Statistical equilibrium of tetrahedra from maximum entropy principle

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Discrete formulations of (quantum) gravity in four spacetime dimensions build space out of tetrahedra. We investigate a statistical mechanical system of tetrahedra from a many-body point of view based on non-local, combinatorial gluing constraints that are modelled as multi-particle interactions. We focus on Gibbs equilibrium states, constructed using Jaynes' principle of constrained maximisation of entropy, which has been shown recently to play an important role in characterising equilibrium in background independent systems. We apply this principle first to classical systems of many tetrahedra using different examples of geometrically motivated constraints. Then for a system of quantum tetrahedra, we show that the quantum statistical partition function of a Gibbs state with respect to some constraint operator can be reinterpreted as a partition function for a quantum field theory of tetrahedra, taking the form of a group field theory.

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I. INTRODUCTION

General relativity has taught us that gravity *is* spacetime geometry, and its observational successes are testimony of the fruitfulness of this lesson, beyond its purely aesthetic appeal. Modern theoretical physics, however, has also provided hints that this continuum description of spacetime and gravity could be emergent, and that some kind of discrete substratum may replace it at the fundamental level [1]. Taking these lessons seriously in the search for quantum gravity has led to non-perturbative, discrete frameworks that aim at constructing quantum theories of geometry, and at showing the emergence of continuum spacetime and GR from discrete foundations. A crucial ingredient in these are geometric objects like polyhedra that can be understood as quantum excitations of geometry. Canonical quantisation of general relativity using Ashtekar variables has led to spin network states [2–4], which admit an interpretation in terms of geometric polyhedra [5, 6]. Cellular complexes of the same type are also the underpinning of covariant spin foam models [7, 8], which have polyhedra dual to spin networks forming their boundary states. In fact, simplicial discretisations have been considered often, originally by Regge [9] with the aim of providing a coordinate-free description of classical spacetime, and are the fundamental mathematical structures of simplicial quantum gravity approaches, like quantum Regge calculus [10] and (causal) dynamical triangulations [11]. Finally, the group field theory framework [12–14] treats polyhedra quite literally as the quanta of spacetime by defining for them a quantum field theory whose interactions represent their gluing and evolution processes; and in doing so, it provides a reformulation of both loop quantum gravity and spin foam models, and of simplicial quantum gravity approaches.

A fully background independent (quantum) statistical mechanical framework [15–20] could be the best way to provide a foundation, and subsequently to analyse such discrete quantum gravity approaches, concerning in particular the emergence of spacetime structures in a continuum approximation [1], treating spacetime itself as a (peculiar) quantum many-body system made of tetrahedra [21]. The definition of such framework poses however many challenges, starting from the identification of a good notion of equilibrium states. Let us say a few words on some of these challenges, and on previous work tackling them.

Presently we are interested in defining Gibbs equilibrium states for a system of an arbitrary but finite number of tetrahedra, with respect to certain gluing constraints motivated from considerations in discrete quantum gravity. As is immediately evident, in such a system there does not exist any notion of a time variable, which begs the question: what notions of equilibrium can a system of many polyhedra admit? In a recent work [22], a statistical mechanics for simplicial degrees of freedom is defined, using the tools provided by a group field theory many-body representation of the same. Therein, general construction schemes are discussed for defining Gibbs states in background independent settings, relevant for both classical and quantum sectors, and independent of any specific underlying framework in which the system is defined. It is suggested that in addition to the Kubo-Martin-Schwinger (KMS) condition [23], the principle of constrained maximisation of entropy as proposed by Jaynes [24, 25], could be the crucial one in such settings [20, 22], allowing for greater generality. The explicit examples provided there however make extensive use of the technical advantages offered by the group field theory formalism. For instance, as an elementary illustration of the utility of Jaynes' principle, an example of a group field theory Gibbs state with respect to a geometric volume operator was presented, and found to naturally support Bose-Einstein condensation to a low-spin phase [22]. In this paper, we tackle the issue of constructing equilibrium states for a system of many tetrahedra on the basis of this principle and in much greater generality, at both classical and quantum level, for general constraints (but giving concrete examples based on a number of geometrically motivated ones), and without relying on specific discrete gravity approaches (but we will see that the group field theory framework emerges naturally in the quantum setting).

The paper is organised as follows. We begin with a discussion of the principle of maximum entropy a la Jaynes while emphasising its role in background independent systems in section II. Focussing first on a system of classical tetrahedra, section III presents its mechanics and statistical mechanics. Disconnected tetrahedra are modelled as 'particles' and its mechanical model is defined via generically non-local, combinatorial 'interactions'. These are gluing constraints in general, as encountered in discrete gravity literature. As a first illustrative example of using the maximum entropy principle in the context of a constrained system related to tetrahedra, we study the case of closure constraint for a single open tetrahedron and construct a Gibbs state with respect to it in section III A. We find that such a state encodes the constraint information partially in a statistical way. Moreover, it is found to be a generalisation of Souriau's Gibbs states to the case of a first class constraint. We define the corresponding statistical system of many free (disconnected) closed tetrahedra, in section III C, then we move from this to the more interesting case of connected tetrahedra, and we consider gluing constraints for the same system which can be interpreted as a definition of their dynamics or 'interactions'. The result of imposing this set of constraints exactly is a labelled triangulation. We use the twisted geometry interpretation of the same constraints to illustrate further the way discrete geometry is (or could be) encoded in the system and to suggest further developments. Section IV discusses the analogous system of many quantum tetrahedra, first outlining its mechanics and statistical mechanics, and subsequently showing that the quantum statistical partition function of a Gibbs state of such a system can be recast in the form of a quantum field theory of tetrahedra, and we show its relation to the group field theory framework.

II. GENERALISED GIBBS STATE FROM ENTROPY MAXIMISATION

As proposed by Jaynes in two seminal papers [24, 25], given our limited knowledge of a system with many underlying degrees of freedom in terms of a set of observable averages $\{\langle \mathcal{O}_a \rangle = Q_a\}$, the least biased statistical distribution (in the sense of not assuming more information about the system than what we actually have) over the microscopic state space of the system is obtained by maximising the information entropy of the said system. By doing this, we are using *only* the amount of information we have access to, not less or more. The resulting distribution is of the Gibbs form, and faithfully encodes our knowledge (and lack thereof) of the microscopics of the system, thus it is the best one can do in order to infer other observable equilibrium properties of the same system.

Consider a finite set $\{\mathcal{O}_a\}_{a=1,2,...}$ of smooth functions $\mathcal{O}_a : \Gamma_{\text{ex}} \to \mathbb{R}$, on a finite-dimensional extended[1] symplectic phase space Γ_{ex}, with Liouville measure $d\lambda$. A statistical (density) state on a phase space is a real-valued, positive and normalised function on it with respect to the measure. Let ρ be a statistical state on Γ_{ex}, such that the statistical averages of $\mathcal{O}_a$ in ρ are fixed,

$$\langle \mathcal{O}_a \rangle_\rho \equiv \int_{\Gamma_{\text{ex}}} d\lambda \, \mathcal{O}_a \, \rho \quad = \quad Q_a \; , \tag{2.1}$$

assuming that the integrals are convergent so that Q_a are well-defined. The state ρ is normalised by definition, so that

$$\langle 1 \rangle_\rho = 1 \; , \tag{2.2}$$

and, its Shannon entropy is

$$S[\rho] = -\langle \ln \rho \rangle_\rho \; . \tag{2.3}$$

Consider maximisation of $S[\rho]$ under the given set of constraints (2.1) and (2.2) [24]. Using the Lagrange multipliers technique, this amounts to finding a stationary solution for the following auxiliary functional

$$L[\rho, \beta_a, \kappa] = S[\rho] - \sum_a \beta_a (\langle \mathcal{O}_a \rangle_\rho - Q_a) - \kappa (\langle 1 \rangle_\rho - 1) \tag{2.4}$$

where $\beta_a, \kappa \in \mathbb{R}$ are Lagrange multipliers. Then, requiring stationarity[2] of L with respect to variations in ρ gives a generalised Gibbs state

$$\rho_{\{\beta_a\}} = \frac{1}{Z_{\{\beta_a\}}} e^{-\sum_a \beta_a \mathcal{O}_a} \tag{2.5}$$

with partition function

$$Z_{\{\beta_a\}} \equiv \int_{\Gamma_{\text{ex}}} d\lambda \, e^{-\sum_a \beta_a \mathcal{O}_a} = e^{1+\kappa} \tag{2.6}$$

where as is usual, normalisation multiplier κ is a function of the rest. The parameters $\{\beta_a\}$ are such that the partition function integral converges.

Analogous arguments hold for finite quantum systems and the above scheme can be implemented straightforwardly [25], as long as the operators under consideration are such that the relevant traces are well-defined on a kinematic (unconstrained) Hilbert space. Statistical states are density operators, i.e. self-adjoint, positive and trace-class operators, on the Hilbert space. Statistical averages for self-adjoint observables $\hat{\mathcal{O}}_a$ are now,

$$\langle \hat{\mathcal{O}}_a \rangle_\rho \equiv \text{Tr}(\hat{\rho} \, \hat{\mathcal{O}}_a) = Q_a \; . \tag{2.7}$$

Following the constrained optimisation problem presented above gives a resultant Gibbs density operator,

$$\hat{\rho}_{\{\beta_a\}} = \frac{1}{Z_{\{\beta_a\}}} e^{-\sum_a \beta_a \hat{\mathcal{O}}_a} \tag{2.8}$$

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[1] By extended we mean that: 1) it is the unconstrained phase space with respect to any constraints under consideration; 2) the system is not equipped with any external time or clock variable, and even if such a variable exists then at this fully parametrized level it is one of the dynamical variables included in the definition of this phase space.

[2] Notice that requiring stationarity of L with variations in Lagrange multipliers implies fulfilment of the constraints (2.1) and (2.2). These two 'equations of motion' of L along with the one determining ρ (coming from stationarity of L with respect to ρ) provide a complete description of the system at hand.

where the state is well-defined as long as the trace for the partition function converges.

The significance of the maximum entropy principle is its applicability to a wide variety of situations. As long as the mathematical description of a given system (in terms of a state space and an observable algebra) is well-defined, and we have access to certain observables $\{\mathcal{O}_a\}$ with which we can define a macrostate $\{Q_a\}$ of the system, the maximum entropy principle can be applied to characterise a notion of statistical equilibrium. Already, the notion of equilibrium is implicit in the existence of the constraints (2.1) which basically say that the system has certain properties that act as good observables to label the state of the system with, because their expectation values remain constant. This feature of remaining constant, which is taken as a starting point of this procedure, could have pointed us already to the fact that there can exist a certain equilibrium description of the system in terms of these variables. As emphasised above, what Jaynes' procedure does is to allow us to find this description in a way that is least biased.

A. Vector-valued temperature

The generalised Gibbs state in (2.5) defines a unique equilibrium distribution labelled by a set of temperatures $\{\beta_a\}$. In fact, we can encode $\{\beta_a\}$ in a multi-component, real vector-valued inverse temperature $\beta \equiv \{\beta_a\}$ and rewrite this state as

$$\rho_\beta = \frac{1}{Z_\beta} e^{-\beta\cdot\mathcal{O}} \tag{2.9}$$

with the vector-valued function $\mathcal{O} = \{\mathcal{O}_a\}$ accordingly defined.

In general, whenever $\beta \cdot \mathcal{O}$ is convex, $\{\beta_a\}$[3] can be understood as generalised (inverse) temperatures, determined by the constraints (2.1), via the equations

$$-\frac{\partial \ln Z_\beta}{\partial \beta_a} = Q_a \tag{2.10}$$

for each a. Other standard thermodynamic relations follow, at least formally. In particular, the partition function $Z_{\{\beta_a\}}$, or equivalently the thermodynamic free energy potential

$$\Phi_\beta := -\ln Z_\beta \tag{2.11}$$

encodes complete thermodynamic information about the system. Quantities Q_a are generalised heats satisfying,

$$dS = \sum_a \beta_a dQ_a \tag{2.12}$$

where

$$S = \sum_a \beta_a Q_a - \Phi_\beta \tag{2.13}$$

is the entropy of state (2.9).

B. Modular flow, stationarity and global equilibrium

Given a generalised Gibbs state as a result of the maximum entropy principle, one can (if one wants) extract a one-parameter modular flow, generated by $-\ln\rho$, in the sense that $\rho\,\omega(X_\beta) = \mathrm{d}\rho$, where X_β is the modular vector field induced by the function $\beta\cdot\mathcal{O}$ (for the particular case above), which plays the role of a generalised modular Hamiltonian, while ω is the symplectic form on Γ_{ex}. By construction, the Gibbs state will be stationary with respect to this flow. In this sense, any such ρ always satisfies stationarity, which is the more traditional characterisation of equilibrium, with respect to its own modular flow[4]. For cases when the Gibbs state is defined by generators of certain transformations, then the modular flow parameter is a rescaling of the parameter of the said transformations by a

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[3] Each individual 'temperature' β_a defines the periodicity in the flow of $\mathcal{O}_a$.
[4] In fact, any faithful algebraic state over a von Neumann algebra defines a 1-parameter Tomita flow with respect to which the state satisfies the KMS condition. The deep significance of this fact for background independent systems was realised in [15, 16].

factor of $1/\beta$. In more general cases when $\mathcal{O}$ is not a generator of some transformation a priori, then the interpretation of its modular flow parameter would depend on the specific case at hand.

Notice that in the special case when the set of constraints defining (2.5) is completely independent, so that the set of the associated flows satisfy $[X_a, X_{a'}] = 0 \;\; \forall a, a'$, the full equilibrium state can be understood as a product of equilibria and consistently written in a factorised form

$$\rho_{\{\beta_a\}} = \prod_a \rho(\beta_a) = \prod_a \frac{1}{Z(\beta_a)} e^{-\beta_a \mathcal{O}_a} \tag{2.14}$$

where $Z_{\{\beta_a\}} = \prod_a Z(\beta_a)$. In this case, stationarity with respect to the evolution induced by $\rho_{\{\beta_a\}}$ is realised only in terms of stationarity with respect to the set of individual modular flows.

In order to define a global notion of equilibrium characterised by a single temperature, we need a prescription for coupling the individual flows [19]. The simplest way to define a global equilibrium is to assume a global temperature $\tilde{\beta}$, relating the individual Lagrange parameters by rescaling

$$\sum_a \beta_a \mathcal{O}_a \longrightarrow \tilde{\beta} \left(\sum_a \frac{\beta_a}{\tilde{\beta}} \mathcal{O}_a \right) \equiv \tilde{\beta} \tilde{\mathcal{O}} \tag{2.15}$$

where $\tilde{\beta} \in \mathbb{R}$. To the state $\rho_{\tilde{\beta}}$ we can formally associate a one-parameter flow in Γ_{ex}, with vector flow $X_{\tilde{\beta}}$ given by a linear combination of the X_a. More generally, the independence of the different flows will be broken by the presence of correlations among the different constraint observables.

C. Remarks

It was emphasised by Jaynes that this information-theoretic manner of defining equilibrium statistical mechanics (and from it, thermodynamics) is to elevate the status of entropy as being more fundamental than even energy. This perspective can be crucially appropriate in settings where energy (and time) are ill-defined or not defined at all. Moreover, this procedure is valid for both classical and quantum settings, and does not technically require any symmetries of the system to be defined a priori, unlike in the more traditional characterisation using the KMS condition. The maximum entropy principle could thus be particularly useful in background independent settings, including both covariant systems on spacetime, and more radical non-spatiotemporal systems like those in discrete quantum gravity (regardless of the specific framework) [20, 22].

Observables $\mathcal{O}$ which characterise a given equilibrium state, in principle, only need to be mathematically well-defined in the given description of the system. Valid choices include a Hamiltonian for time translations in non-relativistic systems [26]; a clock Hamiltonian for evolution in a reference matter variable in deparametrized systems [22]; geometric observables such as 3-volume [22, 24]; and gauge-invariant quantities in presymplectic systems [18]. Even generators of kinematic symmetries such as rotations, or more generally, of 1-parameter subgroups of Lie group actions [27, 28] can be used. The key point is that: (i) these observables need not necessarily encode a physical model of the system and can correspond purely to structural properties such as a kinematic symmetry or a geometric aspect (i.e. they can be physical or structural); and (ii) they need not necessarily be generators of symmetries or physical evolution (including time translation), and can correspond to those properties that are not naturally associated to any transformations of the system (i.e. they can be dynamical or thermodynamical) [22]. Another feature that can be asked of observables $\mathcal{O}$ is that they be gauge-invariant, when gauge symmetries are present. This then ensures that the Gibbs state is defined on the reduced, gauge-invariant state space and in this sense is a physical statistical state of the system. Having clarified these general points, in this work we are mostly interested in gluing constraints in a system of many tetrahedra producing connected configurations and interpreted as a time-independent notion of 'dynamics', adapted to a discrete quantum gravity setting.

III. MANY CLASSICAL TETRAHEDRA

A. Statistical fluctuations in closure

The Jaynes' characterisation of equilibrium allows for a natural group-theoretic generalisation of thermodynamics, whenever the constraint is associated to some (dynamical) symmetry of the system. In this case, the momentum map associated to the Hamiltonian action of the symmetry group on the covariant phase space of the system plays the role of a generalised energy function, comprising the full set of conserved quantities. Moreover, its convexity properties allow for a generalisation of the standard equilibrium thermodynamics [27].

This approach is useful also in our simplicial geometric context. We want to use generalised Gibbs states to define along these lines a statistical characterisation of the tetrahedral geometry in terms of its closure, starting from the extended phase space of a single open tetrahedron. The closure constraint is what allows to interpret geometrically a set of 3d vectors as the normal vectors to the faces of a polyhedron, and thus to fully capture its intrinsic geometry in terms of them. We will base on this our subsequent treatment of a system of many closed tetrahedra (or polyhedra in general).

Consider the symplectic space,

$$\begin{aligned}\Gamma_{\{A_I\}} &= \{(X_I) \in \mathfrak{su}(2)^{*4} \cong \mathbb{R}^{3\times 4} \mid ||X_I|| = A_I\} \\ &\cong S^2_{A_1} \times ... \times S^2_{A_4}\end{aligned} \tag{3.1}$$

where each $S^2_{A_I}$ is a 2-sphere with radius A_I, and $I = 1,2,3,4$. If the four vectors X_I are constrained to sum to zero, the surfaces associated to them (as orthogonal to each of them) close, giving a 4-polyhedron in $\mathbb{R}^3$ with faces of areas $\{A_I\}$[5]. In this example, we consider $\Gamma_{\{A_I\}}$ as the extended phase space of interest, and denote $\Gamma_{\rm ex} \equiv \Gamma_{\{A_I\}}$.

Consider then the diagonal action of the $SU(2)$ Lie group (rotations) on $\Gamma_{\rm ex}$. To this action we can associate a momentum map $J : \Gamma_{\rm ex} \to \mathfrak{su}(2)^*$ defined by,

$$J = \sum_{I=1}^{4} X_I \tag{3.2}$$

where $||X_I|| = A_I$. The symplectic reduction of $\Gamma_{\rm ex}$ with respect to the zero level set $J = 0$, imposes closure of the four faces, resulting in the Kapovich-Millson phase space [29] $\mathcal{S}_4 = \Gamma_{\rm ex}//SU(2) = J^{-1}(0)/SU(2)$ of a closed tetrahedron with given face areas. Space $\Sigma \equiv J^{-1}(0)$ is the constrained submanifold.

We are interested in defining an equilibrium state on $\Gamma_{\rm ex}$ by imposing the closure constraint (only) on average, along the lines described in II. As we shall see below, from a statistical perspective, we can interpret the exact, or 'strong', fulfilment of closure as defining a microcanonical statistical state on $\Gamma_{\rm ex}$ with respect to this constraint. Therefore, defining a generalised Gibbs state on $\Gamma_{\rm ex}$ would naturally correspond to a 'weak' fulfilment of the constraint.

A Gibbs state with respect to closure for an open tetrahedron is defined by maximising the entropy functional (2.3) under normalisation (2.2) and the following three constraints,

$$\langle J_i \rangle_\rho \equiv \int_{\Gamma_{\rm ex}} d\lambda \; \rho \; J_i = 0 \qquad (i = 1,2,3) \tag{3.3}$$

where ρ is a statistical state defined on $\Gamma_{\rm ex}$, and J_i are components of J in a basis of $\mathfrak{su}(2)^*$. Notice that J_i are smooth, real-valued[6] scalar functions on $\Gamma_{\rm ex}$. These are the functions of interest which take on the role of quantities $\mathcal{O}_a$ used in (2.1). We stress again that equation (3.3), for each i, is a weaker condition than imposing closure exactly by $J_i = 0$. Optimising L of equation (2.4) then gives a Gibbs state on $\Gamma_{\rm ex}$ of the form,

$$\rho_\beta = \frac{1}{Z(\beta)} e^{-\beta \cdot J} \tag{3.4}$$

where now the Lagrange multiplier β is a vector in the algebra $\mathfrak{su}(2)$, with components β_i, and $\beta \cdot J = \sum_i \beta_i J_i$ denotes its inner product with $J \in \mathfrak{su}(2)^*$. The equilibrium partition function is given by,

$$Z(\beta) = \int_{\Gamma_{\rm ex}} d\lambda \; e^{-\beta \cdot J} \tag{3.5}$$

where β is such that the integral converges.

Now, recall J being the momentum map corresponding to the diagonal action of $SU(2)$ on $\Gamma_{\rm ex}$. The corresponding co-momentum map $\beta \cdot J \equiv J(\beta)$ then plays the role of the modular Hamiltonian of the system on $\Gamma_{\rm ex}$. Therefore state ρ_β, constructed using the maximum entropy principle is in fact an example of a generalisation of the Gibbs states defined by Souriau [27, 28], to the case of Lie group actions associated to gauge symmetries generated by first class

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[5] Analogous arguments hold for the case of an open d-polyhedron and its associated closure condition.
[6] Real-valued because the algebra $\mathfrak{su}(2)$ under consideration is a vector space over the reals.

constraints. In this case, the vanishing of the associated momentum map is directly related to fulfilling the closure constraint. A detailed analysis of the single tetrahedron thermodynamics is given in [30].

The state ρ_β encodes equilibrium with respect to translations along the integral curves of the vector field ξ_β on Γ_{ex}, defined by the equation $\omega(\xi_\beta) = -\mathrm{d}J(\beta)$, where ω is the symplectic 2-form on Γ_{ex}. It is the fundamental vector field corresponding to vector $\beta \in \mathfrak{su}(2)$. In other words, ρ_β encodes equilibrium with respect to the one-parameter flow characterised by β, which is a generalised vector-valued temperature. This is analogous to the the well-known case of accelerated trajectories on Minkowski spacetime, where thermal equilibrium can be established along Rindler orbits defined by the boost isometry, where β defines the Unruh (inverse) temperature. Another example, in quantum gravity, is that of momentum Gibbs states constructed in group field theory [22].

B. Classical mechanics and statistical mechanics

As we have seen, we can encode the classical intrinsic geometry of a polyhedron by symplectically reducing, with respect to the closure condition, the space Γ_{ex}, to get its Kapovich-Millson space [29],

$$\mathcal{S}_d = \{(X_I) \in \mathfrak{su}(2)^{*d} \mid \sum_I X_I = 0, ||X_I|| = A_I\} \,.$$

In general, the space $\mathcal{S}_d$ is a $(2d-6)$-dimensional symplectic manifold. One could lift the restriction of fixed face areas, thereby adding d degrees of freedom, to get the $(3d-6)$-dimensional space of closed polyhedra modulo rotations. For $d = 4$, this is the 6-dim space of a tetrahedron [31, 32], considered often in discrete quantum gravity contexts. This space corresponds to the possible values of the 6 edge lengths of a tetrahedron, or to the 6 areas of its four faces and two independent areas of parallelograms identified by midpoints of pairs of opposite edges. This space is not symplectic in general, and to get a symplectic manifold from it, one can either remove the d area degrees of freedom to get $\mathcal{S}_d$, or add d number of $U(1)$ degrees of freedom (angle conjugates to the areas), to get the spinor description of the so-called framed polyhedra [33]. Along the lines showed in III A, we can easily extend the statistical description to the case of the framed polyhedron system. However, we are presently more interested in extending the statistical description to a collection of many closed polyhedra.

Let us then consider the space of closed polyhedra with a fixed orientation and extend the phase space description so to encompass the extrinsic geometric degrees of freedom, which we expect to play a role in the description of the coupling leading to a collective model.

The face normal vectors can be seen as elements of the dual algebra $\mathfrak{su}(2)^* \cong \mathbb{R}^3$, which is a Poisson manifold with its Kirillov-Kostant Poisson structure [32]. We add conjugate variables to these $\mathfrak{su}(2)^*$ degrees of freedom (thereby, doubling the dimension) and consider the phase space, $T^*(SU(2)^d/SU(2))$, where the quotient by $SU(2)$ encodes the imposition of the closure.[7]

We further restrict to the case of tetrahedra ($d = 4$). Then the single 'particle' classical phase space under consideration is,

$$\Gamma = T^*(SU(2)^4/SU(2)) \,. \tag{3.6}$$

The extended phase space of an N-particle classical system is given by the direct product space,

$$\Gamma_N = \Gamma^{\times N} \,. \tag{3.7}$$

On this space mechanical models for a system of many tetrahedra can be defined via constraints among such tetrahedra. Typical examples would be non-local, combinatorial gluing constraints, possibly scaled by an amplitude weight. From the point of view of many-body physics, we expect these gluing constraints to be in fact modelled as generic multi-particle interactions, defined in terms of tetrahedron intrinsic and extrinsic geometric degrees of freedom. Different choices of these interactions identify different models of the system.

Thus, the minimal interaction or, better, the key ingredient of such interactions of any many-tetrahedra model are the constraints which glue two faces of any two different tetrahedra. By gluing we mean here the requirement that the areas of these faces match and that their face normals align (with opposite orientation). We will detail below how these conditions are implemented. More stringent conditions, imposing stronger matching of geometric data, as well as more relaxed ones, can also be considered, as will be discussed below. What constitutes as gluing is thus

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[7] In the context of gravity, other choices for the Lie group are $Spin(4)$ and $SL(2,\mathbb{C})$, which could be dealt with in our framework in an entirely analogous manner.

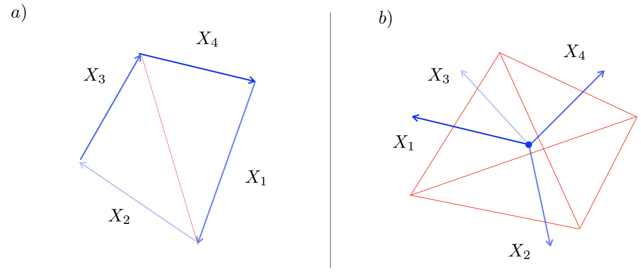

Figure 1. (a) A convex polygon with side vectors X_I. The space of possible polygons in $\mathbb{R}^3$ up to rotations is a $(2d-6)$-dimensional phase space. For non-coplanar normals, the same data define also a unique polyhedron by Minkowski's theorem. (b) For $d = 4$ we get a geometric tetrahedron.

a model-building choice, and so is the choice of which combinatorial pattern of gluings among a given number of tetrahedra is enforced. So, once the system knows how to glue two faces, then the remaining content of a model dictates how the tetrahedra interact non-locally, face-wise, to make simplicial complexes. Again, we will show some such choices below, when illustrating examples of our general framework.

An outline of the ensuing statistical system is as follows. A mechanical model of a system of many classical tetrahedra thus consists of a state space (3.7), an algebra of smooth functions over it and a set of gluing constraints defining the constrained dynamics. Further, a statistical mechanical model is defined by a statistical state (a real-valued, positive and normalised function) on this same system. And equilibrium configurations comprised of collections of geometric tetrahedra can be constructed, at least formally, by using Jaynes' principle in terms of a suitable set of gluing constraints for such mechanical models.

To consider then a system of an arbitrary, variable number of tetrahedra in a statistical setting amounts to including grand-canonical type probability weights, $e^{\mu N}$. Let Z_N be the partition function, which encodes (by definition) all statistical and thermodynamical information about the state ρ_N on Γ_N, $Z_N = \int_{\Gamma_N} d\lambda\ \rho_N$. Then a system with a variable (and arbitrary, possibly infinite) particle number is described by, $Z = \sum_{N\geq 0} e^{\mu N} Z_N$.

C. System of tetrahedra at equilibrium

We now detail the construction of classical Gibbs states for systems of many classical tetrahedra with some concrete examples. The key ingredient is a set of conditions, the 'gluing conditions', which are understood as the constraints which lead from a set of disconnected tetrahedra to an extended simplicial complex. The same gluing process can be encoded in terms of dual graphs, understood as the 1-skeleton of the cellular complex dual to the simplicial complex of interest. The geometry of the initial set of tetrahedra as well as of the resulting simplicial complex can be captured by the $T^*SU(2)$ data introduced above. We will perform our construction in terms of these data first. A more detailed, thus transparent, characterisation of the same (loose notion of) geometry can be obtained in terms of so-called twisted geometry decomposition, which we will connect with at a second stage, to suggest further research directions based on our construction.

Let γ denote an oriented, 4-valent closed graph with L number of oriented links and N number of nodes. Each link ℓ is dressed with $T^*SU(2) \cong SU(2) \times \mathfrak{su}(2)^* \ni (g_\ell, X_\ell)$ data, with variables satisfying invariance under diagonal $SU(2)$ action at each node n. γ is dual to a simplicial complex γ^*, with triangular faces ℓ and tetrahedra n. Geometric closure of each tetrahedron corresponds to $SU(2)$-invariance at the dual node. The source and target nodes (tetrahedra) sharing a directed link (face) ℓ are denoted by $s(\ell)$ and $t(\ell)$ respectively. A state (g_ℓ, X_ℓ) on γ is then an element of $\Gamma_\gamma = T^*SU(2)^L//SU(2)^N$. Such configurations admit a loose notion of discrete geometry in terms of area vectors, normal to the surfaces dual to the links, and identifying a simplicial complex, as we have discussed above. The geometry so-defined is potentially pathological, in the sense that the resulting simplicial complex may not be fully specified in terms of metric data, i.e. its associated edge lengths, as a Regge geometry would be. For our purposes, though, this characterisation suffices to show how a statistical state can be constructed based on encoding gluing and possibly other constraints on the initially disconnected tetrahedra. We will discuss further the purely geometric aspects in the following.

To understand better the gluing process, and the resulting constraints, let us begin with a single closed, classical tetrahedron n with state space Γ of equation (3.6). As mentioned earlier, Γ is the state space where 3d rotations have not been factorised out, which essentially means that each such tetrahedron is equipped with an arbitrary (orthonor-

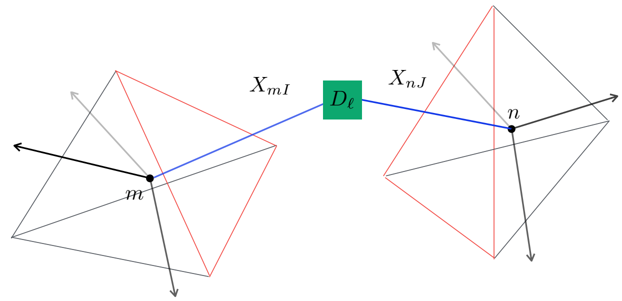

Figure 2. Gluing via constraint $X_{(nJ)} + X_{(mI)} = 0$.

mal) reference frame determining its overall orientation in its $\mathbb{R}^3$ embedding. In the holonomy-flux representation, the four triangular faces $\ell|_n$, of tetrahedron n, are labelled by the four pairs (g_ℓ, X_ℓ), with variables satisfying closure. In the dual picture, we have a single open graph node n with four half-links $\ell|_n$ incident on it. Each half-link is bounded by two nodes, one of which is the central node n, common to all four ℓ. Each ℓ is oriented outward (by choice of convention) from the common node n, which then is the source node for all four half-links. Then in the holonomy-flux parametrisation, each half-link ℓ is labelled by (g_ℓ, X_ℓ).

Let us denote the I^{th} half-link belonging to an open node n by (nI), where $I = 1, 2, 3, 4$. Equivalently, (nI) also denotes the I^{th} face of tetrahedron n. Two tetrahedra n and m are said to be neighbours if at least one pair of faces, (nI) and (mJ), are adjacent, that is the variables assigned to the two faces satisfy the following constraints,

$$g_{(nI)} g_{(mJ)} = e \quad , \quad X_{(nI)} + X_{(mJ)} = 0 \ . \tag{3.8}$$

A given classical state associated to the connected graph γ can then be understood as a result of imposing the constraints (3.8) on pairs of faces in a system of N open nodes, or disconnected tetrahedra. That is, γ is a result of imposing L number each of $SU(2)$-valued and $\mathfrak{su}(2)^*$-valued constraints, which we denote by C and D respectively. This in turn is a total $6L$ number of $\mathbb{R}$-valued (component) constraint functions $\{C_{\ell,a}\,, D_{\ell,a}\}$, for $\ell = 1, 2, ..., L$ and $a = 1, 2, 3$. For instance, creation of a full link $\ell = (nI, mJ)$ involves matching the fluxes, component-wise, by imposing the three constraints $D_{\ell,a} = X^a_{(nI)} + X^a_{(mJ)} = 0$, as well as restricting the conjugate parallel transports to satisfy $C_{\ell,a} = (g_{(nI)} g_{(mJ)})^a - e^a = 0$. Naturally the final combinatorics of γ is determined by which half-links are glued pairwise, which is encoded in which specific pairs of such constraints are imposed on the initial data.

As an example, consider a dipole graph, Figure 3. This can be understood as imposing constraints on pairs of half-links of two open 4-valent nodes. Here $L = 4$, thus we have at hand four constraints D_ℓ on flux variables,

$$\begin{aligned} X_{(11)} + X_{(21)} = 0 \ , \quad X_{(12)} + X_{(22)} = 0 \ , \\ X_{(13)} + X_{(23)} = 0 \ , \quad X_{(14)} + X_{(24)} = 0 \ . \end{aligned} \tag{3.9}$$

This corresponds to a set of 3×4 component constraint equations $D_{\ell,a} = 0$. Similarly for holonomy variables.

As another example, consider a 4-simplex graph made of five 4-valent nodes (see Figure 4). The combinatorics is encoded in the choice of pairs of half links that are glued. Here $L = 10$, corresponding to ten constraints D_ℓ on the flux variables,

$$\begin{aligned} &X_{(12)} + X_{(21)} = 0 \ , \quad X_{(13)} + X_{(31)} = 0 \ , \quad X_{(14)} + X_{(41)} = 0 \ , \quad X_{(15)} + X_{(51)} = 0 \ , \quad X_{(23)} + X_{(32)} = 0 \ , \\ &X_{(24)} + X_{(42)} = 0 \ , \quad X_{(25)} + X_{(52)} = 0 \ , \quad X_{(34)} + X_{(43)} = 0 \ , \quad X_{(35)} + X_{(53)} = 0 \ , \quad X_{(45)} + X_{(54)} = 0 \ . \end{aligned} \tag{3.10}$$

As before, this corresponds to 30 component equations for the flux variables, and another 30 for holonomies.

When these constraints are satisfied exactly, that is $\{C_{\ell,a} = 0\,, D_{\ell,a} = 0\}$ (for all ℓ, a), then this system of N tetrahedra admits a geometric interpretation based on the resultant simplicial complex. But as discussed in the previous section, there is a way of imposing these constraints only on average, that is $\{\langle C_{\ell,a}\rangle_\rho = 0\,, \langle D_{\ell,a}\rangle_\rho = 0\}$.

This statistical manner of weakly imposing the constraints results in a generalised Gibbs state, parametrised by $6L$ number of generalised temperatures,

$$\rho_{\{\alpha_\epsilon, \beta_\epsilon\}} \propto e^{-\sum_{\epsilon=1}^{3L} (\alpha_\epsilon C_\epsilon + \beta_\epsilon D_\epsilon)} \tag{3.11}$$

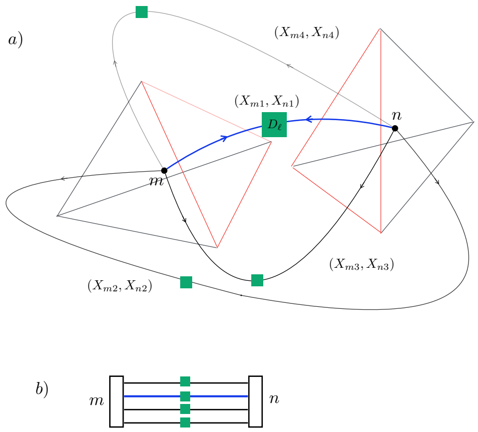

Figure 3. Dipole gluing.

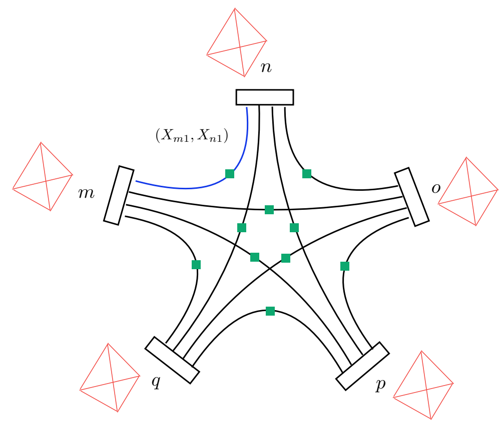

Figure 4. Resultant 4-simplex from combinatorial gluing between faces of five tetrahedra.

where index $\epsilon = 1, 2, ..., 3L$ spans the different components a of the different links ℓ, i.e. all possible pairings ℓ, a. Notice that the constraints $\{C_\epsilon, D_\epsilon\}$ are smooth functions on Γ_N, and $\rho_{\{\alpha_\epsilon,\beta_\epsilon\}}$ is thus a state defined for the N particle system (assuming well-defined normalisation). Since the creation of a full link ℓ requires all three components of a given group- or algebra-valued constraint to be satisfied simultaneously for the given pair of half links, we assign one pair of temperature parameters $(\alpha_\ell, \beta_\ell)$ to each link ℓ, instead of three pairs.

Further, if we understand the combination of gluing constraints associated to a given connected graph γ (and its dual simplicial complex) as a single 'interaction' of the involved tetrahedra, i.e. as a single global constraint on them, then for a set of constraints (corresponding to some γ), we have a single common temperature which encodes the rate of statistical fluctuations over a given geometric configuration on γ^*.

In this sense, we can generally write the Gibbs state in terms of a single temperature parameter for the given graph interaction,

$$\rho_{\beta_\gamma} \propto e^{-\beta_\gamma \sum_{\ell=1}^{L} \frac{(\alpha_\ell \cdot C_\ell + \beta_\ell \cdot D_\ell)}{\beta_\gamma}} = e^{-\beta_\gamma G}, \tag{3.12}$$

where $G \equiv \sum_{\ell=1}^{L} (\alpha_\ell \cdot C_\ell + \beta_\ell \cdot D_\ell)/\beta_\gamma$ indicates the overall constraint associated to the given pattern of gluings corresponding to the graph γ. The above can of course be generalised along the same lines to include different interaction terms, each corresponding to a given pattern of gluings corresponding to a different graph γ. This case would be treated as we have described in section II, discussing Jaynes' principle for multiple constraints.

Such a state is a statistical mixture of configurations where the ones which are glued with the combinatorics of γ (and thus admit a loose geometric interpretation) are weighed exponentially more than those which are not. In this sense it illustrates a statistical way to encode an approximate notion of discrete geometry.

The above example for a given (fixed) 4-simplex graph can be generalised to take into account all the possible realisations of the graph. In general, for any graph γ defined by a rank-n gluing constraint, we should allow for the sum $\sum_{\ell=1}^{L} (\alpha_\ell \cdot C_\ell + \beta_\ell \cdot D_\ell)$ to run over the links L comprised of all possible sets of nodes $n \in N$. Formally, this can be realised by taking

$$\rho_{\beta_\gamma}(N) \propto e^{-\frac{\beta_\gamma}{C(n,N)} \sum_{\{n\}_\gamma} \left(\sum_{\ell \in \gamma_n}^{L} \frac{(\alpha_\ell \cdot C_\ell + \beta_\ell \cdot D_\ell)}{\beta_\gamma} \right)} \tag{3.13}$$

where $C(n, N) = \frac{N!}{(N-n)! \cdot n!}$ is the number of possible ways to create the graph γ out of different sets of n nodes, within the N tetrahedra. The state is now dependent on the specific graph induced by the constraint combinatorial structure, and on the number of nodes n involved in the gluing. Notice that we choose the temperature to be insensitive to the overall multiplicity. Namely, we set β_γ for all realizations of γ's.

Finally, if we allow for the number of tetrahedra N to vary – hence considering $\rho_{\beta_\gamma}(N)$ – we can write a general expression for the partition function of a system of many classical tetrahedra at equilibrium with respect to a rank-n gluing constraint,

$$Z_{\mu,\beta_\gamma} = \sum_N z^N Z_{\beta_\gamma}(N). \tag{3.14}$$

where $z = \exp(\mu\beta_\gamma)$, μ is the Lagrange parameter for N and $Z_{\beta_\gamma}(N)$ is the canonical partition function, written in terms of the Gibbs state in (3.12), for a finite number of tetrahedra.

Now that we have presented the construction of a statistical state for many classical tetrahedra, that involves some set of gluing constraints, imposing a geometric interpretation, we can discuss briefly some model-building strategies that can be followed to construct more examples of interesting simplicial gravity models. Any such model building strategy should be based on a clear understanding of how simplicial geometry is encoded in the data we have used.

A more precise parametrization of the holonomy-flux geometries can also be given in the language of twisted geometries [34, 35]. This relies on the fact that the link space $T^*SU(2)$ can be decomposed as $S^2 \times S^2 \times T^*S^1 \ni (\mathfrak{N}_{s(\ell)}, \mathfrak{N}_{t(\ell)}, A_\ell, \xi_\ell)$, modulo null orbits of the latter, and up to a $\mathbb{Z}_2$ symmetry. The variables are related by the following canonical transformations,

$$g = \mathfrak{n}_s e^{\xi\tau_3} \mathfrak{n}_t^{-1} \quad , \quad X = A\mathfrak{n}_s\tau_3\mathfrak{n}_t^{-1} \tag{3.15}$$

where $\mathfrak{n}_{s,t} \in SU(2)$ are those elements which in the adjoint representation R rotate the vector $z \equiv (0, 0, 1)$ to give $\mathbb{R}^3$ vectors $\mathfrak{N}_{s,t}$ respectively. That is $\mathfrak{N}_{s,t} = R(\mathfrak{n}_{s,t}).z$, or equivalently $\mathfrak{n}_{s,t}\tau_3\mathfrak{n}_{s,t}^{-1} = \sum_{a=1}^3 \mathfrak{N}^a_{s,t}\tau_a$ respectively for s and t. Generators of $\mathfrak{su}(2)$ are $\tau_a = -\frac{i}{2}\sigma_a$, where σ_a are Pauli matrices. Vectors $\mathfrak{N}_{s(\ell)}$ and $\mathfrak{N}_{t(\ell)}$ are unit normals to the face ℓ as seen from two arbitrary, different orthonormal reference frames attached to $s(\ell)$ and $t(\ell)$ respectively. A_ℓ is the area of ℓ, and ξ_ℓ is an angle which encodes (partial[8]) extrinsic curvature information. So, a closed twisted geometry configuration supported on γ is an element of $\times_\ell T^*S^1 \times_n \mathcal{S}_4$, where $\mathcal{S}_4$ is the Kapovich-Millson phase space of a tetrahedron given a set of face areas; each link is labelled with (A_ℓ, ξ_ℓ), and each node with four unit normals $\mathfrak{N}$ (in a given reference frame) that satisfy closure.

A twisted geometry is in general discontinuous across the faces; so is the one described in terms of holonomy-flux variables, because both contain the same information. Face area A_ℓ of a shared triangle is the same as seen from tetrahedron $s(\ell)$ or $t(\ell)$ on either side; but the edge lengths when approaching from either side may different in general. That is, the shape of the triangle ℓ, as seen from the two tetrahedra sharing it, is not constrained to match. If additional shape-matching conditions [36] were satisfied, then we would instead have a proper Regge (metric) geometry on γ^*, which is a subclass of twisted geometries. These shape-matching conditions can be related to the so-called simplicity constraints, which are central in all model building strategies in the context of spin foam models, and whose effect is exactly to enforce geometricity (in the sense of metric and tetrad geometry) on discrete data of the holonomy-flux type, characterising (continuum and discrete) topological BF theories in any dimension.

The gluing constraints in equation (3.8), in twisted geometry variables, take the form of the following constraints

$$\begin{aligned} A_{(nI)} - A_{(mJ)} = 0 \quad &, \quad \xi_{(nI)} + \xi_{(mJ)} = 0 \ , \\ \mathfrak{N}_{s(nI)} - \mathfrak{N}_{t(mJ)} = 0 \quad &, \quad \mathfrak{N}_{t(nI)} - \mathfrak{N}_{s(mJ)} = 0 \ . \end{aligned} \tag{3.16}$$

The result is the same, of course, as in the holonomy-flux case: half-links (nI) and (mJ) which satisfy the above set of six component constraint functions (in either of the parametrisations) are thus glued[9] to form a single link $\ell \equiv (nI, mJ)$. Equivalently, the two faces of the initially disconnected tetrahedra are now adjacent.

The more refined geometric data used in the twisted geometry language allow for a model-building strategy leading for example to statistical states in which only some of the gluing conditions are imposed strongly, while others are imposed only on average. In the same spirit of achieving greater geometrical significance of the statistical state that one ends up with, our construction scheme can be applied with additional constraints, beyond the gluing ones we illustrated above. For instance, starting with the space of twisted geometries on a given simplicial complex (dual to) γ, one could consider imposing (on average) also shape-matching constraints, or simplicity constraints, to encode an approximate notion of a Regge geometry using a Gibbs statistical state. This would be the statistical counterpart of the construction of spin foam models, i.e. discrete gravity path integrals in representation theoretic variables [8, 37, 38], based on the formulation of gravity as a constrained BF theory. This is left to future work.

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[8] The remaining two degrees of freedom of extrinsic curvature are encoded in the normals $\mathfrak{N}_{s(\ell)}$ and $\mathfrak{N}_{t(\ell)}$ [34]. For instance, in the subclass of Regge geometries, ξ_ℓ is proportional to the modulus of the extrinsic curvature [35].

[9] Gluing the two half-links is essentially superposing one over the other in terms of aligning their respective reference frames. This is evident from the constraints for the normal vectors $\mathfrak{N}$ which superposes the target node of one half-link on the source node of the other, and vice-versa.

IV. MANY QUANTUM TETRAHEDRA

A. Quantum mechanics and statistical mechanics

There are several ways of translating the classical construction presented in the previous sections at the quantum level, starting from a quantisation of the geometry of a single tetrahedron [31, 32].

In a quantum setting in general, each closed polyhedron face I is assigned an $SU(2)$ representation data j_I with its associated representation space $\mathcal{H}_{j_I}$, and polyhedron itself with an intertwiner. Quantisation of $T^*(SU(2))^d/SU(2))$ is the full space of d-valent intertwiners, $\bigoplus_{j_I} \text{Inv} \otimes_{I=1}^d \mathcal{H}_{j_I}$. Here $\text{Inv}\otimes_{I=1}^d \mathcal{H}_{j_I}$ is the space of d-valent intertwiners with given fixed spins $\{j_I\}$ i.e. given fixed face areas, corresponding to a quantisation of $\mathcal{S}_d$. A collection of neighbouring quantum polyhedra has been associated to a spin network of arbitrary valence [5], with the labelled nodes and links of the latter being dual to labelled polyhedra and their shared faces respectively. Then for a quantum tetrahedron, the 1-particle Hilbert space is taken to be

$$\mathcal{H} = \bigoplus_{j_I} \text{Inv} \otimes_{I=1}^4 \mathcal{H}_{j_I} \ , \tag{4.1}$$

with quantum states of an N-particle system belonging to

$$\mathcal{H}_N = \mathcal{H}^{\otimes N} \ . \tag{4.2}$$

We can equivalently work with the holonomy representation of the same quantum system, in terms $SU(2)$ group data. This is also the state space of a single gauge-invariant quantum of a group field theory defined on an $SU(2)^4$ base manifold [12, 39],

$$\mathcal{H} = L^2(SU(2)^4/SU(2)) \ . \tag{4.3}$$

A further, equivalent representation could be given in terms of non-commutative Lie algebra (flux) variables [6, 40].

As discussed in section III, mechanical models of N quantum tetrahedra can be defined by a set of gluing constraint operators defined on $\mathcal{H}_N$. The general discussion therein is applicable here also. The basic ingredient of gluing is again to define face sharing conditions. For instance, the classical constraints of equation (3.8) can be implemented by group averaging of wavefunctions [39],

$$\Psi_\gamma(\{g_{(nI)}g^{-1}_{(mJ)}\}) = \prod_{(nI,mJ)|_\gamma} \int_{SU(2)} dh_{(nI,mJ)} \ \psi(\{g_{(nI)}h_{(nI,mJ)}, g_{(mJ)}h_{(nI,mJ)}\}) \tag{4.4}$$

where we have used the notation as introduced in section III C above, and $\psi \in \mathcal{H}_N$ is a wavefunction for a system of generically disconnected N tetrahedra. So, a wavefunction defined over full links (nI, mJ) of a graph γ is a result of averaging over half-links (nI) and (mJ) by $SU(2)$ elements $h_{(nI,mJ)}$. The same can also be implemented in terms of fluxes X, using a non-commutative Fourier transform between holonomy and flux variables [39].

Thus a quantum mechanical model of a system of N tetrahedra consists of the unconstrained Hilbert space $\mathcal{H}_N$, an operator algebra defined over it and a set of gluing constraint operators specifying the model.

Now for a quantum multi-particle system, a Fock space is a suitable home for configurations with varying particle numbers. For bosonic[10] quanta, each N-particle sector is the symmetric projection of the full N-particle Hilbert space, so that the Fock space takes the following form,

$$\mathcal{H}_F = \bigoplus_{N\geq 0} \text{sym}\, \mathcal{H}_N \ . \tag{4.5}$$

The Fock vacuum $|0\rangle$ is the one corresponding to a state with no tetrahedron degrees of freedom.

Then, a system of an arbitrarily large number of quantum tetrahedra is described by the state space $\mathcal{H}_F$, an algebra of operators over it with a special subset of them identified as gluing constraints. Quantum statistical states of tetrahedra are density operators (self-adjoint, positive and trace-class operators) on $\mathcal{H}_F$ [22].

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[10] As for a standard multi-particle system, bosonic statistics corresponds to a symmetry under particle exchange. For the case when a system of quantum tetrahedra is glued appropriately to form a spin network, then this symmetry is interpreted as implementing the graph automorphism of node relabellings.

Let us a consider a system of quantum tetrahedra with a model defined by a (self-adjoint) constraint operator $\hat{C}$ defined on $\mathcal{H}_F$, and a generalised Gibbs state of the form,

$$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{C}} \quad , \quad Z_\beta = \mathrm{Tr}_{\mathcal{H}_F}(e^{-\beta \hat{C}}) \tag{4.6}$$

where β is the Lagrange multiplier for $\langle \hat{C} \rangle = 0$.

We can think of this state as the result of a maximisation of entropy with respect to a series of constraints $\langle \hat{C}_a \rangle = 0$, each one encoding a specific gluing interaction kernel. The form of the Gibbs state in (4.6) then results by coupling the constraints via a global rescaling of the individual temperatures, along the line discussed in (2.15), so to give $\hat{C} = \sum_a \frac{\beta_a}{\beta} \hat{C}_a$.

Each $\hat{C}_a$ encodes a different coupling operator of given rank and combinatorial degeneracy. In this sense, we can think of the individual relative temperatures β_a/β as coupling constants in the sum $\sum_a \frac{\beta_a}{\beta} \hat{C}_a$, giving different weights to the different gluing kernels, within a unique equilibrium distribution.

Particularly, a density operator with a contribution from a grand-canonical weight of the form $e^{\mu \hat{N}}$, corresponds to a statistical state with a varying particle number, where $\hat{N}$ is the number operator associated with the Fock vacuum. The corresponding partition function

$$Z_{\mu,\beta} = \mathrm{Tr}_{\mathcal{H}_F}\left[e^{-\beta(\hat{C} - \mu \hat{N})} \right] = \mathrm{Tr}_{\mathcal{H}_F}\left[e^{-\beta(\sum_a \frac{\beta_a}{\beta} \hat{C}_a - \frac{\tilde{\mu}}{\beta} \hat{N})} \right] \tag{4.7}$$

provides the quantum counterpart of the expression (3.14) in III C.

B. Field theory of quantum tetrahedra

The Hilbert space $\mathcal{H}_F$[11] is generated by a set of ladder operators acting on the cyclic vacuum $|0\rangle$, and satisfying the algebra,

$$[\hat{\varphi}(g_I), \hat{\varphi}^*(g'_J)] = \delta(g_I, g'_J) \tag{4.8}$$

where δ is an identity distribution on the space of smooth, complex-valued L^2 functions on $SU(2)^4$.

This formulation already hints at a second quantised language in terms of quantum fields of tetrahedra. This language can indeed be applied to the whole statistical mechanics framework we have developed, in particular to the partition function obtained in the previous section.

The way to obtain this field-theoretic reformulation is pretty standard. Indeed, the trace in the partition function (4.6) can be evaluated using an overcomplete coherent state basis of (field) coherent states,

$$|\psi\rangle = e^{-\frac{||\psi||^2}{2}} e^{\int_{SU(2)^4} \psi(g_I) \hat{\varphi}^*(g_I)} |0\rangle \, . \tag{4.9}$$

Here the states are labelled by $\psi \in \mathcal{H}$ and $||.||$ is the L^2 norm in $\mathcal{H}$. This gives,

$$Z = \int [D\mu(\psi, \bar{\psi})] \, \langle \psi | \, e^{-\beta \hat{C}} \, |\psi\rangle \, , \tag{4.10}$$

where the resolution of identity is $I = \int [D\mu(\psi, \bar{\psi})] \, |\psi\rangle \langle \psi|$, and the coherent state functional measure [42] is,

$$D\mu(\psi, \bar{\psi}) = [D\psi D\bar{\psi}] \; e^{-||\psi||^2} \, . \tag{4.11}$$

This quantum statistical partition function can be reinterpreted as the partition function for a field theory (restricted to complex-valued L^2 fields) of the underlying quanta, which here are quantum tetrahedra [39]. This can be seen as follows.

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[11] We remark that $\mathcal{H}_F$ is the GNS representation space of the Fock algebraic state associated with a group field theory Weyl algebra [22, 41].

For a generic self-adjoint $\hat{C}(\hat{\varphi},\hat{\varphi}^*)$ as a polynomial function of the generators, and a given (but generic) choice of the operator ordering defining the exponential operator, the integrand of Z can be treated as follows,

$$\begin{aligned}\langle\psi|\, e^{-\beta\hat{C}(\hat{\varphi},\hat{\varphi}^*)}\,|\psi\rangle &= \langle\psi|\sum_{k=0}^{\infty}\frac{(-\beta)^k}{k!}\hat{C}^k\,|\psi\rangle \\ &= \langle\psi|:e^{-\beta\hat{C}}:|\psi\rangle + \langle\psi|:\mathrm{poly}(\hat{\varphi},\hat{\varphi}^*,\beta):|\psi\rangle \end{aligned} \tag{4.12}$$

where to get the second equality, we have used the commutation relations (4.8) on each $\hat{C}^k(\hat{\varphi},\hat{\varphi}^*)$, to collect all normal ordered terms $:\hat{C}^k:$ to get the normal ordered exponential $:e^{-\beta\hat{C}}:$, and the second term is a collection of the remaining terms arising as a result of swapping $\hat{\varphi}$'s and $\hat{\varphi}^*$'s, which will then in general be a normal ordered polynomial function of $\hat{\varphi},\hat{\varphi}^*$ and β. Recalling that coherent states are eigenstates of the annihilation operator, $\hat{\varphi}(g_I)\,|\psi\rangle = \psi(g_I)\,|\psi\rangle$, we have

$$\langle\psi|:e^{-\beta\hat{C}}:|\psi\rangle = e^{-\beta\langle\psi|:\hat{C}:|\psi\rangle} \equiv e^{-\beta C[\bar{\psi},\psi]}\ , \tag{4.13}$$

and, denoting by operator $\hat{A}\equiv\mathrm{poly}(\hat{\varphi},\hat{\varphi}^*,\beta)$,

$$\langle\psi|:\hat{A}(\hat{\varphi},\hat{\varphi}^*,\beta):|\psi\rangle = A[\bar{\psi},\psi,\beta]\ , \tag{4.14}$$

which encodes all higher order quantum corrections[12]. The quantum statistical partition function for a dynamical system of complex-valued L^2 fields ψ defined on the base manifold $SU(2)^4$ can thus be written as,

$$Z = \int[D\mu(\psi,\bar{\psi})]\ \left(e^{-\beta C[\bar{\psi},\psi]} + A[\bar{\psi},\psi,\beta]\right)\ \equiv\ Z_0 + Z_{\mathcal{O}(\hbar)} \tag{4.15}$$

where, by notation $\mathcal{O}(\hbar)$ we mean only that this sector of the full theory encodes all higher orders in quantum corrections relative to Z_0.[13] This complete partition function defines thus a statistical field theory of quantum tetrahedra (or in general, polyhedra with a fixed number of boundary faces), with a combinatorially non-local statistical weight (defined by the functional in the fields appearing in the exponent of the amplitude and a possibly highly non-trivial measure term), i.e. a group field theory. This derivation and interpretation of the foundation of group field theories was suggested in [12], and the present work puts it on more solid grounds. If we are able to either reformulate exactly, or under suitable approximations, Z in terms of a simple exponential measure

$$Z_{\mathrm{eff}} = \int[D\mu(\psi,\bar{\psi})]\ e^{-C_{\mathrm{eff}}[\bar{\psi},\psi,\beta,C,A]} \tag{4.16}$$

then the correspondence with a standard field theory would be even more manifest. Of course, this is directly the result of the rewriting if the simple normal ordering is chosen from the start for the exponential operator of the constraint.

To give an example, consider for instance the mechanical model of quantum tetrahedra defined by a set of gluing constraints $\hat{C}^5$ involving five tetrahedra and having the combinatorial structure of the boundary of a 4-simplex, plus a condition involving only two of them. Then, the statistical quantum field theory as derived from the full quantum statistical system (when the latter is taken to be in a generalised Gibbs state) is

$$Z_0 = \int[D\mu(\psi,\bar{\psi})]\, e^{-\beta C^5[\bar{\psi},\psi]}\ , \tag{4.17}$$

with the statistical weight dictated by

$$C^5[\bar{\psi},\psi] \equiv \int_{SU(2)^{4\times 2}}\bar{\psi}(g^1)K(g^1,g^2)\psi(g^2) + \int_{SU(2)^{4\times 5}}\psi(g^1)\psi(g^2)\psi(g^3)\psi(g^4)\psi(g^5)V(g^1,g^2,g^3,g^4,g^5) + c.c.\ . \tag{4.18}$$

Hence, for the choice of constraint $\hat{C}^5$, the partition function defines a 4d simplicial group field theory model of complex-valued, $SU(2)$-gauge invariant, L^2 fields ψ, defined on the base manifold $SU(2)^4$. The specific choice of

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[12] Notice that the quantum measure A is not necessarily of an exponential form.

[13] Further investigation into the interpretation, significance and consequences of this rewriting of Z in discrete quantum gravity is left for future work.

classical constraints one wants to impose enter the result in terms of the matrix elements, in the Fock space, of the corresponding quantum operators. In the second quantised reformulation, these matrix elements become the convolution kernels for fields, here labelled K and V.

The comparison with existing group field theory models, for topological BF theories, thus in absence of additional geometricity conditions and simply using gluing conditions of holonomy-flux data, shows that these are obtained from our statistical construction, but by starting from a reformulation of the initial gluing constraints. Recall that the constraint equation

$$\hat{C}\,|\psi\rangle = 0 \tag{4.19}$$

can be equivalently recast in the form

$$\hat{\mathbb{P}}_C\,|\psi\rangle = |\phi\rangle \tag{4.20}$$

where $\hat{\mathbb{P}}_C \simeq \delta(\hat{C})$ is the operator projecting on the kernel of $\hat{C}$. Physical states are those $|\phi\rangle$ left invariant by the projector operator (namely $|\phi\rangle \in \mathrm{range}(\hat{\mathbb{P}}_C)$). In particular, we can recast equation (4.20) in the form of a constraint relation, by considering the complementary operator $\hat{\mathbb{Q}}_C \equiv (1_{\mathcal{H}_F} - \hat{\mathbb{P}}_C)$, hence writing

$$\hat{\mathbb{Q}}_C\,|\psi\rangle = 0, \tag{4.21}$$

with physical states now being in the kernel of $\hat{\mathbb{Q}}_C$.[14]

If we now repeat the the derivation of the canonical partition function, to get

$$Z_{\mu,\beta} = \mathrm{Tr}_{\mathcal{H}_F}\left[e^{-\beta(\hat{\mathbb{Q}}_C - \mu\hat{N})}\right] = \mathrm{Tr}_{\mathcal{H}_F}\left[e^{-\beta(1_{\mathcal{H}_F} - \hat{\mathbb{P}}_C - \mu\hat{N})}\right], \tag{4.22}$$

we will end up dealing with a statistical weight expressed in terms of matrix elements of the projector operator,

$$\mathbb{P}_C[\bar{\psi},\psi] \equiv \int_G \bar{\psi}(g^1)\ldots\bar{\psi}(g^m) P_{m+n}(g^1,\ldots,g^m;g^1,\ldots,g^n)\psi(g^1)\psi(g^2)\ldots\psi(g^n). \tag{4.23}$$

This directly gives the group field theory interaction kernels for BF models, which are expressed in terms of products of delta functions whose arguments are the classical gluing constraints.

To summarise, in this statistical formulation we are able to give a more solid foundation to the picture in which a group field theory is a quantum field theory of tetrahedra (or polyhedra, in general), and the kernels of a GFT action[15] originate from non-local many-body interactions (gluing constraints) between the underlying quanta.

V. CONCLUSION

We have investigated the statistical mechanics of classical and quantum tetrahedra, which are candidates for quanta of spacetime geometry in discrete quantum gravity approaches. Particularly, we have focused on the definition of Gibbs equilibrium states, in such a background independent context. They can be defined using Jaynes' principle, which does not rely on the identification of any (time) symmetry or automorphism for characterising the state, but only on the requirement of maximal entropy subject to macroscopic constraints (which are then approximately satisfied in terms of expectation values). Starting with a system of many classical tetrahedra, we have presented its mechanics and statistical mechanics. As a first illustrative example, we have defined a Gibbs state for the case of the closure constraint for a single classical tetrahedron. Already this example shows that, in a constrained system, a Gibbs or a microcanonical state can be used respectively to partially (on average, $\langle C\rangle = 0$) or exactly ($C = 0$) impose the constraints. In other words, the imposition of constraints can be viewed in a novel way in terms of identifying suitable statistical states on the full unconstrained state space. Further, the particular example of a Gibbs state with respect to the closure condition of a tetrahedron is a generalisation of Souriau's Gibbs states to the case of first class constraints. We then consider generalised Gibbs states in a system of (arbitrary) many tetrahedra, with

[14] Trivially, for projectors we have that $\mathrm{range}(I-\mathbb{P}) = \mathrm{null}(\mathbb{P})$, and $\mathrm{null}(I-\mathbb{P}) = \mathrm{range}(\mathbb{P})$.
[15] What we understand better now as a gluing constraint and call C, is customarily called action, and treated also like a Euclidean action (even though it is not associated with any notion of Wick rotating from a Lorentzian action due to the absence of any spatiotemporal structures of the present system) in group field theory literature.

respect to gluing constraints which produce a (approximately, twisted) geometric configuration for connected simplicial complexes, formed by the same tetrahedra. Finally, we have described how our construction translates naturally at the quantum level, in terms of a Hilbert (Fock) space of many quantum tetrahedra and constraint operators acting on them (with the same geometric interpretation). After presenting the quantum statistical mechanics of many tetrahedra, we discuss how the same is recast in the form of a quantum statistical field theory partition function for tetrahedra using a 2nd quantized reformulation and field coherent states (as customary); this corresponds, in fact, to the partition function of a group field theory description of the same system of many quantum tetrahedra.

The statistical framework presented in this work could be used to explore in detail specific examples of simplicial gravity (or group field theory) models with direct or stronger geometric interpretation, and thus of greater interest for quantum gravity. For instance, one could consider the state space of geometric (in the sense of metric) tetrahedra and utilise a generalised Gibbs state to define the partition function with a dynamics encoded by the Regge action. Another interesting direction would be to define a Gibbs density implementing not only gluing constraints but also shape-matching constraints on the twisted geometry space, or simplicity constraints, thus reducing again to a proper Regge geometry from flux-holonomy data. More generally, our framework can be used, starting from any given concrete model, to extract and analyse the thermodynamics and hydrodynamics of the underlying system of (quantum) tetrahedra. It is at this coarse grained level of description, in fact, that we expect a continuum spacetime and geometry, with an approximately gravitational dynamics, to emerge [1, 43].

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