Kerr Black Holes and Nonlinear Radiation Memory

Thomas Mädler[1] ‡ **and Jeffrey Winicour**[2,3]

[1] Núcleo de Astronomía, Facultad de Ingeniería, Universidad Diego Portales, Av. Ejército 441, Santiago, Chile

[2] Department of Physics and Astronomy
University of Pittsburgh, Pittsburgh, PA 15260, USA

[3] Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut,
14476 Golm, Germany

Abstract.

The Minkowski background intrinsic to the Kerr-Schild version of the Kerr metric provides a definition of a boosted spinning black hole. There are two Kerr-Schild versions corresponding to ingoing or outgoing principal null directions. The two corresponding Minkowski backgrounds and their associated boosts differ drastically. This has an important implication for the gravitational memory effect. A prior analysis of the ejection of a non-spinning Schwarzschild black hole showed that the memory effect in the nonlinear regime agrees with the linearized result based upon the retarded Green function only if the ejection velocity corresponds to a boost symmetry of the ingoing Minkowski background. A boost with respect to the outgoing Minkowski background is inconsistent with the absence of ingoing radiation from past null infinity. We show that this results extends to the ejection of a Kerr black hole and apply it to set upper and lower bounds for the memory effect resulting from the collision of two spinning black holes.

1. Introduction

The gravitational memory effect results in a net change in the relative separation of distant particles after a wave passes, determined by the difference between the initial and final radiation strain measured by a gravitational wave detector. The possibility of observable astrophysical consequences of the effect was first studied in linearized gravity where the memory is produced by the ejection of massive particles which escape to infinity, as described by the retarded solution of the linearized Einstein equations [1, 2]. In previous work [3], we showed that this result could also be obtained in linearized theory by considering the transition from an initial state whose exterior was described by a Schwarzschild metric at rest to a final state whose exterior was a boosted exterior Schwarzschild metric. This result was subsequently extended to the nonlinear treatment

‡ Email:thomas.maedler@mail.udp.cl

of the transition from a stationary to boosted Schwarzschild exterior [4, 5]. Here we further extend this treatment of the memory effect to the boosted Kerr metric.

Our linearized treatment of the memory effect was based upon the stationary and boosted versions of the ingoing Kerr-Schild version of the Schwarzschild metric to describe the far field of the initial and final states. In order to extend this result to the nonlinear case three major differences from the linearized theory had to be dealt with.

First, the linearized result was based upon the boost symmetry of the unperturbed Minkowski background. The Kerr-Schild metrics [6, 7] have the form

$$g_{ab} = \eta_{ab} + 2H\ell_a\ell_b \tag{1}$$

comprised of a Minkowski metric η_{ab}, a principal null vector ℓ_a (with respect to both η_{ab} and g_{ab}) and a scalar field H. In the nonlinear case, the time reflection symmetry of the Schwarzschild metric leads to two different choices of a "Minkowski background" η_{ab} depending on whether ℓ^a is chosen to be in the ingoing or outgoing direction. The relation between Boyer-Lindquist coordinates [8] and the ingoing and outgoing versions of Kerr-Schild coordinates is described in Sec. 2.

Second, no analogue of the Green function exists in the nonlinear case to construct a retarded solution. Instead, the retarded solution due to the emission of radiation from an accelerated particle is characterized by the absence of ingoing radiation from past null infinity $\mathcal{I}^-$. The vanishing of radiation memory at $\mathcal{I}^-$ is necessary for the absence of ingoing radiation. This condition allows the ingoing radiation strain, which forms the free characteristic initial data on $\mathcal{I}^-$, to be set to zero. Otherwise, non-zero radiation memory at $\mathcal{I}^-$ would require ingoing radiation. Since an initial stationary Schwarzschild metric has vanishing radiation strain at $\mathcal{I}^-$, the final boosted metric must also have vanishing radiation strain at $\mathcal{I}^-$ if there is no intervening ingoing radiation.

In the nonlinear regime, we found unexpected differences in the boosts associated with the ingoing and outgoing versions of the Kerr-Schild metric. The memory effect due to the ejection of a Schwarzschild black hole is only correctly described by the boost $\mathcal{B}$ associated with the Poincare group of the Minkowski background of the ingoing Kerr-Schild metric. This is because $\mathcal{B}$ belongs to the Lorentz subgroup of the Bondi-Metzner-Sachs (BMS) [9] asymptotic symmetry group at $\mathcal{I}^-$ so that it does not introduce strain at $\mathcal{I}^-$. As a result, the transition from a stationary to boosted state is consistent with the lack of ingoing radiation from $\mathcal{I}^-$. But $\mathcal{B}$ is not a BMS Lorentz symmetry of future null infinity $\mathcal{I}^+$ where it induces a supertranslation member of the BMS group. The strain introduced by this supertranslation results in non-zero radiation memory at $\mathcal{I}^+$. This radiation is in precise agreement with the linearized result based upon the retarded Green function. The results are independent of the details of the intervening radiative period.

Conversely, the boost symmetry of the Minkowski metric associated with the outgoing version of the Kerr-Schild metric is a BMS symmetry at $\mathcal{I}^+$. Consequently, it introduces neither strain nor radiation memory at $\mathcal{I}^+$.

Third, in the nonlinear regime the mass of the final black hole depends upon the energy loss carried off by gravitational waves. This couples the memory effect due to the ejection of a particle to the Christodoulou memory effect [10] due to gravitational radiation.

In extending this approach from the boosted Schwarzschild to the boosted Kerr metric there are further complications, in addition to the algebraic complexity. Unlike the Schwarzschild case, the principal null directions of the Kerr metric are not hypersurface orthogonal. As a result, there is no natural way to construct a null coordinate system in order to study the asymptotic behavior at null infinity. Here we show that there does exist a natural choice of *hyperboloidal* coordinates which provide a spacelike foliation extending asymptotically to null infinity. These hyperboloidal hypersurfaces are the null hypersurfaces of the Minkowski background for the Kerr-Schild version of the Kerr metric, which we abbreviate by the KSK metric. For the ingoing KSK metric (see Sec. 3), the hyperboloids approach $\mathcal{I}^-$ and for the outgoing case (see Sec. 4) they approach $\mathcal{I}^+$. The associated coordinates lead to a straightforward Penrose compactification of null infinity and allow an unambiguous treatment of the asymptotic radiation strain and identification of the Poincare symmetries of the Minkowski background with BMS symmetries. Such curved space hyperboloidal hypersurfaces have been utilized in formulating the Cauchy problem for Einstein's equations in a manner suitable for radiation studies [11]. Their existence for the Kerr exterior was investigated in terms of an asymptotic series expansion in [12]. Here we construct a simple, geometrically natural and purely analytic hyperboloidal foliation that globally covers the entire Kerr exterior.

An additional factor is that the Kerr metric does not have the time reflection symmetry of the Schwarzschild metric, but instead a $(\tau, \varphi) \rightarrow (-\tau, -\varphi)$ symmetry in Boyer-Lindquist coordinates, as described in Sec. 2. This complicates relating asymptotic properties at $\mathcal{I}^-$ and $\mathcal{I}^+$.

Although an exact Kerr exterior is unrealistic in a dynamic spacetime, it is a reasonable far field approximation for the final black hole state in the limit of infinite retarded time at future null infinity, in accordance with the no hair scenario. In Sec. 5, we derive the nonlinear memory effect for the transition from a stationary to boosted Kerr black hole. In Sec. 6, we generalize this result to the collision of two black holes to form a final black hole. We derive upper and lower bounds for the radiation memory resulting from the collision. The bounds depend upon the mass of the final black hole, which involves the Christodoulou memory effect [10] resulting from the loss of energy due to gravitational waves.

Kerr-Schild metrics have played an important role in the construction of exact solutions [13]. Because their metric form (1) is invariant under the Lorentz symmetry of the Minkowski background η_{ab}, the boosted KSK metric has been important in numerical relativity in prescribing initial data for superimposed moving and spinning black holes in a binary orbit [14, 15]. The initial data for numerical simulations are prescribed in terms of the ingoing version of the KSK metric, whose advanced time coordinatization

extends across the future event horizon. The initial black hole velocities are generated by the boost symmetry of the Minkowski background for the ingoing KSK metric. This is analogous to our treatment of radiation memory.

The distinction between the ingoing and outgoing Kerr-Schild metrics and their associated background Minkowski symmetries requires considerable notational care. We retain the notation in our previous papers in which a superscript $(+)$ denoted quantities associated with the advanced time versions of the Schwarzschild metric and a superscript $(-)$ denoted quantities associated with the retarded time version. Corresponding to this notation, we use a superscript $(+)$ for quantities associated with the ingoing version of the KSK metric and a superscript $(-)$ for quantities associated with the outgoing version. As an example, the ingoing principal null vector is denoted by $\ell_a^{(+)}$ and its null rays emanate from past null infinity $\mathcal{I}^-$ and extend across the future event horizon; and the outgoing principal null vector is denoted by $\ell_a^{(-)}$, whose null rays extend to future null infinity $\mathcal{I}^+$. We denote abstract spacetime indices by $a, b, ...$ and coordinate indices by $\alpha, \beta, ...$ We often use the standard comma notation to denote partial derivatives, e.g. $f_{,\alpha} = \partial f / \partial x^\alpha$.

2. The Kerr-Schild Kerr metric and its associated Minkowski backgrounds

The Boyer-Lindquist coordinates [8], which we denote by $(\tau, r, \vartheta, \varphi)$, provide the intermediate connection between the ingoing and outgoing versions of the KSK metric. In these coordinates, the Kerr metric is

$$ds^2 = -d\tau^2 + \Sigma\Big(\frac{dr^2}{\Delta} + d\vartheta^2\Big) + (r^2 + a^2)\sin^2\vartheta d\varphi^2 + \frac{2mr}{\Sigma}(a\sin^2\vartheta d\varphi - d\tau)^2, \quad (2)$$

where m is the mass, a is the specific angular momentum and

$$\Sigma = r^2 + a^2\cos^2\vartheta , \quad \Delta = r^2 - 2mr + a^2 . \tag{3}$$

Note that this substitutes $a \to -a$ in the formulae of [6, 7] to agree with the standard convention that the sense of rotation is in the positive φ direction.

The KSK metric can be expressed in terms of either the ingoing principal null direction $\ell^{(+)a}$ or the outgoing principal null direction $\ell^{(-)a}$. These two forms of the of the metric have different inertial coordinates $x^{(\pm)a} = (t^{(\pm)}, x^{(\pm)}, y^{(\pm)}, z^{(\pm)})$ for their corresponding Minkowski backgrounds $\eta_{ab}^{(\pm)}$. The main details have been worked out by considering the $(\tau, \varphi) \to (-\tau, -\varphi)$ reflection symmetry of the Kerr metric in Boyer-Lindquist coordinates [8]. The coordinate transformations leading from (2) to the ingoing $(+)$ or outgoing $(-)$ Kerr-Schild form involve a generalization of the Schwarzschild tortoise coordinate r^*,

$$r^* = \int \Big(\frac{r^2 + a^2}{r^2 - 2mr + a^2}\Big)dr$$

$$= r + m\ln\Big(\frac{r^2 - 2mr + a^2}{4m^2}\Big) + \frac{m^2}{\sqrt{m^2 - a^2}}\ln\Big(\frac{r - m - \sqrt{m^2 - a^2}}{r - m + \sqrt{m^2 - a^2}}\Big) \quad (4)$$

and the intermediate angles

$$\Phi^{\pm} = \varphi \pm a \int \frac{dr}{\Delta} = \varphi \pm \frac{a}{2\sqrt{m^2 - a^2}} \ln\left(\frac{r - m - \sqrt{m^2 - a^2}}{r - m + \sqrt{m^2 - a^2}}\right) \ . \tag{5}$$

The transformation from Boyer-Lindquist to Kerr-Schild coordinates can then be written compactly as

$$t^{(\pm)} \quad = \tau \pm (r - r^*) \tag{6}$$

$$x^{(\pm)} + iy^{(\pm)} = \sqrt{r^2 + a^2} \sin\vartheta \exp\left\{i[\Phi^{(\pm)} \pm \arctan(a/r)]\right\} \tag{7}$$

$$z^{(\pm)} \quad = r \cos\vartheta. \tag{8}$$

Here $x^{(+)2} + y^{(+)2} = x^{(-)2} + y^{(-)2}$ so we simply denote $x^2 + y^2 = x^{(\pm)2} + y^{(\pm)2}$. Similarly, we denote $z = z^{(\pm)}$. The Boyer-Lindquist radial coordinate r is then determined implicitly by

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1, \tag{9}$$

independent of the choice of background coordinates.

The resulting Kerr Schild metrics in the background inertial coordinates $x^{(\pm)\alpha}$ are

$$g_{\alpha\beta}^{\pm} = \eta_{\alpha\beta}^{\pm} + 2H\ell_{\alpha}^{\pm}\ell_{\beta}^{\pm} \ , \quad H = \frac{mr^3}{r^4 + a^2 z^2}, \tag{10}$$

where the ingoing and outgoing versions of the principal null vectors have components

$$\ell_{\alpha}^{+}(x^{(+)b}) = (\ell_t^{(+)}, \ell_x^{(+)}, \ell_y^{(+)}, \ell_z^{(+)}) = \left(-1, -\frac{rx^{(+)} + ay^{(+)}}{r^2 + a^2}, -\frac{ry^{(+)} - ax^{(+)}}{r^2 + a^2}, -\frac{z}{r}\right), \tag{11}$$

and

$$\ell_{\alpha}^{-}(x^{(-)b}) = (\ell_t^{(-)}, \ell_x^{(-)}, \ell_y^{(-)}, \ell_z^{(-)}) = \left(-1, \frac{rx^{(-)} - ay^{(-)}}{r^2 + a^2}, \frac{ry^{(-)} + ax^{(-)}}{r^2 + a^2}, \frac{z}{r}\right). \tag{12}$$

Recall that $\det(g_{\alpha\beta}^{\pm}) = \det(\eta_{\alpha\beta}^{\pm}) = -1$ for Kerr-Schild metrics in these background coordinates.

From (6) and (7), it follows that the time coordinates of the two Minkowski backgrounds are related by

$$t^{(+)} = t^{(-)} + 2(r - r^*) \tag{13}$$

and the spatial coordinates are related by

$$x^{(+)} + iy^{(+)} = (x^{(-)} + iy^{(-)})e^{i\Psi(r)} \ , \quad z = z^{(+)} = z^{(-)}, \tag{14}$$

where

$$\Psi(r) := \frac{a}{\sqrt{m^2 - a^2}} \ln\left(\frac{r - m - \sqrt{m^2 - a^2}}{r - m + \sqrt{m^2 - a^2}}\right) + 2\arctan(a/r) \ . \tag{15}$$

We set

$$\rho^2 = x^2 + y^2 + z^2, \tag{16}$$

and introduce the standard spherical coordinates $(\rho, \theta, \phi^{(\pm)})$ for the Minkowski backgrounds,

$$x^{(\pm)} + iy^{(\pm)} = \rho\sin\theta\exp[i\phi^{(\pm)}]. \quad z = \rho\cos\theta \ . \tag{17}$$

Here ρ and θ, but not $\phi^{(\pm)}$, are background independent.

3. $\mathcal{I}^-$ and the boost symmetry

In [4], we showed that the linearized memory effect arising from the retarded solution for an ejected particle could be obtained from the boosted version of the advanced time Kerr-Schild-Schwarzschild metric. In that treatment, the boost was a Lorentz symmetry of the linearized Minkowski background. However, this could not be extended unambiguously to the nonlinear case, where there are two different choices of boost symmetry corresponding to the Minkowski backgrounds $\eta_{\alpha\beta}^{(+)}$ or $\eta_{\alpha\beta}^{(-)}$ of the ingoing or outgoing versions of the curved space Kerr-Schild-Schwarzschild metric. In the curved space case, it is not the choice of ingoing or outgoing Kerr-Schild metric (which are algebraically equal) but the choice of boost that leads to the essential result. In particular, because the boost symmetry of the ingoing background $\eta_{\alpha\beta}^{(+)}$ is a BMS symmetry of $\mathcal{I}^-$ it does not produce ingoing radiation strain at $\mathcal{I}^-$ but it does induce a supertranslation at $\mathcal{I}^+$, which leads to outgoing radiation strain.

In the Schwarzschild case, the null hypersurfaces determined by the principal null directions provide a simple approach to construct null infinity. In the Kerr case, this is more complicated because the principal null directions are not hypersurface orthogonal. For this reason, we describe $\mathcal{I}^-$ in the Kerr case by considering the null spherical coordinates associated with the Minkowski background $\eta_{ab}^{(+)}$,

$$\tilde{x}^{(+)\alpha} = (v, \rho, \theta, \phi^{(+)}), \quad v = t^{(+)} + \rho. \tag{18}$$

In these coordinates,

$$\eta_{\alpha\beta}^{(+)} d\tilde{x}^{(+)\alpha} d\tilde{x}^{(+)\beta} = -dv^2 + 2dvd\rho + \rho^2 dq^{(+)2}, \tag{19}$$

where $dq^{(+)2} = d\theta^2 + \sin^2\theta d\phi^{(+)2}$ is the unit sphere metric. The ingoing KSK metric takes the form

$$g_{\alpha\beta} d\tilde{x}^{(+)\alpha} d\tilde{x}^{(+)\alpha} = -dv^2 + 2dvd\rho + \rho^2 dq^{(+)2} + 2H(\ell_\alpha^{(+)} d\tilde{x}^{(+)\alpha})^2, \tag{20}$$

where, using (9) and (11),

$$\ell_\alpha^{(+)} d\tilde{x}^{(+)\alpha} = -dv + (1 - \frac{r}{\rho})d\rho + \frac{a\rho^2 \sin\theta}{r(r^2 + a^2)}(a\cos\theta d\theta + r\sin\theta d\phi^{(+)}). \tag{21}$$

The inverse property of Kerr-Schild metrics,

$$g^{ab} = \eta^{(+)ab} - 2H\ell^{(+)a}\ell^{(+)b}, \tag{22}$$

implies

$$g^{ab}(\partial_a v)\partial_b v = -2H(\ell^{(+)a}\partial_a v)^2. \tag{23}$$

As a result, since $H \geq 0$, the hypersurfaces $v = const$ are spacelike except in the limiting Schwarzschild case, where $a = 0$ and the hypersurfaces are null. Explicitly, (11) leads to

$$\ell^{(+)a}\partial_a v = 1 - \frac{r\rho}{r^2 + a^2} - \frac{a^2 z^2}{r\rho(r^2 + a^2)}, \tag{24}$$

or, using (9),

$$\ell^{(+)a}\partial_a v = 1 - \frac{r}{\rho}. \tag{25}$$

3.1. Compactification of past null infinity

For $a \neq 0$, the hypersurfaces $v = const$ are spacelike hyperboloids in the Kerr geometry which approach $\mathcal{I}^-$ asymptotically. In order to compactify $\mathcal{I}^-$, we replace the hyperboloidal spherical coordinates $\tilde{x}^{(+)\alpha} = (v, \rho, \theta, \phi^{(+)})$ by the compactified coordinates $\hat{x}^{(+)\alpha} = (v, \ell, \theta, \phi^{(+)})$, where $\ell = 1/\rho$. In these coordinates $\mathcal{I}^-$ is given by $\ell = 0$.

Now introduce the conformally rescaled metric denoted by $\hat{g}_{ab} = \ell^2 g_{ab}$. The conformal metric is then given by

$$\hat{g}_{\alpha\beta} d\hat{x}^{(+)\alpha} d\hat{x}^{(+)\beta} = -\ell^2 dv^2 - 2dv d\ell + dq^{(+)2} + 2H(\ell \ell_\alpha^{(+)} d\hat{x}^{\alpha(+)})^2, \tag{26}$$

with

$$\ell \ell_\alpha^{(+)} d\hat{x}^{(+)\alpha} = -\ell dv - [\ell^{-1} - r]d\ell + \frac{a\sin\theta}{r\ell(r^2 + a^2)}(a\cos\theta d\theta + r\sin\theta d\phi^{(+)}). \tag{27}$$

The asymptotic behavior of $\hat{g}_{ab}$ depends upon the asymptotic expansion of the Boyer-Lindquist coordinate r. From (9), r is determined by the quartic equation

$$r^4 - (x^2 + y^2 + z^2 - a^2)r^2 - a^2 z^2 = r^4 - (\rho^2 - a^2)r^2 - a^2 \rho^2 \cos^2\theta = 0. \tag{28}$$

The solution

$$r = \frac{1}{2}\sqrt{2\rho^2 - 2a^2 + 2\sqrt{\rho^4 - 2\rho^2 a^2 + a^4 + 4a^2 z^2}} \tag{29}$$

$$= \frac{1}{2}\sqrt{2\rho^2 - 2a^2 + 2\sqrt{\rho^4 - 2\rho^2 a^2 + a^4 + 4a^2 \rho^2 \cos^2\theta}} \tag{30}$$

has the asymptotic ℓ expansion about $\mathcal{I}^-$

$$r(\ell, \theta) = \ell^{-1} - \frac{a^2 \sin^2\theta}{2}\ell - \frac{a^4 \sin^2\theta(1 - 5\cos^2\theta)}{8}\ell^3 + O(\ell^5), \quad \ell = 1/\rho. \tag{31}$$

As a result, H, $\ell \ell_\alpha^{(+)} d\hat{x}^{(+)\alpha}$ and the conformal metric have the asymptotic ℓ-expansions

$$H = m\ell\left[1 + \frac{a^2}{2}(1 - 3\cos^2\theta)\ell^2 + \frac{a^4}{8}(3 - 30\cos^2\theta + 35\cos^4\theta)\ell^4 + O(\ell^6)\right], \tag{32}$$

$$\ell \ell_\alpha^{(+)} d\hat{x}^\alpha = \ell\left(-dv - \frac{a^2}{2}\sin^2\theta d\ell + a\sin^2\theta d\phi^{(+)}\right) + \ell^2\left(a^2 \sin\theta\cos\theta d\theta\right)$$
$$+ \ell^3\left[\frac{a^4}{8}(5\cos^2\theta - 1)\sin^2\theta d\ell - a^3 \sin^2\theta\cos^2\theta d\phi^{(+)}\right]$$
$$+ \ell^4\left[\frac{a^4}{2}(1 - 3\cos^2\theta)\sin\theta\cos\theta d\theta\right] + O(\ell^5) \tag{33}$$

and

$$\hat{g}_{\alpha\beta} d\hat{x}^{(+)\alpha} d\hat{x}^{(+)\beta} = -2dv d\ell + dq^{(+)2} - \ell^2 dv^2 + 2m\ell^3\left(dv + \frac{a^2}{2}\sin^2\theta d\ell - a\sin^2\theta d\phi^{(+)}\right)^2$$

$$+ \ell^4\left[-4ma^2\left(dv + \frac{a^2}{2}\sin^2\theta d\ell - a\sin^2\theta d\phi^{(+)}\right)\sin\theta\cos\theta d\theta\right] + O(\ell^5), \tag{34}$$

with the determinant $\hat{g} = -\sin^2\theta$ and inverse

$$\hat{g}^{\alpha\beta}\partial_{\hat{x}^{(+)\alpha}}\partial_{\hat{x}^{(+)\beta}} = -2\partial_v\partial_\ell + q^{AB}\partial_{\hat{x}^{(+)A}}\partial_{\hat{x}^{(+)B}} + \ell^2\partial_\ell\partial_\ell - 2m\ell^3\left(\frac{a^2\sin^2\theta}{2}\partial_v + \partial_\ell + a\partial_{\phi^{(+)}}\right)^2$$

$$- 4a^2m\ell^4\left[\left(\frac{a^2\sin^2\theta}{2}\partial_v + \partial_\ell + a\partial_{\phi^{(+)}}\right)\sin\theta\cos\theta\right]\partial_\theta + O(\ell^5). \tag{35}$$

As a result, $\mathcal{I}^-$, given by $\ell = 0$ with $\hat{\nabla}_a\ell|_{\mathcal{I}^-} \neq 0$, has Penrose compactification with metric

$$\hat{g}_{\alpha\beta}d\hat{x}^{(+)\alpha}d\hat{x}^{(+)\beta}\big|_{\mathcal{I}^-} = -2dvd\ell + dq^{(+)2}, \tag{36}$$

i.e. $\mathcal{I}^-$ is a null hypersurface with standard asymptotically Minkowskian geometry consisting of unit sphere cross-sections. In addition, it is straightforward to verify that

$$\hat{\nabla}_a\hat{\nabla}_b\ell|_{\mathcal{I}^-} = 0, \tag{37}$$

so that ℓ is a conformal factor in which the shear and divergence of $\mathcal{I}^-$ vanish. Thus ℓ is a preferred conformal factor for which the compactification of $\mathcal{I}^-$ has the same asymptotic properties as described by a conformal Bondi frame [18]. This allows a simple description of the BMS asymptotic symmetries and other physical properties of $\mathcal{I}^-$.

3.2. Physical properties of $\mathcal{I}^-$

The Lorentz symmetries of the Minkowski background are not symmetries of the KSS metric but they are BMS symmetries of $\mathcal{I}^-$. The remaining BMS symmetries are the supertranslations on $\mathcal{I}^-$, $v \to v + \alpha(\theta, \phi^{(+)})$. The supertranslations with α composed of $\ell = 0$ and $\ell = 1$ spherical harmonics correspond to the Poincare translations of the Minkowski background.

In the hyperboloidal coordinates $(v, \rho, x^{(+)A})$, where $x^{(+)A} = (\theta, \phi^{(+)})$, the strain tensor $\sigma_{AB}(v, x^{(+)A})$ describing the ingoing radiation from $\mathcal{I}^-$ is determined by the asymptotic expansion of the metric according to

$$\hat{g}_{AB} = q_{AB} + 2\ell\sigma_{AB} + O(\ell^2), \tag{38}$$

where $\sigma_{AB}(v, x^{(+)C})$ is trace-free and can be described by the spin-weight-2 function

$$\sigma(v, x^{(+)A}) = q^A q^B \sigma_{AB}(v, x^{(+)A}). \tag{39}$$

Here q^A is the complex polarization dyad associated with the unit sphere metric on $\mathcal{I}^-$,

$$q_{AB} = \frac{1}{2}(q_A\bar{q}_B + \bar{q}_A q_B), \quad q^A\bar{q}_A = 2, \quad q^A q_A = 0, \tag{40}$$

where $q_A = q_{AB}q^B$. For the standard choice of spherical coordinates, we set $q^A\partial_{(+)A} = \partial_\theta + (i/\sin\theta)\partial_{\phi^{(+)}}$. This normalization implies

$$\sigma = \sigma_{\theta\theta} - \frac{\sigma_{\phi^{(+)}\phi^{(+)}}}{\sin^2\theta} + \frac{2i\sigma_{\theta\phi^{(+)}}}{\sin\theta} , \tag{41}$$

which corresponds to the standard plus/cross decomposition, as used in [3, 4, 16]. The normalization used in [5] inadvertently reduced σ by a factor of $1/2$.

In terms of the physical space description in the associated inertial Cartesian coordinates, the polarization dyad q^A has components $q^i = \rho Q^i$ where

$$Q^i = \frac{1}{\rho} x^i_{,(+)A} q^A = (\cos\theta\cos\phi^{(+)} - i\sin\phi^{(+)}, \cos\theta\sin\phi^{(+)} + i\cos\phi^{(+)}, -\sin\theta) , \quad (42)$$

where $Q_i \bar{Q}^j = 2$, $Q_i Q^j = 0$ and $Q_i x^i = 0$. Here we raise and lower the Cartesian indices $i, j, \ldots$ for fields in the Euclidean background according to the example $Q_i = \delta_{ij} Q^j$

In terms of the Euclidean coordinates,

$$\sigma(v, x^{(+)A}) = \lim_{\rho\to\infty} \frac{\rho}{2} Q^i Q^j g_{ij}, \quad (43)$$

where the limit at $\mathcal{I}^-$ is taken holding $(v, x^{(+)A})$ constant. For the unboosted KSK metric,

$$\sigma(v, x^{(+)A}) = \lim_{\rho\to\infty} \rho H (Q^i \ell_i^{(+)})^2 = \lim_{\rho\to\infty} m (Q^i \ell_i^{(+)})^2. \quad (44)$$

A straightforward calculation gives

$$Q^i \ell_i^{(+)} = \frac{\rho a \sin\theta (a\cos\theta + ir)}{r(r^2 + a^2)}. \quad (45)$$

With reference to (31), it follows that $Q^i \ell_i^{(+)} = O(1/\rho)$ so that the radiation strain vanishes at the advanced times $v = const$ picked out by the null cones of $\eta_{ab}^{(+)}$. However, under the supertranslation $v \to v + \alpha(x^{(+)A})$ the radiation strain has the gauge freedom, which in a non-radiative epoch takes the form (cf. [16])

$$\sigma \to \sigma + q^A q^B \eth_{(+)A} \eth_{(+)B} \, \alpha \quad (46)$$

where $\eth_{(+)A}$ is the covariant derivative with respect to the unit sphere metric $dq^{(+)2}$. As a result, *distorted* cross-sections of $\mathcal{I}^-$ have non-vanishing strain.

3.3. The boost symmetry of $\mathcal{I}^-$

The boost symmetry $\mathcal{B}$ of the Minkowski background $\eta_{ab}^{(+)}$ is not an exact symmetry of the Kerr metric but it is an asymptotic BMS symmetry of $\mathcal{I}^-$. Consider now a boost whose 4-velocity has components $v^\mu = \Gamma(1, V^i)$, $\Gamma = 1/\sqrt{1 - V^2}$, for which $\eta_{ab}^{(+)} \to \eta_{ab}^{(+)}$. The boosted coordinates $x_{\mathcal{B}}^{(+)\mu}$ are given by the Lorentz transformation

$$x_{\mathcal{B}}^{(+)\mu} = \Lambda^\mu_\nu x^{(+)\nu} , \quad (47)$$

where for $V^i = V n^i$, with direction cosines n^i,

$$\Lambda^t_{\ t} = \Gamma , \quad (48)$$

$$\Lambda^t_{\ i} = \Lambda^i_{\ t} = -\Gamma V_i , \quad (49)$$

$$\Lambda^i_{\ j} = \delta^i_j + (\Gamma - 1) n^i n_j . \quad (50)$$

The boosted coordinates are

$$t_{\mathcal{B}}^{(+)} = \Gamma(t^{(+)} - V_i x^{(+)i}), \quad (51)$$

$$x_{\mathcal{B}}^{(+)i} = x^{(+)i} + \left[-\Gamma V t^{(+)} + (\Gamma - 1)(n_j x^{(+)j}) \right] n^i , \quad (52)$$

e.g, for a boost in the z-direction, $n^x = n^y = n^z - 1 = 0$, $x_{\mathcal{B}}^{(+)} = x^{(+)}$, $y_{\mathcal{B}}^{(+)} = y^{(+)}$ and

$$t_{\mathcal{B}}^{(+)} = \Gamma(t^{(+)} - Vz^{(+)}), \tag{53}$$

$$z_{\mathcal{B}}^{(+)} = \Gamma(z^{(+)} - Vt^{(+)}) . \tag{54}$$

The background spherical radius ρ transforms as

$$\rho^2 \to \rho_{\mathcal{B}}^2 = x^{(+)\mu}x_\mu^{(+)} + (v_\mu x^{(+)\mu})^2 = -[t^{(+)}]^2 + \rho^2 + \Gamma^2(t^{(+)} - V_i x^{(+)i})^2 . \tag{55}$$

Setting $t^{(+)} = v - \rho$, the large ρ expansion of $\rho_{\mathcal{B}}$ about $\mathcal{I}^-$ holding v constant is

$$\rho_{\mathcal{B}} = \rho\Gamma(1 + V_i\rho^i) - \frac{vV\Gamma(V + n_i\rho^i)}{1 + V_i\rho^i} + O(1/\rho) \;, \tag{56}$$

where $\rho^i = x^{(+)i}/\rho = (\sin\theta\cos\phi^{(+)}, \sin\theta\sin\phi^{(+)}, \cos\theta)$.

The boosted version of the Boyer-Lindquist radial coordinate (29),

$$r_{\mathcal{B}} = r(\rho_{\mathcal{B}}, z_{\mathcal{B}}) = \frac{1}{2}\sqrt{2\rho_{\mathcal{B}}^2 - 2a^2 + 2\sqrt{\rho_{\mathcal{B}}^4 - 2\rho_{\mathcal{B}}^2 a^2 + a^4 + 4a^2 z_{\mathcal{B}}^2}} \;, \tag{57}$$

has the large ρ expansion about $\mathcal{I}^-$ holding $v = t^{(+)} + \rho$ constant

$$r_{\mathcal{B}} = (1 + V_i\rho^i)\Gamma\rho - \frac{vV\Gamma(V + n_i\rho^i)}{1 + V_i\rho^i} + O(1/\rho) = \rho_{\mathcal{B}} + O(1/\rho). \tag{58}$$

This leads to the expansion of the boosted version of the Kerr-Schild function about $\mathcal{I}^-$,

$$H_{\mathcal{B}} = \frac{mr_{\mathcal{B}}^3}{r_{\mathcal{B}}^4 + a^2 z_{\mathcal{B}}^2} = \frac{m}{(1 + V_i\rho^i)\Gamma\rho}\left[1 + \frac{V(V + n_i\rho^i)v}{(1 + V_i\rho^i)^2\rho}\right] + O(\rho^{-2}). \tag{59}$$

It follows from (11) and (31) that the ingoing principle null direction has asymptotic behavior

$$\ell_\alpha^{(+)} = -\nabla_\alpha(t^{(+)} + \rho) + O(1/\rho) . \tag{60}$$

Using the covariant substitutions $-\nabla_\alpha t \to v_\alpha$ and $\nabla_\alpha\rho \to [x_\alpha + (x_\mu^{(+)}v^\mu)v_\alpha]/\rho_{\mathcal{B}}$, its boosted version $L_\alpha := \ell_{\mathcal{B}\,\alpha}^{(+)}$ has asymptotic behavior

$$L_\alpha = v_\alpha - \frac{1}{\rho_{\mathcal{B}}}(x_\alpha^{(+)} + v_\beta x^{(+)\beta}v_\alpha) + O(1/\rho). \tag{61}$$

Setting $t^{(+)} = v - \rho$, the expansion of the boosted version of (45) about $\mathcal{I}^-$, holding v constant, then leads to

$$Q^i L_i = (1 - \frac{1}{\rho_{\mathcal{B}}}v_\beta x^{(+)\beta})Q^i v_i + O(1/\rho) = \left(1 - \frac{1}{\rho_{\mathcal{B}}}\rho\Gamma(1 + V_i\rho^i)\right)Q^i v_i + O(1/\rho) \;, \tag{62}$$

so it follows from (56) that $Q^i L_i = O(1/\rho)$.

Thus, referring to the boosted version of (44),

$$\sigma_{\mathcal{B}}(v, x^{(+)A}) = \lim_{\rho\to\infty}\rho H_{\mathcal{B}}(Q^i L_i)^2 = 0, \tag{63}$$

i.e. the strain at $\mathcal{I}^-$ vanishes for the boosted KSK metric, as expected since the boost is an asymptotic symmetry of $\mathcal{I}^-$.

4. Future null infinity

Following the procedure for treating $\mathcal{I}^-$, we describe $\mathcal{I}^+$ in terms of the KSK metric by considering retarded null spherical coordinates associated with the Minkowski background, $\eta_{ab}^{(-)}$,

$$\tilde{x}^{(-)\alpha} = (u, \rho, \theta, \phi^{(-)}), \quad u = t^{(-)} - \rho. \tag{64}$$

In these coordinates,

$$\eta_{\alpha\beta}^{(-)} d\tilde{x}^{(-)\alpha} d\tilde{x}^{(-)\beta} = -du^2 - 2dud\rho + \rho^2 dq^{(-)2} \tag{65}$$

where $dq^{(-)2} = d\theta^2 + \sin^2\theta d\phi^{(-)2}$, and the outgoing version of the KSK metric has components

$$g_{\alpha\beta} d\tilde{x}^{(-)\alpha} d\tilde{x}^{(-)\beta} = -du^2 - 2dud\rho + \rho^2 dq^{(-)2} + 2H(\ell_\alpha^{(-)} d\tilde{x}^{(-)\alpha})^2. \tag{66}$$

The inverse form of the outgoing KSK metric,

$$g^{ab} = \eta^{(-)ab} - 2H\ell^{(-)a}\ell^{(-)b}, \tag{67}$$

now implies

$$g^{ab}(\partial_a u)\partial_b u = -2H(\ell^{(-)a}\partial_a u)^2. \tag{68}$$

Analogous to the ingoing case, since $H \geq 0$, the hypersurfaces $u = const$ are spacelike hyperbolae which approach $\mathcal{I}^+$, except in the limiting Schwarzschild case where they are null. Explicitly, following the calculation of (25),

$$\ell^{(-)a}\partial_a u = 1 - \frac{r}{\rho}. \tag{69}$$

In order to compactify $\mathcal{I}^+$, we replace the hyperboloidal spherical coordinates $\tilde{x}^{(-)\alpha} = (u, \rho, \theta, \phi^{(-)})$ by the compactified coordinates $\hat{x}^{(-)\alpha} = (u, \ell, \theta, \phi^{(-)})$, where $\ell = 1/\rho$, and $\ell = 0$ at $\mathcal{I}^+$. Again we introduce the conformally rescaled metric $\hat{g}_{ab} = \ell^2 g_{ab}$,

$$\hat{g}_{\alpha\beta} d\hat{x}^{(-)\alpha} d\hat{x}^{(-)\beta} = -\ell^2 du^2 + 2dud\ell + dq^{(-)2} + 2H(\ell\ell_\alpha^{(-)} d\hat{x}^{\alpha(-)})^2, \tag{70}$$

where

$$dq^{(-)2} = d\theta^2 + \sin^2\theta d\phi^{(-)2} \tag{71}$$

and

$$\ell\ell_\alpha^{(-)} d\hat{x}^{(-)\alpha} = -\ell du + [\ell^{-1} - r]d\ell - \frac{a\sin\theta}{r\ell(r^2 + a^2)}(a\cos\theta d\theta - r\sin\theta d\phi^{(-)}). \tag{72}$$

The asymptotic behavior of $\hat{g}_{ab}$ at $\mathcal{I}^+$ follows from the asymptotic ℓ expansion (31) of the Boyer-Lindquist coordinate r which leads to

$$\ell\ell_\alpha^{(-)} d\hat{x}^{(-)\alpha} = \ell\left(-du + \frac{a^2}{2}\sin^2\theta d\ell + a\sin^2\theta d\phi^{(-)}\right) + \ell^2\left(-a^2\sin\theta\cos\theta d\theta\right)$$

$$+ \ell^3\left[-\frac{a^4}{8}(5\cos^2\theta - 1)\sin^2\theta d\ell - a^3\sin^2\theta\cos^2\theta d\phi^{(-)}\right]$$

$$+ \ell^4\left[-\frac{a^4}{2}(1 - 3\cos^2\theta)\sin\theta\cos\theta d\theta\right] + O(\ell^5) \tag{73}$$

so that

$$\hat{g}_{\alpha\beta}d\hat{x}^{(-)\alpha}d\hat{x}^{(-)\beta} = 2dud\ell + dq^{(-)2} - \ell^2 du^2 + 2m\ell^3\Big(-du + \frac{a^2}{2}\sin^2\theta d\ell + a\sin^2\theta d\phi^{(-)}\Big)^2$$
$$+ \ell^4\Big[-4ma^2\Big(-du + \frac{a^2}{2}\sin^2\theta d\ell + a\sin^2\theta d\phi^{(-)}\Big)\sin\theta\cos\theta d\theta\Big] + O(\ell^5)\,, \tag{74}$$

with determinant $\hat{g} = -\sin^2\theta$ and inverse

$$\hat{g}^{\alpha\beta}\partial_{\hat{x}^{(-)\alpha}}\partial_{\hat{x}^{(-)\beta}} = 2\partial_u\partial_\ell + q^{AB}\partial_{\hat{x}^{(-)A}}\partial_{\hat{x}^{(-)B}} + \ell^2\partial_\ell\partial_\ell - 2m\ell^3\Big(\frac{a^2\sin^2\theta}{2}\partial_u - \partial_\ell + a\partial_{\phi^{(-)}}\Big)^2$$
$$+ 4a^2m\ell^4\Big[\Big(\frac{a^2\sin^2\theta}{2}\partial_u - \partial_\ell + a\partial_{\phi^{(-)}}\Big)\sin\theta\cos\theta\Big]\partial_\theta + O(\ell^5). \tag{75}$$

We have

$$\hat{g}_{\alpha\beta}d\hat{x}^{(-)\alpha}d\hat{x}^{(-)\beta}\big|_{\mathcal{I}^+} = 2dud\ell + dq^{(-)2}, \tag{76}$$

i.e. $\mathcal{I}^+$ is a null hypersurface with standard asymptotically Minkowskian geometry consisting of unit sphere cross-sections. In addition, analogous to the case for $\mathcal{I}^-$,

$$\hat{\nabla}_a\hat{\nabla}_b\ell\big|_{\mathcal{I}^+} = 0 \tag{77}$$

so that ℓ is a preferred conformal factor in which the shear and divergence of $\mathcal{I}^+$ vanish.

An important feature is that both $\mathcal{I}^-$ and $\mathcal{I}^+$ have universal conformal structure of unit sphere cross-sections with the same conformal factor ℓ. Moreover, (15) leads to the expansion

$$\Psi = -2ma\ell^2 - \frac{8}{3}m^2a\ell^3 - 2ma(2m^2 - a^2\cos^2\theta)\ell^4 + O(\ell^5) \tag{78}$$

so that $\Psi|_{\ell=0} = 0$. Consequently, (14) and (17) imply that we can set

$$\phi = \phi^{(+)}|_{\ell=0} = \phi^{(-)}|_{\ell=0} \tag{79}$$

and

$$dq^2 = q_{AB}dx^Adx^B = d\theta^2 + \sin^2\theta d\phi^2 = dq^{(+)2}|_{\ell=0} = dq^{(-)2}|_{\ell=0}. \tag{80}$$

Thus we can use a common unit sphere metric q_{AB}, with associated covariant derivative $\eth_A$, common spherical coordinates $x^A = (\theta, \phi)$ and a common polarization dyad $q_{AB} = (1/2)(q_A\bar{q}_B + \bar{q}_Aq_B)$ to describe both the ingoing radiation from $\mathcal{I}^-$ and the outgoing radiation at $\mathcal{I}^+$.

5. Boosts and radiation memory

Analogous to (44), the outgoing radiation strain at $\mathcal{I}^+$ can be described by a spin-weight-2 function $\sigma(u, x^A)$, where u is the retarded time and $x^A = (\theta, \phi)$ are the angular coordinates on $\mathcal{I}^+$ determined by the Minkowski background. In these retarded coordinates adapted to the asymptotic Minkowskian structure, it is given by the limit at $\mathcal{I}^+$ holding u and x^A constant,

$$\sigma(u, x^A) = \lim_{\rho\to\infty}\frac{1}{2\rho}q^Aq^Bg_{AB} = q^Aq^B\sigma_{AB}(u, x^A). \tag{81}$$

The radiation memory $\Delta\sigma(x^C)$ at $\mathcal{I}^+$ measures the change in the radiation strain between infinite future and past retarded time,

$$\Delta\sigma(x^A) = \sigma(u = \infty, x^A) - \sigma(u = -\infty, x^A). \tag{82}$$

In the associated inertial Cartesian coordinates, the dyad q^A has components $q^i = \rho Q^i$ and

$$\sigma(u, x^A) = \lim_{\rho\to\infty} \frac{\rho}{2} Q^i Q^j g_{ij}. \tag{83}$$

For the unboosted KSK metric,

$$\sigma(u, x^A) = \lim_{\rho\to\infty} \rho H (Q^i \ell_i^{(-)})^2 = \lim_{\rho\to\infty} m (Q^i \ell_i^{(-)})^2. \tag{84}$$

A straightforward calculation gives

$$Q^i \ell_i^{(-)} = -\frac{\rho a^2 \sin\theta \cos\theta}{r(r^2 + a^2)} + \frac{i\rho a \sin\theta}{r^2 + a^2}. \tag{85}$$

Again using (31), this implies $Q^i \ell_i^{(-)} = O(1/\rho)$ so that the radiation strain at $\mathcal{I}^+$ of the unboosted KSK metric vanishes.

Consider now a system which is asymptotically described by an unboosted Kerr metric in the retarded past $u = -\infty$ and by a boosted KSK metric in the future $u = \infty$, where the boost $\mathcal{B}$ is a Lorentz symmetry of $\eta_{ab}^{(+)}$. The radiation memory is then given by

$$\Delta\sigma(x^A) = \sigma_{\mathcal{B}}(u = \infty, x^A) - \sigma(u = -\infty, x^A), \tag{86}$$

where $\sigma_{\mathcal{B}}(u = \infty, x^A)$ is the radiation strain of the final boosted state and initially $\sigma(u = -\infty, x^A) = 0$. The final strain $\sigma(u = \infty, x^A)$ may be calculated using either the ingoing or outgoing form of the KSK metric. It is technically simpler to use the ingoing form since the boost $\mathcal{B}$ leaves $\eta_{ab}^{(+)}$ unchanged. The final strain for the boosted version of the KSK metric $g_{\mathcal{B}ab}$, computed in the same frame as the initial strain (83), is then given by

$$\sigma_{\mathcal{B}}(u = \infty, x^A) = \lim_{u\to\infty} \lim_{\rho\to\infty} \frac{\rho}{2} Q^i Q^j g_{\mathcal{B}ij} = \lim_{u\to\infty} \lim_{\rho\to\infty} \rho H_{\mathcal{B}} (Q^i L_i)^2. \tag{87}$$

The leading terms in the $1/\rho$ expansion of $\rho_{\mathcal{B}}$, $r_{\mathcal{B}}$ and $H_{\mathcal{B}}$, given in (56), (58) and (59), are unchanged when the limit at $\mathcal{I}^+$ is taken holding u constant. Thus

$$\sigma_{\mathcal{B}}(u = \infty, x^A) = \lim_{u\to\infty} \lim_{\rho\to\infty} \frac{m\rho}{\rho_{\mathcal{B}}} (Q^i L_i)^2. \tag{88}$$

The key difference here is that the limit at $\mathcal{I}^+$ involves the boosted ingoing principal null direction L_α whose asymptotic behavior (61) leads to

$$Q^i L_i = Q^i v_i \left(1 + \frac{\Gamma(t^{(+)} - V_i x^{(+)i})}{\rho_{\mathcal{B}}} \right) + O(1/\rho). \tag{89}$$

Now, instead of holding the advanced time v constant to take the limit at $\mathcal{I}^-$, we hold the retarded time u constant to take the limit at $\mathcal{I}^+$. Referring to (13)

$$t^{(+)} = t^{(-)} + 2(r - r^*) = u + \rho + 2(r - r^*), \tag{90}$$

where (4) leads to the expansion

$$r^* - r = 2m \ln \left(\frac{\rho}{2m} \right) - \frac{4m^2}{\rho} + O(1/\rho^2). \tag{91}$$

From (55), $\rho_\mathcal{B}$ has the asymptotic behavior, holding u constant,

$$\frac{\rho_\mathcal{B}}{\rho} = \Gamma(1 - \rho^i V_i) + O(1/\rho). \tag{92}$$

As a result, since $\lim_{\rho \to \infty} \ln \rho / \rho = 0$, (90) leads to the limit, holding u constant,

$$\lim_{\rho \to \infty} \frac{t^{(+)}}{\rho_\mathcal{B}} = \lim_{\rho \to \infty} \frac{\rho}{\rho_\mathcal{B}} = \frac{1}{\Gamma(1 - \rho^i V_i)} \tag{93}$$

and (89) leads to

$$\lim_{\rho \to \infty} Q^i L_i = 2Q^i v_i \tag{94}$$

Thus

$$\sigma_\mathcal{B}(u = \infty, x^A) = \lim_{u \to \infty} \lim_{\rho \to \infty} \frac{4m\rho}{\rho_\mathcal{B}} (Q^i v_i)^2 = \frac{4m\Gamma}{(1 - \rho^i V_i)} (Q^i V_i)^2. \tag{95}$$

The resulting radiation memory due to the ejection of a Kerr black hole of mass m is

$$\Delta \sigma = \frac{4m\Gamma}{(1 - \rho^i V_i)} (Q^i V_i)^2. \tag{96}$$

This is in exact agreement with the linearized result [2] and for the nonlinear result [4] for the ejection of a Schwarzschild black hole.

6. Discussion

We have shown that a boost $\mathcal{B}$ of the Minkowski background $\eta_{ab}^{(+)}$ for the ingoing KSK metric leads to a model for the nonlinear memory effect due to the ejection of a Kerr black hole. An initially stationary KSK metric which is followed by an accelerating, radiative interval leads to a final boosted state which is consistent with the absence of ingoing radiation and whose outgoing radiation agrees with the linearized memory effect obtained from the retarded solution [2] and with the nonlinear result for a boosted Schwarzschild metric [4].

Although we have treated the memory effect for a Kerr black hole which is initially at rest and after a radiative interval ends in a boosted state, the result can be generalized. First, the asymptotic Lorentz symmetry at null infinity implies that the memory effect for a transition of a black hole from a rest state to a boosted state with mass m and velocity V^i is is the same as the memory for a black hole of mass m with initial velocity $-V^i$ and zero final velocity. In addition, it is expected, even in the nonlinear theory, that the superposition principle holds for particles at infinite separation since the constraints vanish in that limit. This allows the memory effect to be generalized to a system of particles.

As a simple example, consider two distant Kerr black holes of mass m with initial velocities V^i and $-V^i$ in the z-direction which come to rest in a final state with mass M. According to (96), the memory effect for this system is

$$\Delta\sigma = 4m\Gamma(Q^iV_i)^2(\frac{1}{1+\rho^iV_i} + \frac{1}{1-\rho^iV_i}) = \frac{8m\Gamma V^2\sin^2\theta}{1-V^2\cos^2\theta}. \tag{97}$$

The collision is constrained by the radiative loss in Bondi energy, which requires

$$2m\Gamma > M \tag{98}$$

so that

$$\Delta\sigma > \frac{4MV^2\sin^2\theta}{1-V^2\cos^2\theta} \tag{99}$$

for the collision of initially distant Kerr black holes.

Thus the memory effect has a minimum value determined by the mass of the final black hole. This lower bound is largest when the merger of the two black holes takes place slowly so that there is negligible radiative energy loss and $M \approx 2m\Gamma$.

The memory effect is also constrained by Hawking's area increase law for the event horizon in the merger of two black holes [17]. For the collision of initially distant Schwarzschild black holes,

$$4m^2 < M(M+\sqrt{M^2-A^2}) = M^2(1+\cos\chi), \quad 0 \le \chi \le \pi/2, \tag{100}$$

where $A = M\sin\chi$ is the specific angular momentum of the final Kerr black hole as determined by the initial impact parameter. Thus (100) implies

$$2m < M\sqrt{1+\cos\chi}. \tag{101}$$

As a result, the memory (97) is bounded by

$$\frac{4M\Gamma V^2\sin^2\theta\sqrt{1+\cos\chi}}{1-V^2\cos^2\theta} > \Delta\sigma > \frac{4MV^2\sin^2\theta}{1-V^2\cos^2\theta}. \tag{102}$$

For the case of high radiation efficiency to form a small black hole the memory effect would be small. In such a case, the nonlinear Christodoulou memory [10] due to gravitational wave emission would substantially cancel the boost memory.

The area inequality (100) constrains the efficiency $\mathcal{E}$ of the radiation, as discussed in [17] for the case that the black holes have negligible initial velocities. More generally, a boosted Schwarzschild black hole of mass m has the same area $16\pi m^2$ as at rest but has energy $m\Gamma$. For the above example of two Schwarzschild black holes colliding to form a Kerr black hole, (101) leads to the bound on the radiation efficiency

$$\mathcal{E} = \frac{2m\Gamma - M}{2m\Gamma} = 1 - \frac{M}{2m\Gamma} < 1 - \frac{1}{\Gamma\sqrt{1+\cos\chi}}. \tag{103}$$

For Schwarzschild black holes with negligible initial velocities, i.e. $\Gamma \approx 1$, this inequality allows the highest efficiency when the final black hole has zero spin,

$$\mathcal{E} < 1 - 1/\sqrt{2}, \tag{104}$$

in agreement with Hawking's result [17]. But for high initial velocities, i.e. $\Gamma \gg 1$, this leaves open the possibility $\mathcal{E} \approx 1$ so that the final black hole mass could be small, i.e. $M \approx 0$. In that case the memory effect would be neglible.

The above results only give an upper limit on the efficiency. Numerical simulations of high energy black hole collisions lead to results consistent with the $\Gamma = 0$ case [19, 20]. Although there is no known astrophysical process which would lead to collisions between high velocity black holes, this issue deserves further theoretical and numerical consideration.

In [4], we analyzed how radiation memory affects angular momentum conservation. In a non-radiative regime, where $\partial_u \sigma = 0$ the supertranslation freedom (46) can be used to pick out preferred cross-sections of $\mathcal{I}^+$ by setting the electric component of σ to 0. These preferred cross-section reduce the supertranslation freedom to the translation freedom so that a preferred Poincaré subgroup can be picked out from the BMS group. The same is true in the limits $u \to \pm\infty$, in which the requirement of a finite radiative energy loss implies $\partial_u \sigma \to 0$. Although the electric part of the strain can be gauged away at either $u = +\infty$ or $u = -\infty$, the memory effect $\Delta\sigma$ is gauge invariant and (46) determines a supertranslation shift

$$q^A q^B \eth_A \eth_B \, \alpha(x^C) = \Delta\sigma(x^C). \tag{105}$$

between the preferred Poincaré groups at $u = \pm\infty$. The rotation subgroups picked out by the initial and final preferred Poincaré groups differ by this supertranslation. As a result, the corresponding components of angular momentum intrinsic to the initial and final states differ by supermomenta.

Only the electric part of the strain is affected by supertranslations because α is real and σ is intrinsically complex. The decomposition of the strain into electric and magnetic parts is analogous to the E-mode/B-mode decomposition of electromagnetic waves. There are compelling theoretical arguments that the magnetic part of the memory effect must vanish for realistic physical sources, except for the possibility of primordial gravitational waves [21, 3].

The supertranslation shift between the initial and final preferred Poincaré groups complicates the interpretation of angular momentum flux conservation laws. This could lead to a distinctly general relativistic mechanism for angular momentum loss. Although the intermediate radiative epoch must be treated by numerical methods, the Kerr-Schild model developed here provides a framework for such investigations.

Acknowledgments

References

[1] Zeldovich Y and Polnarev A G 1974 Radiation of gravitational waves by a cluster of superdense stars Sov. Astron. **18** 17

[2] Braginsky V B and Thorne K S 1987 Gravitational wave bursts with memory and experimental prospects Nature (London) **327** 123

[3] Mädler T and Winicour J 2016 The sky pattern of the linearized gravitational memory effect Class. Quantum Grav. **33** 175006

[4] Mädler T and Winicour J 2017 Radiation memory, boosted Schwarzschild spacetimes and supertranslations Class. Quantum Grav. **34** 115009

[5] Mädler T and Winicour J 2018 Boosted Schwarzschild metrics from a Kerr-Schild perspective Class. Quantum Grav. **35** 035009

[6] Kerr R P and Schild A 1965 Some algebraically degenerate solutions of Einstein's gravitational field equations Proc. Symp. Appl. Math. **17** 199

[7] Kerr R P and Schild A 1966 A new class of vacuum solutions of the Einstein field equations Atti degli Convegno Sulla Relativita Generale p. 222 (Firenze) [online as Kerr R P and Schild A Republication of: A new class of vacuum solutions of the Einstein field equations (2009) Gen. Rel. Grav. **41** 2485]

[8] Boyer R H and Lindquist R W 1967 Maximal Analytic Extension of the Kerr Metric J. Math. Phys. **8** 265

[9] Sachs R 1962 Asymptotic symmetries in gravitational theory Phys. Rev. **128** 2851

[10] Christodoulou D 1991 Nonlinear nature of gravitation and gravitational-wave experiments Phys. Rev. Letters **67** 1486

[11] Friedrich H 1983 Cauchy problems for the conformal vacuum field equations in general relativity Comm. Math. Phys. **91** 445

[12] Zenginoğlu A 2008 Hyperboloidal foliations and scri-fixing Class. Quantum Gravity **25** 145002

[13] Kramer D, Stephani H, MacCallum M and Herlt E 1980 *Exact solutions of Einstein's field equations*, ed. Schmutzer E (Cambridge University Press, Cambridge)

[14] Matzner R A, Huq M F and Shoemaker D 1998 Initial data and coordinates for multiple black hole systems Phys. Rev. D **59** 024015

[15] Bonning E, Marronetti P, Neilson D, and Matzner R A 2003 Physics and initial data for multiple black hole spacetimes Phys. Rev. D **68** 044019

[16] Mädler T and Winicour J 2016 Bondi-Sachs formalism Scholarpedia. **11** (12) 33528

[17] Hawking S W 1971 Gravitational radiation from colliding black holes Phvs. Rev. Letters **26** 1344

[18] Tamburino L A and Winicour J 1966 Gravitational Fields in Finite and Conformal Bondi Frames Phys. Rev. **150** 1039

[19] Sperhake U, Cardoso V, Pretorius F, Berti E and González J A 2008 The high-energy collision of two black holes Phys. Rev. Letters **101** 161101

[20] Shibata M, Okawa H and Yamamoto T 2008 High-velocity collision of two black holes Phys. Rev. D **78** 101501

[21] Winicour J 2014 Global aspects of radiation memory Class. Quantum Grav. **31** 205003