A REDUCTION PRINCIPLE FOR FOURIER COEFFICIENTS OF AUTOMORPHIC FORMS

DMITRY GOUREVITCH, HENRIK P. A. GUSTAFSSON, AXEL KLEINSCHMIDT, DANIEL PERSSON, AND SIDDHARTHA SAHI

ABSTRACT. In this paper we analyze a general class of Fourier coefficients of automorphic forms on reductive adelic groups $\mathbf{G}(\mathbb{A}_{\mathbb{K}})$ and their covers. We prove that any such Fourier coefficient is expressible through integrals and sums involving 'Levi-distinguished' Fourier coefficients. By the latter we mean the class of Fourier coefficients obtained by first taking the constant term along the nilradical of a parabolic subgroup, and then further taking a Fourier coefficient corresponding to a \mathbb{K} -distinguished nilpotent orbit in the Levi quotient. In a follow-up paper we use this result to establish explicit formulas for Fourier expansions of automorphic forms attached to minimal and next-to-minimal representations of simply-laced reductive groups.

²⁰¹⁰ Mathematics Subject Classification. 11F30, 11F70, 22E55, 20G45.

Key words and phrases. automorphic function, automorphic representation, Fourier expansion on covers of reductive groups, Whittaker support, nilpotent orbit, wave-front set.

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1. Introduction and main results

1.1. **Introduction.** Let \mathbb{K} be a number field, \mathfrak{o} its ring of integers, and $\mathbb{A} = \mathbb{A}_{\mathbb{K}}$ its ring of adeles. Let \mathbf{G} be a reductive group defined over \mathbb{K} , $\mathbf{G}(\mathbb{A})$ the group of adelic points of \mathbf{G} , and $\Gamma := \mathbf{G}(\mathbb{K})$ the group of \mathbb{K} -points of \mathbf{G} . For now, let $G = \mathbf{G}(\mathbb{A})$ and let η be an automorphic form on G, although we shall also consider finite central extensions and a more general space of functions on $\Gamma \backslash G$ below. Fix a minimal parabolic subgroup \mathbf{B} in \mathbf{G} defined over \mathbb{K} (a Borel subgroup if \mathbf{G} is quasi-split) and let N be the \mathbb{A} -points of the unipotent radical of \mathbf{B} . Consider the (infinite) set of unitary characters $\chi_N : (N \cap \Gamma) \backslash N \to \mathbb{C}^{\times}$. It is well known that the period integral, *i.e.* the constant term, of η with respect to the commutator subgroup N' := [N, N] can be decomposed according to

$$\int_{(N'\cap\Gamma)\backslash N'} \eta(ng)dn = \sum_{\chi_N} \mathcal{W}_{\chi_N}[\eta](g), \tag{1.1}$$

where $\mathcal{W}_{\chi_N}[\eta] \in C^{\infty}(G)$ is the standard Whittaker coefficient corresponding to χ_N given by

$$\mathcal{W}_{\chi_N}[\eta](g) := \int_{(N \cap \Gamma) \setminus N} \eta(ng) \chi_N(n)^{-1} dn, \tag{1.2}$$

which satisfies the N-equivariance property $\mathcal{W}_{\chi_N}[\eta](ng) = \chi_N(n)\mathcal{W}_{\chi_N}[\eta](g)$.

It is natural to ask whether one can recover all of η from its Whittaker coefficients, and not just the constant term (1.1). This is known to be true when η is a cusp form on $GL_n(\mathbb{A})$ for which we have the Piatetski-Shapiro-Shalika formula [PS79, Sha74]:

$$\eta(g) = \sum_{\gamma \in \mathbf{N}_{n-1}(\mathbb{K}) \backslash \operatorname{GL}_{n-1}(\mathbb{K})} \sum_{\chi_N} \mathcal{W}_{\chi_N}[\eta] ((\gamma_1) g), \tag{1.3}$$

where \mathbf{N}_{n-1} is the unipotent radical of a Borel subgroup of GL_{n-1} and the sum, in this case, is over non-degenerate χ_N . Similar results may be stated for non-cusp forms with the inclusion of degenerate χ_N .

For other groups, there are examples of (non-generic) cusp forms for which all Whittaker coefficients vanish, and thus such forms definitely cannot be recovered. For Sp_4 for example, the existence of such forms is shown by [Ike01]. Our first result, Theorem A below, provides a sufficient condition for recovering a form η from its Whittaker coefficients. Another aim of this paper is to provide results for when Whittaker coefficients alone are not sufficient for recovering the automorphic form.

We reserve the name Whittaker coefficient to mean a Fourier coefficient with respect to a maximal unipotent. In general, for any unipotent subgroup U of G and character χ_U on $U \cap \Gamma \setminus U$, the associated (U, χ_U) -Fourier coefficient is given by

$$\mathcal{F}_{U,\chi_U}[\eta](g) := \int_{(U \cap \Gamma) \setminus U} \eta(ug) \chi_U(u)^{-1} du.$$
 (1.4)

This general terminology for a Fourier coefficient is, for example, used in [FGKP18], while in [HS16] the same notion is called a unipotent period. We will use a convenient set of data to describe both the unipotent subgroup U and character χ_U for a very general class of (U, χ_U) -Fourier coefficients using a so called Whittaker pair of elements in the Lie algebra and its dual as explained in §2, following [GGS17, GGS] but with slightly different notation. For the rest of this paper we will only consider this class of Whittaker pair Fourier coefficients, and will call them Fourier coefficients for short. Among others, this class includes Whittaker coefficients, and more generally the cases where U is the unipotent radical of any parabolic subgroup, as well as the notion of Fourier coefficients used in [GRS97, GRS11, Gin06, GH11, JLS16] (which we will call neutral Fourier coefficients as explained below).

When Whittaker coefficients are not enough to recover the automorphic form it is natural to ask what is the 'simplest', in some sense, type of Fourier coefficients that are. Although it is true, for example, that the Fourier coefficients with respect to an abelian unipotent subgroup are enough to recover the original automorphic form, these Fourier coefficients are in general difficult to compute directly, and their computation is in fact another aim of this paper. Whittaker coefficients can be considered 'simple' because they are integrals over maximal unipotent subgroups. For spherical automorphic forms this means that they

are determined by their values on a maximal torus, and are, up to archimedean parts, given by the Casselman–Shalika formula [CS80].

In this paper we define a binary relation over Fourier coefficients (or rather over the associated Whittaker pair data) called *dominates* and show that if A dominates B then A naturally and linearly determines B, and the dimension of the unipotent integration domain for A is smaller than or equal to that of B. It is with this relation that we can give a precise meaning to the word 'simple' used in the question above.

As a simple example we may take the dominated coefficient B to be a Whittaker coefficient with a character supported only on the exponentiated root space of a single simple root, and the dominating coefficient A to be a Fourier coefficient with respect to the unipotent radical of the standard maximal parabolic subgroup obtained from the same root, together with a restriction of the same character. Then, B is indeed linearly determined by A since B can be obtained as a period integral of A over the complementing unipotent subgroup.

These properties of linear determination and monotonicity of the dimension remain true for the transitive closure of the binary relation. Being already reflexive, we can thus consider the associated quasiorder, which gives us the notion of minimal elements, that is, a class of 'simplest' Fourier coefficients. For GL_n the minimal coefficients in this quasiorder (i.e. with the largest unipotent subgroup) are all Whittaker coefficients, meaning that any Fourier coefficient (and of course the automorphic form itself) linearly determines a corresponding Whittaker coefficient. Note that our original question, answered by Piatetski-Shapiro and Shalika for GL_n , is in the opposite direction: when can the class of Whittaker coefficients together linearly determine the original automorphic form?

In general, the minimal coefficients in the above quasiorder are not all Whittaker coefficients; they are instead in the larger class of so called *Levi-distinguished* coefficients which we define below. We show, in Theorem A, that the class of these Levi-distinguished coefficients are enough to recover the original automorphic form.

The complementary notion of Levi-distinguished coefficients, *i.e.* the maximal coefficients in the quasiorder, are so called neutral Fourier coefficients. These are naturally associated to nilpotent orbits \mathcal{O} in the Lie algebra, and their vanishing properties define the global wave-front set of an automorphic form or representation as discussed below. Indeed, it was shown in [GGS17] that any other Fourier coefficient is naturally and linearly determined by a neutral Fourier coefficient which therefore determine the vanishing properties of all Fourier coefficients. This result is now superseded by the properties of the dominance relation introduced in this paper. Another important difference is that previous work in the global setting has primarily focused on showing that the vanishing of one Fourier coefficient implies the vanishing of another, while we in this paper give explicit relations and expressions between different Fourier coefficients.

To summarize, we schematically have that,

$$\mathcal{O} \longleftrightarrow \text{neutral} \succeq \text{any} \succeq \text{Levi-distinguished} \supseteq \text{Whittaker}$$
 (1.5)

where by $A \succeq B$ we mean that A dominates B.

Furthermore, as a refinement of the above question we show that a Fourier coefficient with character associated to a nilpotent orbit \mathcal{O} is linearly determined by Levi-distinguished coefficients associated to orbits equal to or bigger than \mathcal{O} . Note that this statement is also

in the opposite direction of the natural linear determination from the dominance relation. Previous work [GGS] in this direction studied the vanishing properties of Fourier coefficients with characters associated to a maximal orbit in the global wave-front set. Their result is now directly implied by a special case of Theorem C in this paper. In addition, our work has consequences for the question of Eulerianity of Fourier coefficients as we shall explore in [GGK⁺20].

1.2. **Main results.** Let us now explain some of our main results in more detail. We work in a generality slightly wider than linear reductive groups. Namely, we also consider automorphic forms on finite central extensions of $\mathbf{G}(\mathbb{A})$. Let $\mathrm{pr}: G \to \mathbf{G}(\mathbb{A})$ be such an extension. We assume that there exists a section $\mathbf{G}(\mathbb{K}) \to G$, fix such a section and denote its image by Γ . This generality includes the groups defined in [BD01]. By [MW95, Appendix I], for any unipotent subgroup $\mathbf{U} \subset \mathbf{G}$, the projection pr has a canonical section on $\mathbf{U}(\mathbb{A})$. We will always use this section to identify $\mathbf{U}(\mathbb{A})$ with a subgroup of G.

Let $\mathfrak{g} = \mathfrak{g}(\mathbb{K})$ be the Lie algebra of $\mathbf{G}(\mathbb{K}) \cong \Gamma$, and let \mathfrak{g}^* be the dual of \mathfrak{g} as a vector space. We say that a semi-simple element S of \mathfrak{g} is rational if ad(S) has eigenvalues in \mathbb{Q} . A Whittaker pair is an ordered pair $(S, \varphi) \in \mathfrak{g} \times \mathfrak{g}^*$, where S is a rational semi-simple element, and $ad^*(S)\varphi = -2\varphi$. This eigenvalue equation implies that φ is a nilpotent element of \mathfrak{g}^* . Using the Killing form we may identify φ with a unique nilpotent element in \mathfrak{g} which we will denote by $f = f_{\varphi}$.

Let χ be a fixed additive, non-trivial unitary character on \mathbb{A} trivial on \mathbb{K} . A Whittaker pair (S,φ) determines a unipotent subgroup $N_{S,\varphi}$ of G given by (2.5) below, as well as a unitary character χ_{φ} on $N_{S,\varphi}$ defined by $\chi_{\varphi}(n) = \chi(\varphi(\log n))$ for $n \in N_{S,\varphi}$. Note that $\log n$ can be expanded to a finite sum since n is unipotent and that we have extended φ to a functional on $\mathfrak{g}(\mathbb{A}) := \mathfrak{g} \otimes_{\mathbb{K}} \mathbb{A}$ by linearity.

Our results are applicable to automorphic forms, but also to a wider class of functions on G which we will call automorphic functions. The space of automorphic functions, denoted by $C^{\infty}(\Gamma\backslash G)$, consists of functions that are left Γ -invariant, finite under the right action of the preimage in G of $\prod_{\text{finite }\nu} \mathbf{G}(\mathfrak{o}_{\nu})$, and smooth when restricted to the preimage in G of $\prod_{\text{infinite }\nu} \mathbf{G}(\mathbb{K}_{\nu})$. In other words, compared to automorphic forms, we remove the usual requirements of moderate growth, and finiteness under the center \mathfrak{z} of the universal enveloping algebra.

Following [MW87, GRS97, GRS11, GGS17] we attach to each Whittaker pair (S, φ) and automorphic function η on G the Fourier coefficient

$$\mathcal{F}_{S,\varphi}[\eta](g) = \int_{[N_{S,\varphi}]} \eta(ng) \,\chi_{\varphi}(n)^{-1} \,dn. \tag{1.6}$$

where we, for a unipotent subgroup $U \subset G$, denote by [U] the quotient $(U \cap \Gamma) \setminus U$. Here, dn denotes the pushforward to $[N_{S,\varphi}]$ of the Haar measure on $N_{S,\varphi}$, that is normalized such that the pushforward is a probability measure. We will use similar notation in the future without further notice.

Remark 1.2.1. This definition is more general than what is usually referred to as a Fourier coefficient in the literature, cf. [GRS97, GRS11, Gin06, GH11, JLS16]. Note also that the unipotent group $N_{S,\omega}$ is not necessarily the unipotent radical of a parabolic subgroup of

G. For a Jacobson-Morozov \mathfrak{sl}_2 -triple (e, h, f) we define $\varphi_f \in \mathfrak{g}^*$ to be given by Killing form pairing with f, and call the Whittaker pair (h, φ) a neutral Whittaker pair, and the corresponding Fourier coefficient a neutral Fourier coefficient. This is what is called a Fourier coefficient in [GRS97, GRS11, Gin06, GH11, JLS16].

Note that $\mathcal{F}_{S,\varphi}[\eta](g)$ is a smooth function on G in the above sense, but is not invariant under Γ any more. On the other hand, its restriction to the joint centralizer $G_{S,\varphi}$ of S and φ is left $G_{S,\varphi} \cap \Gamma$ -invariant. As shown in [GH11], if η is also \mathfrak{z} -finite and has moderate growth, then the restriction of η to $G_{S,\varphi}$ still has moderate growth, but may stop being \mathfrak{z} -finite.

As further detailed in Definition 2.3.3 below, we denote by WO(η) the set of nilpotent $\mathbf{G}(\mathbb{K})$ -orbits \mathcal{O} in \mathfrak{g}^* such that there exists a non-vanishing neutral Fourier coefficient $\mathcal{F}_{h,\varphi}[\eta]$ with $\varphi \in \mathcal{O}$. It was shown in [GGS17, Theorem C] that if a neutral Fourier coefficient $\mathcal{F}_{h,\varphi}[\eta] = 0$ then $\mathcal{F}_{S,\varphi}[\eta] = 0$ for any Whittaker pair (S,φ) , not necessarily neutral. The set WO is also sometimes called global wave-front set in the literature [JLS16].

In Definition 2.3.1 below we introduce an order relation on \mathbb{K} -rational nilpotent orbits. By Lemma 2.3.2, if $\mathcal{O}' > \mathcal{O}$ under this relation then for any place ν of \mathbb{K} , the closure of \mathcal{O}' in $\mathfrak{g}^*(\mathbb{K}_{\nu})$ (in the local topology) contains \mathcal{O} . We denote the set of maximal elements in $WO(\eta)$ by $WS(\eta)$ and call it the Whittaker support of η .

Because of the many different kinds of Fourier coefficients figuring in this paper, we will also make the following distinctions. If $N_{S,\varphi}$ is the unipotent radical of a minimal parabolic subgroup of G, we say that (S,φ) is a standard Whittaker pair and call the Fourier coefficient $\mathcal{F}_{S,\varphi}$ a Whittaker coefficient. One can show that a nilpotent $\varphi \in \mathfrak{g}^*$ can be completed to a standard Whittaker pair if and only if it is a principal nilpotent element of a Levi subgroup of G defined over \mathbb{K} . For short, we will call such φ a PL nilpotent and call its orbit a PL-orbit. See §2.1 below for further details.

Finally, in §2.2 we will define what we call Levi-distinguished Fourier coefficients. Such a coefficient is defined by a parabolic subgroup $P \subseteq G$ (defined over \mathbb{K}), a Levi decomposition P = LU and a Whittaker pair (H, φ) for L, in which φ is \mathbb{K} -distinguished, i.e. does not belong to the dual Lie algebra of any proper Levi subgroup of L defined over \mathbb{K} . The corresponding Fourier coefficient is given by considering the constant term with respect to U as a function on L, and then taking the Fourier coefficient $\mathcal{F}_{H,\varphi}$. To see that this construction defines a Fourier coefficient on G, we let Z be a rational semi-simple element that commutes with L and has all its non-zero adjoint eigenvalues much bigger than those of H (in absolute value). Then the Levi-distinguished Fourier coefficient is $\mathcal{F}_{H+Z,\varphi}$. By Lemma 2.2.9 below, if φ is a principal nilpotent in L then $\mathcal{F}_{H+Z,\varphi}$ is a Whittaker coefficient.

We will be relating different types of Fourier coefficients using linear operations. To make this statement precise, we introduce the following definition.

Definition 1.2.2. Let $A: X \to X_A$ and $B: X \to X_B$ be linear maps between vector spaces X, X_A and X_B . We say that A is *linearly determined* by B if $\ker(B) \subseteq \ker(A)$.

Now, let $B_i: X \to X_{B_i}$ be a family of linear maps index by a (possibly infinite) set I. We say that A is linearly determined by $\{B_i\}$ if A is linearly determined by $B: X \to \prod_{i \in I} X_{B_i}$ defined by $x \mapsto (B_1(x), B_2(x), \ldots)$. Note that $\ker(B) = \bigcap_{i \in I} \ker(B_i)$.

Remark 1.2.3. It is easy to see that A is linearly determined by B if and only if there exists a linear map $C: X_B \to X_A$ such that $A = C \circ B$. In this paper we will consider

 $X = C^{\infty}(\Gamma \backslash G)$, or a subspace thereof, and the maps A and B to be different Fourier coefficients. In the course of the proofs of our main theorems we will provide explicit formulas for the relevant linear maps C. They involve sums over characters, sums over Γ -translates of the arguments and integrations over unipotent subgroups giving expressions schematically of the form

$$\sum_{S,\varphi} \sum_{\gamma} \int dv \, \mathcal{F}_{S,\varphi}[\eta](\gamma vg) \,. \tag{1.7}$$

In this setting, the notion of *linearly determined by* is equivalent to the notion of *spanned by* used in [HS16].

Our main results can be summarized in the following theorems which are proven in §4.

Theorem A. Let $\eta \in C^{\infty}(\Gamma \backslash G)$. Then

- (i) η is linearly determined by the family of all its Levi-distinguished Fourier coefficients.
- (ii) If all orbits $\mathcal{O} \in WO(\eta)$ are PL-orbits then η , and all its Fourier coefficients, are linearly determined by Whittaker coefficients.
- (iii) Any Fourier coefficient $\mathcal{F}_{S,\varphi}[\eta]$ is linearly determined by the Levi-distinguished Fourier coefficients with characters in orbits which are equal to or bigger than $\Gamma\varphi$.

Theorem A immediately implies the following corollary.

Corollary B. Let $\eta \in C^{\infty}(\Gamma \backslash G)$ be cuspidal, in the sense that the constant term of η with respect to the nilradical of any parabolic subgroup vanishes. Then η is linearly determined by its Fourier coefficients with respect to \mathbb{K} -distinguished orbits.

Remark 1.2.4. One can show that for split simply-laced groups the so-called minimal and the next-to-minimal orbits are always PL. Thus, Theorem A implies that automorphic forms attached to the minimal and next-to-minimal representations of simply-laced groups, as well as all their Fourier coefficients, are linearly determined by Whittaker coefficients. We provide more explicit formulas for this linear determination in a subsequent paper [GGK⁺19].

In order to present our next theorem we will need to introduce some notation.

Notation 1.2.5. For a rational semi-simple $H \in \mathfrak{g}$ and $\lambda \in \mathbb{Q}$ denote by $\mathfrak{g}_{\lambda}^{H}$ the λ -eigenspace of $\mathrm{ad}(H)$. Denote also $\mathfrak{g}_{>\lambda}^{H} := \bigoplus_{\mu > \lambda} \mathfrak{g}_{\lambda}^{H}$, $\mathfrak{g}_{\geq \lambda}^{H} := \mathfrak{g}_{\lambda}^{H} \oplus \mathfrak{g}_{>\lambda}^{H}$, and similarly for $\mathfrak{g}_{<\lambda}^{H}$ and $\mathfrak{g}_{\leq \lambda}^{H}$. For $\varphi \in \mathfrak{g}^{*}$ denote by \mathfrak{g}_{φ} its stabilizer in \mathfrak{g} under the coadjoint action.

As will be further explained in §2, the group cover splits over unipotent subgroups of G which means that we may uniquely identify them with unipotent subgroups of G. Thus, for a nilpotent subalgebra $\mathfrak{v} \subset \mathfrak{g}$, let $\mathfrak{v}(\mathbb{A}) := \mathfrak{v} \otimes_{\mathbb{K}} \mathbb{A}$ and denote by $\operatorname{Exp}(\mathfrak{v})$ the unipotent subgroup of Γ obtained by exponentiation of \mathfrak{v} and by $V := \operatorname{Exp}(\mathfrak{v}(\mathbb{A}))$ the unipotent subgroup of G obtained by exponentiation of $\mathfrak{v}(\mathbb{A})$.

Definition 1.2.6. Let (H, φ) and (S, φ) be two Whittaker pairs with the same φ . We say that (H, φ) dominates (S, φ) if H and S commute and

$$\mathfrak{g}_{\varphi} \cap \mathfrak{g}_{>1}^{H} \subseteq \mathfrak{g}_{>0}^{S-H}. \tag{1.8}$$

The use of the term *dominates* is due to Proposition 4.0.1 below where we show that if (H, φ) dominates (S, φ) then $\mathcal{F}_{S,\varphi}[\eta]$ is linearly determined by $\mathcal{F}_{H,\varphi}[\eta]$ for any automorphic function η . As is shown in the proof there this follows directly and naturally from a deformation of the dominating Whittaker pair to the other.

In contrast, the next theorem shows how to determine $\mathcal{F}_{H,\varphi}[\eta]$ in terms of $\mathcal{F}_{S,\varphi}[\eta]$ when the orbit $\mathbf{G}(\mathbb{K})\varphi$ is an element of $\mathrm{WS}(\eta)$.

Theorem C. Let η be an automorphic function on G. Let (H, φ) and (S, φ) be Whittaker pairs such that (H, φ) dominates (S, φ) and $\mathbf{G}(\mathbb{K})\varphi \in \mathrm{WS}(\eta)$. Denote

$$\mathfrak{v} := \mathfrak{g}_{>1}^H \cap \mathfrak{g}_{<1}^S, \ and \ V := \operatorname{Exp}(\mathfrak{v}(\mathbb{A})). \tag{1.9}$$

(i) If $\mathfrak{g}_1^H = \mathfrak{g}_1^S = 0$ then

$$\mathcal{F}_{H,\varphi}[\eta](g) = \int_{V} \mathcal{F}_{S,\varphi}[\eta](vg) \, dv \,. \tag{1.10}$$

(ii) More generally, denote $\mathfrak{u}:=(\mathfrak{g}_{\geq 1}^S\cap\mathfrak{g}_{>1}^H)/(\mathfrak{g}_{>1}^S\cap\mathfrak{g}_{>1}^H),\ \mathfrak{w}:=(\mathfrak{g}_{\geq 1}^H\cap\mathfrak{g}_{<1}^S)/\mathfrak{v},\ \Omega:=\mathrm{Exp}(\mathfrak{w})\ and\ U:=\mathrm{Exp}(\mathfrak{u}(\mathbb{A})).$ Then

$$\mathcal{F}_{H,\varphi}[\eta](g) = \sum_{w \in \Omega} \int_{V} \int_{[U]} \mathcal{F}_{S,\varphi}[\eta](wvug) \, dudv \,. \tag{1.11}$$

We would like to emphasize that the integral over V is a non-compact adelic integral.

Remark 1.2.7. In [GGS17] (and in Corollary 3.2.2) it is shown that any Whittaker pair (H, φ) is dominated by a neutral pair (h, φ) . In §4 we will show that any Whittaker pair (H, φ) dominates a Levi-distinguished pair (S, φ) . By Lemma 2.2.9 below, if φ is principal in a Levi subgroup (PL), then any Levi-distinguished Fourier coefficient $\mathcal{F}_{S,\varphi}[\eta]$ is a Whittaker coefficient, and thus, if φ is PL and $\mathbf{G}(\mathbb{K})\varphi \in \mathrm{WS}(\eta)$ then any Fourier coefficient $\mathcal{F}_{H,\varphi}[\eta]$ is obtained by an integral transform from a Whittaker coefficient $\mathcal{F}_{S,\varphi}[\eta]$.

1.3. **Structure of the paper.** In §2 we give the definitions of the notions mentioned above, as well as of *Whittaker triples* and *quasi-Fourier coefficients*. These are technical notions defined in [GGS] and widely used in the current paper as well.

In §3 we relate Fourier and quasi-Fourier coefficients corresponding to different Whittaker pairs and triples. To do that we further develop the deformation technique of [GGS17, GGS], making it both more general, more explicit, and better adapted to the global case. The deformation technique is in turn built upon a version of the root-exchange technique of [GRS97, GRS11].

In §4 we provide proofs for the main theorems and in §5 we provide explicit examples in the cases SL_4 , GL_n , Sp_4 and Heisenberg parabolics of arbitrary simply-laced Lie groups.

Two appendices contain additional proofs of lemmas on PL-orbits and an ordering relation on rational nilpotent orbits.

More explicit results can be obtained in the case of Fourier expansions of automorphic forms in minimal and next-to-minimal representations as will be proved in the follow-up paper [GGK⁺19].

1.4. Acknowledgements. The authors are grateful for helpful discussions with Ben Brubaker, David Ginzburg and Stephen D. Miller. We are particularly thankful to Joseph Hundley for sharing with us his insights on nilpotent orbits for exceptional groups. We also thank the Banff International Research Station for Mathematical Innovation and Discovery and the Simons Center for Geometry and Physics for their hospitality during different stages of this project. D.G. was partially supported by ERC StG grant 637912. During his time at Stanford University (when the majority of this paper was written), H.G. was supported by the Knut and Alice Wallenberg Foundation. Later, H.G. was supported by the Swedish Research Council (Vetenskapsrådet), grant nr. 2018-06774. S.S. was partially supported by Simons Foundation grant 509766. D.P. was supported by the Swedish Research Council (Vetenskapsrådet), grant nr. 2018-04760.

2. Definitions

As in the introduction, let \mathbb{K} be a number field, \mathfrak{o} its ring of integers, and let $\mathbb{A} = \mathbb{A}_{\mathbb{K}}$ be its ring of adeles. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and fix a non-trivial additive character $\chi : \mathbb{A} \to \mathbb{T}$, which is trivial on \mathbb{K} . Then χ defines an isomorphism between \mathbb{A} and the character group $\mathbb{A} := \text{Hom}(\mathbb{A}, \mathbb{T})$ via the map $a \mapsto \chi_a$, where $\chi_a(b) = \chi(ab)$ for $a, b \in \mathbb{A}$. Furthermore, this isomorphism restricts to an isomorphism

$$\widehat{\mathbb{A}/\mathbb{K}} := \operatorname{Hom}(\mathbb{A}/\mathbb{K}, \mathbb{T}) \cong \{ r \in \widehat{\mathbb{A}} : r|_{\mathbb{K}} \equiv 1 \} = \{ \chi_a : a \in \mathbb{K} \} \cong \mathbb{K}, \tag{2.1}$$

which means that we may parametrize characters on \mathbb{A} trivial on \mathbb{K} by elements in \mathbb{K} .

Let **G** be a reductive group defined over \mathbb{K} , $\mathbf{G}(\mathbb{A})$ the group of adelic points of **G** and let G be a finite central extension of $\mathbf{G}(\mathbb{A})$. That is,

$$1 \to C \to G \xrightarrow{\text{pr}} \mathbf{G}(\mathbb{A}) \to 1 \tag{2.2}$$

for some finite group C.

We assume that there exists a section $\mathbf{G}(\mathbb{K}) \to G$ of the projection $\mathrm{pr}: G \to \mathbf{G}(\mathbb{A})$. Fix such a section and denote its image in G by Γ . By [MW95, Appendix I], the cover $G \to \mathbf{G}(\mathbb{A})$ canonically splits over unipotent subgroups, and thus we will consider such subgroups as subgroups of G. Let $\mathfrak{g}(\mathbb{K})$ denote the Lie algebra of $\mathbf{G}(\mathbb{K}) \cong \Gamma$ which we will often abbreviate to \mathfrak{g} . Let \mathfrak{v} be a nilpotent subalgebra of \mathfrak{g} and let $\mathfrak{v}(\mathbb{A}) := \mathfrak{v} \otimes_{\mathbb{K}} \mathbb{A}$ be its adelization. As in the introduction, we denote by $\mathrm{Exp}(\mathfrak{v})$ the unipotent subgroup of Γ obtained by exponentiation of \mathfrak{v} using the above split over unipotent subgroups, and we denote by $V := \mathrm{Exp}(\mathfrak{v}(\mathbb{A}))$ the unipotent subgroup of G obtained by exponentiation of $\mathfrak{v}(\mathbb{A})$. We note that $\mathrm{Exp}(\mathfrak{v}) = V \cap \Gamma$ and for later convenience we will denote by [V] the quotient $(V \cap \Gamma) \setminus V$.

To conveniently describe different unipotent subgroups of G and characters on these subgroups we introduce the following notion.

Definition 2.0.1. A Whittaker pair is an ordered pair $(S, \varphi) \in \mathfrak{g} \times \mathfrak{g}^*$ such that S is a rational semi-simple element (that is, a semi-simple element for which the eigenvalues of the adjoint action are in \mathbb{Q}), and $\mathrm{ad}^*(S)\varphi = -2\varphi$.

We will often identify $\varphi \in \mathfrak{g}^*$ with its dual nilpotent element $f = f_{\varphi} \in \mathfrak{g}$ with respect to the Killing form $\langle \,, \, \rangle$. We will say that φ is nilpotent if f_{φ} is a nilpotent element of \mathfrak{g} . Equivalently, $\varphi \in \mathfrak{g}^*$ is nilpotent if and only if the Zariski closure of its coadjoint orbit includes zero. For example, if (S, φ) is a Whittaker pair then φ is nilpotent.

For any rational semi-simple $S \in \mathfrak{g}$ and $i \in \mathbb{Q}$ we introduce the following notation

$$\mathfrak{g}_{i}^{S} := \{ X \in \mathfrak{g} : [S, X] = iX \}, \qquad \mathfrak{g}_{>i}^{S} := \bigoplus_{\substack{j \in \mathbb{Q} \\ j > i}} \mathfrak{g}_{j}^{S}, \qquad \mathfrak{g}_{\geq i}^{S} := \mathfrak{g}_{i}^{S} \oplus \mathfrak{g}_{>i}^{S}, \qquad (2.3)$$

and analogously for $\mathfrak{g}_{\leq i}^S$ and $\mathfrak{g}_{\leq i}^S$, with a similar use of notation for \mathfrak{g}^* .

For any $\varphi \in \mathfrak{g}^*$ let \mathfrak{g}_{φ} be the centralizer of φ in \mathfrak{g} under the coadjoint action and define an anti-symmetric form $\omega_{\varphi} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ by $\omega_{\varphi}(X,Y) = \varphi([X,Y])$. We extend φ and ω_{φ} to a functional and an anti-symmetric form on $\mathfrak{g}(\mathbb{A})$ respectively by linearity. Given a Whittaker pair $(S,\varphi) \in \mathfrak{g} \times \mathfrak{g}^*$, we let $\mathfrak{u} := \mathfrak{g}_{\geq 1}^S$ and define

$$\mathfrak{n}_{S,\varphi} := \{ X \in \mathfrak{u} : \omega_{\varphi}(X,Y) = 0 \text{ for all } Y \in \mathfrak{u} \} \quad \text{and} \quad N_{S,\varphi} := \operatorname{Exp}(\mathfrak{n}_{S,\varphi}(\mathbb{A})) \tag{2.4}$$

which, by Lemma 3.2.5 below, can also be written as

$$\mathfrak{n}_{S,\varphi} = \mathfrak{g}_{>1}^S \oplus (\mathfrak{g}_1^S \cap \mathfrak{g}_{\varphi}). \tag{2.5}$$

Note that $\mathfrak{n}_{S,\varphi}$ is an ideal in \mathfrak{u} with abelian quotient, and that φ defines a character of $\mathfrak{n}_{S,\varphi}$. We define a corresponding character χ_{φ} on $N_{S,\varphi}$, trivial on $N_{S,\varphi} \cap \Gamma$, by

$$\chi_{\varphi}(n) := \chi(\varphi(\log n)) = \chi(\langle f_{\varphi}, \log n \rangle). \tag{2.6}$$

More generally, let $\mathfrak{r} \subseteq \mathfrak{u}$ be any isotropic subspace (not necessarily maximal) with respect to $\omega_{\varphi}|_{\mathfrak{u}}$, that includes $\mathfrak{n}_{S,\varphi}$. Note that $\mathfrak{n}_{S,\varphi} \subseteq \mathfrak{r} \subseteq \mathfrak{u}$, and that $\mathfrak{n}_{S,\varphi}$ and \mathfrak{r} are ideals in \mathfrak{u} . Let $R = \operatorname{Exp}(\mathfrak{r}(\mathbb{A}))$. Then $\chi_{\varphi}^R : R \to \mathbb{T}$ defined by $\chi_{\varphi}^R(r) = \chi(\varphi(\log r))$ is a character of R trivial on $R \cap \Gamma$. Indeed, since \mathfrak{r} is isotropic, we have that $\omega_{\varphi}|_{\mathfrak{r}(\mathbb{A})} = 0$ and thus $\chi_{\varphi}^R \in \operatorname{Hom}(R, \mathbb{T})$, and $\varphi(X) \in \mathbb{K}$ for $X \in \mathfrak{r}(\mathbb{K})$.

Definition 2.0.2. We call a function on G an automorphic function if it satisfies the following properties:

- (i) invariant under the left action of Γ ,
- (ii) finite under the right action of the preimage in G of $\prod_{\text{finite }\nu} \mathbf{G}(\mathfrak{o}_{\nu})$, and
- (iii) smooth when restricted to the preimage in G of $\prod_{\text{infinite }\nu} \mathbf{G}(\mathbb{K}_{\nu})$.

We denote the space of all automorphic functions by $C^{\infty}(\Gamma \backslash G)$.

Definition 2.0.3. Let (S, φ) be a Whittaker pair for \mathfrak{g} and let $R, N_{S,\varphi}, \chi_{\varphi}$ and χ_{φ}^{R} be as above. For an automorphic function η , we define the *Fourier coefficient* of η with respect to the pair (S, φ) to be

$$\mathcal{F}_{S,\varphi}[\eta](g) := \int_{[N_{S,\varphi}]} \eta(ng) \, \chi_{\varphi}(n)^{-1} \, dn. \tag{2.7}$$

We also define its R-Fourier coefficient to be the function

$$\mathcal{F}_{S,\varphi}^{R}[\eta](g) := \int_{[R]} \eta(rg) \, \chi_{\varphi}^{R}(r)^{-1} \, dr. \tag{2.8}$$

Observe that if π denotes a subrepresentation of $C^{\infty}(\Gamma \backslash G)$ that contains η then $\mathcal{F}_{S,\varphi}[\eta]$ and $\mathcal{F}_{S,\varphi}^{R}[\eta]$ are matrix coefficients corresponding to the vector $\eta \in \pi$ and the functional on the space of automorphic functions defined by the integrals above.

Remark 2.0.4. In [GGS17, §6] the integrals (2.7) and (2.8) above are called Whittaker–Fourier coefficients, but in this paper we call them Fourier coefficients for short. The Whittaker coefficients (1.2) are called in [GGS17, §6] principal degenerate Whittaker–Fourier coefficients.

Definition 2.0.5. A Whittaker pair (h,φ) is called *neutral* if either $(h,\varphi) = (0,0)$, or h and the Killing form pairing $f = f_{\varphi}$ with φ can be completed to an \mathfrak{sl}_2 -triple (e,h,f). Equivalently, (h,φ) is called neutral if $\mathrm{ad}^*(\varphi)$ defines an epimorphism $\mathfrak{g}_0^h \twoheadrightarrow (\mathfrak{g}^*)_{-2}^h$, and h can be completed to an \mathfrak{sl}_2 -triple. For more details on \mathfrak{sl}_2 -triples over arbitrary fields of characteristic zero see [Bou75, §11].

Definition 2.0.6. We say that (S, φ, φ') is a Whittaker triple if (S, φ) is a Whittaker pair and $\varphi' \in (\mathfrak{g}^*)_{>-2}^S$.

For a Whittaker triple (S, φ, φ') , let U, R, and $N_{S,\varphi}$ be as in Definition 2.0.3. Note that $\varphi + \varphi'$ defines a character of \mathfrak{r} . Extend it by linearity to a character of $\mathfrak{r}(\mathbb{A})$ and define an automorphic character $\chi_{\varphi+\varphi'}$ of R by $\chi_{\varphi+\varphi'}^R(\exp X) := \chi(\varphi(X) + \varphi'(X))$. For an example for this notation see §5.2 below.

Definition 2.0.7. For an automorphic function f, we define its (S, φ, φ') -quasi Fourier coefficient to be the function

$$\mathcal{F}_{S,\varphi,\varphi'}[\eta](g) := \int_{[N_{S,\varphi}]} \chi_{\varphi+\varphi'}(n)^{-1} \eta(ng) dn.$$
 (2.9)

We also define its $(S, \varphi, \varphi', R)$ -quasi Fourier coefficient to be the function

$$\mathcal{F}_{S,\varphi,\varphi'}^{R}[\eta](g) := \int_{[R]} \chi_{\varphi+\varphi'}^{R}(r)^{-1} \eta(rg) dr. \tag{2.10}$$

Definition 2.0.8. We call a \mathbb{K} -subgroup of \mathbf{G} a *split torus of rank m* if it is isomorphic as a \mathbb{K} -subgroup to GL_1^m . We call a Lie subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$ a \mathbb{K} -Levi subalgebra if it is the centralizer of a split torus.

Remark 2.0.9. We note that the Lie algebra of any split torus is spanned by rational semisimple elements. Consequently, a subalgebra of $\mathfrak{l} \subseteq \mathfrak{g}$ is a \mathbb{K} -Levi subalgebra if and only if it is the centralizer of a rational semisimple element of \mathfrak{g} . Another equivalent condition is that \mathfrak{l} is the Lie algebra of a Levi subgroup of a parabolic subgroup of G defined over \mathbb{K} .

For convenience, we fix a complex embedding $\sigma : \mathbb{K} \hookrightarrow \mathbb{C}$, which allows us to map a Γ -orbit \mathcal{O} in \mathfrak{g} to a $\mathbf{G}(\mathbb{C})$ -orbit in $\mathfrak{g}(\mathbb{C}) := \mathfrak{g} \otimes_{\sigma(\mathbb{K})} \mathbb{C}$. One can show, using [Đok98], that the complex orbit corresponding to \mathcal{O} does not depend on σ . However, we will not need this fact.

2.1. Principal nilpotent elements, PL elements and standard Whittaker pairs.

Definition 2.1.1. We say that a nilpotent orbit under Γ in \mathfrak{g}^* is principal if it is Zariski dense in the nilpotent cone $\mathcal{N}(\mathfrak{g}^*)$. We say that $\varphi \in \mathfrak{g}^*$ is a principal nilpotent element if its orbit is principal.

We say that a nilpotent $\varphi \in \mathfrak{g}^*$ is *principal in a Levi* (or PL for short) if there exists a \mathbb{K} -Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$ and a nilpotent element $f \in \mathfrak{l}$ such that the Killing form pairing

with f defines φ in \mathfrak{g}^* , and a principal nilpotent element of \mathfrak{l}^* . We call a nilpotent Γ -orbit in \mathfrak{g}^* a PL-orbit if it consists of PL elements.

We remark that if G is quasi-split then a nilpotent element $\varphi \in \mathfrak{g}^*$ is principal if and only if it is regular, *i.e.* the dimension of its centralizer equals the rank of \mathfrak{g} .

Lemma 2.1.2. Let \mathfrak{n} be the nilpotent radical of the Lie algebra of a minimal parabolic subgroup P_0 . Then \mathfrak{n} intersects any nilpotent orbit under Γ in \mathfrak{g} .

Proof. Let $f \in \mathfrak{g}$ be nilpotent, and complete to an \mathfrak{sl}_2 -triple (e, h, f). Then h defines a parabolic subgroup P which then includes a minimal parabolic Q_0 . Then Q_0 is conjugate to P_0 under Γ (see [BT65, Thm. 4.13(b)]). Since f lies in the nilpotent radical of the Lie algebra of P, its conjugate will lie in \mathfrak{n} .

Definition 2.1.3. We say that a Whittaker pair (S, φ) is *standard* if $\mathfrak{n}_{S,\varphi}$ is the nilpotent radical of the Lie algebra of a minimal parabolic subgroup of G.

Corollary 2.1.4. A nilpotent element $\varphi \in \mathfrak{g}^*$ is principal if and only if it can be completed to a neutral standard Whittaker pair.

Proof. Let h complete φ to a neutral standard Whittaker pair. Then $\mathfrak{n}_{h,\varphi}=\mathfrak{g}_{>0}^h=\mathfrak{g}_{>0}^h$ is the nilpotent radical of the Lie algebra of a minimal parabolic subgroup, and thus so is $\mathfrak{n}:=\mathfrak{g}_{<1}^h$. Let $f\in\mathfrak{n}$ define φ through the Killing form pairing. Then we have $[f,\mathfrak{g}_{\leq 0}^h]=\mathfrak{n}$ and thus $\Gamma f\cap\mathfrak{n}$ is Zariski open, and thus Zariski dense, in \mathfrak{n} . The statement follows now from Lemma 2.1.2.

Conversely, let $\varphi \in \mathfrak{g}^*$ be a principal nilpotent, and let (e,h,f) be an \mathfrak{sl}_2 -triple such that f defines φ via the Killing form. Let \mathcal{O} denote the complex orbit of f and $\bar{\mathcal{O}}$ denote its Zariski closure. Then $\bar{\mathcal{O}} = \mathcal{N}(\mathfrak{g}) \supset \mathfrak{g}_{<0}^h$. Thus \mathcal{O} is the Richardson orbit for $\mathfrak{g}_{\leq 0}^h$, and thus $\dim \mathcal{O} = 2 \dim \mathfrak{g}_{<0}^h$. Now suppose by way of contradiction that the pair (h,φ) is not standard. Then $\mathfrak{g}_{\leq 0}^h$ is not a minimal \mathbb{K} -parabolic subalgebra, *i.e.* there exists a smaller \mathbb{K} -parabolic subalgebra \mathfrak{p} with nilpotent radical $\mathfrak{n} \supsetneq \mathfrak{g}_{<0}^h$. But $\mathfrak{n} \subset \mathcal{N} = \bar{\mathcal{O}}$, and thus \mathcal{O} is a Richardson orbit for \mathfrak{p} , thus $\dim \mathcal{O} = 2 \dim \mathfrak{n} > 2 \dim \mathfrak{g}_{<0}^h = \dim \mathcal{O}$ - contradiction.

Corollary 2.1.5. A nilpotent $\varphi \in \mathfrak{g}^*$ is PL if and only if it can be completed to a standard Whittaker pair (S, φ) .

Proof. Let (S, φ) be a standard Whittaker pair. Then S = h + Z where (h, φ) is neutral and commutes with Z. Then Z defines a Levi subalgebra \mathfrak{l} , and the Whittaker pair (h, φ) is neutral and standard in \mathfrak{l} . By Corollary 2.1.4, φ is principal in \mathfrak{l} .

Conversely, if φ is principal in \mathfrak{l} and Z defines \mathfrak{l} we let S := TZ + h for $T \in \mathbb{Q}_{>0}$ big enough. Then (S, φ) is a standard Whittaker pair.

Let us remark that in [GGS17] a different definition of principal and PL elements was given. The following lemma states the equivalence of the definitions.

Lemma 2.1.6. Let $\varphi \in \mathfrak{g}^*$ be nilpotent. Then

(i) φ is PL if and only if there exist a maximal split toral subalgebra \mathfrak{a} of \mathfrak{g} and a choice of associated simple roots Π such that $\varphi \in \bigoplus_{\alpha_i \in \Pi} \mathfrak{g}_{\alpha_i}^*$, where $g_{\alpha_i}^*$ denotes the dual of the root space \mathfrak{g}_{α_i} .

(ii) If $\varphi \in \bigoplus_{\alpha_i \in \Pi} \mathfrak{g}_{\alpha_i}^*$ then φ is principal in the Levi subalgebra given by those simple roots α_i for which the projection of φ to $\mathfrak{g}_{\alpha_i}^*$ is non-zero.

Proof. Let $\sum_{\alpha_i \in \Pi} \mathfrak{g}_{\alpha_i}^{\times} \subset \bigoplus_{\alpha_i \in \Pi} \mathfrak{g}_{\alpha_i}^{*}$ denote the subset of vectors with all projections nonzero. It is enough to show that $\varphi \in \mathfrak{g}^*$ is principal if and only if there exist (\mathfrak{a}, Π) as above such that $\varphi \in \sum_{\alpha_i \in \Pi} \mathfrak{g}_{\alpha_i}^{\times}$.

To show that, assume first that φ is principal. Then, by Corollary 2.1.4, φ can be completed to a neutral standard pair (h, φ) . Then h defines a torus and simple roots, and we have $\varphi \in \sum_{\alpha_i \in \Pi} \mathfrak{g}_{\alpha_i}^{\times}$. Conversely, give \mathfrak{a} and Π as above, we let $h := \sum c_i \alpha_i^{\vee}$, where α_i^{\vee} are the coroots given by scalar product with α_i , and c_i are chosen such that $\varphi \in (\mathfrak{g}^*)_{-2}^h$. Then (h, φ) is a standard Whittaker pair. Moreover, φ is a generic element of $(\mathfrak{g}^*)_{-2}^h$ and thus the Jacobson–Morozov theory implies that (h, φ) is a neutral pair.

Remark 2.1.7. Note that for $G = GL_n(\mathbb{A})$ all orbits \mathcal{O} are PL-orbits. In general this is, however, not the case, see Appendix A for details.

2.2. Levi-distinguished Fourier coefficients.

Definition 2.2.1. We say that a nilpotent $f \in \mathfrak{g}$ is \mathbb{K} -distinguished, if it does not belong to a proper \mathbb{K} -Levi subalgebra $\mathfrak{l} \subsetneq \mathfrak{g}$. In this case we will also say that $\varphi \in \mathfrak{g}^*$ given by the Killing form pairing with f is \mathbb{K} -distinguished. We will also say that the orbit of φ is \mathbb{K} -distinguished.

Example 2.2.2. The nilpotent orbits under $\mathrm{Sp}_{2n}(\mathbb{C})$ are given by partitions of 2n such that odd parts have even multiplicity. Each such orbit, except the zero one, decomposes to infinitely many $\mathrm{Sp}_{2n}(\mathbb{Q})$ -orbits - one for each collection of equivalence classes of quadratic forms Q_1, \ldots, Q_k of dimensions $m_1, \ldots m_k$ where k is the number of even parts in the partition and $m_1, \dots m_k$ are the multiplicities of these parts. A complex orbit intersects a proper Levi subalgebra if and only if all parts have multiplicity one (and thus there are no odd parts). To see the "only if" part note that if the partition includes a part k with multiplicity two then the orbit intersects the Levi $GL_k \times Sp_{2(n-k)}$. If k is odd then this Levi is defined over \mathbb{Q} and thus all \mathbb{Q} -distinguished orbits correspond to totally even partitions. If k is even then this Levi is defined over \mathbb{O} if and only if the quadratic form on the multiplicity space of k is anisotropic. Thus, we obtain that a necessary condition for an orbit \mathcal{O} to be \mathbb{Q} -distinguished is that its partition $\lambda(\mathcal{O})$ is totally even, a sufficient condition is that $\lambda(\mathcal{O})$ is multiplicity free, and for totally even partitions with multiplicities there are infinitely many Q-distinguished orbits and at least one not Q-distinguished. For example, for the partition (4,2) all orbits in $\mathfrak{sp}_6(\mathbb{Q})$ are \mathbb{Q} -distinguished, for the partition 2^3 some orbits are Q-distinguished and some are not, and all other partitions do not correspond to Q-distinguished orbits.

Lemma 2.2.3. Every principal nilpotent element is \mathbb{K} -distinguished.

Proof. Let $f \in \mathfrak{g}$ define a principal nilpotent element via the Killing form. Suppose the contrary: f lies in a proper \mathbb{K} -Levi subalgebra \mathfrak{l} of \mathfrak{g} . Let $Z \in \mathfrak{g}$ be a rational semi-simple element that defines \mathfrak{l} . Complete f to an \mathfrak{sl}_2 -triple $\gamma := (e, h, f)$ in \mathfrak{l} . Then $\mathrm{ad}(Z)$ acts by a scalar on every irreducible submodule of the adjoint action of γ on \mathfrak{g} . Since $\mathfrak{l} \neq \mathfrak{g}$, there exists an irreducible submodule V on which $\mathrm{ad}(Z)$ acts by a negative scalar -c. Let v be

a highest weight vector of V of weight d, and let $S := h + c^{-1}(d+2)Z$. Then $v + f \in \mathfrak{g}_{-2}^S$ and thus v + f is nilpotent. Since f is principal, v + f lies in the Zariski closure of Γf . On the other hand, v + f belongs to the affine space $f + \mathfrak{g}^e$, which is called the Slodowy slice to Γf at f, and is transversal to Γf , contradicting the assumption that v + f lies in the Zariski closure of Γf .

Lemma 2.2.4. Let $f \in \mathfrak{g}$ be nilpotent. Then all \mathbb{K} -Levi subalgebras $\mathfrak{l} \subseteq \mathfrak{g}$ such that $f \in \mathfrak{l}$ and f is \mathbb{K} -distinguished in \mathfrak{l} are conjugate by the centralizer of f.

Proof. Complete f to an \mathfrak{sl}_2 -triple $\gamma:=(e,h,f)$ and denote its centralizer by G_γ . Let us show that all \mathbb{K} -Levi subalgebras \mathfrak{l} of \mathfrak{g} that contain γ and in which f is distinguished are conjugate by G_γ . Let \mathfrak{l} be such a subalgebra, $L\subseteq G$ be the corresponding Levi subgroup, and let C denote the maximal split torus of the center of L. Then C is a split torus in G_γ . Let us show that it is a maximal split torus. Let $T\supseteq C$ be a larger split torus in G_γ . Then, the centralizer of T in \mathfrak{g} is a \mathbb{K} -Levi subalgebra that lies in \mathfrak{l} and includes γ , and thus is equal to \mathfrak{l} . Thus T=C.

Since \mathfrak{l} is the centralizer of T in G, T is a maximal split torus of G_{γ} , and all maximal split tori of reductive groups are conjugate (see [Bor91, 15.14]), we get that all the choices of L are conjugate.

Since all the choices of γ are conjugate by the centralizer of f, the lemma follows.

Definition 2.2.5. Let $Z \in \mathfrak{g}$ be a rational-semisimple element and \mathfrak{l} denote its centralizer. Let (h, φ) be a neutral Whittaker pair for \mathfrak{l} , such that the orbit of φ in \mathfrak{l}^* is \mathbb{K} -distinguished. We say that the Whittaker pair $(h + Z, \varphi)$ is *Levi-distinguished* if

$$\mathfrak{g}_{>1}^{h+Z} = \mathfrak{g}_{\geq 2}^{h+Z} = \mathfrak{g}_{>0}^Z \oplus \mathfrak{l}_{\geq 2}^h \text{ and } \mathfrak{g}_1^{h+Z} = \mathfrak{l}_1^h. \tag{2.11}$$

In this case we also say that the Fourier coefficient $\mathcal{F}_{h+Z,\varphi}$ is Levi-distinguished.

Remark 2.2.6. Let (h, φ) be a neutral Whittaker pair for \mathfrak{g} . If φ is \mathbb{K} -distinguished then $\mathcal{F}_{h,\varphi}$ is a Levi-distinguished Fourier coefficient. If a rational semi-simple Z commutes with h and with φ , and φ is \mathbb{K} -distinguished in $\mathfrak{l} := \mathfrak{g}_0^Z$ then $\mathcal{F}_{h+TZ,\varphi}$ is a Levi-distinguished Fourier coefficient for any T bigger than m/M+1, where m is the maximal eigenvalue of h and M is the minimal positive eigenvalue of h. See also Lemma 4.0.8 for further discussion.

Lemma 2.2.7 ([GGS17, Lemma 3.0.2]). For any Whittaker pair (H, φ) there exists $Z \in \mathfrak{g}_0^H$ such that $(H - Z, \varphi)$ is a neutral Whittaker pair.

Remark 2.2.8. In [GGS17] the lemma is proven over a local field, but the proof only used the Jacobson–Morozov theorem, that holds over arbitrary fields of characteristic zero.

Lemma 2.2.9. For any Whittaker pair (H, φ) , the following are equivalent:

- (i) (H,φ) is standard pair
- (ii) (H,φ) is a Levi-distinguished Fourier coefficient, and φ is a PL nilpotent.

Proof. First let (H, φ) be a standard pair. Then by Lemma 2.2.7, H can be decomposed as H = h + Z where (h, φ) is a neutral pair and Z commutes with h and with φ . Let \mathfrak{g} and L denote the centralizers of Z in \mathfrak{g} and G, and $N := N_{H,\varphi}$. Then N is the unipotent radical of a minimal parabolic subgroup of G, and L is a Levi subgroup of G. Thus, $N \cap L$ is the unipotent radical of a minimal parabolic subgroup of L. The Lie algebra of $N \cap L$ is

 $\mathfrak{n}_{H,\varphi} \cap \mathfrak{g}_0^Z = \mathfrak{g}_{\geq 1}^h \cap \mathfrak{g}_0^Z$. Thus, $\operatorname{Exp}(\mathfrak{g}_{\leq -1}^h \cap \mathfrak{g}_0^Z)$ is the unipotent radical of a minimal parabolic subgroup of L. Since φ is given by Killing form pairing with $f \in \mathfrak{g}_{\leq -1}^h \cap \mathfrak{g}_0^Z$, we get by Corollary 2.1.4 that φ is principal in \mathfrak{l} . Replacing Z by tZ with t large enough, we obtain that (H,φ) is a Levi-distinguished pair.

Now, assume that φ is a PL nilpotent, and let $\mathcal{F}_{h+Z,\varphi}$ be a Levi-distinguished pair. Let $\mathfrak{l}=\mathfrak{g}_0^Z$ be the corresponding Levi, and let $f=f_{\varphi}$ be the element of \mathfrak{g} that defines φ . Since f is distinguished in \mathfrak{l} , and principal in some Levi, Lemmas 2.2.3 and 2.2.4 imply that f is principal in \mathfrak{l} . Thus, $\mathfrak{n}_{H,\varphi}\cap\mathfrak{l}$ is the nilpotent radical of the Lie algebra of a minimal parabolic subgroup of L and thus $\mathfrak{n}_{H,\varphi}=\mathfrak{n}_{H,\varphi}\cap\mathfrak{l}\oplus\mathfrak{g}_{>0}^Z$ the nilpotent radical of the Lie algebra of a minimal parabolic subgroup of G. Thus $\mathcal{F}_{H,\varphi}$ is standard Whittaker pair. \square

2.3. Order on nilpotent orbits and Whittaker support.

Definition 2.3.1. We define a partial order on nilpotent orbits in $\mathfrak{g}^* = \mathfrak{g}^*(\mathbb{K})$ to be the transitive closure of the following relation $R: (\mathcal{O}, \mathcal{O}') \in R$ if $\mathcal{O} \neq \mathcal{O}'$ and there exist $\varphi \in \mathcal{O}$, a rational semi-simple $Z \in \mathfrak{g}$ and $\varphi' \in (\mathfrak{g}^*)_{>0}^Z$ such that $\varphi \in (\mathfrak{g}^*)_0^Z$ and $\varphi + \varphi' \in \mathcal{O}'$.

In Appendix B, we study these rational orbits in more detail. In particular, in Corollary B.2 we prove that this is indeed a partial order, i.e. that R is anti-symmetric.

Lemma 2.3.2. If \mathcal{O}' is bigger than \mathcal{O} , i.e. if $(\mathcal{O}, \mathcal{O}') \in R$, then for any place ν of \mathbb{K} , the closure of \mathcal{O}' in $\mathfrak{g}(\mathbb{K}_{\nu})$ (in the local topology) contains \mathcal{O} .

Proof. It is enough to show that for any $Z \in \mathfrak{g}$, $\varphi \in \mathfrak{g}_0^Z$ and $\psi \in \mathfrak{g}_{>0}^Z$, φ lies in the closure of $\mathbf{G}(\mathbb{K}_{\nu})(\varphi + \psi)$. Let $\varepsilon_i \in \mathbb{K}_{\nu}$ be a sequence converging to zero and let $g_i := \exp(-\varepsilon_i Z)$. Then g_i centralize φ , while $g_i \psi \to 0$. Thus $g_i(\varphi + \psi) \to \varphi$.

Definition 2.3.3. For an automorphic function η , we define WO(η) to be the set of nilpotent orbits \mathcal{O} in \mathfrak{g}^* under the coadjoint action of $\mathbf{G}(\mathbb{K})$ such that $\mathcal{F}_{h,\varphi}[\eta] \neq 0$ for some neutral Whittaker pair (h,φ) with $\varphi \in \mathcal{O}$. Using the partial order of Definition 2.3.1, we define the Whittaker support WS(η) to be the set of maximal elements in WO(η).

3. Relating different Fourier coefficients

3.1. Relating different isotropic subspaces. We will now see how $\mathcal{F}_{S,\varphi,\varphi'}$ linearly determines $\mathcal{F}_{S,\varphi,\varphi'}^R$ and vice versa.

Lemma 3.1.1 (cf. [GGS17, Lemma 6.0.2]). Let $\eta \in C^{\infty}(\Gamma \backslash G)$, let (S, φ, φ') be a Whittaker triple, and $N_{S,\varphi}$, U and R be as in Definition 2.0.1. Let \mathfrak{r}^{\perp} denote the orthogonal complement to \mathfrak{r} in \mathfrak{u} under the form ω_{φ} and let $R^{\perp} := \operatorname{Exp}(\mathfrak{r}^{\perp})$. Then,

$$\mathcal{F}_{S,\varphi,\varphi'}^{R}[\eta](g) = \int_{[R/N_{S,\varphi}]} \mathcal{F}_{S,\varphi,\varphi'}[\eta](ug) du$$
(3.1)

and

$$\mathcal{F}_{S,\varphi,\varphi'}[\eta](g) = \sum_{\gamma \in \exp(\mathfrak{u}/\mathfrak{r}^{\perp})} \mathcal{F}_{S,\varphi,\varphi'}^{R}[\eta](\gamma g). \tag{3.2}$$

Proof. We assume that φ is non-zero since otherwise $R = N_{S,\varphi}$. We have that $N_{S,\varphi} \subseteq R$ with $R/N_{S,\varphi}$ abelian which means that (3.1) follows immediately from the definitions of $\mathcal{F}_{S,\varphi,\varphi'}$ and $\mathcal{F}_{S,\varphi,\varphi'}^R$. For (3.2) observe that the function $(\chi_{\varphi}^R)^{-1} \cdot \mathcal{F}_{S,\varphi,\varphi'}[\eta]$ on R is left-invariant under the action of $N_{S,\varphi} \cdot (R \cap \Gamma)$. In other words, we can identify it with a function on

$$(N_{S,\varphi} \cdot (R \cap \Gamma)) \setminus R \cong \operatorname{Exp}(\mathfrak{r}/\mathfrak{n}_{S,\varphi}) \setminus \operatorname{Exp}((\mathfrak{r}/\mathfrak{n}_{S,\varphi})(\mathbb{A})) =: [R/N_{S,\varphi}], \tag{3.3}$$

where the equality follows from the fact that $R/N_{S,\varphi}$ is abelian. Therefore, we have a Fourier series expansion

$$\mathcal{F}_{S,\varphi,\varphi'}[\eta](u) = \sum_{\psi \in [R/N_{S,\varphi}]^{\wedge}} c_{\psi,\chi_{\varphi+\varphi'}^{R}}(\eta)\psi(u)\chi_{\varphi}^{R}(u), \tag{3.4}$$

where $[R/N_{S,\varphi}]^{\wedge}$ denotes the Pontryagin dual group of $[R/N_{S,\varphi}]$ and

$$c_{\psi,\chi_{\varphi}^{R}}(\eta) = \int_{[R]} \psi(u)^{-1} \chi_{\varphi+\varphi'}^{R}(u)^{-1} \eta(u) du.$$
 (3.5)

In particular, denoting by $Id \in G$ the identity element we obtain

$$\mathcal{F}_{S,\varphi,\varphi'}[\eta](\mathrm{Id}) = \sum_{\psi \in [R/N_{S,\varphi}]^{\wedge}} c_{\psi,\chi_{\varphi+\varphi'}^{R}}(\eta). \tag{3.6}$$

Now observe that the map $X \mapsto \omega_{\varphi}(X, \cdot) = \varphi \circ \operatorname{ad}(X)$ induces an isomorphism between $\mathfrak{u}/\mathfrak{r}^{\perp}$ and $(\mathfrak{r}/\mathfrak{n})'$. Hence, according to equations (2.1) and (3.3), we can use the character χ to define a group isomorphism

$$(U \cap \Gamma)/(R^{\perp} \cap \Gamma) \longrightarrow [R/N_{S,\varphi}]^{\wedge}$$

$$u \mapsto \psi_{u},$$

$$(3.7)$$

where

$$\psi_u(r) = \chi(\varphi([X, Y])), \quad u = \exp X \text{ and } r = \exp Y.$$

Hence, for all $u \in U \cap \Gamma$ and $r \in R$ we have

$$\psi_u(r)\chi_{\varphi+\varphi'}^R(r) = \chi(\varphi([X,Y]) + \varphi'([X,Y]))\chi(\varphi(Y) + \varphi'(Y)) = \chi((\varphi+\varphi')(Y+[X,Y]))$$
$$= \chi((\varphi+\varphi')(e^{\operatorname{ad}(X)}(Y))) = \chi_{\varphi+\varphi'}((\operatorname{Ad}(u)Y)) = \chi_{\varphi+\varphi'}^R(uru^{-1}).$$

Here we are taking again $u = \exp X$, $r = \exp Y$ and the middle equality follows from the vanishing of φ on $\mathfrak{g}_{\geq 2}^S$. But now, from formula (3.5) and the fact that f is automorphic, we have

$$\begin{split} c_{\psi_{u},\chi_{\varphi+\varphi'}^{R}}(\eta) &= \int\limits_{[R]} \psi_{u}(r)^{-1}\chi_{\varphi+\varphi'}^{R}(r)^{-1}\eta(r)dr = \int\limits_{[R]} \chi_{\varphi+\varphi'}^{R}(uru^{-1})^{-1}\eta(r)dr. \\ &= \int\limits_{[R]} \chi_{\varphi+\varphi'}^{R}(r)^{-1}\eta(u^{-1}ru)dr = \mathcal{F}_{S,\varphi,\varphi'}^{R}[\eta](u), \end{split}$$

for all $u \in U \cap \Gamma$. Combining this with (3.6) and (3.7) we obtain

$$\mathcal{F}_{S,\varphi,\varphi'}[\eta](\mathrm{Id}) = \sum_{u \in (U \cap \Gamma)/(R^{\perp} \cap \Gamma)} \mathcal{F}_{S,\varphi,\varphi'}^{R}[\eta](u). \tag{3.8}$$

Applying this to η and its right shifts we obtain (3.2).

Corollary 3.1.2. Let $\eta \in C^{\infty}(\Gamma \backslash G)$, let (S, φ, φ') be a Whittaker triple, and $N_{S,\varphi}$, and U be as in Definition 2.0.1. Let $\mathfrak{r}, \mathfrak{r}' \subseteq \mathfrak{u}$ be two isotropic subspaces (not necessarily maximal) that include $\mathfrak{n}_{S,\varphi}$. Assume dim $\mathfrak{r} = \dim \mathfrak{r}'$. Then

$$\mathcal{F}_{S,\varphi,\varphi'}^{R}[\eta](g) = \int_{R/(R \cap R')} \mathcal{F}_{S,\varphi,\varphi'}^{R'}[\eta](ug) \, du \,. \tag{3.9}$$

Note that this is a non-compact, adelic, integral.

This corollary can be seen as a version of the root exchange lemma in [GRS11].

3.2. Relating different Whittaker pairs. Let (H, φ) be a Whittaker pair.

Lemma 3.2.1. Let Z be as in Lemma 2.2.7. Then $(H - Z, \varphi)$ dominates (H, φ) .

Proof. Denote h := H - Z. We have to show that (1.8) holds, *i.e.*

$$\mathfrak{g}_{\varphi} \cap \mathfrak{g}_{>1}^h \subseteq \mathfrak{g}_{>0}^Z. \tag{3.10}$$

Since \mathfrak{g}_{φ} is spanned by lowest weight vectors, we have $\mathfrak{g}_{\varphi} \subseteq \mathfrak{g}_{\leq 0}^h$ and thus $\mathfrak{g}_{\varphi} \cap \mathfrak{g}_{\geq 1}^h = \{0\}$. \square

Corollary 3.2.2. Any Whittaker pair is dominated by a neutral Whittaker pair.

Another example of domination is provided by the following proposition, that immediately follows from [GGS17, Proposition 3.3.3].

Proposition 3.2.3. If φ is a PL nilpotent then there exists $Z \in \mathfrak{g}$ such that $(H + Z, \varphi)$ is a standard Whittaker pair and (H, φ) dominates $(H + Z, \varphi)$.

From now till the end of the section let $Z \in \mathfrak{g}_0^H$ be a rational semi-simple element such that (H,φ) dominates $(H+Z,\varphi)$. We will now consider the deformation of the former Whittaker pair to the latter. For any rational number $t \geq 0$ define

$$H_t := H + tZ, \quad \mathfrak{u}_t := \mathfrak{g}_{\geq 1}^{H_t}, \quad \mathfrak{v}_t := \mathfrak{g}_{> 1}^{H_t}, \text{ and } \mathfrak{w}_t := \mathfrak{g}_1^{H_t}.$$
 (3.11)

Definition 3.2.4. We call $t \geq 0$ regular if $\mathfrak{u}_t = \mathfrak{u}_{t+\varepsilon}$ for any small enough $\varepsilon \in \mathbb{Q}$, or in other words $\mathfrak{w}_t \subset \mathfrak{g}_0^Z$. If t is not regular we call it *critical*. Equivalently, t is critical if $\mathfrak{g}_1^{H_t} \not\subseteq \mathfrak{g}_0^Z$ which we may interpret as something new has entered the 1-eigenspace of H. For convenience, we will say that t = 0 is critical.

We also say that $t \geq 0$ is quasi-critical if either $\mathfrak{g}_1^{H_t} \not\subseteq \mathfrak{g}_0^Z$ or $\mathfrak{g}_2^{H_t} \not\subseteq \mathfrak{g}_0^Z$. We may interpret this as something new has entered either the 1-eigenspace or the 2-eigenspace. The latter is related to new characters being available in the Whittaker pairs.

Note that there are only finitely many critical numbers. Recall the anti-symmetric form ω_{φ} on \mathfrak{g} given by $\omega_{\varphi}(X,Y)=\varphi([X,Y])$.

Lemma 3.2.5 ([GGS17, Lemma 3.2.6]).

- (i) The form ω_{φ} is ad(Z)-invariant.
- (ii) $\operatorname{Ker} \omega_{\varphi} = \mathfrak{g}_{\varphi}$.
- (iii) $\operatorname{Ker}(\omega_{\varphi}|_{\mathfrak{w}_t}) = \operatorname{Ker}(\omega_{\varphi}) \cap \mathfrak{w}_t.$
- (iv) $\operatorname{Ker}(\omega_{\varphi}|_{\mathfrak{u}_t}) = \mathfrak{v}_t \oplus \operatorname{Ker}(\omega_{\varphi}|_{\mathfrak{w}_t}).$
- (v) $\mathfrak{w}_s \cap \mathfrak{g}_{\varphi} \subseteq \mathfrak{u}_t$ for any s < t.

Recall that $\mathfrak{n}_{H_t,\varphi} := \operatorname{Ker}(\omega_{\varphi}|_{\mathfrak{u}_t})$, denote it by \mathfrak{n}_t . and let

$$\mathfrak{l}_t := (\mathfrak{u}_t \cap \mathfrak{g}_{<0}^Z) + \mathfrak{n}_t \text{ and } \mathfrak{r}_t := (\mathfrak{u}_t \cap \mathfrak{g}_{>0}^Z) + \mathfrak{n}_t.$$
(3.12)

We note that \mathfrak{l}_t and \mathfrak{r}_t are nilpotent subalgebras. The choice of notation for them comes from 'left' and 'right'.

Lemma 3.2.6. For any $t \geq 0$ we have

- (i) l_t and \mathfrak{r}_t are ideals in \mathfrak{u}_t and $[\mathfrak{l}_t,\mathfrak{r}_t] \subseteq \mathfrak{l}_t \cap \mathfrak{r}_t = \mathfrak{n}_t$.
- (ii) \mathfrak{l}_t and \mathfrak{r}_t are isotropic subspaces of \mathfrak{u}_t , and the natural projections $\mathfrak{l}_t/\mathfrak{n}_t \to \mathfrak{u}_t/\mathfrak{r}_t^{\perp}$ and $\mathfrak{r}_t/\mathfrak{n}_t \to \mathfrak{u}_t/\mathfrak{l}_t^{\perp}$ are isomorphisms. Furthermore, $\mathfrak{l}_t = \mathfrak{g}_1^{H_t} \cap \mathfrak{g}_{\leq 0}^Z \oplus \mathfrak{n}_t$.
- (iii) Suppose that $0 \le s < t$, and all the elements of the interval (s,t) are regular. Then

$$\mathfrak{v}_t \oplus (\mathfrak{w}_t \cap \mathfrak{g}_{\leq 0}^Z) = \mathfrak{v}_s \oplus (\mathfrak{w}_s \cap \mathfrak{g}_{>0}^Z) \tag{3.13}$$

$$\mathfrak{l}_t = \mathfrak{r}_s + (\mathfrak{w}_t \cap \mathfrak{g}_{\varphi}) \text{ and } \mathfrak{r}_s \cap (\mathfrak{w}_t \cap \mathfrak{g}_{\varphi}) = \mathfrak{w}_0 \cap \mathfrak{g}_0^Z \cap \mathfrak{g}_{\varphi}.$$
(3.14)

Moreover, \mathfrak{r}_s is an ideal in \mathfrak{l}_t and the quotient is commutative.

Proof. It is easy to see that \mathfrak{v}_t is an ideal in \mathfrak{u}_t with commutative quotient, and that $\mathfrak{v}_t \subseteq \mathfrak{l}_t \cap \mathfrak{r}_t = \mathfrak{n}_t$. This proves (i). For the first part of (ii), note that $q_t := (\mathfrak{l}_t + \mathfrak{r}_t)/\mathfrak{n}_t$ is a symplectic space in which the projections of \mathfrak{l}_t and \mathfrak{r}_t are complementary Lagrangians.

For the second part, we have by Lemma 3.2.5 that $\mathfrak{g}_1^{H_t} \cap \mathfrak{g}_{\varphi} \subseteq \mathfrak{g}_{>0}^Z$ and thus,

$$\mathfrak{l}_t = \mathfrak{v}_t \oplus (\mathfrak{w}_t \cap \mathfrak{g}_{\leq 0}^Z) \oplus (\mathfrak{w}_t \cap \mathfrak{g}_{\varphi}). \tag{3.15}$$

For part (iii) note that

$$\mathfrak{v}_s = (\mathfrak{v}_s \cap \mathfrak{g}_{>0}^Z) \oplus (\mathfrak{v}_s \cap \mathfrak{g}_{<0}^Z), \qquad (3.16)$$

$$\mathfrak{v}_t = (\mathfrak{v}_t \cap \mathfrak{g}_{\leq 0}^Z) \oplus (\mathfrak{v}_t \cap \mathfrak{g}_{\geq 0}^Z), \qquad (3.17)$$

$$\mathfrak{v}_t \cap \mathfrak{g}_{>0}^Z = (\mathfrak{w}_s \cap \mathfrak{g}_{>0}^Z) \oplus (\mathfrak{v}_s \cap \mathfrak{g}_{>0}^Z), \qquad (3.18)$$

$$\mathfrak{v}_s \cap \mathfrak{g}_{<0}^Z = (\mathfrak{w}_t \cap \mathfrak{g}_{<0}^Z) \oplus (\mathfrak{v}_t \cap \mathfrak{g}_{<0}^Z). \tag{3.19}$$

This implies (3.13). By Lemma 3.2.5 we have

$$\mathfrak{n}_s = \mathfrak{v}_s \oplus (\mathfrak{g}_{\varphi} \cap \mathfrak{w}_s) \subseteq \mathfrak{v}_s \oplus (\mathfrak{w}_s \cap \mathfrak{g}_{\geq 0}^Z),$$
(3.20)

and thus

$$\mathfrak{r}_s = \mathfrak{v}_s \oplus (\mathfrak{w}_s \cap \mathfrak{g}_{>0}^Z) \oplus (\mathfrak{w}_0 \cap \mathfrak{g}_0^Z \cap \mathfrak{g}_{\varphi}) \text{ and } \mathfrak{r}_s \cap (\mathfrak{w}_t \cap \mathfrak{g}_{\varphi}) = \mathfrak{w}_0 \cap \mathfrak{g}_0^Z \cap \mathfrak{g}_{\varphi}.$$
 (3.21)

Hence,
$$(3.13)$$
 and (3.15) imply (3.14) , and the rest is straightforward.

Using Lemma 2.2.7, choose an \mathfrak{sl}_2 -triple $(e_{\varphi}, h, f_{\varphi})$ in \mathfrak{g}_0^Z such that h commutes with H and with Z, and φ is given by the Killing form pairing with $f = f_{\varphi}$. Let $L_t := \operatorname{Exp}(\mathfrak{l}_t)$, $R_t := \operatorname{Exp}(\mathfrak{r}_t)$. From Lemmas 3.2.6 and 3.1.1 we get

Lemma 3.2.7. Let $t \geq s \geq 0$ and $\varphi' \in (\mathfrak{g}^*)_{>-2}^{H_t} \cap (\mathfrak{g}^*)_{>-2}^{H_s}$. Assume that there are no critical values in (s,t). Then

(i) $\mathcal{F}_{H_t,\varphi,\varphi'}$, $\mathcal{F}_{H_t,\varphi,\varphi'}^{L_t}$, and $\mathcal{F}_{H_t,\varphi,\varphi'}^{R_t}$ linearly determine each other. In particular,

$$\mathcal{F}_{H_t,\varphi,\varphi'}^{L_t}[\eta](g) = \int_{V} \mathcal{F}_{H_t,\varphi,\varphi'}^{R_t}[\eta](vg) \, dv \,, \tag{3.22}$$

where $\mathfrak{v} := (\mathfrak{g}_{\geq 1}^{H_t} \cap \mathfrak{g}_{\leq 0}^Z)/(\mathfrak{g}_{> 1}^{H_t} \cap \mathfrak{g}_{\leq 0}^Z)$ and $V = \operatorname{Exp}(\mathfrak{v}(\mathbb{A}))$. (ii) $\mathcal{F}_{H_t,\varphi,\varphi'}$ is linearly determined by $\mathcal{F}_{H_s,\varphi,\varphi'}$. Moreover, $\mathcal{F}_{H_s,\varphi,\varphi'}$ is linearly determined by the set

$$\{\mathcal{F}_{H_t,\varphi,\varphi'+\psi'}\mid \psi'\in (\mathfrak{g}^*)_{-1}^{H_t}\cap (\mathfrak{g}^*)^{e_{\varphi}}\cap (\mathfrak{g}^*)_{<0}^Z\}.$$

(iii) Let $\psi \in (\mathfrak{g}^*)_{>-2}^{H_s} \cap (\mathfrak{g}^*)_{-2}^{H_t}$ Then $\mathcal{F}_{H_s,\varphi,\psi+\varphi'}$ is linearly determined by the set

$$\{\mathcal{F}_{H_t,\varphi+\psi,\varphi'+\psi'} \mid \psi' \in (\mathfrak{g}^*)_{-1}^{H_t}\}.$$

(iv) Let $\psi \in (\mathfrak{g}^*)_{-2}^{H_s} \cap (\mathfrak{g}^*)_{-2}^{H_t}$ Then $\mathcal{F}_{H_t, \omega, \psi + \omega'}$ is linearly determined by the set

$$\{\mathcal{F}_{H_s,\varphi+\psi,\varphi'+\psi'} \mid \psi' \in (\mathfrak{g}^*)^{H_s}_{-1}\}.$$

Proof. Part (i) follows from Lemmas 3.1.1 and 3.2.6(ii), and Corollary 3.1.2.

For part (ii), note first that by Lemma 3.2.6, $\mathfrak{r}_s \subseteq \mathfrak{l}_t$ with commutative quotient ($\mathfrak{w}_t \cap$ $\mathfrak{g}_{\varphi})/(\mathfrak{w}_0 \cap \mathfrak{g}_0^Z \cap \mathfrak{g}_{\varphi})$, and let $B := [L_t/R_s]$ denote the corresponding compact commutative group. Then $\mathcal{F}_{H_s,\varphi,\varphi'}^{L_t}$ is obtained from $\mathcal{F}_{H_s,\varphi,\varphi'}^{R_s}$ just by integration over B.

To obtain $\mathcal{F}^{R_s}_{H_s,\varphi,\varphi'}$ we decompose it into Fourier series on B, similar to the proof of Lemma 3.1.1. Characters of B are given by

$$((\mathfrak{w}_t \cap \mathfrak{g}_{\varphi})/(\mathfrak{w}_0 \cap \mathfrak{g}_0^Z \cap \mathfrak{g}_{\varphi}))^* \cong (\mathfrak{g}^*)_{-1}^{H_t} \cap (\mathfrak{g}^*)_{-1}^{e_{\varphi}} \cap (\mathfrak{g}^*)_{<0}^Z$$

and the Fourier series coefficient corresponding to each ψ' in this space is $\mathcal{F}^{L_t}_{H_t,\varphi,\varphi'+\psi'}$.

For part (iii), note that v_t is an ideal in l_t with commutative quotient. Together with (3.14) this implies that \mathfrak{v}_t is an ideal in \mathfrak{r}_s with commutative quotient. Denote V:= $\operatorname{Exp}(\mathfrak{v}_t(\mathbb{A}))$ and define a new coefficient I by

$$I^f(g):=\int_{[V]}\chi_{\varphi+\psi+\varphi'}(n)^{-1}f(ng)dn.$$

Then I is linearly determined by the set

$$\{\mathcal{F}_{H_t,\varphi+\psi,\varphi'+\psi'} | \psi' \in (\mathfrak{g}^*)^{H_t}_{-1}\}.$$

Finally, from (3.13) we see that $\mathcal{F}_{H_s,\varphi,\psi+\varphi'}^{R_s}$ is obtained from I by integration.

Part (iv) is proven in a similar way. Namely, denote $V' := \text{Exp}(\mathfrak{v}_s)$ and define a new coefficient J by

$$J^f(g) := \int_{[V']} \chi_{\varphi + \psi + \varphi'}(n)^{-1} f(ng) dn.$$

Then J is linearly determined by the set

$$\{\mathcal{F}_{H_t,\varphi+\psi,\varphi'+\psi'} | \psi' \in (\mathfrak{g}^*)_{-1}^{H_s} \}.$$

On the other hand, from (3.14) we see that $\mathcal{F}_{H_t,\varphi,\psi+\varphi'}^{L_t}$ is obtained from J by integration. \square

Proposition 3.2.8. Let $H_t = H + tZ$ as above, $s \ge 0$ and let $\varphi' \ne 0 \in (\mathfrak{g}^*)_{>-2}^{H_s} \cap (\mathfrak{g}^*)^{e_{\varphi}} \cap (\mathfrak{g}^*)_{<0}^{Z}$. Then $\mathcal{F}_{H_s,\varphi,\varphi'}$ is linearly determined by the set

$$\{\mathcal{F}_{H_t,\Phi,\Phi'} \mid t > s \text{ critical }, \Phi \in (\mathfrak{g}^*)_{-2}^{H_t}, \Phi' \in (\mathfrak{g}^*)_{>-2}^{H_t} \text{ and } \Gamma\Phi > \Gamma\varphi\},$$
 (3.23)

where $\Gamma\Phi > \Gamma\varphi$ means strictly bigger by the order relation given in Definition 2.3.1.

Note that there are finitely many critical values t.

Proof. Since $\varphi' \in \mathfrak{g}_{\leq 0}^Z$ there exist t > s, $\psi \in (\mathfrak{g}^*)_{-2}^{H_t}$ and $\eta \in (\mathfrak{g}^*)_{>-2}^{H_t}$ such that $\psi \neq 0$ and $\varphi' = \psi + \eta$. Let t be the smallest such t, and since $[Z, e_{\varphi}] = 0$ we have that $\psi, \eta \in (\mathfrak{g}^*)^{e_{\varphi}}$.

Let $a_0 := s$, let a_1, \ldots, a_{m-1} be the critical values between s and t and $a_m := t$. We prove the statement by induction on m.

The base case is m=1, *i.e.* there are no critical values between s and t. Then Lemma 3.2.7(iii) implies that $\mathcal{F}_{H_s,\varphi,\varphi'}$ is linearly determined by the set

$$\{\mathcal{F}_{H_t,\varphi+\psi,\eta+\psi'} | \psi' \in (\mathfrak{g}^*)_{-1}^{H_t} \}.$$

Denote $\Phi := \varphi + \psi$. Note that Φ belongs to the Slodowy slice to $\Gamma \varphi$ at φ since $\psi \in (\mathfrak{g}^*)^{e_{\varphi}}$, and thus $\Gamma \Phi > \Gamma \varphi$. For each ψ' denote $\Phi' := \eta + \psi'$ and note that

$$\mathcal{F}_{H_t,\varphi+\psi,\eta+\psi'} = \mathcal{F}_{H_t,\Phi,\Phi'}.$$

The induction step easily follows from the base using Lemma 3.2.7(ii).

Lemma 3.2.9 ([GGS, Lemma 4.2.4]). Let $\psi \in (\mathfrak{g}^*)_{-2}^H \cap (\mathfrak{g}^*)_{>0}^Z$. Assume that $\varphi + \psi \in \mathbf{G}(\mathbb{C})\varphi$. Then $\varphi + \psi \in \mathbf{G}(\mathbb{C})_H\varphi$.

3.3. Conjugations and translations.

Lemma 3.3.1. Let (S, φ, ψ) be a Whittaker triple, η an automorphic function and $\gamma \in \Gamma$. Then,

$$\mathcal{F}_{S,\varphi,\psi}[\eta](g) = \mathcal{F}_{\mathrm{Ad}(\gamma)S,\mathrm{Ad}^*(\gamma)\varphi,\mathrm{Ad}^*(\gamma)\psi}[\eta](\gamma g). \tag{3.24}$$

Proof. We have that $\chi_{\varphi+\psi}(u) = \chi_{\mathrm{Ad}^*(\gamma)(\varphi+\psi)}(\mathrm{Ad}(\gamma)u)$. Indeed, the right-hand side equals

$$\chi\Big(\big(\mathrm{Ad}^*(\gamma)(\varphi+\psi)\big)(\mathrm{Ad}(\gamma)u)\Big) = \chi\big((\varphi+\psi)(\mathrm{Ad}(\gamma^{-1})\,\mathrm{Ad}(\gamma)u)\big) = \chi_{\varphi+\psi}(u). \tag{3.25}$$

We also have that $\mathfrak{g}_{\lambda}^{\mathrm{Ad}(\gamma)S} = \mathrm{Ad}(\gamma)\mathfrak{g}_{\lambda}^{S}$ since, for $x \in \mathfrak{g}$, $[\mathrm{Ad}(\gamma)S,\mathrm{Ad}(\gamma)x] = \mathrm{Ad}(\gamma)[S,x]$. Similarly, $\mathfrak{g}_{\mathrm{Ad}^{*}(\gamma)\varphi} = \mathrm{Ad}(\gamma)\mathfrak{g}_{\varphi}$ and thus, $\mathfrak{n}_{\mathrm{Ad}(\gamma)S,\mathrm{Ad}^{*}(\gamma)\varphi} = \mathrm{Ad}(\gamma)\mathfrak{n}_{S,\varphi}$.

Hence, using the automorphic invariance of η , the right-hand side of (3.24) equals

$$\int_{[N_{\mathrm{Ad}(\gamma)S,\mathrm{Ad}^*(\gamma)}]} \eta(\gamma^{-1}u\gamma g) \chi_{\mathrm{Ad}^*(\gamma)(\varphi+\psi)}(u)^{-1} du = \int_{[N_{\mathrm{Ad}(\gamma)S,\mathrm{Ad}^*(\gamma)}]} \eta(u'g) \chi_{\mathrm{Ad}^*(\gamma)(\varphi+\psi)}(\mathrm{Ad}(\gamma)u')^{-1} du'.$$

By the arguments above, this equals $\mathcal{F}_{S,\varphi,\psi}[\eta](g)$.

4. Proof of the main results

Proposition 4.0.1. Let (H, φ) and (S, φ) be Whittaker pairs such that (H, φ) dominates (S, φ) . Then $\mathcal{F}_{S,\varphi}$ is linearly determined by $\mathcal{F}_{H,\varphi}$ and $\dim \mathfrak{n}_{H,\varphi} \leq \dim \mathfrak{n}_{S,\varphi}$.

Note that this is in the other direction compared to the statement of Theorem C and is much easier to prove.

Proof. Let Z := S - H, and for any $t \ge 0$ let $H_t := H + tZ$. Let t_1, \ldots, t_k be all the critical values of t between 0 and 1. Let $t_0 := 0$ and $t_{k+1} := 1$. By Lemma 3.2.7(ii), for any $0 \le i \le k$, $\mathcal{F}_{H_{t_i},\varphi}$ linearly determines $\mathcal{F}_{H_{t_{i+1}},\varphi}$. Since $H_{t_0} = H$ and $H_{t_{k+1}} = S$, the first statement of the proposition follows. At all these critical values, the different Fourier coefficients are related by either a root exchange as described in Corollary 3.1.2 for which the dimension of the unipotent subgroup is unchanged, or by a further expansion as in Lemma 3.1.1 for which the dimension is increasing (with t). This proves the second statement of the proposition.

It was shown in Corollary 3.2.2 that any Whittaker pair (S, φ) is dominated by a neutral pair (h, φ) .

Corollary 4.0.2. $\mathcal{F}_{S,\varphi}$ is linearly determined by $\mathcal{F}_{h,\varphi}$ where (h,φ) is a neutral pair.

Let (H, φ) be a Whittaker pair. Using Lemma 2.2.7, decompose H = h + Z, where (h, φ) is a neutral pair, and Z commutes with h and φ .

Definition 4.0.3. Denote by $in(H,\varphi)$ the number

$$\dim \mathfrak{g}_{<1}^h \cap \mathfrak{g}_{\geq 1}^{h+Z} + \dim \mathfrak{g}_{<2}^h \cap \mathfrak{g}_{\geq 2}^{h+Z}. \tag{4.1}$$

Note that this number is different from an analogous number in [GGS].

Let us now show that $in(H,\varphi)$ depends only on (H,φ) and does not depend on the decomposition H=h+Z.

Lemma 4.0.4 ([GGS, Lemma 4.2.7]). Let $\tilde{h} \in \mathfrak{g}_H$ be another neutral element for f. Then there exists a nilpotent element $X \in \mathfrak{g}_H$ such that $\exp(\operatorname{ad}(X))(h) = \tilde{h}$.

Here, \mathfrak{g}_H denotes the centralizer of H.

Corollary 4.0.5. The number $in(H,\varphi)$ depends only on (H,φ) and not on h. In fact, $in(H,\varphi)$ depends only on $(H,\mathbf{G}_H(\mathbb{C})\varphi)$.

Proof. If $H = \tilde{h} + \tilde{Z}$ is another decomposition as above, then by Lemma 4.0.4, $\tilde{h} = \operatorname{Ad}(\gamma)h$ for some $\gamma \in \Gamma_H$. Then $\tilde{Z} = \operatorname{Ad}(\gamma)Z$ and

$$\dim \mathfrak{g}_{<\lambda}^{\mathrm{Ad}(\gamma)h} \cap \mathfrak{g}_{>\lambda}^{\mathrm{Ad}(\gamma)(h+Z)} = \dim \mathfrak{g}_{<\lambda}^h \cap \mathfrak{g}_{>\lambda}^{h+Z}, \tag{4.2}$$

which proves that $in(H,\varphi)$ does not depend on the choice of h.

For the second statement, let $\varphi' = \operatorname{Ad}^*(\gamma)\varphi$, with $\gamma \in \Gamma_H$. Since h is neutral to φ , $\operatorname{Ad}(\gamma)h$ is neutral to φ' and $H = \operatorname{Ad}(\gamma)h + \operatorname{Ad}(\gamma)Z$ where $\operatorname{Ad}(\gamma)Z$ commutes with $\operatorname{Ad}(\gamma)h$ and $\operatorname{Ad}^*(\gamma)\varphi$. By the same argument as above, $\operatorname{in}(H, \operatorname{Ad}^*(\gamma)\varphi) = \operatorname{in}(H, \varphi)$.

Let $C \subseteq \Gamma$ denote the centralizer of (h, φ) . Let A denote a maximal split torus of C such that its Lie algebra \mathfrak{a} includes Z, and let M denote the centralizer of \mathfrak{a} in G. Then M is a Levi subgroup of G, \mathfrak{m} includes h, Z and φ , and φ is \mathbb{K} -distinguished in \mathfrak{m} . Let z be a rational semi-simple element of \mathfrak{a} that is generic in the sense that its centralizer is M.

Lemma 4.0.6. As an element of \mathfrak{m} , φ is \mathbb{K} -distinguished.

Proof. Let \mathfrak{l} be the Lie algebra of a Levi subgroup of M defined over \mathbb{K} such that $\varphi \in \mathfrak{l}^*$. We have to show that L = M. By replacing L by its conjugate we can assume $h \in \mathfrak{l}$, and that there exists a rational semi-simple element $z' \in \mathfrak{m}$ such that \mathfrak{l} is the centralizer of z'. Then z' commutes with h and φ and we have to show that z' is central in \mathfrak{m} .

Indeed, $z' \in \mathfrak{m} \cap \mathfrak{c} = \mathfrak{a}$. Now, any $X \in \mathfrak{m}$ commutes with z, and thus with any element of \mathfrak{a} , since z is generic in \mathfrak{a} . Thus \mathfrak{a} lies in the center of \mathfrak{m} and thus z' is central.

Note that the eigenvalues of the adjoint action of any Lie algebra element are symmetric around zero. Let N be a positive integer that is bigger than the ratio of the maximal eigenvalue of ad(z) by the minimal positive eigenvalue of ad(Z). Let

$$Z' := NZ + z. \tag{4.3}$$

From our choice of N we have

$$\mathfrak{g}_{>0}^{Z'} = \mathfrak{g}_{>0}^Z \oplus (\mathfrak{g}_0^Z \cap \mathfrak{g}_{>0}^z) \text{ and } \mathfrak{g}_0^{Z'} = \mathfrak{g}_0^z = \mathfrak{m} \subseteq \mathfrak{g}_0^Z.$$
 (4.4)

That $\mathfrak{m} \subseteq \mathfrak{g}_0^Z$ follows from the fact that M is the centralizer of z which equals the centralizer of \mathfrak{a} and \mathfrak{a} includes Z.

Lemma 4.0.7. For rational T > 0, (H, φ) dominates $(H + TZ', \varphi)$, that is, H, φ and TZ' commute, and satisfy (1.8).

Proof. By construction H = h + Z, φ and Z commute, and since $h, Z, \varphi \in \mathfrak{m}$ they commute with z. Thus, Z' commutes with H and φ . Furthermore, $\mathfrak{g}_{\varphi} \cap \mathfrak{g}_{\geq 1}^H \subseteq \mathfrak{g}_{\leq 0}^h \cap \mathfrak{g}_{\geq 1}^H \subseteq \mathfrak{g}_{>0}^Z \subseteq \mathfrak{g}_{>0}^{Z'} = \mathfrak{g}_{>0}^{TZ'}$.

Lemma 4.0.8. For a fixed $\lambda \in \mathbb{Q}$, and a rational T > 0 large enough,

$$\mathfrak{g}_{>1}^{H+TZ'} = \mathfrak{g}_{\geq 2}^{H+TZ'} = \mathfrak{g}_{>0}^{Z'} \oplus (\mathfrak{g}_0^{Z'} \cap \mathfrak{g}_{>1}^{H+TZ'}) = \mathfrak{g}_{>0}^{Z'} \oplus \mathfrak{m}_{\geq 2}^h \text{ and } \mathfrak{g}_{\lambda}^{H+TZ'} = \mathfrak{m}_{\lambda}^H = \mathfrak{m}_{\lambda}^h. \quad (4.5)$$

The Fourier coefficient $\mathcal{F}_{H+TZ',\varphi}$ is then Levi-distinguished.

Proof. For large enough T, we have that $\mathfrak{g}_{>1}^{H+TZ'} \cap \mathfrak{g}_{<0}^{Z'} = \{0\}$ and $\mathfrak{g}_{>1}^{H+TZ'} \cap \mathfrak{g}_{>0}^{Z'} = \mathfrak{g}_{>0}^{Z'}$. Thus $\mathfrak{g}_{>1}^{H+TZ'} = \mathfrak{g}_{>1}^{H+TZ'} \cap (\mathfrak{g}_{<0}^{Z'} \oplus \mathfrak{g}_{0}^{Z'} \oplus \mathfrak{g}_{>0}^{Z'}) = \mathfrak{g}_{>0}^{Z'} \oplus (\mathfrak{g}_{0}^{Z'} \cap \mathfrak{g}_{>1}^{H+TZ'})$. Since H = h + Z and $\mathfrak{g}_{0}^{Z'} = \mathfrak{m} \subseteq \mathfrak{g}_{0}^{Z}$ we have that $\mathfrak{g}_{0}^{Z'} \cap \mathfrak{g}_{>1}^{H+TZ'} = \mathfrak{g}_{0}^{Z'} \cap \mathfrak{g}_{>1}^{h}$ and since h is neutral $\mathfrak{g}_{>1}^{h} = \mathfrak{g}_{\geq 2}^{h}$. Now $\mathfrak{g}_{0}^{Z'} = \mathfrak{m}$ and thus, $\mathfrak{g}_{>1}^{H+TZ'} = \mathfrak{g}_{>0}^{Z'} \oplus (\mathfrak{g}_{0}^{Z'} \cap \mathfrak{g}_{\geq 2}^{h}) = \mathfrak{g}_{>0}^{Z'} \oplus \mathfrak{m}_{\geq 2}^{h}$. Doing the same manipulations for $\mathfrak{g}_{>2}^{H+TZ'}$ one ends up with the same result, proving the equality $\mathfrak{g}_{>1}^{H+TZ'} = \mathfrak{g}_{>2}^{H+TZ'}$.

Now, for any fixed $\lambda \in \mathbb{Q}$ and a large enough T, we have that $\mathfrak{g}_{\lambda}^{H+TZ'} = \mathfrak{g}_{\lambda}^{H} \cap \mathfrak{g}_{0}^{Z'} = \mathfrak{g}_{\lambda}^{H} \cap \mathfrak{m} = \mathfrak{m}_{\lambda}^{H}$. Again, since H = h + Z and $\mathfrak{m} \subseteq \mathfrak{g}_{0}^{Z}$, we get that $\mathfrak{m}_{\lambda}^{H} = \mathfrak{m}_{\lambda}^{h}$.

Since H+TZ'=h+Z+TZ', the semi-simple element denoted by Z in Definition 2.2.5 is here Z+TZ', which, for large enough T has the centralizer $\mathfrak{g}_0^Z\cap\mathfrak{g}_0^{Z'}=\mathfrak{g}_0^{Z'}=\mathfrak{m}$. By Lemma 4.0.6, φ is \mathbb{K} -distinguished in \mathfrak{m} . Since $\mathfrak{g}_{>0}^Z\subseteq\mathfrak{g}_{>0}^{Z'}$ we have that $\mathfrak{g}_{>0}^{Z+TZ'}=\mathfrak{g}_{>0}^{Z'}$ and thus (4.5) implies (2.11) which means that $\mathcal{F}_{H+TZ',\varphi}$ is Levi-distinguished.

Lemma 4.0.9. Let (H, φ, φ') be a Whittaker triple such that the pair (H, φ) is either neutral or Levi-distinguished. Then $\mathcal{F}_{H,\varphi,\varphi'} = \mathcal{F}_{H,\varphi}$.

Proof. If (H,φ) is neutral set h:=H. If (H,φ) is Levi-distinguished decompose H=h+Zwhere (h,φ) is a neutral pair and Z commutes with it. In both cases we have $\mathfrak{g}_{>1}^H = \mathfrak{g}_{>2}^H$, and $\mathfrak{g}_1^H\subseteq\mathfrak{g}_1^h$. Note that \mathfrak{g}_{φ} is spanned by lowest weight vectors and thus $\mathfrak{g}_{\varphi}\subseteq\mathfrak{g}_{\leq 0}^h$. Thus $\mathfrak{g}_{\varphi} \cap \mathfrak{g}_1^H = 0$. By Lemma 3.2.5 this implies that $\mathfrak{n}_{H,\varphi} = \mathfrak{g}_{>1}^H = \mathfrak{g}_{>2}^H$. Since $\varphi' \in (\overline{\mathfrak{g}^*})_{>-2}^H$, it vanishes on $\mathfrak{g}_{\geq 2}^H$ and thus $\mathcal{F}_{h,\varphi,\varphi'} = \mathcal{F}_{h,\varphi}$.

4.1. **Proof of Theorem A.** We will prove a more general theorem.

Theorem 4.1.1. Let η be an automorphic function on a reductive group G. Then, any quasi-Fourier coefficient $\mathcal{F}_{S,\varphi,\varphi'}[\eta]$ is linearly determined by the Levi-distinguished Fourier coefficients with characters in orbits which are equal or bigger than $G\varphi$.

In particular, if all non-PL coefficients of η vanish,

Proof. Choose h, Z, z, Z' as above and let $H_t := H + tZ'$. Choose a large enough T from

Lemma 4.0.8. Recall that $t \geq 0$ is quasi-critical if either $\mathfrak{g}_1^{H_t} \not\subseteq \mathfrak{g}_0^Z$ or $\mathfrak{g}_2^{H_t} \not\subseteq \mathfrak{g}_0^Z$. If there are no quasi-critical $t \in (0,T]$ then by Lemma 3.2.7(ii), $\mathcal{F}_{H,\varphi,\varphi'}$ is linearly determined by the set of all $\mathcal{F}_{H+TZ',\varphi,\varphi'+\psi}$ with $\psi \in (\mathfrak{g}^*)^{H+TZ'}_{-1} \cap (\mathfrak{g}^*)^e \cap (\mathfrak{g}^*)^{Z'}_{<0}$. By Lemma 4.0.8, $\mathcal{F}_{H+TZ',\varphi}$ is Levi-distinguished, and thus, by Lemma 4.0.9, we have $\mathcal{F}_{H+TZ',\varphi,\varphi'+\psi} = \mathcal{F}_{H+TZ',\varphi}$. Thus, $\mathcal{F}_{H,\varphi,\varphi'}$ is linearly determined by $\mathcal{F}_{H+TZ',\varphi}$ which is Levi-distinguished.

Now assume that there are quasi-critical numbers in (0,T] and let s be the smallest one. Let $H_s := H + sZ'$.

Since s is the first quasi-critical value we have that $(\mathfrak{g}^*)_{>-2}^H \subseteq (\mathfrak{g}^*)_{\geq -2}^{H_s}$ because this is the first point where something new may enter the -2-eigenspace. Decompose $\varphi' = \psi + \varphi''$ where $\psi \in (\mathfrak{g}^*)_{-2}^{H_s}$ and $\varphi'' \in (\mathfrak{g}^*)_{>-2}^{H_s}$. By Lemma 3.2.7(iii), $\mathcal{F}_{H,\varphi,\varphi'}$ is linearly determined by

$$\{\mathcal{F}_{H_s,\varphi+\psi,\varphi''+\psi''} \mid \psi'' \in (\mathfrak{g}^*)_{-1}^{H_s}\}. \tag{4.6}$$

Now, we repeat the procedure for each triple $\mathcal{F}_{H_s,\varphi+\psi,\varphi''+\psi''}$ and so on. To see that the algorithm terminates, note that $\psi \in (\mathfrak{g}^*)_{\leq 0}^{Z'}$ and thus the orbit of $\varphi + \psi$ is bigger than or equal to the orbit of φ .

Suppose the orbits are the same. Then, by Corollary 4.0.5, $\operatorname{in}(H_s, \varphi + \psi) = \operatorname{in}(H_s, \varphi)$. From (4.4) we see that $\mathfrak{g}_{>0}^{Z+sZ'} \cap \mathfrak{g}_0^{Z'} = \{0\}$ and $\mathfrak{g}_{>0}^{Z+sZ'} \cap \mathfrak{g}_{<0}^{Z'} = \mathfrak{g}_{>0}^{Z+sZ'} \cap \mathfrak{g}_{<0}^{Z'} \cap \mathfrak{g}_{>0}^{Z} = \{0\}$ which means that $\mathfrak{g}_{>0}^{Z+sZ'} \subseteq \mathfrak{g}_{>0}^{Z'}$ and thus

$$\mathfrak{g}_{<1}^h \cap \mathfrak{g}_{>1}^{h+Z} \subseteq \mathfrak{g}_{<1}^h \cap \mathfrak{g}_{>1}^{h+Z+sZ'}$$
 and $\mathfrak{g}_{<2}^h \cap \mathfrak{g}_{>2}^{h+Z} \subseteq \mathfrak{g}_{<2}^h \cap \mathfrak{g}_{>2}^{h+Z+sZ'}$. (4.7)

Since s is quasi-critical, one of the inclusions in (4.7) is strict and thus in $(H_s, \varphi) > \operatorname{in}(H, \varphi)$. Thus we get that either $\Gamma(\varphi + \psi) > \Gamma\varphi$ or $\operatorname{in}(H_s, \varphi + \psi) > \operatorname{in}(H, \varphi)$. Since both the orbit dimensions and the indices are bounded by $\dim \mathfrak{g}$, the algorithm eventually terminates.

Finally, by Lemma 2.2.9, the Levi distinguished Fourier coefficients of PL elements are Whittaker coefficients. This proves the second part of the statement.

Theorem A follows, since $\eta = \mathcal{F}_{0,0}[\eta]$.

4.2. **Proof of Corollary B.** Corollary B follows immediately from Theorem A and the next lemma.

Lemma 4.2.1. Let $\eta \in C^{\infty}(\Gamma \backslash G)$, and assume that the constant term $c_U(\eta) := \int_{[U]} \eta(u) du$ vanishes for any $U \subset G$ which is a unipotent radical of a proper parabolic subgroup. Let $\mathcal{F}_{S,\varphi}(\eta)$ be a non-vanishing Levi-distinguished Fourier coefficient. Then the orbit $\Gamma \varphi \in \mathfrak{g}^*$ is \mathbb{K} -distinguished.

Proof. Recall that by Definition 2.2.5, there is a decomposition S = h + Z such that (h, φ) be a neutral Whittaker pair for $\mathfrak{l} := \mathfrak{g}^Z$, the orbit of φ in \mathfrak{l}^* is \mathbb{K} -distinguished, and

$$\mathfrak{g}_{>1}^{h+Z} = \mathfrak{g}_{>2}^{h+Z} = \mathfrak{g}_{>0}^Z \oplus \mathfrak{l}_{\geq 2}^h \text{ and } \mathfrak{g}_1^{h+Z} = \mathfrak{l}_1^h.$$
 (4.8)

Let $\mathfrak{p} := \mathfrak{g}_{\geq 0}^Z$, P be the corresponding parabolic subgroup, and U be the unipotent radical of P. By (4.8), $\mathcal{F}_{S,\varphi}(\eta) = \mathcal{F}_{h,\varphi}(c_U(\eta))$, where we view $c_U(\eta)$ as an element of $C^{\infty}(\Gamma \backslash G)$. Since $\mathcal{F}_{S,\varphi}(\eta)$ does not vanish, neither does $c_U(\eta)$ and thus P = G. Thus L = G and thus the orbit $\Gamma \varphi \in \mathfrak{g}^*$ is \mathbb{K} -distinguished.

4.3. Proof of Theorem C.

Proposition 4.3.1. Let (H, φ, φ') be a Whittaker triple and let η be an automorphic function with $\mathcal{F}_{H,\varphi,\varphi'}[\eta] \neq 0$. Then there exists $\mathcal{O} \in WS(\eta)$ such that $\mathcal{O} \geq \Gamma \varphi$.

Proof. By Theorem 4.1.1, $\mathcal{F}_{H,\varphi,\varphi'}$ is linearly determined by Fourier coefficients corresponding to orbits bigger than or equal to $\Gamma\varphi$. By Corollary 4.0.2, these are in turn linearly determined by neutral Fourier coefficients corresponding to the same orbits. Since $\mathcal{F}_{H,\varphi,\varphi'}[\eta] \neq 0$, some of these neutral Fourier coefficients of η do not vanish.

Let us now adapt the assumption and the notation of Theorem C. Let Z := S - H and let $H_t := H + tZ$. Let $0 < t_1 < \cdots < t_n < 1$ be all the critical values between 0 and 1. Let $t_0 := 0$ and $t_{n+1} := 1$. Lastly, for each t_i , let $R := R_{t_i}$ and $L := L_{t_i}$ be defined as in (3.12).

Lemma 4.3.2. We have $\mathcal{F}_{H_{t,:},\varphi}^{R}[\eta] = \mathcal{F}_{H_{t,:},\varphi}^{L}[\eta]$.

Proof. Let $f \in \mathfrak{g}$ be the unique nilpotent element such that φ is given by Killing form pairing with f. Complete f to an \mathfrak{sl}_2 -triple (e,h,f) such that h commutes with S and H. Denote $H_j := H_{t_j}$ for any j, and $\mathfrak{c} := (\mathfrak{g}^*)_{-1}^{H_{i+1}} \cap (\mathfrak{g}^*)^e \cap (\mathfrak{g}^*)_{<0}^Z$. Arguing as in the proof of Lemma 3.2.7(ii), we obtain

$$\mathcal{F}^R_{H_i,arphi}[\eta] = \sum_{arphi' \in \mathfrak{c}} \mathcal{F}^L_{H_{i+1},arphi,arphi'}[\eta].$$

We have to show that for any non-zero $\varphi' \in \mathfrak{c}$, we have $\mathcal{F}^L_{H_{i+1},\varphi,\varphi'}[\eta] = 0$. This follows from Lemma 3.1.1, Proposition 3.2.8, Proposition 4.3.1, and the condition that $\Gamma \varphi \in \mathrm{WS}(\eta)$. \square

Proof of Theorem C. With the notation introduced above, let

$$\mathfrak{v}_i := (\mathfrak{g}_{>1}^{H_{t_i}} \cap \mathfrak{g}_{<0}^Z) / (\mathfrak{g}_{>1}^{H_{t_i}} \cap \mathfrak{g}_{<0}^Z) \quad \text{and} \quad V_i = \operatorname{Exp}(\mathfrak{v}_i).$$
 (4.9)

By Lemma 3.1.1 we have

$$\mathcal{F}_{H_{t_i},\varphi}^L[\eta](g) = \int_{V_i} \mathcal{F}_{H_{t_i},\varphi}^R[\eta](v_i g) \, dv_i \,. \tag{4.10}$$

Using Lemma 4.3.2 we obtain

$$\mathcal{F}_{H,\varphi}^{R}[\eta](g) = \int_{V_1} \dots \int_{V_{n-1}} \int_{V_n} \mathcal{F}_{S,\varphi}^{L}[\eta](v_n \dots v_0 g) dv.$$

$$(4.11)$$

Since $\mathfrak{v} = \bigoplus_{i=1}^n (\mathfrak{g}_1^{H_{t_i}} \cap \mathfrak{g}_{\leq 0}^Z)$, and as a commutative Lie algebra $\mathfrak{g}_1^{H_{t_i}} \cap \mathfrak{g}_{\leq 0}^Z$ is naturally isomorphic to \mathfrak{v}_i , the group V is glued from V_i . Thus

$$\int_{V_1} \dots \int_{V_{n-1}} \int_{V_n} \mathcal{F}_{S,\varphi}^L[\eta](v_n \dots v_0 g) dv = \int_{V} \mathcal{F}_{S,\varphi}^L[\eta](vg) dv.$$

$$(4.12)$$

To prove part (i) note from (3.12) that if $\mathfrak{g}_1^H = \mathfrak{g}_1^S = 0$ then $\mathcal{F}_{H,\varphi} = \mathcal{F}_{H,\varphi}^R$ and $\mathcal{F}_{S,\varphi} = \mathcal{F}_{S,\varphi}^L$, and thus part (i) follows from (4.11) and (4.12).

For part (ii), note that \mathfrak{u} and \mathfrak{w} as defined in the statement are given by

$$\mathfrak{u}=(\mathfrak{g}_{\geq 1}^S\cap\mathfrak{g}_{<0}^Z)/(\mathfrak{g}_{>1}^S\cap\mathfrak{g}_{<0}^Z) \text{ and } \mathfrak{w}=(\mathfrak{g}_{\geq 1}^H\cap\mathfrak{g}_{<0}^Z)/(\mathfrak{g}_{>1}^H\cap\mathfrak{g}_{<0}^Z).$$

Thus Lemmas 3.1.1 and 3.2.6 imply

$$\mathcal{F}_{S,\varphi}^{L}[\eta](g) = \int_{[U]} \mathcal{F}_{S,\varphi}[\eta](ug)du, \text{ and } \mathcal{F}_{H,\varphi}[\eta](g) = \sum_{w \in \Omega} \mathcal{F}_{H,\varphi}^{R}[\eta](wg).$$
 (4.13)

Applying (4.11), (4.12), and (4.13) to shifts of η we obtain

$$\mathcal{F}_{H,\varphi}[\eta](g) = \sum_{w \in \Omega} \mathcal{F}_{H,\varphi}^{R}[\eta](wg) = \sum_{w \in \Omega} \int_{V} \mathcal{F}_{S,\varphi}^{L}[\eta](vwg)dv$$
$$= \sum_{w \in \Omega} \int_{V} \int_{[U]} \mathcal{F}_{S,\varphi}[\eta](uvwg)dudv. \tag{4.14}$$

5. Applications and examples

In this section we will illustrate how to apply the framework introduced in this paper to compute certain Fourier coefficients in detail. We begin in §5.1 to consider the case when G is split and simply-laced and $P \subset G$ is a parabolic subgroup with unipotent radical U isomorphic to a Heisenberg group. We use the technique of this paper in order to express any automorphic function on G in terms of its Fourier coefficients with respect to U. In §5.2 we then give an example of a Whittaker triple and a quasi-Fourier coefficient for the group $G = \mathrm{SL}_4$. In §5.3 we demonstrate the statement and the proof of Theorem A for $G = \mathrm{GL}_n$. In §5.4 we demonstrate Theorem A and Corollary B for $G = \mathrm{Sp}_4$.

As many examples below are built on classical groups, we shall use matrix notation and denote by e_{ij} the elementary matrix with a 1 at position (i, j) and zeroes elsewhere.

5.1. Fourier expansions along Heisenberg parabolics. Let G be split and simply-laced, and let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of a maximal split torus. Fix a choice of positive roots. For any simple root α define $S_{\alpha} \in \mathfrak{h}$ by $\alpha(S_{\alpha}) := 2$ and $\beta(S_{\alpha}) = 0$ for any other simple root β .

Definition 5.1.1. We say that a simple root α is a Heisenberg root if $\mathfrak{g}_{>0}^{S_{\alpha}}$ is a Heisenberg Lie algebra, or, equivalently, if $g_4^{S_{\alpha}}$ has dimension one.

Lemma 5.1.2. If \mathfrak{g} is simple of type A_n , there are no Heisenberg roots. If \mathfrak{g} is simple of type D_n or E_n then there exists a unique Heisenberg root, and this is the unique simple root satisfying $\langle \alpha, \alpha_{\max} \rangle = 1$, where α_{\max} denotes the highest root.

Proof. Let \mathfrak{g} be simple and let α be a Heisenberg root. Then $g_4^{S_{\alpha}}$ has to be the highest weight space of the adjoint representation, *i.e.* the root space of α_{\max} . Since $g_4^{S_{\alpha}}$ is one-dimensional, $\alpha_{\max} - \beta$ is not a root for any simple root $\beta \neq \alpha$. Thus $\alpha_{\max} - \alpha$ is a root, and thus $\langle \alpha, \alpha_{\max} \rangle = 1$. The roots α with this property are precisely the nodes in the affine Dynkin diagram, that are connected to the affine node (corresponding to $-\alpha_{\max}$).

Checking the affine Dynkin diagrams (see [Bou68, Tables IV-VII]), we see that there is a unique simple root α with this property in types D_n and E_n , and these roots are indeed Heisenberg. In the Bourbaki notation, these roots are α_2 for D_n and E_6 , α_1 for E_7 and α_8 for E_8 . In type A_n there are two roots with this property but none of them is Heisenberg. In fact, $\mathfrak{g}_{>0}^{S_{\beta}}$ is abelian for any simple root β in type A_n . This is so, since in type A_n , α_{\max} is the sum of all simple roots (with all coefficients being 1).

Notation 5.1.3. Let α be a root. Define $h_{\alpha} := \alpha^{\vee} \in \mathfrak{h}$ by requiring for all roots β

$$\beta(h_{\alpha}) = 2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = \langle \alpha, \beta \rangle. \tag{5.1}$$

Denote also by $\mathfrak{g}_{-\alpha}^{\times}$ the set of non-zero covectors in the dual root space $\mathfrak{g}_{-\alpha}^{*}$.

Note that for $\beta \neq \pm \alpha$, $\beta(h_{\alpha}) \in \{-1, 0, 1\}$. By [Hum78, Proposition II.8.3], (h_{α}, φ) is a neutral pair for any $\varphi \in \mathfrak{g}_{-\alpha}^{\times}$.

Notation 5.1.4. For any Heisenberg root α , let $\Omega_{\alpha} \subset \Gamma$ be the abelian subgroup obtained by exponentiation of the abelian Lie algebra given by the direct sum of the root spaces of negative roots β satisfying $\langle \alpha, \beta \rangle = 1$. Let

$$\Psi_{\alpha} := \{ \text{ root } \varepsilon \mid \langle \varepsilon, \alpha \rangle \le 0, \varepsilon(S_{\alpha}) = 2 \}.$$
 (5.2)

Note that all the roots in Ψ_{α} have to be positive.

In this subsection we use our technique to deduce the following proposition, that will be used in the sequel paper [GGK⁺19].

Proposition 5.1.5. Let α be a Heisenberg root. Let $\gamma_{\alpha} \in \Gamma$ be a representative of a Weyl group element that conjugates α to α_{\max} , where α_{\max} denotes the maximal root of the component of \mathfrak{g} corresponding to α . Then we have

$$\eta(g) = \sum_{\varphi \in (\mathfrak{g}^*)_{-2}^{S_{\alpha}}} \mathcal{F}_{S_{\alpha}, \varphi}[\eta](g) + \sum_{\varphi \in \mathfrak{g}_{-\alpha}^{\times}} \sum_{\omega \in \Omega_{\alpha}} \sum_{\psi \in \bigoplus_{\varepsilon \in \Psi_{\alpha}} \mathfrak{g}_{-\varepsilon}^*} \mathcal{F}_{S_{\alpha}, \varphi + \psi}[\eta](\omega \gamma_{\alpha} g).$$
 (5.3)

For the proof we will need the following lemma.

Lemma 5.1.6. Let α be a Heisenberg root. Then for any $\varphi \in \mathfrak{g}_{-\alpha}^{\times}$ we have

$$\mathcal{F}_{h_{\alpha},\varphi}[\eta](g) = \sum_{\omega \in \Omega_{\alpha}} \sum_{\psi \in \bigoplus_{\varepsilon \in \Psi_{\alpha}} \mathfrak{g}_{-\varepsilon}^{*}} \mathcal{F}_{S_{\alpha},\varphi+\psi}[\eta](\omega g).$$
 (5.4)

Proof. Consider the deformation $(1-t)h_{\alpha} + tS_{\alpha}$. By Lemma 3.1.1, we have

$$\mathcal{F}_{h_{\alpha},\varphi}[\eta](g) = \sum_{\omega \in \Omega_{\alpha}} \mathcal{F}_{h_{\alpha},\varphi}^{R_0}[\eta](\omega g).$$
 (5.5)

Then, the critical values are 1/2 and 2/3, and the quasi-critical values are 1/3 and 1. At 1/3, we have no Whittaker triple entries yet and thus nothing moves into the -2-eigenspace. At 1/2, we get contributions in the third component of the Whittaker triple from the root spaces of all the roots ε with $\langle \varepsilon, \alpha \rangle = 0$ and $\varepsilon(S_{\alpha}) = -2$. At t = 2/3 we also get all the negative roots with $\langle \varepsilon, \alpha \rangle = 1$ and $\varepsilon(S_{\alpha}) = -2$. This means that we would get contributions from all these root spaces in the third component of the Whittaker triple. At t = 1 the Whittaker triple becomes a Whittaker pair and thus we obtain

$$\mathcal{F}_{h_{\alpha},\varphi}[\eta](g) = \sum_{\omega \in \Omega_{\alpha}} \mathcal{F}_{h_{\alpha},\varphi}^{R_{0}}[\eta](\omega g) = \sum_{\omega \in \Omega_{\alpha}} \sum_{\psi \in \bigoplus_{\varepsilon \in \Psi_{\alpha}} \mathfrak{g}_{-\varepsilon}^{*}} \mathcal{F}_{S_{\alpha},\varphi+\psi}[\eta](\omega g). \tag{5.6}$$

Proof of Proposition 5.1.5. By the conditions, the Lie algebra $\mathfrak{g}_{>0}^{S_{\alpha}}$ is a Heisenberg Lie algebra, with center $\mathfrak{g}_{4}^{S_{\alpha}}$, and abelian quotient $\mathfrak{g}_{2}^{S_{\alpha}}$. We restrict η to the exponential of the center and decompose to Fourier series. The constant term with respect to the center $\mathfrak{g}_{4}^{S_{\alpha}}$ is $\mathcal{F}_{S_{\alpha}/3,0}[\eta]$, and the other terms are $\mathcal{F}_{S_{\alpha}/2,\varphi}[\eta]$ for $\varphi \neq 0 \in (\mathfrak{g}^{*})_{-4}^{S_{\alpha}} = (\mathfrak{g}^{*})_{-2}^{S_{\alpha}/2}$. We remark that this constant term can be denoted $\mathcal{F}_{cS_{\alpha},0}[\eta]$ for any $1/4 \leq c < 1/2$ but not for c = 1/2 since 0 defines a zero form on the 1-eigenspace, and thus $\mathfrak{n}_{cS_{\alpha},0} = \mathfrak{g}_{\geq 1}^{cS_{\alpha}} = \mathfrak{g}_{\geq c^{-1}}^{S_{\alpha}}$ and $\mathfrak{n}_{S_{\alpha}/2,0} = \mathfrak{g}_{\geq 2}^{S_{\alpha}}$. Note also that $(\mathfrak{g}^{*})_{-2}^{S_{\alpha}/2} = \mathfrak{g}_{-\alpha_{\max}}^{\times}$. Altogether we have

$$\eta(g) = \mathcal{F}_{S_{\alpha}/3,0}[\eta](g) + \sum_{\varphi \in \mathfrak{g}_{-\alpha_{\max}}^{\times}} \mathcal{F}_{S_{\alpha}/2,\varphi}[\eta](g).$$
 (5.7)

Note that γ_{α} conjugates $S_{\alpha}/2$ to h_{α} . Thus, by Lemma 3.3.1, we have

$$\mathcal{F}_{S_{\alpha}/2,\varphi}[\eta](g) = \mathcal{F}_{h_{\alpha},\mathrm{Ad}^*(\gamma_{\alpha})\varphi}[\eta](\gamma_{\alpha}g)$$

and

$$\sum_{\varphi \in \mathfrak{g}_{-\alpha_{\max}}^{\times}} \mathcal{F}_{S_{\alpha}/2,\varphi}[\eta](g) = \sum_{\varphi \in \mathfrak{g}_{-\alpha}^{\times}} \mathcal{F}_{h_{\alpha},\varphi}[\eta](\gamma_{\alpha}g).$$
 (5.8)

We restrict the constant term of (5.7) to the abelian quotient of $\text{Exp}(\mathfrak{g}_{>0}^{S_{\alpha}})$, decompose to Fourier series and obtain

$$\mathcal{F}_{S_{\alpha}/3,0}[\eta](g) = \sum_{\varphi \in (\mathfrak{g}^*)_{-2}^{S_{\alpha}}} \mathcal{F}_{S_{\alpha},\varphi}[\eta](g).$$
 (5.9)

Formula (5.3) follows now from (5.7), (5.8), (5.4) and (5.9).

Remark 5.1.7. Here we elaborate a little on the structure of the Fourier expansion (5.3) and comment on the relation to previous works on Heisenberg expansions. The semisimple element S_{α} defines a Heisenberg parabolic subgroup $P_{\alpha} \subset G$ with Levi decomposition $P_{\alpha} = LU$. The Lie algebra $\mathfrak{p}_{\alpha} \subset \mathfrak{g}$ of P_{α} exhibiting the following grading

$$\mathfrak{p}_{\alpha} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \tag{5.10}$$

where the subscripts indicate the values of the inner products $\langle \cdot, \alpha_{\max} \rangle$. Thus \mathfrak{g}_1 is spanned by all roots ε such that $\langle \varepsilon, \alpha_{\max} \rangle = 1$. Equivalently, these are all roots ε such that $\alpha_{\max} - \varepsilon$ is also a root. Notice that all roots in \mathfrak{g}_1 are positive and the only simple root satisfying the condition $\langle \varepsilon, \alpha_{\max} \rangle = 1$ is α itself. Since only $\varepsilon = \alpha_{\max}$ satisfies $\langle \varepsilon, \alpha_{\max} \rangle = 2$ the space \mathfrak{g}_2 is one-dimensional, spanned by $E_{\alpha_{\max}}$. The zeroth subspace \mathfrak{g}_0 is the Lie algebra of the Levi $L \subset P$. The subspace $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is thus the Heisenberg nilpotent subalgebra with center \mathfrak{g}_2 . Notice that $\sum_{\gamma \in \Psi_{\alpha}} \mathfrak{g}_{\gamma}$ is a Lagrangian subspace of \mathfrak{g}_1 . Indeed, \mathfrak{g}_1 has a canonical Lagrangian decomposition

$$\mathfrak{g}_1 = \sum_{\gamma \in \Psi_\alpha} \mathfrak{g}_\gamma \oplus \sum_{\gamma \in \Psi_\alpha^\perp} \mathfrak{g}_\gamma, \tag{5.11}$$

where Ψ_{α}^{\perp} is the orthogonal complement

$$\Psi_{\alpha}^{\perp} = \{ \text{ root } \varepsilon \mid \langle \varepsilon, \alpha \rangle \ge 1, \langle \varepsilon, \alpha_{\text{max}} \rangle = 1 \}.$$
 (5.12)

Note that the root α belongs to Ψ_{α}^{\perp} . The Fourier expansion (5.3) thus corresponds to the standard non-abelian Fourier expansion along the Heisenberg unipotent U, which exhibits a sum over the center \mathfrak{g}_2 along with a sum over a Lagrangian subspace Ψ_{α} of \mathfrak{g}_1 . The choice of Lagrangian decomposition is usually referred to as a choice of "polarization". Similar kinds of expansions have been treated in several places in the literature; see [KS90, KPW02, PP09, BKN⁺10, FGKP18] for a sample. In the notation of the original paper by Kazhdan and Savin [KS90], the space Ψ_{α} corresponds to Π_{α}^* while Ψ_{α}^{\perp} corresponds to Π_{α} .

5.2. Whittaker triples. We will now illustrate what type of quasi-Fourier coefficients we are able to describe using Whittaker triples that are not captured by Whittaker pairs in an example for $G = SL_4$.

Let (S, φ, ψ) be the Whittaker triple with $S = \frac{1}{3} \operatorname{diag}(3, 1, -1, -3)$, $\varphi = e_{41}$ and $\psi = me_{31} + ne_{42}$, where $m, n \in \mathbb{K}$ and e_{ij} denote elementary matrices. The S-eigenvalues for the different elementary matrices can be illustrated by the following matrix

$$\begin{pmatrix}
0 & 2/3 & 4/3 & 2 \\
-2/3 & 0 & 2/3 & 4/3 \\
-4/3 & -2/3 & 0 & 2/3 \\
-2 & -4/3 & -2/3 & 0
\end{pmatrix},$$
(5.13)

from which we may read out that φ has eigenvalue -2 while ψ has eigenvalue -4/3. As seen from this matrix we get the following unipotent subgroup (independent of ψ)

$$N_{S,\varphi} = \left\{ \begin{pmatrix} 1 & 0 & x_2 & x_1 \\ 1 & 0 & x_3 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} : x_1, x_2, x_3 \in \mathbb{A} \right\}, \tag{5.14}$$

and the corresponding Fourier coefficient of an automorphic function η can be expressed as

$$\mathcal{F}_{S,\varphi,\psi}[\eta](g) = \int_{(\mathbb{K}\backslash\mathbb{A})^3} \eta\left(\begin{pmatrix} 1 & 0 & x_2 & x_1 \\ 1 & 0 & x_3 \\ & 1 & 0 \\ & & 1 \end{pmatrix}g\right) \chi(x_1 + mx_2 + nx_3)^{-1} d^3x, \qquad (5.15)$$

where we recall that χ is a fixed non-trivial character on \mathbb{A} trivial on \mathbb{K} .

From this example we see that we require Whittaker triples in addition to Whittaker pairs if we want to construct Fourier coefficients with characters that are not only supported on x_1 but also on x_2 and x_3 .

5.3. The case of GL_n . Let $G := GL_n$, $G := GL_n(\mathbb{A})$, and $\Gamma := GL_n(\mathbb{K})$. In this section we will follow the proof of Theorem A to present any automorphic function $\eta \in C^{\infty}(\Gamma \setminus G)$ as a countable linear combination of its Whittaker coefficients. We will show that our proof amounts in this case morally to the same decomposition as in [PS79, Sha74].

In [PS79, Sha74], η is first restricted to the mirabolic nilradical, *i.e.*

$$U = \left\{ \left(\begin{array}{cc} \mathrm{Id}_{n-1} & * \\ 0 & 1 \end{array} \right) \right\},\,$$

and decomposed into Fourier series with respect to U. Our algorithm does the same thing, but in several steps. First let $(h, \varphi) = (0, 0)$. Let $N \gg 0$,

$$z_1 := \operatorname{diag}(0, -1, -N, \dots, -N^{n-3}, -N^{n-2}),$$

and consider the deformation $S_t := tz_1$. Under this deformation, the first thing that happens is that the highest root space (spanned by e_{1n}) enters $\mathfrak{g}_1^{S_t}$. At this point η decomposes into a sum of quasi-Whittaker coefficients. At the next step e_{1n} enters $\mathfrak{g}_2^{S_t}$, and the quasi-Whittaker coefficients become Whittaker coefficients. For the constant term, we continue with the same deformation, until e_{2n} enters. For the non-constant term we have to change the deformation into something that will commute with φ . The φ can be identified with ae_{n1} under the trace form, for some $a \in \mathbb{K}^{\times}$. We take the deformation by

$$z_2 := \operatorname{diag}(-N^{n-2}, -1, -N, \dots, -N^{n-4}, -N^{n-3}, -N^{n-2}),$$

and continue in the same way. Eventually, all of U enters and all possible characters (including the trivial one) appear.

Let us now analyze the summands. The constant term is $\mathcal{F}_{tz_1,0}$ for $t = 2/(N^{n-2} - N^{n-3})$, and we can continue the deformation along z_1 . Any non-trivial character of U can be conjugated using GL_{n-1} (embedded into the upper left corner) to the one given by $e_{n,n-1}$. We can now choose the deformation

$$z_3 := \operatorname{diag}(-1, -N, \dots, -N^{n-4}, -N^{n-3}, -N^{n-3}).$$

In the same way as above, it will give a decomposition of $\mathcal{F}_{tz_1,0}$ into Fourier series with respect to the column n-1, *i.e.*

$$U' = \left\{ \left(\begin{array}{ccc} \mathrm{Id}_{n-2} & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \right\}.$$

Continuing in this way we obtain

$$\eta(g) = \sum_{x \in 2^{[n-1]}} \sum_{\gamma \in \Gamma_x} \mathcal{F}_{S,\varphi_x}[\eta](\gamma g), \tag{5.16}$$

where [n-1] denotes the set $\{1,\ldots,n-1\}$, $2^{[n-1]}$ denotes the set of all its subsets, $S=\operatorname{diag}(n-1,n-3,\ldots,3-n,1-n)$, and for any $x\in 2^{[n-1]}$, $\psi_x:=\sum_{i\in x}e_{i+1,i}$ and Γ_x is a certain subset of Γ .

For cuspidal η and $x \neq [n-1]$, we have $\mathcal{F}_{S,\varphi_x}[\eta] = 0$ and (5.16) becomes the formula in [PS79, Sha74]. If η is minimal then $\mathcal{F}_{S,\varphi_x}[\eta] = 0$ for |x| > 1 and if η is next-to-minimal then $\mathcal{F}_{S,\varphi_x}[\eta] = 0$ for |x| > 2. These cases were computed in [AGK⁺18], motivated by applications in string theory.

5.4. **Examples for** Sp_4 . Let $G := \operatorname{Sp}_4(\mathbb{A})$, $\Gamma := \operatorname{Sp}_4(\mathbb{K})$ and let $\eta \in C^{\infty}(\Gamma \backslash G)$. In this section we express η in terms of its Levi-distinguished Fourier coefficients, providing an example for Theorem A. Let $\mathfrak{g} := \operatorname{Lie}(\Gamma)$, realized in \mathfrak{gl}_4 by the 2×2 block matrices

$$\begin{pmatrix}
A & B = B^t \\
C = C^t & -A^t
\end{pmatrix}.$$
(5.17)

Let $\mathfrak{n} \subset \mathfrak{g}$ be the maximal unipotent subalgebra spanned by the matrices $e_{12} - e_{43}, e_{13}, e_{24}, e_{14} + e_{23}$ and let $N := \operatorname{Exp}(\mathfrak{n}(\mathbb{A}))$. For any $a, b \in \mathbb{K}$ denote by $\chi_{a,b}$ the character of \mathfrak{n} given by $\chi_{a,b}(e_{12} - e_{43}) = a$ and $\chi_{a,b}(e_{24}) = b$, and let $\mathcal{W}_{a,b}$ denote the corresponding Whittaker coefficient. Let $\mathfrak{u} \subset \mathfrak{n}$ be the Siegel nilradical, i.e. the normal commutative subalgebra spanned by the matrices $e_{13}, e_{24}, e_{14} + e_{23}$ and let $U := \operatorname{Exp}(\mathfrak{u}(\mathbb{A}))$. Let L denote the Siegel Levi subgroup of Γ given by $\operatorname{diag}(g, (g^t)^{-1})$, where $g \in \operatorname{GL}_2(\mathbb{A})$. Using the trace form on \mathfrak{g} , we can identify \mathfrak{u}^* with the nilradical $\bar{\mathfrak{u}}$ of the opposite parabolic, i.e. with the space of matrices of the form (5.17) with A = B = 0. Note that $\bar{\mathfrak{u}} \cong \operatorname{Sym}^2(\mathbb{K}^2)$, and L acts on it by the standard action on symmetric forms. For any $\varphi \in \mathfrak{u}^* \cong \bar{\mathfrak{u}} \cong \operatorname{Sym}^2(\mathbb{K}^2)$, denote by $\mathcal{F}_{\mathfrak{u},\varphi}$ the corresponding parabolic Fourier coefficient.

Since U is abelian, the Fourier decomposition on it gives

$$\eta = \sum_{\varphi \in \mathfrak{u}^*} \mathcal{F}_{\mathfrak{u},\varphi}[\eta] \,. \tag{5.18}$$

We now decompose this sum into three different terms, by the rank of φ , viewed as a quadratic form. Let us first analyze the constant term $\mathcal{F}_{\mathfrak{u},0}[\eta]$. We restrict it to L, and decompose to Fourier series on the abelian group $N \cap L$. We obtain

$$\mathcal{F}_{\mathfrak{u},0}[\eta] = \sum_{a \in \mathbb{K}} \mathcal{W}_{a,0}[\eta] \,. \tag{5.19}$$

Next, any φ of rank one is conjugate under L to $\varphi_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. This φ_1 is normalized by N, and thus we can again decompose $\mathcal{F}_{\mathfrak{u},\varphi}[\eta]$ on $N \cap L$. We obtain

$$\mathcal{F}_{\mathfrak{u},\varphi_1}[\eta] = \sum_{a \in \mathbb{K}} \mathcal{W}_{a,1}[\eta] \,. \tag{5.20}$$

The non-degenerate forms (i.e. those of rank two) can be divided into two subsets: split and non-split. All the split ones are conjugate under L to $\varphi_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let $w \in \Gamma$ denote a representative for the Weyl group element given by the simple reflection with

respect to the long simple root $\alpha_2 = 2\varepsilon_2$, e.g. $w = \text{diag}(1, 1, 1, -1)\sigma_{24}$, where σ_{24} is the permutation matrix on indices 2 and 4. Then $\mathfrak{u}^w = \text{Span}(e_{12} - e_{43}, e_{13}, e_{42})$, and φ_2^w equals the restriction to \mathfrak{u}^w of $\chi_{1,0}$. Using Corollary 3.1.2, we can express $\mathcal{F}_{\mathfrak{u}^w,\chi_{1,0}}$ through $\mathcal{F}_{\mathfrak{u}',\chi_{1,0}}$, where $\mathfrak{u}' = \text{Span}(e_{12} - e_{43}, e_{13}, e_{24}) \subset \mathfrak{n}$. The integration will be over elements matrices of the form $v_x := \text{Id} + xe_{24} \in G$. Using Fourier expansion by the remaining coordinate of $e_{14} + e_{23} \in \mathfrak{n}$, we obtain

$$\mathcal{F}_{\mathfrak{u},\varphi}[\eta](g) = \int_{x \in \mathbb{A}} \mathcal{W}_{1,a}[\eta](v_x w g). \tag{5.21}$$

Finally, let $X \subset \bar{u} \cong \mathfrak{u}^*$ denote the set of anisotropic non-degenerate forms. For $\varphi \in X$, we have no expression of $\mathcal{F}_{\mathfrak{u},\varphi}[\eta]$ in terms of Whittaker coefficients. However, any $\varphi \in X$ is \mathbb{K} -distinguished. Indeed, let $h := \mathrm{Id} \in \mathfrak{l}$. Then (h,φ) is a neutral pair, and its centralizer is anisotropic. By Lemma 4.0.6 applied to (h,φ) and Z := 0, φ is \mathbb{K} -distinguished.

Combining (5.18)–(5.21) we obtain the following theorem, that exemplifies Theorem A.

Theorem 5.4.1. For any $\eta \in C^{\infty}(\Gamma \backslash G)$ and $g \in G$, $\eta(g)$ equals

$$\sum_{\varphi \in X} \mathcal{F}_{\mathfrak{u},\varphi}[\eta](g) + \sum_{a \in \mathbb{K}} \left(\sum_{\gamma \in L/O(1,1)} \int_{x \in \mathbb{A}} \mathcal{W}_{1,a}[\eta](v_x w \gamma g) + \sum_{\gamma \in L/(N \cap L)} \mathcal{W}_{a,1}[\eta](\gamma g) + \mathcal{W}_{a,0}[\eta](g) \right),$$

where $O(1,1) \subset L$ denotes the stabilizer of the split form φ_2 .

If η is cuspidal then $W_{0,a}[\eta] = W_{a,0}[\eta] = 0$. If η is non-generic η , then $W_{1,a}[\eta] = W_{a,1}[\eta] = 0$, unless a = 0. Thus Theorem 5.4.1 implies the following corollary.

Corollary 5.4.2. Let $\eta \in C^{\infty}(\Gamma \backslash G)$ and $g \in G$.

(i) If η is cuspidal then

$$\eta(g) = \sum_{\varphi \in X} \mathcal{F}_{\mathfrak{u},\varphi}[\eta](g) + \sum_{a \in \mathbb{K}^{\times}} \left(\sum_{\gamma \in L/\mathcal{O}(1,1)_{x \in \mathbb{A}}} \int_{x \in \mathbb{A}} \mathcal{W}_{1,a}[\eta](v_{x}w\gamma g) + \sum_{\gamma \in L/(N \cap L)} \mathcal{W}_{a,1}[\eta](\gamma g) \right).$$

(ii) If η is non-generic then

$$\eta(g) = \sum_{\varphi \in X} \mathcal{F}_{\mathfrak{u},\varphi}[\eta](g) + \sum_{\gamma \in L/\mathcal{O}(1,1)} \int\limits_{x \in \mathbb{A}} \mathcal{W}_{1,0}[\eta](v_x w \gamma g) + \sum_{\gamma \in L/(N \cap L)} \mathcal{W}_{0,1}[\eta](\gamma g) + \sum_{a \in \mathbb{K}} \mathcal{W}_{a,0}[\eta](g) \,.$$

(iii) If η is cuspidal and non-generic then $\eta = \sum_{\varphi \in X} \mathcal{F}_{\mathfrak{u},\varphi}[\eta]$.

Corollary 5.4.2(i) is an explicit example for Corollary B.

APPENDIX A. ON PL-ORBITS

A complex orbit is a PL-orbit if and only if its Bala-Carter label has no parenthesis. In particular, all complex minimal and next-to-minimal orbits are PL. The classification of PL orbits of complex classical groups in terms of the corresponding partitions is given in [GS15, §6].

The classification of rational PL-orbits is a more complicated task. In this subsection we discuss the PL property for small \mathbb{K} -rational orbits of simple split groups. A complex orbit $\mathcal{O}_{\mathbb{C}}$ may include several or even infinitely many rational orbits. If $\mathcal{O}_{\mathbb{C}}$ is non-PL then all

its rational orbits are non-PL. If $\mathcal{O}_{\mathbb{C}}$ is PL then it includes at least one rational PL-orbit, but can also include non-PL rational orbits. In type A_n , all rational orbits are PL. Let us now describe the PL properties of minimal and next-to-minimal orbits. Here, minimal and next-to-minimal refers to the closure order on the complex orbits, which might be coarser than the order defined in Definition 2.3.1.

All minimal rational orbits are PL. Indeed, for classical groups it is easy to establish the Levi in which they are principal: for $SO_{n+1,n}$ it is $SO_{2,1} \times (GL_1)^{n-1}$, for Sp_{2n} it is $Sp_2 \times (GL_1)^{n-1}$ and for $SO_{n,n}$ it is $SO_{2,2} \times (GL_1)^{n-2}$. For exceptional groups, the rational minimal orbit is unique and thus PL. This uniqueness was explained to us by Joseph Hundley. Let us now deal with the next-to-minimal orbits.

Lemma A.1. All next-to-minimal rational orbits for $SO_{n,n}$ and $SO_{n+1,n}$ are PL.

Proof. One can give a the classification of the rational orbits in the spirit of the classification of real orbits given in [CM93, §9.3]. Namely, a \mathbb{K} -rational orbit with a given partition is defined by a collection of quadratic forms Q_{2i+1} on multiplicity spaces of the odd parts. If we add a hyperbolic form to the direct sum of these forms we get the initial form, which is also hyperbolic. Here, a hyperbolic form is a direct multiple of the 2-dimensional quadratic form given by H(x,y) = xy. By Witt's cancelation theorem this implies that the direct sum of the forms on multiplicity spaces of the odd parts is hyperbolic.

An orbit for $SO_{n,n}$ is PL if and only if all Q_{2i+1} are hyperbolic, except Q_{2j+1} for a single index $j \geq 1$, which is a direct sum of a hyperbolic form and a one-dimensional quadratic form. For $SO_{n,n}$ there are two next-to-minimal partitions. One of them is 2^41^{2n-8} . For it, Q_1 has to be hyperbolic. The other next-to-minimal partition is 31^{2n-3} . Thus Q_3 is one-dimensional. Now, note that $H^n = Q_3 \oplus -Q_3 \oplus H^{n-1}$. Thus, $Q_3 \oplus Q_1 = Q_3 \oplus -Q_3 \oplus H^{n-1}$ and thus $Q_1 = (-Q_3) \oplus H^{n-1}$, i.e. Q_1 is a direct sum of a hyperbolic form and a one-dimensional quadratic form.

Similarly, it is easy to see that the next-to-minimal orbits for $SO_{n+1,n}$ are principal in Levis isomorphic to $(GL_2)^2 \times (GL_1)^{n-4}$ or $SO_{2,1} \times (GL_1)^{n-1}$.

However, $\operatorname{Sp}_{2n}(\mathbb{K})$ has infinitely many rational next-to-minimal orbits, already for n=2. Moreover, by [Ike01] there exist cuspidal next-to-minimal representations of $\operatorname{Sp}_4(\mathbb{A})$. Note that cuspidal non-generic automorphic forms cannot be determined by their Whittaker coefficients, since the latter coefficients have to vanish on such forms. See [Gin06, §4] for a discussion of cuspidal representations, in particular those of $\operatorname{Sp}_4(\mathbb{A})$.

As for the exceptional groups, Joseph Hundley showed that the next-to-minimal orbit is unique, and thus PL, for E_6 , E_7 , E_8 and G_2 [Hun].

The group F_4 has infinitely many rational next-to-minimal orbits. We expect that infinitely many of them are not PL.

APPENDIX B. SOME GEOMETRIC LEMMAS

Lemma B.1. Let $Z \in \mathfrak{g}$ be rational semi-simple, let $\varphi \in \mathfrak{g}_0^Z$ and $\varphi' \in \mathfrak{g}_{>0}^Z$. Assume that φ is conjugate to $\varphi + \varphi'$ by $\mathbf{G}(\mathbb{C})$. Then there exist $X \in \mathfrak{g}_{>0}^Z$ such that $\mathrm{ad}^*(X)(\varphi) = \varphi'$ and $v \in \mathrm{Exp}(\mathfrak{g}_{>0}^Z)$ such that $\mathrm{Ad}^*(v)(\varphi) = \varphi + \varphi'$.

Proof. Decompose $\varphi' = \sum_{i=1}^k \varphi_i'$ where $\varphi_i' \in (\mathfrak{g}^*)_{\lambda_i}^Z$ and $\lambda_1 < \lambda_2 < \dots < \lambda_k \in \mathbb{Q}_{>0}$ are all the positive eigenvalues of Z.

Let us first construct X. For any $t \in \mathbb{R}$, we have the following identity in $\mathfrak{g}^*(\mathbb{C})$:

$$\mathrm{Ad}^*(\exp(tZ))(\varphi + \varphi') = \varphi + \sum_{i=1}^k \mathrm{Ad}^*(\exp(t\lambda_i))\varphi_i'. \tag{B.1}$$

Thus, $\varphi + \sum_i \operatorname{Ad}^*(\exp(t\lambda_i))\varphi_i' \in G(\mathbb{C})\varphi$. Differentiating by t at 0 we obtain that $\sum_i \lambda_i \varphi_i'$ lies in the tangent space to the orbit $\mathbf{G}(\mathbb{C})\varphi$ at φ . This tangent space is the image of φ under the coadjoint action. Thus there exists $Y_{\mathbb{C}} \in \mathfrak{g}(\mathbb{C})$ with $\operatorname{ad}^*(Y)(\varphi) = \sum_i \lambda_i \varphi_i'$. Since both φ and $\sum_i \lambda_i \varphi_i'$ lie in the \mathbb{K} -points \mathfrak{g}^* , there exists $Y \in \mathfrak{g}$ with the same property. Decompose $Y = Y' + \sum_i Y_i$ with $Y_i \in \mathfrak{g}_{\lambda_i}^Z$. Since φ commutes with Z, we obtain $\operatorname{ad}^*(Y_i)(\varphi) = \lambda_i \varphi_i'$. Now we take $X := \sum_i \lambda_i^{-1} Y_i \in \mathfrak{g}_{\geq 0}^Z$.

We now prove the existence of v by descending induction on the maximal index i such that $\varphi' \in \mathfrak{g}_{>\lambda_i}^Z$. The base case i = k has $\varphi' = 0$. For the induction step, let i < k such that $\varphi' \in \mathfrak{g}_{>\lambda_i}^Z$. Then $\mathrm{Ad}^*(\exp(-X))(\varphi + \varphi') = \varphi + \psi$, where $\psi \in \mathfrak{g}_{>\lambda_{i+1}}^Z$. By the induction hypothesis, $\varphi + \psi \in \mathrm{Ad}^*(\exp(\mathfrak{g}_{>0}^Z))\varphi$.

Corollary B.2. The relation R of Definition 2.3.1 is indeed an order relation.

Proof. We have to show that if $(\mathcal{O}, \mathcal{O}') \in R$ then $(\mathcal{O}', \mathcal{O}) \notin R$. Suppose the contrary. Then by Lemma 2.3.2 the complexifications $\mathcal{O}'_{\mathbb{C}}$ and $\mathcal{O}_{\mathbb{C}}$ coincide. Moreover, because of the above assumption there exist a rational semi-simple $Z \in \mathfrak{g}$, $\varphi \in \mathcal{O} \cap \mathfrak{g}_0^Z$, and $\psi \in \mathfrak{g}_{>0}^Z$ such that $\varphi + \psi \in \mathcal{O}_{\mathbb{C}}$, but $\varphi + \psi \notin \mathcal{O}$. This contradicts Lemma B.1.

For future applications, we will need the following generalization of Lemma B.1.

Lemma B.3. Let $Z, S \in \mathfrak{g}$ be commuting rational semi-simple elements, let $q \in \mathbb{Q}$ and let $\varphi \in \mathfrak{g}_0^Z \cap \mathfrak{g}_q^S$ and $\varphi' \in \mathfrak{g}_{\geq 0}^Z \cap \mathfrak{g}_q^S$. Assume that φ is conjugate to $\varphi + \varphi'$ by $\mathbf{G}(\mathbb{C})$. Then there exist $X \in \mathfrak{g}_{\geq 0}^Z \cap \mathfrak{g}_0^S$ such that $\mathrm{ad}^*(X)(\varphi) = \varphi'$ and $v \in \mathrm{Exp}(\mathfrak{g}_{\geq 0}^Z \cap \mathfrak{g}_0^S)$ such that $\mathrm{Ad}^*(v)(\varphi) = \varphi + \varphi'$.

Proof. To construct X we proceed in the same way as in the proof of Lemma B.1, and then decompose it with respect to eigenspaces of S and take projection on the 0 eigenspace. Then we construct v in the same way as in the proof of Lemma B.1.

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DMITRY GOUREVITCH, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, WEIZMANN INSTITUTE OF SCIENCE, POB 26, REHOVOT 76100, ISRAEL

E-mail address: dmitry.gourevitch@weizmann.ac.il URL: http://www.wisdom.weizmann.ac.il/~dimagur

HENRIK GUSTAFSSON

Until September 15, 2019:

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CA 94305-2125. Since September 16, 2019:

SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NJ 08540.

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, PISCATAWAY, NJ 08854.

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF GOTHENBURG AND CHALMERS UNIVERSITY OF TECHNOLOGY, SE-412 96 GOTHENBURG, SWEDEN.

 $E ext{-}mail\ address: gustafsson@ias.edu} \ URL: \ {\tt http://hgustafsson.se}$

AXEL KLEINSCHMIDT, Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut), AM MÜHLENBERG 1, DE-14476 POTSDAM, GERMANY, AND International Solvay Institutes, ULB-CAMPUS PLAINE CP231, BE-1050, BRUSSELS, BELGIUM

E-mail address: axel.kleinschmidt@aei.mpg.de

Daniel Persson, Chalmers University of Technology, Department of Mathematical Sciences, SE-412 96 Gothenburg, Sweden

 $E ext{-}mail\ address: daniel.persson@chalmers.se}$

SIDDHARTHA SAHI, DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, HILL CENTER - BUSCH CAMPUS, 110 FRELINGHUYSEN ROAD PISCATAWAY, NJ 08854-8019, USA

 $E ext{-}mail\ address: sahi@math.rugers.edu}$