

Poor man's scaling: anisotropic Kondo and Coqblin–Schrieffer models

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We discuss Kondo effect for a general model, describing a quantum impurity with degenerate energy levels, interacting with a gas of itinerant electrons, and derive scaling equation to the second order for such a model. We show how the scaling equation for the spin-anisotropic Kondo model with the power law density of states (DOS) for itinerant electrons follows from the general scaling equation. We introduce the anisotropic Coqblin–Schrieffer model, apply the general method to derive scaling equation for that model for the power law DOS, and integrate the derived equation analytically.

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I. INTRODUCTION

Observed under appropriate conditions logarithmic increase (with the decreasing of temperature) of the scattering of the itinerant electrons by an isolated magnetic impurity was explained in 1964 in a seminal paper by Kondo, entitled "Resistance Minimum in Dilute Magnetic Alloy"¹. Soon after it became clear that the phenomenon is manifested not only in resistivity, but in nearly all thermodynamic and kinetic properties². The theoretical analysis of this effect turned out to be very fruitful, and led to the appearance of many approaches and techniques, which became paradigms in many totally different fields of physics. One of such approaches was the so called poor man's scaling, pioneered by Anderson³.

Though initially only magnetic impurity scattering was considered, later it became understood that similar effect can appear in case of a general quantum impurity. Such a general model was thoroughly reviewed in 1998 by Cox & Zawadovski⁴.

Recently the models where the density of states (DOS) of itinerant electrons in the vicinity of the Fermi level is the power function of energy has attracted a lot of interest^{5–11}. The scaling was generalized to be applicable to such systems in the work by Withoff and Fradkin¹².

Following the long line of works where spin-anisotropic Kondo model was studied^{3,10,13,14}, we decided to revisit the problem of scaling equation for the Kondo effect in general⁴, and introduce and study the anisotropic Coqblin–Schrieffer (CS) model¹⁵ in particular. Notice that it is known that the spin anisotropy can substantially change the physics of the Kondo effect in comparison with the isotropic case^{4,16–18}. The CS model, though being well studied previously^{2,19–24}, draw a lot of attention recently in connection with the studies of quantum dots²⁵, heavy fermions²⁶ and ultra-cold gases^{27,28}.

The rest of the paper is constructed as follows. We formulate in Section II a poor man's scaling equation to second order for a general model, describing a quantum impurity embedded into a gas of itinerant electrons. We also generalize the equation to the case of power law be-

havior of the DOS in the vicinity of the Fermi level. In Section III we show how the obtained earlier scaling equation for the spin-anisotropic Kondo model follow from the general scaling equation. In Section IV the XXZ CS model is introduced and scaling equation for this model is derived. Then everything is generalized to the case of the anisotropic CS model. The scaling equation both in the particular case of the XXZ CS model and in the general case of the anisotropic CS model are integrated analytically. We conclude in Section V. Some mathematical spin-offs are presented in the Appendix.

II. PERTURBATION THEORY

A. Kondo effect as explained by Kondo

The Hamiltonian we start from is⁴

$$\begin{aligned} H &= H_0 + V \\ &= \sum_{\mathbf{k}\alpha} \epsilon_{\mathbf{k}} c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\alpha} + \sum_{\substack{\mathbf{k}\mathbf{k}' \\ \alpha\beta, ab}} V_{\beta\alpha, ba} X_{ba} c_{\mathbf{k}'\beta}^\dagger c_{\mathbf{k}\alpha}, \end{aligned} \quad (1)$$

where $c_{\mathbf{k}\alpha}^\dagger$ and $c_{\mathbf{k}\alpha}$ are electron creation and annihilation operators of itinerant electron with wave vector \mathbf{k} and internal quantum number α , $\epsilon_{\mathbf{k}}$ is the energy of the electron; $X_{ba} = |b\rangle\langle a|$, where $|a\rangle, |b\rangle$ are the internal states of the scattering system, is the Hubbard X -operator.

In this paper we use the old-fashioned on-the-energy-shell perturbation theory²⁹ following the paper by Kondo; similar approach was applied to the Anderson model by Haldane³⁰. For the Hamiltonian (1) the transition probability per unit time from the initial state of the whole system m to the final state n is given to the second Born approximation by¹

$$W_{n\leftarrow m} = 2\pi\delta(E_n - E_m)|T_{nm}|^2, \quad (2)$$

where

$$T_{nm} = V_{nm} + \sum_{\ell} \frac{V_{n\ell}V_{\ell m}}{E_m - E_{\ell}} + \dots \quad (3)$$

is the scattering matrix. Explicitly Eq. (3) takes the form¹

$$T_{\beta\alpha,ba} = V_{\beta\alpha,ba} + \sum_{\gamma,c} V_{\beta\gamma,bc} V_{\gamma\alpha,ca} \sum_{\epsilon_{\mathbf{q}} > 0} \frac{1}{\epsilon - \epsilon_{\mathbf{q}}} - \sum_{\gamma,c} V_{\gamma\alpha,bc} V_{\beta\gamma,ca} \sum_{\epsilon_{\mathbf{q}} < 0} \frac{1}{\epsilon_{\mathbf{q}} - \epsilon}. \quad (4)$$

Writing Eq. (4) we assumed that in the state m all the electron states below the Fermi surface (corresponding to $\epsilon_{\mathbf{k}} = 0$) are occupied, and there is an additional electron with the wave vector \mathbf{k} and internal quantum number α ; the scattering system is in the state $|a\rangle$. In the state n all the electron states below the Fermi surface are again occupied, and there is an additional electron with the wave vector \mathbf{k}' ($\epsilon_{\mathbf{k}'} = \epsilon_{\mathbf{k}} = \epsilon$) and internal quantum number β ; the scattering system is in the state $|b\rangle$.

In the R.H.S. of Eq. (4) the second term describes the processes when the electron with $\mathbf{k}\alpha$ is first scattered to the unoccupied state $\mathbf{q}\gamma$ and then to $\mathbf{k}'\beta$, and the third term describes the processes when an electron from an occupied state $\mathbf{q}\gamma$ is first scattered to $\mathbf{k}'\beta$ and then the electron with $\mathbf{k}\alpha$ fills up the state $\mathbf{q}\gamma$ which is now empty¹.

Equation (4) clearly explains the connection between the dynamics of the scattering system and the Kondo effect. For static impurity, Eq. (3) takes the form

$$T_{\beta\alpha} = V_{\beta\alpha} + \sum_{\gamma,\mathbf{q}} \frac{V_{\beta\gamma}V_{\gamma\alpha}}{\epsilon - \epsilon_{\mathbf{q}}} + \dots \quad (5)$$

Because all the integrals with respect to energy in perturbation series terms are understood in the Principal Value sense, the denominator going to zero is by itself not a problem, and the second order terms just gives a correction to the matrix element of the order of the ratio of the scattering energy V to the band width (we consider electron band $\epsilon \in [-D_0, D_0]$ and assume that the ratio is V/D is small). In addition, this correction is only weakly ϵ -dependent. On the other hand, each of the second order terms in Eq. (4) contains large logarithmic multiplier $\ln(\epsilon/D_0)$, because of strongly asymmetric range of integration²; due to the existence of the impurity quantum numbers, these terms do not add up to Eq. (5).

B. Scaling equation to second order

Equation (4), as it is written down, allows to calculate scattering at high temperatures. However, it allows more – to obtain a scaling equation for the Kondo model in the framework of the approach pioneered by Anderson^{2,3},

which allows to selectively sum up the infinite perturbation series, using explicitly only the first two terms of such an expansion, as presented in Eq. (3).

We are interested only in the matrix elements between the electron states at a distance from the Fermi energy much less than the band width. The brilliant idea of Anderson, applied to the present situation, consists in reducing the band width of the itinerant electrons from $[-D_0, D_0]$ to $[-D_0 - dD, D_0 + dD]$ ($dD < 0$) and taking into account the terms which corresponded to summation in Eq. (4) with respect to the electron states in energy intervals $[-D_0, -D_0 - dD]$ and $[D_0 + dD, D_0]$ by renormalizing $V_{\beta\alpha,ba}$. Notice that such renormalization can be performed provided that $|\epsilon| \ll D_0$ and hence can be discarded in the denominators of the second order terms in Eq. (4). In our particular case, like in general, renormalization is the reduction of the Hilbert space \mathcal{H} and changing the Hamiltonian H so as to keep the physical observable T constant. Thus we obtain

$$dV_{\beta\alpha,ba} = \rho \sum_{\gamma,c} [V_{\beta\gamma,bc}V_{\gamma\alpha,ca} - V_{\gamma\alpha,bc}V_{\beta\gamma,ca}] \frac{dD}{D_0}, \quad (6)$$

where ρ is the density of states of itinerant electrons (assumed to be constant).

Poor man's scaling consists in changing Eq. (6) to

$$dV_{\beta\alpha,ba}(D) = \rho \sum_{\gamma,c} [V_{\beta\gamma,bc}(D)V_{\gamma\alpha,ca}(D) - V_{\gamma\alpha,bc}(D)V_{\beta\gamma,ca}(D)] \frac{dD}{D}, \quad (7)$$

where D is now a running parameter. From Eq. (7) we obtain the scaling equation

$$\frac{dV_{\beta\alpha,ba}}{d \ln \Lambda} = \rho \sum_{\gamma,c} [V_{\beta\gamma,bc}V_{\gamma\alpha,ca} - V_{\gamma\alpha,bc}V_{\beta\gamma,ca}], \quad (8)$$

where $\Lambda = D/D_0$ (actually, the change of (implied) argument of each matrix element $D \rightarrow \Lambda$ in Eq. (8) was done for no reason), and, of course, Eq. (1) should be now understood as

$$H = \sum_{\mathbf{k}\alpha} \epsilon_{\mathbf{k}} c_{\mathbf{k}\alpha}^{\dagger} c_{\mathbf{k}\alpha} + \sum_{\substack{\mathbf{k}\mathbf{k}' \\ \alpha\beta,ab}} V_{\beta\alpha,ba}(\Lambda) X_{ba} c_{\mathbf{k}'\beta}^{\dagger} c_{\mathbf{k}\alpha}. \quad (9)$$

Let the matrix $V_{\beta\alpha,ba}$ is presented as a sum of direct products of matrices, acting in ab and $\alpha\beta$ spaces respectively

$$V = \sum_{p\pi} G_p \otimes \Gamma_{\pi} c_{p\pi}, \quad (10)$$

where the set of matrices $\{G_p\}$ is closed with respect to commutation and hence generates some Lie algebra g ; so is the set of matrices $\{\Gamma_p\}$ (Lie algebra γ).

With the help of Eq. (10) we can write down Eq. (8)

in a more transparent form

$$\begin{aligned} & \sum_{p\pi} G_p \otimes \Gamma_\pi \frac{dc_{p\pi}}{d\ln\Lambda} \\ &= \frac{1}{2}\rho \sum_{s\sigma t\tau} [G_s, G_t] \otimes [\Gamma_\sigma, \Gamma_\tau] c_{s\sigma} c_{t\tau}. \end{aligned} \quad (11)$$

Introducing structure constants of the Lie algebras g and γ as f_{st}^p and $\varphi_{\sigma\tau}^\pi$

$$[G_s, G_t] = i \sum_p f_{st}^p G_p, \quad [\Gamma_\sigma, \Gamma_\tau] = i \sum_\pi \varphi_{\sigma\tau}^\pi \Gamma_\pi, \quad (12)$$

we can write down Eq. (8) in an even more transparent form

$$\frac{dc_{p\pi}}{d\ln\Lambda} = -\frac{1}{2}\rho \sum_{s\sigma t\tau} f_{st}^p \varphi_{\sigma\tau}^\pi c_{s\sigma} c_{t\tau}. \quad (13)$$

Typically, the $\{G_p\}$ ($\{\Gamma_\pi\}$) appear as infinitesimal operators of some Lie group G (Γ), and hence are Hermitian. Actually, this can be said the other way round. Assuming that the matrices $\{G_p\}$ ($\{\Gamma_\pi\}$) are Hermitian, we see that the algebra g (γ) is real, and, hence, by Lie's third theorem is the Lie algebra of some simply connected Lie group³¹. Anyway, if $\{G_p\}$ ($\{\Gamma_\pi\}$) are Hermitian, the matrix $c_{p\pi}$ is real.

Consider an important particular case when $\gamma \equiv g$. (Matrices $\{G_p\}$ and $\{\Gamma_\pi\}$ not necessarily realize the same representation of the algebra, but have the same commutation relations; to emphasize that we will designate $\{\Gamma_\pi\}$ as $\{\Gamma_p\}$.) If we assume that the matrix $c_{p\pi}$, in addition to being real, is symmetric, it can be diagonalized by a unitary transformation of the generators (such transformation does not change the commutation relations), and the matrix keeps its diagonal form in the process of renormalization. Thus Eq. (10) can be "reduced to the principal axes", that is to the form

$$V = \sum_p G_p \otimes \Gamma_p c_p, \quad (14)$$

and Eq. (13) takes the form

$$\frac{dc_p}{d\ln\Lambda} = -\frac{1}{2}\rho \sum_{st} (f_{st}^p)^2 c_s c_t. \quad (15)$$

Equation (15) will be solved in Section III. General analysis of the equation for possible three-dimensional Lie algebras will be presented in Appendix A.

C. Power law DOS

The results of the previous Section can be easily generalized to the case when the electron dispersion law determines the power law dependence of the DOS upon the energy

$$\rho(\epsilon) = C|\epsilon|^r, \quad \text{if } |\epsilon| < D_0, \quad (16)$$

where r can be either positive or negative¹². (We consider in this paper only particle-hole symmetric DOS; the influence of high particle-hole asymmetry on Kondo effect was studied in Ref. 32.) Notice that $r = 0$ corresponds to the previously considered case of flat DOS. In this case instead of Eq. (8) we obtain

$$\frac{dV_{\beta\alpha,ba}}{d\ln\Lambda} = rV_{\beta\alpha,ba} + G \sum_{\gamma,c} [V_{\beta\gamma,bc}V_{\gamma\alpha,ca} - V_{\gamma\alpha,bc}V_{\beta\gamma,ca}], \quad (17)$$

where $G = CD_0^r$ is the DOS at the original band edges. Eq. (17) is invariant with respect to simultaneous change of sign of r , of all matrix elements of V and of the direction of flow. Hence further on we consider explicitly only the case of positive r .

The appearance of the linear term in the R.H.S. of Eq. (17) demands explanation^{12,33}. To get scaling equation, in addition to integrating out the states at the band edges, another procedure is needed, both for the case of flat DOS, and for the case of power law DOS. We did not mention it in the previous Section, because it does not change the equation, but in the case of power law DOS it does. After integrating out the states at the band edges we have to restore the original band width, which demands decrease of the unit of energy by a factor of D_0/D . To keep the DOS constant we should increase the unit of volume by the factor $(D_0/D)^{r+1}$. One should understand that in Eq. (1), V has units of energy multiplied by volume, so after all the rescalings the perturbation is multiplied by the factor of $(D/D_0)^r$, which explains the appearance of the linear term.

To make Eq. (17) look similar to Eq. (8) we introduce new variables $\lambda = (D/D_0)^r$ and $\tilde{V} = V/r\lambda$, after which we can write down Eq. (17) as

$$\frac{d\tilde{V}_{\beta\alpha,ba}}{d\lambda} = G \sum_{\gamma,c} [\tilde{V}_{\beta\gamma,bc}\tilde{V}_{\gamma\alpha,ca} - \tilde{V}_{\gamma\alpha,bc}\tilde{V}_{\beta\gamma,ca}]. \quad (18)$$

D. What is the scaling parameter?

Previously in this Section and in Section II we studied how the effective perturbation at a given energy changes, when the cut-off changes? Alternatively, we can ask ourselves: How does the effective perturbation at a given cut-off change, when the energy changes? That is we are interested in

$$dT_{\beta\alpha,ba}(\epsilon) \equiv T_{\beta\alpha,ba}(\epsilon + d\epsilon) - T_{\beta\alpha,ba}(\epsilon) \quad (19)$$

($d\epsilon$ can be of any sign). Looking at Eq. (4) we understand that

$$\begin{aligned} dT_{\beta\alpha,ba}(\epsilon) &= \sum_{\gamma,c} V_{\beta\gamma,bc} V_{\gamma\alpha,ca} \sum_{-d\epsilon < \epsilon_{\mathbf{q}} < 0} \frac{1}{\epsilon - \epsilon_{\mathbf{q}}} \\ &- \sum_{\gamma,c} V_{\gamma\alpha,bc} V_{\beta\gamma,ca} \sum_{0 < \epsilon_{\mathbf{q}} < d\epsilon} \frac{1}{\epsilon_{\mathbf{q}} - \epsilon} \\ &= \rho \sum_{\gamma,c} [V_{\beta\gamma,bc} V_{\gamma\alpha,ca} - V_{\gamma\alpha,bc} V_{\beta\gamma,ca}] \frac{d\epsilon}{\epsilon}. \end{aligned} \quad (20)$$

(for the sake of definiteness we have chosen $d\epsilon > 0$). Scaling equation we obtain by changing V in the last line of Eq. (20) to T . Thus we recover Eq. (8), only this time $\Lambda = |\epsilon|/D_0$. Similarly, for the power law DOS, we recover Eq. (17), with $\lambda = (|\epsilon|/D_0)^r$.

III. THE SPIN-ANISOTROPIC KONDO MODEL

To see how the general scaling equation is applied let us consider the following spin-anisotropic model (summation with respect to any repeated Cartesian index is implied)

$$H = \sum_{\mathbf{k}\alpha} \epsilon_{\mathbf{k}} c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\alpha} + \sum_{\mathbf{k}\mathbf{k}'\alpha\beta} J_{ij} S^i \sigma_{\alpha\beta}^j c_{\mathbf{k}'\alpha}^\dagger c_{\mathbf{k}\beta}, \quad (21)$$

where S^x, S^y, S^z are the impurity spin operators, $\sigma^x, \sigma^y, \sigma^z$ are the Pauli matrices, and J_{ij} is the anisotropic exchange coupling matrix.

The Hamiltonian (21) appears under many different names. Our opinion is that if we want to choose an eponymic name, the model should be called after T. Kasuya³⁴. However, in line with the tradition we keep the name Kondo model, given because of important contribution to the derivation and analysis of the model made by J. Kondo³⁵.

Taking into account the commutation relations

$$[S^i, S^j] = i\epsilon_{ijk} S^k, \quad [\sigma^i, \sigma^j] = 2i\epsilon_{ijk} \sigma^k, \quad (22)$$

where ϵ is Levi-Civita symbol, from Eq. (18) we obtain the scaling equation¹⁰

$$\frac{d\tilde{J}_{mn}}{d\lambda} = -\epsilon_{ikm} \epsilon_{jln} \tilde{J}_{ij} \tilde{J}_{kl}. \quad (23)$$

(In this Section we measure J and \tilde{J} in units of $r/2G$.)

We assume that the microscopic tensor J_{ik} entering into the Hamiltonian (21) is symmetric (no spin-orbit interaction³⁶). Analysis of the Hamiltonian in the presence of spin-orbit interaction see in Appendix B. In this case the tensor can be reduced to principal axes by rotation of the coordinate system, and it keeps its diagonal form in the process of renormalization. So we can write down the interaction in a diagonal in cartesian indices (though non explicitly rotation covariant) form

$$V = J_i S^i \sigma^i. \quad (24)$$

The scaling equation for this interaction is¹⁰

$$\frac{d\tilde{J}_i}{d\lambda} = -\tilde{J}_j \tilde{J}_k, \quad (25)$$

where i, j, k are all different.

The general solution of Eq. (25) (and, hence, of scaling equation) is written in terms of elliptic functions¹⁰

$$\begin{aligned} J_\alpha &= A\lambda \cdot \text{ns}(A\lambda + \psi, k) \\ J_\beta &= A\lambda \cdot \text{cs}(A\lambda + \psi, k) \\ J_\gamma &= A\lambda \cdot \text{ds}(A\lambda + \psi, k), \end{aligned} \quad (26)$$

where $\{\alpha, \beta, \gamma\}$ is an arbitrary permutation of $\{x, y, z\}$. Using the language of geometry, we say that each flow line lies on the surface of a special cone

$$(1 - k^2)J_\alpha^2 + k^2 J_\beta^2 - J_\gamma^2 = 0. \quad (27)$$

(For $k = 0$ or $k = 1$ the special cone is a pair of planes.) Flow line passing through any given point with the coordinates $J_x^{(0)}, J_y^{(0)}, J_z^{(0)}$ is described by Eq. (26) with α corresponding to the Cartesian component with maximal $|J^{(0)}|$, and β corresponding to the Cartesian component with minimal $|J^{(0)}|$. The parameter k for the cone, the flow line belongs to, is obviously found by substituting $J_x^{(0)}, J_y^{(0)}, J_z^{(0)}$ into Eq. (27). Thus the whole phase space is divided into 3 domains, touching each other, each domain is defined by Eq. (27) with k changing between 0 and 1 and corresponds to all the special cones with the axis along one of the Cartesian axes.

Detailed analysis of the solution (26) was presented earlier¹⁰. Here we would like to discuss only the finite asymptotics of the solution $\lambda \rightarrow 0$. For $\psi \neq 0, 2K(k)$ (where K is the complete elliptic integral of the first kind), $(J_x, J_y, J_z) \rightarrow (0, 0, 0)$, which corresponds to a trivial fixed point. For $\psi = 0$, $(J_x, J_y, J_z) \rightarrow (1, 1, 1)$, and for $\psi = 2K(k)$, $(J_\alpha, J_\beta, J_\gamma) \rightarrow (-1, -1, 1)$. The latter asymptotics correspond to four non-trivial fixed points of the scaling equation

$$\frac{1}{r} \frac{dJ_i}{d \ln \Lambda} = J_i - J_j J_k. \quad (28)$$

IV. THE ANISOTROPIC CS MODEL

A. The CS model

The CS model¹⁵ is represented by the Hamiltonian

$$\begin{aligned} H &= \sum_m \epsilon c_m^\dagger c_m + J \sum_{mm'} X_{mm'} c_m^\dagger c_{m'} \\ &- (J/N) \sum_{mm'} X_{mm'} c_m^\dagger c_{m'}, \end{aligned} \quad (29)$$

where quantum number m changes from 1 to N . (To avoid cluttering, we omit in the equations in this Section

the wave vector indices.) We changed sign of the constant J with respect to the original paper¹⁵, so that the Hamiltonian (29) would look like the Hamiltonian (21). The last term in the R.H.S. of Eq. (29) (and in similar equations further on) is crossed-out to show that it does not give any contribution to the scaling equation, which for the Hamiltonian (29) (and the constant DOS) has the form²

$$\frac{dJ}{d \ln \Lambda} = -N\rho J^2. \quad (30)$$

For $N = 2$ the model coincides with the spin-isotropic Kondo model.

B. The XXZ CS model

If we demand that interaction (24) has $SU(2)$ symmetry, it takes the form

$$V = J\vec{S} \cdot \vec{\sigma}. \quad (31)$$

If we reduce the symmetry to $U(1)$, interaction (24) takes the form

$$V = J_x(S^x\sigma^x + S^y\sigma^y) + J_z S^z\sigma^z. \quad (32)$$

we will call such exchange interaction the XXZ Kondo model.

Equation (32) can be written down using Hubbard operators

$$\begin{aligned} V = & J_x \left(X_{+-} c_{-}^{\dagger} c_{+} + X_{-+} c_{+}^{\dagger} c_{-} \right) \\ & + J_z \left(X_{++} c_{+}^{\dagger} c_{+} + X_{--} c_{-}^{\dagger} c_{-} \right) \\ & - \frac{1}{2} J_z \left(X_{++} + X_{--} \right) \left(c_{+}^{\dagger} c_{+} + c_{-}^{\dagger} c_{-} \right). \end{aligned} \quad (33)$$

Motivated by Eq. (33) we suggest the following Hamiltonian for arbitrary N , which we for obvious reasons will call the XXZ CS model,

$$\begin{aligned} H = & \sum_m \epsilon c_m^{\dagger} c_m + J_x \sum_{m \neq m'} X_{mm'} c_{m'}^{\dagger} c_m \\ & + J_z \sum_m X_{mm} c_m^{\dagger} c_m - \frac{J_z}{N} \sum_{mm'} X_{mm} c_{m'}^{\dagger} c_{m'}. \end{aligned} \quad (34)$$

(An alternative motivation for introducing the model can be found in Appendix C.) Further on in this Section we'll present the calculations only for the case of constant DOS, and only the final results will be written down for the case of the power law DOS.

For the interaction (34) scaling equation (11) becomes

$$\begin{aligned} \frac{dJ_x}{d \ln \Lambda} &= -(N-2)\rho J_x^2 - 2\rho J_x J_z \\ \frac{dJ_z}{d \ln \Lambda} &= -N\rho J_x^2. \end{aligned} \quad (35)$$

For the case of isotropic CS model and for the case $N = 2$, Eq. (35) is reduced to the well established results².

C. Integration of the scaling equation for the XXZ CS model

Dividing two equations in (35) by each other we obtain homogeneous differential equations of the first degree, which can be easily integrated

$$J_z = \frac{C}{\sqrt[N+2]{|w-1|^N |w + \frac{2}{N}|^2}} \quad (36)$$

$$J_x = wJ_z, \quad (37)$$

where C is an arbitrary constant. Substituting the solution (36), (37) into Eq. (35) we obtain

$$\frac{dw}{d \ln \Lambda} = \frac{CN\rho w(w-1)\left(w + \frac{2}{N}\right)}{\sqrt[N+2]{|w-1|^N |w + \frac{2}{N}|^2}}. \quad (38)$$

(For $N = 2$ Eq. (38) can be easily integrated in terms of elementary transcendental functions, otherwise in quadratures.) Eq. (38) has three fixed points: $w = 1$, $w = -2/N$, and $w = 0$. The first two are stable for $C > 0$, the last one is stable for $C < 0$. (One should keep in mind that Λ decreases in the process of evolution.)

At the phase plane with the coordinates J_x, J_z the fixed point $w = 1$ turns into the flow line $J_x = J_z$, and the fixed point $w = -2/N$ turns into the flow line $J_x = -(2/N)J_z$. Both lines are attractors for $J_z > 0$, and repellers (serving as the phase boundaries) for $J_z < 0$. The $w = 0$ fixed point turns into the line of fixed points $J_x = 0$, stable for $J_z < 0$, and unstable for $J_z > 0$. (The half-line $J_x = 0, J_z > 0$ serves as the phase boundary.) Notice that $J_x = J_z = 0$ is a degenerate fixed point³⁷.

Stable fixed point correspond to phases of Eq. (35), characterized by the asymptotic behaviour of the flow lines. The part of the phase plane $J_x > -(2/N)J_z, 0$ is characterized by the attractor $J_x = J_z > 0$, and will be called the Kondo phase I. Notice that in this phase, the $SU(N)$ symmetry, which is absent for the microscopic Hamiltonian, is being recovered in the process of scaling. The part $J_x < -J_z, 0$ is characterized by the attractor $J_x = -(2/N)J_z < 0$, and will be called the Kondo phase II. In the part $-(2/N)J_z \geq J_x \geq J_z$ the flow lines are attracted to the fixed points at the axis $J_x = 0$.

A flow line can reach the fixed point $w = 0$ only in the end of infinitely long evolution, which is a common situation for a fixed point. Hence Ising model is obtained only in the infrared limit. However, in a Kondo phase a flow line reaches fixed point $w = 1$ or $w = -2/N$ after finite evolution (the R.H.S. of Eq. (38) being non-analytic function of w at these points). Singularity of J_x, J_z at finite value of Λ is another indication of the limited applicability (in the Kondo phases) of the truncated to second order scaling equation.

To the general solution (36) we should add singular

solutions

$$\begin{aligned} J_x &= J_z = \frac{1}{N\rho \ln \Lambda + C_1} \\ J_x &= -\frac{2J_z}{N} = \frac{-1}{2\rho \ln \Lambda + C_2}. \end{aligned} \quad (39)$$

Once again we see that evolution starting in one of the Kondo phases hits the singular point at finite value of $\ln \Lambda$.

Equation (36) allows us to plot the flow diagram of the scaling equation, which is presented on Fig. 1. For the sake of definiteness we have chosen $N = 4$. Notice that the flow diagram is qualitatively similar to that of the XXZ Kondo model².

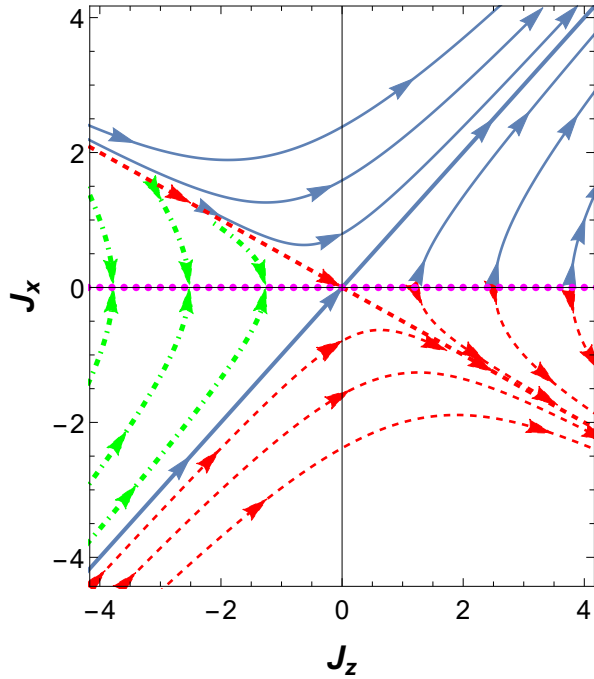


FIG. 1: (color online) Flow diagram for the scaling equation (35) for $N = 4$. Solid (blue) flow lines belong to the Kondo phase I. Dashed (red) flow lines belong to the Kondo phase II. Dashed dotted (green) flow lines belong to the Ising phase. Thick (magenta) dots on the J_z axis are the fixed points.

For the case of the power law DOS Eqs. (36) and (38) become

$$J_z = \frac{C\lambda}{N+2\sqrt{|w-1|^N |w+\frac{2}{N}|^2}} \quad (40)$$

and

$$r \frac{dw}{d\lambda} = \frac{CNG\omega(w-1)(w+\frac{2}{N})}{N+2\sqrt{|w-1|^N |w+\frac{2}{N}|^2}} \quad (41)$$

(Eq. (39) is changed similarly).

Eqs. (40) and (41) is a rigorous but a bit formal result. In particular, it demands some effort to extract out of them the fixed points of the original scaling equation

$$\begin{aligned} r \frac{dJ_x}{d\lambda} &= rJ_x - (N-2)GJ_x^2 - 2GJ_xJ_z \\ r \frac{dJ_z}{d\lambda} &= rJ_z - NGJ_x^2, \end{aligned} \quad (42)$$

which can be easily found by inspection of Eq. (42). The equation has a trivial fixed point $(J_x^*, J_z^*) = (0, 0)$, which is stable for $r > 0$ and unstable for $r < 0$, and two semi-stable (critical) non-trivial fixed points

$$\begin{aligned} J_x^* &= \frac{2-N \pm \sqrt{(N-2)^2 + 8Nr^2}}{4NG} \\ J_z^* &= \frac{NGJ_x^{*2}}{r}. \end{aligned} \quad (43)$$

D. The anisotropic CS model

To formulate the general anisotropic CS model let us return to the spin-anisotropic Kondo model. The interaction can be written down using Hubbard operators

$$\begin{aligned} V &= \frac{J_x}{4} (X_{+-} + X_{-+}) (c_+^\dagger c_- + c_-^\dagger c_+) \\ &\quad - \frac{J_y}{4} (X_{+-} - X_{-+}) (c_+^\dagger c_- - c_-^\dagger c_+) \\ &\quad + J_z (X_{++} c_+^\dagger c_+ + X_{--} c_-^\dagger c_-) \\ &\quad - \frac{J_z}{2} (X_{++} + X_{--}) (c_+^\dagger c_+ + c_-^\dagger c_-). \end{aligned} \quad (44)$$

Motivated by Eq. (44) we suggest the following interaction for arbitrary N

$$\begin{aligned} V &= \frac{J_x}{2} \sum_{m \neq m'} X_{mm'} (c_{m'}^\dagger c_m + c_m^\dagger c_{m'}) \\ &\quad + \frac{J_y}{2} \sum_{m \neq m'} X_{mm'} (c_{m'}^\dagger c_m - c_m^\dagger c_{m'}) \\ &\quad + J_z \sum_m X_{mm} c_m^\dagger c_m - \frac{J_z}{N} \sum_{mm'} X_{mm} c_{m'}^\dagger c_{m'}. \end{aligned} \quad (45)$$

For this interaction scaling equation (11) becomes

$$\begin{aligned} \frac{dJ_x}{d \ln \Lambda} &= -(N-2)\rho J_x J_y - 2\rho J_y J_z \\ \frac{dJ_y}{d \ln \Lambda} &= -(N-2)\rho J_x J_y - 2\rho J_x J_z \\ \frac{dJ_z}{d \ln \Lambda} &= -N\rho J_x J_y. \end{aligned} \quad (46)$$

Like in the previous Subsection we obtain after inte-

gration

$$J_z = \frac{C_1}{N^{+2}\sqrt{|w-1|^N|w+\frac{2}{N}|^2}} = \frac{C_2}{N^{+2}\sqrt{|v-1|^N|v+\frac{2}{N}|^2}} \quad (47)$$

$$J_x = wJ_z, \quad J_y = vJ_z. \quad (48)$$

Analog of Eq. (38) can be presented as

$$\frac{dw}{d \ln \Lambda} = \frac{C_1 N \rho v w (w-1) (w + \frac{2}{N})}{N^{+2}\sqrt{|w-1|^N|w+\frac{2}{N}|^2}}. \quad (49)$$

Thus in the phase space with the coordinates J_x, J_y, J_z there are 4 Kondo phases, each defined by the attractor of all the flow lines, given by one of the vectors: $(1, 1, 1)$, $(-\frac{2}{N}, -\frac{2}{N}, 1)$, $(-1, \frac{2}{N}, -1)$, $(\frac{2}{N}, -1, -1)$. In addition there are 3 Ising phases, each defined by the fixed points of all the flow lines, lying on one of the lines: $J_x = J_y = 0$, $J_x = J_z = 0$, $J_y = J_z = 0$. Further analysis of the phase diagram we postpone until later.

For the case of the power law DOS Eqs. (47) and (49) become

$$J_z = \frac{C_1 \lambda}{N^{+2}\sqrt{|w-1|^N|w+\frac{2}{N}|^2}} = \frac{C_2 \lambda}{N^{+2}\sqrt{|v-1|^N|v+\frac{2}{N}|^2}} \quad (50)$$

and

$$r \frac{dw}{d \lambda} = \frac{C_1 N G v (w-1) (w + \frac{2}{N})}{N^{+2}\sqrt{|w-1|^N|w+\frac{2}{N}|^2}}. \quad (51)$$

In the end, notice that it would be interesting to apply the approach presented in this paper to the case, where the exchange is influenced by Rashba³⁸ or Dzyaloshinsky-Morya-Kondo interaction³⁶, to the case when the DOS has a logarithmic singularity³⁹, to the multi-channel Kondo model⁴⁰, and also to the problem of competition between the Kondo effect and RKKY interaction⁴¹. Another possibly interesting field for application of the presented approach is the models when orbital and spin degrees of freedom of the impurity co-exist⁴².

V. CONCLUSIONS

In the present contribution we derive the poor man's scaling equation to the second order for a general model, describing a quantum impurity with degenerate energy levels embedded into a gas of itinerant electrons, both for flat and for the power law DOS. We show how the obtained previously scaling equations for spin-anisotropic Kondo model follow from the general scaling equation.

We introduce the anisotropic CS model, and the XXZ CS model as its particular case. We apply the general scaling equation to derive scaling equations for these models. We integrate analytically the scaling equation both in the particular case of the XXZ CS model and in the general case of the anisotropic CS model.

L_1 :	$f_{23}^1 = 1$	$f_{31}^2 = 1$	$f_{12}^3 = 1$	
L_2 :	$f_{23}^1 = 1$	$f_{31}^2 = 1$	$f_{12}^3 = -1$	
$L_3(\alpha)$:	$f_{23}^1 = 1$	$f_{23}^2 = \alpha$	$f_{31}^2 = 1$	$(\alpha \geq 0)$
$L_4(\alpha)$:	$f_{23}^1 = 1$	$f_{23}^2 = \alpha$	$f_{31}^2 = -1$	$(\alpha \geq 0)$
L_5 :	$f_{23}^2 = 1$	$f_{31}^1 = -1$		
L_6 :	$f_{23}^1 = 1$			
L_7 :	$f_{23}^2 = 1$			

TABLE I: Possible real three-dimensional Lie algebras. In each case the remaining structure constants f_{st}^p except those appearing in the table are zero; we have omitted zero algebra.

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Appendix A: Real three-dimensional Lie algebras

Every Lie Algebra over a real three-dimensional vector space is isomorphic with one of the following algebras appearing in the Table I⁴³: The algebra L_1 is isomorphic to $su(2)$ algebra; the algebra L_2 gives the same scaling equation as L_1 . We will write down explicitly only scaling equation (15) for the algebras L_3, L_4 . Such equation (we assume $\rho = 1$) is

$$\begin{aligned} \frac{dc_1}{d \ln \Lambda} &= -c_2 c_3 \\ \frac{dc_2}{d \ln \Lambda} &= -c_1 c_3 - \alpha^2 c_2 c_3 \\ \frac{dc_3}{d \ln \Lambda} &= 0. \end{aligned} \quad (A1)$$

Equation (A1) can be easily solved in terms of exponential functions. Scaling equations for the algebras L_5, L_6, L_7 are even simpler.

It is worth pointing to the Lie groups $g_3(\alpha)$ and $g_4(\alpha)$, which have L_3 and L_4 as their Lie algebras. (The group g_1 can be represented by the group of rotations which keep the form $f_1 = x_1 x_2 + y_1 y_2 + z_1 z_2$ invariant, and the group g_2 - by the group of "rotations" which keep the form $f_2 = x_1 x_2 + y_1 y_2 - z_1 z_2$ invariant). It is shown in

Ref.⁴³ that the group $h_3(\alpha)$ can be represented by the matrices of the form

$$\begin{pmatrix} f'(z) & f(z) & 0 \\ -f(z) & f'(z) - \alpha f(z) & 0 \\ y & x & 1 \end{pmatrix}, \quad (\text{A2})$$

where $f(z) = \beta^{-1} \exp(\frac{1}{2}\alpha z) \sinh(\beta z)$ ($\beta = \frac{1}{2}\sqrt{\alpha^2 - 4}$) if $\alpha > 2$, $f(z) = z \exp x$ if $\alpha = 2$ and $f(z) = \gamma^{-1} \exp(\frac{1}{2}\alpha z) \sin(\gamma z)$ ($\gamma = \frac{1}{2}\sqrt{4 - \alpha^2}$) if $0 \leq \alpha < 2$. The group $h_4(\alpha)$ can be represented by the matrices of the form

$$\begin{pmatrix} f'(z) - \alpha f(z) & f(z) & 0 \\ f(z) & f'(z) & 0 \\ x & y & 1 \end{pmatrix}, \quad (\text{A3})$$

where $f(z) = \delta^{-1} \exp(\frac{1}{2}\alpha z) \sinh(\delta z)$ ($\delta = \frac{1}{2}\sqrt{\alpha^2 + 4}$).

Appendix B: The Dzyaloshinskii-Moriya-Kondo interaction

In the presence of spin-orbit interaction the Hamiltonian (21) can be written as³⁶

$$H = \sum_{\mathbf{k}\alpha} \epsilon_{\mathbf{k}} c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\alpha} + \sum_{\mathbf{k}\mathbf{k}'\alpha\beta} J_{ij} S^i \sigma_{\alpha\beta}^j c_{\mathbf{k}'\alpha}^\dagger c_{\mathbf{k}\beta} + \sum_{\mathbf{k}\mathbf{k}'\alpha\beta} \vec{D} \cdot [\vec{S} \times \vec{\sigma}_{\alpha\beta}] c_{\mathbf{k}'\alpha}^\dagger c_{\mathbf{k}\beta}, \quad (\text{B1})$$

where we keep the matrix J_{ij} symmetric but introduce the Dzyaloshinskii-Moriya^{44,45} (DM) interaction; \vec{D} is the DM vector. Mathematically, while writing down Eq. (B1) we just presented arbitrary asymmetric matrix as a sum of symmetric and antisymmetric ones. We just mention here, that if we impose $U(1)$ symmetry, in an appropriate coordinate system the matrix J_{ij} will be written as $J_{ij} = \text{diag}(J_x, J_x, J_z)$, and the vector \vec{D} as $\vec{D} = (0, 0, D_z)$ ³⁶.

Appendix C: The CS model revisited

To warm up, let us start from the Kondo model. Historically, first isotropic case was studied, then the XXZ

model, and then the completely anisotropic model. Let us mentally inverse the process, and try to understand how the XXZ model could have appeared as a renormalizable particular case of the Hamiltonian (24).

First, a general mathematical statement. Because the scaling equation (15) obviously keeps the symmetry of the Hamiltonian, we can obtain renormalizable particular case of the general Hamiltonian (10) by imposing on it some symmetry, and considering the most general Hamiltonian compatible with the chosen symmetry.

Now back to Kondo model. Let us impose on the Hamiltonian (24) the $U(1)$ symmetry. The group $U(1)$ has only 3 bilinear invariants, which all appear in Eq. (33). (We took into account additionally symmetry of any spin Hamiltonian with respect to inversion, which in our case means symmetry with respect to interchange of $+$ and $-$.) The condition $\langle V \rangle = 0$ demands the relation between the coefficients of two of them, and we recover Eq. (33).

Now comes the CS model. It can be obtained from the Hamiltonian

$$H = \sum_m \epsilon c_m^\dagger c_m + \sum_{mm'} J_{m'm} X_{mm'} c_m^\dagger c_m - \sum_{mm'} J'_{m'm} X_{m'm'} c_m^\dagger c_m, \quad (\text{C1})$$

by imposing upon it $SU(N)$ symmetry. In fact, if we introduce matrices \hat{C} and \hat{X} , with the matrix elements $c_m^\dagger c_m$ and $X_{m'm}$ respectively, the interaction should contain only the bilinear combinations of the matrix elements that are invariants of the symmetry group. For the group $SU(N)$ there are only two of them: $\text{Tr}(\hat{C} \cdot \hat{X})$ and $\text{Tr}\hat{C} \cdot \text{Tr}\hat{X}$. If we impose additional condition¹⁵ $\langle V \rangle = 0$, to remove from the Hamiltonian the direct (potential) term, we exactly reproduce Eq. (29).

It is tempting to assume that the Hamiltonian (34) can be obtained by imposing on the Hamiltonian (C1) the symmetry $U(N-1)$ (the maximal subgroup of $SU(N)$)⁴⁶. However we are unable to either prove or disprove this assumption.

In general, the group theory approach suggests the way to get the particular cases of the Hamiltonian (C1), which are renormalizable. We should just choose some subgroup of $SU(N)$ as the symmetry group of the Hamiltonian (C1), and write down a general linear combination of all the bilinear invariants.

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