

Hankel-Norm Approximation of Large-Scale Descriptor Systems

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Abstract

The Hankel-norm approximation is a model reduction method which provides the best approximation in the Hankel semi-norm. In this paper the computation of the optimal Hankel-norm approximation is generalized to the case of linear time-invariant continuous-time descriptor systems. An efficient algorithm is developed by refining the generalized balanced truncation square root method. For a wide practical usage, adaptations of the introduced algorithm towards stable computations and sparse systems are made as well as an approach for a projection-free algorithm. To show the approximation behavior of the introduced method, numerical examples are presented.

Keywords: model order reduction, Hankel singular values, linear systems, differential-algebraic equations

1 Introduction

Many different real-world applications, like chemical processes, electrical circuits and networks, or computational fluid dynamics, naturally lead to models, described by systems of differential-algebraic equations. Since experiments can be very costly, time-consuming, and expensive, these models are used for simulations and the design of controllers. The modeling process often results in linear time-invariant continuous-time descriptor systems of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

with $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$. Here, $u(t) \in \mathbb{R}^m$ are the inputs of the system, which influence the generalized states $x(t) \in \mathbb{R}^n$ to get the desired outputs $y(t) \in \mathbb{R}^p$. Throughout this paper, it is assumed that the matrix pencil $\lambda E - A$ is regular, i.e., there exists at least one $\lambda \in \mathbb{C}$ such that $\det(\lambda E - A) \neq 0$. In this case, and with the initial condition $Ex(0) = 0$, the input-output behavior of the system (1) in the frequency domain can be described via the system's transfer function

$$G(s) = C(sE - A)^{-1}B + D. \tag{2}$$

The quintuple (E, A, B, C, D) , consisting of matrices from (1), defines a realization of (1) and its transfer function (2). Usually, the numbers of inputs and outputs are very small in contrast to the number of differential-algebraic equations and generalized states n , which quickly enlarges due to

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different reasons, e.g., the model shall provide a required accuracy. Because of that, the usage of complete models often reaches the limits of computational resources like memory and computation time. Since the acquired data for the model usually contain a huge amount of redundancies, it is possible to approximate the original model by a new system with a much smaller order. The task of model reduction is to construct a reduced-order descriptor system

$$\begin{aligned}\hat{E}\hat{x}(t) &= \hat{A}\hat{x}(t) + \hat{B}u(t), \\ \hat{y}(t) &= \hat{C}\hat{x}(t) + \hat{D}u(t),\end{aligned}\tag{3}$$

of order $r \ll n$, such that the input-output behavior of the original system (1) is approximated.

Many model reduction techniques were originally developed for the standard system case, where the descriptor term E is the identity matrix I_n (or at least nonsingular). But in recent years, quite a few of these methods have been extended to the case of descriptor systems with singular E matrices. There are different approaches for the construction of (3), e.g., matrix equations can be used to determine a measure for truncatable states [6], or the transfer function can be approximated by rational interpolation [12]. A special technique of model reduction is the computation of the optimal Hankel-norm approximation (HNA). This technique actually provides a best approximation in the Hankel semi-norm. Based on the work of Adamjan, Arov, and Krein about the approximation of Hankel matrices [1], an algorithm for the computation of the HNA for standard systems was introduced by Glover in [11].

A generalization of the HNA to the descriptor system case was already mentioned by Cao, Saltik, and Weiland in [10]. They are using the Weierstrass canonical form for an explicit construction of reduced decoupled subsystems. The main problem of this method is the computation of the Weierstrass canonical form which is numerically costly and unstable. Also additional conditions, like C-controllability and C-observability of the system, have to be assumed.

In this paper, a new efficient algorithm for the computation of the generalized Hankel-norm approximation (GHNA) will be proposed. Our main contributions are twofold:

1. We generalize the concept of all-pass transfer functions to descriptor systems (Theorem 1).
2. We derive new and reliable numerical implementations of the GHNA that also allow the application of the Hankel-norm approximation method to large-scale problems with sparse coefficient matrices as they arise, e.g., from systems with dynamics described by semi-discretized unsteady partial differential equations.

Therefore, in Section 2 the mathematical background of linear descriptor systems is recalled. Then, the HNA method for the standard system case is introduced in the first part of Section 3. Afterwards, the generalized balanced truncation is reviewed and used for the construction of the new GHNA method. The numerical difficulties and adjustments are discussed in Section 4 for usable implementations of the method. Two different implementations of the method are then tested on numerical examples in Section 5. In Section 6, the conclusions of this paper can be found.

2 Mathematical Basics

For regular matrix pencils $\lambda E - A$, the Weierstrass canonical form always exists: there are invertible matrices $W, T \in \mathbb{C}^{n \times n}$ such that

$$W(\lambda E - A)T = \lambda \begin{bmatrix} I_{n_f} & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_{n_\infty} \end{bmatrix},\tag{4}$$

where J and N are both in Jordan canonical form, J is regular, and N is nilpotent with index ν . The numbers n_f and n_∞ are the dimensions of the deflating subspaces corresponding to the finite and infinite eigenvalues of $\lambda E - A$, respectively. Then, the spectral projectors onto the left and right deflating subspaces corresponding to the finite eigenvalues of the matrix pencil $\lambda E - A$ can be defined as

$$P_\ell = W \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} W^{-1} \quad \text{and} \quad P_r = T^{-1} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} T,\tag{5}$$

with W and T from the Weierstrass canonical form (4).

Another necessary assumption is the c -stability of the matrix pencil $\lambda E - A$, i.e., the matrix pencil $\lambda E - A$ is regular and all finite eigenvalues of $\lambda E - A$ lie in the open left half-plane. In this case, the proper controllability and observability Gramians are defined as the unique, positive semidefinite solutions of the projected generalized continuous-time Lyapunov equations

$$E\mathcal{G}_{pc}A^T + A\mathcal{G}_{pc}E^T + P_\ell BB^T P_\ell^T = 0, \quad \mathcal{G}_{pc} = P_r \mathcal{G}_{pc} P_r^T, \quad (6)$$

$$E^T \mathcal{G}_{po}A + A^T \mathcal{G}_{po}E + P_r^T C^T C P_r = 0, \quad \mathcal{G}_{po} = P_\ell^T \mathcal{G}_{po} P_\ell, \quad (7)$$

with P_ℓ and P_r the spectral projectors corresponding to the finite eigenvalues (5); see [18]. Furthermore, the improper controllability and observability Gramians are given as the unique, positive semidefinite solutions of the projected generalized discrete-time Lyapunov equations

$$A\mathcal{G}_{ic}A^T - E\mathcal{G}_{ic}E^T - Q_\ell BB^T Q_\ell^T = 0, \quad \mathcal{G}_{ic} = Q_r \mathcal{G}_{ic} Q_r^T, \quad (8)$$

$$A^T \mathcal{G}_{io}A - E^T \mathcal{G}_{io}E - Q_r^T C^T C Q_r = 0, \quad \mathcal{G}_{io} = Q_\ell^T \mathcal{G}_{io} Q_\ell, \quad (9)$$

with $Q_\ell = I_n - P_\ell$ and $Q_r = I_n - P_r$ the spectral projectors onto the left and right deflating subspaces corresponding to the infinite eigenvalues of the matrix pencil $\lambda E - A$; see [18].

Using the system Gramians, the set of Hankel singular values is defined in the following; see [14].

Definition 1. The square roots of the n_f largest eigenvalues of $\mathcal{G}_{pc}E^T \mathcal{G}_{po}E$ denoted by $\varsigma_1 \geq \varsigma_2 \geq \dots \geq \varsigma_{n_f}$ are the proper Hankel singular values of (1). The square roots of the n_∞ largest eigenvalues of $\mathcal{G}_{ic}A^T \mathcal{G}_{io}A$ denoted by $\theta_1 \geq \theta_2 \geq \dots \geq \theta_{n_\infty}$ are the improper Hankel singular values of (1).

In case of a non-singular descriptor term E , the proper Hankel singular values are the classical Hankel singular values of the system. Therefore, an equivalent energy interpretation of the proper Hankel singular values exists which proposes the truncation of states corresponding to small proper Hankel singular values, which are difficult to control and observe. Unfortunately, this does not hold for the improper Hankel singular values since these correspond to the constraints of the system. The truncation of non-zero improper Hankel singular values may result in physically meaningless systems.

There exist diverse concepts of controllability and observability for descriptor systems. For this paper, we restrict ourselves to the following definitions; see, e.g., [18].

Definition 2. System (1) is called:

1. R-controllable if

$$\text{rank} [\lambda E - A, \quad B] = n \quad \text{for all } \lambda \in \mathbb{C}.$$

2. C-controllable if the system is R-controllable and

$$\text{rank} [E, \quad B] = n.$$

3. R-observable if

$$\text{rank} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n \quad \text{for all } \lambda \in \mathbb{C}.$$

4. C-observable if the system is R-observable and

$$\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n.$$

The relation between these controllability, observability notions and the system Gramians is given in [19, Theorem 2.3]. Especially, all proper Hankel singular values are non-zero if and only if the system is R-controllable and R-observable.

The mapping from past inputs $u_- : (-\infty, 0] \rightarrow \mathbb{R}^m$ to future outputs $y_+ : (0, +\infty] \rightarrow \mathbb{R}^p$ is described by the Hankel operator $y_+ = \mathcal{H}u_-$. A generalization of this operator to the case of descriptor systems can be found in [10]. The measure of the influence of past inputs on future outputs in the \mathcal{L}_2 -norm leads to the definition of the Hankel semi-norm for descriptor systems.

Definition 3. The Hankel semi-norm of a system G is given by

$$\|G\|_H = \sup_{u_- \in \mathcal{W}_2^{\nu-1}(-\infty, 0]} \frac{\|y_+\|_{\mathcal{L}_2}}{\|u_-\|_{\mathcal{L}_2}}, \quad (10)$$

where $\mathcal{W}_2^{\nu-1}(-\infty, 0]$ denotes the Sobolev space of $\nu - 1$ times weakly differentiable functions w.r.t. the \mathcal{L}_2 inner product on the interval $(-\infty, 0]$ and $\|\cdot\|_{\mathcal{L}_2}$ the \mathcal{L}_2 -norm.

In case of an invertible descriptor term E , the Hankel semi-norm (10) simplifies to

$$\|G\|_H = \varsigma_{\max}(G),$$

where $\varsigma_{\max}(G)$ is the largest Hankel singular value of the system G .

3 Generalized Hankel-Norm Approximation

3.1 Algorithm for Standard Systems

First, the algorithm for the standard system case, introduced by Glover in [11], is considered. Therefor, a balanced minimal realization of the given standard system $(I_{n_{\min}}, A, B, C, D)$ is assumed, where n_{\min} is the McMillan degree of the system, i.e., the order of its minimal realization. The computation is usually done by the balanced truncation square root method. Since the resulting system is balanced and minimal, the system Gramians are equal and diagonal

$$\mathcal{G}_{pc} = \mathcal{G}_{po} = \text{diag}(\varsigma_1, \varsigma_2, \dots, \varsigma_{n_{\min}}),$$

with $\varsigma_1, \dots, \varsigma_{n_{\min}}$ all non-zero Hankel singular values of the system. Next, the system is partitioned by the order r such that

$$\varsigma_1 \geq \dots \geq \varsigma_r > \varsigma_{r+1} = \dots = \varsigma_{r+k+1} > \varsigma_{r+k+2} \geq \dots \geq \varsigma_{n_{\min}},$$

with $k \geq 1$ being the multiplicity of the $(r + 1)$ -st Hankel singular value. The Gramians are reordered to separate the block with the $(r + 1)$ -st Hankel singular value as

$$\check{\mathcal{G}}_{pc} = \check{\mathcal{G}}_{po} = \begin{bmatrix} \check{\Sigma} & \\ & \varsigma_{r+1} I_k \end{bmatrix}, \quad (11)$$

with $\check{\Sigma} = \text{diag}(\varsigma_1, \dots, \varsigma_r, \varsigma_{r+k+2}, \dots, \varsigma_{n_{\min}})$. Accordingly to (11), the remaining system matrices have to be permuted and partitioned

$$\check{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \check{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \check{C} = [C_1, \quad C_2].$$

Then, the partitioned system is transformed by the following formulas

$$\begin{aligned} \tilde{A} &= \Gamma^{-1}(\varsigma_{r+1}^2 A_{11}^T + \check{\Sigma} A_{11} \check{\Sigma} + \varsigma_{r+1} C_1^T U B_1^T), \\ \tilde{B} &= \Gamma^{-1}(\check{\Sigma} B_1 - \varsigma_{r+1} C_1^T U), \\ \tilde{C} &= C_1 \check{\Sigma} - \varsigma_{r+1} U B_1^T, \\ \tilde{D} &= D + \varsigma_{r+1} U, \end{aligned} \quad (12)$$

with $\Gamma = \check{\Sigma}^2 - \varsigma_{r+1}^2 I_{n_{\min}-k}$ and $U = (C_2^T)^\dagger B_2$. Here, M^\dagger denotes the Moore-Penrose pseudo-inverse of a matrix M . This system is constructed such that the error transfer function $\mathcal{E} = G - \tilde{G}$ is scaled all-pass with \tilde{G} the transfer function of (12), i.e., it holds

$$\mathcal{E}(s)\mathcal{E}^T(-s) = \varsigma_{r+1}^2 I_p, \quad (13)$$

for all $s \in \mathbb{C}$ that are not poles of $\mathcal{E}(s)$ or $\mathcal{E}^T(-s)$. In this case, the approximation error satisfies

$$\|\mathcal{E}\|_H = \|\mathcal{E}\|_{\mathcal{L}_\infty} = \varsigma_{r+1}. \quad (14)$$

The transfer function \tilde{G} of (12) has exactly $n_{\min} - k - r$ unstable poles. As last step, an additive decomposition of \tilde{G} is computed such that $\tilde{G} = G_h + G_+$, where G_+ is the anti-stable part of order $n_{\min} - k - r$ and G_h is the stable part of order r . Since the Hankel semi-norm only depends on the stable part of the system, the error (14) in the Hankel semi-norm does not change if the unstable part is removed, such that

$$\|G - G_h\|_H = \varsigma_{r+1}. \quad (15)$$

3.2 Computing a Balanced Realization for Descriptor Systems

As for the standard system case, for descriptor systems, a balanced conditionally minimal realization is needed. The term ‘‘conditionally’’ minimal means that the order of the system is minimal except of the reduction of the index-1 parts in E , see [17]. The computation is done using the generalized balanced truncation square root method (GBT(SR)). The basic idea of this method is the computation of a balanced realization and the truncation of unnecessary states.

Definition 4. A realization of a descriptor system (1) is called balanced if

$$\mathcal{G}_{pc} = \mathcal{G}_{po} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{G}_{ic} = \mathcal{G}_{io} = \begin{bmatrix} 0 & 0 \\ 0 & \Theta \end{bmatrix}$$

hold, with the proper Hankel singular values $\Sigma = \text{diag}(\varsigma_1, \dots, \varsigma_{n_f})$ and the improper Hankel singular values $\Theta = \text{diag}(\theta_1, \dots, \theta_{n_\infty})$.

The truncation of the states is made with respect to the computed Hankel singular values. The proper Hankel singular values have the same meaning as the classical Hankel singular values in the standard case, i.e., states corresponding to small proper Hankel singular values are difficult to control and observe at the same time and can be omitted. In case of the improper Hankel singular values, only zeros can be truncated since the truncation of non-zero improper Hankel singular values results in physically meaningless approximations [14]. The number of non-zero improper Hankel singular values is equal to the rank of the matrix $\mathcal{G}_{ic}A^T\mathcal{G}_{io}A$, which can in fact be bounded by

$$\text{rank}(\mathcal{G}_{ic}A^T\mathcal{G}_{io}A) \leq \min(\nu m, \nu p, n_\infty), \quad (16)$$

with ν , index of the system, m , number of inputs, p , number of outputs, and n_∞ , dimension of the deflating subspace corresponding to the infinite eigenvalues of $\lambda E - A$. So for large n_∞ and usually small ν , the descriptor system (1) can be reduced significantly by the truncation of zero improper Hankel singular values.

One method to compute the balanced truncation of a descriptor system is the square root method. Therefor, consider the skinny singular value decompositions

$$L_p^T E R_p = [U_1, \quad U_2] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}, \quad (17)$$

$$L_i^T A R_i = U_3 \Theta_3 V_3^T, \quad (18)$$

with $\mathcal{G}_{pc} = R_p R_p^T$, $\mathcal{G}_{po} = L_p L_p^T$, $\mathcal{G}_{ic} = R_i R_i^T$, and $\mathcal{G}_{io} = L_i L_i^T$. The matrices $[U_1, \quad U_2]$, $[V_1, \quad V_2]$, U_3 and V_3 have orthonormal columns and the diagonal matrices Σ_1 , Σ_2 , and Θ_3 contain the non-zero proper and improper Hankel singular values, respectively. The partition of the proper Hankel singular values is chosen such that Σ_1 contains all the desired Hankel singular values and Σ_2 the undesired ones. By using the singular value decompositions in (17) and (18), the following transformation matrices can be defined

$$W_\ell = \begin{bmatrix} L_p U_1 \Sigma_1^{-\frac{1}{2}}, & L_i U_3 \Theta_3^{-\frac{1}{2}} \end{bmatrix} \in \mathbb{R}^{n \times \ell}, \quad (19)$$

$$T_\ell = \begin{bmatrix} R_p V_1 \Sigma_1^{-\frac{1}{2}}, & R_i V_3 \Theta_3^{-\frac{1}{2}} \end{bmatrix} \in \mathbb{R}^{n \times \ell},$$

where $\ell = \ell_f + \ell_\infty$ is the sum of the number of desired proper Hankel singular values ℓ_f and the non-zero improper Hankel singular values ℓ_∞ . The transformed realization

$$(\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{D}) = (W_\ell^T E T_\ell, W_\ell^T A T_\ell, W_\ell^T B, C T_\ell, D) \quad (20)$$

is of order ℓ and balanced with the set of Hankel singular values contained in Σ_1 and Θ_3 . The resulting matrix pencil $\lambda\hat{E} - \hat{A}$ is a resembling of the Weierstrass canonical form (4), such that

$$\hat{E} = \begin{bmatrix} I_{\ell_f} & 0 \\ 0 & E_\infty \end{bmatrix} \quad \text{and} \quad \hat{A} = \begin{bmatrix} A_f & 0 \\ 0 & I_{\ell_\infty} \end{bmatrix} \quad (21)$$

hold, where $A_f \in \mathbb{R}^{\ell_f \times \ell_f}$ is non-singular and $E_\infty \in \mathbb{R}^{\ell_\infty \times \ell_\infty}$ is nilpotent with index ν .

Due to the reason that only the zero improper Hankel singular values have been truncated, the polynomial part of the system G has not changed. So it can be shown that the same error bound as for the classical balanced truncation method holds. Let \hat{G} be the transformed descriptor system (20), then it holds

$$\|G - \hat{G}\|_{\mathcal{H}_\infty} \leq 2 \sum_{k=\ell_f+1}^{n_f} \varsigma_k(G),$$

with $\varsigma_k(G)$ the k -th proper Hankel singular value of G .

3.3 Hankel-Norm Approximation of Descriptor Systems

As for the standard case, the GHNA method for descriptor systems is based on the construction of an error system with all-pass transfer function (13). The following theorem provides an algebraic characterization of descriptor systems with all-pass transfer functions.

Theorem 1. *Let (E, A, B, C, D) be a realization of a descriptor system (1) with a regular matrix pencil $\lambda E - A$, the same number of inputs and outputs, $m = p$, the system's transfer function $G(s)$ and $\varsigma > 0$ a real constant. Also, it is assumed that the descriptor system is R -controllable and R -observable. Then $G(s)$ is all-pass, i.e., $G(s)G^T(-s) = \varsigma^2 I_m$ holds, if and only if the following conditions are satisfied:*

1. There are symmetric matrices \mathcal{G}_{pc} and \mathcal{G}_{po} with

$$\mathcal{G}_{pc} = P_r \mathcal{G}_{pc} P_r^T, \quad (22)$$

$$\mathcal{G}_{po} = P_\ell^T \mathcal{G}_{po} P_\ell. \quad (23)$$

2. The matrices \mathcal{G}_{pc} and \mathcal{G}_{po} are the solutions of the projected generalized continuous-time Lyapunov equations

$$E \mathcal{G}_{pc} A^T + A \mathcal{G}_{pc} E^T + P_\ell B B^T P_\ell^T = 0, \quad (24)$$

$$E^T \mathcal{G}_{po} A + A^T \mathcal{G}_{po} E + P_r^T C^T C P_r = 0. \quad (25)$$

3. The proper Hankel singular values satisfy

$$\mathcal{G}_{pc} E^T \mathcal{G}_{po} E = \varsigma^2 P_r, \quad (26)$$

$$\mathcal{G}_{po} E \mathcal{G}_{pc} E^T = \varsigma^2 P_\ell^T. \quad (27)$$

4. Let $G(s) = G_{sp}(s) + P(s)$ be decomposed into the strictly proper part G_{sp} and the polynomial part P . Then it holds $P(s) = \sum_{k=0}^{\infty} M_k s^k$ with

$$M_0 M_0^T = \varsigma^2 I_m, \quad (28)$$

$$M_k = 0 \quad \text{for} \quad k \geq 1. \quad (29)$$

5. Also, the following constraints hold

$$M_0^T C P_r + B^T \mathcal{G}_{po} E = 0, \quad (30)$$

$$M_0 B^T P_\ell^T + C \mathcal{G}_{pc} E^T = 0. \quad (31)$$

Proof. The proof can be found in the Appendix. \square

Theorem 1 can be used to derive more general construction formulas for all-pass error systems, see [8] for generalized formulas for an invertible E matrix. Also, it shows that the improper part of the system should not change. For the development of an algorithm, the structure of the reduced-order model (21), obtained from the generalized balanced truncation, can be exploited. So, let the matrices \hat{B} and \hat{C} be partitioned accordingly to (21) as

$$\hat{B} = \begin{bmatrix} B_f \\ B_\infty \end{bmatrix} \quad \text{and} \quad \hat{C} = [B_f, \quad B_\infty].$$

Using this block partition, the system $(\hat{E}, \hat{A}, \hat{B}, \hat{C}, D)$ automatically decouples into its slow subsystem

$$\begin{aligned} \dot{x}_f(t) &= A_f x_f(t) + B_f u(t), \\ y_f(t) &= C_f x_f(t) \end{aligned} \tag{32}$$

and its fast subsystem

$$\begin{aligned} E_\infty \dot{x}_\infty(t) &= x_\infty(t) + B_\infty u(t), \\ y_\infty(t) &= C_\infty x_\infty(t) + D u(t). \end{aligned} \tag{33}$$

First, the fast subsystem (33) is considered. Since the GBT(SR) was used to compute the system (33), there are no zero improper Hankel singular values anymore. As mentioned before and considering Theorem 1, there is no meaningful further reduction concerning the improper Hankel singular values without generating physically meaningless results, so the fast subsystem stays unchanged.

Now, let us consider the slow subsystem (32). It is easy to see that (32) is in standard form. Also beneficial properties, resulting from the applied balanced truncation method, still hold for this subsystem which means it is stable and balanced.

Let the original system be decomposed as $G = G_{sp} + P$ into its slow subsystem G_{sp} and its fast subsystem P . By the truncation of only zero proper Hankel singular values, the system (32) is a minimal realization of the original slow subsystem G_{sp} . Now, the standard HNA method, mentioned in the previous section, can be applied to (32). As result, an r -th order HNA is computed

$$\begin{aligned} E_h \dot{x}_h(t) &= A_h x_h(t) + B_h u(t), \\ y_h(t) &= C_h x_h(t) + D_h u(t), \end{aligned} \tag{34}$$

where E_h results from avoiding disadvantageous scaling of the matrices A_h and B_h . More general transformation formulas for invertible E matrices have been developed in [8]. To get an optimal HNA of the descriptor system (1), now the computed HNA (34) and the reduced-order fast subsystem (33) are coupled

$$\begin{aligned} \begin{bmatrix} E_h & 0 \\ 0 & E_\infty \end{bmatrix} \dot{\hat{x}}(t) &= \begin{bmatrix} A_h & 0 \\ 0 & I_{\ell_\infty} \end{bmatrix} \hat{x}(t) + \begin{bmatrix} B_h \\ B_\infty \end{bmatrix} u(t), \\ \hat{y}(t) &= [C_h, \quad C_\infty] \hat{x}(t) + (D_h + D)u(t). \end{aligned} \tag{35}$$

In the following theorem, the properties of the resulting GHNA are summarized.

Theorem 2. *Let G be a c -stable descriptor system (1) with a regular matrix pencil. The ℓ -th order generalized Hankel-norm approximation (35), with its transfer function \hat{G} and $\ell = r + \ell_\infty$, has the following properties:*

1. *The realization of \hat{G} is conditionally minimal and c -stable.*
2. *The absolute error in the Hankel semi-norm is given by*

$$\|G - \hat{G}\|_H = \varsigma_{r+1}(G),$$

where $\varsigma_{r+1}(G)$ is the $(r + 1)$ -st proper Hankel singular value of G .

3. The absolute error in the \mathcal{H}_∞ -norm can be bounded by

$$\|G - \hat{G}\|_{\mathcal{H}_\infty} \leq 2 \sum_{k=r+1}^{n_f} \varsigma_k(G),$$

where $\varsigma_k(G)$ is the k -th proper Hankel singular values of G .

Proof. Let $G = G_{sp} + P$ be the original system and $\tilde{G} = G_b + P_b$ the balanced, conditionally minimal realization obtained by the GBT(SR) method. Here G_{sp} , G_b denote the slow subsystems and P , P_b the fast ones. The GHNA is constructed by

$$\hat{G} = G_h + P_b, \quad (36)$$

where G_h is the r -th order HNA (34) of the standard system G_b .

First, consider part 1. The balanced realization \tilde{G} is conditionally minimal and c-stable. So by construction (36), both of these properties are transferred to the GHNA.

Now consider the error formulas in 2. and 3. Therefor, let $\mathcal{E} = G - \hat{G}$ be the error system of the GHNA. Then it holds

$$\begin{aligned} \mathcal{E} &= G - \hat{G} \\ &= G_{sp} + P - G_h - P_b \\ &= G_b + P_b - G_h - P_b \\ &= G_b - G_h, \end{aligned}$$

since the balanced realization \tilde{G} is conditionally minimal and therefor, $G_b = G_{sp}$ and $P_b = P$. Using the error bound of the standard method (15) one obtains

$$\|G - \hat{G}\|_H = \|G_b - G_h\|_H = \varsigma_{r+1}(G_b) = \varsigma_{r+1}(G).$$

Using the same approach, the error in the \mathcal{H}_∞ -norm is given by

$$\|G - \hat{G}\|_{\mathcal{H}_\infty} = \|G_b - G_h\|_{\mathcal{H}_\infty} \leq 2 \sum_{k=r+1}^{n_f} \varsigma_k(G),$$

if the \mathcal{H}_∞ -norm error bound for the standard r -th order HNA from [2] is used. \square

In Algorithm 1, the complete GHNA method is summarized.

4 Numerical Methods for GHNA

4.1 Approximate GHNA

The GHNA method can quickly become numerically unstable. This problem arises from the transformation formulas (12) for the construction of a scaled all-pass error transfer function. It is easy to see that the diagonal matrix $\Gamma = \check{\Sigma}^2 - \varsigma_{r+1}^2 I_{n_{\min} - k}$ can lead to large numerical errors for small proper Hankel singular values in further computations. This happens if either the chosen value ς_{r+1} or the remaining proper Hankel singular values in $\check{\Sigma}$ are very small. One preventive measure was the usage of the descriptor system structure (34) to avoid unnecessary scaling by Γ . In further considerations, only the case of too small remaining Hankel singular values is treated.

Small proper Hankel singular values can arise from numerical errors during the computation of the minimal realization. Therefor, one approach to solve this problem is to compute a smaller balanced truncation of the slow subsystem than the minimal realization such that too small proper Hankel singular values are cut off. In this case, an additional error is made since the balanced realization is only an approximation of the original system. To get a measure for the additional

Algorithm 1 Generalized Hankel-Norm Approximation (GHNA) Method

- 1: Solve the continuous-time Lyapunov equations (6) and (7) for the Cholesky factorizations $\mathcal{G}_{pc} = R_p R_p^T$ and $\mathcal{G}_{po} = L_p L_p^T$.
- 2: Solve the discrete-time Lyapunov equations (8) and (9) for the Cholesky factorizations $\mathcal{G}_{ic} = R_i R_i^T$ and $\mathcal{G}_{io} = L_i L_i^T$.
- 3: Compute the two skinny singular value decompositions

$$L_p^T E R_p = U_1 \Sigma V_1^T \quad \text{and} \quad L_i^T A R_i = U_2 \Theta V_2^T.$$

- 4: Compute the transformation matrices

$$\begin{aligned} W_p &= L_p U_1 \Sigma^{-\frac{1}{2}}, & T_p &= R_p V_1 \Sigma^{-\frac{1}{2}}, \\ W_i &= L_i U_2 \Theta^{-\frac{1}{2}}, & T_i &= R_i V_2 \Theta^{-\frac{1}{2}}. \end{aligned}$$

- 5: Compute the minimal balanced realization of the slow subsystem

$$(I_{\ell_f}, A_f, B_f, C_f, 0) = (W_p^T E T_p, W_p^T A T_p, W_p^T B, C T_p, 0).$$

- 6: Choose the proper Hankel singular value ς_{r+1} .
- 7: Permute and partition the Gramians of the slow subsystem

$$\check{\mathcal{G}}_{pc} = \check{\mathcal{G}}_{po} = \text{diag}(\check{\Sigma}, \varsigma_{r+1} I_k),$$

and the corresponding system matrices

$$\check{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \check{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \check{C} = [C_1, \quad C_2].$$

- 8: Compute the all-pass transformation

$$\begin{aligned} \tilde{E} &= \check{\Sigma}^2 - \varsigma_{r+1}^2 I_{\ell_f - k}, \\ \tilde{A} &= \varsigma_{r+1}^2 A_{11}^T + \check{\Sigma} A_{11} \check{\Sigma} + \varsigma_{r+1} C_1^T U B_1^T, \\ \tilde{B} &= \check{\Sigma} B_1 - \varsigma_{r+1} C_1^T U, \\ \tilde{C} &= C_1 \check{\Sigma} - \varsigma_{r+1} U B_1^T, \\ \tilde{D} &= \varsigma_{r+1} U, \end{aligned}$$

with $U = (C_2^T)^\dagger B_2$.

- 9: Compute the additive decomposition

$$\tilde{G}(s) = \tilde{C}(s\tilde{E} - \tilde{A})\tilde{B} + \tilde{D} = G_h(s) + F(s),$$

where F is anti-stable and G_h stable with the realization $(E_h, A_h, B_h, C_h, D_h)$.

- 10: Compute the balanced realization of the fast subsystem

$$(E_\infty, I_{\ell_\infty}, B_\infty, C_\infty, D) = (W_i^T E T_i, W_i^T A T_i, W_i^T B, C T_i, D).$$

- 11: Couple the resulting subsystems

$$(\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{D}) = \left(\begin{bmatrix} E_h & 0 \\ 0 & E_\infty \end{bmatrix}, \begin{bmatrix} A_h & 0 \\ 0 & I_{\ell_\infty} \end{bmatrix}, \begin{bmatrix} B_h \\ B_\infty \end{bmatrix}, [C_h, \quad C_\infty], D_h + D \right).$$

error, let G_b be the computed balanced truncation of order n_b of the slow subsystem G_{sp} . Then it has been shown in [11] that in the Hankel semi-norm it holds

$$\|G_{sp} - G_b\|_H \leq 2 \sum_{k=n_b+1}^{n_f} \varsigma_k(G_{sp}), \quad (37)$$

with n_f the order of the slow subsystem G_{sp} . For the overall error, let $G = G_{sp} + P$ be the original descriptor system and $\tilde{G} = G_b + P_b$ the balanced realization with G_b of order n_b . The generalized Hankel-norm approximation is denoted by $\hat{G} = G_h + P_b$, where the r -th order standard Hankel-norm approximation G_h was computed from the balanced realization G_b . Using (37) one obtains

$$\begin{aligned} \|G - \hat{G}\|_H &= \|G_{sp} + P - G_h - P_b\|_H \\ &= \|G_{sp} - G_h\|_H \\ &= \|G_{sp} - G_b + G_b - G_h\|_H \\ &\leq \|G_b - G_h\|_H + \|G_{sp} - G_b\|_H \\ &\leq \varsigma_{r+1}(G_b) + 2 \sum_{k=n_b+1}^{n_f} \varsigma_k(G_{sp}). \end{aligned} \quad (38)$$

Since balancing the system does not change the Hankel singular values, the Hankel singular values of G_b and G_{sp} are also the proper Hankel singular values of G . The resulting error can be bounded by

$$\|G - \hat{G}\|_H \leq \varsigma_{r+1}(G) + 2 \sum_{k=n_b+1}^{n_f} \varsigma_k(G).$$

Concerning the \mathcal{H}_∞ -norm, the approach in (38) can be used to get

$$\|G - \hat{G}\|_{\mathcal{H}_\infty} \leq 2 \sum_{k=r+1}^{n_f} \varsigma_k(G),$$

which is the same error bound as for the exact method.

This approximate version of the GHNA takes advantage of the use of the GBT(SR) method in form of the adaptive choice of the order n_b . It is possible to choose the order n_b with respect to the proper Hankel singular value ς_{r+1} such that

$$2 \sum_{k=n_b+1}^{n_f} \varsigma_k(G) \ll \varsigma_{r+1}(G).$$

In this case, the resulting additional error becomes negligible small concerning the original Hankel semi-norm error. But the corresponding matrix Γ leads to a better conditioned problem. The algorithmic adjustments in the implementation of the GHNA method are small, since only the truncation of non-zero proper Hankel singular values has to be allowed in the generalized balanced truncation method. In this case, the Σ_2 term in (17) with the undesired proper Hankel singular values is not zero and only the matrices U_1 , Σ_1 , and V_1 are used for further computations.

Another advantage of the approximate algorithm can be found in the computation of the balanced truncation. The GBT(SR) method needs to scale the transformation matrices (19) using the inverse remaining Hankel singular values which is more accurate if the small proper Hankel singular values are truncated. Also in the sense of computational costs, this approximate method has advantages. The further steps of the algorithm, i.e., the all-pass transformation and additive decomposition, are extremely costly for large matrices in terms of computational time and memory usage. Therefore, it is advantageous to already have a small balanced realization for the further computations.

4.2 Application to Sparse Systems

A frequently appearing case in practice is the model reduction of large-scale sparse descriptor systems. In this case, the system matrices E and A from the descriptor system (1) are in a large-scale sparse form, i.e., the dimension n is large, the matrices can be stored using $\mathcal{O}(n)$ memory and the matrix-vector multiplication can be computed in $\mathcal{O}(n)$ effort. Often such matrices result from the discretization of partial differential equations.

The transformation into a balanced realization does not preserve the sparsity of the system matrices. Therefore, the GHNA method can only be adapted to sparse systems in the first two steps. This concerns the computation of the solutions of the generalized projected Lyapunov equations (6)–(9). It has been observed that the eigenvalues of the symmetric positive semidefinite solutions of Lyapunov equations with low-rank right-hand sides generally decay rapidly. The same result holds for the generalized projected Lyapunov equations [21]. Therefore, the system Gramians can be approximated by low-rank Cholesky factorizations, e.g., $\mathcal{G}_{pc} \approx Z_{pc}Z_{pc}^T$ with $Z_{pc} \in \mathbb{R}^{n \times k}$ and $k \ll n$.

For the proper system Gramians, the computation is done by adapting existing low-rank methods, e.g., Krylov subspace methods or low-rank ADI methods. In this case, the right-hand side has to be replaced by the projected form from the Lyapunov equations (6), (7). Additionally, it is recommended to project the solution back into the corresponding subspace after some steps of the methods due to a drift-off effect.

In contrast to this, for the improper system Gramians full-rank factorizations can be constructed explicitly such that $G_{ic} = Z_{ic}Z_{ic}^T$ and $G_{io} = Z_{io}Z_{io}^T$, with

$$\begin{aligned} Z_{ic} &= [Q_r A^{-1} B, \quad A^{-1} E Q_r A^{-1} B, \quad \dots, \quad (A^{-1} E)^{\nu-1} Q_r A^{-1} B], \\ Z_{io} &= [Q_\ell^T A^{-T} C^T, \quad A^{-T} E^T Q_\ell^T A^{-T} C^T, \quad \dots, \quad (A^{-T} E^T)^{\nu-1} Q_\ell^T A^{-T} C^T]; \end{aligned}$$

see [21] for more details. Thereby, the size of the full-rank factorizations is bounded by the number of inputs m or outputs p times the system's index ν . This corresponds to the overall bound of the non-zero improper Hankel singular values (16).

Still for using these methods, the spectral projections P_ℓ , P_r , Q_ℓ and Q_r have to be computed. But for many problems, these spectral projections can be applied by exploiting the special structure of the problem; see [21] for some examples.

4.3 The Projection-Free Approach

In case of unstructured problems, there are no explicit construction formulas for the spectral projectors P_ℓ , P_r , Q_ℓ and Q_r , so they have to be explicitly computed for the use in the generalized projected Lyapunov equations (6)–(9). But as for the GBT(SR) method, an alternative approach to the use of spectral projectors can be given; see [19].

As already used in the GHNA algorithm, the GBT method can be interpreted as a decoupling of the original system into the slow and fast subsystems and the individual reduction of both. Therefore, consider the following generalized block triangular form. There are orthogonal matrices $U, V \in \mathbb{R}^{n \times n}$ such that

$$E = V \begin{bmatrix} E_f & E_u \\ 0 & E_\infty \end{bmatrix} U^T \quad \text{and} \quad A = V \begin{bmatrix} A_f & A_u \\ 0 & A_\infty \end{bmatrix} U^T,$$

where the matrix pencil $\lambda E_f - A_f$ contains all the finite eigenvalues of $\lambda E - A$ and the matrix pencil $\lambda E_\infty - A_\infty$ has only infinite eigenvalues. For the computation of a block diagonalization of the system, the coupled Sylvester equations

$$E_f Y - Z E_\infty = -E_u, \tag{39}$$

$$A_f Y - Z A_\infty = -A_u, \tag{40}$$

have to be solved for Y and Z ; see [5]. Using all of these matrices for the restricted system equivalence transformation

$$W_{dec} = V \begin{bmatrix} I_{n_f} & 0 \\ -Z^T & I_{n_\infty} \end{bmatrix}, \quad T_{dec} = U \begin{bmatrix} I_{n_f} & Y \\ 0 & I_{n_\infty} \end{bmatrix}$$

of the original descriptor system (1), one obtains

$$\begin{aligned} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} \dot{\tilde{x}}(t) &= \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} B_f \\ B_\infty \end{bmatrix} u(t), \\ y(t) &= [C_f \quad C_\infty] \tilde{x}(t) + Du(t), \end{aligned} \quad (41)$$

where the remaining matrices are constructed as

$$\begin{aligned} V^T B &= \begin{bmatrix} B_u \\ B_\infty \end{bmatrix}, \quad B_f = B_u - ZB_\infty, \\ CU &= \begin{bmatrix} C_f \\ C_u \end{bmatrix}, \quad C_\infty = C_f Y + C_u. \end{aligned} \quad (42)$$

Obviously, the realization in (41) decouples into the fast and slow subsystems of (1). Since the spectral projectors of the subsystems are identity matrices, the corresponding Lyapunov equations (6)–(9) simplify to

$$\begin{aligned} E_f X_{pc} A_f^T + A_f X_{pc} E_f^T + B_f B_f^T &= 0, \\ E_f^T X_{po} A_f + A_f^T X_{po} E_f + C_f^T C_f &= 0, \end{aligned}$$

for the slow subsystem and

$$\begin{aligned} A_\infty X_{ic} A_\infty^T - E_\infty X_{ic} E_\infty^T - B_\infty B_\infty^T &= 0, \\ A_\infty^T X_{io} A_\infty - E_\infty^T X_{io} E_\infty - C_\infty^T C_\infty &= 0, \end{aligned}$$

for the fast subsystem. These Lyapunov equations can be computed without the spectral projections. The matrices X_{pc} and X_{po} correspond to the parts of the proper controllability and observability Gramians, which contain the potentially non-zero proper Hankel singular values. The same holds for X_{ic} , X_{io} and the improper system Gramians. For the rest of the algorithm, only the transformations have to be restricted to the subsystems.

The projection-free approach is implemented in the version 3.0 of the MORLAB toolbox; see [7]. In this special implementation, the block diagonalization of the system is done by using a block transformation approach based on the following generalization of Theorem 4.1 from [13].

Theorem 3. *Let $\Gamma \subset \mathbb{C}$ be a region in the complex plane which contains n_1 eigenvalues of the matrix pencil $\lambda E - A$. Let $Q, Z \in \mathbb{R}^{n \times n}$ be orthogonal matrices that transform the matrix pencil $\lambda E - A$ into the upper block triangular form:*

$$Q^T (\lambda E - A) Z = \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} (\lambda E - A) [Z_1, Z_2] = \begin{bmatrix} \lambda E_{11}^{(1)} - A_{11}^{(1)} & \lambda E_{12}^{(1)} - A_{12}^{(1)} \\ 0 & \lambda E_{22}^{(1)} - A_{22}^{(1)} \end{bmatrix},$$

with $\Lambda(A_{11}^{(1)}, E_{11}^{(1)}) \subseteq \Gamma$ and $\Lambda(A_{11}^{(1)}, E_{11}^{(1)}) \cap \Lambda(A_{22}^{(1)}, E_{22}^{(1)}) = \emptyset$. Similarly, let $U, V \in \mathbb{R}^{n \times n}$ be orthogonal matrices that transform the matrix pencil $\lambda E - A$ into the upper block triangular form:

$$U^T (\lambda E - A) V = \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} (\lambda E - A) [V_1, V_2] = \begin{bmatrix} \lambda E_{11}^{(2)} - A_{11}^{(2)} & \lambda E_{12}^{(2)} - A_{12}^{(2)} \\ 0 & \lambda E_{22}^{(2)} - A_{22}^{(2)} \end{bmatrix},$$

with $\Lambda(A_{22}^{(2)}, E_{22}^{(2)}) \subseteq \Gamma$ and $\Lambda(A_{11}^{(2)}, E_{11}^{(2)}) \cap \Lambda(A_{22}^{(2)}, E_{22}^{(2)}) = \emptyset$. Then

$$X = [U_2, \quad Q_2] \quad \text{and} \quad Y = [Z_1, \quad V_1]$$

are transformation matrices, such that $X^T (\lambda E - A) Y$ has a block diagonal structure where the upper block contains the n_1 eigenvalues lying inside Γ and the lower block has the remaining $n - n_1$ eigenvalues of $\lambda E - A$ outside of Γ .

Proof. The proof can be found in [22, Section 5.2]. □

In contrast to the approach above, it is not necessary to compute the solution of the coupled Sylvester equations and, due to the block orthogonal structure of the transformation matrices, the right-hand sides are usually better conditioned than (42). In MORLAB, the right matrix pencil disk function method is used to generate the block transformation matrices, see [22] for more details on the implementation. Additionally, Theorem 3 can be used to compute the additive decomposition in step 9 of Algorithm 1 by separating the eigenvalues with negative and positive real-parts.

5 Numerical Examples

Two examples have been chosen to demonstrate the introduced GHNA method. All the computations were done on a machine with one Intel(R) Core(TM) i7-6700 CPU processor running at 3.40GHz and equipped with 8 GB total main memory. The computer is running on Ubuntu 16.04.4 LTS and uses MATLAB 9.1.0.441655 (R2016b).

5.1 Semi-Discretized Stokes Equation

First, the method is tested on a large-scale sparse example. The Stokes equation describes the flow of fluids at very low velocities without convection and coincides with the linearization of the Navier-Stokes equation around the zero-state. The spatial discretization of the Stokes equation by the finite volume method leads to a descriptor system of the form

$$\begin{aligned} \dot{v}_h(t) &= A_{11}v_h(t) + A_{12}p_h(t) + B_1u(t), \\ 0 &= A_{12}^T v_h(t) + B_2u(t), \\ y(t) &= C_1v_h(t) + C_2p_h(t), \end{aligned} \quad (43)$$

where v_h and p_h are the semi-discretized vectors of velocity and pressure, respectively, and the matrices B_1 , B_2 , C_1 , C_2 are all vectors. For matrix pencils like in (43) the spectral projectors P_ℓ and P_r are given by explicit construction formulas

$$\begin{aligned} P_\ell &= \begin{bmatrix} \Pi & -\Pi A_{11} A_{12} (A_{12}^T A_{12})^{-1} \\ 0 & 0 \end{bmatrix}, \\ P_r &= \begin{bmatrix} \Pi & 0 \\ -(A_{12}^T A_{12})^{-1} A_{12}^T A_{11} \Pi & 0 \end{bmatrix}, \end{aligned}$$

where $\Pi = I_{n_v} - A_{12}(A_{12}^T A_{12})^{-1} A_{12}^T$ is the orthogonal projector onto the kernel of A_{12}^T along the image of A_{12} ; see [20]. The generation of data is based on the test example 3.3 in [16]. The Stokes equation was discretized on a uniform staggered grid of 80×80 points which leads to a descriptor system of the size $n = 19,039$, where the matrix pencil $\lambda E - A$ has $n_f = 6,241$ finite and $n_\infty = 12,798$ infinite eigenvalues. The data was generated to get a full-rank A_{12} such that the system (43) is of index 2.

For the computation, the implementation of the GHNA method was adjusted to the sparse system case, as described in Section 4.2, and for the solution of the projected continuous-time Lyapunov equations (6) and (7), the solvers from version 1.0.1 of the M-M.E.S.S. toolbox have been used [15]. See the demo file `bt_mor_DAE2.m` in [15] for the used parameter settings. With these adjusted solvers, the two iterations for the low-rank factors quickly converged after 31 and 32 iteration steps as shown in Figure 1. An approximation of the non-zero proper Hankel singular values has been computed and plotted in Figure 2 using the low-rank factorizations of the proper system Gramians.

As mentioned before, it is numerically more stable using a balanced truncation of the slow subsystem than the minimal realization. For this reason, a tolerance for the allowed proper Hankel singular values was computed as $\log(n) \cdot \epsilon$ and multiplied with the largest proper Hankel singular value, with n the order of the system and ϵ the machine epsilon. The resulting bound is also shown in Figure 2 and the computed balanced realization is of order 21.

To compute a fourth order standard Hankel-norm approximation of the slow subsystem, the fifth proper Hankel singular value $\varsigma_5 = 1.8370 \cdot 10^{-6}$ was chosen. The additive decomposition of the

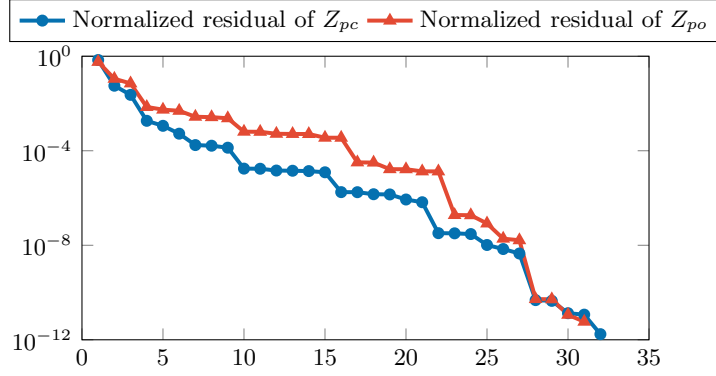


Figure 1: Convergence of the low-rank ADI iterations for the low-rank factorizations $\mathcal{G}_{pc} = Z_{pc}Z_{pc}^T$ and $\mathcal{G}_{po} = Z_{po}Z_{po}^T$ (Stokes example).

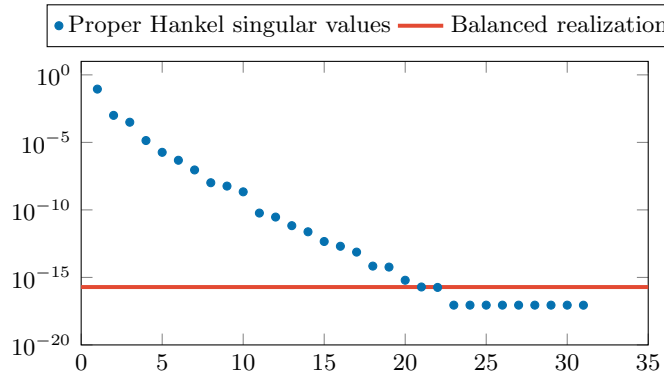


Figure 2: Computed proper Hankel singular values and the tolerance for the balanced realization (Stokes example).

transformed realization (12) was made by using the `ml_adtf_dss` routine from version 3.0 of the MORLAB toolbox [7]. The projected generalized discrete-time Lyapunov equations (8) and (9) were constructed as shown in Section 4.2. In contrast to the continuous-time case, every iteration step was reprojected since the iteration converges after 2 steps at maximum. As result only one non-zero improper Hankel singular value $\theta_1 = 5.3046 \cdot 10^{-18}$ was computed. This implies that the reduced-order system would be of index 1. In this case, the fast subsystem (33) is equivalent to a feed-through term of the form $-C_\infty B_\infty = -1.875 \cdot 10^{-17}$. Since this value is negligible small compared to the resulting feed-through term $\hat{D} = \zeta_5$ from the GHNA method, the state corresponding to this improper Hankel singular value was truncated, too.

To sum up, the original semi-discretized Stokes equation is approximated by a GHNA of order 4 ($r = 4, \ell_\infty = 0$). The error of the original and reduced-order transfer functions in the spectral norm can be seen in Figure 3. Additionally, the corresponding \mathcal{H}_∞ error bound as well as a reduced-order model of the same order computed by the GBT(SR) method are plotted to get an impression of the approximation behavior. The shown error behavior of the GHNA is typical. Since the reduced-order model is based on an all-pass error transfer function, the error behavior becomes nearly all-pass if the influence of the anti-stable part is negligible small. Also, the error of the GHNA approaches the chosen proper Hankel singular value ζ_5 , which is exactly the error of the approximation in the Hankel semi-norm.

Further examples and tests of the sparse implementation of the GHNA method can be found in [22].

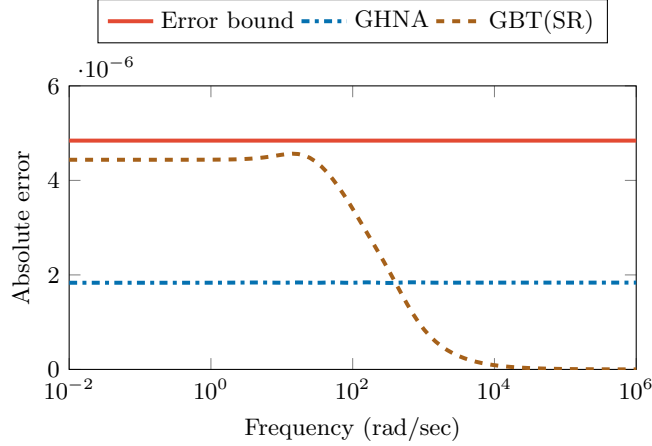


Figure 3: Absolute error of the GHNA and GBT(SR) transfer functions in the spectral norm with the corresponding error bound (Stokes example).

5.2 A Damped Mass-Spring System

As a second example, a damped mass-spring system with a holonomic constraint is considered here. The detailed construction of the system can be found in [14]. The vibrations of the resulting system are described by a system of second-order equations

$$\begin{aligned} M\ddot{p}(t) &= Kp(t) + D\dot{p}(t) - G^T\lambda(t) + B_u u(t), \\ 0 &= Gp(t), \\ y(t) &= C_p p(t), \end{aligned} \quad (44)$$

where $p(t)$ is the position vector, $\lambda(t) \in \mathbb{R}$ is the Lagrange multiplier, $K, D \in \mathbb{R}^{g \times g}$ are the tridiagonal stiffness and damping matrices, $M = \text{diag}(m_1, \dots, m_g)$ is the mass matrix and $G = [1, 0, \dots, 0, -1]$ is the constraint matrix. The input matrix is given by $B_u = e_1$ and three positions of masses are measured by $C_p = [e_1, e_2, e_{g-1}]^T$, where e_i is the i -th column of I_g .

For the application of the GHNA method, the system (44) has to be rewritten in first-order form. Therefore, the velocity vector $v(t) = \dot{p}(t)$ is introduced and all states are collected in $x(t) = [p(t)^T, v(t)^T, \lambda(t)]^T$, such that the system (44) can be rewritten in the form

$$\begin{aligned} \begin{bmatrix} I_g & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) &= \begin{bmatrix} 0 & I_g & 0 \\ K & D & -G^T \\ G & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ B_u \\ 0 \end{bmatrix} u(t), \\ y(t) &= [C_p \quad 0 \quad 0] x(t). \end{aligned} \quad (45)$$

This linearization is an index-3 descriptor system. The number of masses was chosen as $g = 1500$, which leads to $n = 3001$ states in the linearized system (45). For the computation of the GHNA, the `ml_hna_dss` method from version 3.0 of the MORLAB toolbox has been used [7]. In this function, the projection-free approach from Section 4.3 is implemented as mentioned there. For the computation of the additive decompositions, the right matrix pencil disk function is used and the generalized Lyapunov equations are solved via the matrix sign function method; see, for example, [3] and [4]. More details on handling descriptor systems with the MORLAB toolbox can be found in [9]. The computed proper Hankel singular values and the used bound for the minimal realization of the system can be seen in Figure 4.

The computed reduced-order model is of order 6 ($r = 6, \ell_\infty = 0$). So also in this case, the reduced-order model does not contain algebraic constraints anymore, which means the \hat{E} matrix is regular. The absolute error of the GHNA is plotted in Figure 5 with the corresponding \mathcal{H}_∞ error bound and the error of the GBT(SR) reduced-order model for comparison.

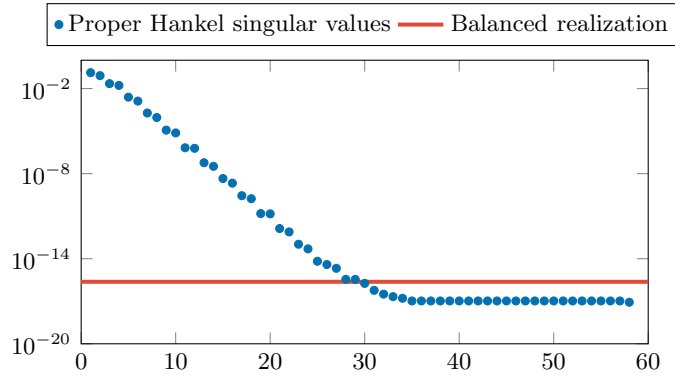


Figure 4: Computed proper Hankel singular values and the tolerance for the balanced realization (mass-spring example).

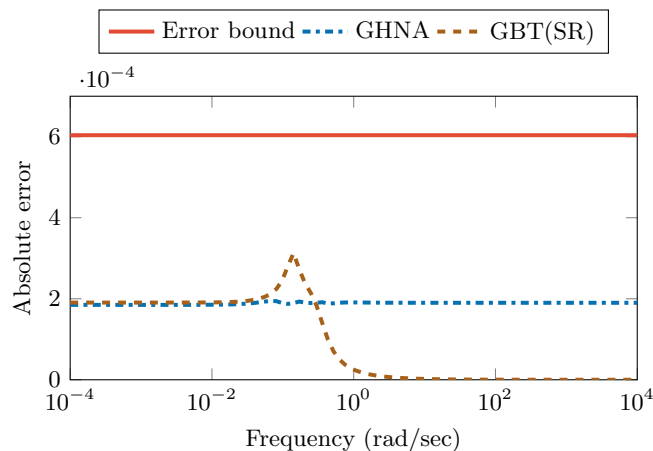


Figure 5: Absolute error of the GHNA and GBT(SR) transfer functions in the spectral norm with the corresponding error bound (mass-spring example).

6 Conclusion

An algebraic characterization of descriptor systems with all-pass transfer function was proven and based on this explanation, an efficient algorithm for the computation of the generalized Hankel-norm approximation was developed by exploiting the generalized balanced truncation square root method. To get a numerically more stable algorithm, an approximate version of the Hankel-norm approximation was introduced. For an efficient practical usage, the introduced method was considered for sparse large-scale systems as well as for unstructured dense systems. The approximation behavior of the method was shown on large- and medium-scale examples.

In contrast to the approach of Cao, Saltik, and Weiland [10], the method, introduced in this paper, has several numerical advantages. It has a more stable and efficient computational behavior, due to the fact that the Weierstrass canonical form does not have to be computed. Also, the introduced method can be applied to more general descriptor systems since C-controllability and C-observability were not assumed.

Acknowledgment

This work was supported by the German Research Foundation (DFG) priority program 1897: “Calm, Smooth and Smart – Novel Approaches for Influencing Vibrations by Means of Deliberately Introduced Dissipation” and the German Research Foundation (DFG) research training group 2297 “MathCoRe”, Magdeburg.

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Appendix

For the proof of Theorem 1, the following lemma is used.

Lemma 1. *Each all-pass transfer function $G(s) \in \mathbb{C}^{m \times m}$ is proper.*

Proof. From the definition of all-pass transfer functions it follows that the product

$$G(s)G^T(-s) = I_m$$

has to be proper.

Improper Case:

First, let’s assume that G is an improper transfer function. The entries of $G(s)$ are rational polynomials with real coefficients. Since G is improper at least one entry of G must have a higher numerator polynomial degree than the denominator. Also, one can observe that for $G^T(-s)$ the entries of the matrix are only transposed and coefficients of odd polynomial order change their signs. In the single-input single-output (SISO) case $G(s)$ is given by

$$G(s) = \frac{n(s)}{d(s)},$$

with $\deg(n) > \deg(d)$. Let the numerator and denominator of the para-Hermitian function be denoted by $\tilde{n}(s)$ and $\tilde{d}(s)$. In this case, it is obvious that for the product it holds

$$2 \cdot \deg(n) = \deg(n\tilde{n}) > \deg(\tilde{d}\tilde{d}) = 2 \cdot \deg(d).$$

So the product is always improper.

In the multi-input multi-output (MIMO) case, it is assumed for simplicity that $m = 2$ and that the denominator is equal for all entries and can be factored out such that

$$G(s) = \frac{1}{d(s)} \begin{bmatrix} n_{11}(s) & n_{12}(s) \\ n_{21}(s) & n_{22}(s) \end{bmatrix} \quad \text{and} \quad G^T(-s) = \frac{1}{\tilde{d}(s)} \begin{bmatrix} \tilde{n}_{11}(s) & \tilde{n}_{21}(s) \\ \tilde{n}_{12}(s) & \tilde{n}_{22}(s) \end{bmatrix}.$$

The resulting product is then

$$G(s)G^T(-s) = \frac{1}{d(s)\tilde{d}(s)} \begin{bmatrix} n_{11}(s)\tilde{n}_{11}(s) + n_{12}\tilde{n}_{12} & n_{11}(s)\tilde{n}_{21}(s) + n_{12}\tilde{n}_{22} \\ n_{21}(s)\tilde{n}_{11}(s) + n_{22}\tilde{n}_{12} & n_{21}(s)\tilde{n}_{21}(s) + n_{22}\tilde{n}_{22} \end{bmatrix}.$$

If only one of the product entries would have a higher polynomial degree than the denominator the argumentation from the SISO case would follow. Therefore, we can assume w.l.o.g. that

$$\deg(n_{11}) = \deg(n_{12}) = \deg(d) + 1 = g + 1.$$

We concentrate on the $(1, 1)$ entry of the matrix product. For the resulting polynomial degrees it holds

$$\begin{aligned}\deg(n_{11}\tilde{n}_{11}) &= 2g + 2 \\ \deg(n_{12}\tilde{n}_{12}) &= 2g + 2 \\ \deg(n_{11}\tilde{n}_{11} + n_{12}\tilde{n}_{12}) &\leq 2g + 2 \\ \deg(d\tilde{d}) &= 2g\end{aligned}$$

To get a proper product transfer function, we need that the two highest coefficients in $n_{11}\tilde{n}_{11} + n_{12}\tilde{n}_{12}$ cancel out. If we now develop the polynomials with

$$\begin{aligned}n_{11}(s) &= \sum_{k=0}^{g+1} n_{11,k} s^k, & n_{12}(s) &= \sum_{k=0}^{g+1} n_{12,k} s^k, \\ \tilde{n}_{11}(s) &= \sum_{k=0}^{g+1} \tilde{n}_{11,k} s^k, & \tilde{n}_{12}(s) &= \sum_{k=0}^{g+1} \tilde{n}_{12,k} s^k,\end{aligned}$$

we get that for the first coefficients

$$n_{11,g+1}\tilde{n}_{11,g+1} = -n_{12,g+1}\tilde{n}_{12,g+1}$$

has to hold, with $|n_{11,g+1}| = |\tilde{n}_{11,g+1}|$ and $|n_{12,g+1}| = |\tilde{n}_{12,g+1}|$. Now, if $g + 1$ is even we get

$$n_{11,g+1} = \tilde{n}_{11,g+1}, \quad n_{12,g+1} = \tilde{n}_{12,g+1} \quad \Rightarrow \quad n_{11,g+1}^2 = -\tilde{n}_{12,g+1}^2,$$

and if $g + 1$ is odd

$$n_{11,g+1} = -\tilde{n}_{11,g+1}, \quad n_{12,g+1} = -\tilde{n}_{12,g+1}, \quad \Rightarrow \quad -n_{11,g+1}^2 = \tilde{n}_{12,g+1}^2.$$

Both cases are a contradiction to the condition that the coefficients are real and non-zero. Therefore, an all-pass transfer function cannot be improper.

Strictly Proper Case: Now, let's assume that G is a strictly proper transfer function. Using the same argumentation as in the improper case, we get that the product of a strictly proper transfer function with its para-Hermitian is also strictly proper. \square

Now, Theorem 1 can be proven.

Proof. At first, we can assume w.l.o.g. that $\varsigma = 1$, since the system can be scaled to that case by $\tilde{B} = \varsigma^{-\frac{1}{2}}B$, $\tilde{C} = \varsigma^{-\frac{1}{2}}C$ and $\tilde{D} = \varsigma^{-1}D$.

" \Rightarrow ":

Assume the transfer function $G(s)$ is all-pass. With the previous lemma it follows that $G(s)$ has to be proper. If we consider now the decomposition of the transfer function into its strictly proper and polynomial part $G(s) = G_{sp}(s) + P(s)$, the polynomial one must satisfy

$$P(s) = \sum_{k=1}^{\infty} M_k s^k,$$

with $M_k = 0$ for all $k \geq 1$. In this case, it holds

$$\lim_{s \rightarrow \infty} G(s) = M_0,$$

and with the definition of all-pass transfer functions we get

$$M_0 M_0^T = G(s) G^T(-s) = I_m.$$

So the expressions (28) and (29) hold. Since the matrix pencil $\lambda E - A$ is assumed to be regular, there are non-singular matrices $Q, Z \in \mathbb{R}^{n \times n}$, which transform the matrix pencil into the following block diagonal structure

$$Q(\lambda E - A)Z = \lambda \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} - \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix},$$

where $\lambda E_f - A_f$ contains all the finite eigenvalues of $\lambda E - A$ and $\lambda E_\infty - A_\infty$ contains only infinite eigenvalues. These transformation matrices can be used on the complete system as a restricted system equivalence transformation

$$\begin{aligned} & (QEZ, QAZ, QB, CZ, D) \\ &= \left(\begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix}, \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix}, \begin{bmatrix} B_f \\ B_\infty \end{bmatrix}, [C_f \ C_\infty], D \right). \end{aligned} \quad (46)$$

This system decouples into its slow

$$\begin{aligned} E_f \dot{x}_f(t) &= A_f x_f(t) + B_f u(t), \\ y_f(t) &= C_f x_f(t), \end{aligned}$$

and fast subsystem

$$\begin{aligned} E_\infty \dot{x}_\infty(t) &= A_\infty x_\infty(t) + B_\infty u(t), \\ y_\infty(t) &= C_\infty x_\infty(t) + Du(t). \end{aligned}$$

The slow subsystem corresponds to the strictly proper part of the transfer function and the fast subsystem to the polynomial part. Then, the constant part of the transfer function is then given by

$$M_0 = D - C_\infty A_\infty^{-1} B_\infty,$$

and the transfer function $G(s)$ has another realization of the form

$$G(s) = C_f (sE_f - A_f)^{-1} B_f + M_0,$$

with invertible matrix E_f .

Now, we can use the definition of an all-pass transfer function, i.e., $G(s)G^T(-s) = I_m$ to get the relation $G^{-1}(s) = G^T(-s)$, which can be written as

$$\begin{aligned} G^{-1}(s) &= M_0^{-1} - M_0^{-1} C_f (sE_f - A_f + B_f M_0^{-1} C_f)^{-1} B_f M_0^{-1} \\ &= G^T(-s) \\ &= M_0^T + B_f^T (-sE_f^T - A_f^T)^{-1} C_f^T. \end{aligned}$$

The equality $M_0^{-1} = M_0^T$ was already proven above. From the R -controllability and R -observability assumption together with the regularity of E_f , it follows that there exist invertible matrices $T, W \in \mathbb{R}^{n \times n}$ which transform the realization of the inverse transfer function into the realization of the para-Hermitian one, with

$$E_f^T = W E_f T, \quad (47)$$

$$-A_f^T = W (A_f - B_f M_0^T C_f) T, \quad (48)$$

$$C_f^T = W B_f M_0^T, \quad (49)$$

$$B_f^T = M_0^T C_f T. \quad (50)$$

Now, these expressions can be reformulated. From (49) we obtain

$$\begin{aligned} & C_f^T = W B_f M_0^T \\ \iff & W^{-1} C_f^T = B_f M_0^T \\ \iff & W^{-1} C_f^T M_0 = B_f \\ \iff & B_f^T = M_0^T C_f W^{-T}. \end{aligned}$$

From (50) we get

$$\begin{aligned}
& B_f^T = M_0^T C_f T \\
\iff & B_f^T T^{-1} = M_0^T C_f \\
\iff & M_0 B_f^T T^{-1} = C_f \\
\iff & C_f^T = T^{-T} B_f M_0^T.
\end{aligned}$$

The equation (48) can be reformulated as

$$\begin{aligned}
& -A_f^T = W(A_f - B_f M_0^T C_f) T \\
\iff & -W^{-1} A_f^T T^{-1} = A_f - B_f M_0^T C_f \\
\iff & A_f = -W^{-1} A_f^T T^{-1} + B_f M_0^T C_f \\
\iff & -A_f^T = T^{-T} A_f W^{-T} - C_f^T M_0 B_f^T \\
& = T^{-T} (A_f - B_f M_0^T C_f) W^{-T}.
\end{aligned}$$

And as last one, for (47) it holds

$$\begin{aligned}
& E_f^T = W E_f T \\
\iff & W^{-1} E_f^T T^{-1} = E_f \\
\iff & E_f^T = T^{-T} E_f W^{-T}.
\end{aligned}$$

Therefor, T and W^{-T} as well as T^{-1} and W^T satisfy the same set of equations, which means that $W = T^{-T}$. Using this, the expressions (47)–(50) are equivalent to

$$E_f^T = T^{-T} E_f T, \quad (51)$$

$$-A_f^T = T^{-T} (A_f - B_f M_0^T C_f) T, \quad (52)$$

$$C_f^T = T^{-T} B_f M_0^T, \quad (53)$$

$$B_f^T = M_0^T C_f T. \quad (54)$$

The expressions (51), (52) and (54) give the T as solution of the following system of matrix equations

$$\begin{aligned}
A_f T + T^T A_f^T - B_f B_f^T &= 0, \\
E_f T &= T^T E_f^T.
\end{aligned}$$

By setting the symmetric matrix $\tilde{G}_{pc} = -T E_f^{-T} = -E_f^{-1} T^T$ the equation system can be rewritten as

$$A_f \tilde{G}_{pc} E_f^T + E_f \tilde{G}_{pc} A_f^T + B_f B_f^T = 0. \quad (55)$$

Analogously, it follows

$$A_f^T \tilde{G}_{po} E_f + E_f^T \tilde{G}_{po} A_f + C_f^T C_f = 0. \quad (56)$$

with the symmetric matrix $\tilde{G}_{po} = -T^{-1} E_f^{-1} = -E_f^{-T} T^{-T}$.

For the matrices \tilde{G}_{pc} and \tilde{G}_{po} , the following matrix product is considered

$$\tilde{G}_{pc} E_f^T \tilde{G}_{po} E_f = (-T E_f^{-T}) E_f^T (-T^{-1} E_f^{-1}) E_f = T T^{-1} = I_{n_f}, \quad (57)$$

and also

$$\tilde{G}_{po} E_f \tilde{G}_{pc} E_f^T = (-T^{-1} E_f^{-1}) E_f (-T E_f^{-T}) E_f^T = T T^{-1} = I_{n_f}. \quad (58)$$

Additionally, from (53) it follows

$$\begin{aligned}
& T^T C_f^T = B_f M_0^T \\
\iff & M_0 B_f^T - C_f T = 0 \\
\iff & M_0 B_f^T + C_f G_{pc} E_f^T = 0,
\end{aligned} \tag{59}$$

and from (54) we obtain

$$M_0^T C_f + B_f^T \tilde{G}_{po} E_f = 0. \tag{60}$$

With the last step all conditions are satisfied on the realization with invertible matrix E_f . In the next step, the original dimension of the system has to be rebuilt by using the block diagonal structure (46). This is done by applying appropriate spectral projectors of the deflating subspaces corresponding to the finite eigenvalues of $\lambda E - A$. In case of a system in the form (46), the left and right spectral projectors are given by

$$\tilde{P}_\ell = \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{P}_r = \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix}.$$

Since the matrices \tilde{G}_{pc} and \tilde{G}_{po} are only determined by the system parts corresponding to the finite eigenvalues, they have to be expended accordingly to the spectral projectors by

$$\tilde{G}_{pc} \rightarrow \begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \tilde{G}_{po} \rightarrow \begin{bmatrix} \tilde{G}_{po} & 0 \\ 0 & 0 \end{bmatrix}.$$

Using this, the equation (55) is equivalent to

$$\begin{aligned}
& \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix} \begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix}^T + \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} \begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix}^T \\
& + \tilde{P}_\ell \begin{bmatrix} B_f \\ B_\infty \end{bmatrix} \begin{bmatrix} B_f \\ B_\infty \end{bmatrix}^T \tilde{P}_\ell^T = 0
\end{aligned} \tag{61}$$

and equation (56) to

$$\begin{aligned}
& \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix}^T \begin{bmatrix} \tilde{G}_{po} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} + \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix}^T \begin{bmatrix} \tilde{G}_{po} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix} \\
& + \tilde{P}_r^T [C_f \ C_\infty]^T [C_f \ C_\infty] \tilde{P}_r = 0.
\end{aligned} \tag{62}$$

Also, the matrix product in (57) becomes

$$\begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix}^T \begin{bmatrix} \tilde{G}_{po} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} = \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix},$$

as well as (58) with

$$\begin{bmatrix} \tilde{G}_{po} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} \begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix}^T = \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix}.$$

The constraint (60) becomes

$$M_0 \begin{bmatrix} B_f \\ B_\infty \end{bmatrix}^T \tilde{P}_\ell^T + [C_f \ C_\infty] \begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix}^T = 0,$$

and (59) is equivalent to

$$M_0^T [C_f \ C_\infty] \tilde{P}_r + \begin{bmatrix} B_f \\ B_\infty \end{bmatrix}^T \begin{bmatrix} \tilde{G}_{po} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} = 0.$$

As last step in this part, the realization has to be back-transformed into the original one. By multiplying (61) from the left with Q^{-1} and from the right with Q^{-T} we get

$$\begin{aligned}
& Q^{-1} \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix} \begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix}^T Q^{-T} \\
& + Q^{-1} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} \begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix}^T Q^{-T} \\
& + Q^{-1} \tilde{P}_\ell \begin{bmatrix} B_f \\ B_\infty \end{bmatrix} \begin{bmatrix} B_f \\ B_\infty \end{bmatrix}^T \tilde{P}_\ell^T Q^{-T} \\
& = Q^{-1} \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix} Z^{-1} Z \begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} Z^T Z^{-T} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix}^T Q^{-T} \\
& + Q^{-1} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} Z^{-1} Z \begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} Z^T Z^{-T} \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix}^T Q^{-T} \\
& + Q^{-1} \tilde{P}_\ell Q Q^{-1} \begin{bmatrix} B_f \\ B_\infty \end{bmatrix} \begin{bmatrix} B_f \\ B_\infty \end{bmatrix}^T Q^{-T} Q^{-T} \tilde{P}_\ell^T Q^{-T} \\
& = A \mathcal{G}_{pc} E^T + E \mathcal{G}_{pc} A^T + P_\ell B B^T P_\ell^T \\
& = 0,
\end{aligned}$$

with the spectral projection

$$P_\ell = Q^{-1} \tilde{P}_\ell Q = Q^{-1} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} Q,$$

and the symmetric matrix

$$\mathcal{G}_{pc} = Z \begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} Z^T.$$

These are the conditions (22) and (24). Analogously, the conditions (23) and (25) can be shown by multiplying (62) with Z^{-T} from the left and Z^{-1} from the right. For the condition (22) it holds

$$\begin{aligned}
P_r \mathcal{G}_{pc} P_r^T & = Z \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} Z^{-1} Z \begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} Z^T Z^{-T} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} Z^T \\
& = Z \begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} Z^T \\
& = \mathcal{G}_{pc},
\end{aligned}$$

and for (23)

$$\begin{aligned}
P_\ell^T \mathcal{G}_{po} P_\ell & = Q^T \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} Q^{-T} Q^T \begin{bmatrix} \tilde{G}_{po} & 0 \\ 0 & 0 \end{bmatrix} Q Q^{-1} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} Q \\
& = Q^T \begin{bmatrix} \tilde{G}_{po} & 0 \\ 0 & 0 \end{bmatrix} Q \\
& = \mathcal{G}_{po}.
\end{aligned}$$

The condition (26) for the proper Hankel singular values is then

$$\begin{aligned}
\mathcal{G}_{pc}E^T\mathcal{G}_{po}E &= Z \begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} Z^T Z^{-T} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix}^T Q^{-T} \\
&\quad \times Q^T \begin{bmatrix} \tilde{G}_{po} & 0 \\ 0 & 0 \end{bmatrix} Q Q^{-1} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} Z^{-1} \\
&= Z \begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix}^T \begin{bmatrix} \tilde{G}_{po} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} Z^{-1} \\
&= Z \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} Z^{-1} \\
&= P_r,
\end{aligned}$$

and also the dual condition (27) can be shown this way. For the additional constraint (30) it holds

$$\begin{aligned}
M_0^T C P_r + B^T \mathcal{G}_{po} E &= M_0^T \begin{bmatrix} C_f & C_\infty \end{bmatrix} Z^{-1} Z \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} Z^{-1} \\
&\quad + \begin{bmatrix} B_f \\ B_\infty \end{bmatrix}^T Q^{-T} Q^T \begin{bmatrix} \tilde{G}_{po} & 0 \\ 0 & 0 \end{bmatrix} Q Q^{-1} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} Z^{-1} \\
&= \left(M_0^T \begin{bmatrix} C_f & C_\infty \end{bmatrix} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} \right. \\
&\quad \left. + \begin{bmatrix} B_f \\ B_\infty \end{bmatrix}^T \begin{bmatrix} \tilde{G}_{po} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} \right) Z^{-1} \\
&= 0,
\end{aligned}$$

and for the dual one (31) it works the same. Hence, all conditions of the characterization are fulfilled.

" \Leftarrow ":

Now, it is assumed that the conditions (22)–(31) hold. It has to be shown that the resulting transfer function of the linear descriptor system is all-pass. Therefor, a reformulation of (24) is considered

$$\begin{aligned}
P_\ell B B^T P_\ell^T &= -A \mathcal{G}_{pc} E^T - E \mathcal{G}_{pc} A^T \\
&= -A \mathcal{G}_{pc} E^T - E \mathcal{G}_{pc} A^T + s E \mathcal{G}_{pc} E^T - s E \mathcal{G}_{pc} E^T \\
&= (sE - A) \mathcal{G}_{pc} E^T + E \mathcal{G}_{pc} (-sE^T - A^T)
\end{aligned}$$

The right-hand side of this expression shall be transformed into the form of a transfer function and its para-Hermitian. It holds

$$\begin{aligned}
&(sE - A)^{-1} P_\ell B B^T P_\ell^T (-sE^T - A^T)^{-1} \\
&= \mathcal{G}_{pc} E^T (-sE^T - A^T)^{-1} + (sE - A)^{-1} E \mathcal{G}_{pc} \\
\Rightarrow &C P_r (sE - A)^{-1} P_\ell B B^T P_\ell^T (-sE^T - A^T)^{-1} P_r^T C^T \\
&= C P_r \mathcal{G}_{pc} E^T (-sE^T - A^T)^{-1} P_r^T C^T + C P_r (sE - A)^{-1} E \mathcal{G}_{pc} P_r^T C^T.
\end{aligned}$$

In those parts with the symmetric matrix \mathcal{G}_{pc} , there is an additional spectral projector. Using the definition of P_r and \mathcal{G}_{pc} from the previous direction one obtains

$$\begin{aligned}
P_r \mathcal{G}_{pc} &= Z \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} Z^{-1} Z \begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} Z^T \\
&= Z \begin{bmatrix} \tilde{G}_{pc} & 0 \\ 0 & 0 \end{bmatrix} Z^T \\
&= \mathcal{G}_{pc}.
\end{aligned}$$

Hence, it holds

$$\begin{aligned} & CP_r(sE - A)^{-1}P_\ell BB^T P_\ell^T (-sE^T - A^T)^{-1}P_r^T C^T \\ &= CG_{pc}E^T(-sE^T - A^T)^{-1}P_r^T C^T + CP_r(sE - A)^{-1}EG_{pc}C^T. \end{aligned}$$

Now, the additional constraint (31) leads to

$$\begin{aligned} & CP_r(sE - A)^{-1}P_\ell BB^T P_\ell^T (-sE^T - A^T)^{-1}P_r^T C^T \\ &= -M_0 B^T P_\ell^T (-sE^T - A^T)^{-1}P_r^T C^T - CP_r(sE - A)^{-1}P_\ell B M_0^T \end{aligned}$$

and, inserting the definition of the spectral projectors, we get on the left-hand side

$$\begin{aligned} & CP_r(sE - A)^{-1}P_\ell BB^T P_\ell^T (-sE^T - A^T)^{-1}P_r^T C^T \\ &= CZ \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} Z^{-1}(sE - A)^{-1}Q^{-1} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} QB \\ &\quad \times B^T Q^T \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} Q^{-T}(-sE^T - A^T)^{-1}Z^{-T} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} Z^T C^T \\ &= CZ \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} (sQEZ - QAZ)^{-1} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} QB \\ &\quad \times B^T Q^T \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} (-sZ^T E^T Q^T - Z^T A^T Q^T)^{-1} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} Z^T C^T \\ &= [C_f \quad C_\infty] \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} \left(s \begin{bmatrix} E_f & 0 \\ 0 & E_\infty \end{bmatrix} - \begin{bmatrix} A_f & 0 \\ 0 & A_\infty \end{bmatrix} \right)^{-1} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_f \\ B_\infty \end{bmatrix} \\ &\quad \times [B_f^T \quad B_\infty^T] \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} \left(-s \begin{bmatrix} E_f^T & 0 \\ 0 & E_\infty^T \end{bmatrix} \begin{bmatrix} A_f^T & 0 \\ 0 & A_\infty^T \end{bmatrix} \right)^{-1} \begin{bmatrix} I_{n_f} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_f^T \\ C_\infty^T \end{bmatrix} \\ &= (C_f(sE_f - A_f)^{-1}B_f + 0 \cdot (sE_\infty - A_\infty)^{-1} \cdot 0) \\ &\quad \times (B_f^T(-sE_f^T - A_f^T)^{-1}C_f^T + 0 \cdot (-sE_\infty^T - A_\infty^T)^{-1} \cdot 0) \\ &= C_f(sE_f - A_f)^{-1}B_f B_f^T (-sE_f^T - A_f^T)^{-1}C_f^T. \end{aligned}$$

For the right hand-side it holds

$$\begin{aligned} & -M_0 B^T P_\ell^T (-sE^T - A^T)^{-1}P_r^T C^T - CP_r(sE - A)^{-1}P_\ell B M_0^T \\ &= -M_0 B_f^T (-sE_f^T - A_f^T)^{-1}C_f^T - C_f(sE_f - A_f)^{-1}B_f M_0^T. \end{aligned}$$

Using the above expressions, the all-pass condition is satisfied

$$\begin{aligned} G(s)G^T(-s) &= (C(sE - A)^{-1}B + D)(B^T(sE^T - A^T)^{-1}C^T + D^T) \\ &= (C_f(sE_f - A_f)^{-1}B_f + M_0)(B_f^T(sE_f^T - A_f^T)^{-1}C_f^T + M_0^T) \\ &= C_f(sE_f - A_f)^{-1}B_f B_f^T (sE_f^T - A_f^T)^{-1}C_f^T \\ &\quad + M_0 B_f^T (sE_f^T - A_f^T)^{-1}C_f^T + C_f(sE_f - A_f)^{-1}B_f M_0^T + M_0 M_0^T \\ &= M_0 M_0^T \\ &= I_m. \end{aligned}$$

□