P}^{\mathcal{A}} = T^{\mathcal{A}5}$ where 5 is the extra dimension of an $so(d,2)$ vector as compared to an $so(d,1)$ one $\eta^{55} = -1$. Here $\mathcal{A}, \mathcal{B}, \ldots = 0, \ldots, d$ are the indices of the AdS Lorentz algebra $so(d,1)$.

[33]Another reason for the relevance of the AdS-Lorentz interpretation is that $\text{IL}^{\mathcal{AB}}\text{I} = \text{L}^{\mathcal{AB}}$ and $\text{IP}^{\mathcal{A}}\text{I} = -\text{P}^{\mathcal{A}}$ if we define $\text{I}P^a\text{I} = -K^a$ etc. Such automorphism of the AdS-algebra is used in the study of higher spin fields in AdS, see e.g. [89]. If we define $\text{I}P^a\text{I} = K^a$ etc., then the stability algebra is $so(d-1,2)$, which can be interpreted as the conformal algebra in $(d-1)$-dimension. Such definition is more physical since it is the $so(d-1,2)$ subalgebra that would admit supersymmetric extensions once $so(d,2)$ does for lower d. Nevertheless, below we mostly use the $so(d,1)$-interpretation.

from which it follows

$$\mathcal{J}^{\mathcal{ABCD}} = \mathsf{L}^{[\mathcal{AB}}\mathsf{L}^{\mathcal{CD}]}, \qquad\qquad \mathcal{J} = -\frac{1}{2}\mathsf{L}_{\mathcal{AB}}\mathsf{L}^{\mathcal{AB}} + \frac{d}{4}(d-2). \qquad (\text{B.2})$$

This is exactly the ideal that defines the HSA of the generalized free field, but in one dimension lower, cf. (4.17) and [62]. The conformal weight of this fictitious generalized free field in $(d-1)$ dimensions is $(d-2)/2$ or $d/2$.[34]

Thanks to the fact that the weight of this fictitious generalized free field is generic the Lorentz subalgebra belongs to a one-parameter family of algebras. Therefore, the Lorentz subalgebra can be deformed. In particular, the second Hochschild cohomology group is not empty, $HH^2(L(hs), L(hs)) \neq 0$.

Deformation of the double. That the I-stable subalgebra $L(hs)$ admits a one-parameter family of deformations is an indication that the I-extended algebra $D(hs)$ also does. It seems to be no general theorem, however, that would allow one to directly construct such a deformation.[35] The following three justifications are helpful. (1) In the case of the smash-product of the Weyl algebra by a finite group of symplectic reflections (which is the case that many HSA's can be reduced to) it can be shown that such deformations do exist and it is even possible to explicitly construct them, see [46, 47]. Note that the Weyl algebra itself is rigid and therefore extending it with $\mathbb{Z}_2$ is crucial for the deformation. (2) For many algebras there is a duality [96] between Hochschild homology and cohomology and we can explicitly construct the cycle that the sought for Hochschild two-cocycle is dual to (see below). (3) At least for the algebras we are interested in this paper there is a simple oscillator realization and in section 6 we construct the deformed double $D_\hbar(hs)$ explicitly.

Dual cycle. Cochains act in a natural way on chains, so that the latter form a module over the former [97]. As different from cocycles, cycles are usually easier to find. Then, if the algebra falls into the class of algebras for which the Hochschild cohomology $HH^\bullet(A)$ is dual to the homology $HH_\bullet(A)$, one can compute the dimension of various $HH^\bullet(A)$ from those of $HH_\bullet(A)$. Another usage of nontrivial cycles is to test whether a given cocycle is nontrivial since the chain differential is dual to the cochain differential with respect to the natural pairing. We will construct a cycle for $D(hs)$, which implies that there is a dual cocycle.

Note first, that the HSA hs of some free field[36] is determined by a certain two-sided ideal $\mathcal{J}$ of $U(so(d,2))$. For the free fields obeying the $\square$-type equation the ideal contains the generator described by the Young diagram $\square\square$. Taken together with the fixed value of the Casimir operator this means that the AdS-momentum squares to a constant:

$$\mathsf{P}_{\mathcal{A}}\mathsf{P}^{\mathcal{A}} = M^2. \qquad (\text{B.3})$$

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[34]The value of the Casimir operator is $\Delta(\Delta - (d-1))$. Notice that both the roots are above the unitarity bound $(d-1)/2 - 1$.

[35]Indeed, the same argument applies to the hs itself, while it is usually rigid, as different from $D(hs)$.

[36]Here we avoid generalized free fields at generic value of the conformal weight.

For example, for the HSA of the free boson CFT we find [74] (B.1e)[37]

$$\mathsf{P}_{\mathcal{A}}\mathsf{P}^{\mathcal{A}} = -\frac{(d-2)}{2}\,. \tag{B.4}$$

Now, consider the two-chain[38]

$$\gamma = 1 \otimes \mathsf{P}_{\mathcal{A}} \otimes \mathsf{P}^{\mathcal{A}}\,. \tag{B.5}$$

It is a nontrivial cycle of hs with values in the representation hs^I that is twisted by $\mathtt{I}$:

$$\partial\gamma = \mathsf{P}_{\mathcal{A}} \otimes \mathsf{P}^{\mathcal{A}} - 1 \otimes \mathsf{P}_{\mathcal{A}}\mathsf{P}^{\mathcal{A}} + \mathtt{I}(\mathsf{P}^{\mathcal{A}}) \otimes \mathsf{P}_{\mathcal{A}} = 0\,. \tag{B.6}$$

Here we used (B.3) and the fact that the complex is normalized, i.e., $M^2 \sim 0$ when it appears in any of the factors except the first one. In this case it is easy to uplift the cycle from the normalized complex to the original one. Indeed,

$$\gamma' = 1 \otimes \mathsf{P}_{\mathcal{A}} \otimes \mathsf{P}^{\mathcal{A}} + M^2(1 \otimes 1 \otimes 1) \tag{B.7}$$

is closed as it is. Therefore, γ' represents a class in $HH_2(hs, hs^I)$. This cycle can also be uplifted to the cycle of the full double $D(hs)$

$$\gamma' = \mathtt{I} \otimes \mathsf{P}_{\mathcal{A}} \otimes \mathsf{P}^{\mathcal{A}} + M^2(\mathtt{I} \otimes 1 \otimes 1)\,, \tag{B.8}$$

representing thus an element of $HH_2(D(hs), D(hs))$. It should be dual to a nontrivial cocycle ϕ representing an element of $HH^2(D(hs), D(hs))$. This is the cocycle (5.5) we need to deform $D(hs)$.

Let us also consider the special case of three dimensions. Firstly, we can replace (B.8) with an equivalent two-cycle

$$\gamma' = \mathsf{L}_{\mathcal{AB}} \otimes \mathsf{P}^{\mathcal{A}} \otimes \mathsf{P}^{\mathcal{B}} + \frac{1}{4}1 \otimes \mathsf{L}_{\mathcal{AB}} \otimes \mathsf{L}^{\mathcal{AB}} - \frac{1}{2}C_L 1 \otimes 1 \otimes 1\,, \qquad \partial\gamma' = 0\,, \tag{B.9}$$

where $C_L = -\frac{1}{2}\mathsf{L}_{\mathcal{AB}}\mathsf{L}^{\mathcal{AB}}$ is the value of the Casimir operator of the AdS-Lorentz subalgebra, see e.g. (B.2). Secondly, in the $sl(2,\mathbb{C})$-spinorial language the generators T^{AB} of $so(3,2)$ decompose into $\mathsf{P}_{\alpha\dot\alpha}$, and $\mathsf{L}_{\alpha\beta}$, $\mathsf{L}_{\dot\alpha\dot\beta}$,[39] the latter being (anti-)selfdual components of $\mathsf{L}_{\mathcal{AB}}$. Then, (B.9) reduces to the two independent cycles:

$$\gamma' = \mathsf{L}^{\alpha\beta} \otimes \mathsf{P}_{\alpha\dot\beta} \otimes \mathsf{P}_{\beta}{}^{\dot\beta} + \frac{1}{2}1 \otimes \mathsf{L}_{\alpha\beta} \otimes \mathsf{L}^{\alpha\beta} - c_L 1 \otimes 1 \otimes 1\,, \qquad \partial\gamma' = 0\,, \tag{B.10}$$

---

[37]This is the AdS-base rewriting of $P^a P_a = 0$, $K^a K_a = 0$, and $\boldsymbol{C}_2 - \frac{1}{4}(d^2 - 4) = 0$.

[38]The Hochschild differential acts as (note the twist by $\mathtt{I}$)

$$\partial(c_0 \otimes c_1 \otimes \cdots \otimes c_k) = c_0 c_1 \otimes c_2 \cdots \otimes c_k - c_0 \otimes c_1 c_2 \otimes \cdots \otimes c_k + \cdots + (-)^k I(c_k)c_0 \otimes c_1 \cdots \otimes c_{k-1}\,.$$

The arguments $c_1, \ldots, c_k$ are assumed to take values in the quotient space hs/K, where $K \subset hs$ is the base field. In practice this means that $K \sim 0$ for all the factors except for the first one. Such complex is called *normalized* and it is known to have the same homology, $HH_\bullet(hs/K, hs^I) \sim HH_\bullet(hs, hs^I)$.

[39]Here, α, β and $\dot\alpha, \dot\beta$ are the indices of the fundamental representation of $sl(2,\mathbb{C})$ and its conjugate. The dictionary between the vectorial and spinorial languages is via the σ-matrices, e.g. $\mathsf{P}_{\mathcal{A}} = \sigma_{\mathcal{A}}^{\alpha\dot\alpha}\mathsf{P}_{\alpha\dot\alpha}$.

where $c_L = -\frac{1}{2}\mathsf{L}_{\alpha\beta}\mathsf{L}^{\alpha\beta} = -3/4$ and the second cycle is obtained by complex conjugation. These two cycles imply that there are two independent deformations of the free boson HSA and free fermion HSA (which is the same) in three dimensions. In section 6 we provide a full description of the deformed algebra for the examples of interest. In order to better understand the reason for the extended higher spin algebras to admit a deformation it would be instructive to study the representation theory of the deformed algebras and its field-theoretical realizations.

C   Sketch of the proof

We need to check that m_n defined in section 3 do solve the Maurer-Cartan equation

$$\delta m_n + \sum_{i+j=n+2} m_i \circ m_j = 0. \tag{C.1}$$

Due to the specific form of m_n (with arguments from A_{-1} on the left) there are fewer equations to be checked. Firstly, one can restrict oneself to the sector with three A_{-1} factors and $n-2$ factors in A_0, i.e., the arguments in (C.1) are permutations of $abcuv\ldots w$. Secondly, the nontrivial equations can be parameterized by the position of c:

$$E_k(a, b, \ldots, u, c, \overbrace{v, \ldots, w}^{k}) = \delta m_n + \sum_{i+j=n+2} m_i \circ m_j \Big|_{a,b,\ldots,u,c,v,\ldots,w} = 0. \tag{C.2}$$

The differential δ is very simple for most of k's, $k = 1, \ldots, n-3$:

$$\delta m_n(a, b, \ldots, u, c, v, \ldots, w) = -m_n(a, b, \ldots, uc, v, \ldots, w) + m_n(a, b, \ldots, u, cv, \ldots, w) \tag{C.3}$$

and contains four terms for the maximal $k = n-2$

$$\delta m_n(a, b, c, v, \ldots, w) = -a m_n(b, c, v, \ldots, w) + m_n(ab, c, v, \ldots, w) + \\ - m_n(a, bc, v, \ldots, w) + m_n(a, b, cv, \ldots, w). \tag{C.4}$$

The rationale for the recursive formula given in the main text is that the differential (C.3) annihilates those components of m_n that have too many multiplicative arguments on the right. Therefore, one can start at $k = 1$, to which only m_3 and m_{n-1} contribute:

$$E_1(a, b, \ldots, u, c, w) = - m_n(a, b, \ldots, uc, w) + m_n(a, b, \ldots, u, cv) + \\ - m_{n-1}(a, b, \ldots, m_3(u, c, w)) + m_3(m_{n-1}(a, b, \ldots, u), c, v) = 0.$$

This equation determines the part of f_n that has no multiplicative arguments at all. Using $m_{n-1} = f_{n-1}(a, b, \ldots)u$ and explicit form of m_3, one observes that $f_n = \phi_1(f_{n-1}(a, b, \ldots), u)$, i.e., $m_n = f_n(a, b, \ldots, u)w$, up to the terms with more direct factors. Next, one should proceed to $k = 2$ and fix the part of f_n that has one direct factor. At each order one will get the equations that are supposed to be true for $m_{3,\ldots,k+2}$. The trick here is that E_k contains Gerstenhaber products of $m_3, m_{n-1}, \ldots, m_{k+2}, m_{n-k}$ and

the lowest f_{n-k} always enters with the same arguments, i.e., can be treated as a single variable. Eventually, E_k can be reduced to equations for $m_3, \ldots, m_{k+2}$ irrespective of n. For example, $E_1 = 0$ is solved by $f_n = \phi_1(f_{n-1}, \bullet)$ irrespective of what f_{n-1} is, but the same time this fixes the lowest component of f_{n-1} itself, and so on.

A non-recursive proof is based on manipulations with the trees. Let us recall that f_n is a sum over all terms that are depicted by trees (with one branch)[40]

$$f_n(a, b, u, \ldots, w) \ni \qquad \qquad \text{(C.5)}$$

To deal with more complicated trees we introduce an order. A tree is called ordered if it does not contain vertices of the form

$$\qquad \qquad \text{(C.6)}$$

i.e., in the actual expression any $\phi_{m+1}(\bullet, \bullet)$ does not have any factors on the left, e.g. $a\phi_{m+1}(b, c)$, where a can be any expression possibly containing several factors and other ϕ's. The bad vertices can be ordered via

$$\qquad \qquad \text{(C.7)}$$

Equation E_k contains several terms, those coming from δm_n are already ordered (except for E_{n-2}). Also, only good vertices arise when m_i is inserted into m_j as an (right) argument of some ϕ_k:

$$\qquad \qquad \text{(C.8)}$$

The only source of bad vertices is when m_i in inserted into an argument of m_j that corresponds to a multiplicative argument (simple vertex):

$$\qquad \qquad \text{(C.9)}$$

---

[40]Recall that the (green) dots correspond to some ϕ_{m+1}, while the simple vertices are mapped into insertions of multiplicative arguments on the right.

These terms need to be reordered and will eventually generate (with the opposite sign) all the trees with one branch (C.5) or two branches (C.8) that are already present. Therefore, the Maurer-Cartan equation is indeed satisfied. It would be interesting to find an appropriate configuration space where the proof would reduce to the Stokes theorem.

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