

FOURIER COEFFICIENTS AND SMALL AUTOMORPHIC REPRESENTATIONS

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ABSTRACT. In this paper we analyze Fourier coefficients of automorphic forms on adelic reductive groups $G(\mathbb{A})$. Let π be an automorphic representation of $G(\mathbb{A})$. It is well-known that Fourier coefficients of automorphic forms can be organized into nilpotent orbits \mathcal{O} of G . We prove that any Fourier coefficient $\mathcal{F}_{\mathcal{O}}$ attached to π is linearly determined by so-called ‘Levi-distinguished’ coefficients associated with orbits which are equal or larger than \mathcal{O} . When G is split and simply-laced, and π is a minimal or next-to-minimal automorphic representation of $G(\mathbb{A})$, we prove that any $\eta \in \pi$ is completely determined by its standard Whittaker coefficients with respect to the unipotent radical of a fixed Borel subgroup, analogously to the Piatetski-Shapiro–Shalika formula for cusp forms on GL_n . In this setting we also derive explicit formulas expressing any maximal parabolic Fourier coefficient in terms of (possibly degenerate) standard Whittaker coefficients for all simply-laced groups. We provide detailed examples for when G is of type D_5 , E_6 , E_7 or E_8 with potential applications to scattering amplitudes in string theory.

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CONTENTS

| | |
|--|----|
| 1. Introduction and main results | 3 |
| 1.1. Introduction | 3 |
| 1.2. Main results | 5 |
| 1.3. Motivation from string theory | 10 |
| 1.4. Structure of the paper | 12 |
| 1.5. Acknowledgements | 13 |
| 2. Definitions | 14 |
| 2.1. Levi-distinguished Fourier coefficients | 16 |
| 2.2. Order on nilpotent orbits and Whittaker support | 18 |
| 2.3. Minimal and next-to-minimal representations | 18 |
| 3. Relating different Fourier coefficients | 19 |
| 3.1. Relating different isotropic subspaces | 19 |
| 3.2. Relating different Whittaker pairs | 20 |
| 3.3. Conjugations and translations | 24 |
| 4. General reductive groups | 24 |
| 4.1. Proof of Theorem A | 26 |
| 4.2. Proof of Theorem B | 27 |
| 4.3. On PL-orbits | 28 |
| 4.4. Some geometric lemmas | 29 |
| 5. Small automorphic functions on simply-laced Lie algebras | 31 |
| 5.1. Proof of Theorem C | 31 |
| 5.2. Proof of Theorem D | 33 |
| 5.3. Proof of Theorem E | 34 |
| 5.4. Expressing the form itself through Whittaker coefficients | 35 |
| 5.5. Comparison with related results in the literature | 38 |
| 5.6. Proof of Proposition 5.4.4 for type E_n | 40 |
| 6. Detailed examples | 42 |
| 6.1. Whittaker triples | 42 |
| 6.2. Examples for $SO_{5,5}$ | 43 |
| 6.3. Examples for E_6 | 46 |
| 6.4. Examples for E_7 | 48 |
| 6.5. Examples for E_8 | 49 |
| References | 50 |

1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. Let \mathbb{K} be a number field and \mathbb{A} its ring of adeles. Let G be a reductive group defined over \mathbb{K} , $G(\mathbb{A})$ the group of adelic points of G and η be an automorphic form on $G(\mathbb{A})$. Fix a minimal parabolic subgroup B (a Borel subgroup if G is quasi-split) in G and let N be its unipotent radical. Consider the (infinite) set of unitary characters $\chi_N : N(\mathbb{K})\backslash N(\mathbb{A}) \rightarrow \mathbb{C}^\times$. It is well-known that the constant term of η with respect to $[N, N]$ can be decomposed according to

$$\int_{[N, N](\mathbb{K})\backslash [N, N](\mathbb{A})} \eta(ng)dn = \sum_{\chi_N} \mathcal{W}_{\chi_N}[\eta](g), \quad (1.1)$$

where $\mathcal{W}_{\chi_N} \in C^\infty(G(\mathbb{A}))$ is the standard Whittaker coefficient corresponding to χ_N given by

$$\mathcal{W}_{\chi_N}[\eta](g) := \int_{N(\mathbb{K})\backslash N(\mathbb{A})} \eta(ng)\chi_N(n)^{-1}dn. \quad (1.2)$$

This is N -equivariant, $\mathcal{W}_{\chi_N}[\eta](ng) = \chi_N(n)\mathcal{W}_{\chi_N}[\eta](g)$. If η is spherical (i.e. $\eta(gk) = \eta(g)$ for k in the maximal compact subgroup of $K \subset G$) then, by the Iwasawa decomposition $g = nak$, $\mathcal{W}_{\chi_N}[\eta](g)$ is determined by its restriction to the maximal torus $T \subset G$.

If η is an Eisenstein series induced from a character $\mu = \Pi_\nu \mu_\nu$ of a Borel subgroup $B \subset G$, and χ_N is *generic*, the Whittaker coefficient is well-known to be Eulerian,

$$\mathcal{W}_{\chi_N}[\eta] = \Pi_\nu \mathcal{W}_{\chi_N, \nu}[\mu], \quad (1.3)$$

where the local factors are given by so-called *Jacquet integrals*

$$\mathcal{W}_{\chi_N, \nu}[\mu] = \int_{N(\mathbb{K}_\nu)} \mu_\nu(n)\chi_{N, \nu}(n)^{-1}dn. \quad (1.4)$$

This is a powerful result, as for each finite place ν these integrals are explicitly computable using the *Casselman–Shalika formula* [CS80].

It is natural to ask whether one can recover all of η from its Whittaker coefficients, and not just the constant term of η with respect to $[N, N]$. This is known to be true when η is a cusp form on $\mathrm{GL}_n(\mathbb{A})$ for which we have the Piatetski–Shapiro–Shalika formula [PS79, Sha74]:

$$\eta(g) = \sum_{\gamma \in N_{n-1}(\mathbb{K})\backslash \mathrm{GL}_{n-1}(\mathbb{K})} \sum_{\chi_N} \mathcal{W}_{\chi_N}[\eta]((\begin{smallmatrix} \gamma & \\ & 1 \end{smallmatrix})g), \quad (1.5)$$

where N_{n-1} is the unipotent radical of a Borel subgroup of GL_{n-1} . On the other hand, all Whittaker coefficients of non-generic cusp forms vanish, and thus such forms definitely cannot be recovered. By [Ike01] such forms exist on Sp_4 . Our first result, Theorem A below, provides a sufficient condition for recovering a form η from its Whittaker coefficients.

It is also natural to consider more general Fourier coefficients with respect to unipotent radicals U of arbitrary parabolic subgroups $P = LU \subseteq G$. Consider a set of unitary characters $\chi_U : U(\mathbb{K})\backslash U(\mathbb{A}) \rightarrow \mathbb{C}^\times$. We have the associated *Fourier coefficient* of an

automorphic form η on G given by:

$$\mathcal{F}_{\chi_U}[\eta](g) := \int_{U(\mathbb{K}) \backslash U(\mathbb{A})} \eta(ug) \chi_U(u)^{-1} du. \quad (1.6)$$

By construction this is U -equivariant and can be viewed as a function $\mathcal{F}_{\chi_U} : C(\mathbb{K}) \backslash C(\mathbb{A}) \rightarrow \mathbb{C}$, where C is the stabilizer of χ_U inside the Levi L . In the special case when P is a minimal parabolic, the Fourier coefficient \mathcal{F}_{χ_U} coincides with the Whittaker coefficient (1.2). As already stressed above, when U is non-abelian, the coefficient \mathcal{F}_{χ_U} only captures a part of the Fourier expansion of η . To reconstruct η from its coefficients one must consider the derived series of U :

$$U^{(i+1)} = [U^{(i)}, U^{(i)}], \quad U^{(0)} = U. \quad (1.7)$$

This series will terminate after finitely many steps since U is unipotent. A unitary character $\chi_{U^{(i)}}$ is trivial on $U^{(i+1)}$ and the complete non-abelian Fourier expansion of η with respect to U takes the form

$$\eta = \mathcal{F}_0[\eta] + \sum_{\chi_{U^{(0)}} \neq 1} \mathcal{F}_{\chi_{U^{(0)}}}[\eta] + \sum_{\chi_{U^{(1)}} \neq 1} \mathcal{F}_{\chi_{U^{(1)}}}[\eta] + \cdots + \sum_{\chi_{U^{(i_0-1)}} \neq 1} \mathcal{F}_{\chi_{U^{(i_0-1)}}}[\eta],$$

where i_0 is the smallest integer for which $U^{(i_0)} = \mathbf{1}$.

In general it is a hard problem to obtain explicit formulas for arbitrary Fourier coefficients $\mathcal{F}_{\chi_{U^{(i)}}}$ in the series above; in particular, they are generically non-Eulerian and no analogue of the Casselman–Shalika formula exists. However, for special choices of data, that is, choice of automorphic form η , unipotent $U^{(i)}$ and character $\chi_{U^{(i)}}$, the coefficients $\mathcal{F}_{\chi_{U^{(i)}}}[\eta]$ may simplify.

In this paper we will prove many results concerning Fourier coefficients of the form discussed above, as well as more general ones. In particular, we prove that in a large class of cases, the coefficients $\mathcal{F}_{\chi_{U^{(i)}}}$ are linearly determined by the simpler Whittaker coefficients \mathcal{W}_{χ_N} which allows us to compute $\mathcal{F}_{\chi_{U^{(i)}}}$ explicitly. The emphasis on the reduction to Whittaker coefficients is due to the fact that it is known how to compute them explicitly and they take simple forms for small representations [FKP14].

Our results are strongest in the case when G is a split simply-laced group and η is a so-called *minimal* or *next-to-minimal* automorphic form. This means that all Fourier coefficients attached to nilpotents outside of a union of Zariski closures of minimal or next-to-minimal nilpotent orbits vanish. We refer to §2.3 below for the precise definitions. A sufficient condition for this is that one of the local components of the representation generated by η is minimal or next-to-minimal, see Lemma 2.2.4 below. For minimal representations, this condition is also shown to be necessary under some additional assumptions on G , see [GS05, KS15].

The minimal representations have been extensively studied in the literature, in particular due to their crucial role in establishing functoriality in the form of theta correspondences. Moreover in a series of works [GRS11, GRS97, Gin06, Gin14], π_{\min} was used to construct global Eulerian integrals. Next-to-minimal representations have not been as extensively analyzed though in recent years this has started to change, partly due to their importance in understanding scattering amplitudes in string theory [GMV15, Pio10, FKP14, GKP16, FGKP18]; see §1.3 below for more details on this connection.

To achieve our goal we will use several notions of Fourier coefficients. We define these in §2, following [GG17, GGS] but with slightly different notation. In [GG17, GGS] it was shown that there exist G -equivariant epimorphisms between different spaces of Fourier coefficients, thus determining their vanishing properties in terms of nilpotent orbits. In this paper we determine exact relations (instead of only showing the existence of such) between different types of Fourier coefficients, and in particular reduce Fourier coefficients that are difficult to compute into more manageable coefficients such as the known Whittaker coefficients with respect to the unipotent radical of a minimal parabolic subgroup.

1.2. Main results. Let us now explain some of our main results in more detail. To this end we need to briefly introduce some terminology. Denote by \mathfrak{g} the \mathbb{K} -points of the Lie algebra of G . A *Whittaker pair* is an ordered pair $(S, \varphi) \in \mathfrak{g} \times \mathfrak{g}^*$, where S is a semi-simple element with eigenvalues of $\text{ad}(S)$ in \mathbb{Q} , and $\text{ad}^*(S)(\varphi) = -2\varphi$. This implies that φ is necessarily nilpotent and corresponds to a unique nilpotent element $f = f_\varphi \in \mathfrak{g}$ by the Killing form pairing. Each Whittaker pair (S, φ) defines a unipotent subgroup $N_{S, \varphi} \subset G$ given by (2.2) below and a unitary character χ_φ on $N_{S, \varphi}$ by $\chi_\varphi(n) = \chi(\varphi(\log n))$ for $n \in N_{S, \varphi}(\mathbb{A})$.

Our results are applicable to a wide space of functions on $G(\mathbb{A})$, that we denote by $C^\infty(G(\mathbb{K}) \backslash G(\mathbb{A}))$ and call the space of automorphic function. This space consists of functions f that are left $G(\mathbb{K})$ -invariant, finite under the right action of $K_f := \prod_{\text{finite } \nu} G(\mathcal{O}_\nu)$, and smooth when restricted to $G_\infty := \prod_{\text{infinite } \nu} G(\mathbb{K}_\nu)$. In other words, we remove the usual requirements of moderate growth and finiteness under the center \mathfrak{z} of the universal enveloping algebra.

Following [MW87, GRS97, GRS11, GG17] we attach to each Whittaker pair (S, φ) and automorphic function η on G the following Fourier coefficient

$$\mathcal{F}_{S, \varphi}[\eta](g) = \int_{N_{S, \varphi}(\mathbb{K}) \backslash N_{S, \varphi}(\mathbb{A})} \eta(ng) \chi_\varphi(n)^{-1} dn. \quad (1.8)$$

Remark 1.2.1. This definition is more general than what is usually referred to as a Fourier coefficient in the literature, cf. [GRS97, GRS11, Gin06, GH11].

Note that $\mathcal{F}_{S, \varphi}[\eta](g)$ is a smooth function on $G(\mathbb{A})$ in the above sense, but is not invariant under $G(\mathbb{K})$ any more. On the other hand, its restriction to the joint centralizer $G_{S, \varphi}$ of S and φ is left $G_{S, \varphi}(\mathbb{K})$ -invariant. As shown in [GH11], if η is in addition \mathfrak{z} -finite and has moderate growth, then the restriction of η to $G_{S, \varphi}(\mathbb{A})$ still has moderate growth, but may stop being \mathfrak{z} -finite.

Note also that the unipotent group $N_{S, \varphi}$ is not necessarily the unipotent radical of a parabolic subgroup of G ; see the discussion of the derived series in §1.1. Consider, for example, the case of $G = E_8$ and let $P = LU \subset E_8$ be the Heisenberg parabolic such that the Levi is $L = E_7 \times \text{GL}_1$ and the unipotent radical U is the 57-dimensional Heisenberg group with one-dimensional center $C = [U, U]$. Then the Fourier coefficient $\mathcal{F}_{S, \varphi}$ includes the “non-abelian” coefficient corresponding to $N_{S, \varphi} = C$ and χ_φ a non-trivial character on C . This case is relevant for applications to physics; see §1.3 below.

If a Whittaker pair (h, φ) corresponds to a Jacobson–Morozov \mathfrak{sl}_2 -triple $(f_\varphi, h, e_\varphi)$ we say that it is a *neutral* Whittaker pair, and call the corresponding Fourier coefficient *neutral Fourier coefficient*. This is what is usually called a Fourier coefficient in the literature.

The *global wave front set* of η , denoted $\text{WO}(\eta)$, is defined as the set of nilpotent orbits \mathcal{O} such that there exists a neutral pair (h, φ) with non-vanishing $\mathcal{F}_{h, \varphi}[\eta]$ and $\varphi \in \mathcal{O}$, see Definition 2.2.3 below. It was shown in [GGS17, Theorem C] that if $\mathcal{F}_{h, \varphi}[\eta] = 0$ then $\mathcal{F}_{S, \varphi}[\eta] = 0$ for any Whittaker pair (S, φ) , not necessarily neutral.

Because of the many different kinds of Fourier coefficients figuring in this paper, we will also make the following distinctions. If S is such that $N_{S, \varphi}$ is the unipotent radical of a minimal parabolic subgroup of G , independent of φ , we say that (S, φ) is a *standard Whittaker pair* and call the Fourier coefficient $\mathcal{F}_{S, \varphi}$ a (*standard*) *Whittaker coefficient* denoted by $\mathcal{W}_{S, \varphi}$. If S corresponds to our fixed minimal parabolic subgroup B we may simply write \mathcal{W}_φ as in (1.2). Another special case of a standard Whittaker pair is a *principal Whittaker pair* as introduced in Definition 2.0.5 further restricting S , which is then also called principal. There we also define what it means for a Whittaker pair (S, φ) or a character φ to be *principal in a Levi subgroup* (or a *PL-pair*) with the corresponding Fourier coefficient $\mathcal{F}_{S, \varphi}$ being called a *PL-coefficient*.

Finally, in §2.1 we will define what we call *Levi-distinguished Fourier coefficients*. Such a coefficient is defined by a parabolic subgroup $P \subseteq G$ (defined over \mathbb{K}), a Levi decomposition $P = LU$ and a Whittaker pair (H, φ) for L , in which φ is \mathbb{K} -*distinguished*, i.e. does not belong to the dual Lie algebra of any Levi subgroup of L defined over \mathbb{K} . The corresponding Fourier coefficient is given by considering the constant term with respect to U as a function on L , and then taking the Fourier coefficient $\mathcal{F}_{H, \varphi}$. To see that this construction defines a Fourier coefficient on G , we let Z be a rational semi-simple element that commutes with L and has all its non-zero adjoint eigenvalues much bigger than those of H (in absolute value). Then the Levi-distinguished Fourier coefficient is $\mathcal{F}_{H+Z, \varphi}$. By Lemma 2.1.9 below, if φ is a principal nilpotent in L then $\mathcal{F}_{H+Z, \varphi}$ is a Whittaker coefficient.

Our main results can be summarized in the following theorems which are proven in §§4,5.

Theorem A. *Let η be an automorphic function on a reductive group G . Then, any Fourier coefficient $\mathcal{F}_{S, \varphi}[\eta]$ is linearly determined by the Levi-distinguished Fourier coefficients with characters in orbits which are equal to or bigger than $G\varphi$.*

In particular, if all non-PL coefficients of η vanish, then all Fourier coefficients are linearly determined by Whittaker coefficients $\mathcal{W}_{\varphi'}[\eta]$.

We refer to Definition 2.2.1 below for our order relation on \mathbb{K} -rational nilpotent orbits. The term ‘linearly determined’ is explained in Definition 2.0.9 below. It includes taking sums over characters, sums over $G(\mathbb{K})$ -translates of the arguments and integrations over unipotent subgroups giving expressions schematically on the form

$$\sum_{\varphi'} \sum_{\gamma} \int du \mathcal{W}_{\varphi'}(\gamma u g) \quad (1.9)$$

and similarly for other Levi-distinguished Fourier coefficients.

One can show that for simply-laced groups the minimal and the next-to-minimal orbits are always PL. Thus, Theorem A implies that minimal and next-to-minimal forms on simply-laced groups, as well as all their Fourier coefficients are linearly determined by Whittaker coefficients. We provide explicit formulas for this determination in Theorems C, D, E below.

In order to present our next theorems we will need to introduce some notation.

Notation 1.2.2. For a rational semi-simple $H \in \mathfrak{g}$ and $\lambda \in \mathbb{Q}$ denote by \mathfrak{g}_λ^H the i -eigenspace of $\text{ad}(H)$. Denote also $\mathfrak{g}_{>\lambda}^H := \bigoplus_{\mu>\lambda} \mathfrak{g}_\mu^H$, $\mathfrak{g}_{\geq\lambda}^H := \mathfrak{g}_\lambda^H \oplus \mathfrak{g}_{>\lambda}^H$, and similarly for $\mathfrak{g}_{<\lambda}^H$ and $\mathfrak{g}_{\leq\lambda}^H$. For $\varphi \in \mathfrak{g}^*$ denote by \mathfrak{g}_φ its stabilizer in \mathfrak{g} under the coadjoint action.

Definition 1.2.3. Let (H, φ) be a Whittaker pair, and let $Z \in \mathfrak{g}(\mathbb{K})$ be a rational semi-simple element that commutes with H and φ and satisfies

$$\mathfrak{g}_\varphi \cap \mathfrak{g}_{\geq 1}^H \subseteq \mathfrak{g}_{\geq 0}^Z \quad (1.10)$$

We will say that (H, φ) *dominates* $(H + Z, \varphi)$.

In Proposition 4.0.1 below we show that if (H, φ) dominates (S, φ) then $\mathcal{F}_{H, \varphi}$ linearly determines $\mathcal{F}_{S, \varphi}$. The next theorem gives a sufficient condition for $\mathcal{F}_{S, \varphi}$ to determine $\mathcal{F}_{H, \varphi}$.

Theorem B. *Let (H, φ) and (S, φ) be Whittaker pairs such that (H, φ) dominates (S, φ) . Denote*

$$\mathfrak{v} := \mathfrak{g}_{>1}^H \cap \mathfrak{g}_{<1}^S, \text{ and } V := \text{Exp}(\mathfrak{v}). \quad (1.11)$$

Let η be an automorphic function on G , and assume that the orbit of φ is maximal in $\text{WO}(\eta)$.

(i) If $\mathfrak{g}_1^H = \mathfrak{g}_1^S = 0$ then

$$\mathcal{F}_{H, \varphi}[\eta](g) = \int_{V(\mathbb{A})} \mathcal{F}_{S, \varphi}[\eta](vg) \, dv. \quad (1.12)$$

(ii) More generally, denote

$$\mathfrak{u} := (\mathfrak{g}_{\geq 1}^S \cap \mathfrak{g}_{>1}^H) / (\mathfrak{g}_{>1}^S \cap \mathfrak{g}_{>1}^H), \quad \mathfrak{w} := (\mathfrak{g}_{\geq 1}^H \cap \mathfrak{g}_{<1}^S) / \mathfrak{v}, \quad U := \text{Exp}(\mathfrak{u}), \quad W := \text{Exp}(\mathfrak{w}) \quad (1.13)$$

Then

$$\mathcal{F}_{H, \varphi}[\eta](g) = \sum_{w \in W(\mathbb{K})} \int_{V(\mathbb{A})} \int_{U(\mathbb{K}) \backslash U(\mathbb{A})} \mathcal{F}_{S, \varphi}[\eta](wvug) \, dudv. \quad (1.14)$$

In [GGS17] (and in Corollary 3.2.2) it is shown that any Whittaker pair (H, φ) is dominated by a neutral pair (h, φ) . In §4 below we show that any Whittaker pair (H, φ) dominates a Levi-distinguished pair (S, φ) . Note that if φ is principal in a Levi subgroup (PL), then any Levi-distinguished Fourier coefficient is a Whittaker coefficient, and thus, if φ is PL and is maximal for η then any Fourier coefficient $\mathcal{F}_{H, \varphi}[\eta]$ is obtained by an integral transform from a Whittaker coefficient $\mathcal{W}_{S, \varphi}[\eta]$. For the remaining theorems we will consider minimal and next-to-minimal automorphic functions on a split simply-laced group G of rank r . We define those notions in §2.3 below.

For the following theorems we assume G to be split, choose a split maximal torus and a set of positive roots and let α be a simple root. We are interested in Fourier coefficients with respect to the unipotent radical U_α of the maximal parabolic subgroup P_α where $\text{Lie } U_\alpha$ is spanned by the Chevalley generators of positive roots with non-zero α -component. Letting (S_α, φ) be a Whittaker pair where $S_\alpha \in \mathfrak{h}$ is defined by $\alpha(S_\alpha) = 2$ and $\alpha_i(S_\alpha) = 0$ for all other simple roots, we get that $N_{S_\alpha, \varphi} = U_\alpha$, the unipotent of the maximal parabolic P_α , independent of φ . This means that, for a Whittaker pair (S_α, φ) , the Fourier coefficient $\mathcal{F}_{S_\alpha, \varphi}$ is the usual Fourier coefficient with respect to the unipotent subgroup U_α and the

character χ_φ . Let also β be the only simple root that is not orthogonal to α , and I denote the set of indices for all the remaining simple roots α_i .

Theorem C. *Let η_{\min} be a minimal automorphic function on a simply-laced split group G and (S_α, φ) a Whittaker pair with S_α as above. Depending on the orbit of φ , we have the following statements for the corresponding Fourier coefficient.*

(i) *If φ is minimal, then*

$$\mathcal{F}_{S_\alpha, \varphi}[\eta_{\min}](g) = \mathcal{W}_{\varphi'}[\eta_{\min}](\gamma_0 g) \quad (1.15)$$

where γ_0 is an element in $G(\mathbb{K})$ that conjugates φ to an element φ' of weight $-\alpha$.

(ii) *If φ is not minimal and not zero then $\mathcal{F}_{S_\alpha, \varphi}[\eta_{\min}] = 0$.*

Together with the result from [MW95] for computing the constant term (that is, $\varphi = 0$) in maximal parabolics, this exhausts all possibilities for φ . We also obtained an expression for the automorphic function itself. For any root δ denote by \mathfrak{g}_δ^* the corresponding subspace of \mathfrak{g}^* and by $\mathfrak{g}_\delta^\times$ the set of non-zero elements of this subspace. Note that \mathfrak{g}_δ^* is a one dimensional linear space over \mathbb{K} .

Theorem D. *Let η_{\min} be a minimal automorphic function on a simply-laced split group G . If the Dynkin diagram of G has no components of type E_8 then*

$$\eta_{\min}(g) = \mathcal{W}_0[\eta_{\min}](g) + \sum_{i=1}^{\text{rk}(G)} \sum_{\gamma \in \Lambda_i(\mathbb{K})} \sum_{\varphi \in \mathfrak{g}_{-\alpha_i}^\times} \mathcal{W}_\varphi[\eta_{\min}](\gamma g) \quad (1.16)$$

where, for each i , Λ_i is a subquotient of a Levi subgroup of G that is determined in the proof. If the Dynkin diagram of G has k components of type E_8 then we get k additional terms, each accounting for the non-abelian part of the maximal parabolic of the α_8 of the corresponding component. We have

$$\eta_{\min}(g) = \mathcal{W}_0[\eta_{\min}](g) + \sum_{i=1}^{\text{rk}(G)} \sum_{\gamma \in \Lambda_i(\mathbb{K})} \sum_{\varphi \in \mathfrak{g}_{-\alpha_i}^\times} \mathcal{W}_\varphi[\eta_{\min}](\gamma g) + \sum_{j=1}^k \sum_{\varphi \in \mathfrak{g}_{-\alpha_8^j}^\times} \sum_{w \in W_j(\mathbb{K})} \mathcal{W}_\varphi[\eta_{\min}](s_j w g), \quad (1.17)$$

where α_8^j is the 8-th root of E_8^j , the j -th E_8 -component of G , in the Bourbaki labeling; $s_j \in G(\mathbb{K})$ is an element that normalizes the Cartan and conjugates the highest root of E_8^j to α_8^j , and W_j is a subquotient of a certain unipotent subgroup of E_8^j that is determined in the proof.

Theorem E. *Let η_{ntm} be a next-to-minimal automorphic function on a simply-laced split group G and (S_α, φ) a Whittaker pair with S_α as above. Depending on the orbit of φ , we have the following statements for the corresponding Fourier coefficient.*

(i) *If φ is minimal, then*

$$\mathcal{F}_{S_\alpha, \varphi}[\eta_{\text{ntm}}](g) = \mathcal{W}_{\varphi'}[\eta_{\text{ntm}}](\gamma_0 g) + \sum_{i \in I} \sum_{\gamma \in \Gamma_i(\mathbb{K})} \sum_{\psi \in \mathfrak{g}_{-\alpha_i}^\times} \mathcal{W}_{\varphi' + \psi}[\eta_{\text{ntm}}](\gamma \gamma_0 g) \quad (1.18)$$

where γ_0 is an element in $G(\mathbb{K})$ that conjugates φ to $\varphi' \in \mathfrak{g}_{-\alpha}^*$ and Γ_i are certain subsets of Levi subgroups of G that are determined in the proof. Recall that I is the set of indices for the simple roots that are orthogonal to α .

(ii) If φ is next-to-minimal, then

$$\mathcal{F}_{S_\alpha, \varphi}[\eta_{\text{ntm}}](g) = \int_{V(\mathbb{A})} \mathcal{W}_{\varphi'}[\eta_{\text{ntm}}](v\gamma_0 g) dv \quad (1.19)$$

where γ_0 is an element in $G(\mathbb{K})$ that conjugates φ to $\varphi' \in \bigoplus_{i=1}^r \mathfrak{g}_{-\alpha_i}^*$ and $V = \text{Exp}(\mathfrak{v})$ with

$$\mathfrak{v} = \mathfrak{g}_{>1}^{S'} \cap \bar{\mathfrak{b}} \quad (1.20)$$

where $S' = \gamma_0 S_\alpha \gamma_0^{-1}$ and $\bar{\mathfrak{b}}$ is the Lie algebra of the negative Borel spanned by \mathfrak{h} and the Chevalley generators for negative roots.

(iii) If φ is not in the closure of any complex next-to-minimal orbit, then $\mathcal{F}_{S_\alpha, \varphi}[\eta_{\text{ntm}}] = 0$.

Colloquially, we will write the condition in (iii) as φ being in an orbit larger than next-to-minimal.

Remark 1.2.4. It is interesting to ask which Fourier coefficients are Eulerian [Gin06, Gin14]. The expectation based on the reduction formula of [FKP14] and explicit examples checked there is that Whittaker coefficients $\mathcal{W}_\varphi[\eta]$ of an Eisenstein series η on a group G are Eulerian if the orbit of φ is maximal in $\text{WO}(\eta)$. In general, the reduction formula expresses $\mathcal{W}_\varphi[\eta]$ through a sum of *generic* Whittaker coefficients on a semi-simple group determined by φ . If φ belongs to a maximal orbit in $\text{WO}(\eta)$, this sum collapses to a single term in all known examples and since generic Whittaker coefficients on the subgroup are Eulerian this implies the same for $\mathcal{W}_\varphi[\eta]$. By Theorem B this implies that any parabolic Fourier coefficient associated to φ is Eulerian. By *parabolic* Fourier coefficient associated to φ we mean $\mathcal{F}_{S, \varphi}$ such that all the eigenvalues of S are even integers.

With this logic one obtains also from (1.15) that the Fourier coefficient $\mathcal{F}_{S_\alpha, \varphi}[\eta_{\text{min}}]$ of an Eisenstein series in the minimal representation calculated in the unipotent of a maximal parabolic determined by α should be Eulerian for simply-laced split groups. By contrast, if φ does not belong to a maximal orbit of η the Whittaker coefficients and Fourier coefficients are not expected to be Eulerian and this is also evident from formula (1.18) showing a Fourier of a next-to-minimal automorphic function η for a minimal character φ .

Theorem F. *Let η_{ntm} be a next-to-minimal automorphic function on a simply-laced split group G . Then*

- (i) η_{ntm} is linearly determined by its Whittaker coefficients.
- (ii) If G is of type A_n for $n > 2$ or G is of type D_n and the set $\text{WO}(\eta_{\text{ntm}})$ lies in the Zariski closure of a single next-to-minimal complex orbit \mathcal{O} then

$$\eta_{\text{ntm}}(g) = \mathcal{F}_{S_\alpha, 0}[\eta_{\text{ntm}}](g) + \sum_{\tilde{\gamma} \in \Gamma_{\varphi_0}} \left(\mathcal{W}_{\varphi_0}[\eta_{\text{ntm}}](\tilde{\gamma}g) + \sum_{i \in I} \sum_{\gamma \in \Gamma_i} \sum_{\psi \in \mathfrak{g}_{-\alpha_i}^\times} \mathcal{W}_{\varphi_0 + \psi}[\eta_{\text{ntm}}](\gamma\tilde{\gamma}g) \right), \quad (1.21)$$

where α is α_n in type A_n and either α_1 or α_n in type D_n , depending on \mathcal{O} ; $\varphi_0 \in \mathfrak{g}_{-\alpha}^\times$ is a fixed non-zero element, Γ_{φ_0} is the quotient of the Levi subgroup of G given by S_α by the stabilizer of φ_0 , and the rest of the notation is as in Theorem E.

(iii) If G is of type E_n for $n \in \{6, 7\}$ we have

$$\begin{aligned} \eta_{\text{ntm}}(g) = \mathcal{F}_{S_\alpha, 0}[\eta_{\text{ntm}}](g) &+ \sum_{\tilde{\gamma} \in \Gamma_{\varphi_0}} \left(\mathcal{W}_{\varphi_0}[\eta_{\text{ntm}}](\tilde{\gamma}g) + \sum_{i \in I} \sum_{\gamma \in \Gamma_i} \sum_{\psi \in \mathfrak{g}_{-\alpha_i}^\times} \mathcal{W}_{\varphi_0 + \psi}[\eta_{\text{ntm}}](\gamma\tilde{\gamma}g) \right) \\ &+ \sum_{\tilde{\gamma} \in \Gamma_{\varphi_0}} \sum_{\gamma \in \Lambda_{V(\mathbb{A})}} \int \mathcal{W}_{\varphi_0 + \psi_0}[\eta_{\text{ntm}}](v\gamma\tilde{\gamma}g) dv, \end{aligned} \quad (1.22)$$

where $\alpha = \alpha_n$, ψ_0 is a fixed element of $\mathfrak{g}_{-\alpha_{\max}}^\times$, Λ is the quotient of the Levi subgroup given by S_α by the stabilizer of $\varphi_0 + \psi_0$ and the rest of the notation is as above.

(iv) If G is of type E_8 we have

$$\begin{aligned} \eta_{\text{ntm}}(g) = \mathcal{F}_{S_\alpha, 0}[\eta_{\text{ntm}}](g) &+ \sum_{\tilde{\gamma} \in \Gamma_{\varphi_0}} \left(\mathcal{W}_{\varphi_0}[\eta_{\text{ntm}}](\tilde{\gamma}g) + \sum_{i \in I} \sum_{\gamma \in \Gamma_i} \sum_{\psi \in \mathfrak{g}_{-\alpha_i}^\times} \mathcal{W}_{\varphi_0 + \psi}[\eta_{\text{ntm}}](\gamma\tilde{\gamma}g) \right) \\ &+ \sum_{\tilde{\gamma} \in \Lambda_{V(\mathbb{A})}} \int \mathcal{W}_{\varphi_0 + \psi_0}[\eta_{\text{ntm}}](v\tilde{\gamma}g) dv + \sum_{w \in W(\mathbb{K})} \left(\sum_{c \in \mathbb{K}^\times} \mathcal{W}_{c\varphi_0}[\eta_{\text{ntm}}](wsg) + \right. \\ &\left. \sum_{c \in \mathbb{K}^\times} \sum_{i \in I} \sum_{\gamma \in \Gamma_i} \sum_{\psi \in \mathfrak{g}_{-\alpha_i}^\times} \mathcal{W}_{c\varphi_0 + \psi}[\eta_{\text{ntm}}](\gamma wsg) + \sum_{\tilde{\gamma} \in \mathcal{M}_{V(\mathbb{A})}} \int \mathcal{W}_{\varphi_0 + \psi'_0}[\eta_{\text{ntm}}](v\tilde{\gamma}wsg) \right), \end{aligned} \quad (1.23)$$

where $\alpha = \alpha_8$, ψ'_0 is a fixed element of $\mathfrak{g}_{\alpha_8 + \alpha_7 - \alpha_{\max}}^\times$, \mathcal{M} the quotient of the Levi subgroup given by $\alpha_1, \dots, \alpha_6$ by the stabilizer of $\varphi_0 + \psi'_0$ and the rest of the notation is as above.

The assumption in type D_n is justified by the conjecture that all the maximal orbits in $\text{WO}(\eta)$ lie in the same complex orbit (see [Gin06]). Using Lemma 2.2.4 below on the connection of Fourier coefficients of local components we obtain

Corollary G. *Let π be an irreducible representation of $G(\mathbb{A})$ and let $\pi = \bigotimes \pi_\nu$ be the decomposition of π to local components. Suppose that there exists ν such that π_ν is minimal or next-to-minimal. Then π cannot be realized in cuspidal automorphic forms on $G(\mathbb{A})$.*

Remark 1.2.5. Theorems C, D and E generalize the results of [MS12, AGK⁺18] from SL_n to arbitrary simply-laced split Lie groups G . Together with theorem F they provide explicit expressions for the complete Fourier expansions of next-to-minimal automorphic forms on all split simply-laced groups and we shall compare these to other results available in the literature in §5.5.

1.3. Motivation from string theory. The results of this paper have applications in string theory. In short, string theory predicts certain quantum corrections to Einstein's general theory of relativity. These quantum corrections come in the form of an expansion in curvature tensors and their derivatives. The first non-trivial correction is of fourth order in the Riemann tensor, denoted schematically \mathcal{R}^4 , and has a coefficient which is a function

TABLE 1. Table of Cremmer–Julia symmetry groups $E_n(\mathbb{R})$, $n = d+1$, with compact subgroup $K_n(\mathbb{R})$ and U-duality groups $E_n(\mathbb{Z})$ for compactifications of IIB string theory on a d -dimensional torus T^d to $D = 10 - d$ dimensions.

| d | $E_{d+1}(\mathbb{R})$ | $K_{d+1}(\mathbb{R})$ | $E_{d+1}(\mathbb{Z})$ |
|-----|--|---|--|
| 0 | $\mathrm{SL}_2(\mathbb{R})$ | $\mathrm{SO}_2(\mathbb{R})$ | $\mathrm{SL}_2(\mathbb{Z})$ |
| 1 | $\mathrm{GL}_2(\mathbb{R})$ | $\mathrm{SO}_2(\mathbb{R})$ | $\mathrm{SL}_2(\mathbb{Z})$ |
| 2 | $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_3(\mathbb{R})$ | $\mathrm{SO}_2(\mathbb{R}) \times \mathrm{SO}_2(\mathbb{R})$ | $\mathrm{SL}_3(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ |
| 3 | $\mathrm{SL}_5(\mathbb{R})$ | $\mathrm{SO}_5(\mathbb{R})$ | $\mathrm{SL}_5(\mathbb{Z})$ |
| 4 | $\mathrm{SO}_{5,5}(\mathbb{R})$ | $(\mathrm{SO}_5(\mathbb{R}) \times \mathrm{SO}_5(\mathbb{R}))/\mathbb{Z}_2$ | $\mathrm{SO}_{5,5}(\mathbb{Z})$ |
| 5 | $E_6(\mathbb{R})$ | $\mathrm{USp}_8(\mathbb{R})/\mathbb{Z}_2$ | $E_6(\mathbb{Z})$ |
| 6 | $E_7(\mathbb{R})$ | $\mathrm{SU}_8(\mathbb{R})/\mathbb{Z}_2$ | $E_7(\mathbb{Z})$ |
| 7 | $E_8(\mathbb{R})$ | $\mathrm{Spin}_{16}(\mathbb{R})/\mathbb{Z}_2$ | $E_8(\mathbb{Z})$ |

$\eta_n : E_n/K_n \rightarrow C$, where G_n/K_n is a particular symmetric space, the classical moduli space of the theory. The parameter $n = d + 1$ encodes the number of spacetime dimensions d that have been compactified on a torus T^d . The groups E_n are all split real forms of rank n complex Lie groups (see table 1).

In the full quantum theory the classical symmetry $E_n(\mathbb{R})$ is broken to an arithmetic subgroup $E_n(\mathbb{Z})$, called the *U-duality group*, which is the Chevalley group of integer points of E_n . Thus, the coefficient functions η_n are really functions on the double coset $E_n(\mathbb{Z}) \backslash E_n(\mathbb{R}) / K_n$ and in certain cases they can be uniquely determined. For the two leading order quantum corrections, corresponding to \mathcal{R}^4 and $\partial^4 \mathcal{R}^4$, the coefficient functions η_n are respectively attached to the minimal and next-to-minimal automorphic representations of E_n [Pio10, GMV15]. Fourier expanding η_n with respect to various unipotent subgroups $U \subset E_n$ reveals interesting information about perturbative and non-perturbative quantum effects. Of particular interest are the cases when U is the unipotent radical of a maximal parabolic $P_\alpha \subset G$ corresponding to a simple root α at an “extreme” node (or end node) in the Dynkin diagram. Consider the sequence of groups E_n displayed in table 1, and the associated Dynkin diagram in “Bourbaki labelling”. The extreme simple roots are then α_1, α_2 and α_n (this is slightly modified for the low rank cases where the Dynkin diagram becomes disconnected). Fourier expanding the automorphic form η with respect to the corresponding maximal parabolics then have the following interpretations (see figure 1 for the associated labelled Dynkin diagrams):

- $P = P_{\alpha_1}$: **String perturbation limit.** In this case the constant term of the Fourier expansion corresponds to perturbative terms (tree level, one-loop etc.) with respect to an expansion around small string coupling, $g_s \rightarrow 0$. The non-constant Fourier coefficients encode non-perturbative effects of the order e^{-1/g_s} and e^{-1/g_s^2} arising from so-called D-instantons and NS5-instantons.
- $P = P_{\alpha_2}$: **M-theory limit.** This is an expansion in the limit of large volume of the M-theory torus T^{d+1} . The non-perturbative effects arise from M2- and M5-brane instantons.
- $P = P_{\alpha_n}$: **Decompactification limit.** This is an expansion in the limit of large volume of a single circle S^1 in the torus T^d (or T^{d+1} in the M-theory picture). The

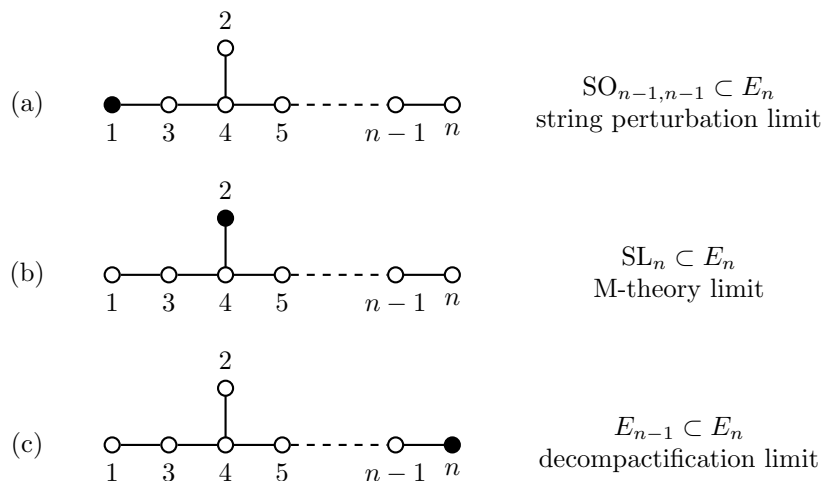


FIGURE 1. The various string theory limits associated with different maximal parabolic subgroups P_α . Roots are labeled in the Bourbaki ordering.

non-perturbative effects encoded in the non-constant Fourier coefficients correspond to so called BPS-instantons and Kaluza–Klein instantons.

For the reasons presented above, it is of interest in string theory to have general techniques for explicitly calculating Fourier coefficients of automorphic forms with respect to arbitrary unipotent subgroups.

In string theory the abelian and non-abelian Fourier coefficients of the type defined in (1.6) typically reveal different types of non-perturbative effects (see for instance [PP09, BKN⁺10, Per12]). The archimedean and non-archimedean parts of the adelic integrals have different interpretations in terms of combinatorial properties of instantons and the instanton action, respectively. For example, in the simplest case of an Eisenstein series on SL_2 the non-archimedean part is a divisor sum $\sigma_k(n) = \sum_{d|n} d^k$ and corresponds to properties of D-instantons [GG97, GG98, KV98, MNS00] (see also [FGKP18] for a detailed discussion in the present context). Theorem F provides explicit expressions for the Fourier coefficients of the automorphic coupling of the next-to-minimal $\partial^4 \mathcal{R}^4$ higher derivative correction in the decompactification limit.

1.4. Structure of the paper. In §2 we give the definitions of the notions mentioned above, as well as of *Whittaker triples* and *quasi-Fourier coefficients*. These are technical notions defined in [GGS] and widely used in the current paper as well.

In §3 we relate Fourier and quasi-Fourier coefficients corresponding to different Whittaker pairs and triples. To do that we further develop the deformation technique of [GGS17, GGS], making it both more general, more explicit, and better adapted to the global case. The deformation technique is in turn based on the root-exchange technique of [GRS97, GRS11].

In §4 we prove Theorems A and B. We first apply §3 to prove Theorem A. Then, we deduce Theorem B from Theorem A using similar methods. In §4.3 we describe the PL property of

minimal and next-to-minimal orbits, for the benefit of the reader. The statements in §4.3 that concern exceptional groups are due to Joseph Hundley, and are given without proofs since they are not used in the following. In §4.4 we prove some geometric lemmas that are used in §5. We keep these lemmas in §4 since they hold in full generality.

In §5 we deduce Theorems D-E from Theorem B and §4.4. We do not use Theorem A, though this theorem gives an existence proof for formulas as in Theorems D-E, as well as an algorithm to obtain similar formulas. However, in §5 we find several shortcuts that lead to more compact formulas. Denote by L_α the centralizer of S_α . We first deduce from §4.4 that any minimal $\varphi \in (\mathfrak{g}^*)_{-2}^{S_\alpha}$ can be conjugated into $\mathfrak{g}_{-\alpha}^\times$ using L_α (Corollary 5.1.3). This, together with Theorem B, implies Theorem C(i). Part (ii) follows from the definition of minimality and Corollary 4.0.2, that says that any Fourier coefficient is linearly determined by a neutral Fourier coefficient corresponding to the same orbit.

To prove Theorem D, assume first that G has a maximal parabolic subgroup P_α with abelian unipotent radical U_α . In this case we decompose the form η_{\min} into Fourier series with respect to U_α . Each Fourier coefficient is of the form $\mathcal{F}_{S_\alpha, \varphi}$. For $\varphi = 0$, we show that the restriction of this coefficient to L_α is minimal and use the theorem for L_α (by induction on rank). For non-zero and non-minimal φ , $\mathcal{F}_{S_\alpha, \varphi}$ vanishes by C(ii). For minimal φ the expressions for $\mathcal{F}_{S_\alpha, \varphi}$ are given by Theorem C(i). We group them together using Corollary 5.1.3. If G does not have a maximal parabolic subgroup with an abelian unipotent radical then G is a product of components of type E_8 and thus has a maximal parabolic subgroup P_α for which the unipotent radical U_α is the 57-dimensional Heisenberg group. We then decompose η_{\min} into Fourier series with respect to the center of U_α . The expression for the constant terms is obtained in the same way as above. The other terms, that are also called *non-abelian terms*, are neutral Fourier coefficients $\mathcal{F}_{1/2S_\alpha, \varphi}[\eta_{\min}]$ and the expression for them follows from Theorem B.

Theorem E(ii) and E(iii) follow from Theorem B and Corollary 4.0.2 respectively. To prove E(i) we restrict $\mathcal{F}_{S_\alpha, \varphi}[\eta_{\text{ntm}}]$ to L_α , show that it is a minimal automorphic function and apply Theorem D. In §5.4 we obtain a full expression for η_{ntm} using the same strategy as in the proof of Theorem D. However, we need two additional components. One is Proposition 5.4.4 that describes the action of L_α on next-to-minimal elements of $(\mathfrak{g}^*)_{-2}^{S_\alpha}$. The other is an expression for non-abelian terms $\mathcal{F}_{1/2S_\alpha, \varphi}[\eta_{\text{ntm}}]$ for next-to-minimal η_{ntm} . Our strategy for obtaining this expression is the same as the strategy for the proof of Theorem E(i). Finally, in §5.5 we compare our expressions for η_{ntm} to the results of [BP17, GKP16, KP04].

In §6 we provide examples to Theorems D-E for groups of type D_5 , E_6 , E_7 and E_8 computing the expansions of automorphic function and Fourier coefficients with respect to different parabolic subgroups of interest in string theory.

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2. DEFINITIONS

Let \mathbb{K} be a number field and let $\mathbb{A} = \mathbb{A}_{\mathbb{K}}$ be its ring of adèles. In this section we let χ be a non-trivial unitary character of \mathbb{A} , which is trivial on \mathbb{K} . Then χ defines an isomorphism between \mathbb{A} and $\hat{\mathbb{A}}$ via the map $a \mapsto \chi_a$, where $\chi_a(b) = \chi(ab)$ for all $b \in \mathbb{A}$. This isomorphism restricts to an isomorphism

$$\widehat{\mathbb{A}/\mathbb{K}} \cong \{r \in \hat{\mathbb{A}} : |r|_{\mathbb{K}} \equiv 1\} = \{\chi_a : a \in \mathbb{K}\} \cong \mathbb{K}. \quad (2.1)$$

Let \mathfrak{g} denote the Lie algebra of G . By abuse of notation we will also denote by \mathfrak{g} the \mathbb{K} -points of this Lie algebra.

Definition 2.0.1. A *Whittaker pair* is an ordered pair $(S, \varphi) \in \mathfrak{g} \times \mathfrak{g}^*$ such that S is a rational semi-simple element (that is, with eigenvalues of the adjoint action $\text{ad}(S)$ in \mathbb{Q}), and $\text{ad}^*(S)(\varphi) = -2\varphi$. Note that $\varphi \in (\mathfrak{g}^*)_{-2}^S$ is necessary nilpotent.

Given a Whittaker pair (S, φ) on \mathfrak{g} , we set $\mathfrak{u} = \mathfrak{g}_{\geq 1}^S = \mathfrak{g}_{> 1}^S \oplus \mathfrak{g}_1^S$ and $\mathfrak{n}_{S, \varphi} = \{X \in \mathfrak{u} : \omega_{\varphi}(X, Y) = 0 \text{ for all } Y \in \mathfrak{u}\}$ to be the radical of the form $\omega_{\varphi}|_{\mathfrak{u}}$, where $\omega_{\varphi}(X, Y) = \varphi([X, Y])$. According to Lemma 3.2.5 below one can show that

$$\mathfrak{n}_{S, \varphi} = \mathfrak{g}_{> 1}^S \oplus \mathfrak{g}_1^S \cap \mathfrak{g}_{\varphi} \quad (2.2)$$

where \mathfrak{g}_1^S is the 1-eigenspace of S in \mathfrak{g} , $\mathfrak{g}_{> 1}^S$ is the direct sum of eigenspaces with eigenvalues > 1 , and \mathfrak{g}_{φ} the centralizer of φ in \mathfrak{g} under the coadjoint action. Let $\mathfrak{l} \subseteq \mathfrak{u}$ be any isotropic subspace with respect to $\omega_{\varphi}|_{\mathfrak{u}}$ that includes $\mathfrak{n}_{S, \varphi}$. Note that $\mathfrak{n}_{S, \varphi} \subseteq \mathfrak{l} \subseteq \mathfrak{u}$ and $\mathfrak{n}_{S, \varphi}, \mathfrak{l}$ are ideals in \mathfrak{u} . Let $U = \exp \mathfrak{u}$, $N_{S, \varphi} = \exp \mathfrak{n}_{S, \varphi}$ and $L = \exp \mathfrak{l}$. Observe that we can extend φ to a linear functional on $\mathfrak{g}(\mathbb{A})$ by linearity and, furthermore, the character $\chi_{\varphi}^L(\exp X) = \chi(\varphi(X))$ defined on $L(\mathbb{A})$ is automorphic, that is, it is trivial on $L(\mathbb{K})$. We will denote its restriction to $N_{S, \varphi}(\mathbb{A})$ simply by χ_{φ} .

We will often identify φ with its dual nilpotent element $f = f_{\varphi} \in \mathfrak{g}$ with respect to the Killing form $\langle \cdot, \cdot \rangle$ or with its corresponding character $\chi_{\varphi}(n) = \chi(\varphi(\log n)) = \chi(\langle f, \log n \rangle) = \chi_f(n)$, sometimes calling φ itself a character. For a subgroup $U \subseteq G$ we denote by $[U]$ the quotient $U(\mathbb{K}) \backslash U(\mathbb{A})$.

Definition 2.0.2. Let (S, φ) be a Whittaker pair for \mathfrak{g} and let $L, N_{S, \varphi}, \chi_{\varphi}$ and χ_{φ}^L be as above. For an automorphic function η , we define the *Fourier coefficient* of η with respect to the pair (S, φ) to be

$$\mathcal{F}_{S, \varphi}[\eta](g) := \int_{[N_{S, \varphi}]} \eta(n g) \chi_{\varphi}(n)^{-1} dn. \quad (2.3)$$

We also define its *L-Fourier coefficient* to be the function

$$\mathcal{F}_{S, \varphi}^L[\eta](g) := \int_{[L]} \eta(l g) \chi_{\varphi}^L(l)^{-1} dl. \quad (2.4)$$

Observe that $\mathcal{F}_{S,\varphi}[\eta]$ and $\mathcal{F}_{S,\varphi}^L[\eta]$ are matrix coefficients corresponding to the vector $\eta \in \pi$ and the functional on the space of automorphic functions defined by the integrals above.

Definition 2.0.3. A Whittaker pair (H, φ) is called a neutral Whittaker pair if either $(H, \varphi) = (0, 0)$, or H can be completed to an \mathfrak{sl}_2 -triple (e, H, f) such that φ is the Killing form pairing with f . Equivalently, the coadjoint action on φ defines an epimorphism $\mathfrak{g}_0^H \rightarrow (\mathfrak{g}^*)_{-2}^H$, and also H can be completed to an \mathfrak{sl}_2 -triple. For more details on \mathfrak{sl}_2 -triples over arbitrary fields of characteristic zero see [Bou75, §11].

Remark 2.0.4. If $(f_\varphi, h, e_\varphi)$ is an \mathfrak{sl}_2 -triple associated with the principal nilpotent orbit then $N_{h,\varphi}$ is a maximal unipotent subgroup, and the Fourier coefficient $\mathcal{F}_{h,\varphi}[\eta]$ is a Whittaker coefficient $\mathcal{W}_{h,\varphi}[\eta]$. Recall that a principal nilpotent element ψ is a nilpotent element whose centralizer in \mathfrak{g} is minimal. If G is quasi-split then this is equivalent to $\dim \mathfrak{g}_\psi = \text{rk } \mathfrak{g}$.

Definition 2.0.5. We call S *principal* if it can be completed to a neutral pair (S, ψ) such that ψ is a principal nilpotent element and we call a Whittaker pair (S, φ) *principal* if S is principal. Note that this implies that S defines a minimal parabolic subgroup, which means that a principal pair is a special case of a standard pair.

A nilpotent element φ is called a *PL-element*, where PL stands for *principal in a Levi*, if it can be completed to a Whittaker pair (S, φ) where S is principal. Note that this is equivalent to φ being a principal nilpotent element of the Lie algebra of some rational Levi subgroup $L \subseteq G$. It is also equivalent to the statement that φ defines a character of the nilradical of a (rational) minimal parabolic subgroup of G , see e.g. [GGS17, §3.3]. The pair (S, φ) is then said to be a *PL-pair* and the corresponding Fourier coefficient $\mathcal{F}_{S,\varphi}$ a *PL-coefficient*.

A nilpotent orbit \mathcal{O} which contains a PL nilpotent element (or, equivalently, consists of PL nilpotent elements) is called a *PL-orbit*.

Remark 2.0.6. (i) In [GGS17, §6] the integrals (2.3) and (2.4) above are called Whittaker–Fourier coefficients, but in this paper we call them Fourier coefficients for short. The (standard) Whittaker coefficients are called in [GGS17, §6] principal degenerate Whittaker–Fourier coefficients.

(ii) Note that for $G = \text{GL}_n$ all orbits \mathcal{O} are PL-orbits. In general this is, however, not the case, see §4.3 below.

Definition 2.0.7. We say that (S, φ, φ') is a *Whittaker triple* if (S, φ) is a Whittaker pair and $\varphi' \in (\mathfrak{g}^*)_{>-2}^S$.

For a Whittaker triple (S, φ, φ') , let U, L , and $N_{S,\varphi}$ be as in Definition 2.0.2. Note that $\varphi + \varphi'$ defines a character of \mathfrak{l} . Extend it by linearity to a character of $\mathfrak{l}(\mathbb{A})$ and define an automorphic character $\chi_{\varphi+\varphi'}$ of $L(\mathbb{A})$ by $\chi_{\varphi+\varphi'}^L(\exp X) := \chi(\varphi(X) + \varphi'(X))$.

Definition 2.0.8. For an automorphic function f , we define its (S, φ, φ') -*quasi Fourier coefficient* to be the function

$$\mathcal{F}_{S,\varphi,\varphi'}[\eta](g) := \int_{[N_{S,\varphi}]} \chi_{\varphi+\varphi'}(n)^{-1} \eta(n g) dn. \quad (2.5)$$

We also define its $(S, \varphi, \varphi', L)$ -quasi Fourier coefficient to be the function

$$\mathcal{F}_{S, \varphi, \varphi'}^L[\eta](g) := \int_{[L]} \chi_{\varphi + \varphi'}^L(l)^{-1} \eta(lg) dl. \quad (2.6)$$

Definition 2.0.9. We say that $\mathcal{F}_{S, \varphi, \varphi'}$ linearly determines $\mathcal{F}_{H, \psi, \psi'}$ if there exists a linear operator \mathcal{L} on $C^\infty(G(\mathbb{A}))$ such that $\mathcal{F}_{H, \psi, \psi'} = \mathcal{L} \circ \mathcal{F}_{S, \varphi, \varphi'}$.

We say that $\mathcal{F}_{H, \psi, \psi'}$ is linearly determined by a set $\{\mathcal{F}_{S_i, \varphi_i, \varphi'_i} \mid i \in I\}$ if I is finite or countable and there exists a set of linear operators \mathcal{L}_i such that $\mathcal{F}_{H, \psi, \psi'} = \sum_i \mathcal{L}_i \circ \mathcal{F}_{S_i, \varphi_i, \varphi'_i}$.

2.1. Levi-distinguished Fourier coefficients.

Definition 2.1.1. We call a \mathbb{K} -subgroup of G a split torus of rank m if it is isomorphic as a \mathbb{K} -subgroup to GL_1^m . We call a Lie subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$ a \mathbb{K} -Levi subalgebra if it is the centralizer of a split torus.

We say that a nilpotent $f \in \mathfrak{g}$ is \mathbb{K} -distinguished, if it does not belong to a proper \mathbb{K} -Levi subalgebra $\mathfrak{l} \subsetneq \mathfrak{g}$. In this case we will also say that $\varphi \in \mathfrak{g}^*$ given by the Killing form pairing with f is \mathbb{K} -distinguished. We will also say that the orbit of φ is \mathbb{K} -distinguished.

Remark 2.1.2. We note that the Lie algebra of any split torus is spanned by rational semisimple elements. Consequently, a subalgebra of $\mathfrak{l} \subseteq \mathfrak{g}$ is a \mathbb{K} -Levi subalgebra if and only if it is the centralizer of a rational semisimple element of \mathfrak{g} . Another equivalent condition is that \mathfrak{l} is the Lie algebra of a Levi subgroup of a parabolic subgroup of G defined over \mathbb{K} .

It is easy to see that all principal nilpotent elements are distinguished.

Example 2.1.3. The nilpotent orbits of $\mathrm{Sp}_{2n}(\mathbb{C})$ are given by partitions of $2n$ such that odd parts have even multiplicity. Each such orbit, except the zero one, decomposes to infinitely many $\mathrm{Sp}_{2n}(\mathbb{Q})$ -orbits - one for each collection of equivalence classes of quadratic forms Q_1, \dots, Q_k of dimensions m_1, \dots, m_k where k is the number of even parts in the partition and m_1, \dots, m_k are the multiplicities of these parts. A complex orbit intersects a proper Levi subalgebra if and only if all parts have multiplicity one (and thus there are no odd parts). To see the “only if” part note that if the partition includes a part k with multiplicity two then the orbit intersects the Levi $\mathrm{GL}_k \times \mathrm{Sp}_{2(n-k)}$. If k is odd then this Levi is defined over \mathbb{Q} and thus all \mathbb{Q} -distinguished orbits correspond to totally even partitions. If k is even then this Levi is defined over \mathbb{Q} if and only if the quadratic form on the multiplicity space of k is (positive or negative) definite. Thus, we obtain that a necessary condition for an orbit \mathcal{O} to be \mathbb{Q} -distinguished is that its partition $\lambda(\mathcal{O})$ is totally even, a sufficient condition is that $\lambda(\mathcal{O})$ is multiplicity free, and for totally even partitions with multiplicities there are infinitely many \mathbb{Q} -distinguished orbits and at least one not \mathbb{Q} -distinguished. For example, for the partition $(4, 2)$ all orbits in $\mathfrak{sp}_6(\mathbb{Q})$ are \mathbb{Q} -distinguished, for the partition 2^3 some orbits are \mathbb{Q} -distinguished and some are not, and all other partitions do not correspond to \mathbb{Q} -distinguished orbits.

Lemma 2.1.4. *Let $f \in \mathfrak{g}$ be nilpotent. Then all \mathbb{K} -Levi subalgebras $\mathfrak{l} \subseteq \mathfrak{g}$ such that $f \in \mathfrak{l}$ and f is \mathbb{K} -distinguished in \mathfrak{l} are conjugate by the centralizer of f .*

Proof. Complete f to an \mathfrak{sl}_2 -triple $\gamma := (e, h, f)$ and denote its centralizer by G_γ . Let us show that all \mathbb{K} -Levi subalgebras \mathfrak{l} of \mathfrak{g} that contain γ and in which f is distinguished are

conjugate by G_γ . Let \mathfrak{l} be such a subalgebra, $L \subseteq G$ be the corresponding Levi subgroup, and let C denote the maximal split torus of the center of L . Then C is a split torus in G_γ . Let us show that it is a maximal split torus. Let $T \supseteq C$ be a larger split torus in G_γ . Then, the centralizer of T in \mathfrak{g} is a \mathbb{K} -Levi subalgebra that lies in \mathfrak{l} and includes γ , and thus is equal to \mathfrak{l} . Thus $T = C$.

Since \mathfrak{l} is the centralizer of T in G , T is a maximal split torus of G_γ , and all maximal split tori of reductive groups are conjugate, we get that all the choices of L are conjugate.

Since all the choices of γ are conjugate by the centralizer of f , the lemma follows. \square

Definition 2.1.5. Let $Z \in \mathfrak{g}$ be a rational-semisimple element and \mathfrak{l} denote its centralizer. Let (h, φ) be a neutral Whittaker pair for \mathfrak{l} , such that the orbit of φ in \mathfrak{l}^* is \mathbb{K} -distinguished. We call the Fourier coefficient $\mathcal{F}_{h+Z, \varphi}$ a *Levi-distinguished* Fourier coefficient if

$$\mathfrak{g}_{>1}^{h+Z} = \mathfrak{g}_{\geq 2}^{h+Z} = \mathfrak{g}_{>0}^Z \oplus \mathfrak{l}_{\geq 2}^h \text{ and } \mathfrak{g}_1^{h+Z} = \mathfrak{l}_1^h. \quad (2.7)$$

Remark 2.1.6. Let (h, φ) be a neutral Whittaker pair for \mathfrak{g} . If φ is \mathbb{K} -distinguished then $\mathcal{F}_{h, \varphi}$ is a Levi-distinguished Fourier coefficient. If a rational semi-simple Z commutes with h and with φ and φ is \mathbb{K} -distinguished in $\mathfrak{l} := \mathfrak{g}_0^Z$ then $\mathcal{F}_{h+TZ, \varphi}$ is a Levi-distinguished Fourier coefficient for any T bigger than $m/M + 1$, where m is the maximal eigenvalue of h and M is the minimal positive eigenvalue of Z . See also Lemma 4.0.8 for further discussion.

Lemma 2.1.7 ([GG17, Lemma 3.0.2]). *For any Whittaker pair (H, φ) there exists $z \in \mathfrak{g}_0^H$ such that $(H - z, \varphi)$ is a neutral Whittaker pair.*

Remark 2.1.8. In [GG17] the lemma is proven over a local field, but all we use in the proof is the Jacobson-Morozov theorem, that holds over arbitrary fields of characteristic zero.

Lemma 2.1.9. *For any Whittaker pair (H, φ) , the followings are equivalent:*

- (a) *The Fourier coefficient $\mathcal{F}_{H, \varphi}$ is a Whittaker coefficient*
- (b) *The $\mathcal{F}_{H, \varphi}$ is a Levi-distinguished Fourier coefficient, and φ is a PL nilpotent.*

Proof. First let $\mathcal{F}_{H, \varphi}$ be a Whittaker coefficient. Then by Lemma 2.1.7, H can be decomposed as $H = h + Z$ where (h, φ) is a neutral pair and Z commutes with h and with φ . Let \mathfrak{l} and L denote the centralizers of Z in \mathfrak{g} and G , and $N := N_{H, \varphi}$. Then N is a maximal parabolic unipotent of G , and L is a Levi subgroup of G . Thus, $N \cap L$ is maximal unipotent subgroup in L . The Lie algebra of $N \cap L$ is $\mathfrak{n}_{H, \varphi} \cap \mathfrak{g}_0^Z = \mathfrak{g}_{\geq 1}^h \cap \mathfrak{g}_0^Z$. Thus, $\text{Exp}(\mathfrak{g}_{\leq -1}^h \cap \mathfrak{g}_0^Z)$ is also maximal unipotent in L . Since φ is given by Killing form pairing with $f \in \mathfrak{g}_{\leq -1}^h \cap \mathfrak{g}_0^Z$, we get that φ is principal in \mathfrak{l} . Replacing Z by tZ with t large enough, we obtain that $\mathcal{F}_{H, \varphi}$ is a Levi-distinguished coefficient.

Now, assume that φ is a PL nilpotent, and let $\mathcal{F}_{h+Z, \varphi}$ be a Levi-distinguished Fourier coefficient. Let $\mathfrak{l} = \mathfrak{g}_0^Z$ be the corresponding Levi, and let $f = f_\varphi$ be the element of \mathfrak{g} that defines φ . Since f is distinguished in \mathfrak{l} , and principal in some Levi, Lemma 2.1.4 implies that f is principal in \mathfrak{l} . Thus, $\mathfrak{n}_{H, \varphi} \cap \mathfrak{l}$ is a maximal nilpotent subalgebra of \mathfrak{l} and thus $\mathfrak{n}_{H, \varphi} = \mathfrak{n}_{H, \varphi} \cap \mathfrak{l} \oplus \mathfrak{g}_{>0}^Z$ is a maximal nilpotent subalgebra of \mathfrak{g} . Thus $\mathcal{F}_{H, \varphi}$ is a Whittaker coefficient. \square

2.2. Order on nilpotent orbits and Whittaker support.

Definition 2.2.1. We define a partial order on nilpotent orbits in $\mathfrak{g}^* = \mathfrak{g}^*(\mathbb{K})$ to be the transitive closure of the following relation R : $(\mathcal{O}, \mathcal{O}') \in R$ if $\mathcal{O} \neq \mathcal{O}'$ and there exist $\varphi \in \mathcal{O}$, $\varphi' \in \mathcal{O}'$ and a rational semi-simple $Z \in \mathfrak{g}$ such that $\varphi \in (\mathfrak{g}^*)_0^Z$ and $\varphi' - \varphi \in (\mathfrak{g}^*)_{>0}^Z$.

In Corollary 4.4.5 below we prove that this is indeed a partial order, *i.e.* that R is anti-symmetric.

Note however that we will base statements such as φ being minimal or next-to-minimal on the coarser ordering of complex orbits as detailed further in §2.3.

Lemma 2.2.2. *If \mathcal{O}' is bigger than \mathcal{O} then for any place ν of \mathbb{K} , the closure of \mathcal{O}' in $\mathfrak{g}(\mathbb{K}_\nu)$ (in the local topology) includes \mathcal{O} .*

Proof. It is enough to show that for any $Z \in \mathfrak{g}$, $\varphi \in \mathfrak{g}_0^Z$ and $\psi \in \mathfrak{g}_{>0}^Z$, φ lies in the closure of $G(\mathbb{K}_\nu)(\varphi + \psi)$. Let $\varepsilon_i \in \mathbb{K}_\nu$ be a sequence converging to zero and let $g_i := \exp(-\varepsilon_i Z)$. Then g_i centralize φ , while $g_i \psi \rightarrow 0$. Thus $g_i(\varphi + \psi) \rightarrow \varphi$. \square

Definition 2.2.3. For an automorphic function η , we define $\text{WO}(\eta)$ to be the set of nilpotent orbits \mathcal{O} in \mathfrak{g}^* such that $\mathcal{F}_{h,\varphi}[\eta] \neq 0$ for some neutral Whittaker pair (h, φ) with $\varphi \in \mathcal{O}$. We define the *Whittaker support* $\text{WS}(\eta)$ to be the set of maximal elements in $\text{WO}(\eta)$.

The following well known lemma relates these notions to the local notion of wave-front set. For a survey on this notion, and its relation to degenerate Whittaker models we refer the reader to [GS18, §4].

Lemma 2.2.4. *Suppose that η is an automorphic form in the classical sense, and that it generates an irreducible representation π of $G(\mathbb{A})$. Let $\pi = \bigotimes_\nu \pi_\nu$ be the decomposition of π to local factors. Let $\mathcal{O} \in \text{WO}(\eta)$. Then, for any ν , there exists an orbit \mathcal{O}'_ν in the wave-front set of π_ν such that \mathcal{O} lies in the Zariski closure of \mathcal{O}'_ν . Moreover, if ν is non-archimedean, then \mathcal{O} lies in the closure of \mathcal{O}'_ν in the topology of $\mathfrak{g}^*(\mathbb{K}_\nu)$.*

Proof. Acting by G on the argument of η we can assume that there exists a neutral pair (h, φ) with $\varphi \in \mathcal{O}$ such that $\mathcal{F}_{h,\varphi}[\eta](1) \neq 0$. Moreover, decomposing η to a sum of pure tensors, and replacing η by one of the summands, we can assume that η is a pure tensor and $\mathcal{F}_{h,\varphi}[\eta](1) \neq 0$ still holds. Let $\eta = \bigotimes'_\mu v_\mu$ be the decomposition of η to local factors. Consider the functional ξ on π_ν given by $\xi(v) := \mathcal{F}_{h,\varphi}(v \otimes (\bigotimes'_{\mu \neq \nu} v_\mu))(1)$. Substituting the vector v_ν we see that this functional is non-zero. It is easy to see that this ξ is $(N_{h,\varphi}(\mathbb{K}_\nu), \chi_\varphi)$ -equivariant. The theorem follows now from [MW87, Proposition I.11] and [Var14] for non-archimedean ν , and from [Ros95, Theorem D] and [Mat87] for archimedean ν . \square

2.3. Minimal and next-to-minimal representations. We call a non-zero complex orbit in $\mathfrak{g}^*(\mathbb{C})$ *minimal* if its Zariski closure consists of itself and of the zero element. We call a non-zero complex orbit *next-to-minimal*, or shortly *ntm*, if it is not minimal and its Zariski closure consists of itself, of minimal orbits and of the zero element. We call a rational element or a rational orbit *minimal/next-to-minimal* if its complex orbit is minimal/next-to-minimal.

We say that an automorphic function η is *minimal* if $\text{WS}(\eta)$ consists of minimal orbits. By [GGS17, Theorem C] (or by Proposition 4.0.1 below), this implies that $\mathcal{F}_{H,\varphi}[\eta] = 0$ for any Whittaker pair (H, φ) with φ non-zero and non-minimal. We call a non-trivial representation of $G(\mathbb{A})$ in automorphic functions *minimal* if all the forms in this representation are minimal or constant.

We say that an automorphic function η is *next-to-minimal* if $\text{WS}(\eta)$ consists of next-to-minimal orbits. Again, by [GGS17, Theorem C] (or by Proposition 4.0.1 below), this implies that $\mathcal{F}_{H,\varphi}[\eta] = 0$ for any Whittaker pair (H, φ) with φ higher than next-to-minimal. We call a representation π of $G(\mathbb{A})$ in automorphic functions *next-to-minimal* if it is not trivial and not minimal, and all the forms in this representation are next-to-minimal, minimal or constant. By Lemma 2.2.4, if π consists of automorphic forms in the classical sense, is non-trivial, irreducible and has a minimal local factor then it is minimal. Similarly, if it has a next-to-minimal local factor then it is minimal or next-to-minimal.

Remark 2.3.1. Let $\mathfrak{g} = \bigoplus_{i=1}^k \mathfrak{g}_i$, with \mathfrak{g}_i simple. Note that the minimal orbits of \mathfrak{g} are of the form $\times_{i=1}^j \{0\} \times \mathcal{O} \times \times_{i=j+1}^k \{0\}$, with \mathcal{O} a minimal orbit. The next-to-minimal orbits of \mathfrak{g} are either of the same form with \mathcal{O} next-to-minimal, or of the form $\times_{i=1}^{j-1} \{0\} \times \mathcal{O} \times \times_{i=j+1}^{l-1} \{0\} \times \mathcal{O}' \times \times_{i=l+1}^k \{0\}$, where \mathcal{O} and \mathcal{O}' are minimal orbits in \mathfrak{g}_j and \mathfrak{g}_l respectively.

3. RELATING DIFFERENT FOURIER COEFFICIENTS

3.1. Relating different isotropic subspaces. We will now see how $\mathcal{F}_{S,\varphi,\varphi'}$ linearly determines $\mathcal{F}_{S,\varphi,\varphi'}^L$ and vice versa.

Lemma 3.1.1 (cf. [GGS17, Lemma 6.0.2]). *Let $\eta \in \pi$ and (S, φ, φ') be a Whittaker triple, and $N_{S,\varphi}$, U and L as in Definition 2.0.1. Let \mathfrak{t}^\perp denote the orthogonal complement to \mathfrak{t} in \mathfrak{u} under the form ω_φ and let $L^\perp := \text{Exp}(\mathfrak{t}^\perp)$. Then,*

$$\mathcal{F}_{S,\varphi,\varphi'}^L[\eta](g) = \int_{[L/N_{S,\varphi}]} \mathcal{F}_{S,\varphi,\varphi'}[\eta](ug) du \quad (3.1)$$

and

$$\mathcal{F}_{S,\varphi,\varphi'}[\eta](g) = \sum_{\gamma \in (U/L^\perp)(\mathbb{K})} \mathcal{F}_{S,\varphi,\varphi'}^L[\eta](\gamma g). \quad (3.2)$$

Proof. We assume that φ is non-zero since otherwise $L = N_{S,\varphi}$. We have that $N_{S,\varphi} \subseteq L$ with $L/N_{S,\varphi}$ abelian which means that (3.1) follows immediately from the definitions of $\mathcal{F}_{S,\varphi,\varphi'}$ and $\mathcal{F}_{S,\varphi,\varphi'}^L$. For (3.2) observe that the function $(\chi_\varphi^L)^{-1} \cdot \mathcal{F}_{S,\varphi,\varphi'}[\eta]$ on L is left-invariant under the action of $N_{S,\varphi}(\mathbb{A})L(\mathbb{K})$. In other words, we can identify it with a function on

$$N_{S,\varphi}(\mathbb{A})L(\mathbb{K}) \backslash L(\mathbb{A}) \cong (L/N_{S,\varphi})(\mathbb{K}) \backslash (L/N_{S,\varphi})(\mathbb{A}) =: [L/N_{S,\varphi}], \quad (3.3)$$

where the equality follows from the fact that $L/N_{S,\varphi}$ is abelian. Therefore, we have a Fourier series expansion

$$\mathcal{F}_{S,\varphi,\varphi'}[\eta](l) = \sum_{\psi \in [L/N_{S,\varphi}]^\wedge} c_{\psi, \chi_{\varphi+\varphi'}^L}(\eta) \psi(l) \chi_\varphi^L(l), \quad (3.4)$$

where $[L/N_{S,\varphi}]^\wedge$ denotes the Pontryagin dual group of $[L/N_{S,\varphi}]$ and

$$c_{\psi,\chi_\varphi^L}(\eta) = \int_{[L]} \psi(l)^{-1} \chi_{\varphi+\varphi'}^L(l)^{-1} \eta(l) dl. \quad (3.5)$$

In particular

$$\mathcal{F}_{S,\varphi,\varphi'}[\eta](e) = \sum_{\psi \in [L/N_{S,\varphi}]^\wedge} c_{\psi,\chi_{\varphi+\varphi'}^L}(\eta). \quad (3.6)$$

Now observe that the map $X \mapsto \omega_\varphi(X, \cdot) = \varphi \circ \text{ad}(X)$ induces an isomorphism between $\mathfrak{u}/\mathfrak{l}^\perp$ and $(\mathfrak{l}/\mathfrak{n})'$. Hence, according to equations (2.1) and (3.3), we can use the character χ to define a group isomorphism

$$\begin{aligned} (U/L^\perp)(\mathbb{K}) &\longrightarrow [L/N_{S,\varphi}]^\wedge \\ u &\longmapsto \psi_u, \end{aligned} \quad (3.7)$$

where

$$\psi_u(l) = \chi(\varphi([X, Y])), \quad u = \exp X \quad \text{and} \quad l = \exp Y.$$

Hence, for all $u \in U(\mathbb{K})$ and $l \in L$ we have

$$\begin{aligned} \psi_u(l) \chi_{\varphi+\varphi'}^L(l) &= \chi(\varphi([X, Y]) + \varphi'([X, Y])) \chi(\varphi(Y) + \varphi'(Y)) = \chi((\varphi + \varphi')(Y + [X, Y])) \\ &= \chi((\varphi + \varphi')(e^{\text{ad}(X)}(Y))) = \chi_{\varphi+\varphi'}((\text{Ad}(u)Y)) = \chi_{\varphi+\varphi'}^L(ulu^{-1}). \end{aligned}$$

Here we are taking again $u = \exp X$, $l = \exp Y$ and the middle equality follows from the vanishing of φ on $\mathfrak{g}_{>2}^S$. But now, from formula (3.5) and the fact that f is automorphic, we have

$$\begin{aligned} c_{\psi_u, \chi_{\varphi+\varphi'}^L}(\eta) &= \int_{[L]} \psi_u(l)^{-1} \chi_{\varphi+\varphi'}^L(l)^{-1} \eta(l) dl = \int_{[L]} \chi_{\varphi+\varphi'}^L(ulu^{-1})^{-1} \eta(l) dl. \\ &= \int_{[L]} \chi_{\varphi+\varphi'}^L(l)^{-1} \eta(u^{-1}lu) dl = \mathcal{F}_{S,\varphi,\varphi'}^L[\eta](u), \end{aligned}$$

for all $u \in U(\mathbb{K})$. Combining this with (3.6,3.7) we obtain

$$\mathcal{F}_{S,\varphi,\varphi'}[\eta](e) = \sum_{u \in (U/L^\perp)(\mathbb{K})} \mathcal{F}_{S,\varphi,\varphi'}^L[\eta](u). \quad (3.8)$$

Applying this to right shifts of η we obtain (3.2). \square

3.2. Relating different Whittaker pairs. Let (H, φ) be a Whittaker pair.

Lemma 3.2.1. *Let z be as in Lemma 2.1.7. Then $(H - z, \varphi)$ dominates (H, φ) .*

Proof. Denote $h := H - z$. We have to show that (1.10) holds, i.e.

$$\mathfrak{g}_\varphi \cap \mathfrak{g}_{\geq 1}^h \subseteq \mathfrak{g}_{\geq 0}^z \quad (3.9)$$

Since \mathfrak{g}_φ is spanned by lowest weight vectors, we have $\mathfrak{g}_\varphi \subseteq \mathfrak{g}_{\leq 0}^h$ and thus $\mathfrak{g}_\varphi \cap \mathfrak{g}_{\geq 1}^h = \{0\}$. \square

Corollary 3.2.2. *Any Whittaker pair is dominated by a neutral Whittaker pair.*

Another example of domination is provided by the following proposition, that immediately follows from [GGS17, Proposition 3.3.3].

Proposition 3.2.3. *If φ is a PL nilpotent then there exists $Z \in \mathfrak{g}$ such that $(H + Z, \varphi)$ is a standard Whittaker pair and (H, φ) dominates $(H + Z, \varphi)$.*

From now till the end of the section let $Z \in \mathfrak{g}_0^H$ be a rational semi-simple element such that (H, φ) dominates $(H + Z, \varphi)$.

For any rational number $t \geq 0$ define

$$H_t := H + tZ, \quad \mathfrak{u}_t := \mathfrak{g}_{\geq 1}^{H_t}, \quad \mathfrak{v}_t := \mathfrak{g}_{> 1}^{H_t}, \quad \text{and} \quad \mathfrak{w}_t := \mathfrak{g}_1^{H_t}. \quad (3.10)$$

Definition 3.2.4. We call $t \geq 0$ *regular* if $\mathfrak{u}_t = \mathfrak{u}_{t+\varepsilon}$ for any small enough $\varepsilon \in \mathbb{Q}$, or in other words $\mathfrak{w}_t \subset \mathfrak{g}_0^Z$. If t is not regular we call it *critical*. Equivalently, t is critical if $\mathfrak{g}_1^{H_t} \not\subset \mathfrak{g}_0^Z$ which we may interpret as something new has entered the 1-eigenspace of H . For convenience, we will say that $t = 0$ is critical.

We also say that $t \geq 0$ is *quasi-critical* if either $\mathfrak{g}_1^{H_t} \not\subset \mathfrak{g}_0^Z$ or $\mathfrak{g}_2^{H_t} \not\subset \mathfrak{g}_0^Z$. We may interpret this as something new has entered either the 1-eigenspace or the 2-eigenspace. The latter is related to new characters being available in the Whittaker pairs.

Note that there are only finitely many critical numbers. Recall the anti-symmetric form ω_φ on \mathfrak{g} given by $\omega_\varphi(X, Y) = \varphi([X, Y])$.

Lemma 3.2.5 ([GGS17, Lemma 3.2.6]).

- (i) *The form ω_φ is $\text{ad}(Z)$ -invariant.*
- (ii) $\text{Ker } \omega_\varphi = \mathfrak{g}_\varphi$.
- (iii) $\text{Ker}(\omega_\varphi|_{\mathfrak{w}_t}) = \text{Ker}(\omega_\varphi) \cap \mathfrak{w}_t$.
- (iv) $\text{Ker}(\omega_\varphi|_{\mathfrak{u}_t}) = \mathfrak{v}_t \oplus \text{Ker}(\omega_\varphi|_{\mathfrak{w}_t})$.
- (v) $\mathfrak{w}_s \cap \mathfrak{g}_\varphi \subseteq \mathfrak{u}_t$ for any $s < t$.

Recall that $\mathfrak{n}_{H_t, \varphi} := \text{Ker}(\omega_\varphi|_{\mathfrak{u}_t})$, denote it by \mathfrak{n}_t . and let

$$\mathfrak{l}_t := (\mathfrak{u}_t \cap \mathfrak{g}_{< 0}^Z) + \mathfrak{n}_t \quad \text{and} \quad \mathfrak{r}_t := (\mathfrak{u}_t \cap \mathfrak{g}_{> 0}^Z) + \mathfrak{n}_t. \quad (3.11)$$

Lemma 3.2.6. *For any $t \geq 0$ we have*

- (i) \mathfrak{l}_t and \mathfrak{r}_t are ideals in \mathfrak{u}_t and $[\mathfrak{l}_t, \mathfrak{r}_t] \subseteq \mathfrak{l}_t \cap \mathfrak{r}_t = \mathfrak{n}_t$.
- (ii) *The natural projections $\mathfrak{l}_t/\mathfrak{n}_t \rightarrow \mathfrak{u}_t/\mathfrak{r}_t^\perp$ and $\mathfrak{r}_t/\mathfrak{n}_t \rightarrow \mathfrak{u}_t/\mathfrak{l}_t^\perp$ are isomorphisms.*
Furthermore, $\mathfrak{l}_t = \mathfrak{g}_1^{H_t} \cap \mathfrak{g}_{< 0}^Z \oplus \mathfrak{n}_t$.
- (iii) *Suppose that $0 \leq s < t$, and all the elements of (s, t) are regular. Then*

$$\mathfrak{v}_t \oplus (\mathfrak{w}_t \cap \mathfrak{g}_{< 0}^Z) = \mathfrak{v}_s \oplus (\mathfrak{w}_s \cap \mathfrak{g}_{> 0}^Z) \quad (3.12)$$

$$\mathfrak{l}_t = \mathfrak{r}_s + (\mathfrak{w}_t \cap \mathfrak{g}_\varphi) \quad \text{and} \quad \mathfrak{r}_s \cap (\mathfrak{w}_t \cap \mathfrak{g}_\varphi) = \mathfrak{w}_0 \cap \mathfrak{g}_0^Z \cap \mathfrak{g}_\varphi. \quad (3.13)$$

Moreover, \mathfrak{r}_t is an ideal in \mathfrak{l}_s and the quotient is commutative.

Proof. It is easy to see that \mathfrak{v}_t is an ideal in \mathfrak{u}_t with commutative quotient, and that $\mathfrak{v}_t \subseteq \mathfrak{l}_t \cap \mathfrak{r}_t = \mathfrak{n}_t$. This proves (i). For the first part of (ii), note that $q_t := (\mathfrak{l}_t + \mathfrak{r}_t)/\mathfrak{n}_t$ is a symplectic space in which the projections of \mathfrak{l}_t and \mathfrak{r}_t are complementary Lagrangians.

For the second part, we have by Lemma 3.2.5 that $\mathfrak{g}_1^{H_t} \cap \mathfrak{g}_\varphi \subseteq \mathfrak{g}_{\geq 0}^Z$ and thus,

$$\mathfrak{l}_t = \mathfrak{v}_t \oplus (\mathfrak{w}_t \cap \mathfrak{g}_{< 0}^Z) \oplus (\mathfrak{w}_t \cap \mathfrak{g}_\varphi). \quad (3.14)$$

For (iii) note that

$$\mathfrak{v}_s = (\mathfrak{v}_s \cap \mathfrak{g}_{\geq 0}^Z) \oplus (\mathfrak{v}_s \cap \mathfrak{g}_{< 0}^Z) \quad (3.15)$$

$$\mathfrak{v}_t = (\mathfrak{v}_t \cap \mathfrak{g}_{< 0}^Z) \oplus (\mathfrak{v}_t \cap \mathfrak{g}_{\geq 0}^Z) \quad (3.16)$$

$$\mathfrak{v}_t \cap \mathfrak{g}_{\geq 0}^Z = (\mathfrak{w}_s \cap \mathfrak{g}_{> 0}^Z) \oplus (\mathfrak{v}_s \cap \mathfrak{g}_{\geq 0}^Z) \quad (3.17)$$

$$\mathfrak{v}_s \cap \mathfrak{g}_{< 0}^Z = (\mathfrak{w}_t \cap \mathfrak{g}_{< 0}^Z) \oplus (\mathfrak{v}_t \cap \mathfrak{g}_{< 0}^Z) \quad (3.18)$$

This implies (3.12). By Lemma 3.2.5 we have

$$\mathfrak{n}_s = \mathfrak{v}_s \oplus (\mathfrak{g}_\varphi \cap \mathfrak{w}_s) \subseteq \mathfrak{v}_s \oplus (\mathfrak{w}_s \cap \mathfrak{g}_{\geq 0}^Z), \quad (3.19)$$

and thus

$$\mathfrak{r}_s = \mathfrak{v}_s \oplus (\mathfrak{w}_s \cap \mathfrak{g}_{> 0}^Z) \oplus (\mathfrak{w}_0 \cap \mathfrak{g}_0^Z \cap \mathfrak{g}_\varphi) \text{ and } \mathfrak{r}_s \cap (\mathfrak{w}_t \cap \mathfrak{g}_\varphi) = \mathfrak{w}_0 \cap \mathfrak{g}_0^Z \cap \mathfrak{g}_\varphi. \quad (3.20)$$

Hence, (3.12) and (3.14) imply (3.13), and the rest is straightforward. \square

Using Lemma 2.1.7, choose an \mathfrak{sl}_2 -triple $(e_\varphi, h, f_\varphi)$ in \mathfrak{g}_0^Z such that h commutes with H and with Z , and φ is given by the Killing form pairing with $f = f_\varphi$. Let $L_t := \text{Exp}(\mathfrak{l}_t)$, $R_t := \text{Exp}(\mathfrak{r}_t)$. From Lemmas 3.2.6 and 3.1.1 we get

Lemma 3.2.7. *Let $t \geq s \geq 0$ and $\varphi' \in (\mathfrak{g}^*)_{> -2}^{H_t} \cap (\mathfrak{g}^*)_{> -2}^{H_s}$. Assume that there are no critical values in (s, t) . Then*

(i) $\mathcal{F}_{H_t, \varphi, \varphi'}$, $\mathcal{F}_{H_t, \varphi, \varphi'}^{L_t}$, and $\mathcal{F}_{H_t, \varphi, \varphi'}^{R_t}$ linearly determine each other. In particular,

$$\mathcal{F}_{H_t, \varphi, \varphi'}^{L_t}[\eta](g) = \int_{V(\mathbb{A})} \mathcal{F}_{H_t, \varphi, \varphi'}^{R_t}[\eta](vg) dv \quad (3.21)$$

where $\mathfrak{v} := (\mathfrak{g}_{\geq 1}^{H_t} \cap \mathfrak{g}_{< 0}^Z) / (\mathfrak{g}_{> 1}^{H_t} \cap \mathfrak{g}_{< 0}^Z)$ and $V = \text{Exp}(\mathfrak{v})$.

(ii) $\mathcal{F}_{H_t, \varphi, \varphi'}$ is linearly determined by $\mathcal{F}_{H_s, \varphi, \varphi'}$. Moreover, $\mathcal{F}_{H_s, \varphi, \varphi'}$ is linearly determined by the set

$$\{\mathcal{F}_{H_t, \varphi, \varphi' + \psi'} \mid \psi' \in (\mathfrak{g}^*)_{-1}^{H_t} \cap (\mathfrak{g}^*)^e \cap (\mathfrak{g}^*)_{< 0}^Z\}.$$

(iii) Let $\psi \in (\mathfrak{g}^*)_{> -2}^{H_s} \cap (\mathfrak{g}^*)_{-2}^{H_t}$. Then $\mathcal{F}_{H_s, \varphi, \psi + \varphi'}$ is linearly determined by the set

$$\{\mathcal{F}_{H_t, \varphi + \psi, \varphi' + \psi'} \mid \psi' \in (\mathfrak{g}^*)_{-1}^{H_t}\}.$$

(iv) Let $\psi \in (\mathfrak{g}^*)_{-2}^{H_s} \cap (\mathfrak{g}^*)_{> -2}^{H_t}$. Then $\mathcal{F}_{H_t, \varphi, \psi + \varphi'}$ is linearly determined by the set

$$\{\mathcal{F}_{H_s, \varphi + \psi, \varphi' + \psi'} \mid \psi' \in (\mathfrak{g}^*)_{-1}^{H_s}\}.$$

Proof. Part (i) follows from Lemmas 3.1.1 and 3.2.6(ii).

For part (ii), note first that by Lemma 3.2.6, $\mathfrak{r}_s \subseteq \mathfrak{l}_t$ with commutative quotient $(\mathfrak{w}_t \cap \mathfrak{g}_\varphi) / (\mathfrak{w}_0 \cap \mathfrak{g}_0^Z \cap \mathfrak{g}_\varphi)$, and let $B := [L_t/R_s]$ denote the corresponding compact commutative group. Then $\mathcal{F}_{H_t, \varphi, \varphi'}^{L_t}$ is obtained from $\mathcal{F}_{H_s, \varphi, \varphi'}^{R_s}$ just by integration over B .

To obtain $\mathcal{F}_{H_s, \varphi, \varphi'}^{R_s}$ we decompose it into Fourier series on B , similar to the proof of Lemma 3.1.1. Characters of B are given by

$$((\mathfrak{w}_t \cap \mathfrak{g}_\varphi) / (\mathfrak{w}_0 \cap \mathfrak{g}_0^Z \cap \mathfrak{g}_\varphi))^* \cong (\mathfrak{g}^*)_{-1}^{H_t} \cap (\mathfrak{g}^*)^e \cap (\mathfrak{g}^*)_{<0}^Z$$

and the Fourier series coefficient corresponding to each ψ' in this space is $\mathcal{F}_{H_t, \varphi, \varphi' + \psi'}^{L_t}$.

For part (iii), note that \mathfrak{v}_t is an ideal in \mathfrak{l}_t with commutative quotient. Together with (3.13) this implies that \mathfrak{v}_t is an ideal in \mathfrak{r}_s with commutative quotient. Denote $V := \text{Exp}(\mathfrak{v}_t)$ and define a new coefficient I by

$$I^f(g) := \int_{[V]} \chi_{\varphi + \psi + \varphi'}(n)^{-1} f/ng) dn.$$

Then I is linearly determined by the set

$$\{\mathcal{F}_{H_t, \varphi + \psi, \varphi' + \psi'} \mid \psi' \in (\mathfrak{g}^*)_{-1}^{H_t}\}.$$

Finally, from (3.12) we see that $\mathcal{F}_{H_s, \varphi, \psi + \varphi'}^{R_s}$ is obtained from I by integration.

Part (iv) is proven in a similar way. Namely, denote $V' := \text{Exp}(\mathfrak{v}_s)$ and define a new coefficient J by

$$J^f(g) := \int_{[V']} \chi_{\varphi + \psi + \varphi'}(n)^{-1} f/ng) dn.$$

Then J is linearly determined by the set

$$\{\mathcal{F}_{H_t, \varphi + \psi, \varphi' + \psi'} \mid \psi' \in (\mathfrak{g}^*)_{-1}^{H_s}\}.$$

On the other hand, from (3.13) we see that $\mathcal{F}_{H_t, \varphi, \psi + \varphi'}^{L_t}$ is obtained from J by integration. \square

Note that (3.21) is a special case of the root exchange lemma in [GRS11].

Proposition 3.2.8. *Let $H_t = H + tZ$ as above, $s \geq 0$ and let $\varphi' \neq 0 \in (\mathfrak{g}^*)_{>-2}^{H_s} \cap (\mathfrak{g}^*)^e \cap (\mathfrak{g}^*)_{<0}^Z$. Then $\mathcal{F}_{H_s, \varphi, \varphi'}$ is linearly determined by the set*

$$\{\mathcal{F}_{H_t, \Phi, \Phi'} \mid t > s \text{ critical, } \Phi \in (\mathfrak{g}^*)_{-2}^{H_t}, \Phi' \in (\mathfrak{g}^*)_{>-2}^{H_t} \text{ and } G(\mathbb{K})\Phi > G(\mathbb{K})\varphi\}, \quad (3.22)$$

where $G(\mathbb{K})\Phi > G(\mathbb{K})\varphi$ means strictly bigger by the order relation given in Definition 2.2.1.

Note that there are finitely many critical values t .

Proof. Since $\varphi' \in \mathfrak{g}_{<0}^Z$ there exist $t > s$, $\psi \in (\mathfrak{g}^*)_{-2}^{H_t}$ and $\eta \in (\mathfrak{g}^*)_{>-2}^{H_t}$ such that $\psi \neq 0$ and $\varphi' = \psi + \eta$. Let t be the smallest such t , and since $[Z, e] = 0$ we have that $\psi, \eta \in (\mathfrak{g}^*)^e$.

Let $a_0 := s$, let a_1, \dots, a_{m-1} be the critical values between s and t and $a_m := t$. We prove the statement by induction on m .

The base case is $m = 1$, i.e. there are no critical values between s and t . Then Lemma 3.2.7(iii) implies that $\mathcal{F}_{H_s, \varphi, \varphi'}$ is linearly determined by the set

$$\{\mathcal{F}_{H_t, \varphi + \psi, \eta + \psi'} \mid \psi' \in (\mathfrak{g}^*)_{-1}^{H_t}\}.$$

Denote $\Phi := \varphi + \psi$. Note that Φ belongs to the Slodowy slice to $G\varphi$ at φ since $\psi \in (\mathfrak{g}^*)^e$, and thus $G(\mathbb{K})\Phi > G(\mathbb{K})\varphi$. For each ψ' denote $\Phi' := \eta + \psi'$ and note that $\mathcal{F}_{H_t, \varphi + \psi, \eta + \psi'} = \mathcal{F}_{H_t, \Phi, \Phi'}$.

The induction step easily follows from the base using Lemma 3.2.7(ii). \square

Lemma 3.2.9 ([[GGG](#), Lemma 4.2.4]). *Let $\psi \in (\mathfrak{g}^*)_{-2}^H \cap (\mathfrak{g}^*)_{>0}^Z$. Assume that $\varphi + \psi \in G(\mathbb{C})\varphi$. Then $\varphi + \psi \in G(\mathbb{C})_H\varphi$.*

3.3. Conjugations and translations.

Lemma 3.3.1. *Let (S, φ, ψ) be a Whittaker triple, η an automorphic function and $\gamma \in G(\mathbb{K})$. Then,*

$$\mathcal{F}_{S, \varphi, \psi}[\eta](g) = \mathcal{F}_{\text{Ad}(\gamma)S, \text{Ad}^*(\gamma)\varphi, \text{Ad}^*(\gamma)\psi}[\eta](\gamma g). \quad (3.23)$$

Proof. We have that $\chi_{\varphi + \psi}(u) = \chi_{\text{Ad}^*(\gamma)(\varphi + \psi)}(\text{Ad}(\gamma)u)$. Indeed, the right-hand side equals

$$\chi\left((\text{Ad}^*(\gamma)(\varphi + \psi))(\text{Ad}(\gamma)u)\right) = \chi((\varphi + \psi)(\text{Ad}(\gamma^{-1})\text{Ad}(\gamma)u)) = \chi_{\varphi + \psi}(u) \quad (3.24)$$

We also have that $\text{Ad}(\gamma)\mathfrak{g}_\lambda^{\text{Ad}(\gamma)S} = \mathfrak{g}_\lambda^S$ since, for $x \in \mathfrak{g}$, $[\text{Ad}(\gamma)S, \text{Ad}(\gamma)x] = \text{Ad}(\gamma)[S, x]$. Similarly, $\text{Ad}(\gamma)\mathfrak{n}_{\text{Ad}^*(\gamma)\varphi} = \mathfrak{n}_\varphi$ and thus, $\text{Ad}(\gamma)\mathfrak{n}_{\text{Ad}(\gamma)S, \text{Ad}^*(\gamma)\varphi} = \mathfrak{n}_{S, \varphi}$.

Hence, using the automorphic invariance of η , the right-hand side of (3.23) equals

$$\int_{[N_{\text{Ad}(\gamma)S, \text{Ad}^*(\gamma)}]} \eta(\gamma^{-1}u\gamma g) \chi_{\text{Ad}^*(\gamma)(\varphi + \psi)}(u)^{-1} du = \int_{[\text{Ad}(\gamma)N_{\text{Ad}(\gamma)S, \text{Ad}^*(\gamma)}]} \eta(u'g) \chi_{\text{Ad}^*(\gamma)(\varphi + \psi)}(\text{Ad}(\gamma)u')^{-1} du'$$

where we have used the usual short-hand notation $[N] = N(\mathbb{K}) \backslash N(\mathbb{A})$. By the arguments above, this equals $\mathcal{F}_{S, \varphi, \psi}[\eta](g)$. \square

4. GENERAL REDUCTIVE GROUPS

Proposition 4.0.1. *Let (H, φ) and (S, φ) be Whittaker pairs such that (H, φ) dominates (S, φ) . Then $\mathcal{F}_{S, \varphi}$ is linearly determined by $\mathcal{F}_{H, \varphi}$.*

Note that this is in the other direction than the statement of Theorem [B](#) and is much easier to prove.

Proof. Let $Z := S - H$, and for any $t \geq 0$ let $H_t := H + tZ$. Let t_1, \dots, t_k be all the critical values of t between 0 and 1. Let $t_0 := 0$ and $t_{k+1} := 1$. By Lemma [3.2.7\(ii\)](#), for any $0 \leq i \leq k$, $\mathcal{F}_{H_{t_i}, \varphi}$ linearly determines $\mathcal{F}_{H_{t_{i+1}}, \varphi}$. Since $H_{t_0} = H$ and $H_{t_{k+1}} = S$, the proposition follows. \square

It was shown in Corollary [3.2.2](#) that any Whittaker pair (S, φ) is dominated by a neutral pair (h, φ) .

Corollary 4.0.2. *$\mathcal{F}_{S, \varphi}$ is linearly determined by $\mathcal{F}_{h, \varphi}$ where (h, φ) is a neutral pair.*

Let (H, φ) be a Whittaker pair. Using Lemma [2.1.7](#), decompose $H = h + Z$, where (h, φ) is a neutral pair, and Z commutes with h and φ .

Definition 4.0.3. Denote by $\text{in}(H, \varphi)$ the number

$$\dim \mathfrak{g}_{<1}^h \cap \mathfrak{g}_{\geq 1}^{h+Z} + \dim \mathfrak{g}_{<2}^h \cap \mathfrak{g}_{\geq 2}^{h+Z} \quad (4.1)$$

Note that this number is different from an analogous number in [[GGG](#)].

Let us now show that $\text{in}(H, \varphi)$ depends only on (H, φ) and does not depend on the decomposition $H = h + Z$.

Lemma 4.0.4 ([GGS, Lemma 4.2.7]). *Let $\tilde{h} \in \mathfrak{g}_H$ be another neutral element for f . Then there exists a nilpotent element $X \in \mathfrak{g}_H$ such that $\exp(\text{ad}(X))(h) = \tilde{h}$.*

Corollary 4.0.5. *The number $\text{in}(H, \varphi)$ depends only on (H, φ) and not on h . In fact, $\text{in}(H, \varphi)$ depends only on $(H, G_H(\mathbb{C})\varphi)$.*

Proof. If $H = \tilde{h} + \tilde{Z}$ is another decomposition as above, then by Lemma 4.0.4, $\tilde{h} = \text{Ad}(\gamma)h$ for some $\gamma \in G_H(\mathbb{K})$. Then $\tilde{Z} = \text{Ad}(\gamma)Z$ and

$$\dim \mathfrak{g}_{<\lambda}^{\text{Ad}(\gamma)h} \cap \mathfrak{g}_{\geq\lambda}^{\text{Ad}(\gamma)(h+Z)} = \dim \mathfrak{g}_{<\lambda}^h \cap \mathfrak{g}_{\geq\lambda}^{h+Z} \quad (4.2)$$

which proves that $\text{in}(H, \varphi)$ does not depend on the choice of h .

For the second statement, let $\varphi' = \text{Ad}^*(\gamma)\varphi$, with $\gamma \in G_H(\mathbb{K})$. Since h is neutral to φ , $\text{Ad}(\gamma)h$ is neutral to φ' and $H = \text{Ad}(\gamma)h + \text{Ad}(\gamma)Z$ where $\text{Ad}(\gamma)Z$ commutes with $\text{Ad}(\gamma)h$ and $\text{Ad}^*(\gamma)\varphi$. By the same argument as above, $\text{in}(H, \text{Ad}^*(\gamma)\varphi) = \text{in}(H, \varphi)$. \square

Let $C \subseteq G(\mathbb{K})$ denote the centralizer of (h, φ) . Let A denote a maximally split torus of C such that its Lie algebra \mathfrak{a} includes Z , and let M denote the centralizer of \mathfrak{a} in G . Then M is a Levi subgroup of G , \mathfrak{m} includes h, Z and φ , and φ is \mathbb{K} -distinguished in \mathfrak{m} . Let z be a rational semi-simple element of \mathfrak{a} that is generic in the sense that its centralizer is M .

Lemma 4.0.6. *As an element of \mathfrak{m} , φ is \mathbb{K} -distinguished.*

Proof. Let \mathfrak{l} be the Lie algebra of a Levi subgroup of M defined over \mathbb{K} such that $\varphi \in \mathfrak{l}^*$. We have to show that $L = M$. By replacing L by its conjugate we can assume $h \in \mathfrak{l}$, and that there exists a rational semi-simple element $z' \in \mathfrak{m}$ such that \mathfrak{l} is the centralizer of z' . Then z' commutes with h and φ and we have to show that z' is central in \mathfrak{m} .

Indeed, $z' \in \mathfrak{m} \cap \mathfrak{c} = \mathfrak{a}$. Now, any $X \in \mathfrak{m}$ commutes with z , and thus with any element of \mathfrak{a} , since z is generic in \mathfrak{a} . Thus \mathfrak{a} lies in the center of \mathfrak{m} and thus z' is central. \square

Note that the eigenvalues of the adjoint action of any Lie algebra element are symmetric around zero. Let N be a positive integer that is bigger than the ratio of the maximal eigenvalue of $\text{ad}(z)$ by the minimal positive eigenvalue of $\text{ad}(Z)$. Let

$$Z' := NZ + z. \quad (4.3)$$

From our choice of N we have

$$\mathfrak{g}_{>0}^{Z'} = \mathfrak{g}_{>0}^Z \oplus (\mathfrak{g}_0^Z \cap \mathfrak{g}_{>0}^z) \text{ and } \mathfrak{g}_0^{Z'} = \mathfrak{g}_0^z = \mathfrak{m} \subseteq \mathfrak{g}_0^Z. \quad (4.4)$$

That $\mathfrak{m} \subseteq \mathfrak{g}_0^Z$ follows from the fact that M is the centralizer of z which equals the centralizer of \mathfrak{a} and \mathfrak{a} includes Z .

Lemma 4.0.7. *For rational $T > 0$, (H, φ) dominates $(H + TZ', \varphi)$, that is, H, φ and TZ' commute, and satisfy (1.10).*

Proof. By construction $H = h + Z$, φ and Z commute, and since $h, Z, \varphi \in \mathfrak{m}$ they commute with z . Thus, Z' commutes with H and φ . Furthermore, $\mathfrak{g}_\varphi \cap \mathfrak{g}_{\geq 1}^H \subseteq \mathfrak{g}_{\leq 0}^h \cap \mathfrak{g}_{\geq 1}^H \subseteq \mathfrak{g}_{>0}^Z \subseteq \mathfrak{g}_{>0}^{Z'} = \mathfrak{g}_{>0}^{TZ'}$. \square

Lemma 4.0.8. *For a fixed $\lambda \in \mathbb{Q}$, and a rational $T > 0$ large enough,*

$$\mathfrak{g}_{>1}^{H+TZ'} = \mathfrak{g}_{\geq 2}^{H+TZ'} = \mathfrak{g}_{>0}^{Z'} \oplus (\mathfrak{g}_0^{Z'} \cap \mathfrak{g}_{>1}^{H+TZ'}) = \mathfrak{g}_{>0}^{Z'} \oplus \mathfrak{m}_{\geq 2}^h \text{ and } \mathfrak{g}_\lambda^{H+TZ'} = \mathfrak{m}_\lambda^H = \mathfrak{m}_\lambda^h. \quad (4.5)$$

The Fourier coefficient $\mathcal{F}_{H+TZ',\varphi}$ is then Levi-distinguished.

Proof. For large enough T , we have that $\mathfrak{g}_{>1}^{H+TZ'} \cap \mathfrak{g}_{<0}^{Z'} = \{0\}$ and $\mathfrak{g}_{>1}^{H+TZ'} \cap \mathfrak{g}_{>0}^{Z'} = \mathfrak{g}_{>0}^{Z'}$. Thus $\mathfrak{g}_{>1}^{H+TZ'} = \mathfrak{g}_{>1}^{H+TZ'} \cap (\mathfrak{g}_{<0}^{Z'} \oplus \mathfrak{g}_0^{Z'} \oplus \mathfrak{g}_{>0}^{Z'}) = \mathfrak{g}_{>0}^{Z'} \oplus (\mathfrak{g}_0^{Z'} \cap \mathfrak{g}_{>1}^{H+TZ'})$. Since $H = h + Z$ and $\mathfrak{g}_0^{Z'} = \mathfrak{m} \subseteq \mathfrak{g}_0^Z$ we have that $\mathfrak{g}_0^{Z'} \cap \mathfrak{g}_{>1}^{H+TZ'} = \mathfrak{g}_0^{Z'} \cap \mathfrak{g}_{>1}^h$ and since h is neutral $\mathfrak{g}_{>1}^h = \mathfrak{g}_{\geq 2}^h$. Now $\mathfrak{g}_0^{Z'} = \mathfrak{m}$ and thus, $\mathfrak{g}_{>1}^{H+TZ'} = \mathfrak{g}_{>0}^{Z'} \oplus (\mathfrak{g}_0^{Z'} \cap \mathfrak{g}_{\geq 2}^h) = \mathfrak{g}_{>0}^{Z'} \oplus \mathfrak{m}_{\geq 2}^h$. Doing the same manipulations for $\mathfrak{g}_{\geq 2}^{H+TZ'}$ one ends up with the same result, proving the equality $\mathfrak{g}_{>1}^{H+TZ'} = \mathfrak{g}_{\geq 2}^{H+TZ'}$.

Now, for any fixed $\lambda \in \mathbb{Q}$ and a large enough T , we have that $\mathfrak{g}_\lambda^{H+TZ'} = \mathfrak{g}_\lambda^{H+TZ'} \cap \mathfrak{g}_0^{Z'} = \mathfrak{g}_\lambda^H \cap \mathfrak{m} = \mathfrak{m}_\lambda^H$. Again, since $H = h + Z$ and $\mathfrak{m} \subseteq \mathfrak{g}_0^Z$, we get that $\mathfrak{m}_\lambda^H = \mathfrak{m}_\lambda^h$.

Since $H + TZ' = h + Z + TZ'$, the semi-simple element denoted by Z in Definition 2.1.5 is here $Z + TZ'$, which, for large enough T has the centralizer $\mathfrak{g}_0^Z \cap \mathfrak{g}_0^{Z'} = \mathfrak{g}_0^{Z'} = \mathfrak{m}$. By Lemma 4.0.6, φ is \mathbb{K} -distinguished in \mathfrak{m} . Since $\mathfrak{g}_{>0}^Z \subseteq \mathfrak{g}_{>0}^{Z'}$ we have that $\mathfrak{g}_{>0}^{Z+TZ'} = \mathfrak{g}_{>0}^{Z'}$ and thus (4.5) implies (2.7) which means that $\mathcal{F}_{H+TZ',\varphi}$ is Levi-distinguished. \square

Lemma 4.0.9. *Let (H, φ, φ') be a Whittaker triple such that the pair (H, φ) is either neutral or Levi-distinguished. Then $\mathcal{F}_{H,\varphi,\varphi'} = \mathcal{F}_{H,\varphi}$.*

Proof. If (H, φ) is neutral set $h := H$. If (H, φ) is Levi-distinguished decompose $H = h + Z$ where (h, φ) is a neutral pair and Z commutes with it. In both cases we have $\mathfrak{g}_{>1}^H = \mathfrak{g}_{\geq 2}^H$, and $\mathfrak{g}_1^H \subseteq \mathfrak{g}_1^h$. Note that \mathfrak{g}_φ is spanned by lowest weight vectors and thus $\mathfrak{g}_\varphi \subseteq \mathfrak{g}_{\leq 0}^h$. Thus $\mathfrak{g}_\varphi \cap \mathfrak{g}_1^H = 0$. By Lemma 3.2.5 this implies that $\mathfrak{n}_{H,\varphi} = \mathfrak{g}_{>1}^H = \mathfrak{g}_{\geq 2}^H$. Since $\varphi' \in (\mathfrak{g}^*)_{>-2}^H$, it vanishes on $\mathfrak{g}_{\geq 2}^H$ and thus $\mathcal{F}_{h,\varphi,\varphi'} = \mathcal{F}_{h,\varphi}$. \square

4.1. Proof of Theorem A. We will prove a more general theorem.

Theorem 4.1.1. *Let η be an automorphic function on a reductive group G . Then, any quasi-Fourier coefficient $\mathcal{F}_{S,\varphi,\varphi'}[\eta]$ is linearly determined by the Levi-distinguished Fourier coefficients with characters in orbits which are equal or bigger than $G\varphi$.*

In particular, if all non-PL coefficients of η vanish, then all Fourier coefficients are linearly determined by Whittaker coefficients $\mathcal{W}_\psi[\eta]$.

Proof. Choose h, Z, z, Z' as above and let $H_t := H + tZ'$. Choose a large enough T from Lemma 4.0.8. Recall that $t \geq 0$ is quasi-critical if either $\mathfrak{g}_1^{H_t} \not\subseteq \mathfrak{g}_0^Z$ or $\mathfrak{g}_2^{H_t} \not\subseteq \mathfrak{g}_0^Z$.

If there are no quasi-critical $t \in (0, T]$ then by Lemma 3.2.7(ii), $\mathcal{F}_{H,\varphi,\varphi'}$ is linearly determined by the set of all $\mathcal{F}_{H+TZ',\varphi,\varphi'+\psi}$ with $\psi \in (\mathfrak{g}^*)_{-1}^{H+TZ'} \cap (\mathfrak{g}^*)^e \cap (\mathfrak{g}^*)_{<0}^{Z'}$. By Lemma 4.0.8, $\mathcal{F}_{H+TZ',\varphi}$ is Levi-distinguished, and thus, by Lemma 4.0.9, we have $\mathcal{F}_{H+TZ',\varphi,\varphi'+\psi} = \mathcal{F}_{H+TZ',\varphi}$. Thus, $\mathcal{F}_{H,\varphi,\varphi'}$ is linearly determined by $\mathcal{F}_{H+TZ',\varphi}$ which is Levi-distinguished.

Now assume that there are quasi-critical numbers in $(0, T]$ and let s be the smallest one. Let $H_s := H + sZ'$.

Since s is the first quasi-critical value we have that $(\mathfrak{g}^*)_{>-2}^H \subseteq (\mathfrak{g}^*)_{\geq -2}^{H_s}$ because this is the first point where something new may enter the -2 -eigenspace. Decompose $\varphi' = \psi + \varphi''$

where $\psi \in (\mathfrak{g}^*)_{-2}^{H_s}$ and $\varphi'' \in (\mathfrak{g}^*)_{>-2}^{H_s}$. By Lemma 3.2.7(iii), $\mathcal{F}_{H_s, \varphi, \varphi'}$ is linearly determined by

$$\{\mathcal{F}_{H_s, \varphi + \psi, \varphi'' + \psi''} \mid \psi'' \in (\mathfrak{g}^*)_{-1}^{H_s}\}. \quad (4.6)$$

Now, we repeat the procedure for each triple $\mathcal{F}_{H_s, \varphi + \psi, \varphi'' + \psi''}$ and so on. To see that the algorithm terminates, note that $\psi \in (\mathfrak{g}^*)_{<0}^{Z'}$ and thus the orbit of $\varphi + \psi$ is bigger than or equal to the orbit of φ .

Suppose the orbits are the same. Then, by Corollary 4.0.5, $\text{in}(H_s, \varphi + \psi) = \text{in}(H_s, \varphi)$.

From (4.4) we see that $\mathfrak{g}_{>0}^{Z+sZ'} \cap \mathfrak{g}_0^{Z'} = \{0\}$ and $\mathfrak{g}_{>0}^{Z+sZ'} \cap \mathfrak{g}_{<0}^{Z'} = \mathfrak{g}_{>0}^{Z+sZ'} \cap \mathfrak{g}_{<0}^{Z'} \cap \mathfrak{g}_{>0}^Z = \{0\}$ which means that $\mathfrak{g}_{>0}^{Z+sZ'} \subseteq \mathfrak{g}_{>0}^{Z'}$ and thus

$$\mathfrak{g}_{<1}^h \cap \mathfrak{g}_{\geq 1}^{h+Z} \subseteq \mathfrak{g}_{<1}^h \cap \mathfrak{g}_{\geq 1}^{h+Z+sZ'} \quad \text{and} \quad \mathfrak{g}_{<2}^h \cap \mathfrak{g}_{\geq 2}^{h+Z} \subseteq \mathfrak{g}_{<2}^h \cap \mathfrak{g}_{\geq 2}^{h+Z+sZ'} \quad (4.7)$$

Since s is quasi-critical, one of the inclusions in (4.7) is strict and thus $\text{in}(H_s, \varphi) > \text{in}(H, \varphi)$.

Thus we get that either $G(\mathbb{K})(\varphi + \psi) > G(\mathbb{K})\varphi$ or $\text{in}(H_s, \varphi + \psi) > \text{in}(H, \varphi)$. Since both the orbit dimensions and the indices are bounded by $\dim \mathfrak{g}$, the algorithm eventually terminates.

Finally, by Lemma 2.1.9, the Levi distinguished Fourier coefficients of PL elements are Whittaker coefficients. This proves the second part of the statement. \square

4.2. Proof of Theorem B.

Proposition 4.2.1. *Let (H, φ, φ') be a Whittaker triple and let η be an automorphic function with $\mathcal{F}_{H, \varphi, \varphi'}[\eta] \neq 0$. Then there exists $\mathcal{O} \in \text{WS}(\eta)$ such that $\mathcal{O} \geq G(\mathbb{K})\varphi$.*

Proof. By Theorem 4.1.1, $\mathcal{F}_{H, \varphi, \varphi'}$ is linearly determined by Fourier coefficients corresponding to orbits bigger than or equal to $G(\mathbb{K})\varphi$. By Corollary 4.0.2, these are in turn linearly determined by neutral Fourier coefficients corresponding to the same orbits. Since $\mathcal{F}_{H, \varphi, \varphi'}[\eta] \neq 0$, some of these neutral Fourier coefficients of η do not vanish. \square

Let us now adapt the assumption and the notation of Theorem B. Let $Z := S - H$ and let $H_t := H + tZ$. Let $0 < t_1 < \dots < t_n < 1$ be all the critical values between 0 and 1. Let $t_0 := 0$ and $t_{n+1} := 1$. Lastly, let, for each t_i , R and L be defined as in (3.11).

Lemma 4.2.2. *We have $\mathcal{F}_{H_{t_i}, \varphi}^R[\eta] = \mathcal{F}_{H_{t_{i+1}}, \varphi}^L[\eta]$.*

Proof. Let $f \in \mathfrak{g}$ be the unique nilpotent element such that φ is given by Killing form pairing with f . Complete f to an \mathfrak{sl}_2 -triple (e, h, f) such that h commutes with S and H . Denote $H_j := H_{t_j}$ for any j , and $\mathfrak{c} := (\mathfrak{g}^*)_{-1}^{H_{i+1}} \cap (\mathfrak{g}^*)^e \cap (\mathfrak{g}^*)_{<0}^Z$. Arguing as in the proof of Lemma 3.2.7(ii), we obtain

$$\mathcal{F}_{H_i, \varphi}^R[\eta] = \sum_{\varphi' \in \mathfrak{c}} \mathcal{F}_{H_{i+1}, \varphi, \varphi'}^L[\eta].$$

We have to show that for any non-zero $\varphi' \in \mathfrak{c}$, we have $\mathcal{F}_{H_{i+1}, \varphi, \varphi'}^L[\eta] = 0$. This follows from Lemma 3.1.1, Proposition 3.2.8, Proposition 4.2.1, and the condition that $G(\mathbb{K})\varphi \in \text{WS}(\eta)$. \square

Proof of Theorem B. Let $S = H + Z$ and

$$\mathfrak{v}_i := (\mathfrak{g}_{\geq 1}^{H_{t_i}} \cap \mathfrak{g}_{< 0}^Z) / (\mathfrak{g}_{> 1}^{H_{t_i}} \cap \mathfrak{g}_{< 0}^Z) \quad \text{and} \quad V_i = \text{Exp}(\mathfrak{v}_i). \quad (4.8)$$

By Lemma 3.1.1 we have

$$\mathcal{F}_{H_{t_i}, \varphi}^L[\eta](g) = \int_{V_i(\mathbb{A})} \mathcal{F}_{H_{t_i}, \varphi}^R[\eta](v_i g) dv_i \quad (4.9)$$

Using Lemma 4.2.2 we obtain

$$\mathcal{F}_{H, \varphi}^R[\eta](g) = \int_{V_1(\mathbb{A})} \dots \int_{V_{n-1}(\mathbb{A})} \int_{V_n(\mathbb{A})} \mathcal{F}_{S, \varphi}^L[\eta](v_n \dots v_0 g) dv \quad (4.10)$$

Since $\mathfrak{v} = \bigoplus_{i=1}^n (\mathfrak{g}_1^{H_{t_i}} \cap \mathfrak{g}_{< 0}^Z)$, and as a commutative Lie algebra $\mathfrak{g}_1^{H_{t_i}} \cap \mathfrak{g}_{< 0}^Z$ is naturally isomorphic to \mathfrak{v}_i , the group V is glued from V_i . Thus

$$\int_{V_1(\mathbb{A})} \dots \int_{V_{n-1}(\mathbb{A})} \int_{V_n(\mathbb{A})} \mathcal{F}_{S, \varphi}^L[\eta](v_n \dots v_0 g) dv = \int_{V(\mathbb{A})} \mathcal{F}_{S, \varphi}^L[\eta](v g) dv. \quad (4.11)$$

To prove part (i) note from (3.11) that if $\mathfrak{g}_1^H = \mathfrak{g}_1^S = 0$ then $\mathcal{F}_{H, \varphi} = \mathcal{F}_{H, \varphi}^R$ and $\mathcal{F}_{S, \varphi} = \mathcal{F}_{S, \varphi}^L$, and thus part (i) follows from (4.10) and (4.11).

For part (ii), note that \mathfrak{u} and \mathfrak{w} as defined in the statement are equal to $\mathfrak{u} = (\mathfrak{g}_{\geq 1}^S \cap \mathfrak{g}_{< 0}^Z) / (\mathfrak{g}_{> 1}^S \cap \mathfrak{g}_{< 0}^Z)$ and $\mathfrak{w} = (\mathfrak{g}_{\geq 1}^H \cap \mathfrak{g}_{< 0}^Z) / (\mathfrak{g}_{> 1}^H \cap \mathfrak{g}_{< 0}^Z)$. Thus Lemmas 3.1.1 and 3.2.6 imply

$$\mathcal{F}_{S, \varphi}^L[\eta](g) = \int_{[U]} \mathcal{F}_{S, \varphi}[\eta](u g) du, \quad \text{and} \quad \mathcal{F}_{H, \varphi}[\eta](g) = \sum_{w \in W(\mathbb{K})} \mathcal{F}_{H, \varphi}^R[\eta](w g) \quad (4.12)$$

Applying (4.10), (4.11), and (4.12) to shifts of η we obtain

$$\begin{aligned} \mathcal{F}_{H, \varphi}[\eta](g) &= \sum_{w \in W(\mathbb{K})} \mathcal{F}_{H, \varphi}^R[\eta](w g) = \sum_{w \in W(\mathbb{K})} \int_{V(\mathbb{A})} \mathcal{F}_{S, \varphi}^L[\eta](v w g) dv = \\ &= \sum_{w \in W(\mathbb{K})} \int_{V(\mathbb{A})} \int_{[U]} \mathcal{F}_{S, \varphi}[\eta](u v w g) du dv \quad (4.13) \end{aligned}$$

□

4.3. On PL-orbits. A complex orbit is a PL-orbit if and only if its Bala-Carter label has no parenthesis. In particular, all complex minimal and next-to-minimal orbits are PL. The classification of PL orbits of complex classical groups in terms of the corresponding partitions is given in [GS15, §6].

The classification of rational PL-orbits is a more complicated task. In this subsection we discuss the PL property for small \mathbb{K} -rational orbits of simple split groups. A complex orbit $\mathcal{O}_{\mathbb{C}}$ may include several or even infinitely many rational orbits. If $\mathcal{O}_{\mathbb{C}}$ is non-PL then all its rational orbits are non-PL. If $\mathcal{O}_{\mathbb{C}}$ is PL then it includes at least one rational PL-orbit, but

can also include non-PL rational orbits. In type A_n , all rational orbits are PL. Let us now describe the PL properties of minimal and next-to-minimal orbits.

All minimal rational orbits are PL. Indeed, for classical groups it is easy to establish the Levi in which they are principal: for $\mathrm{SO}_{n+1,n}$ it is $\mathrm{SO}_{2,1} \times (\mathrm{GL}_1)^{n-1}$, for Sp_{2n} it is $\mathrm{Sp}_2 \times (\mathrm{GL}_1)^{n-1}$ and for $\mathrm{SO}_{n,n}$ it is $\mathrm{SO}_{2,2} \times (\mathrm{GL}_1)^{n-2}$. For exceptional groups, the rational minimal orbit is unique and thus PL. This uniqueness was explained to us by Joseph Hundley.

Let us now deal with the next-to-minimal orbits.

Lemma 4.3.1. *All next-to-minimal rational orbits of $\mathrm{SO}_{n,n}$ and $\mathrm{SO}_{n+1,n}$ are PL.*

Proof. One can give a classification of the rational orbits in the spirit of the classification of real orbits given in [CM93, §9.3]. Namely, a \mathbb{K} -rational orbit with a given partition is defined by a collection of quadratic forms Q_{2i+1} on multiplicity spaces of the odd parts. If we add a hyperbolic form to the direct sum of these forms we get the initial form, which is also hyperbolic. Here, a hyperbolic form is a direct multiple of the 2-dimensional quadratic form given by $H(x, y) = xy$. By Witt's cancelation theorem this implies that the direct sum of the forms on multiplicity spaces of the odd parts is hyperbolic.

An orbit in $\mathrm{SO}_{n,n}$ is PL if and only if all Q_{2i+1} are hyperbolic, except Q_{2j+1} for a single index $j \geq 1$, which is a direct sum of a hyperbolic form and a one-dimensional quadratic form. For $\mathrm{SO}_{n,n}$ there are two next-to-minimal partitions. One of them is $2^4, 1^{2n-8}$. For it, Q_1 has to be hyperbolic. The other next-to-minimal partition is $3, 1^{2n-3}$. Thus Q_3 is one-dimensional. Now, note that $H^n = Q_3 \oplus -Q_3 \oplus H^{n-1}$. Thus, $Q_3 \oplus Q_1 = Q_3 \oplus -Q_3 \oplus H^{n-1}$ and thus $Q_1 = (-Q_3) \oplus H^{n-1}$, i.e. Q_1 is a direct sum of a hyperbolic form and a one-dimensional quadratic form.

Similarly, it is easy to see that the next-to-minimal orbits for $\mathrm{SO}_{n+1,n}$ are principal in Levis isomorphic to $(\mathrm{GL}_2)^2 \times (\mathrm{GL}_1)^{n-4}$ or $\mathrm{SO}_{2,1} \times (\mathrm{GL}_1)^{n-1}$. \square

However, $\mathrm{Sp}_{2n}(\mathbb{K})$ has infinitely many rational next-to-minimal orbits, already for $n = 2$. Moreover, by [Ike01] there exist cuspidal next-to-minimal representations of $\mathrm{Sp}_4(\mathbb{A})$. Note that cuspidal non-generic automorphic forms cannot be determined by their Whittaker coefficients, since the latter coefficients have to vanish on such forms. See [Gin06, §4] for a discussion of cuspidal representations, in particular those of $\mathrm{Sp}_4(\mathbb{A})$.

As for the exceptional groups, Joseph Hundley showed that the next-to-minimal orbit is unique, and thus PL, for E_6, E_7, E_8 and G_2 .

The group F_4 has infinitely many rational next-to-minimal orbits. We expect that infinitely many of them are not PL.

4.4. Some geometric lemmas.

Lemma 4.4.1. *Let $Z \in \mathfrak{g}$ be rational semi-simple, let $\varphi \in \mathfrak{g}_0^Z$ and $\varphi' \in \mathfrak{g}_{>0}^Z$. Assume that φ is conjugate to $\varphi + \varphi'$ by $G(\mathbb{C})$. Then there exists $X \in \mathfrak{g}_{>0}^Z$ such that $\mathrm{ad}^*(X)(\varphi) = \varphi'$.*

Recall that we often refer to $\mathfrak{g}(\mathbb{K})$ as \mathfrak{g} .

Proof. Decompose $\varphi' = \sum_{i=1}^k \varphi'_i$ where $\varphi'_i \in (\mathfrak{g}^*)_{\lambda_i}^Z$ and $\lambda_1 < \lambda_2 < \dots < \lambda_k$ are positive eigenvalues of Z with $\lambda_i \in \mathbb{Q}$. Then, for any $t \in \mathbb{R}$, we have the following identity in $\mathfrak{g}^*(\mathbb{R})$:

$$\exp(tZ)(\varphi + \varphi') = \varphi + \sum_{i=1}^k \exp(t\lambda_i)\varphi'_i \quad (4.14)$$

Thus, $\varphi + \sum_i \exp(t\lambda_i)\varphi'_i \in G(\mathbb{C})\varphi$. Differentiating by t at 0 we obtain that $\sum_i \lambda_i \varphi'_i$ lies in the tangent space to the orbit $G(\mathbb{C})\varphi$ at φ . This tangent space is the image of φ under the coadjoint action. Thus there exists $Y \in \mathfrak{g}$ with $\text{ad}^*(Y)(\varphi) = \sum_i \lambda_i \varphi'_i$. Decompose $Y = Y' + \sum_i Y_i$ with $Y_i \in \mathfrak{g}_{\lambda_i}^Z$. Since φ commutes with Z , we obtain $\text{ad}^*(Y_i)(\varphi) = \lambda_i \varphi'_i$. Now we take $X := \sum_i \lambda_i^{-1} Y_i \in \mathfrak{g}_{>0}^Z$. \square

Corollary 4.4.2. *Let (H, φ, φ') be a Whittaker triple, and let $S \in \mathfrak{g}_0^H$ form Whittaker pairs (S, φ) and $(S, \varphi + \varphi')$. Assume that φ is conjugate to $\varphi + \varphi'$ under $G(\mathbb{C})$. Then there exists $X \in \mathfrak{g}_0^S \cap \mathfrak{g}_{<0}^{S-H}$ such that $\text{ad}^*(X)(\varphi) = \varphi'$.*

Proof. Let $Z := H - S$. By Lemma 4.4.1, there exists $Y \in \mathfrak{g}_{>0}^Z$ with $\text{ad}^*(Y)(\varphi) = \varphi'$. Decompose $Y = Y_- + X + Y_+$ with $Y_- \in \mathfrak{g}_{<0}^S$, $X \in \mathfrak{g}_0^S$, and $Y_+ \in \mathfrak{g}_{>0}^S$. Since $[Z, S] = 0$, and $Y \in \mathfrak{g}_{>0}^Z$, we get $X \in \mathfrak{g}_{>0}^Z$ and thus $X \in \mathfrak{g}_0^Z \cap \mathfrak{g}_{<0}^{S-H}$. Since $\text{ad}^*(Y)(\varphi) = \varphi'$, and $\varphi, \varphi' \in (\mathfrak{g}^*)_{-2}^S$, we have $\text{ad}^*(Y_-)(\varphi) = \text{ad}^*(Y_+)(\varphi) = 0$ and $\text{ad}^*(X)(\varphi) = \varphi'$. \square

Corollary 4.4.3. *Let $Z \in \mathfrak{g}$ be rational semi-simple, let $\varphi \in \mathfrak{g}_0^Z$ and $\varphi' \in \mathfrak{g}_{>0}^Z$. Assume that φ is conjugate to $\varphi + \varphi'$ by $G(\mathbb{C})$. Then there exists $v \in \text{Exp}(\mathfrak{g}_{>0}^Z)$ such that $v(\varphi) = \varphi + \varphi'$.*

Proof. Decompose $\varphi' = \sum_{i=1}^k \varphi'_i$ where $\varphi'_i \in \mathfrak{g}_{\lambda_i}^Z$ and $\lambda_1 < \lambda_2 < \dots < \lambda_k$ are all the positive eigenvalues of Z . We prove the corollary by descending induction on the maximal index i such that $\varphi' \in \mathfrak{g}_{>\lambda_i}^Z$. The base case $i = k$ is obvious. For the induction step, let $i < k$ such that $\varphi' \in \mathfrak{g}_{>\lambda_i}^Z$ and let X be as in Lemma 4.4.1. Note that X is nilpotent. Then $\exp(X)(\varphi + \varphi') = \varphi + \psi$, where $\psi \in \mathfrak{g}_{>\lambda_{i+1}}^Z$. By the induction hypothesis, $\varphi + \psi \in \text{Exp}(\mathfrak{g}_{>0}^Z)\varphi$. \square

In other words, $\varphi + \varphi'$ can be conjugated to φ using a unipotent conjugation.

In the same way, but using Corollary 4.4.2 in place of Lemma 4.4.1, one proves the following more elaborate version of this corollary.

Corollary 4.4.4. *Let $S, Z \in \mathfrak{g}$ be rational semi-simple commuting elements, let $\varphi \in \mathfrak{g}_0^Z \cap \mathfrak{g}_{-2}^S$ and $\varphi' \in \mathfrak{g}_{>0}^Z \cap \mathfrak{g}_{-2}^S$. Assume that φ is conjugate to $\varphi + \varphi'$ by $G(\mathbb{C})$. Then there exists $v \in \text{Exp}(\mathfrak{g}_{>0}^Z \cap \mathfrak{g}_0^S)$ such that $v(\varphi) = \varphi + \varphi'$.*

Corollary 4.4.5. *The relation defined in Definition 2.2.1 is indeed an order relation.*

Proof. We have to show that if \mathcal{O}'' is bigger than \mathcal{O} then \mathcal{O} cannot be bigger than \mathcal{O}'' . Suppose the contrary. Then by Lemma 2.2.2 the complexifications $\mathcal{O}''_{\mathbb{C}}$ and $\mathcal{O}_{\mathbb{C}}$ coincide. Moreover, because of the above assumption there exist a rational semi-simple $Z \in \mathfrak{g}$, $\varphi \in \mathcal{O} \cap \mathfrak{g}_0^Z$, and $\psi \in \mathfrak{g}_{>0}^Z$ such that $\varphi + \psi \in \mathcal{O}_{\mathbb{C}}$, but $\varphi + \psi \notin \mathcal{O}$. This contradicts Corollary 4.4.3. \square

Lemma 4.4.6 ([Hum78, Proposition II.8.3]). *Assume that G is split, and fix a maximal split torus T . Let \mathfrak{h} be the Lie algebra of T , α be a root, and let $\varphi \in \mathfrak{g}_{-\alpha}^\times$. Define $h_\alpha \in \mathfrak{h}$ by*

$$\beta(h_\alpha) = 2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}. \quad (4.15)$$

Then (h_α, φ) is a neutral pair.

Note that if \mathfrak{g} is simply-laced and $\beta \neq \pm\alpha$ then $\beta(h_\alpha) \in \{-1, 0, 1\}$.

5. SMALL AUTOMORPHIC FUNCTIONS ON SIMPLY-LACED LIE ALGEBRAS

For the whole section we assume that G is split and the Dynkin diagram of \mathfrak{g} is simply-laced, i.e. all the connected components have types A , D , or E . As in §1.2, let, for any root δ , \mathfrak{g}_δ^* denote the corresponding root-subspace of \mathfrak{g}^* and $\mathfrak{g}_\delta^\times$ the set of non-zero elements of this subspace.

Lemma 5.0.1. *If $[\mathfrak{g}, \mathfrak{g}]$ is simple then any two roots are Weyl-conjugate.*

Proof. Any root is Weyl-conjugate to a simple root, and any two simple roots in a connected simply-laced diagram are Weyl-conjugate. \square

Corollary 5.0.2. *For any root δ , any $\varphi \in \mathfrak{g}_\delta^\times$ lies in a minimal orbit.*

Corollary 5.0.3. *Assume that $[\mathfrak{g}, \mathfrak{g}]$ is simple of type A or E . Then any two pairs of orthogonal roots are Weyl-conjugate.*

Proof. By Lemma 5.0.1, we can assume that both pairs include the highest root. Since the diagram consisting of roots orthogonal to the highest one is still connected, the stabilizer of the highest root acts transitively on it. \square

5.1. Proof of Theorem C. Throughout the subsection fix a simple root α . Define $S_\alpha \in \mathfrak{h}$ by $\alpha(S_\alpha) = 2$ and $\gamma(S_\alpha) = 0$ for any other simple root γ . Also, define $h_\alpha \in \mathfrak{h}$ by $\gamma(h_\alpha) = 2\langle \alpha, \gamma \rangle / \langle \alpha, \alpha \rangle$ as in (4.15) for any root γ . Note that by Lemma 4.4.6, (h_α, φ) is a neutral Whittaker pair for any $\varphi \in \mathfrak{g}_{-\alpha}^\times$.

As mentioned in the introduction, if a Fourier coefficient $\mathcal{F}_{S, \varphi}$ is a Whittaker coefficient, i.e. $N_{S, \varphi}$ is the unipotent radical of a Borel subgroup, we will denote it by $\mathcal{W}_{S, \varphi}$, where we may drop the S if it corresponds to a fixed choice of Borel subgroup and simple roots. In other words, we define $S_B \in \mathfrak{h}$ by $S_B(\gamma) = 2$ for any simple root γ and write $\mathcal{W}_{S_B, \varphi} = \mathcal{W}_\varphi$.

Lemma 5.1.1. *If η is a minimal automorphic function and $\varphi \in \mathfrak{g}_{-\alpha}^\times$, then*

- (i) $\mathcal{F}_{S_\alpha, \varphi}[\eta] = \mathcal{W}_\varphi[\eta]$.
- (ii) Let $\bar{\mathfrak{b}}$ denote the opposite of the standard Borel subalgebra, and let

$$W := \text{Exp}(\mathfrak{g}_{\geq 1}^{h_\alpha} \cap \bar{\mathfrak{b}}). \quad (5.1)$$

Then

$$\mathcal{F}_{h_\alpha, \varphi}[\eta](g) = \sum_{w \in W(\mathbb{K})} \mathcal{W}_\varphi[\eta](wg).$$

Proof. We will use Theorem B. We have $\mathfrak{g}_1^S = \{0\} = \mathfrak{g}_{\geq 1}^{S_\alpha} \cap \mathfrak{g}_{< 1}^S$, which implies (i). For (ii) we note in addition that $\mathfrak{g}_{> 1}^{h_\alpha} = \mathfrak{g}_{\geq 2}^{h_\alpha} = \mathfrak{g}_2^{h_\alpha} = \mathfrak{g}_\alpha$ and thus $\mathfrak{g}_{> 1}^{h_\alpha} \cap \mathfrak{g}_{< 1}^S = \{0\}$. Finally, $\mathfrak{g}_{< 1}^S = \bar{\mathfrak{b}}$. \square

Let L_α denote the Levi subgroup of the parabolic subgroup P_α of G . Denote by M_α the stabilizer in L_α of the space \mathfrak{g}_α^* (as an element of the projective space of \mathfrak{g}^*).

Lemma 5.1.2. *Any root δ with $\delta(S_\alpha) = -2$ can be conjugated to $-\alpha$ using the Weyl group of L_α .*

Proof. We can assume that \mathfrak{g} is simple. This statement can be proved using the language of minuscule representations, i.e., representations such that the Weyl group has a single orbit on the weights of the representations. These are given for example in [Bou75]. We thus need to show that all representations of the Levi L_α for a simple root α of a simply-laced root system are minuscule when acting on the first internal Chevalley module $U_\alpha/[U_\alpha, U_\alpha]$. This can be done by inspection case-by-case.

Case A_n : For any simple root α , the semi-simple part of L_α is of Cartan type A (if the root α is at the end of the Dynkin diagram) or of type AA (when α is in the middle). In both cases the first internal Chevalley module is a fundamental representation (type A) or a product of two fundamental representations (type AA). Since vector representations of type A are minuscule, the claim is true.

Case D_n : Depending on which simple root α one chooses, the internal Chevalley modules are exterior powers of fundamental representations of type A factors in L_α , or fundamental representation or spinor representations of type D factors in L_α . All the representations are minuscule.

Case E_6 : All Levi L_α of E_6 are of Cartan types A or D or products thereof (up to abelian factors). The representations arising as internal Chevalley modules are all minuscule by inspection as they are exterior powers of type A fundamental representations or spinor representations of type D.

Case E_7 : There is one new case beyond the representation types above. In the new case the Levi contains E_6 and the representation of E_6 that arises in the internal Chevalley module is the 27-dimensional one. This is also a minuscule representation by inspection.

Case E_8 : There is again only one new case to consider when the Levi L_α contains the factor E_7 . The E_7 representation arising in the internal Chevalley module is the 56-dimensional one which is minuscule as well. \square

Corollary 5.1.3. *The set of minimal elements in $(\mathfrak{g}^*)_{-2}^{S_\alpha}$ is $L_\alpha(\mathbb{K})(\mathfrak{g}_{-\alpha}^\times)$.*

Proof. Let z be a generic element of \mathfrak{h} that is 0 on α and negative on other positive roots. Decompose $(\mathfrak{g}^*)_{-2}^{S_\alpha} = \bigoplus_{i=0}^k V_k$ by eigenvectors of z , with eigenvalues $0 = t_0 < t_1 < \dots < t_k$. Note that $V_0 = \mathfrak{g}_{-\alpha}^*$. Let $X \in (\mathfrak{g}^*)_{-2}^{S_\alpha}$ be a minimal element and $X = \sum_i X_i$ its decomposition by eigenvalues of z . By Lemma 5.1.2, we can assume, by replacing X by its $L_\alpha(\mathbb{K})$ -conjugate, that $X_0 \neq 0$. By Corollary 4.4.4 X is conjugate to X_0 using L_α . \square

Proof of Theorem C. Part (ii) follows from Proposition 4.0.1 and the minimality of η .

Part (i) follows from Lemma 5.1.1(i), Lemma 3.3.1 and Corollary 5.1.3. \square

5.2. Proof of Theorem D. Let η be a minimal automorphic function. For any simple root α denote by m_α the maximal multiplicity of α in other roots. As above, let L_α be the Levi subgroup of P_α and M_α the stabilizer of \mathfrak{g}_α^* in L_α .

Proposition 5.2.1. *Let α be a simple root with $m_\alpha = 1$. Then*

$$\eta(g) = \mathcal{F}_{S_{\alpha,0}}[\eta](g) + \sum_{\gamma \in L_\alpha(\mathbb{K})/M_\alpha(\mathbb{K})} \sum_{\varphi \in \mathfrak{g}_{-\alpha}^\times} \mathcal{W}_\varphi[\eta](\gamma g). \quad (5.2)$$

Proof. Since $m_\alpha = 1$, the group U_α is abelian. Decompose η to Fourier series on U_α . The coefficients in the Fourier series will be given by $\mathcal{F}_{S_{\alpha,\varphi'}}[\eta]$ with $\varphi' \in (\mathfrak{g}^*)_{-2}^{S_\alpha}$. Note that this coefficient vanishes unless φ' is minimal or zero, and that by Corollary 5.1.3, all minimal $\varphi' \in (\mathfrak{g}^*)_{-2}^{S_\alpha}$ can be conjugated into $\mathfrak{g}_{-\alpha}^\times$ using $L_\alpha(\mathbb{K})$. Thus we have

$$\eta(g) = \sum_{\varphi' \in (\mathfrak{g}^*)_{-2}^{S_\alpha}} \mathcal{F}_{S_{\alpha,\varphi'}}[\eta](g) = \mathcal{F}_{S_{\alpha,0}}[\eta](g) + \sum_{\gamma \in L_\alpha(\mathbb{K})/M_\alpha(\mathbb{K})} \sum_{\varphi \in \mathfrak{g}_{-\alpha}^\times} \mathcal{F}_{S_{\alpha,\varphi}}[\eta](\gamma g) \quad (5.3)$$

Lemma 5.1.1(i) and the minimality of η imply that $\mathcal{F}_{S_{\alpha,\varphi}}[\eta](\gamma g) = \mathcal{W}_\varphi[\eta](\gamma g)$. \square

Lemma 5.2.2. *Let $\mathfrak{l} \subset \mathfrak{g}$ be a \mathbb{K} -Levi subalgebra, and let \mathcal{O} be the minimal nilpotent orbit in \mathfrak{g} . Then $\mathcal{O} \cap \mathfrak{l}$ is either empty or the minimal orbit of \mathfrak{l} .*

Proof. Suppose the contrary. Let $\mathcal{O}_\mathfrak{l}$ denote the minimal orbit of \mathfrak{l} . Then $\mathcal{O}_\mathfrak{l}$ lies in the Zariski closure of $\mathcal{O} \cap \mathfrak{l}$. Thus there exists an \mathfrak{sl}_2 triple (e, h, f) in \mathfrak{l} such that $f \in \mathcal{O}_\mathfrak{l}$, and the Slodowy slice $f + \mathfrak{l}^e$ to $\mathcal{O}_\mathfrak{l}$ at f intersects \mathcal{O} . Namely, there exists a non-zero $X \in \mathfrak{l}^e$ with $f + X \in \mathcal{O}$. This contradicts the minimality of \mathcal{O} , since $f + \mathfrak{l}^e$ is transversal to the orbit of f . \square

Lemma 5.2.3. *For any simple root α , the restriction $\mathcal{F}_{S_{\alpha,0}}[\eta]|_{L_\alpha}$ is a minimal or a trivial automorphic function of L_α .*

Proof. Suppose that there exists a Whittaker pair (H, φ) for L_α with $\varphi \neq 0$ such that $\mathcal{F}_{H,\varphi}[\mathcal{F}_{S_{\alpha,0}}[\eta]] \neq 0$. Then, for T big enough, we have $\mathcal{F}_{H,\varphi}[\mathcal{F}_{S_{\alpha,0}}[\eta]] = \mathcal{F}_{H+TS_{\alpha,\varphi}}[\eta]$. Thus, the orbit of φ is minimal in \mathfrak{g}^* and thus, by Lemma 5.2.2 also in \mathfrak{l}_α^* . \square

Proof of Theorem D. The proof is by induction on the rank of G . The base case of rank 1 group is the classical Fourier series decomposition. For the induction step assume first that \mathfrak{g} does not have any simple components of type E_8 . Note that in this case there exists an extreme simple root α with $m_\alpha = 1$. By Proposition 5.2.1 we have

$$\eta(g) = \mathcal{F}_{S_{\alpha,0}}[\eta](g) + \sum_{\gamma \in L_\alpha(\mathbb{K})/M_\alpha(\mathbb{K})} \sum_{\varphi \in \mathfrak{g}_\alpha^\times} \mathcal{W}_\varphi[\eta](\gamma g) \quad (5.4)$$

By Lemma 5.2.3, $\mathcal{F}_{S_{\alpha,0}}[\eta]$ is in a minimal representation of L_α .

As before, let $S_B \in \mathfrak{h}$ denote the element that is 2 on all positive roots. Note that for any $\varphi \in (\mathfrak{l}_\alpha^*)_{-2}^{S_B}$, we have $\mathcal{W}'_\varphi[\mathcal{F}_{S_{\alpha,0}}[\eta]] = \mathcal{W}_\varphi[\eta]$ where the prime denotes a Whittaker coefficient with respect to L_α .

Enumerate the roots such that α is the last one. For any $1 \leq i \leq \text{rk}(G)$ denote by L_i the Levi subgroup given by the simple roots $\alpha_1, \dots, \alpha_{i-1}$, and by M_i the stabilizer in L_i of the space $\mathfrak{g}_{\alpha_i}^*$ (as an element of the projective space of \mathfrak{g}^*).

From the induction hypothesis we obtain

$$\eta(g) = \mathcal{W}_0[\eta](g) + \sum_{i=1}^{\text{rk}(G)} \sum_{\gamma \in L_i(\mathbb{K})/M_i(\mathbb{K})} \sum_{\varphi \in \mathfrak{g}_{-\alpha_i}^\times} \mathcal{W}_\varphi[\eta](\gamma g) \quad (5.5)$$

Let us now deal with the remaining case in which \mathfrak{g} has a simple component of type E_8 . Fix such a component and let α denote the 8-th root of this component in the Bourbaki labeling, and α_{\max} denote the highest root of this component. Then $m_\alpha = 2$, and $(\mathfrak{g}^*)_{-4}^{S_\alpha} = \mathfrak{g}_{-\alpha_{\max}}^*$. Recall that $\mathcal{F}_{0,0}[\eta] = \eta$. We now make the following deformation. Let $H_t := tS_\alpha$, for any rational $t \in [0, 1]$. Then the only critical values are $1/4$ and $1/2$ and the only quasi-critical values at eigenvalue 2 are $1/2$ and 1. Thus we get

$$\eta(g) = \mathcal{F}_{S_\alpha/4,0}[\eta](g) + \sum_{\varphi' \in \mathfrak{g}_{-\alpha_{\max}}^\times} \mathcal{F}_{S_\alpha/4,0,\varphi'}[\eta](g) \quad (5.6)$$

It is easy to see from the definitions that, with φ' non-zero, $\mathcal{F}_{S_\alpha/4,0,\varphi'}[\eta] = \mathcal{F}_{S_\alpha/2,\varphi'}[\eta]$. Note also that, by Lemma 4.4.6, $(S_\alpha/2, \varphi')$ is a neutral Whittaker pair. Conjugating α_{\max} to α using the normalizer of the Cartan, we reduce the computation of $\mathcal{F}_{S_\alpha/2,\varphi'}[\eta]$ to the formula for $\mathcal{F}_{h_\alpha,\varphi'}[\eta]$ given in Lemma 5.1.1(ii).

For $\mathcal{F}_{S_\alpha/4,0}[\eta]$, we proceed as in Proposition 5.2.1 and obtain

$$\mathcal{F}_{S_\alpha/4,0}[\eta](g) = \mathcal{F}_{S_\alpha,0}[\eta](g) + \sum_{\gamma \in L_\alpha(\mathbb{K})/M_\alpha(\mathbb{K})} \sum_{\varphi \in \mathfrak{g}_{-\alpha}^\times} \mathcal{W}_\varphi[\eta](\gamma g) \quad (5.7)$$

For the constant term $\mathcal{F}_{S_\alpha,0}[\eta]$ we obtain a formula from the induction hypothesis. Altogether, we get

$$\eta(g) = \mathcal{W}_0[\eta](g) + \sum_{i=1}^{\text{rk}(G)} \sum_{\gamma \in L_i(\mathbb{K})/M_i(\mathbb{K})} \sum_{\varphi \in \mathfrak{g}_{-\alpha_i}^\times} \mathcal{W}_\varphi[\eta](\gamma g) + \sum_{j=1}^k \sum_{\varphi \in \mathfrak{g}_{-\alpha_8^j}^\times} \sum_{w \in W_j(\mathbb{K})} \mathcal{W}_\varphi[\eta](s_j w g), \quad (5.8)$$

where α_8^j is the 8-th root of E_8^j , the j -th E_8 -component of G , in the Bourbaki labeling; s_j is a representative of a Weyl group element conjugating the highest root of E_8^j to α_8^j , and W_j are as in (5.1). \square

5.3. Proof of Theorem E. Suppose that $\text{rk}(\mathfrak{g}) > 2$. Let η be a next-to-minimal automorphic function. Let α be a simple root and let $\varphi' \in \mathfrak{g}_{-\alpha}^\times$.

Lemma 5.3.1. *Let $\gamma \neq \alpha$ be a positive root, and let $\psi \in \mathfrak{g}_{-\gamma}^\times$. Let \mathcal{O} denote the orbit of $\psi + \varphi'$. Then the possible values of $\langle \alpha, \gamma \rangle$ are 1, 0, or -1 , and \mathcal{O} has Bala-Carter label \mathfrak{l} , where \mathfrak{l} is a Levi subalgebra of type A_1 , $A_1 \times A_1$, or A_2 , respectively.*

In particular, \mathcal{O} is minimal if $\langle \alpha, \gamma \rangle > 0$, \mathcal{O} is next to minimal if $\langle \alpha, \gamma \rangle = 0$ and \mathcal{O} is neither minimal nor next to minimal if $\langle \alpha, \gamma \rangle < 0$.

Proof. Let $\mathfrak{h}' \subset \mathfrak{h}$ be the simultaneous kernel of α and γ , and let \mathfrak{l} be its centralizer in \mathfrak{g} . Then \mathfrak{h}' has codimension at most 2 in \mathfrak{h} , hence \mathfrak{l} is a Levi subalgebra of semisimple

rank ≤ 2 whose roots include α and γ . The rest of the lemma is a straightforward rank 2 calculation. \square

Notation 5.3.2. Denote by Δ_α the set of simple roots orthogonal to α . Define $S \in \mathfrak{h}$ to be 0 on any simple root $\delta \in \Delta_\alpha$, and 2 on other simple roots. Let $Z := S_\alpha - h_\alpha$. Note that Z vanishes on simple roots in Δ_α and on α and is 1 on other simple roots.

Proposition 5.3.3. *We have $\mathcal{F}_{S_\alpha, \varphi'}[\eta] = \mathcal{F}_{S, \varphi'}[\eta]$.*

Proof. Note that S_α dominates S , and that $\mathfrak{g}_1^{S_\alpha} = \mathfrak{g}_1^S = \mathfrak{g}_{>1}^{S_\alpha} \cap \mathfrak{g}_{<1}^S = \{0\}$. Thus the statement follows from Theorem B. \square

Let $G' \subset G$ be the Levi subgroup given by Δ_α .

Proposition 5.3.4. *The restriction $\mathcal{F}_{S, \varphi'}[\eta]|_{G'}$ is a minimal or a constant automorphic function on G' .*

For the proof we will need the following geometric lemma.

Lemma 5.3.5. *Let $\psi \in \mathfrak{g}^{*}$ be nilpotent such that $\varphi' + \psi$ belongs to a next-to-minimal orbit in \mathfrak{g}^* . Then ψ belongs to the minimal orbit of \mathfrak{g}^{*} .*

Proof. Clearly $\psi \neq 0$. If the orbit of ψ is not minimal then it belongs to the Slodowy slice of some element ψ' of the minimal orbit of $(\mathfrak{g}')^*$. Then $\varphi' + \psi'$ belongs to a next-to-minimal orbit of \mathfrak{g}^* , and $\varphi' + \psi$ belongs to the Slodowy slice of $\varphi' + \psi'$ and thus lies in an orbit that is higher than next-to-minimal. \square

Proof of Proposition 5.3.4. Suppose that there exists a Whittaker pair (H, ψ) with $\psi \neq 0$ such that $\mathcal{F}_{H, \psi}[\mathcal{F}_{S, \varphi'}[\eta]] \neq 0$. Then, for T big enough, we have $\mathcal{F}_{H, \psi}[\mathcal{F}_{S, \varphi'}[\eta]] = \mathcal{F}_{S+TZ+H, \varphi'+\psi}[\eta]$. By Proposition 4.0.1 and Lemma 5.3.5, ψ lies in the minimal orbit of \mathfrak{g}^{*} . \square

Proof of Theorem E. Part (iii) follows from Proposition 4.0.1, since η is a next-to-minimal form.

For part (ii), let $z \in \mathfrak{h}_\varphi$ be a rational semi-simple element such that $\mathfrak{g}_1^{S_\alpha+z} = 0$, $\mathfrak{g}_{>1}^{S_\alpha+z}$ is the nilpotent radical of a Borel subalgebra \mathfrak{b} of \mathfrak{g} , and (S_α, φ) dominates $(S_\alpha + z, \varphi)$. Such z exists by Lemmas 4.0.7, 4.0.8 and 2.1.9, since the next-to-minimal orbit is PL. Conjugating \mathfrak{b} to the standard one, and applying Theorem B(i) we obtain the statement. In §6 we give explicit examples for Z (or rather conjugations) that minimize the dimension of V .

For part (i), note that Corollary 5.1.3 implies that, conjugating by $\gamma \in L_\alpha(\mathbb{K})$, we can assume that $\varphi \in \mathfrak{g}_{<1}^\times$. By Proposition 5.3.3, $\mathcal{F}_{S_\alpha, \varphi}[\eta] = \mathcal{F}_{S, \varphi}[\eta]$. By Proposition 5.3.4, $\mathcal{F}_{S, \varphi}[\eta]|_{G'}$ is a minimal or constant form on G' . The statement follows now from Theorem D together with the fact that the G' -Whittaker coefficients $\mathcal{W}'_\psi[\eta']$ where $\eta'(g') = \mathcal{F}_{S, \varphi}[\eta](g')$ are, in fact, G -Whittaker coefficients $\mathcal{W}_{\varphi+\psi}[\eta]$ due to the extra integral in $\mathcal{F}_{S, \varphi}[\eta]$. \square

5.4. Expressing the form itself through Whittaker coefficients. Let η be a next-to-minimal automorphic function. If \mathfrak{g} has a component which is not of type E_8 then there exists a maximal parabolic P_α with an abelian nilradical. Using Fourier transform on this nilradical we obtain

$$\eta(g) = \mathcal{F}_{S_\alpha,0}[\eta](g) + \sum_{\text{minimal } \varphi \in (\mathfrak{g}^*)_{-2}^{S_\alpha}} \mathcal{F}_{S_\alpha,\varphi}[\eta](g) + \sum_{\text{ntm } \varphi \in (\mathfrak{g}^*)_{-2}^{S_\alpha}} \mathcal{F}_{S_\alpha,\varphi}[\eta](g) \quad (5.9)$$

Theorem E provides the expressions for all the terms on the right-hand side except the constant term. Similarly to Lemma 5.2.3, one shows that the restriction of the constant term $\mathcal{F}_{S_\alpha,0}[\eta](g)$ to the Levi subgroup L_α is next-to-minimal or minimal or constant. Using induction on the rank of G we can obtain an expression for this constant term in terms of Whittaker coefficients.

Suppose now that \mathfrak{g} is a product of components of type E_8 , and let α be the 8th root α_8 of one of the components. As in the proof of Theorem D, we have

$$\eta(g) = \mathcal{F}_{S_\alpha/4,0}[\eta](g) + \sum_{\varphi' \in \mathfrak{g}_{-\alpha_{\max}}^\times} \mathcal{F}_{S_\alpha/4,0,\varphi'}[\eta](g) \quad (5.10)$$

For $\mathcal{F}_{S_\alpha/4,0}[\eta]$, we have

$$\mathcal{F}_{S_\alpha/4,0}[\eta](g) = \mathcal{F}_{S_\alpha,0}[\eta](g) + \sum_{\text{minimal } \varphi \in (\mathfrak{g}^*)_{-2}^{S_\alpha}} \mathcal{F}_{S_\alpha,\varphi}[\eta](g) + \sum_{\text{ntm } \varphi \in (\mathfrak{g}^*)_{-2}^{S_\alpha}} \mathcal{F}_{S_\alpha,\varphi}[\eta](g) \quad (5.11)$$

The last terms in (5.10) can be evaluated as follows. As in the proof of Theorem D, we have, for $\varphi' \in \mathfrak{g}_{-\alpha_{\max}}^\times$, that $\mathcal{F}_{S_\alpha/4,0,\varphi'}[\eta] = \mathcal{F}_{S_\alpha/2,\varphi'}[\eta]$. We conjugate $\mathcal{F}_{S_\alpha/2,\varphi'}[\eta]$ to $\mathcal{F}_{h_\alpha,\varphi}[\eta]$ with $\varphi \in \mathfrak{g}_{-\alpha}^\times$. Then, we consider the deformation $(1-t)h_\alpha + tS_\alpha$. First we express $\mathcal{F}_{h_\alpha,\varphi}[\eta]$ through $\mathcal{F}_{h_\alpha,\varphi}^R[\eta]$ using summation over $W(\mathbb{K})$, where $W(\mathbb{K})$ is as in Lemma 5.1.1(ii).

Then, the critical values are $1/2$ and $2/3$, and the quasi-critical values at eigenvalue 2 are $1/3$ and 1 . At $1/3$, only one root joins, namely α_{\max} , but we have no Whittaker triple entries in $(\mathfrak{g}^*)_{-\alpha_{\max}}$ or otherwise that could move into the -2 -eigenspace. At $1/2$, we get all the roots from the set

$$\Phi_\alpha := \left\{ \sum_{i=1}^6 c_i \alpha_i + 2\alpha_7 + \alpha_8 \right\} = \{ \varepsilon \text{ positive root} \mid \langle \varepsilon, \alpha \rangle = 0, S_\alpha(\varepsilon) = 2 \} \quad (5.12)$$

containing 27 roots, and at $t = 2/3$ we also get the root $\delta + \alpha_7$. This means that we would get contributions from all these root spaces in the third component of the Whittaker triple. Let $\bar{\Phi}_\alpha = \Phi_\alpha \cup \{ \delta + \alpha_7 \}$.

At $t = 1$ all these 28 roots join $\alpha = \alpha_8$ and thus we obtain, for any $\varphi \in \mathfrak{g}_{-\alpha}^\times$,

$$\mathcal{F}_{h_\alpha,\varphi}^R[\eta](g) = \mathcal{F}_{S_\alpha,\varphi}[\eta](g) + \sum_{\text{ntm } \psi \in \varphi + \bigoplus_{\varepsilon \in \bar{\Phi}_\alpha} \mathfrak{g}_{-\varepsilon}^*} \mathcal{F}_{S_\alpha,\psi}[\eta](g) \quad (5.13)$$

As discussed above, we already have formulas for all the expressions in the right hand side. This finishes the proof of (i). To prove the rest of the theorem we will now group the expressions in the above formulas.

Let $T \subset G$ be the split torus corresponding to our fixed Cartan subalgebra \mathfrak{h} .

Lemma 5.4.1. *For any simple root α , $T(\mathbb{K})$ acts transitively on $\mathfrak{g}_\alpha^\times$.*

Proof. It is enough to consider \mathfrak{g} to be simple and simply-laced with $\text{rk } \mathfrak{g} \geq 2$, and to find a coroot β^\vee for \mathfrak{g} such that $\beta^\vee(\alpha) = -1$. This is satisfied by any β^\vee such that β is a simple root adjacent to α in the Dynkin diagram. \square

Remark 5.4.2. Note that one may also use similar rescaling to simplify some of the expressions in Theorems **D** and **E** whenever we have the necessary degrees of freedom.

For types A_n and E_n , let \mathcal{O}_{ntm} denote the only complex next-to-minimal orbit, and, for type D_n , let it denote one of the two next-to-minimal orbits.

Notation 5.4.3. For the remainder of this section we denote by α be the extreme root given in the Bourbaki labeling by: $\alpha = \alpha_n$ in types A_n, E_n , and $\alpha = \alpha_1$ in type D_n if $\alpha_1 + \alpha_{\max} \notin \mathcal{O}_{\text{ntm}}$ and α_n if $\alpha_1 + \alpha_{\max} \in \mathcal{O}_{\text{ntm}}$. Denote by $X := \mathcal{O}_{\text{ntm}} \cap (\mathfrak{g}^*)_{-2}^{S_\alpha}$. If \mathfrak{g} is of type E_n we let δ be the highest root with $\delta(S_\alpha) = 2$ and $\langle \alpha, \delta \rangle = 0$, i.e. δ is α_{\max} except for E_8 where it is $\delta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7 + \alpha_8$.

Proposition 5.4.4. *If \mathfrak{g} is of type A_n or D_n then $X = \emptyset$. If \mathfrak{g} is of type E_n then L_α acts transitively on X .*

The complex version of this proposition follows from [MS12, §5]. This implies the proposition in types A_n and D_n . We prove the case of E_n in §5.6 below.

For any root ε denote by Γ_ε the quotient $L_\varepsilon(\mathbb{K})/M_\varepsilon(\mathbb{K})$. Fix $\varphi_0 \in \mathfrak{g}_{-\alpha}^\times$ and let Γ_{φ_0} denote the quotient of $L_\alpha(\mathbb{K})$ by the stabilizer of φ_0 in $L_\alpha(\mathbb{K})$. In type D_n we assume that $\text{WS}(\eta) = \mathcal{O}_{\text{ntm}} \cap \mathfrak{g}^*$. From Corollary 5.1.3, Lemma 5.4.1, Proposition 5.4.4, and (5.9) we obtain for A_n and D_n :

$$\begin{aligned} \eta(g) &= \mathcal{F}_{S_\alpha, 0}[\eta](g) + \sum_{\tilde{\gamma} \in \Gamma_{\varphi_0}} \mathcal{F}_{S_\alpha, \varphi_0}[\eta](\tilde{\gamma}g) \\ &= \mathcal{F}_{S_\alpha, 0}[\eta](g) + \sum_{\tilde{\gamma} \in \Gamma_{\varphi_0}} \left(\mathcal{W}_{\varphi_0}[\eta](\tilde{\gamma}g) + \sum_{i \in I} \sum_{\gamma \in \Gamma_i} \sum_{\psi \in \mathfrak{g}_{-\alpha_i}^\times} \mathcal{W}_{\varphi_0 + \psi}[\eta](\gamma\tilde{\gamma}g) \right) \end{aligned} \quad (5.14)$$

The second equality follows from Theorem **E**.

For E_n , fix $\psi_0 \in \mathfrak{g}_{-\delta}^\times$ with δ as in Notation 5.4.3 and denote by Λ the quotient of $L_\alpha(\mathbb{K})$ by the subgroup that stabilizes $\varphi_0 + \psi_0$. For E_6 and E_7 we have $\delta = \alpha_{\max}$ and we obtain from Corollary 5.1.3, Lemma 5.4.1, Proposition 5.4.4, and (5.9):

$$\eta(g) = \mathcal{F}_{S_\alpha, 0}[\eta](g) + \sum_{\tilde{\gamma} \in \Gamma_{\varphi_0}} \mathcal{F}_{S_\alpha, \varphi_0}[\eta](\tilde{\gamma}g) + \sum_{\gamma \in \Lambda} \mathcal{F}_{S_\alpha, \varphi_0 + \psi_0}[\eta](\gamma g) \quad (5.15)$$

Using Theorem **E** we rewrite this as

$$\begin{aligned} \eta(g) &= \mathcal{F}_{S_\alpha, 0}[\eta](g) + \sum_{\tilde{\gamma} \in \Gamma_{\varphi_0}} \left(\mathcal{W}_{\varphi_0}[\eta](\tilde{\gamma}g) + \sum_{i \in I} \sum_{\gamma \in \Gamma_i} \sum_{\psi \in \mathfrak{g}_{-\alpha_i}^\times} \mathcal{W}_{\varphi_0 + \psi}[\eta](\gamma\tilde{\gamma}g) \right) \\ &\quad + \sum_{\tilde{\gamma} \in \Gamma_{\varphi_0}} \sum_{\gamma \in \Lambda} \int_{V(\mathbb{A})} \mathcal{W}_{\varphi_0 + \psi_0}[\eta](v\gamma\tilde{\gamma}g) dv \end{aligned} \quad (5.16)$$

One can also obtain expressions for E_8 in the same way. From formulas (5.10, 5.11) and the discussion after them, we have

$$\eta(g) = \mathcal{F}_{S_\alpha, 0}[\eta](g) + \sum_{\text{minimal } \varphi \in (\mathfrak{g}^*)_{-2}^{S_\alpha}} \mathcal{F}_{S_\alpha, \varphi}[\eta](g) + \sum_{\text{ntm } \varphi \in (\mathfrak{g}^*)_{-2}^{S_\alpha}} \mathcal{F}_{S_\alpha, \varphi}[\eta](g) + \sum_{\varphi \in \mathfrak{g}_{-\alpha}^\times} \mathcal{F}_{h_\alpha, \varphi}[\eta](sg), \quad (5.17)$$

where s is a representative of a Weyl group element that conjugates α_{\max} to α , and

$$\mathcal{F}_{h_\alpha, \varphi}[\eta](sg) = \sum_{w \in W(\mathbb{K})} \left(\mathcal{F}_{S_\alpha, \varphi}[\eta](wsg) + \sum_{\text{ntm } \psi \in \varphi + \bigoplus_{\varepsilon \in \overline{\Phi}_\alpha} \mathfrak{g}_{-\varepsilon}^*} \mathcal{F}_{S_\alpha, \psi}[\eta](wsg) \right), \quad (5.18)$$

where $W(\mathbb{K})$ is as in Lemma 5.1.1(ii).

Fix $\varphi_0 \in \mathfrak{g}_{-\alpha}^\times$ and $\psi_0 \in \mathfrak{g}_{-\delta}^\times$, where $\delta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7 + \alpha_8$ as in Notation 5.4.3. Denote by M the E_6 -type Levi subgroup generated by the roots $\alpha_1, \dots, \alpha_6$, and by \mathcal{M} the quotient of $M(\mathbb{K})$ by the centralizer of $\varphi_0 + \psi_0$.

Lemma 5.4.5 (See §5.6 below). *The group $M(\mathbb{K})$ acts transitively on the set of next-to-minimal elements in $\mathfrak{g}_{-\alpha}^\times + \bigoplus_{\varepsilon \in \overline{\Phi}_\alpha} \mathfrak{g}_{-\varepsilon}^*$.*

From this lemma and (5.18) we obtain

$$\sum_{\varphi \in \mathfrak{g}_{-\alpha}^\times} \mathcal{F}_{h_\alpha, \varphi}[\eta](sg) = \sum_{w \in W(\mathbb{K})} \left(\sum_{c \in \mathbb{K}^\times} \mathcal{F}_{S_\alpha, c\varphi_0}[\eta](wsg) + \sum_{\gamma \in \mathcal{M}} \mathcal{F}_{S_\alpha, \varphi_0 + \psi_0}[\eta](\gamma wsg) \right) \quad (5.19)$$

Using Corollary 5.1.3, Lemma 5.4.1 and Proposition 5.4.4 we obtain

$$\begin{aligned} \eta(g) &= \mathcal{F}_{S_\alpha, 0}[\eta](g) + \sum_{\tilde{\gamma} \in \Gamma_{\varphi_0}} \mathcal{F}_{S_\alpha, \varphi_0}[\eta](\tilde{\gamma}g) + \sum_{\tilde{\gamma} \in \Lambda} \mathcal{F}_{S_\alpha, \varphi_0 + \psi_0}[\eta](\tilde{\gamma}g) \\ &\quad + \sum_{w \in W(\mathbb{K})} \left(\sum_{c \in \mathbb{K}^\times} \mathcal{F}_{S_\alpha, c\varphi_0}[\eta](wsg) + \sum_{\gamma \in \mathcal{M}} \mathcal{F}_{S_\alpha, \varphi_0 + \psi_0}[\eta](\gamma wsg) \right), \end{aligned} \quad (5.20)$$

Using Theorem E we deduce from this

$$\begin{aligned} \eta(g) &= \mathcal{F}_{S_\alpha, 0}[\eta](g) + \sum_{\tilde{\gamma} \in \Gamma_{\varphi_0}} \left(\mathcal{W}_{\varphi_0}[\eta](\tilde{\gamma}g) + \sum_{i \in I} \sum_{\gamma \in \Gamma_i} \sum_{\psi \in \mathfrak{g}_{-\alpha_i}^\times} \mathcal{W}_{\varphi_0 + \psi}[\eta](\gamma \tilde{\gamma}g) \right) \\ &\quad + \sum_{\tilde{\gamma} \in \Lambda_V(\mathbb{A})} \int \mathcal{W}_{\varphi_0 + \psi_0}[\eta](v \tilde{\gamma}g) dv + \sum_{w \in W(\mathbb{K})} \left(\sum_{c \in \mathbb{K}^\times} \mathcal{W}_{c\varphi_0}[\eta](wsg) + \right. \\ &\quad \left. \sum_{c \in \mathbb{K}^\times} \sum_{i \in I} \sum_{\gamma \in \Gamma_i} \sum_{\psi \in \mathfrak{g}_{-\alpha_i}^\times} \mathcal{W}_{c\varphi_0 + \psi}[\eta](\gamma wsg) + \sum_{\tilde{\gamma} \in \mathcal{M}_V(\mathbb{A})} \int \mathcal{W}_{\varphi_0 + \psi_0}[\eta](v \tilde{\gamma}wsg) \right) \end{aligned} \quad (5.21)$$

5.5. Comparison with related results in the literature. Various works have determined similar Fourier coefficients of small representations in special cases and we now briefly compare our results to them, with a particular emphasis on the E_8 expansions.

We begin with the example of a minimal automorphic form η on E_8 with the expansion determined in (5.8) that was also studied by Ginzburg–Rallis–Soudry [GRS11] and by Kazhdan–Polishchuk [KP04]. In the former paper, the authors showed that the constant

term of η with respect to the center of the Heisenberg unipotent U of E_8 was given by a single Levi (i.e. E_7) orbit of a Fourier coefficient $\mathcal{F}_{\psi_{\alpha_8}}$ on U , where ψ_{α_8} is a character on U supported only on the single simple root α_8 . This corresponds precisely to the second term in equation (5.8), but we have taken one step further in determining $\mathcal{F}_{\psi_{\alpha_8}}$ explicitly in terms of maximally degenerate Whittaker coefficients $\mathcal{W}_\varphi[\eta]$.

In [KP04], the authors give an explicit form of the full non-abelian Fourier expansion of η with respect to U and our result (5.8) is perfectly consistent with theirs. Kazhdan and Polishchuk have, however, a different approach, where they first determine the local contributions (spherical vectors) to the Fourier coefficients and then assemble them together into a global automorphic functional. To connect the two results one must therefore evaluate the Whittaker coefficients in (5.8) and extract their contributions at each local place. For the abelian terms, this has in fact already been done in [GKP16] and by combining those results with ours one achieves perfect agreement with [KP04]. It remains to evaluate explicitly the non-abelian term in (5.8) and extract its Euler product. It would be of particular interest to see if one can reproduce the cubic phase in the spherical vectors of [KP04] in this way.

Next we turn to the Fourier expansion of an E_8 automorphic form in the next-to-minimal representation given in (5.21) that has been studied previously by Bossard–Pioline [BP17]. According to the discussion in §1.3 the decomposition in (5.21) corresponds to the decompactification limit and an expression for the abelian part of the Fourier expansion for the next-to-minimal spherical Eisenstein series on E_8 was given in [BP17, Eq. (3.15)] that we reproduce here for convenience

$$\begin{aligned}
\eta &= \mathcal{F}_{S_{\alpha,0}}[\eta] + 16\pi\xi(4)R^4 \sum_{\substack{\Gamma \in \Lambda_\alpha \\ \Gamma \times \Gamma = 0}} \sigma_8(\Gamma) \frac{K_4(2\pi R|Z(\Gamma)|)}{|Z(\Gamma)|^4} e^{2\pi i \langle \Gamma, a \rangle} \\
&+ 16\pi\xi(3)R \sum_{\substack{\Gamma \in \Lambda_\alpha \\ \Gamma \times \Gamma = 0}} \sigma_2(\Gamma) (\gcd \Gamma)^2 \eta_{\min}^{E_6} \frac{K_1(2\pi R|Z(\Gamma)|)}{|Z(\Gamma)|^3} e^{2\pi i \langle \Gamma, a \rangle} \\
&+ 16\pi R^{-5} \sum_{\substack{\Gamma \in \Lambda_\alpha \\ \Gamma \times \Gamma \neq 0, I_4'(\Gamma) = 0}} \sum_{n|\Gamma} n^{d+1} \sigma_3\left(\frac{\Gamma \times \Gamma}{n^2}\right) \frac{B_{5/2,3/2}(R^2|Z(\Gamma)|^2, R^2\sqrt{\Delta(\Gamma)})}{\Delta(\Gamma)^{3/4}} e^{2\pi i \langle \Gamma, a \rangle} + \dots
\end{aligned} \tag{5.22}$$

Here, explicit coordinates on $E_8/(\text{Spin}_{16}/\mathbb{Z}_2)$ adapted to the E_7 parabolic are used. Specifically, R is a coordinate for the GL_1 factor in the Levi and a denotes (axionic) coordinates on the 56-dimensional abelian part of the unipotent. Λ_α is a lattice in this 56-dimensional representation of E_7 and the coordinates on the E_7 factor of the Levi enter implicitly through the functions $Z(\Gamma)$ and $\Delta(\Gamma)$. We do not require their precise form for the present comparison. K_s denotes the modified Bessel function and $\eta_{\min}^{E_6}$ a spherical vector in the minimal representation of E_6 .

We now establish that (5.22) and (5.21) are compatible. The Fourier expansion in (5.22) is written in terms of sums over charges Γ in the integral lattice Λ_α in the 56-dimensional unipotent and thus resembles structurally (5.17) above as the space $(\mathfrak{g}^*)_{-2}^{S_\alpha}$ represents the space of characters on this unipotent. The Fourier mode for a ‘charge’ Γ is given by $e^{2\pi i \langle \Gamma, a \rangle}$ and is the character on $(\mathfrak{g})_2^{S_\alpha}$. Besides the constant term $\mathcal{F}_{S_{\alpha,0}}[\eta]$ there is a sum

over characters in the minimal and next-to-minimal orbits within $(\mathfrak{g}^*)_{-2}^{S_\alpha}$; the last term in our (5.17) is a non-abelian term that was not determined in [BP17].

Minimal characters correspond to charges Γ such that they satisfy the (rank-one) condition $\Gamma \times \Gamma = 0$ in the notation of [BP17] and looking at (5.22) we see that there are two contributions from such charges. These correspond exactly to the two terms in the parenthesis in the first line of our (5.21): The first term represents the purely minimal charges while the second term in our equation is the second line of (5.22) where a minimal charge is combined with a minimal automorphic form on E_6 . Expanding this minimal automorphic form on E_6 leads to Whittaker coefficients of the form $\mathcal{W}_{\varphi_0+\psi}$ as they are given in the second term in the parenthesis of the first line in (5.21). The sums over Γ_α and Γ_i in our expression correspond to the E_7 orbits of such charges Γ . The second line in our formula (5.21) containing a non-compact integral over Whittaker coefficient $\mathcal{W}_{\varphi_0+\psi}$ corresponds to the last line in (5.22) where a similar integrated Whittaker coefficient $B_{5/2,3/2}$ appears. The non-abelian terms in the last line of (5.21) have not been determined in [BP17] and are given by the ellipses in (5.22).

5.6. Proof of Proposition 5.4.4 for type E_n .

Notation 5.6.1. Denote by Ψ_α the set of all (positive) roots ε that satisfy $\varepsilon(S_\alpha) = 2$, and by Φ_α the set of all roots in Ψ_α that are orthogonal to α . Denote also $z := S_\alpha - h_\alpha$ and $\mathfrak{a} := (\mathfrak{l}_\alpha)_{<0}^z$, and $A := \text{Exp}(\mathfrak{a})$.

Note that $\mathfrak{a} = (\mathfrak{l}_\alpha)_{1^\alpha}^{h_\alpha}$ and $\mathfrak{g}_{-\alpha}^\times \subset (\mathfrak{g}^*)_{0}^z$.

Lemma 5.6.2. *Let $\varphi \in \mathfrak{g}_{-\alpha}^\times$ and $\psi \in (\mathfrak{g}^*)_{-2}^{S_\alpha} \cap (\mathfrak{g}^*)_{-1}^{h_\alpha} \subset (\mathfrak{g}^*)_{<0}^z$. Then there exists $v \in A$ such that $v(\varphi) = \varphi + \psi$.*

Proof. Case 1. $\psi \in \mathfrak{g}_{-\varepsilon}^*$ for some $\varepsilon \in \Psi_\alpha$:

By the assumption that $\psi \in (\mathfrak{g}^*)_{-1}^{h_\alpha}$ and Lemma 5.3.1, $\varphi + \psi$ is conjugate to φ over \mathbb{C} . By Corollary 4.4.4, there exists $v \in A$ such that $v(\varphi) = \varphi + \psi$.

Case 2. General:

We can assume $\psi \neq 0$. Let $H \in \mathfrak{h}$ be a generic element that has negative integer values on all positive roots. Note that $\mathfrak{a} \subset \mathfrak{g}_{>0}^H$. Decompose $\psi = \sum_{i>0} \psi_i$, where $\psi_i \in (\mathfrak{g}^*)_i^H$. We prove the lemma by descending induction on the minimal j for which $\psi_j \neq 0$. The base of the induction follows from Case 1. For the induction step, let j be minimal with $\psi_j \neq 0$. By Case 1, there exists $v_1 \in A$ with $v_1(\varphi) = \varphi - \psi_j$. Then $v_1(\varphi + \psi) = \varphi + \sum_{i>j} \psi'_i$, for some $\psi'_i \in (\mathfrak{g}^*)_i^H$. By the induction hypothesis, there exists $v_2 \in A$ such that $v_2(\varphi) = v_1(\varphi + \psi)$. Take $v := v_1^{-1}v_2$. □

Lemma 5.6.3. *The stabilizer of α in the Weyl group of L_α acts transitively on Φ_α .*

Proof. Note that the roots in $\varepsilon \in \Phi_\alpha$ are exactly the roots satisfying $\varepsilon(S_\beta) = 4$, where β is the only simple root not orthogonal to α . In other words, Φ_α is the set of roots of the L_β -module $(\mathfrak{g}^*)_{-4}^{S_\beta}$. It is enough to check that this module is a minuscule representation of $L_\alpha \cap L_\beta$. The isomorphic module $\mathfrak{g}_{-4}^{S_\beta}$ is described in [MS12, §5] where it is called of the second internal Chevalley module. Let us verify case-by-case that this module is minuscule.

Case E_6 : the Cartan type of $L_\alpha \cap L_\beta$ is A_4 , and $\mathfrak{g}_{-4}^{S_\beta}$ is the standard representation.

Case E_7 : the Cartan type of $L_\alpha \cap L_\beta$ is D_5 , and $\mathfrak{g}_{-4}^{S_\beta}$ is the standard representation.

Case E_8 : the Cartan type of $L_\alpha \cap L_\beta$ is E_6 , and $\mathfrak{g}_{-4}^{S_\beta}$ is the 27-dimensional representation. \square

Denote $Y = \mathfrak{g}_{-\alpha}^\times + \mathfrak{g}_{-\delta}^\times$, i.e. the complement to the coordinate axes in $\mathfrak{g}_{-\alpha}^* \oplus \mathfrak{g}_{-\delta}^*$. Let T be the split torus corresponding to our fixed Cartan subalgebra \mathfrak{h} .

Lemma 5.6.4. $T(\mathbb{K})$ acts transitively on Y .

Proof. It is enough to show that there exist coroots λ, μ such that $\langle \lambda, \alpha \rangle = \langle \mu, \delta \rangle = 1$ and $\langle \lambda, \delta \rangle = \langle \mu, \alpha \rangle = 0$. For E_6 we take $\lambda := \alpha_5^\vee + \alpha_6^\vee$, $\mu := \alpha_2^\vee$. For E_7 we take $\lambda := \alpha_6^\vee + \alpha_7^\vee$, $\mu := \alpha_1^\vee$. For E_8 we take $\lambda := \alpha_6^\vee + \alpha_7^\vee + \alpha_8^\vee$, $\mu := \alpha_6^\vee$. \square

Proof of Proposition 5.4.4. We have $Y \subset X$ by Lemma 5.3.1. Since $T \subset L_\alpha$, it is enough to show that $X \subseteq L_\alpha(\mathbb{K})Y$. Let $\varphi \in X$. Decompose $\varphi = \sum_{\varepsilon \in \Psi_\alpha} \varphi_\varepsilon$, where $\varphi_\varepsilon \in \mathfrak{g}_{-\varepsilon}^*$. Let

$$F := \{\varepsilon \in \Psi_\alpha \mid \varphi_\varepsilon \neq 0\}.$$

By Lemma 5.1.2, we can assume $\alpha \in F$. Using Lemma 5.6.2, we can assume that for any other $\varepsilon \in F$ we have $\langle \alpha, \varepsilon \rangle \leq 0$. Note that this implies that either $F \setminus (F \cap \Phi_\alpha) = \{\alpha\}$ or \mathfrak{g} is of type E_8 and $F \setminus (F \cap \Phi_\alpha) \subseteq \{\alpha, \delta + \alpha_7\}$ by verification on the root systems. In both cases, Lemma 5.3.1 implies that $F \cap \Phi_\alpha$ is not empty. By Lemma 5.6.3 we can assume $\delta \in F$.

If \mathfrak{g} is of type E_8 and $\delta + \alpha_7 \in F$ then we use the action of $\text{Exp}(\mathfrak{g}_{-\alpha_7})$ and the term φ_δ to cancel out the term $\varphi_{\delta + \alpha_7}$. At this point we have $\{\alpha, \delta\} \subseteq F \subseteq \Phi_\alpha \cup \{\alpha\}$ or $\mathfrak{g} = E_8$ and $\{\alpha, \delta\} \subseteq F \subseteq \Phi_\alpha \cup \{\alpha, \alpha + \alpha_7\}$.

Let $S_B \in \mathfrak{h}$ be 2 on all simple roots, and let $Z = S_\alpha + \left(\frac{\delta(S_B)}{2} - 1\right)z - S_B$, where z is as in Notation 5.6.1. Then $\varphi_\alpha + \varphi_\delta \in (\mathfrak{g}^*)_0^Z \cap (\mathfrak{g}^*)_{-2}^{S_\alpha}$ and $\varphi - \varphi_\alpha - \varphi_\delta \in (\mathfrak{g}^*)_{>0}^Z \cap (\mathfrak{g}^*)_{-2}^{S_\alpha}$. The statement follows now from Corollary 4.4.4. \square

To prove Lemma 5.4.5 used for type E_8 where $\overline{\Phi}_\alpha = \Phi_\alpha \cup \{\delta + \alpha_7\}$, we will use the following.

Lemma 5.6.5. For any $\varepsilon \in \overline{\Phi}_\alpha$, $\varepsilon - \alpha$ is not a root.

Proof. $\varepsilon - \alpha$ does not include $\alpha = \alpha_8$ and includes α_7 with coefficient 2 or 3. Since all the roots in E_7 include α_7 with multiplicity in $\{-1, 0, 1\}$, $\varepsilon - \alpha$ is not a root. \square

Proof of Lemma 5.4.5. Denote the set of next-to-minimal elements in $\mathfrak{g}_{-\alpha}^\times + \bigoplus_{\varepsilon \in \overline{\Phi}_\alpha} \mathfrak{g}_{-\varepsilon}^*$ by \mathfrak{X} . First of all, note that $M(\mathbb{K})$ preserves \mathfrak{X} since $\overline{\Phi}_\alpha$ is the set of roots on which $S_\alpha - h_\alpha$ is at least 2 and S_α is 2, and M is the joint centralizer of h_α and S_α .

Now, let $x \in \mathfrak{X}$ and decompose it to a sum of root covectors $x = x_\alpha + \sum_{\varepsilon \in \overline{\Phi}_\alpha} x_\varepsilon$ with $x_\varepsilon \in \mathfrak{g}_{-\varepsilon}^*$. Let $F := \{\varepsilon \in \overline{\Phi}_\alpha \mid x_\varepsilon \neq 0\}$. By Lemma 5.3.1 F intersects Φ_α and thus, by Lemma 5.1.2, we can assume $\delta \in F$. Define $Z \in \mathfrak{h}$ by $(\sum c_i \alpha_i)(Z) = 2c_7 - c_6$. Then $\alpha(Z) = \delta(Z) = 0$ and $\varepsilon(Z) > 0$ for any $\varepsilon \neq \delta \in \overline{\Phi}_\alpha$. Applying Corollary 4.4.2 to $S := S_\alpha$ and $H := S_\alpha - Z$, we obtain that there exists a nilpotent $X \in (\mathfrak{l}_\alpha)_{<0}^Z$ with

$$\text{ad}^*(X)(x_\alpha + x_\delta) = x - x_\alpha - x_\delta \tag{5.23}$$

Decompose X to a sum of root vectors $X = \sum_{\lambda} X_{\lambda}$, $X_{\lambda} \in \mathfrak{g}_{-\lambda}$. We drop from X the X_{λ} that commute with both x_{α} and x_{δ} and (5.23) still holds. Let Ψ be the set of all roots with $X_{\lambda} \neq 0$. We would like to show that Ψ lies in the span of the first six simple roots. Assume the contrary, i.e. there exists $\lambda = \sum c_i \alpha_i \in \Psi$ with $c_7 \neq 0$. Since $X \in \mathfrak{l}_{\alpha}$, $c_8 = 0$ and thus $c_7 = \pm 1$. Also, there are three possibilities:

1. $\alpha + \lambda \in \overline{\Phi}_{\alpha}$
2. $\alpha + \lambda$ is a root and $\alpha + \lambda = \delta + \mu$ for some $\mu \in \Psi$.
3. $\alpha + \lambda$ is not a root and $\delta + \lambda \in \overline{\Phi}_{\alpha}$

Lemma 5.6.5 excludes the first possibility. Assume now that the second possibility holds. Then λ is a positive root, and thus $c_7 = 1$. Since $X \in (\mathfrak{l}_{\alpha})_{<0}^Z$, $2c_7 > c_6$, i.e. $c_6 \in \{0, 1\}$. Thus, $\delta - \lambda - \alpha$ is a root in E_7 of the form $\alpha_7 + 3\alpha_6 + \dots$ or $\alpha_7 + 4\alpha_6 + \dots$. By inspecting the root system we see that there are no such roots in E_7 . If the third possibility holds then $\lambda \neq \alpha_7$ and $\delta + \lambda \in \overline{\Phi}_{\alpha}$. This implies $\delta + \lambda \in \Phi_{\alpha}$. By definition of Φ_{α} this implies $c_7 = 0$.

Altogether, we showed that X lies in the Lie algebra \mathfrak{m} of M . Thus $X \in \mathfrak{m}_{<0}^Z$. But $\mathfrak{m}_{<0}^Z = \mathfrak{m}_{-1}^Z$. Let $y := \text{Exp}(-X)x - x_{\alpha} - x_{\delta}$. Note that all the roots of y still lie in $\overline{\Phi}_{\alpha} \setminus \{\delta\}$, since $X \in \mathfrak{m}$. Thus $x_{\alpha} + x_{\delta} + y \in \mathfrak{X}$. By the same argument as above, there exists $Y \in \mathfrak{m}_{-1}^Z$ such that $\text{ad}^*(Y)(x_{\alpha} + x_{\delta}) = y$.

Since Z is at least 1 on $\overline{\Phi}_{\alpha} \setminus \{\delta\}$ and $\text{ad}^*(X)$ lowers the Z -eigenvalues by 1, we get that $y \in (\mathfrak{g}^*)_{\leq -2}^Z$. However, $\text{ad}^*(Y)(x_{\alpha} + x_{\delta}) \in (\mathfrak{g}^*)_{-1}^Z$ and thus $y = 0$.

Thus $\text{Exp}(-X)x = x_{\alpha} + x_{\delta}$, i.e. we can conjugate x using $\text{Exp}(-X) \in M(\mathbb{K})$ into $Y = \mathfrak{g}_{-\alpha}^{\times} + \mathfrak{g}_{-\delta}^{\times}$. By Lemma 5.6.4 the torus acts transitively on Y . \square

6. DETAILED EXAMPLES

In this section we will illustrate how to use the framework introduced above to compute certain Fourier coefficients in detail, many of which are of particular interest in string theory. In particular, we will in §6.2 show examples for D_5 with detailed steps and deformations that reproduce the results of Theorems C, D and E, while in the following sections we will illustrate how to apply these theorems to different examples.

As in previous sections we will here often identify φ of a Whittaker pair (S, φ) with its Killing form dual f_{φ} and its corresponding character χ_{φ} . Since it is convenient to specify a Cartan element $S \in \mathfrak{h}$ by how the simple roots act on S we introduce the notation $S = (S_1, \dots, S_r)_{\omega^{\vee}} = \sum_{i=1}^r S_i \omega_i^{\vee}$ where r is the rank of G and ω_i^{\vee} are the fundamental coweights which give that $\alpha_j(S) = S_j$. We will also use the following notation for Chevalley generators. Let (n_1, \dots, n_r) be a tuple of non-negative integers and let $E_{n_1 \dots n_r}$ denote the Chevalley generator $E_{n_1 \alpha_1 + \dots + n_r \alpha_r}$ and $F_{n_1 \dots n_r}$ the Chevalley generator $E_{-(n_1 \alpha_1 + \dots + n_r \alpha_r)}$. For the examples we will be considering, n_i will always be single digit and we therefore omit delimiters.

6.1. Whittaker triples. We will now illustrate what type of Fourier coefficients we are able to describe using Whittaker triples that are not captured by Whittaker pairs in an example for $G = \text{SL}_4$.

Let (S, φ, ψ) be the Whittaker triple with $S = \frac{2}{3}(1, 1, 1)_{\omega^{\vee}}$, $\varphi = F_{111}$ and $\psi = mF_{110}$ with $m \in \mathbb{K}$. The S -eigenvalues for the different Chevalley generators can be illustrated by

the following matrix

$$S : \begin{pmatrix} & 2/3 & 4/3 & 2 \\ -2/3 & & 2/3 & 4/3 \\ -4/3 & -2/3 & & 2/3 \\ -2 & -4/3 & -2/3 & \end{pmatrix}. \quad (6.1)$$

As seen from this matrix we get the following unipotent subgroup (independent of φ)

$$N_{S,\varphi} = \left\{ \begin{pmatrix} 1 & 0 & x_1 & x_2 \\ & 1 & 0 & x_3 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\}, \quad (6.2)$$

and the corresponding Fourier coefficient of an automorphic function η on G can be expressed as

$$\mathcal{F}_{S,\varphi,\psi}[\eta](g) = \int_{(\mathbb{K}\backslash\mathbb{A})^3} \eta\left(\begin{pmatrix} 1 & 0 & x_1 & x_2 \\ & 1 & 0 & x_3 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} g\right) \chi(mx_1 + x_2)^{-1} d^3x. \quad (6.3)$$

From this little example we see that we require Whittaker triples in addition to Whittaker pairs if we want to construct characters that are not only sensitive to x_2 in the -2 -eigenspace but also, for instance, to x_1 which is in the $-4/3$ -eigenspace. A similar construction can be made for x_3 .

6.2. Examples for $\mathrm{SO}_{5,5}$. For D_5 we use the conventional labelling of the roots shown in Figure 2.

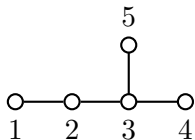


FIGURE 2. Root labels used for D_5 .

The (complex) nilpotent orbits for D_5 are labeled by certain integer partitions of 10 with a partial ordering illustrated in the Hasse diagram of Figure 3 where $\mathcal{O}_{1^{10}}$ is the trivial orbit and $\mathcal{O}_{2^{21}6}$ the minimal orbit. There is no unique next-to-minimal orbit and both $\mathcal{O}_{2^{41}2}$ and $\mathcal{O}_{3^{17}}$ occur as maximal orbits in wave-front sets arising in string theory.

Example 6.2.1. We will now consider an automorphic function η_{ntm} in an automorphic representation π of $\mathrm{SO}_{5,5}$ with $\mathrm{WO}(\pi) \subset \{\mathcal{O}_{1^{10}}, \mathcal{O}_{2^{21}6}, \mathcal{O}_{2^{41}2}\}$ corresponding to the closure of one of the next-to-minimal orbits that we choose for this example. We will compute Fourier coefficients of η_{ntm} with respect to the unipotent radical of the maximal parabolic subgroup associated to the root α_1 , which is the string perturbation limit discussed in §1.3. Specifically, we will consider the pair (S, φ) where $S = S_{\bullet\infty} = (2, 0, 0, 0, 0)_{\omega^\vee}$ and $\varphi = mE_{-\alpha_1}$ with non-zero $m \in \mathbb{K}$.

According to [MS12], there are three complex character variety orbits for this parabolic subgroup which intersect the orbits $\mathcal{O}_{1^{10}}$, $\mathcal{O}_{\mathrm{min}} = \mathcal{O}_{2^{21}6}$ and $\mathcal{O}_{3^{17}}$ with the above φ being in the minimal orbit $\mathcal{O}_{2^{21}6}$.

We make a deformation $H_t = S + tZ$ using $Z = (0, 2000, 200, 20, 2)$ that was chosen to nicely separate the numerous eigenspaces corresponding to quasi-critical values. There are

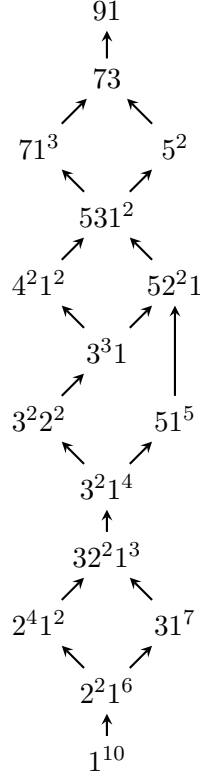


FIGURE 3. Hasse diagram of nilpotent orbits for D_5 . The only non-special orbit is $5 2^2 1$.

25 quasi-critical values t_i for this deformation in the interval $[0, 1]$ also counting 0 and 1, the exact values of which will not be of importance to this discussion. For each of the first six critical values, $\mathfrak{n}_{H_t, \varphi}$ gets enlarged by a one-dimensional subspace generated by E_{01211} , E_{01111} , E_{01110} , E_{01101} , E_{01100} and E_{01000} respectively. Repeated use of Lemma 3.2.7 (ii) gives, for the sixth critical value t_6 , that

$$\mathcal{F}_{S, \varphi}[\eta_{\text{ntm}}](g) = \sum_{\psi} \mathcal{F}_{H_{t_6}, \varphi, \psi}[\eta_{\text{ntm}}](g) \quad (6.4)$$

where the sum is over $\psi \in \langle F_{01211}, F_{01111}, F_{01110}, F_{01101}, F_{01100}, F_{01000} \rangle(\mathbb{K})$ which is the \mathbb{K} -span of these elements.

As we continue the deformation, the same generators will successively enter the 2-eigenspace for H_t , where, according to Lemma 3.2.7 (iii), each corresponding character φ' in the sum over ψ should be moved to the Whittaker pair as $\varphi + \varphi'$. For example, we would, at the seventh critical value t_7 , have that

$$\mathcal{F}_{S, \varphi}[\eta_{\text{ntm}}](g) = \sum_{\varphi'} \sum_{\psi} \mathcal{F}_{H_{t_7}, \varphi + \varphi', \psi}[\eta_{\text{ntm}}](g) \quad (6.5)$$

where the sums are over $\varphi' \in \langle F_{01211} \rangle(\mathbb{K})$ and $\psi \in \langle F_{01111}, F_{01110}, F_{01101}, F_{01100}, F_{01000} \rangle(\mathbb{K})$.

Normally, we would then be unable to continue with the same deformation Z since it would not commute with $\varphi + \varphi'$. However, in our case we have that $\varphi + \varphi'$ is in the orbit $\mathcal{O}_{3^2 1^4}$ unless $\varphi' = 0$, which means that $\mathcal{F}_{H_{t_7}, \varphi + \varphi', y}[\eta_{\text{ntm}}]$ vanishes unless $\varphi' = 0$. The same arguments follows for the remaining generators, and at the twelfth critical value t_{12} we get that

$$\mathcal{F}_{S, \varphi}[\eta_{\text{ntm}}](g) = \mathcal{F}_{H_{t_{12}}, \varphi}[\eta_{\text{ntm}}](g) \quad (6.6)$$

Let $S_{\bullet\bullet\bullet} = (2, 2, 0, 0, 0)$. Then $N_{H_{t_{12}}, \varphi} = N_{S_{\bullet\bullet\bullet}, \varphi}$ and thus $\mathcal{F}_{S, \varphi}[\eta_{\text{ntm}}](g) = \mathcal{F}_{S_{\bullet\bullet\bullet}, \varphi}[\eta_{\text{ntm}}](g)$ illustrating Proposition 5.3.3. Let $G' = G_{\bullet\bullet\bullet}$ be the reductive group of type A_3 given by the simple roots α_3, α_4 and α_5 . The elements of G' leave both φ and S invariant under the (co)adjoint action. According to Proposition 5.3.4 $\eta'_{\text{ntm}}(g') = \mathcal{F}_{S_{\bullet\bullet\bullet}, \varphi}[\eta_{\text{ntm}}](g')$ is attached to a minimal representation of G' . The expansion of such automorphic forms were studied in [GKP16] and is given by repeated use of Proposition 5.2.1 which was used to prove Theorem D and which we will now illustrate. As noted in the proof of Theorem E, the Whittaker coefficients of $\eta'_{\text{ntm}}(g')$ on G' become Whittaker coefficients on the original group G when taking the integral in $\mathcal{F}_{S_{\bullet\bullet\bullet}, \varphi}[\eta_{\text{ntm}}](g')$ into account.

To simplify the calculation we will now build upon the above notation for a selection of subgroups and semi-simple elements. Let therefore x denote a selection of simple roots, which we mark as filled nodes in the Dynkin diagram, for example $x = \infty\bullet$. Then, let G_x be the semi-simple subgroup obtained by the corresponding subsystem of roots keeping the choice of simple roots, and, for a simple root α in the selection x , let P_x^α be the maximal parabolic subgroup of G_x obtained from α . Let also L_x^α denote the Levi subgroup of P_x^α , M_x^α the stabilizer of $\langle F_\alpha \rangle$ in L_x^α , and $\Gamma_x^\alpha(\mathbb{K}) = L_x^\alpha(\mathbb{K})/M_x^\alpha(\mathbb{K})$. Instead of explicitly showing α we may mark it in the selection x like so: $\Gamma_{\infty\bullet}^{\alpha_5}(\mathbb{K}) = \Gamma_{\infty\bullet}^{\alpha_5}(\mathbb{K})$.

Repeatedly using Proposition 5.2.1, we get that $\mathcal{F}_{S, \varphi}[\eta_{\text{ntm}}](g)$ equals

$$\begin{aligned} \mathcal{F}_{S_{\bullet\bullet\bullet}, \varphi}[\eta_{\text{ntm}}](g) &= \mathcal{F}_{S_{\bullet\bullet\bullet}, \varphi}[\eta_{\text{ntm}}](g) + \sum_{\gamma \in \Gamma_{\infty\bullet}^{\alpha_5}(\mathbb{K})} \sum_{\varphi' \in \mathfrak{g}_{-\alpha_5}^\times} \mathcal{W}_{\varphi + \varphi'}[\eta_{\text{ntm}}](\gamma g) \\ \mathcal{F}_{S_{\bullet\bullet\bullet}, \varphi}[\eta_{\text{ntm}}](g) &= \mathcal{F}_{S_{\bullet\bullet\bullet}, \varphi}[\eta_{\text{ntm}}](g) + \sum_{\gamma \in \Gamma_{\infty\bullet}^{\alpha_3}(\mathbb{K})} \sum_{\varphi' \in \mathfrak{g}_{-\alpha_3}^\times} \mathcal{W}_{\varphi + \varphi'}[\eta_{\text{ntm}}](\gamma g) \\ \mathcal{F}_{S_{\bullet\bullet\bullet}, \varphi}[\eta_{\text{ntm}}](g) &= \mathcal{W}_\varphi[\eta_{\text{ntm}}](g) + \sum_{\varphi' \in \mathfrak{g}_{-\alpha_4}^\times} \mathcal{W}_{\varphi + \varphi'}[\eta_{\text{ntm}}](g), \end{aligned} \quad (6.7)$$

where we recall that $\mathfrak{g}_{-\alpha_i}^\times = \langle F_{\alpha_i} \rangle(\mathbb{K}) \setminus \{0\}$. As explained in the proof of Theorem E, the Whittaker coefficients on each successive G' in the induction become Whittaker coefficients on G when taking the integration from $\mathcal{F}_{S_x, \varphi}$ into account.

Altogether, the Fourier coefficient in the 8-dimensional unipotent associated with $S = S_{\bullet\bullet\bullet}$ and $\varphi = mE_{-\alpha_1}$ is

$$\mathcal{F}_{S, \varphi}[\eta_{\text{ntm}}](g) = \mathcal{W}_\varphi[\eta_{\text{ntm}}](g) + \sum_{i=3}^5 \sum_{\substack{\gamma \in \Gamma_i(\mathbb{K}) \\ \varphi' \in \mathfrak{g}_{-\alpha_i}^\times}} \mathcal{W}_{\varphi + \varphi'}[\eta_{\text{ntm}}](\gamma g), \quad \Gamma_i(\mathbb{K}) = \begin{cases} \Gamma_{\infty\bullet}(\mathbb{K}) & i = 3 \\ \{1\} & i = 4 \\ \Gamma_{\bullet\bullet\bullet}(\mathbb{K}) & i = 5 \end{cases} \quad (6.8)$$

which illustrates Theorem E and how to obtain the Γ_i . We note that a formula for this expansion was also determined in [GMV15, Eq. (4.83)] based on theta lifts.

Example 6.2.2. Similar to how the expansion of a minimal automorphic function on A_3 played a role in the above calculation, the expansion of a minimal automorphic function on $\mathrm{SO}_{5,5}(\mathbb{A})$ will play a role in the E_7 calculations of §6.4. Therefore we will now compute the full expansion for a minimal automorphic function η_{\min} on $\mathrm{SO}_{5,5}(\mathbb{A})$ by expanding along the same abelian unipotent radical as above. By repeated use of Proposition 5.2.1 we get that

$$\begin{aligned}
\eta_{\min}(g) &= \mathcal{F}_{S_{\bullet\bullet\bullet\bullet},0}[\eta_{\min}](g) + \sum_{\gamma \in \Gamma_{\bullet\bullet\bullet\bullet}(\mathbb{K})} \sum_{\varphi \in \mathfrak{g}_{-\alpha_1}^\times} \mathcal{W}_\varphi[\eta_{\min}](\gamma g) \\
\mathcal{F}_{S_{\bullet\bullet\bullet\bullet},0}[\eta_{\min}](g) &= \mathcal{F}_{S_{\bullet\bullet\bullet\bullet},0}[\eta_{\min}](g) + \sum_{\gamma \in \Gamma_{\bullet\bullet\bullet\bullet}(\mathbb{K})} \sum_{\varphi \in \mathfrak{g}_{-\alpha_2}^\times} \mathcal{W}_\varphi[\eta_{\min}](\gamma g) \\
\mathcal{F}_{S_{\bullet\bullet\bullet\bullet},0}[\eta_{\min}](g) &= \mathcal{F}_{S_{\bullet\bullet\bullet\bullet},0}[\eta_{\min}](g) + \sum_{\gamma \in \Gamma_{\bullet\bullet\bullet\bullet}(\mathbb{K})} \sum_{\varphi \in \mathfrak{g}_{-\alpha_5}^\times} \mathcal{W}_\varphi[\eta_{\min}](\gamma g) \\
\mathcal{F}_{S_{\bullet\bullet\bullet\bullet},0}[\eta_{\min}](g) &= \mathcal{F}_{S_{\bullet\bullet\bullet\bullet},0}[\eta_{\min}](g) + \sum_{\gamma \in \Gamma_{\bullet\bullet\bullet\bullet}(\mathbb{K})} \sum_{\varphi \in \mathfrak{g}_{-\alpha_3}^\times} \mathcal{W}_\varphi[\eta_{\min}](\gamma g) \\
\mathcal{F}_{S_{\bullet\bullet\bullet\bullet},0}[\eta_{\min}](g) &= \mathcal{W}_0[\eta_{\min}](g) + \sum_{\varphi \in \mathfrak{g}_{-\alpha_4}^\times} \mathcal{W}_\varphi[\eta_{\min}](g)
\end{aligned} \tag{6.9}$$

In summary,

$$\eta_{\min}(g) = \mathcal{W}_0[\eta_{\min}](g) + \sum_{i=1}^5 \sum_{\gamma \in \Gamma_i(\mathbb{K})} \sum_{\varphi \in \mathfrak{g}_{-\alpha_i}^\times} \mathcal{W}_\varphi[\eta_{\min}](\gamma g), \quad \Gamma_i(\mathbb{K}) = \begin{cases} \Gamma_{\bullet\bullet\bullet\bullet}(\mathbb{K}) & i = 1 \\ \Gamma_{\bullet\bullet\bullet\bullet}(\mathbb{K}) & i = 2 \\ \Gamma_{\bullet\bullet\bullet\bullet}(\mathbb{K}) & i = 3 \\ \{1\} & i = 4 \\ \Gamma_{\bullet\bullet\bullet\bullet}(\mathbb{K}) & i = 5 \end{cases} \tag{6.10}$$

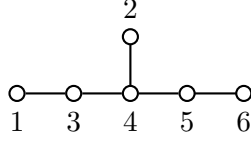
which illustrates Theorem D.

6.3. Examples for E_6 . In this section we will consider a next-to-minimal automorphic function η_{ntm} on $E_6(\mathbb{A})$. We will first compute maximal parabolic Fourier coefficients with respect to P_{α_2} corresponding to the M-theory limit discussed in §1.3, and then we will find a full expansion of η_{ntm} by considering the maximal parabolic P_{α_6} which will be used in one of the E_8 examples in §6.5.

Example 6.3.1. Let (S, φ_{\min}) be the Whittaker pair where $S = S_{\bullet\bullet\bullet\bullet} = (0, 2, 0, 0, 0, 0)_{\omega^\vee}$ and $\varphi_{\min} = mE_{-\alpha_2} \in \mathcal{O}_{\min}$ with non-zero $m \in \mathbb{K}$. Using Proposition 5.3.3 we have that

$$\mathcal{F}_{S, \varphi_{\min}}[\eta_{\mathrm{ntm}}](g) = \mathcal{F}_{S_{\bullet\bullet\bullet\bullet}, \varphi_{\min}}[\eta_{\mathrm{ntm}}](g). \tag{6.11}$$

According to Proposition 5.3.4, $\mathcal{F}_{S_{\bullet\bullet\bullet\bullet}, \varphi}[\eta_{\mathrm{ntm}}](g)$ is minimal on $G_{\bullet\bullet\bullet\bullet}$ which is the reductive group obtained from the root system of G by removing α_2 and α_4 , and is of

FIGURE 4. Root labels used for E_6 .

type $A_2 \times A_2$. Using Proposition 5.2.1 repeatedly we get

$$\begin{aligned}
\mathcal{F}_{S, \varphi_{\min}}[\eta_{\text{ntm}}](g) &= \mathcal{F}_{S_{\bullet\bullet\bullet\circ\circ}, \varphi_{\min}}[\eta_{\text{ntm}}](g) + \sum_{\gamma \in \Gamma_{\bullet\bullet\bullet\circ\circ}(\mathbb{K})} \sum_{\varphi' \in \mathfrak{g}_{-\alpha_3}^{\times}} \mathcal{W}_{\varphi_{\min} + \varphi'}[\eta](\gamma g) \\
\mathcal{F}_{S_{\bullet\bullet\bullet\circ\circ}, \varphi_{\min}}[\eta_{\text{ntm}}](g) &= \mathcal{F}_{S_{\bullet\bullet\bullet\circ\circ}, \varphi_{\min}}[\eta_{\text{ntm}}](g) + \sum_{\varphi' \in \mathfrak{g}_{-\alpha_1}^{\times}} \mathcal{W}_{\varphi_{\min} + \varphi'}[\eta](g) \\
\mathcal{F}_{S_{\bullet\bullet\bullet\circ\circ}, \varphi_{\min}}[\eta_{\text{ntm}}](g) &= \mathcal{F}_{S_{\bullet\bullet\bullet\circ\circ}, \varphi_{\min}}[\eta_{\text{ntm}}](g) + \sum_{\gamma \in \Gamma_{\circ\circ\circ\bullet\circ}(\mathbb{K})} \sum_{\varphi' \in \mathfrak{g}_{-\alpha_5}^{\times}} \mathcal{W}_{\varphi_{\min} + \varphi'}[\eta](\gamma g) \\
\mathcal{F}_{S_{\bullet\bullet\bullet\circ\circ}, \varphi_{\min}}[\eta_{\text{ntm}}](g) &= \mathcal{W}_{\varphi_{\min}}[\eta_{\text{ntm}}](g) + \sum_{\varphi' \in \mathfrak{g}_{-\alpha_6}^{\times}} \mathcal{W}_{\varphi_{\min} + \varphi'}[\eta](g).
\end{aligned} \tag{6.12}$$

Note that $\Gamma_{\bullet\bullet\bullet\circ\circ}(\mathbb{K}) = \Gamma_{\circ\circ\circ\bullet\circ}(\mathbb{K})$. To summarize,

$$\begin{aligned}
\mathcal{F}_{S, \varphi_{\min}}[\eta_{\text{ntm}}](g) &= \mathcal{W}_{\varphi_{\min}}[\eta_{\text{ntm}}](g) + \sum_{i \in \{1, 3, 5, 6\}} \sum_{\gamma \in \Gamma_i(\mathbb{K})} \sum_{\varphi' \in \mathfrak{g}_{-\alpha_i}^{\times}} \mathcal{W}_{\varphi_{\min} + \varphi'}[\eta](\gamma g), \\
\Gamma_i(\mathbb{K}) &= \begin{cases} \{1\} & i = 1, 6 \\ \Gamma_{\bullet\bullet\bullet\circ\circ}(\mathbb{K}) & i = 3 \\ \Gamma_{\circ\circ\circ\bullet\circ}(\mathbb{K}) & i = 5. \end{cases}
\end{aligned} \tag{6.13}$$

Example 6.3.2. We will now write the full expansion of η_{\min} using Theorem D starting with the simple root α_6 , or, equivalently, an expansion along the abelian unipotent radical of P_{α_6} . We get that

$$\begin{aligned}
\eta_{\min}(g) &= \mathcal{W}_0[\eta_{\min}](g) + \sum_{i=1}^6 \sum_{\gamma \in \Gamma_i(\mathbb{K})} \sum_{\varphi' \in \mathfrak{g}_{-\alpha_i}^{\times}} \mathcal{W}_{\varphi'}[\eta_{\min}](\gamma g) \\
\Gamma_i &= \begin{cases} \Gamma_{\bullet\bullet\bullet\bullet\circ} & i = 6 \\ \Gamma_{\bullet\bullet\bullet\circ\circ} & i = 5 \\ \Gamma_{\bullet\bullet\circ\circ\circ} & i = 2 \\ \Gamma_{\bullet\circ\circ\circ\circ} & i = 4 \\ \Gamma_{\circ\bullet\circ\circ\circ} & i = 3 \\ \{1\} & i = 1. \end{cases}
\end{aligned} \tag{6.14}$$

The Γ_i are obtained in a similar way as in the steps shown in (6.7), (6.9) and (6.12).

6.4. **Examples for E_7 .** We will now consider a next-to-minimal automorphic function η_{ntm} on $E_7(\mathbb{A})$ and compute its Fourier coefficients with respect to the unipotent radical of the parabolic subgroup P_{α_7} corresponding to the decompactification limit in the string theory discussed in §1.3.

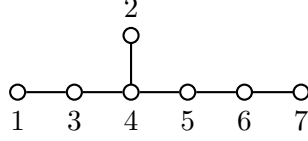


FIGURE 5. Root labels used for E_7 .

Example 6.4.1. Specifically, we will first consider the Whittaker pair (S, φ) where $S = S_{\infty\infty\infty} = (0, 0, 0, 0, 0, 0, 2)_{\omega^\vee}$ and $\varphi_{\min} = mE_{-\alpha_7} \in \mathcal{O}_{\min}$ with non-zero $m \in \mathbb{K}$. The unique complex minimal orbit \mathcal{O}_{\min} is often described by a Bala–Carter label as \mathcal{O}_{A_1} . Using Proposition 5.3.3 we have that

$$\mathcal{F}_{S, \varphi_{\min}}[\eta_{\text{ntm}}](g) = \mathcal{F}_{S_{\infty\infty\infty}, \varphi_{\min}}[\eta_{\text{ntm}}](g) \quad (6.15)$$

where $S_{\infty\infty\infty} = (0, 0, 0, 0, 0, 2, 2)_{\omega^\vee}$.

According to Proposition 5.3.4 we have that the right-hand side is attached to a minimal representation of $G_{\infty\infty\infty}$ of type D_5 whose expansion is given by (6.10) although with differently labeled roots. We get that

$$\mathcal{F}_{S, \varphi_{\min}}[\eta_{\text{ntm}}](g) = \mathcal{W}_{\varphi_{\min}}[\eta_{\text{ntm}}](g) + \sum_{i=1}^5 \sum_{\gamma \in \Gamma_i(\mathbb{K})} \sum_{\varphi' \in \mathfrak{g}_{-\alpha_i}^\times} \mathcal{W}_{\varphi_{\min} + \varphi'}[\eta_{\text{ntm}}](\gamma g), \quad (6.16)$$

$$\Gamma_i(\mathbb{K}) = \begin{cases} \Gamma_{\bullet\bullet\bullet\infty}(\mathbb{K}) & i = 1 \\ \Gamma_{\bullet\bullet\bullet\bullet}(\mathbb{K}) & i = 3 \\ \Gamma_{\infty\bullet\bullet\infty}(\mathbb{K}) & i = 2 \\ \Gamma_{\infty\bullet\bullet\bullet}(\mathbb{K}) & i = 4 \\ \{1\} & i = 5. \end{cases}$$

This further illustrates Theorem E and the pattern that emerges for how to obtain Γ_i .

Example 6.4.2. There is also a unique complex next-to-minimal orbit $\mathcal{O}_{\text{ntm}} = \mathcal{O}_{2A_1}$ and we will now consider the same $S = S_{\infty\infty\infty}$ as above but now with $\varphi_{\text{ntm}} = m_1 F_{0000001} + m_2 F_{2234321} \in \mathcal{O}_{\text{ntm}}$ with non-zero $m_1, m_2 \in \mathbb{K}$.

Since φ_{ntm} in a Whittaker pair $(S, \varphi_{\text{ntm}})$ will still be present after a deformation, although possibly with new contributions, we will not be able to make a deformation to our fixed Borel subgroup since φ_{ntm} is not supported on only our corresponding choice of simple roots. Therefore we first make a conjugation using Lemma 3.3.1 and $w = w_4 w_5 w_6 w_7 w_2 w_4 w_5 w_6 w_1 w_3 w_4 w_5 w_2 w_4 w_3 w_1$ where $w_i = e^{E_{\alpha_i}} e^{-F_{\alpha_i}} e^{E_{\alpha_i}}$ for which $w S w^{-1} = (0, 2, 2, -2, 0, 0, 0)_{\omega^\vee}$ and $w \varphi_{\text{ntm}} w^{-1} = m_1 F_{0100000} + m_2 F_{0010000}$ giving

$$\mathcal{F}_{S, \varphi_{\text{ntm}}}[\eta_{\text{ntm}}](g) = \mathcal{F}_{w S w^{-1}, w \varphi_{\text{ntm}} w^{-1}}[\eta_{\text{ntm}}](wg). \quad (6.17)$$

Using Theorem E (ii) we get that

$$\mathcal{F}_{S, \varphi_{\text{ntm}}}[\eta_{\text{ntm}}](g) = \int_{V(\mathbb{A})} \mathcal{W}_{w\varphi_{\text{ntm}}w^{-1}}[\eta_{\text{ntm}}](vvg) dv \quad (6.18)$$

where $V = \text{Exp}(\langle F_{0001000}, F_{0001100}, F_{0001110}, F_{0001111} \rangle)$. The Weyl word above was chosen such that the dimension of V is minimized.

6.5. Examples for E_8 . Similar to the E_7 examples above, we will here consider η_{ntm} in a next-to-minimal representation of E_8 and its maximal parabolic Fourier coefficients with respect to P_{α_8} corresponding to the decompactification limit in the string theory discussed in §1.3.

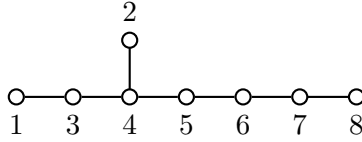


FIGURE 6. Root labels used for E_8 .

Example 6.5.1. Firstly, let $(S, \varphi_{\text{min}})$ be the Whittaker pair with $S = S_{\bullet\bullet\bullet\bullet\bullet\bullet} = (0, 0, 0, 0, 0, 0, 0, 2)_{\omega^\vee}$ and $\varphi_{\text{min}} = mE_{-\alpha_8} \in \mathcal{O}_{\text{min}} = \mathcal{O}_{A_1}$. From Proposition 5.3.3 we get that

$$\mathcal{F}_{S, \varphi_{\text{min}}}[\eta_{\text{ntm}}](g) = \mathcal{F}_{S_{\bullet\bullet\bullet\bullet\bullet\bullet}, \varphi_{\text{min}}}[\eta_{\text{ntm}}](g), \quad (6.19)$$

where the right-hand side, according to Proposition 5.3.4, is in a minimal representation of $G_{\bullet\bullet\bullet\bullet\bullet\bullet}$ of type E_6 the expansion which was found in (6.14). We get that

$$\mathcal{F}_{S, \varphi_{\text{min}}}[\eta_{\text{ntm}}](g) = \mathcal{W}_{\varphi_{\text{min}}}[\eta_{\text{ntm}}](g) + \sum_{i=1}^6 \sum_{\gamma \in \Gamma_i(\mathbb{K})} \sum_{\varphi' \in \mathfrak{g}_{-\alpha_i}^\times} \mathcal{W}_{\varphi_{\text{min}} + \varphi'}[\eta_{\text{ntm}}](\gamma g) \quad (6.20)$$

$$\Gamma_i = \begin{cases} \Gamma_{\bullet\bullet\bullet\bullet\bullet\bullet} & i = 6 \\ \Gamma_{\bullet\bullet\bullet\bullet\bullet\bullet} & i = 5 \\ \Gamma_{\bullet\bullet\bullet\bullet\bullet\bullet} & i = 2 \\ \Gamma_{\bullet\bullet\bullet\bullet\bullet\bullet} & i = 4 \\ \Gamma_{\bullet\bullet\bullet\bullet\bullet\bullet} & i = 3 \\ \{1\} & i = 1. \end{cases}$$

Example 6.5.2. Secondly, let $(S, \varphi_{\text{ntm}})$ be the Whittaker pair where $S = S_{\bullet\bullet\bullet\bullet\bullet\bullet}$ as above, but $\varphi_{\text{ntm}} = m_1 F_{00000001} + m_2 F_{23465321} \in \mathcal{O}_{\text{ntm}} = \mathcal{O}_{2A_1}$ with non-zero $m_1, m_2 \in \mathbb{K}$. Let also $w = w_4 w_5 w_6 w_7 w_8 w_2 w_4 w_5 w_6 w_7 w_1 w_3 w_4 w_5 w_6 w_2 w_4 w_5 w_3 w_4 w_1 w_3 w_2 w_4 w_5$ where $w_i = e^{E_{\alpha_i}} e^{-F_{\alpha_i}} e^{E_{\alpha_i}}$ for which $wSw^{-1} = (0, 2, 2, -2, 0, 0, 0, 0)_{\omega^\vee}$ and $w\varphi_{\text{ntm}}w^{-1} = m_1 F_{01000000} + m_2 F_{00100000}$. From Theorem E (ii) we then get that

$$\mathcal{F}_{S, \varphi_{\text{ntm}}}[\eta_{\text{ntm}}](g) = \int_{V(\mathbb{A})} \mathcal{W}_{w\varphi_{\text{ntm}}w^{-1}}[\eta_{\text{ntm}}](vvg) dv \quad (6.21)$$

where $V = \text{Exp}(\langle F_{00010000}, F_{00011000}, F_{00011100}, F_{00011110}, F_{00011111} \rangle)$. The Weyl word above was chosen such that the dimension of V is minimized.

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