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Analyticity Properties of Scattering Amplitude in Theories with Compactified Space Dimensions

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Abstract

We consider a massive, neutral, scalar field theory of mass m_0 in a five dimensional flat spacetime. Subsequently, one spatial dimension is compactified on a circle, S^1, of radius R. The resulting theory is defined in the manifold, $R^{3,1} \otimes S^1$. The mass spectrum is a state of lowest mass, m_0, and a tower of massive Kaluza-Klein states. The analyticity property of the elastic scattering amplitude is investigated in the Lehmann-Symanzik-Zimmermann (LSZ) formulation of this theory. In the context of nonrelativistic potential scattering, for the $R^3 \otimes S^1$ spatial geometry, it was shown that the forward scattering amplitude does not satisfy analyticity properties in some cases for a class of potentials. If the same result is valid in relativistic quantum field theory then the consequences will be far reaching. We show that the forward elastic scattering amplitude of the theory, in the LSZ axiomatic approach, satisfies forward dispersion relations. The importance of the unitarity constraint on the S-matrix, is exhibited in displaying the properties of the absorptive part of the amplitude.

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1. Introduction

The analyticity property of scattering amplitude is a cardinal attribute and this has been derived in the frameworks of general relativistic quantum field theories. The scattering amplitude, $F(s,t)$, is an analytic function of the center of mass energy squared, s, for momentum transfer squared, t. The dispersion relations in s have been proved when t is within the Lehmann-Martin ellipse. This result has been derived from the axiomatic approach of Lehmann-Symanzik-Zimmenmann (LSZ) [1] and in the general frameworks of axiomatic formulation of field theories [2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15]. The underlying structure of such theories are locality, mircocausality, Lorentz invariance to mention a few. It is generally accepted that if the dispersion relations are violated one might question the validity of the axioms of general field theories since these are the pillars on which the structure of such field theories rest. Moreover, there are host of rigorous results, derived in the form of theorems, which have been tested against experimental data. One of the most important result is the celebrated Froissart-Martin bound [16, 17] that the total cross section is bounded from above: $\sigma_t \le \frac{4\pi}{t_0}(log s)^2$ where t_0 is determined from the first principles for a given scattering process. The high energy data for hadronic total cross sections respect this bound over a wide range of energies accessible by accelerators.
It is well known that the scattering amplitude in nonrelativistic potential scattering enjoys certain analyticity properties in energy k for a large class of potentials. This topic has been studied quite thoroughly in the past [18, 19, 20]. One intriguing point is, in contrast to relativistic quantum field theories (QFT), that the analyticity of the amplitudes in potential scattering is not so intimately related to a principle like microcausality as is the case with QFT. Moreover, the principle of microcausality has its root in the existence of the limiting velocity in the special theory of relativity i.e. velocity of light, c, is the limiting velocity. Moreover, there is an intimate relationship between microcausality, as postulated in quantum field theories, and the analyticity properties of scattering amplitudes. On the other hand, when we consider nonrelativistic quantum mechanical scattering in potential models the theory is invariant under the Galilean transformations. If we encounter a situation, in potential scattering, where the amplitude fails to satisfy analyticity in the momentum, k, it is not so much a matter of serious concern as would be the case for a relativistic QFT, especially the one which sastifies LSZ or Wightman axioms.
It is now generally accepted that theories which exist in higher spacetime dimensions, $\hat{D} > 4$, might play important roles in our attempt to unify fundamental interactions. We mention, in this context, supersymmetric theories, supergravity theories and the string theories which have been investigated intensively over past several decades. It is generally postulated in the context of such higher dimensional theories that some of the spatial dimensions be compactified so that one eventually constructs an effective four dimensional theory of fundamental interactions to describe and understand physics in the domain of present accessible energies. It is proposed, in certain sce-

narios, that signatures of the extra spatial dimensions might be observed in current high energy colliders [21, 22]. Consequently, there has been a lot of phenomenological studies to investigate and build models for possible experimental observations of the decompactified dimensions at the present high energy accelerator such as LHC. Indeed, the scale of the extra compact dimensions is extracted from the LHC experiments and it puts the compactification scale to be higher than 600 GeV.

Khuri [23], motivated by the large compactification scenario, had set out to study analyticity properties of the scattering amplitude in potential scattering where a spatial dimension is compactified on a circle; the so called S^1 compactification. He discovered that the amplitude does not always satisfy the analyticity properties. On the other hand the analyticity properties of the amplitude, in the context of potential scattering, were investigated (for $d = 3$) with noncompact spatial coordinates [18, 19, 20] i.e. there was no S^1 compactification. It was shown that the amplitude satisfied the dispersion relations. In fact it was shown by Khuri [23], through counter examples, within the framework of perturbation theory, how the analyticity of the forward scattering amplitude breaks down in the presence of S^1 compactification for a class of nonrelativistic potential models under certain circumstances as we shall describe later.

The purpose of this investigation is to study the analyticity properties of the scattering amplitude in a field theory with an S^1 compactified spatial dimension. It is an analog of the Kaluza-Klein (KK) compactification although gravitational interaction is not incorporated. Moreover, after compactification, we retain the entire tower of KK states. For sake of simplicity and to bring out the essential features, we consider a single, neutral, massive scalar field of mass m_0 in $D = 5$. On compactification to $D = 4$, not only we have a massive scalar field with mass m_0 as of the 5-dimensional theory but also we have tower of massive scalars as a consequence of compactification. Recently, we have studied analyticity properties and the high energy behavior of the four point function for a massive, neutral scalar field in higher dimensions, $D > 4$ [24]. It was shown, in the LSZ formalism, that the scattering amplitude has desire attributes in the following sense: (i) We proved the generalization of the Jost-Lehmann-Dyson theorem for the causal function and retarded function [25, 26] for the $D > 4$ case [27]. (ii) Subsequently, we showed the existence of the Lehmann-Martin ellipse for such a theory. (iii) Thus a dispersion relation can be written for the amplitude in s for fixed t when the momentum transfer squared lies inside Lehmann-Martin ellipse [28, 29]. (iv) The analog of Martin's theorem can be derived in the sense that the scattering amplitude is analytic the product domain $D_s \otimes D_t$ where D_s is the cut s-plane and D_t is a domain in the t-plane such that the scattering amplitude is analytic inside a disk, $|t| < \tilde{R}$, $\tilde{R}$ is radius of the disk and it is independent of s. Thus the partial wave expansion converges inside this bigger domain. (v) We also derived the analog of Jin-Martin [30] upper bound on the scattering amplitude which states that the fixed t dispersion relation in s does not require more than two subtractions. (vi) Therefore, a generalized Froissart-Martin bound was be proved.

Our principal goal, in this investigation, is to examine the analyticity property of the four point function derived from the compactified theory in $D = 4$, which originated from $D = 5$ field theory, through the S^1 compactification. In other words whether the scattering amplitude possesses the desired analyticity properties as was derived for the higher dimensional (uncompactified) theory. We shall specifically, investigate analyticity properties of the amplitude for scattering of KK states in the forward direction in view of Khuri's result for scattering of such states in potential scattering with a spatial S^1 compactification. We argue in sequel and prove that the forward scattering amplitude satisfies desired analyticity property in s and hence we are able to write down the dispersion relations.

We work within the LSZ formalism for the effective four dimensional theory obtained from the $D = 5$ theory after S^1 compactification. The main conclusion is that, under certain assumptions, the elastic forward scattering amplitude for the scattering of Kaluza-Klein states satisfies dispersion relation when a five dimensional massive theory is described on the manifold $R^{3,1} \otimes S^1$.

The paper is organized as follows. The next section (Section 2) is devoted to a brief review of the contents of Khuri's paper [23]. The third section contains the essentials of LSZ axiomatic approach to field theory defined over a five dimensional spacetime. We define the retarded product of field operators (R-product) and elucidate their properties. We introduce kinematical variables which are used in this paper. We also introduce notations and conventions followed in this article. The next section (Section 4), deals with the definitions, conventions and kinematics. Then we consider scattering of the zero modes (the fields which carry lowest mass) of the 4-dimensional theory. This problem is similar to scattering of equal mass scalars in a four dimensional theory. However, there are some subtleties since we have to account for the complete set of physical intermediate states (including entire KK tower) in certain spectral representations. In section 4.2.2 deals with elastic scattering of the lowest mass state with an excited KK state. This is the case of unequal mass scattering where one state carries KK charge (KK momentum) and the other state has no KK momentum. The fifth section is devoted to elastic scattering of two states carrying KK charges. Therefore, some caution is to be exercised in deriving analyticity property of the amplitude. We assume throughout this investigation that: (i) all particles are stable. (ii) There are no bound states. (iii) The KK charge (the discrete momentum along the compact S^1 direction) is conserved. Our conclusion is that the forward scattering amplitude possesses nice analyticity properties. Thus if we accept the axioms of LSZ and adopt the standard procedures to investigate analyticity properties of the forward scattering amplitudes we arrive at the same result as is known for field theories defined in flat Minkowski spacetime and we show that the forward scattering amplitude satisfies the dispersion relation. The sixth section summarizes our results and contains discussion.

2. Non-relativistic Potential Scattering for $R^3 \otimes S^1$ Geometry

Khuri [23] envisaged scattering of a particle in a space with $R^3 \otimes S^1$ topology. We provide a brief account of his work and incorporate his important conclusions. We refer the original paper to the interested reader. The notations of [23] will be followed. The compactified coordinate is a circle of radius R and it is assumed the radius is *small* i.e. $\frac{1}{R} >> 1$ where dimensionless units were used. The potential, $V(r, \Phi)$, is such that is periodic in the angular coordinate, Φ, of S^1; $\mathbf{r} \in \mathbf{R^3}$ and $r = |\mathbf{r}|$. The potential $V(r, \Phi)$ belongs to a broad class such that for large r these class of potentials fall off like $e^{-\mu r}/r$ as $r \to \infty$; $\mu > 0$, carrying dimension of inverse length. Moreover, $V(r, \Phi) = V(r, \Phi + 2\pi)$. The scattering amplitude depends on three variables - the momentum of the particle, k, the scattering angle θ, and an integer n which appears due to the periodicity of the Φ-coordinate. Thus forward scattering amplitude is denoted by $T_{nn}(K)$, where $K^2 = k^2 + \frac{n^2}{R^2}$. The starting point is the Schrödinger equation

$$\left[\nabla^2 + \frac{1}{R^2}\frac{\partial^2}{\partial \mathbf{\Phi}^2} + K^2 - V(r, \Phi)\right]\Psi(\mathbf{r}, \Phi) = 0 \tag{1}$$

The free plane wave solutions are

$$\Psi_0(\mathbf{x}, \Phi) = \frac{1}{(2\pi)^2} e^{i\mathbf{k}.\mathbf{x}} e^{in\Phi} \tag{2}$$

and $n \in \mathbf{Z}$. The total energy is defined to be

$$\mathbf{K}^2 = k^2 + \frac{n^2}{R^2} \tag{3}$$

The free Green's function (in the presence of a compact coordinate) assumes the following form

$$G_0(\mathbf{K}; \mathbf{x}, \Phi : \mathbf{x}', \Phi') = -\frac{1}{(2\pi)^4} \sum_{n=-\infty}^{n=+\infty} \int d^3p \frac{e^{i\mathbf{p}.(\mathbf{x}-\mathbf{x}')} e^{in.(\Phi-\Phi')}}{[p^2 + \frac{n^2}{R^2} - \mathbf{K}^2 - i\epsilon]} \tag{4}$$

The free Green's function satisfies the free Schrödinger equation

$$\left[\nabla^2 + \frac{1}{R^2}\frac{\partial^2}{\partial \mathbf{\Phi}^2} + K^2\right] G_0(\mathbf{K}; \mathbf{x}, \Phi : \mathbf{x}', \Phi') = \delta^3(\mathbf{x} - \mathbf{x}')\delta(\Phi - \Phi) \tag{5}$$

The d^3p integration can be performed in the expression (4) leading to

$$G_0(\mathbf{K}; \mathbf{x} - \mathbf{x}'; \Phi - \Phi') = -\frac{1}{(8\pi^2)} \sum_{n=-\infty}^{n=+\infty} \frac{e^{i\sqrt{K^2 - (n^2/R^2)}|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} e^{in(\Phi - \Phi')} \tag{6}$$

Khuri introduced the prescription that $\sqrt{K^2 - n^2/R^2}$ is defined in such a way that when $n^2/R^2 > K^2$

$$i\sqrt{K^2 - n^2/R^2} \to -\sqrt{n^2/R^2 - K^2}, \quad n^2 > K^2R^2 \tag{7}$$

Note that the series expansion for $G_0(\mathbf{K}; \mathbf{x}-\mathbf{x}'; \Phi-\Phi')$ as expressed in (6) is strongly damped for large enough $|n|$. A careful analysis, as was carried out in ref. [23], shows that the Green's function is well defined and bounded, except for $|\mathbf{x}-\mathbf{x}'| \to 0$, similar to the properties of Green's functions in potential scattering for a fixed K^2. Khuri [23]expressed the scattering integral equation for the potential $V(r, \Phi)$ as

$$\Psi_{k,n}(\mathbf{x}, \Phi) = e^{i\mathbf{k}.\mathbf{x}} e^{in\Phi} + \int_0^{2\pi} d\Phi' \int d^3\mathbf{x}' G_0(\mathbf{K}; |\mathbf{x}-\mathbf{x}'|; |\Phi-\Phi'|) V(\mathbf{x}', \Phi') \Psi_{k,n}(\mathbf{x}', \Phi') \tag{8}$$

The expression for the scattering amplitude is extracted from the large $|\mathbf{x}|$ limit when one looks at the asymptotic behavior of the wave function,

$$\Psi_{\mathbf{k},n} \to e^{\mathbf{k}.\mathbf{x}} e^{in\Phi} + \sum_{m=-[KR]}^{+[KR]} T(\mathbf{k}', m : \mathbf{k}, n) \frac{e^{ik'_{mn}|\mathbf{x}|}}{|\mathbf{x}|} e^{im\Phi} \tag{9}$$

where $[KR]$ is the largest integer less than KR and

$$k'_{mn} = \sqrt{k^2 + \frac{n^2}{R^2} - \frac{m^2}{R^2}} \tag{10}$$

He also identifies a conservation rule: $K^2 = k^2 + (n^2/R^2) = k'^2 + (m^2/R^2)$. Thus it is argued that that the scattered wave has only $(2[KR] + 1)$ components and those states with $(m^2/(R^2) > k^2 + (n^2/R^2)$ are exponentially damped for large $|\mathbf{x}|$ and consequently these do not appear in the scattered wave (see eq. (7)). Now the scattering amplitude is extracted from equations (8) and (9) to be

$$T(\mathbf{k}', n'; \mathbf{k}, n) = -\frac{1}{8\pi^2} \int d^3\mathbf{x}' \int_0^{2\pi} d\Phi' e^{-i\mathbf{k}'.\mathbf{x}'} e^{-in'\Phi'} V(\mathbf{x}', \Phi') \Psi_{\mathbf{k},n}(\mathbf{x}', \Phi') \tag{11}$$

The condition, $k'^2 + n'^2/R^2 = k^2 + n^2/R^2$ is to be satisfied. Thus the scattering amplitude describes the process where incoming wave $|\mathbf{k}, n>$ is scattered to final state $|\mathbf{k}', n'>$.

Remark: Reader should pay attention to the expression for the discussion of scattering processes in relativistic QFT in sequel and note the similarities and differences in subsequent sections.

Formally, the amplitude assumes the following form for the full Green's function

$$\begin{aligned} T(\mathbf{k}', n'; \mathbf{k}, n) - T_B = \quad & -\frac{1}{8\pi^2} \int .... \int d^3\mathbf{x} d^3\mathbf{x}' d\Phi d\Phi' e^{-i(\mathbf{k}'.\mathbf{x}' + n'\Phi')} V(\mathbf{x}', \Phi') \\ & G(\mathbf{K}; \mathbf{x}', \mathbf{x}; \Phi', \Phi) V(\mathbf{x}, \Phi) e^{i(\mathbf{k}.\mathbf{x} + n\Phi)} \end{aligned} \tag{12}$$

Here T_B is the Born term.

$$T_B = -\frac{1}{8\pi^2}\int d^3x \int_0^{2\pi} e^{i(\mathbf{k}-\mathbf{k}).\mathbf{x}} V(x,\Phi) e^{i(n-n')\Phi} \tag{13}$$

Full Green's function satisfied an equation with the full Hamiltonian

$$\Big[\nabla^2 + \frac{1}{R^2}\frac{\partial^2}{\partial \mathbf{\Phi}^2} + K^2 - V(\mathbf{x},\Phi)\Big] G(\mathbf{K};\mathbf{x},\mathbf{x}',\Phi,\Phi') = \delta^3(\mathbf{x}-\mathbf{x}')\delta(\Phi-\Phi') \tag{14}$$

This is the starting point of computing scattering amplitude perturbatively in potential scattering [18]. Khuri [23] proceeds to study the analyticity properties of the amplitude and it is a parallel development similar to investigations done in the past. In the context of theory with a compact space dimension, he analyses an amplitude like $T_{nn}(K)$ to the second order in the Born approximation.
Khuri explicitly computes the second Born term $T^{(2)}$ for the forward amplitude, for the choice $n = 1$. He has discovered that the analyticity of the forward amplitude breaks down with a counter example; where $T_{nn}(k)$ does not satisfy dispersion relations for a class of Yukawa-type potentials of the form

$$V(r,\Phi) = u_0(r) + 2\sum_{m=1}^{N} u_m(r) cos(m\Phi) \tag{15}$$

where $u_m(r) = \lambda_m \frac{e^{-\mu r}}{r}$. Khuri noted an important feature of his studies that in the case when scattering theory was applied perturbatively in R^3 space the resulting amplitude satisfied analyticity properties for similar Yukawa-type potentials [19, 20]. Thus there has been concerns[2] when non-analyticity was discovered in the non-relativistic quantum mechanics in the space $R^3 \otimes S^1$ by Khuri.
We shall describe the framework of our investigation in the next section. We remark in passing that the analyticity of scattering amplitude in nonrelativitic scattering is not such a profound property as in the relativistic QFT although the analyticity in non-relativistic potential scattering has been investigated quite thoroughly in the past [18]. However, it is to be noted that in absence a limiting velocity (in the relativistic case velocity of light, c, profoundly influences the study of the analyticity of amplitudes) the microcausality is not enforced in nonrelatvistic processes. As we shall show (and as has been emphasized in many classic books on Quantum Field Theories) there is, indeed, a deep connection between microcausality and analyticity. When a spatial dimension is compactified on S^1, the coordinate on the circle is periodic; we can define concept of microcausality. We shall keep this aspect in mind and we shall undertake a systematic study of the analyticity of scattering amplitude in the sequel.

[2] Andre Martin brought the work of Khuri [23] to my attention and persuaded me to undertake this invetstigation.

3. Field Theory in Five Dimensional Spacetime

Let us consider a neutral, scalar field theory with mass, m_0, in flat five dimensional Minkowski space $R^{4,1}$. It is assumed that the particle is stable and there are no bound states. The notation is that the spacetime coordinates are denoted as $\hat{x}$ and all operators are denoted with a *hat* when they are defined in the five dimensional space where the spatial coordinates are noncompact.The LSZ axioms are [1]:
A1. The states of the system are represented in a Hilbert space, $\hat{\mathcal{H}}$. All the physical observables are self-adjoint operators in the Hilbert space, $\hat{\mathcal{H}}$.
A2. The theory is invariant under inhomogeneous Lorentz transformations.
A3. The energy-momentum of the states are defined. It follows from the requirements of Lorentz and translation invariance that we can construct a representation of the orthochronous Lorentz group. The representation corresponds to unitary operators, $\hat{U}(\hat{a},\hat{\Lambda})$, and the theory is invariant under these transformations. Thus there are hermitian operators corresponding to spacetime translations, denoted as $\hat{P}_{\hat{\mu}}$, with $\hat{\mu}=0,1,2,3,4$ which have following properties:

$$\left[\hat{P}_{\hat{\mu}},\hat{P}_{\hat{\nu}}\right]=0 \tag{16}$$

If $\hat{\mathcal{F}}(\hat{x})$ is any Heisenberg operator then its commutator with $\hat{P}_{\hat{\mu}}$ is

$$\left[\hat{P}_{\hat{\mu}},\hat{\mathcal{F}}(\hat{x})\right]=i\hat{\partial}_{\hat{\mu}}\hat{\mathcal{F}}(\hat{x}) \tag{17}$$

It is assumed that the operator does not explicitly depend on spacetime coordinates. If one chooses a representation where the translation operators, $\hat{P}_{\hat{\mu}}$, are diagonal and the basis vectors $|\hat{p},\hat{\alpha}>$ span the Hilbert space, $\hat{\mathcal{H}}$, such that

$$\hat{P}_{\hat{\mu}}|\hat{p},\hat{\alpha}>=\hat{p}_{\hat{\mu}}|\hat{p},\hat{\alpha}> \tag{18}$$

then we are in a position to make more precise statements:
• Existence of the vacuum: there is a unique invariant vacuum state $|0>$ which has the property

$$\hat{U}(\hat{a},\hat{\Lambda})|0>=|0> \tag{19}$$

The vacuum is unique and is Poincaré invariant.
• The eigenvalue of $\hat{P}_{\hat{\mu}}$, $\hat{p}_{\hat{\mu}}$, is light-like, with $\hat{p}_0>0$. We are concerned only with massive stated in this discussion. If we implement infinitesimal Poincaré transformation on the vacuum state then

$$\hat{P}_{\hat{\mu}}|0>=0, \quad \text{and} \quad \hat{M}_{\hat{\mu}\hat{\nu}}|0>=0 \tag{20}$$

from above postulates. Note that $\hat{M}_{\hat{\mu}\hat{\nu}}$ are the generators of Lorentz transformations.
A4. The locality of theory implies that a (bosonic) local operator at spacetime point

$\hat{x}^{\hat{\mu}}$ commutes with another (bosonic) local operator at $\hat{x}'^{\hat{\mu}}$ when their separation is spacelike i.e. if $(\hat{x}-\hat{x}')^2 < 0$. Our Minkowski metric convention is as follows: the inner product of two 5-vectors is given by $\hat{x}.\hat{y} = \hat{x}^0\hat{y}^0 - \hat{x}^1\hat{y}^1 - ... - \hat{x}^4\hat{y}^4$. Since we are dealing with a neutral scalar field, for the field operator $\hat{\phi}(\hat{x})$: $\hat{\phi}(\hat{x})^\dagger = \hat{\phi}(\hat{x})$ i.e. $\hat{\phi}(\hat{x})$ is hermitian. By definition it transforms as a scalar under inhomogeneous Lorentz transformations

$$\hat{U}(\hat{a},\hat{\Lambda})\hat{\phi}(\hat{x})\hat{U}(\hat{a},\hat{\Lambda})^{-1} = \hat{\phi}(\hat{\Lambda}\hat{x}+\hat{a}) \tag{21}$$

The micro causality, for two local field operators, is stated to be

$$\left[\hat{\phi}(\hat{x}),\hat{\phi}(\hat{x}')\right] = 0, \qquad for \;\; (\hat{x}-\hat{x}')^2 < 0 \tag{22}$$

It is well known that, in the LSZ formalism, we are concerned with vacuum expectation values of time ordered products of operators as well as with the the retarded product of fields. The requirements of the above listed axioms lead to certain relationship, for example, between vacuum expectation values of R-products of operators. These are termed as linear relations and the importance of the above listed axioms is manifested through these relations. In contrast, unitarity imposes nonlinear constraints on amplitude. For example, if we expand an amplitude in partial waves, unitarity demands certain positivity conditions to be satisfied by the partial wave amplitudes.

We summarize below some of the important aspects of LSZ formalism as we utilize them through out the present investigation.

(i) The asymptotic condition: According to LSZ the field theory accounts for the asymptotic observables. These correspond to particles of definite mass, charge and spin etc. $\hat{\phi}^{in}(\hat{x})$ represents the free field in the remote past and a Fock space is generated by the field operator. The physical observable can be expressed in terms of these fields.

(ii) $\hat{\phi}(\hat{x})$ is the interacting field. LSZ technique incorporates a prescription to relate the interacting field, $\hat{\phi}(\hat{x})$, with $\hat{\phi}^{in}(\hat{x})$; consequently, the asymptotic fields are defined with a suitable limiting procedure. Thus we introduce the notion of the adiabatic switching off of the interaction. A cutoff adiabatic function is postulated such that this function controls the interactions. It is **1** at finite interval of time and it has a smooth limit of passing to zero as $|t| \to \infty$. It is argued that when adiabatic switching is removed we can define the physical observables.

(iii) The fields $\hat{\phi}^{in}(\hat{x})$ and $\hat{\phi}(\hat{x})$ are related as follows:

$$\hat{x}_0 \to -\infty \qquad \hat{\phi}(\hat{x}) \to \hat{Z}^{1/2}\hat{\phi}^{in}(\hat{x}) \tag{23}$$

By the first postulate, $\hat{\phi}^{in}(\hat{x})$ creates free particle states. However, in general $\hat{\phi}(\hat{x})$ will create multi particle states besides the single particle one since it is the interacting field. Moreover, $< 1|\hat{\phi}^{in}(\hat{x})|0 >$ and $< 1|\hat{\phi}(\hat{x})|0 >$ carry same functional dependence in $\hat{x}$. If the factor of $\hat{Z}$ were not the scaling relation between the two fields (23), then canonical commutation relation for each of the two fields (i.e. $\hat{\phi}^{in}(\hat{x})$ and $\hat{\phi}(\hat{x})$) will be the same. Thus in the absence of $\hat{Z}$ the two theories will be identical. Moreover, the postulate of asymptotic condition states that in the remote future

$$\hat{x}_0 \to \infty \quad \hat{\phi}(\hat{x}) \to \hat{Z}^{1/2}\hat{\phi}^{out}(\hat{x}). \tag{24}$$

We may as well construct a Fock space utilizing $\hat{\phi}(\hat{x})^{out}$ as we could with $\hat{\phi}(\hat{x})^{in}$ Furthermore, the vacuum is unique for $\hat{\phi}^{in}$, $\hat{\phi}^{out}$ and $\hat{\phi}(\hat{x})$. The normalizable single particle states are the same i.e. $\hat{\phi}^{in}|0>=\hat{\phi}^{out}|0>$. We do not display $\hat{Z}$ from now on. If at all any need arises, $\hat{Z}$ can be introduced in the relevant expressions.
We define creation and annihilation operators for $\hat{\phi}^{in}$, $\hat{\phi}^{out}$. We recall that $\hat{\phi}(\hat{x})$ is not a free field. Wheheas the fields $\hat{\phi}^{in,out}(\hat{x})$ satisfy the free field equations $[\Box_5+m_0^2]\hat{\phi}^{in,out}(\hat{x})=0$; the interacting field satisfies an equation of motion which is endowed with a source current: $[\Box_5+m_0^2]\hat{\phi}(\hat{x})]=\hat{j}(\hat{x})$. We may use the plane wave basis for simplicity in certain computations; however, in a more formal approach, it is desirable to use wave packets.
The relevant vacuum expectation values of the products of operators in LSZ formalism are either the time ordered products (the T-products) or the retarded products (the R-products). We shall mostly use the R-products and we use them extensively throughout this investigation. It is defined as

$$\begin{aligned} R\ \hat{\phi}(\hat{x}_0)\hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n) = \quad & (-1)^n\sum_P \theta(\hat{x}_{00}-\hat{x}_{10})\theta(\hat{x}_{10}-\hat{x}_{20})...\theta(\hat{x}_{n-10}-\hat{x}_{n0}) \\ & [[...[\hat{\phi}(\hat{x}),\hat{\phi}_{i_1}(\hat{x}_{i_1})],\hat{\phi}_{i_2}(\hat{x}_{i_2})]..],\hat{\phi}_{i_n}(\hat{x}_{i_n})] \end{aligned} \tag{25}$$

note that $R\hat{\phi}(\hat{x})=\hat{\phi}(\hat{x})$ and P stands for all the permutations $i_1,....i_n$ of $1,2...n$. The R-product is hermitian for hermitian fields $\hat{\phi}_i(\hat{x}_i)$ and the product is symmetric under exchange of any fields $\hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n)$. Notice that the field $\hat{\phi}(\hat{x})$ is kept where it is located in its position. We list below some of the important properties of the R-product for future use [5]:
(i) $R\ \hat{\phi}(\hat{x}_0)\hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n)\neq 0$ only if $\hat{x}_{00}>$ max $\{\hat{x}_{10},..\hat{.}x_{n0}\}$.
(ii) Another important property of the R-product is that

$$R\ \hat{\phi}(\hat{x}_0)\hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n)=0 \tag{26}$$

whenever the time component $\hat{x}_{00}$, appearing in the argument of $\hat{\phi}(\hat{x}_0)$ whose position is held fix, is less than time component of any of the four vectors $(\hat{x}_1,...\hat{x}_n)$ appearing in the arguments of $\hat{\phi}(\hat{x}_1)...\hat{\phi}(\hat{x}_n)$.
(iii) We recall that under a Lorentz transformation $\hat{U}(\hat{\Lambda},0)$

$$\hat{\phi}(\hat{x}_i)\to\hat{\phi}(\hat{\Lambda}\hat{x}_i)=\hat{U}(\hat{\Lambda},0)\hat{\phi}(\hat{x}_i)\hat{U}(\hat{\Lambda},0)^{-1} \tag{27}$$

Therefore,

$$R\ \hat{\phi}(\hat{\Lambda}\hat{x})\hat{\phi}(\hat{\Lambda}\hat{x}_i)...\hat{\phi}_n(\hat{\Lambda}\hat{x}_n)=\hat{U}(\hat{\Lambda},0)R\ \phi(x)\phi_1(x_1)...\phi_n(x_n)U(\Lambda,0)^{-1} \tag{28}$$

And

$$\hat{\phi}_i(\hat{x}_i) \to \hat{\phi}_i(\hat{x}_i + \hat{a}) = e^{i\hat{a}.\hat{P}}\hat{\phi}_i(\hat{x}_i)e^{-i\hat{a}.\hat{P}} \tag{29}$$

under spacetime translations. Consequently,

$$R\ \hat{\phi}(\hat{x}+\hat{a})\hat{\phi}(\hat{x}_i+\hat{a})...\hat{\phi}_n(\hat{x}_n+\hat{a}) = e^{i\hat{a}.\hat{P}}R\ \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n)e^{-i\hat{a}.\hat{P}} \tag{30}$$

Therefore, the vacuum expectation value of the R-product dependents only on difference between pair of coordinates: in other words it depends on the following set of coordinate differences: $\hat{\xi}_1 = \hat{x}_1 - \hat{x}_0, \hat{\xi}_2 = \hat{x}_2 - \hat{x}_1 ... \hat{\xi}_n = \hat{x}_{n-1} - \hat{x}_n$ as a consequence of translational invariance.
(iv) The retarded property of R-function and the asymptotic conditions lead to the following relations.

$$[R\ \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n), \hat{\phi}_l^{in}(\hat{y}_l)] = i\int d^5\hat{y}'_l\Delta(\hat{y}_l - \hat{y}'_l)(\Box_{5\hat{y}'} + \hat{m}_l^2)R\ \hat{\phi}(\hat{x})\hat{\phi}_1(\hat{x}_1)...\hat{\phi}_n(\hat{x}_n)\hat{\phi}_l(\hat{y}'_l) \tag{31}$$

Note: here $\hat{m}_l$ stands for the mass of a field in five dimensions. We may define 'in' and 'out' states in terms of the creation operators associated with 'in' and 'out' fields as follows

$$|\hat{k}_1, \hat{k}_2, ...\hat{k}_n\ in >= \hat{a}^\dagger_{in}(\hat{\mathbf{k}}_1)\hat{a}^\dagger_{in}(\hat{\mathbf{k}}_2)...\hat{a}^\dagger_{in}(\hat{\mathbf{k}}_n)|0> \tag{32}$$

$$|\hat{k}_1, \hat{k}_2, ...\hat{k}_n\ out >= \hat{a}^\dagger_{out}(\hat{\mathbf{k}}_1)\hat{a}^\dagger_{out}(\hat{\mathbf{k}}_2)...\hat{a}^\dagger_{out}(\hat{\mathbf{k}}_n)|0> \tag{33}$$

We can construct a complete set of states either starting from 'in' field operators or the 'out' field operators and each complete set will span the Hilbert space, $\hat{\mathcal{H}}$. Therefore, a unitary operator will relate the two sets of states in this Hilbert space. This is a heuristic way of introducing the concept of the S-matrix. We shall define S-matrix elements through LSZ reduction technique in subsequent section.
We shall not distinguish between notations like $\hat{\phi}^{out,in}$ or $\hat{\phi}_{out,in}$ and therefore, there might be use of the sloppy notation in this regard.
We record the following important remark *en passant*: The generic matrix element $< \hat{\alpha}|\hat{\phi}(\hat{x}_1)\hat{\phi}(\hat{x}_2)...|\hat{\beta} >$ is not an ordinary function but a distribution. Thus it is to be always understood as smeared with a Schwarz type test function $f \in \mathcal{S}$. The test function is infinitely differentiable and it goes to zero along with all its derivatives faster than any power of its argument. We shall formally derive expressions for scattering amplitudes and the absorptive parts by employing the LSZ technique. It is to be understood that these are generalized functions and such matrix elements are properly defined with smeared out test functions.
We obtain below the expression for the Källen-Lehmann representation for the five dimensional theory. It will help us to transparently expose, as we shall recall in the next section, the consequences of S^1 compactification. Let us consider the vacuum expectation value (VEV) of the commutator of two fields in the $D = 5$ theory: $< 0|[\hat{\phi}(\hat{x}), \hat{\phi}(\hat{y})]|0 >$. We introduce a complete set of states between product of the

fields after opening up the commutator. Thus we arrive at the following expression by adopting the standard arguments,

$$< 0|[\hat{\phi}(\hat{x}),\hat{\phi}(\hat{y})]|0 >= \sum_{\hat{\alpha}}\Big(< 0|\hat{\phi}(0)\hat{\alpha} > e^{-i\hat{p}_{\hat{\alpha}}.(\hat{x}-\hat{y})} < \hat{\alpha}|\hat{\phi}(0)|0 > -(\hat{x}\leftrightarrow\hat{y})\Big) \quad (34)$$

Let us define

$$\hat{\rho}(\hat{q}) = (2\pi)^4\sum_{\hat{\alpha}}\delta^5(\hat{q}-\hat{p}_{\hat{\alpha}})| < 0|\hat{\phi}(0)|\hat{\alpha} > |^2 \quad (35)$$

Note that $\hat{\rho}(\hat{q})$ is positive, and $\hat{\rho} = 0$ when $\hat{q}$ is not in the light cone. It is also Lorentz invariant. Thus we write

$$\hat{\rho}(\hat{q}) = \hat{\sigma}(\hat{q}^2)\theta(\hat{q}_0), \quad \hat{\sigma}(\hat{q}^2) = 0, \quad if \quad \hat{q}^2 < 0 \quad (36)$$

This is a positive measure. We may separate the expression for the VEV of the commutator (34) into two parts: the single particle state contribution and the rest. Moreover, we use the asymptotic state condition to arrive at

$$< 0|[\hat{\phi}(\hat{x}),\hat{\phi}(\hat{y})]|0 >= i\hat{Z}\hat{\Delta}(\hat{x},\hat{y};m_0) + i\int_{\hat{m}_1^2}^{\infty} d\hat{m}'^2\hat{\Delta}(\hat{x},\hat{y};\hat{m}') \quad (37)$$

where $\hat{\Delta}(\hat{x},\hat{y};m_0)$ is the VEV of the free field commutator, m_0 is the mass of the scalar. $\hat{m}_1^2 > \hat{M}^2$, the multiple particle threshold.
We are in a position to study several attributes of scattering amplitudes in the five dimensional theory such as proving existence of the Lehmann-Martin ellipse, give a proof of fixed t dispersion relation to mention a few. However, these properties have been derived in a general setting recently [24] for D-dimensional theories. The purpose of incorporating the expression for the VEV of the commutator of two fields in the 5-dimensional theory is to provide a prelude to the modification of similar expressions when we compactify the theory on S^1 as we shall see in the next section.

4. Compactification of the Scalar Field Theory: $\mathbf{R^{4,1}} \rightarrow \mathbf{R^{3,1}} \otimes \mathbf{S^1}$

We consider S^1 compactification of a spatial coordinate of the five dimensional theory. Let us decompose the five dimensional spacetime coordinates, $\hat{x}^{\hat{\mu}}$, as follows:

$$\hat{x}^{\hat{\mu}} = (x^\mu, y), \mu = 0, 1, 2, 3 \quad (38)$$

where x^μ are the four dimensional Minkowski space coordinates; y is the spatial coordinate defined on S^1 such that $y + 2\pi R = y$, R being the radius of compactification. We shall capture the essential features of the S^1 compactification when a neutral scalar field (in $D = 5$) of mass m_0 is described in the geometry $R^{3,1} \otimes S^1$. Let us consider as a first step, some properties of the asymptotic fields such as the 'in' and

’out’ field , $\hat{\phi}^{in,out}(\hat{x})$. The equation of motion is $[\Box_5 + m_0^2]\hat{\phi}^{in,out}(\hat{x}) = 0$. We expand the fields as follows

$$\hat{\phi}^{in,out}(\hat{x}) = \hat{\phi}^{in,out}(x,y) = \phi_0^{in,out}(x) + \sum_{n=-\infty}^{+n=\infty} \phi_n^{in,out}(x) e^{\frac{iny}{R}} \tag{39}$$

Note that $\phi_0^{in,out}(x)$ has no y-dependence and it is the so called *zero mode*. The terms in rest of the series satisfy periodicity in y. We can decompose the five dimensional $\Box_5$ as sum of a four dimensional $\Box$ and a $\frac{\partial}{\partial y^2}$ term. The equation of motion is

$$[\Box - \frac{\partial}{\partial y^2} + m_n^2]\phi_n^{in,out}(x,y) = 0 \tag{40}$$

where $\phi_n^{in,out}(x,y) = \phi_n^{in,out} e^{\frac{iny}{R}}$ and $n = 0$ term has no y-dependence being $\phi_0(x)$. Here $m_n^2 = m_o^2 + \frac{n^2}{R^2}$. Thus we have tower of massive states. The momentum associated in the y-direction is $q_n = n/R$ and is quantized in the units of $1/R$ and it is an additive conserved quantum number. We term it as Kaluza-Klein (KK) charge although there is no gravitational interaction in the five dimensional theory; we still call it KK reduction. For the interacting field $\hat{\phi}(\hat{x})$, we can adopt a similar mode expansion.

$$\hat{\phi}(\hat{x}) = \hat{\phi}(x,y) = \phi_0(x) + \sum_{n=-\infty}^{n=+\infty} \phi_n(x) e^{\frac{iny}{R}} \tag{41}$$

The equation of motion for the interaction fields in endowed with a source term. Thus source current would be expanded as is the expansion (41). Each field $\phi_n(x)$ will have a current, $J_n(x)$ associated with it and source current will be expanded as

$$\hat{j}(x,y) = j_0(x) + \sum_{n=-\infty}^{n=+\infty} J_n(x) e^{iny/R} \tag{42}$$

Note that the set of currents, $\{J_n(x)\}$, are the source currents associated with the tower of interacting fields $\{\phi_n(x)\}$, $n \neq 0$. These fields carry the discrete KK charge, n. Therefore, $J_n(x)$ also carries the same KK charge. We should keep this aspect in mind when we consider matrix element of such currents between states. In future, we might not explicitly display the charge of the current; however, it becomes quite obvious in the context.
The zero modes, $\phi_0^{in,out}$, create their Fock spaces. Similarly, each of the fields $\phi_n^{in,out}(x)$ create their Fock spaces as well. For example a state with spatial momentum, $\mathbf{p}$, energy, p_0 and discrete momentum q_n (in y-direction) is created by

$$A^{\dagger,in}(\mathbf{p}, q_n)|0>= |p, q_n>_{in}, \quad p_0 > 0 \tag{43}$$

and similar orthogonality relation holds for an *out* state. We may recall that in the five dimensional theory, we started with, there was only one neutral scalar field in

the spectrum. As a consequence of the S^1 compactification, the resulting spectrum consists of a massive neutral scalar of mass m_0^2 and a tower of 'charged' massive field. Moreover, each level in this tower is characterized by a mass and a 'charge', (m_n, q_n), respectively; the zero mode has $q_n = 0$. Let us consider the Hilbert space of the compactified theory, keeping in mind the above remarks.
The Decomposition of the Hilbert space $\hat{\mathcal{H}}$: The Hilbert space associated with the five dimensional theory is $\hat{\mathcal{H}}$. It is now decomposed as a direct sum of Hilbert spaces where each one is characterized by its q_n quantum number of the compactified theory

$$\hat{\mathcal{H}} = \sum \oplus \mathcal{H}_n \tag{44}$$

Thus $\mathcal{H}_0$ is the Hilbert space which is spanned by states built from the creation operators $\{a^\dagger(\mathbf{k})\}$ which acting on the vacuum create the complete set of states that span $\mathcal{H}_0$. A single particle state is $a^{\dagger,in}(\mathbf{k})|0>=|\mathbf{k}>_{in}$ and multiparticle states are created using the procedure out lines in (32) and (33) starting from *in* field. If we consider a field $\phi_n^{in}(x,y)$ with a charge q_n, we can create a Fock space through the set of creation operators $\{A^{\dagger,in}(\mathbf{p},q_n)\}$. Moreover, two state vectors with different 'charges' are orthogonal: for example

$$<\mathbf{p}, q_{n'}; in|\mathbf{p}', q_{n'}; in>= \delta^3(\mathbf{p}-\mathbf{p}')\delta_{n,n'} \tag{45}$$

We could as well create a Fock space utilizing *out* fields.
Remark: We have stated earlier and repeat here that we assume that there are no bound states in the theory and all particles are stable. There exists a possibility that a particle with charge $2n$ and mass $m_{2n}^2 = m_0^2 + \frac{4n^2}{R^2}$ could be a bound state of two particles of charge n and masses m_n each under certain circumstances. We have excluded such possibilities from the present investigation.
Now we can adopt the LSZ formalism for the four dimensional spacetime with an extra compact dimension. If we keep in mind the steps introduced above, it is possible to envisage field operators $\phi_n^{in}(x)$ and $\phi_n^{out}(x)$ for each of the fields for a given KK charge, n. Therefore, each Hilbert space, $\mathcal{H}_n$ will be spanned by the state vectors (say for 'in' states) created by operators $a^{\dagger,in}(\mathbf{k})$, for $n=0$ and $A^{\dagger,in}(\mathbf{p},q_n)$, for $n \neq 0$. Moreover, we are in a position to define corresponding set of *interacting fields* $\{\phi_n(x)\}$ which will interpolate into 'in' and 'out' fields in the asymptotic limits designated by their KK charges.
Remark: Note that in (39) sum over $\{n\}$ runs over positive and negative integers. If there is a parity symmetry, $y \to -y$, under which the field is invariant we can reduce the sum to positive n only. However, since q_n is an additive discrete quantum number, a state with $q_n > 0$ could be designated as a particle and the corresponding state $q_n < 0$ can be interpreted as its antiparticle. Thus a two particle state $|p, q_n > |p, -q_n >, \;\; q_n > 0 \; and \; p_0 > 0$ is a particle antiparticle state, $q_n = 0$. For example, it could be two particle state of ϕ_0 satisfying energy momentum conservation, especially if they appear as intermediate states. We shall keep this fact in mind for

future references.
Let us momentarily return to the Källen-Lehmann representation (34) in the present context and utilize the expansion (41) in the expression for the VEV of the commutator of two fields defined in $D = 5$: $< 0|[\hat{\phi}(\hat{x}), \hat{\phi}(\hat{x}')]|0 >$

$$< 0|[\hat{\phi}(x,y), \hat{\phi}(x',y')]|0 >=< 0|[\phi_0(x) + \sum_{-\infty}^{+\infty} \phi_n(x,y),\ \phi_0(x') + \sum_{-\infty}^{+\infty} \phi_l(x',y')]|0 > \quad (46)$$

The VEV of a commutator of two fields given by the spectral representation (34) will be decomposed into sum of several commutators whose VEV will appear:

$$< 0|[\phi_0(x), \phi_0(x')]|0 >, \quad < 0|[\phi_n(x), \phi_{-n}(x')]|0 >, ... \quad (47)$$

Since the vacuum carries zero KK charge, $q_n = 0$, the commutator of two fields (with $n \neq 0$) should give rise to zero-charge and consequently, only ϕ_n and ϕ_{-n} commutators will appear. Moreover, commutator of fields with different q_n vanish since the operators act on states of different Hilbert spaces. Thus we already note the consequences of compactification. When we wish to evaluate the VEV and insert complete set of intermediate states in the product of two operators after opening up the commutators, we note that all states of the entire KK tower can appear as intermediate states as long as they respect all conservation laws. This will be an important feature in all our computations in what follows.

4.1 Conventions, Definitions and Kinematics

We have stated earlier that our goal is to study the analyticity property of the four point amplitude in the *foward* direction. So far we have laid down the requisite procedures for compactification and we have outlined the structure of the Hilbert space in the compactified theory. We defined *in* and *out* fields in each of the sectors. Thus we can apply the LSZ reduction technique to derive expressions for the scattering amplitudes keeping in mind the energy momentum conservation rules and conservation of the KK charge.
We adopt the following notations: the field associate with the zero mode (earlier denoted as ϕ_0) is denoted as ϕ. If we consider scattering of four such particles for the process $a + b \to c + d$, all being ϕ fields we shall denote it as $\phi_a + \phi_b \to \phi_c + \phi_d$. The four momenta of ϕ particles will be denoted as k, in the preceding reaction it will be denoted as $k_a + k_b \to k_c + k_d$. The creation and annihilation operators (say for the 'in' fields) are: $a^{\dagger,in}(\mathbf{k})$ and $a^{in}(\mathbf{k})$ respectively. Any field which belongs to the KK tower is denoted by χ_n, n being the KK charge it carries and we always denote four momentum of a KK particle as p_μ. *For conveniences, we use the notation* $q_n = n$ *from now on which amounts to adopting the convention that* $R = 1$. For any (say 'in') field belonging to KK-tower, creation and annihilation operators are denoted respectively as $A^{\dagger,in}(\mathbf{p}, q_n)$ and $A^{in}(\mathbf{p}, q_n)$. Sometimes, we might not explicitly exhibit presence of the KK charge in a reaction; however, it will stated in a context whenever

required.There will be three types of scattering processes.
(i) $q_n = 0$ sector: the reaction involves only four ϕ fields: $\phi + \phi \to \phi + \phi$.
(ii) Scattering of a ϕ field with a χ field such as $\phi + \chi_n \to \phi + \chi_n$. Since KK charge is conserved, by assumption, the initial and final state particles are described by the χ fields with the same charge.
(iii) The scattering of four χ fields. They could be of two types: (a) Elastic scattering where $\chi_n + \chi_m \to \chi_n + \chi_m$. Here the initial particles carry KK charges n and m and final particles also carry same charges. (b) Inelastic scattering like $\chi_n + \chi_m \to \chi_{n'} + \chi_{m'}$. The total KK charge conservation implies $n + m = n' + m'$.
Of course in all reactions, total energy momentum conservation is to be guaranteed. Let us consider a generic 4-body reaction

$$\tilde{a} + \tilde{b} \to \tilde{c} + \tilde{d} \tag{48}$$

The particles $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ (the corresponding fields being $\tilde{\phi}_a, \tilde{\phi}_b, \tilde{\phi}_c, \tilde{\phi}_d$) respectively carrying momenta $\tilde{p}_a, \tilde{p}_b, \tilde{p}_c, \tilde{p}_d$; these particles may correspond to the KK zero modes (with KK momentum $q = 0$) or particles might carry nonzero KK charge. We shall consider *only elastic scatterings*. The Lorentz invariant Mandelstam variables are

$$s = (\tilde{p}_a+\tilde{p}_b)^2 = (\tilde{p}_c+\tilde{p}_d)^2,\ t = (\tilde{p}_a-\tilde{p}_d)^2 = (\tilde{p}_b-\tilde{p}_c)^2,\ u = (\tilde{p}_a-\tilde{p}_c)^2 = (\tilde{p}_b-\tilde{p}_d)^2 \tag{49}$$

and $\sum \tilde{p}_a^2 + \tilde{p}_b^2 + \tilde{p}_c^2 + \tilde{p}_d^2 = m_a^2 + m_b^2 + m_c^2 + m_d^2$. We shall maintain independent identities of the four particles which will facilitate the computation of the four point function utilizing the LSZ reduction technique. We list below some relevant (kinematic) variables which we need for our future discussions:

$$\mathbf{M}_a^2,\ \ \mathbf{M}_b^2,\ \ \mathbf{M}_c^2,\ \ \mathbf{M}_d^2 \tag{50}$$

These correspond to lowest mass two or more particle states which carry the same quantum number as that of particle a, b, c and d respectively. We also define six more variables to be

$$(\mathbf{M}_{ab}, \mathbf{M}_{cd}),\ \ (\mathbf{M}_{ac}, \mathbf{M}_{bd}),\ \ (\mathbf{M}_{ad}, \mathbf{M}_{bc}) \tag{51}$$

The variable $\mathbf{M}_{ab}$ carries the same quantum number as (a *and* b) and it corresponds to two or more particle states. Similar definition holds for the other five variables introduced above. We define two types of thresholds: (i) the physical threshold, s_{phys}, and s_{thr}. In absence of anomalous thresholds (and equal mass scattering) $s_{thr} = s_{phys}$ for a reaction to proceed in the s channel. Similarly, we may define u_{phys} and u_{thr} which will be useful when we discuss dispersion relations. We assume from now on that $s_{thr} = s_{phys}$ and $u_{thr} = u_{phys}$. We shall outline how a four point function is obtained in LSZ approach. Normally one starts with $|\tilde{p}_d, \tilde{p}_c\ out >$ and $|\tilde{p}_b, \tilde{p}_a\ in >$ and considers the matrix element $< \tilde{p}_d, \tilde{p}_c\ out|\tilde{p}_b, \tilde{p}_a\ in >$. Then we subtract out the matrix element $< \tilde{p}_d, \tilde{p}_c\ in|\tilde{p}_b, \tilde{p}_a\ in >$ to define the S-matrix element.

$$\begin{aligned} < \tilde{p}_d, \tilde{p}_d\ out|\tilde{p}_b, \tilde{p}_a\ in >= \quad & \delta^3(\tilde{\mathbf{p}}_d - \tilde{\mathbf{p}}_b)\delta^3(\tilde{\mathbf{p}}_c - \tilde{\mathbf{p}}_a) - \frac{i}{(2\pi)^3}\int d^4x \int d^4x' \\ & e^{-i(\tilde{p}_a.x - \tilde{p}_c x')} K_x K_{x'} < \tilde{p}_d\ out|R(x', x)|\tilde{p}_b\ in > \end{aligned} \tag{52}$$

where K_x and $K_{x'}$ are the four dimensional Klein-Gordon operators and

$$R(x,x') = -\theta(x_0 - x'_o)[\tilde{\phi}_a(x), \tilde{\phi}_c(x')] \tag{53}$$

We have reduced fields associated with a and c in (52). In the next step we may reduce all the four fields and in such a reduction we shall get VEV of the R-product of four fields which will be operated upon by four K-G operators. However, the latter form of LSZ reduction (when all fields are reduced) is not generally utilized when we want to investigate the analyticity property of the amplitude in the present context. In particular our intent is to write the forward dispersion relation. Thus we abandon the idea of reducing all the four fields in this article.
Remark: Note that on the right hand side of the requation (52) the operators act on $R\tilde{\phi}_a(x)\tilde{\phi}(x')_c$ and there is a θ-function in the definition of the R-product. Consequently, the action of $K_xK_{x'}$ on $\tilde{\phi}_a(x)\tilde{\phi}_c(x')$ will produce a term $R\tilde{j}_a(x)\tilde{j}_c(x')$. In addition the operation of the two K-G operators will give rise to δ-functions and derivatives of δ-functions and some equal time commutators i.e. there will terms whose coefficients are $\delta(x_0 - x'_0)$. When we consider fourier transforms of the derivatives of these δ-function derivative terms they will be transformed to momentum variables. However, the amplitude is a function of Lorentz invariant quantities. Thus one will get only finite polynomials of such variables, as has been argued by Symanzik [31]. His arguments is that in a local quantum field theory only finite number of derivatives of δ-functions can appear. Moreover, in addition, there are some equal time commutators and many of them vanish when we invoke locality arguments. Therefore, we shall use the relation

$$K_xK_{x'}R\tilde{\phi}(x), \tilde{\phi}_c(x') = R\tilde{j}_a(x)\tilde{j}_c(x') \tag{54}$$

keeping in mind that there are derivatives of δ-functions and some equal time commutation relations which might be present on the right hand side of the above equation. Moreover, since the derivative terms give rise to polynomials in Lorentz invariant variables, the analyticity properties of the amplitude are not affected due to the presence of such terms. This will be understood whenever we write an equation like (54). This argument might be repeated later on several occasions in order to remind the reader that the presence of the extra terms, as alluded to, do not affect the analyticity properties of the amplitude. The polynomial boundedness of the scattering amplitude has been proved subsequently in the frameworks of general quantum field theories by Epstein, Glaser and Martin [32].

4.2.1 Scattering of n = 0 Scalars States

We study the analyticity properties of the scattering amplitude of the KK zero modes of mass m_0. Thus two neutral massive scalars elastically scatter. The amplitude is

$$< k_d, k_c\ out|k_b, k_a\ in > - < k_d, k_c\ in|k_b, k_a\ in >= 2\pi\delta^3(\mathbf{k}_a + \mathbf{k}_b - \mathbf{k}_c - \mathbf{k}_d)F(k_a, k_b, k_c, k_d) \tag{55}$$

The notation is as follows: although we are considering scattering of identical, equal mass, neutral particles; it is convenient the label them. k_a and k_b are the four momenta of incoming particles (a and b respectively), k_c and k_d are the momenta of outgoing particles. All external particles are on mass shell. $F(k_a, k_b, k_c, k_d)$ is the scattering amplitude depending on Lorentz invariant variables defined in (49). We apply the LSZ reduction technique to derive the expression for the four point amplitude

$$< k_d, k_c\ out|k_b, k_a\ in >= \quad 4k_a^0 k_b^0 \delta^3(\mathbf{k}_d - \mathbf{k}_b)\delta^3(\mathbf{k}_a - \mathbf{k}_c) - \frac{i}{(2\pi)^3}\int d^4x \int d^4x' e^{-i(k_a.x - k_c.x')} K_x K_{x'} < k_d\ out|R(x';x)|k_b\ in > \tag{56}$$

where

$$R(x';x) = -i\theta(x_0 - x'_0)[\phi_a(x), \phi_c(x')] \tag{57}$$

We have reduced particles a and c in the above equation. K_x and $K_{x'}$ are the Klein-Gordon (K-G) operators. This equation is similar to eq. (52) except that we are now considering the zero modes of KK states. We shall resort to the form of (56) and abandon that form of the amplitude where all four fields are reduced. The essential remarks are in order in the sequel:

(i) The K-G operators in (56) act on $R(x';x)$ in the following ways. When the first K-G operator acts on the θ-function it will give rise to a δ-function and also it will produce a derivative of the δ-function. K-G, with $x-$derivative acting on $\phi_a(x)$ lead to the source current since the interacting field satisfies the equation of motion $(\Box_x + m_0^2)\phi_a(x) = j_a(x)$. As we have invoked the arguments of Symanzik earlier, K_x and $K_{x'}$ acting on $R\phi_a(x)\phi_c(x') = Rj_a(x)j_c(x'')$ up to derivatives of δ-functions. Their presence do not affect the analyticity properties of the amplitude.

(ii) Thus the operation of the two K-G operators leaves us with R-product of two source currents and some extra terms whose nature have been noted earlier. When we write

$$(\Box_x + m_0^2)(\Box_{x'} + m_0^2)(R(x';x)) = Rj_a(x)j_c(x') \tag{58}$$

it is understood that we are omitting presence of extra terms alluded to above. In a strict sense the equality is not valid. If we consider an arbitrary matrix element between $Rj_a(x)j_c(x')$ it can be brought to a form $Rj_a(x/2)j_c(-x/2)$ using the translation operations in those matrix elements as is well known. Thus we may introduce three (generalized) functions [5, 6]

$$F_R(q) = \int_{\infty}^{+\infty} d^4z e^{iq.z}\theta(z_0) < Q_f|[j_a(z/2), j_c(-z/2)]|Q_i > \tag{59}$$

$$F_A(q) = -\int_{\infty}^{+\infty} d^4z e^{iq.z}\theta(-z_0) < Q_f|[j_a(z/2), j_c(-z/2)]|Q_i > \tag{60}$$

and

$$F_C(q) = \int_{-\infty}^{+\infty} d^4z e^{iq.z} < Q_f|[j_a(x), j_c(x)]|Q_i > \tag{61}$$

The functions (59) - (62) are known respectively as the retarded, advanced and causal functions. Here $|Q_i>$ and $|>Q_f$ are states which carry four momenta and these momenta are held fixed and we treat them as parameters as we shall note in ensuing discussions. It is evident from above equations, (59), (60) and (61), that

$$F_C(q) = F_R(q) - F_A(q) \tag{62}$$

Notice that F_C is expressed as commutator of two currents. Then let us open up the commutator and thus we get $j_a(x)j_c(x') - j_c(x')j_a(x)$ between the two states. Let us introduce two complete set of physical states: $\sum_n |\mathcal{P}_n\tilde{\alpha}_n >< \mathcal{P}_n\tilde{\alpha}_n| = \mathbf{1}$ and $\sum_{n'} |\bar{\mathcal{P}}_{n'}\tilde{\beta}_{n'} >< \bar{\mathcal{P}}_{n'}\tilde{\beta}_{n'}| = \mathbf{1}$. Here $\{\tilde{\alpha}_n, \tilde{\beta}_{n'}\}$ stand for quantum numbers that are permitted for the physical intermediate states. However, the situation slightly different in this scenario in contrast to the well known prescriptions in the study of the analyticity domains of the amplitudes when we consider a theory with only a single neutral scalar field. The intermediate states contribute from the entire Hilbert space which is direct sum of the disjoint Hilbert space each one of which is designated by a KK charge. Thus the intermediate physical states are such that their KK charge is zero in the scattering process of zero charge KK particles. Here it is assumed that $|Q_i>$ and $|Q_f>$ are states with $n = 0$. An intermediate state could be a two particle or multiparticle state with total zero KK charge. Now we can express (61) as

$$\int d^4z e^{iq.z}\Big[\sum_n \Big(\int d^4\mathcal{P}_n < Q_f|j_a(\frac{z}{2})|\mathcal{P}_n\tilde{\alpha}_n >< \mathcal{P}_n\tilde{\alpha}_n|j_c(-\frac{z}{2})|Q_i>\Big)$$
$$-\sum_{n'}\Big(\int d^4\bar{\mathcal{P}}_{n'} < Q_f|j_c(-\frac{z}{2})|\bar{\mathcal{P}}_{n'}\tilde{\beta}_{n'} >< \bar{\mathcal{P}}_{n'}\tilde{\beta}_{n'}|j_a(\frac{z}{2})|Q_i>\Big)\Big] \tag{63}$$

If we use spacetime translation on each of the matrix elements in (63) the z-dependence in the arguments of currents disappear i.e. they become $j_a(0)$ and $j_c(0)$; moreover an energy momentum conserving δ-function appears. As a consequence $\mathcal{P}_n = \frac{(Q_i+Q_f)}{2} - q$ and $\bar{\mathcal{P}}_{n'} = \frac{(Q_i+Q_f)}{2} + q$. Therefore,

$$F_C(q) = \sum_n\Big(< Q_f|j_a(0)|\mathcal{P}_n = \frac{(Q_i+Q_f)}{2} - q, \tilde{\alpha}_n >< \tilde{\alpha}_n, \mathcal{P}_n = \frac{(Q_i+Q_f)}{2} - q|j_c(0)|Q_i>\Big)$$
$$-\sum_{n'}\Big(< Q_f|j_c(0)|\bar{\mathcal{P}}_{n'} = \frac{(Q_i+Q_f)}{2} + q, \tilde{\beta}_{n'} >< \tilde{\beta}_{n'}, \bar{\mathcal{P}}_{n'} = \frac{(Q_i+Q_f)}{2} + q|j_a(0)|Q_i>\Big) \tag{64}$$

A few explanatory remarks are in order: (i) In the sum over intermediate states the lowest mass two particle state will be $4m_0^2$. All higher KK charged intermediate states have higher thresholds since they should appear with zero-sum charges. Although we have not specified $|Q_i>$ and $|Q_f>$ to have zero KK charge, for the problem at hand (i.e. $\phi\phi \to \phi\phi$). These two states (when we consider scattering amplitude) will be particle states with zero KK charge. (ii) The second point to note is that entire tower of KK states will not contribute as intermediate states in the above expressions. In fact, for the scattering amplitude one term will be identified as absorptive part of the s-channel amplitude whereas the other term is the u-channel absorptive amplitude. Thus the contributions from KK towers will have finite number of terms. We shall provide a more convincing argument when we consider the elastic scattering of states with with nonzero KK charges (see section 5.1).

Let us consider the contributions of the multiparticle states from zero-KK-charge sector (states belong to $\mathcal{H}_{q_n=0}$ space) to the above expressions. We define

$$2A_s(q) = \sum_{n'} \Big(< Q_f|j(0)_a|\bar{\mathcal{P}}_{n'} = \frac{(Q_i+Q_f)}{2} + q, \tilde{\beta}_{n'} > \times < \tilde{\beta}'_n, \bar{\bar{P}}_n = \frac{(Q_i+Q_f)}{2} + q|j_c(0)|Q_i > \Big) = 0 \tag{65}$$

and

$$2A_u = \sum_{n} \Big(< Q_f|j_c(0)|\mathcal{P}_n = \frac{(Q_i+Q_f)}{2} - q, \tilde{\alpha}_n > \times < \tilde{\alpha}_n, \mathcal{P}_n = \frac{(Q_i+Q_f)}{2} - q|j_l(0)|Q_i > \Big) = 0 \tag{66}$$

Note that the Fourier transform of $F_C(q)$, $\bar{F_C}(z)$, vanishes outside the light cone as a consequence of causality argument. We note that with above definition of A_u and A_s

$$F_C(q) = \frac{1}{2}(A_u(q) - A_s(q)) \tag{67}$$

Moreover, $F_C(q)$ will vanish as function of q wherever, both $A_s(q)$ and $A_u(q)$ vanish simultaneously. We also remind that the intermediate states are physical states and their four momenta lies in the forward light cone, V^+. Consequently,

$$(\frac{Q_i+Q_f}{2} + q)^2 \geq 0, \quad (\frac{Q_i+Q_f}{2})_0 + q_0 \geq 0 \tag{68}$$

and

$$(\frac{Q_i+Q_f}{2} - q)^2 \geq 0, \quad (\frac{Q_i+Q_f}{2})_0 - q_0 \geq 0 \tag{69}$$

These two equations, (68) and (69), imply that there ought to be minimum mass parameters in each of the cases satisfying the conditions, (i) $(\frac{Q_i+Q_f}{2} + q)^2 \geq {\mathcal{M}_+}^2$ and (ii) $(\frac{Q_i+Q_f}{2} - q)^2 \geq {\mathcal{M}_-}^2$. If either of the conditions (i) or (ii) is satisfied then one of the matrix element will be nonvanishing and hence $F_C(q) \neq 0$; as $A_s(q)$ or $A_u(q)$ will not vanish.

The content of the above brief discussion is well known for quite sometime. The essential points to note is that, microcausality and Lorentz invariance impose constraints on $F_C(q), F_R(q)$ and $F_A(q)$ in that the locations of the singularities in the complex q-plane are identified. Therefore, it is possible to derive the analyticity properties

of the scattering amplitude starting from here in the following steps. (i) The Jost-Lehmann-Dyson theorem allows us to find the location of singularities in the retarded function in the q-variable. Consequently, the existence of small and large Lehmann ellipses is derived in the next step.
(ii) The fixed t dispersion relations can be proved when t lies within Lehmann ellipse.
(iii) A host of known results can be derived for the field theory defined on $R^{3,1} \otimes S^1$.
(iv) For example, if we choose $Q_i = k_b$ and $Q_f = k_d$ then we shall derive expressions for the scattering amplitude. Moreover, we shall be able to obtain expressions for A_s and A_u and relate them to absorptive parts. It can be shown, in this case, that there is a region in the q variable where $A_s(q) = A_u(q)$ for real q and q lies in an unphysical kinematical region. This is the coincidence region. The crossing symmetry of the amplitude is proved using the theory of several complex variables. Then the technique of enlarging the domain of holomorphy in the theory of several complex variables is utilized to prove the crossing symmetry. Our intent is not to address the issues related to crossing symmetry in this article.
The summary of this subsection, i.e. **4.2.1**, is that for the zero mode field of a compactified field theory ($R^{3,1} \otimes S^1$) the scattering amplitude satisfies the analytic properties of known massive, neutral, scalar field theory as expected. This conclusion was also reached by Khuri [23] in his model in the $n = 0$ sector of potential scattering.

4.2.2 Elastic Scattering of Scalar $n = 0$ State and $n \neq 0$ State.

This subsection is devoted to study the elastic scattering of an $n = 0$ particle with an $n \neq 0$ particle. We shall utilize the formalism developed in the Subsection 4.2 and take into account the necessary modifications required for this case.
First thing to note is that the initial states are the $n = 0$ KK zero mode particle and the $n \neq 0$ KK particle which are denoted by ϕ_a and χ_b respectively with initial four momenta, k_a and p_b. Similarly, the outgoing particles are denoted by ϕ_c and χ_d with final momenta, k_c and p_d. The masses of ϕ and χ respectively are m_0^2 and $m_n^2 = m_0^2 + \frac{n^2}{R^2}$ and we are focusing only on the elastic process. The source currents associated with the interacting fields $\phi(x)$ and $\chi(x)$ are denoted by $j(x)$ and $J(x)$ respectively. Note that the initial and the final $c.m.$ momenta, $\mathbf{k}$, are the same for this reaction.
The amplitude is defined as

$$< p_d, k_c \; out|p_b, k_a \; in > - < p_d, k_c \; in|p_b, k_a \; in >= 2\pi\delta^3(\mathbf{k}_a + \mathbf{p}_b - \mathbf{k}_c - \mathbf{p}_d)F(k_a, p_b, k_c, p_d) \quad (70)$$

$$< p_d, k_c \; out|p_b, k_a \; in >= \quad 4k_a^0 k_b^0 \delta^3(\mathbf{p}_d - \mathbf{p}_b)\delta^3(\mathbf{k}_a - \mathbf{k}_c) - \frac{i}{(2\pi)^3}\int d^4x \int d^4x' e^{-i(k_a.x - k_c.x')} K_x K_{x'} < k_d \; out|R(x';x)|k_b \; in > \quad (71)$$

Here we have chosen to reduce the states ϕ_a and ϕ_c and the two states with $n \neq 0$ are not reduced. Thus $R(x';x)$ has the same definition as in the previous subsection, 4.2.1, (see (57)). Therefore, we may write the matrix element as before and introduce the three analogous functions, $F_R(q)$, $F_A(q)$ and $F_C(q)$ where the the commutator of the source current $[j_a(z/2), j_c(-z/2)]$ is sandwiched between two fixed arbitrary states $|Q_i>$ and $|Q_f>$. When we discuss the properties of the scattering amplitude we should keep in mind that the unreduced initial and final states states have $n \neq 0$. *Remark*: It is convenient to assign states $|Q_i>$ and $|Q_f>$ nonzero n quantum number but same value of n since the total KK charge is conserved for the initial and final configurations; and note that ϕ states carry $n = 0$ KK charge. In view of above remarks let us examine the structure of $F_C(q)$ matrix element. As before, we open up the commutator $[j_a(x), j_c(x')]$ introduce a complete set of states between the products of the two current. Let us write down the relevant equations for the problem at hand

$$\int d^4z e^{iq.z}\Big[\sum_n \Big(\int d^4\mathcal{P}_n < Q_f|j_a(\frac{z}{2)}|\mathcal{P}_n\tilde{\alpha}_n >< \mathcal{P}_n\tilde{\alpha}_n|j_c(-\frac{z}{2})|Q_i > \Big) \\ -\sum_{n'} \Big(\int d^4\bar{\mathcal{P}}_{n'} < Q_f|j_c(-\frac{z}{2})|\bar{\mathcal{P}}_{n'}\tilde{\beta}_{n'} >< \bar{\mathcal{P}}_{n'}\tilde{\beta}_{n'}|j_a(\frac{z}{2})|Q_i > \Big)\Big] \quad (72)$$

If we use spacetime translation on each of the matrix elements in (63) the z-dependence in the arguments of currents disappear i.e. they become $j_a(0)$ and $j_c(0)$; moreover an energy momentum conserving δ-function appears after the d^4z integration. As a consequence, $\mathcal{P}_n = \frac{(Q_i+Q_f)}{2} - q$ and $\bar{\mathcal{P}}_{n'} = \frac{(Q_i+Q_f)}{2} + q$. Therefore,

$$F_C(q) = \sum_n \Big(< Q_f|j_a(0)|\mathcal{P}_n = \frac{(Q_i + Q_f)}{2} - q, \tilde{\alpha}_n >< \tilde{\alpha}_n, \mathcal{P}_n = \frac{(Q_i + Q_f)}{2} - q|j_c(0)|Q_i > \Big) \\ -\sum_{n'} \Big(< Q_f|j_c(0)|\bar{\mathcal{P}}_{n'} = \frac{(Q_i + Q_f)}{2} + q, \tilde{\beta}_{n'} >< \tilde{\beta}_{n'}, \bar{\mathcal{P}}_{n'} = \frac{(Q_i + Q_f)}{2} + q|j_a(0)|Q_i > \Big) \quad (73)$$

Remark: (i) We have introduced a complete set of physical states as we had in (63). We recall that we have assigned nonzero (and the same) KK charges to the two states in the matrix element. Moreover, KK charge is an additive conserved quantum number. (ii) Therefore, the admissible physical intermediate states appearing in equations (72) and (73) must carry n-units of KK charge. (iii) There are several possibilities: (a) A zero-mode state ($n = 0$) with a state of single KK state carrying n-unit of KK charge. The attributes of these intermediate states can be understood from (51). (b) Several combinations of KK tower states where some may carry $n = 0$ KK charge; however, sum total of the charges of the multiparticle states must add up to 'n'. We would like to draw attention to the fact that such contributions arise due to the presence of KK tower of states. However, there will be only finite number of such intermediate states in the sum since these multi particle states are physical and they conserve energy momentum. This argument is intuitively sound. We mention in passing that our main purpose is to investigate the analyticity property of the amplitude for elastic scattering of states carrying nonzero KK charges. The problem at hand (in this subsection) is that we have one particle with KK charge $n = 0$ and

other one has $n \neq 0$.
We may correspondingly define $A_u(q)$ and $A_s(q)$ in analogy with equations (65) and (66). We intend to argue, in this case, $F_C(q)$ vanishes when each of the two terms in (73) vanish for certain values of q. In order that $F_C(q)$ is nonzero, one of the matrix elements $A_u(q)$ or $A_s(q)$ should be nonzero. On this occasion, we also have constraints

$$(\frac{Q_i + Q_f}{2} + q)^2 \geq 0, \quad (\frac{Q_i + Q_f}{2})_0 + q_0 \geq 0 \tag{74}$$

and

$$(\frac{Q_i + Q_f}{2} - q)^2 \geq 0, \quad (\frac{Q_i + Q_f}{2})_0 - q_0 \geq 0 \tag{75}$$

as was derived in the previous case. We may follows the arguments presented in 4.2.1 that the analyticity properties of this amplitude can be studied even when some of the particles carry a nonzero KK charge. There is not much of a complication in the elastic scattering of two unequal mass particles as is the case here. Therefore, the conclusions drawn at the end of subsection 4.2.1 hold good. Moreover, we are allowed to write a fixed t dispersion relation and do not foresee any difficulty. However, to do so, we have to prove the existence of the Lehmann ellipses which does not seem to be a very challenging problem.Thus the forward dispersion relation can be proved in the case analysed in this subsection

$$(n = 0) + (n \neq 0) \to (n = 0) + (n \neq 0) \tag{76}$$

This scattering process was not addressed by Khuri. The conclusion of this subsection is that the scattering amplitude, considered in this subsection, will satisfy forward dispersion relation. In fact the analyticity hold for nonforward direction as long as t lies inside the corresponding Lehmann-Martin ellipse.

5. Elastic Scatting of States with nonzero Kaluza-Klein Charges

The elastic scattering of two particles carrying nonzero Kaluza-Klein charges are studied here. We repeat some of the assumptions alluded to in the beginning. These are neutral, massive, scalar particles. They are termed as neutral in the sense that they do not carry electric charge (more generally they do not carry any charge associated due to a local gauge symmetry). Of course, the states considered in this section, carry the KK charges. These particles are stable and there are no bound states in the theory. If we consider two incoming particles with KK charges m and n their masses are respectively $m_0^2 + \frac{m^2}{R^2}$ and $m_0^2 + \frac{n^2}{R^2}$. We assume, without any loss of generality, that each of the particles carries same KK charge, $n > 0$, in order to simplify the computations and consequently, all four participating particles are of equal mass. It will not affect any of the conclusions if we considered elastic scattering of two KK particles with different charges as will be evident later. In fact, the prescription laid down so far is adequate to handle elastic scattering of particles with unequal KK charges and hence unequal masses. Indeed, Khuri [23] has derived the result for the case $(n)+(n) \rightarrow (n)+(n)$ for forward scattering. He had chosen the *special case* of $n=1$ in order to demonstrate through a counter example that the dispersion relation is violated for that case. Our goal is to study the analyticity property of forward amplitude in this context. The initial incoming pair of particles are denoted by χ_a and χ_b and they carry four momenta p_a and p_b respectively. The outgoing particle are χ_c and χ_d and carry four momenta p_c and p_d. Our first step is to define the scattering amplitude for this reaction and proceed systematically

$$
\begin{aligned}
< p_d, p_c\ out|p_b, p_a\ in >= \quad & 4p_a^0 p_b^0 \delta^3(\mathbf{p}_d - \mathbf{p}_b)\delta^3(\mathbf{p}_a - \mathbf{p}_c) - \frac{i}{(2\pi)^3}\int d^4x \int d^4x' e^{-i(p_a.x - p_c.x')} \\
& \tilde{K}_x \tilde{K}_{x'} < p_d\ out|R(x';x)|p_b\ in > \qquad (77)
\end{aligned}
$$

where

$$
R(x';x) = -\theta(x_0 - x_0')[\chi_a(x), \chi_c(x')] \qquad (78)
$$

and $\tilde{K}_x = (\Box + m_n^2)$. We let the two K-G operators act on $\bar{R}(x;x')$ in the VEV and resulting equation is

$$
\begin{aligned}
< p_d, p_c\ out|p_b, p_a\ in >= \quad & < p_d, p_c\ in|p_b, p_a\ in > - \frac{1}{(2\pi)^3}\int d^4x \int d^4x' e^{-i(p_a.x - p_c.x')} \\
& < p_d|\theta(x_0' - x_0)[J_c(x'), J_a(x)]|p_b > \qquad (79)
\end{aligned}
$$

Here $J_a(x)$ and $J_c(x')$ are the source currents associated with the fields $\chi_a(x)$ and $\chi_c(x')$ respectively. We arrive at (79) from (77) with the understanding that the *R.H.S.* of (79) contains additional terms as discussed earlier; however, these terms do not affect the study of the analyticity properties of the amplitude. We have mentioned in the previous section that we shall explore the consequences unitarity in this section and the purpose will be transparent presently. Let us define the **T**-matrix as follows:

$$
\mathbf{S} = \mathbf{1} - i\mathbf{T} \qquad (80)
$$

The unitarity of the S-matrix, $\mathbf{S}\mathbf{S}^\dagger = \mathbf{S}^\dagger\mathbf{S} = \mathbf{1}$ yields

$$
(\mathbf{T}^\dagger - \mathbf{T}) = i\mathbf{T}^\dagger\mathbf{T} \qquad (81)
$$

In the present context, we consider the matrix element for the reaction $a+b \rightarrow c+d$. Note that on *L.H.S* of (81) it is taken between $\mathbf{T}^\dagger - \mathbf{T}$. We introduce a complete set of physical states between $\mathbf{T}^\dagger\mathbf{T}$. For the elastic case with all particles of KK charge, n, the unitarity relation is

$$
< p_d, p_c\ in|\mathbf{T}^\dagger - \mathbf{T}|p_b, p_a\ in >= i\sum_{\mathcal{N}} < p_d, p_c\ in|\mathbf{T}^\dagger|\mathcal{N} >< \mathcal{N}|\mathbf{T}|p_b, p_c\ in > \qquad (82)
$$

The complete of states stands for $|\mathcal{N} >= |p_N; \tilde{\alpha}_N >$; p_N is the momentum of a physical state and $\tilde{\alpha}_N$ stands for all other quantum numbers and some times we do not explicitly display its presence in the matrix elements. The unitarity relation reads,

$$
T^*(p_a, p_b; p_c, p_d) - T(p_d, p_c; p_b, p_a) = 2\pi i \sum_N \delta(p_d + p_c - p_N) T^*(N; p_c, p_d) T(N; p_b, p_a) \qquad (83)
$$

We arrive at an expression like the second term of the *R.H.S* of (77) after reducing two fields. If we reduce a single field as the first step (as is worked out in text books) there will be a single K-G operator acting on the field and eventually we obtain matrix element of only a single current. The *R.H.S.* of (83) has matrix element like (for example) $p_a + p_b \to p_N$. Thus we can express it as [3] [33]

$$\delta(p_N - p_a - p_b) T(N : p_b, p_a) = (2\pi)^{3/2} < N\ out|J_a(0)|p_b > \delta(p_N - p_a - p_b) \quad (84)$$

After carrying out the computations we arrive at

$$\begin{aligned} T(p_d, p_c; p_b, p_a) - T^*(p_d, p_c; p_b, p_a) = & \sum_N \Big[\delta(p_d + p_c - p_N) T(p_d, p_c; N) T^*(N; p_b, p_a) - \\ & \delta(p_a - p_c - p_N) T(p_d, -p_c; N) T^*(p_d, -p_c; N) \Big] \end{aligned} \quad (85)$$

Let consider the scattering amplitude for the reaction under considerations.

$$F(s,t) = i \int d^4x e^{i(p_a+p_c)\cdot\frac{x}{2}} \theta(x_0) < p_d|[J_a(x/2), J_c(-x'/2)]|p_b > \quad (86)$$

We define below the imaginary part of this amplitude, $F(s,t)$ and evaluate it

$$\begin{aligned} Im\ F(s,t) = & \frac{1}{2i}(F - F^*) \\ & = \frac{1}{2} \int d^4x e^{i(p_a+p_c)\cdot\frac{x}{2}} < p_d|[J_a(x/2), J_c(-x/2)]|p_b > \end{aligned} \quad (87)$$

Note that F^* is invariant under interchange $p_b \to p_d$ and also $p_d \to p_b$; moreover, $\theta(x_0) + \theta(-x_0) = 1$. We open up the commutator of the two currents in (87). Then introduce a complete set of physical states $\sum_{\mathcal{N}} |\mathcal{N} >< \mathcal{N}| = 1$. Next we implement translation operations in each of the (expanded) matrix elements to express arguments of each current as $J_a(0)$ and $J_c(0)$ and finally integrate over d^4x to get the δ-functions. As a consequence (87) assumes the form

$$\begin{aligned} F(p_d, p_c; p_b, p_a) - F^*(p_b, p_a; p_c, p_d) = & 2\pi i \sum_N \Big[\delta(p_d + p_c - p_N) F(p_d, p_c; N) F^*(p_a, p_b; N) \\ & - \delta(p_a - p_c - p_N) F(p_d, -p_a; N) F^*(p_b, -p_c; N) \Big] \end{aligned} \quad (88)$$

This is the generalized unitarity relation and all the external particles are on the mass shell. Notice that the first term on the *R.H.S* of the above equation is identical in form to the *R.H.S.* of (85); the unitarity relation for **T**-matrix. The first term in (88) has the following interpretation: the presence of the δ-function and total energy momentum conservation implies $p_d + p_c = p_N = p_a + p_b$. We identify it as the s-channel process $p_a + p_b \to p_c + p_d$.

[3] We adopt the arguments and procedures of Gasiorowicz in these derivations

Let us examine the second term of (88). Recall that the unitarity holds for the S-matrix when all external particles are on shell (as is true for the T-matrix). The presence of the δ-function in the expression ensures that the intermediate physical states will contribute for

$$p_b + (-p_c) = p_N = p_d + (-p_a) \tag{89}$$

The masses of the intermediate states must satisfy

$$\mathcal{M}_N^2 = p_N^2 = (p_b - p_c)^2 \tag{90}$$

It becomes physically transparent if we choose the Lorentz frame where particle $'b'$ is at rest i.e. $p_b = (m_b, \mathbf{0})$; thus

$$\mathcal{M}_N^2 = 2m_b(m_b - p_c^0), \quad p_c^0 > 0 \tag{91}$$

since $m_b = m_c$ and $p_c^0 = \sqrt{m_c^2 + \mathbf{p}_c^2} = \sqrt{m_b^2 + \mathbf{p}_c^2}$; $\mathcal{M}_N^2 < 0$ in this case. We recall that all particles carry the same KK charge n and hence the mass is $m_b^2 = m_n^2 = m_0^2 + \frac{n^2}{R^2}$. The intermediate state must carry that quantum number. In conclusion, the second term of (88) does not contribute to the s-channel reaction. There is an important implication of the generalized unitarity equation: Let us look at the crossed channel reaction

$$p_b + (-p_c) \to p_d + (-p_a); \quad -p_a^0 > 0, \; and \; -p_c^0 > 0 \tag{92}$$

Here p_b and p_c are incoming (hence the negative sign for p_c) and p_d and p_a are outgoing. The second matrix element in (88) contributes to the above process in the configurations of the four momenta of these particles as noted in (92); whereas the first term in that equation does not if we follow the arguments for the s-channel process.

Remark: We notice the glimpses of crossing symmetry here. As we have argued earlier in subsection 4.2.1. Indeed, the starting point will be to define $F_C(q)$ and look for the coincidence region. Notice that q is related to physical momenta of external particles when $|Q_i>$ and $|Q_f>$ are identified with the momenta of the 'unreduced' fields. Indeed, we could proceed to prove crossing symmetry for the scattering process; however, it is not our present goal.

Important observations are in order:

(i) We could ask whether entire Kaluza-Klein tower of states would appear as intermediate states in the unitarity equation [4]. It is obvious from the unitarity equation (88) that for the s-channel process, due to the presence of the energy momentum conserving δ-function, $p_n^2 = \mathcal{M}_n^2 = (p_a + p_b)^2$; consequently, not all states of the infinite KK tower will contribute to the reaction in this, (s), channel. Therefore the sum would terminate after finite number of terms, even for very large s as long as it

[4] I would like to thank Luis Alvarez Gaume for raising this issue.

is finite. Same argument also holds for the crossed channel process.
(ii) We could proceed to prove crossing symmetry from this point. However, our goal, in this investigation, is to specifically examine whether the forward scattering amplitude satisfies in the present case. Thus we focus on the forward amplitude. It is convenient to adopt the prescriptions of Symanzik [31] to this end.

5.1 Dispersion Relation for Forward Scattering Amplitude

We intend to study the analyticity property of the elastic scattering amplitude in the forward direction in this subsection. This is of paramount interest to us since it was shown by Khuri, in the case of potential scattering in a quantum mechanical model with $R^3 \otimes S^1$ geometry, that the elastic forward scattering amplitude $T(k,k;n,n)$ does not have analyticity property in contrast to the case with potentials with noncompact coordinates. We utilize the formalism developed in Section 4 and in this section. The configuration for the forward scattering amplitude is $p_c = p_a$ and $p_d = p_b$. Moreover, all the particles are scalars and of equal mass, $m_n^2 = m_0^2 + \frac{n^2}{R^2}$. Recall that the amplitude is a function of the Lorentz invariant variables, s and $t = 0$. The forward amplitude is

$$F(p_b, p_a; p_b, p_a) = \int d^4x e^{ip_a.x} (\Box_x + m_n^2)^2 < p_b|R\chi_a(x)\chi_a(0)|p_b > \qquad (93)$$

leading to

$$F(p_b, p_a; p_b, p_a) = \int d^4x e^{ip_a.x} < p_b|RJ_a(x)J_a(0)|p_b > \qquad (94)$$

We remind that in the nonforward case the reader that we had an exponential term $exp(i(p_a + p_c).x/2)$ and there were two K-G operators $K_x K_{x'}$ acting on two currents $RJ_c(x)J_a(0)$ while we considered nonforward amplitude.
We shall adopt the procedure of Symanzik who proved the dispersion relations for forward scattering amplitude of pion-nucleon scattering. Indeed, this is a simpler case of equal mass scattering. Alternatively, one could adopt mathematically more rigorous formulation of Bogoliubov; however, we have resorted to the approach of [31]. We go to a frame where particle $'b'$ is at rest, $\mathbf{p}_b = 0$. Introduce the Lorentz invariant variable, $\omega = \frac{p_a.p_b}{m_n}$ which is the incident energy of $'a'$ in this frame. The amplitude, (93) takes the form

$$F(p_b, p_a : p_b, p_a) = i \int_0^\infty dt \int d^3\mathbf{x} e^{ip_a^0 x^0 - i\sqrt{(p_a^0)^2 - m_n^2}\hat{\mathbf{e}}.\mathbf{x}} \tilde{f}(\mathbf{x}, x_0) \qquad (95)$$

where $\hat{\mathbf{e}}$ is the unit vector along direction of the three momentum $\mathbf{p}_a$. We can read off expression for $\tilde{f}$ from (93); at this stage it is sufficient to note that that $\tilde{f}(\mathbf{x}, x_0)$ vanishes unless $x_0 > |\mathbf{x}|$ due to microcausality arguments (see more discussions below). We may carry out the angular integration and the resulting expression is

$$F(p_b, p_a; p_b, p_a) = \int_0^\infty \mathcal{F}(\omega, r) dr \qquad (96)$$

where

$$\mathcal{F}(\omega,r) = \qquad 4\pi i r^2 \frac{sin\sqrt{\omega^2-m_n^2}re^{i\omega r}}{\sqrt{\omega^2-m_n^2}r}\times$$
$$\int_r^{\infty} dte^{i\omega(r-t)} < p_b|[J_a(x),J_a(0)]|p_b > \qquad (97)$$

Notice that $\mathcal{F}(\omega,r)$ is analytic function of ω in the upper half ω-plane, i.e. for complex ω, $Im\ \omega \geq 0$. The following features of the $R.H.S.$ of (97) are noteworthy.
(i) It appears that there might be a branch point at $\omega = \pm m_n$. However, note that there is really no branch point at $\omega = \pm m_n$ since, as we observe, $\frac{sin\sqrt{\omega^2-m_n^2}re^{i\omega r}}{r\sqrt{\omega^2-m_n^2}}$ is an even function of $r\sqrt{\omega^2-m_n^2}$. (ii) When $\omega < m_n$ we might apprehend the about the large r behavior of $sin\sqrt{\omega^2-m_n^2}r$ in the complex upper half ω plane; however, the presence of $e^{i\omega r}$ dispels any such doubt. We have used, loosely, the equality $(\Box_x+m_n^2)(R\chi_a(x)\chi_a(0)) = RJ_a(x)J_a(0)$. However, as alluded to earlier the equality is up to the presence of finite number of derivatives of δ-functions. Thus the amplitude, in the case of forward scattering, might have additional terms on $R.H.S.$ which are polynomials in s, in this Lorentz frame ω (see more discussions on this point later). In what follows, we shall incorporate the essential arguments of Symanzik and give some modified steps of his derivation rather than repeat the entire technical details he provided for the $\pi - N$ scattering. Let us assume that large ω behavior of the forward amplitude requires no subtractions and therefore, for large ω, the integral over ω vanishes in the dispersion integral expressed as

$$\mathcal{F}(\omega,r) = \frac{1}{\pi}\int_{\infty}^{+\infty}\frac{\text{Im }\mathcal{F}(\omega',r)}{\omega'-\omega-i\epsilon}d\omega' \qquad (98)$$

We conclude from the expression for $\mathcal{F}(\omega,r)$, (97), that Im $\mathcal{F}(r,\omega) = -$Im $\mathcal{F}(-\omega,r)$; consequently, the amplitude is an even function of energy, ω. We may rewrite (98) as

$$\mathcal{F}(\omega,r) = \frac{1}{\pi}\int_0^{+\infty}\text{Im }\mathcal{F}(\omega',r)\Big[\frac{1}{\omega'-\omega-i\epsilon}+\frac{1}{\omega'-\omega+i\epsilon}\Big] \qquad (99)$$

Note, from the definition of $\mathcal{F}(\omega,r)$, (98), that the integration over r is to be completed if we want to derive dispersion relation for the amplitude $F(p_b,p_a;p_b,p_a)$. Thus there arises the issue of interchange of integrations over r and ω. Eventually, we are to compute the forward absorptive amplitude, (Im F), as given in (97). We arrive at the following expression from (94)

$$\text{Im }F(p_b,p_a;p_b,p_a) = \frac{1}{2}\int d^4xe^{p_a.x} < p_b|[J_a(x),J_b(0)]| > \qquad (100)$$

The angular integration for $e^{ip_a.x}$ can be carried out in the Lorentz frame of our choice leading to an expression analogous to (97). Thus

$$\text{Im } F(p_b, p_a; p_b, p_a) = \frac{1}{2}\int dr 4\pi r \frac{sin\sqrt{\omega^2 - m_n^2}r}{\sqrt{\omega^2 - m_n^2}} \times \int_{-\infty}^{+\infty} e^{i\omega t} < p_b|[J_a(x), J_a(0)]|p_b > dt \quad (101)$$

We recall that in deriving the generalized unitarity relation for the nonforward amplitude, we had opened up the commutator of the two currents and we inserted complete set of states in the products of currents in the each term. Notice that in the present context, the initial and the final states are identical; this is the route to derive the optical theorem. However, our goal is different here. As has been our practice earlier, we use translation operation to get rid of the x-dependence in the argument of one of the current. Consequently, a factor of $e^{i\mathbf{p}_N.\mathbf{x}}$ or $e^{-i\mathbf{p}_N.\mathbf{x}}$ would appear where $\mathbf{p}_N$ is the momentum of the physical intermediate state (we are still in the same Lorentz frame). Therefore, as before, the angular integration can be carried out resulting the factors depending on r only

$$\text{Im } \mathcal{F}(\omega, r) = 2\pi r \frac{sin\sqrt{\omega^2 - m_n^2}r}{\sqrt{\omega^2 - m_n^2}} \sum_N \frac{sin|\mathbf{p}_N|r}{|\mathbf{p}_N|r} | < p_b|J_a(0)|N > |^2 \Big[\delta(\omega + p_N^0 - m_n) - \delta(\omega + m_n - p_N^0)\Big] \quad (102)$$

Here p_N is the four momentum of the intermediate state (recall we used completeness relation $\sum_{\mathcal{N}} |\mathcal{N} >< \mathcal{N}| = \mathbf{1}$ where $\mathcal{N}$ stood for intermediates permitted by energy momentum conservation and collection of all discrete quantum numbers such that KK charge is conserved). The same logic applies here. Notice that the source current J_a carries KK charge of n units. Thus the state $J_a(0)|n>$ has to be such that its KK charge is also n unit since $< p_b|$ carries n unit of KK charge. If we carry out d^4x integration in (100) then we shall get an expression similar to (87), now in the forward direction.

$$\text{Im } F(p_b, p_a; p_b, p_a) = \frac{1}{2}(2\pi)^4 \sum_N | < p_b|J_a(0)|p_b > |^2 \Big[\delta^4(p_b + p_a - p_N) - \delta^4(p_b - p_a + p_N)\Big] \quad (103)$$

It is worthwhile to draw the similarities of the present investigation with the work of Symanzik [31] who proved dispersion relation for the forward scattering amplitude of pion-nucleon scattering. We shall discuss the situation where there is departure of the present work from that of [31]. The isospin quantum numbers of pion and nucleon were not accounted for and both the particles were taken to be spinless. Consequently, the complications due to the nucleon spin were not encountered in that situation. Thus the problem was reduced to the study of the scattering of two unequal mass spinless particles. However, the nucleon was assigned an additive conserved quantum number. Symanzik reduced the two nucleons of initial and final states when he implemented LSZ formalism. In the process of computation of the amplitude, while introducing complete set of states between product of currents in the matrix elements all the conservation laws were accounted for. The dispersion integral was obtained keeping in mind the issues alluded to above. At that juncture, he argued that, at high energy, the amplitude will have, at most, polynomial growth. The proof of Jin-Martin [30] theorem that the elastic amplitude needs no more than two subtractions was not known in 1957. Moreover, the general results of Epstein, Glaser and Martin [32] appeared much later.

In order to contrast our work with Symanzik's, we note the important feature that all the particles carry additive discrete quantum charge, n, and these are equal mass bosons. Consequently, once we reduce $'a'$ and $'c'$,see (103), (before considering the case of forward scattering) the two unreduced states ($'b'$ and $'d'$) each carry n-units of charge. Thus the intermediate states sandwiching between one current with either $|b>$ or $<d|$ must respect the desired conservation laws. We have not proved the Jin-Martin [30] bound in this article. Nonetheless it will suffice if the amplitude has at most a polynomial growth in s for large s. We can write a subtracted dispersion relation in such a case. Notice that in the case of pion-nucleon scattering, there is a stable nucleon carrying the baryon number and the one-particle intermediate state is the nucleon. For the case at hand the intermediate state is to carry two units of n-charge. If the particles of mass $\sqrt{m_0^2+4n^2/R^2}$ appears as a pole in the amplitude, we can account for this pole as the presence of nucleon pole was taken care of in the pion-nucleon process.

Notice that, keeping the above remarks in mind, the rest of the computation could be developed in parallel to the case of the pion-nucleon scattering. The interchange of r and ω integrations in defining the Im $\mathcal{F}$ in the dispersion integral can be justified if we adopt arguments of Symanzik and Gasiorowicz [31, 33]. On the other hand, Bogoliubov introduced a prescription to obtain the dispersion relation for the scattering amplitude. It is now obvious that if we follow either of the procedures Symanzik [31] or Bogoliubov [15] the forward scattering amplitude for elastic the process

$$p_a(n)+p_b(n)\rightarrow p_a(n)+p_b(n) \tag{104}$$

satisfies dispersion relation in ω and hence in s. In view of the above discussions, the forward scattering amplitude might admit a pole with KK charge of $2n$ units. Thus the presence of possible pole term does not affect the ensuing argument about writing a dispersion relation. Now, if we resort to Mandelstam variables s and u and recall $u=4m_n^2-s$ for $t=0$, the dispersion relation can be written down in the familiar form

$$\mathrm{F(s,t=0)}=\frac{1}{\pi}\int_{s_{thr}}^{\infty}ds'\frac{\mathrm{Im}\ F(s',t=0)}{s'-s}+\frac{1}{\pi}\int_{u_{thr}}^{\infty}du'\frac{\mathrm{Im}\ F(u',t=0)}{u'-u} \tag{105}$$

Therefore, it is demonstrated under the stated assumptions that the imaginary part of the amplitude is tends to zero as $s \to \infty$ (i.e. sufficiently convergent) the forward amplitude satisfies an unsubtracted dispersion relation. If the amplitude i is polynomially bounded as s tends to large values, we can introduce finite number of subtractions as the analyticity property will continue to hold. In case we invoke Jin-Martin [30] upper bound for the amplitude then the forward amplitude will need *no more* than *two* subtractions (see discussion for more details).

The important conclusion of our investigation is that the forward scattering amplitude for elastic scattering of KK states with $n > 0$, in a scalar field theory defined in a compactified spacetime, i.e. $R^{3,1} \otimes S^1$; satisfies dispersion relation in s in the forward direction. This conclusion is similar to the case of a massive, neutral, scalar field theory defines in a flat four dimensional spacetime, $R^{3,1}$.

Our main conclusion may be stated as a theorem.

Theorem: A massive neutral scalar field theory is defined on $R^{4,1}$ and the manifold is compactified to $R^{3,1} \otimes S^1$ subsequently. The spectrum of the states are obtained by the Kaluza-Klein dimensional reduction. The forward elastic scattering amplitude for scattering of the Kaluza-Klein states on the manifold $R^{3,1)} \otimes S^1$ satisfies dispersion relations.

6 Summary and Discussions

We summarize our results in this section and discuss their implications. Our principal goal of this article is to study the analyticity property of the forward scattering amplitude for a five dimensional scalar field theory which is compactified to $R^{3,1} \otimes S^1$. The interest in this problem arose from a work of Khuri [23] in potential scattering in a spatial geometry of $R^3 \otimes S^1$. He adopted the Green's function technique and employed perturbation theory in order to compute the scattering amplitude. The main conclusion was that for a class of Yukawa-type potentials, the forward scattering amplitude failed to satisfy the dispersion relation in the second order for the case when the discrete quantum number, n, associated with the periodic coordinate of S^1, is nonzero. Indeed, there were some concerns if dispersion relation is invalidated in relativistic quantum field theories.

I have worked in the axiomatic frameworks of LSZ (Lehmann-Symanzik- Zimmermann). I considered a neutral, scalar field of mass m_0 in $D = 5$ and compactified the spacetime to $R^{3,1} \otimes S^1$. The resulting theory is Lorentz invariant in four dimensional spacetime. It has a massive, neutral scalar field of mass m_0 (*mass of the zero mode*) in addition to a tower of KK states. I presented a brief outline of Khuri's result in Section 2. I developed the systematic prescription to study the field theory in the $D = 4$ spacetime with a compact dimension. The sections 4, 4.2.1, 4.2.2 were devoted to setting up the frame work. The first case was to consider elastic scattering of $n = 0$ particles. This problem has been studied long ago in the frameworks of axiomatic field theory. The only departure, from the standard case, is the presence of KK towers. We argued qualitatively that the entire tower of KK states do not appear as intermediate states when we derive spectral representations for absorptive parts. It is to be noted that, for this case, we can derive the analyticity properties rigorously for the forward amplitude as well as for the nonforward amplitude. Moreover, we can prove the existence Lehmann ellipses and thus write down fixed t dispersion relation as long as $|t|$ lies inside the Lehmann-Martin ellipse. In fact, Khuri [23], in his analysis, concluded that the amplitude for $n = 0$ sector satisfies analyticity properties in his potential model. Next we considered elastic scattering of the $n = 0$ state with a KK state with nonzero charge. This is a case of unequal mass scattering. We adopted the same prescription as we developed for elastic scattering of $n = 0$ states. In the brief subsection 4.2.2 we outlined the argument that this amplitude would satisfy desired and expected analyticity property in the forward direction. Indeed, fixed t dispersion relations can be written down as well. This case has not been studied by Khuri. In the study of the two cases analyzed in Section 4.2.1 and 4.2.2, the unitarity of the S-matrix was not explicitly invoked in the computations. It was sufficient to require Lorentz invariance and microcausality to arrive at those conclusions.

I investigated analyticity property for the forward elastic scattering amplitude of states with nonzero KK charges. In principe, one can study elastic scattering of two states carrying KK charges (l, n). The final particles will also carry the same charge since we focus on elastic process. However, I considered a simpler scenario, without any loss of generality, when initial (two) particles carry the same KK charge, $n > 0$. If these charges were different, say (l, n), then it will be elastic scattering of two unequal mass particles of respectively masses, m_l and m_n. We do not foresee any obstacles to prove forward dispersion relations in the unequal charge elastic scattering processes. For the equal charge elastic scattering (and hence equal mass elastic scattering) case; first we derived an expression for the nonforward scattering amplitude starting from the LSZ reduction particles with equal charges . Our goal is to derive dispersion relation. Therefore, we extracted the *imaginary* part of the amplitude and took the opportunity to invoke generalized unitarity relation. Thus we derived an expression for product of currents and we inserted a complete set of states between the two currents in the expression for the matrix element. The complete set of physical states included contributions of the entire KK tower in the matrix element as long as it respects energy momentum conservation and discrete charge conservation. To recollect, the S-matrix is obtained when the four external particles are on the mass shell and all the intermediate states are physical states. One important conclusion is that energy momentum conservation (and mass shell condition) ensures that *entire KK tower does not contribute* to the sum of intermediate states. As long as s is finite, we may choose it to be very large, there is a cut off on the contribution of KK towers. The second important point to note that we see a glimpse of crossing symmetry (see remarks after (88)) in the following way: The first term in that equation contributes to the s-channel process and the second term does not contribute from energy momentum conservation consideration. The second term, to be interpreted as *crossed channel reaction*, contributed to that reaction and the first term does not. These conclusions hold good for physical processes. That the two amplitudes coincide in the unphysical kinematical region for real momentum variable is well known. We have neither proved crossing symmetry here nor we had any intentions to prove it here. Finally, we show that the forward scattering amplitude satisfies dispersion relations. We invoked arguments of Symanzik in proving the forward dispersion relation. Our conclusion is that in a relativistic quantum field theory defined on a spacetime geometry, $R^{3,1} \otimes S^1$, the forward scattering amplitude for elastic scattering scattering of KK states (with nonzero charge) the dispersion relation is satisfied.

Now we proceed to discuss some aspects of this investigation. We have assumed existence of stable particles in the entire spectrum of the the theory defined on $R^{3,1} \otimes S^1$ geometry. Our arguments is based on the conservation of KK discrete charge $q_n = \frac{n}{R}$; it is the momentum along the compatified direction. We have also assumed the absence of bound states. The charge, q_n is like a global charge and is assumed to be conserved. This conservation law does not originate from a local gauge invariance. At the moment we have no strong argument about absence of bound states. If we had considered a five dimensional theory with gravity and had utilized KK technique to reduce it to a theory in flat Minkowski space with geometry $R^{3,1} \otimes S^1$, there will a $U(1)$ gauge field coupled to the resulting KK scalars in the spectrum. The gauge field appears in the standard dimensional reduction of a five dimensional theory (with gravity) coupled to a massive neutral scalar field. The spectrum of the flat four dimensional theory is a (zero mode) scalar, the tower of KK states and a $U(1)$ gauge field couple to all KK states. In this scenario, the charge could be associate with a $U(1)$ gauge *charge*. However, we cannot use LSZ procedure in the *five* dimensional curved space. Nonetheless in the compactified theory, defined in a flat space with an S^1 coordinate the LSZ formalism can be incorporated. In this scenario, with one $U(1)$ gauge field, may be we could have bound states with very low binding energy (like BPS states); although BPS states appear in spontaneously broken symmetry theory (like BPS monopoles). We mention in passing that the present investigation of analyticity of amplitude is carried out for S^1 compactification. S^1 compactification does not play any special role. We can compactify a flat space $\hat{D}$ dimensional field theory to $D = 4$ theory on a $\hat{D} - 4$ dimensional torus [34, 35]. The line of arguments followed in this work can be suitably generalized to study the analyticity properties of corresponding theory. At this stage we offer no further remarks and leave the details for future investigations.

Khuri [23] was motivated by the large extra dimension scenario to undertake the problem. He had raised the question what will be the consequences of his conclusions (in the potential scattering model) if indeed the dispersion relation is not valid at LHC energies. However, in the field theory, under the consideration, the forward dispersion relation holds good. Nevertheless it is important to ask if there is a large radius compactified theory which is accessible at LHC energy what are consequences from rigorous field theoretic perspectives?

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References

1. H. Lehmann, K. Symanzik and W. Zimmermann, Nuovo Cimento, **1**, 205 (1955)
2. A. Martin, Scattering Theory: unitarity, analyticity and crossing, Springer-Verlag, Berlin-Heidelberg-New York, (1969).
3. A. Martin and F. Cheung, Analyticity properties and bounds of the scattering amplitudes, Gordon and Breach, New York (1970).
4. C. Itzykson and J.-B. Zubber, Quantum Field Theory; Dover Publications, Mineola, New York, 2008.
5. M. Froissart, in Dispersion Relations and their Connection with Causality (Academic, New York); Varrena Summer School Lectures, 1964.
6. H. Lehmann, Varrena Lecture Notes, Nuovo Cimen. Supplemento, **14**, 153 (1959) *series X.*
7. G. Sommer, Fortschritte. Phys. **18**, 577 (1970)
8. R. J. Eden, Rev. Mod. Phys. **43**, 15 (1971)
9. S. M. Roy, Phys. Rep. **C5**, 125 (1972).
10. A. S. Wightman, Phys. Rev. **101**, 860 (1956).
11. R. Jost, The General Theory of Quantized Fields, American Mathematical Society, Providence, Rhodes Island, 1965.
12. J. F. Streater, Rep. Prog. Phys. **38**, 771 (1975)
13. L. Klein, Dispersion Relations and Abstract Approach to Field Theory Field Theory, Gordon and Breach, Publisher Inc, New York, 1961.
14. S. S. Schweber, An Introduction to Relativistic Quantum Field Theory,Raw, Peterson and Company, Evaston, Illinois,1961.
15. N. N. Bogolibov, A. A. Logunov, A. I. Oksak, I. T. Todorov, General Principles of Quantum Field Theory, Klwer Academic Publisher, Dordrecht/Boston/London, 1990.
16. M. Froissart, Phys. Rev. **123**, 1053 (1961)(1961).
17. A. Martin, Phys. Rev. **129**, 1432 (1963); Nuovo Cim. **42A**, 930 (1966).
18. M. L. Goldgerber, K. M. Watson, Collision Theory, Weyl Publication, 1964.

19. N. Khuri, Phys. Rev. **107**, 1148 (1957)

20. D. Wong, Phys. Rev. **107**, 350 (1957)

21. A. Antoniadis and K. Beneki, Mod. Phys. Lett. A **30**, 1502002 (2015) for a recent review.

22. D. Luest and T. R. Taylor, Mod. Phys. Lett. A **30**, 15040015 (2015) for a recent review.

23. N. N. Khuri, Ann. Phys. **242**, 332 (1995)

24. J. Maharana, J. Math. Phys. **58**, 012302 (2017).

25. R. Jost and H. Lehmann, Nuovo Cimen. **5**, 1598 (1957).

26. F. J. Dyson, Phys. Rev. **110**, 1460 (1958).

27. J. Maharana, Phys. Lett. **B 764**, 212 (2017).

28. H. Lehmann, Nuovo Cimen. **10**, 579(1958).

29. A. Martin, Nuovo. Cimen. **42**, 930 (1966).

30. Y. S. Jin and A. Martin, Phys. Rev. **135**, B1369 (1964).

31. K. Symanzik, Phys. Rev. **105**, 743 (1957).

32. H. Epstein, V. Glaser and A. Martin, Commun. Mathe. Phys. **13**, 257 (1969)

33. S. Gasiorowicz, Fortschritte der Physik, **8**, 665 (1960).

34. J. Scherk and J. H. Schwarz, Nucl. Phys. **B81**, 118 (1974).

35. J. Maharana and J. H. Schwarz, Nucl. Phys. **B390**, 3 (1993).