Towards the n-point one-loop superstring amplitude II: Worldsheet functions and their duality to kinematics

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This is the second installment of a series of three papers in which we describe a method to determine higher-point correlation functions in one-loop open-superstring amplitudes from first principles. In this second part, we study worldsheet functions defined on a genus-one surface built from the coefficient functions of the Kronecker–Einsenstein series. We construct two classes of worldsheet functions whose properties lead to several simplifying features within our description of one-loop correlators with the pure-spinor formalism. The first class is described by functions with prescribed monodromies, whose characteristic shuffle-symmetry property leads to a Lie-polynomial structure when multiplied by the local superfields from part I of this series. The second class is given by so-called generalized elliptic integrands (GEIs) that are constructed using the same combinatorial patterns of the BRST pseudo-invariant superfields from part I. Both of them lead to compact and combinatorially rich expressions for the correlators in part III. The identities obeyed by the two classes of worldsheet functions exhibit striking parallels with those of the superfield kinematics. We will refer to this phenomenon as a duality between worldsheet functions and kinematics.

December 2018

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1. Introduction

This is the second part of a series of papers [1] (henceforth referred to as part I, II and III) in the quest of deriving the one-loop correlators of massless open- and closed-superstring states using the pure-spinor formalism [2,3]. As detailed in the introduction of part I, the goal of these papers is to determine the correlators from first principles including gauge invariance, supersymmetry, locality and single-valuedness. The present work is dedicated to the implication of single-valuedness on how the correlators may depend meromorphically on the punctures on a genus-one worldsheet. The key results are the following

i) We present a bootstrap program to construct worldsheet functions for the correlators that share the differential structure and relations of their superspace kinematics. These parallels will be referred to as a duality between kinematics and worldsheet functions, and they endow one-loop amplitudes of the open superstring with a double-copy structure [4].

ii) We establish the notion of generalized elliptic integrands (GEIs) which mirror the combinatorics of BRST invariant kinematic factors in the spirit of the duality between kinematics and worldsheet functions.

These results will come to fruition in the assembly of one-loop correlators in part III, also see appendix C for their representation that manifests their double-copy structure. Since we will often refer to section and equation numbers from the papers I and III, these numbers will be prefixed by the roman numerals I and III accordingly.

2. Worldsheet functions at one loop

This section introduces the elementary worldsheet functions used in part III as building blocks of multiparticle genus-one amplitudes. These functions are meromorphic and defined as the coefficients of a recent expansion [5] of the classical Kronecker–Eisenstein series [6,7]. They are quasi-periodic under $z \to z + \tau$ and therefore live on the universal cover of an elliptic curve.

However, our goal is to study string scattering amplitudes that require functions on an elliptic curve. For this purpose, we will later on consider meromorphic functions defined on an enlarged space parameterized by the standard vertex-insertion coordinates z_i and the *loop momentum* ℓ^m (with vector indices $m, n, p, \ldots = 0, 1, \ldots, 9$ of the ten-dimensional

Lorentz group). Following the chiral-splitting formalism [8,9,10], ℓ^m represents certain zero modes associated with the worldsheet field $x^m(z,\overline{z})$, cf. (I.2.24). The interplay between z_j and ℓ^m will then lead to the definition of *generalized elliptic integrands* (GEIs) [4], which become doubly-periodic under $z \to z+1$ and $z \to z+\tau$ upon integration of loop momenta. The properties and explicit construction of GEIs will be the subject of the subsequent discussions.

As reviewed in more detail in section I.2.2, chiral splitting allows to derive open- and closed-string amplitudes from the same function $\mathcal{K}_n(\ell)$ of the kinematic data. Open string n-point amplitudes at one loop descend from worldsheets of cylinder- and Moebius-strip topologies with punctures z_j on the boundary,

$$\mathcal{A}_n = \sum_{\rm top} C_{\rm top} \int_{D_{\rm top}} d\tau \, dz_2 \, dz_3 \, \ldots \, dz_n \int d^D\ell \; |\mathcal{I}_n(\ell)| \, \langle \mathcal{K}_n(\ell) \rangle \, , \tag{2.1}$$

see [11] for the integration domains $D_{\rm top}$ and the associated color factors $C_{\rm top}$. Closed-string one-loop amplitudes in turn are given by

$$\mathcal{M}_n = \int_{\mathcal{F}} d^2\tau \, d^2z_2 \, d^2z_3 \, \ldots \, d^2z_n \int d^D\ell \; |\mathcal{I}_n(\ell)|^2 \, \langle \mathcal{K}_n(\ell) \rangle \, \langle \tilde{\mathcal{K}}_n(-\ell) \rangle \, , \tag{2.2}$$

where $\mathcal{F}$ denotes the fundamental domain for the modular parameters τ of the torus worldsheet. As a universal part of the underlying correlation functions, both (2.1) and (2.2) involve the Koba–Nielsen factor (with $s_{ij} \equiv k_i \cdot k_j$ and conventions where $2\alpha' = 1$ for open and $\alpha' = 2$ for closed strings)

$$\mathcal{I}_n(\ell) \equiv \exp\Big(\sum_{i<j}^{n} s_{ij} \log \theta_1(z_{ij},\tau) + \sum_{j=1}^{n} z_j (\ell \cdot k_j) + \frac{\tau}{4\pi i} \ell^2 \Big) \, . \tag{2.3}$$

The leftover factors of $\mathcal{K}_n(\ell)$ in the loop integrands carry the dependence on the superspace polarizations and are referred to as *correlators*, see part III for their construction. The brackets $\langle \ldots \rangle$ in the above integrands denote the zero-mode integration of the spinor variables λ^α and θ^α of the pure-spinor formalism [2], and the odd Jacobi theta function in (2.3) is defined by ($q \equiv e^{2\pi i \tau}$)

$$\theta_1(z,\tau) \equiv 2 q^{1/8} \sin(\pi z) \prod_{n=1}^{\infty} (1-q^n) \big(1 - q^n e^{2\pi i z}\big) \big(1 - q^n e^{-2\pi i z}\big) \, . \tag{2.4}$$

Note that the open-string worldsheets relevant to (2.1) can be obtained from a torus via suitable involutions [12,12], that is why the subsequent periodicity requirements will be tailored to the torus topology.

2.1. The Kronecker–Eisenstein series

Our starting point to describe the dependence of the correlators $\mathcal{K}_n(\ell)$ on the worldsheet punctures is the Kronecker–Eisenstein series $F(z,\alpha,\tau)$ [6,7]. Its Laurent series in the second variable defines meromorphic functions $g^{(n)}(z,\tau)$ [5],

$$F(z,\alpha,\tau) \equiv \frac{\theta_1'(0,\tau)\theta_1(z+\alpha,\tau)}{\theta_1(\alpha,\tau)\theta_1(z,\tau)} \equiv \sum_{n=0}^{\infty} \alpha^{n-1} g^{(n)}(z,\tau) \ . \tag{2.5}$$

The simplest instances of these functions are $g^{(0)}(z,\tau)=1$ and ($\partial \equiv \frac{\partial}{\partial z}$)

$$g^{(1)}(z,\tau) = \partial \log \theta_1(z,\tau) \ , \qquad g^{(2)}(z,\tau) = \frac{1}{2}\Big[(\partial \log \theta_1(z,\tau))^2 - \wp(z,\tau)\Big] \, , \tag{2.6}$$

where $\wp(z,\tau) = -\partial^2 \log\theta_1(z,\tau) - \mathrm{G}_2(\tau)$ is the Weierstrass function and $\mathrm{G}_{2k}(\tau)$ denotes the holomorphic Eisenstein series[1]

$$\mathrm{G}_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}} \frac{1}{(m\tau+n)^{2k}} = -g^{(2k)}(0,\tau) \, . \tag{2.7}$$

See the appendix B for the explicit expansions of $g^{(n)}(z,\tau)$ for $n\leq 5$ in terms of Jacobi theta functions.

It is important to note that the function $g^{(1)}(z,\tau)$ has a simple pole $\sim \frac{1}{z}$ at the origin while all $g^{(n)}(z,\tau)$ for $n\geq 2$ are non-singular[2] as $z\to 0$. Furthermore, the heat equation $4\pi i \partial_\tau \theta_1(z,\tau) = \partial^2\theta_1(z,\tau)$ implies that

$$\frac{\partial}{\partial\tau} \log\theta_1(z,\tau) = \frac{1}{2\pi i}\Big\{ g^{(2)}(z,\tau) - \frac{1}{2}\mathrm{G}_2(\tau)\Big\} \, . \tag{2.8}$$

Similarly, one can obtain the τ-derivatives of the above $g^{(n)}$ from the mixed heat equation

$$\frac{\partial}{\partial\tau} F(z,\alpha,\tau) = \frac{1}{2\pi i}\frac{\partial^2 F(z,\alpha,\tau)}{\partial z\,\partial\alpha} \, , \qquad \frac{\partial}{\partial \tau} g^{(n)}(z,\tau) = \frac{n}{2\pi i}\,\partial g^{(n+1)}(z,\tau) \, , \tag{2.9}$$

and these relations will be instrumental when analyzing boundary terms with respect to τ in one-loop correlators later on.

2.1.1. Monodromies of the $g^{(n)}$-functions

[1] Note that the lattice-sum representation (2.7) of G_2 is not absolutely convergent and requires the specification of a summation prescription $\mathrm{G}_2(\tau) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{n^2} + \sum_{m\in\mathbb{Z}\setminus\{0\}}\sum_{n\in\mathbb{Z}} \frac{1}{(m\tau+n)^2}$.

[2] Note, however, that $g^{(k)}(z,\tau)$ for $k\geq 2$ have a simple pole at $z=\tau$ and in fact at all lattice points $z=m\tau+n$ with $m,n\in\mathbb{Z}$ and $m\neq 0$.

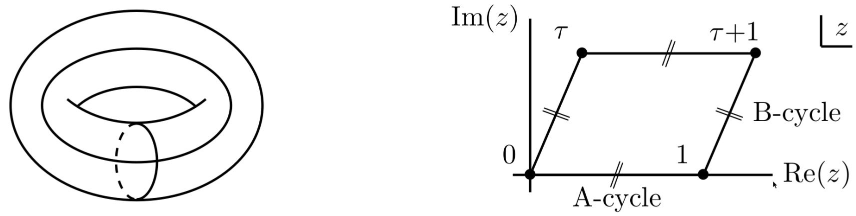

Fig. 1 Parameterization of the torus through the lattice $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with an identification of points z with their translates $z{+}1$ and $z{+}\tau$ along the A- and B-cycle.

In the parameterization of the torus depicted in fig. 1, translations around the A- and B-cycle amount to shifts by 1 and τ, respectively. The quasi-periodicity of the Jacobi theta function (2.4),

$$\theta(z{+}1,\tau) = -\theta(z,\tau) \ , \qquad \theta(z{+}\tau,\tau) = -e^{-i\pi\tau-2\pi i z}\theta(z,\tau) \, , \tag{2.10}$$

results in the following monodromies of the Kronecker–Eisenstein series (2.5) [6,7]

$$\begin{aligned} F(z+1,\alpha,\tau) &= F(z,\alpha,\tau) \, , \\ F(z+\tau,\alpha,\tau) &= e^{-2\pi i\alpha}F(z,\alpha,\tau) \, . \end{aligned} \tag{2.11}$$

It then follows from the expansion (2.5) that the functions $g^{(n)}(z,\tau)$ are single-valued around the A-cycle but have non-trivial B-cycle monodromy,

$$\begin{aligned} g^{(n)}(z+1,\tau) &= g^{(n)}(z,\tau) \, , \\ g^{(n)}(z+\tau,\tau) &= \sum_{k=0}^{n} \frac{(-2\pi i)^k}{k!} g^{(n-k)}(z,\tau) \, . \end{aligned} \tag{2.12}$$

For instance,

$$g^{(1)}(z+\tau,\tau) = -2\pi i \ , \qquad g^{(2)}(z+\tau,\tau) = -2\pi i g^{(1)}(z,\tau) + \frac{1}{2}(2\pi i)^2 \ . \tag{2.13}$$

From now on, in order to compactly represent the dependence on the external punctures $z_1, z_2, \ldots, z_n$ in string correlators, we will use the shorthand

$$g^{(n)}_{ij} \equiv g^{(n)}(z_i - z_j, \tau) \, . \tag{2.14}$$

2.1.2. Weight counting

The integrand of n-point one-loop open-string amplitudes (2.1) can be written in terms of loop momenta, holomorphic Eisenstein series (2.7) excluding G_2 and the above $g^{(m)}_{ij}$ (possibly including their z-derivatives) [13,14]. As a necessary condition for modular invariance of the closed-string amplitude (2.2), the overall powers of $\ell, g_{ij}^{(m)}$ and G_k have to obey the following selection rule: Once we assign the following weights to these constituents,

term	$2\pi i$	ℓ	∂_{z_j}	G_k	$g_{ij}^{(m)}$
weight	1	1	1	k	m

each term in the n-point open-string correlator $\mathcal{K}_n(\ell)$ must have weight $n-4$. The notion of weight in the table is conserved in each term of the monodromies (2.12), and the same will hold in the subsequent Fay relations and total derivatives.

2.2. Fay identities

In the subsequent discussions of one-loop open-string correlators, the Fay identity [15]

$$F(z_1,\alpha_1,\tau)F(z_2,\alpha_2,\tau) = F(z_1,\alpha_1+\alpha_2,\tau)F(z_2-z_1,\alpha_2,\tau) + (1\leftrightarrow 2) \tag{2.15}$$

plays a crucial role when expanded in terms of its coefficient functions from (2.5) [14],

$$\begin{aligned} g_{12}^{(n)} g_{23}^{(m)} = -g_{13}^{(m+n)} &+ \sum_{j=0}^{n} (-1)^j \binom{m-1+j}{j} g_{13}^{(n-j)} g_{23}^{(m+j)} \\ &+ \sum_{j=0}^{m} (-1)^j \binom{n-1+j}{j} g_{13}^{(m-j)} g_{12}^{(n+j)} \, . \end{aligned} \tag{2.16}$$

Its simplest instance can be viewed as the one-loop counterpart of the tree-level partial fraction identity $(z_{12}z_{23})^{-1} + \mathrm{cyc}(1,2,3) = 0$,

$$g_{12}^{(1)} g_{23}^{(1)} + g_{12}^{(2)} + \mathrm{cyc}(1,2,3) = 0 \, . \tag{2.17}$$

Additional instances relevant to the worldsheet functions that appear in one-loop correlators for up to and including nine points are given by

$$\begin{aligned} g_{12}^{(1)} g_{23}^{(2)} &= g_{13}^{(1)} g_{23}^{(2)} + g_{12}^{(1)} g_{13}^{(2)} - g_{13}^{(1)} g_{12}^{(2)} + g_{12}^{(3)} - g_{13}^{(3)} - 2 g_{23}^{(3)} \, , \\ g_{12}^{(2)} g_{23}^{(2)} &= g_{12}^{(2)} g_{13}^{(2)} + g_{13}^{(2)} g_{23}^{(2)} - 2 g_{13}^{(1)} g_{12}^{(3)} - 2 g_{13}^{(1)} g_{23}^{(3)} + 3 g_{12}^{(4)} - g_{13}^{(4)} + 3 g_{23}^{(4)} \, , \\ g_{12}^{(1)} g_{23}^{(3)} &= -g_{12}^{(2)} g_{13}^{(2)} + g_{13}^{(1)} g_{12}^{(3)} + g_{12}^{(1)} g_{13}^{(3)} + g_{13}^{(1)} g_{23}^{(3)} - g_{12}^{(4)} - g_{13}^{(4)} - 3 g_{23}^{(4)} \, , \\ g_{12}^{(2)} g_{23}^{(3)} &= -g_{13}^{(5)} + 6 g_{23}^{(5)} - 4 g_{12}^{(5)} + g_{13}^{(2)} g_{23}^{(3)} - 3 g_{13}^{(1)} g_{23}^{(4)} + g_{13}^{(3)} g_{12}^{(2)} - 2 g_{13}^{(2)} g_{12}^{(3)} + 3 g_{13}^{(1)} g_{12}^{(4)} \, , \\ g_{12}^{(1)} g_{23}^{(4)} &= -g_{13}^{(5)} - 4 g_{23}^{(5)} + g_{12}^{(5)} + g_{13}^{(1)} g_{23}^{(4)} + g_{13}^{(4)} g_{12}^{(1)} - g_{13}^{(3)} g_{12}^{(2)} + g_{13}^{(2)} g_{12}^{(3)} - g_{13}^{(1)} g_{12}^{(4)} \, . \end{aligned} \tag{2.18}$$

Note that the label 2 (corresponding to z_2) appears twice in the monomials of the left-hand side in the above identities while appearing at most once in the monomials of the right-hand side. This property can be exploited to rewrite arbitrary products of $g_{ij}^{(n)}$-functions in a canonical way. Since any repeated label can be eliminated this way, for convenience in a product $g_{ij}^{(n)} g_{jk}^{(m)}$ one can use the Fay identities if the repeated label j is the smallest among i, j and k (which can be obtained from a relabeling of (2.18)). In addition, Fay identities involving z-derivatives of $g^{(n)}(z, \tau)$ are easy to obtain from (2.17) and (2.18), and can be similarly written in a canonical way.

Linear combinations of the above Fay identities can be used to derive identities involving Eisenstein series G_n. For instance, from $g_{ii}^{(3)} = 0$ and $g_{ii}^{(4)} = -\mathrm{G}_4$, the limit $z_3 \to z_1$ of the expressions (2.18) for $g_{12}^{(2)} g_{23}^{(2)} + 2 g_{12}^{(1)} g_{23}^{(3)}$ and $g_{12}^{(2)} g_{23}^{(3)} + 3 g_{12}^{(1)} g_{23}^{(4)}$ implies

$$2 g_{12}^{(4)} + g_{12}^{(2)} g_{12}^{(2)} - 2 g_{12}^{(1)} g_{12}^{(3)} - 3\mathrm{G}_4 = 0\,, \quad 5 g_{12}^{(5)} + g_{12}^{(2)} g_{12}^{(3)} - 3 g_{12}^{(1)} g_{12}^{(4)} - 3\mathrm{G}_4 g_{12}^{(1)} = 0\,, \tag{2.19}$$

and similar relations can be obtained at higher weights. The weight-four identity in (2.19) will often be used in proposing an expression for the eight-point correlator, see section III.3.5.

2.3. Total derivatives

Correlators $\mathcal{K}_n(\ell)$ are always accompanied by the Koba–Nielsen factor $\mathcal{I}_n(\ell)$ given by (2.3), when they enter open- and closed-string amplitudes, see (2.1) and (2.2). One can show that its derivatives with respect to worldsheet positions z_i and modulus τ are given by

$$\frac{\partial}{\partial z_i} \mathcal{I}_n(\ell) = \big(\ell \cdot k_i + \sum_{j \neq i}^{n} s_{ij} g_{ij}^{(1)}\big) \mathcal{I}_n(\ell)\,, \tag{2.20}$$

$$\frac{\partial}{\partial \tau} \mathcal{I}_n(\ell) = \frac{1}{2\pi i} \Big(\frac{1}{2}\ell^2 + \sum_{i<j}^{n} s_{ij} g_{ij}^{(2)}\Big) \mathcal{I}_n(\ell)\,, \tag{2.21}$$

where (2.8) and $\sum_{i<j}^{n} s_{ij} = 0$ have been used in (2.21). Given the integrations over z_j and τ in the amplitudes (2.1) and (2.2), one can therefore set the following total derivatives to zero within one-loop correlators,

$$\Big(\ell \cdot k_i + \sum_{j \neq i}^{n} s_{ij} g_{ij}^{(1)}\Big) f(z, \tau, \ldots) + \frac{\partial f(z, \tau, \ldots)}{\partial z_i} \cong 0\,, \qquad \forall\, f(z, \tau, \ldots)\,, \tag{2.22}$$

$$\Big(\frac{1}{2}\ell^2 + \sum_{1 \leq i<j}^{n} s_{ij} g_{ij}^{(2)}\Big) f(z, \tau, \ldots) + 2\pi i \frac{\partial f(z, \tau, \ldots)}{\partial \tau} \cong 0, \qquad \forall\, f(z, \tau, \ldots)\,, \tag{2.23}$$

where $f(z,\tau,\ldots)$ is an arbitrary function on the worldsheet.

The absence of boundary terms w.r.t. z_j follows from the short-distance behavior[3] $|\mathcal{I}_n(\ell)| \rightarrow |z_{ij}|^{s_{ij}}$ of the Koba–Nielsen factor (2.3) as $z_i \rightarrow z_j$. It is well known from discussions of the anomaly cancellation in the open superstring that the boundaries of moduli space can give non-vanishing contributions from individual worldsheet topologies [16,17]. Hence, blindly discarding total derivatives w.r.t. the modulus τ would generically lead to inconsistencies. However, when summing over the different worldsheet topologies these inconsistencies are canceled for the gauge group $SO(32)$; since this will always be the case for the open superstring we may freely discard total derivatives in τ.

3. Generalized elliptic integrands

When using the chiral-splitting method [8,9,10] to handle the joint zero mode ℓ^m of[4] $\partial x^m(z)$ and $\overline{\partial} x^m(\overline{z})$, superstring scattering integrands of (2.1) and (2.2) involve a loop-momentum dependent Koba–Nielsen factor (2.3). As explained in [10], the integrands of superstring amplitudes containing the loop momentum ℓ^m do not need to be single-valued as functions of z_i. Instead, it is sufficient to attain single-valuedness after the loop momentum is integrated out. Here "single-valued" is used in its conventional sense; it refers to functions $f(z_i)$ left invariant as the coordinates z_i are transported around the A and B homology cycles of fig. 1. In this work, the chiral-splitting method will be used but the concept of single-valuedness will be extended to invariant functions of (z_i, ℓ^m) under a simultaneous variation of both z_i and ℓ^m along the cycles. Let us now present the reasoning that motivated this idea.

3.1. Motivating and defining generalized elliptic integrands

As we will see in section III.3.2, the evaluation of the five-point one-loop amplitude of the open superstring using the standard rules of the pure-spinor formalism (and some mild assumptions) gives rise to the following integrand:

$$\mathcal{K}_5(\ell) = \ell_m V_1 T^m_{2,3,4,5} + \big[V_{12} T_{3,4,5} g^{(1)}_{12} + (2 \leftrightarrow 3,4,5)\big] + \big[V_1 T_{23,4,5} g^{(1)}_{23} + (2,3|2,3,4,5)\big] \, . \quad (3.1)$$

[3] The cancellation of $|z_{ij}|^{s_{ij}}$ as $z_i \rightarrow z_j$ is obvious in the kinematic region where $\mathrm{Re}(s_{ij}) > 0$ and otherwise follows from analytic continuation.

[4] In the pure-spinor formalism, the worldsheet fields $\partial x^m(z)$ and $\overline{\partial} x^m(\overline{z})$ enter the vertex operators in their spacetime-supersymmetric combinations $\Pi^m(z)$ and $\overline{\Pi}^m(z)$ [2].

The kinematic factors $V_1, V_{12}, T^m_{2,3,4,5}, T_{3,4,5}$ in pure-spinor superspace [18] are reviewed in section I.4. Throughout this work, the notation $+(a_1,\ldots,a_p|a_1,\ldots,a_{p+q})$ instructs to sum over all ordered combinations of p the labels a_i taken from the set $\{a_1,a_2,\ldots,a_{p+q}\}$, leading for instance to a total of six permutations of $V_1T_{23,4,5}g^{(1)}_{23}$ in (3.1).

Having obtained (3.1), it was natural to ask about its B-cycle monodromies using the relations (2.13). Ignoring the term with the loop momentum for a moment, it is easy to see that the correlator (3.1) changes by $-2\pi i\big[V_{12}T_{3,4,5} + (2 \leftrightarrow 3,4,5)\big]$ as z_1 goes around the B-cycle. Recalling the vanishing of $k^m_1 V_1 T^m_{2,3,4,5} + \big[V_{12}T_{3,4,5} + (2 \leftrightarrow 3,4,5)\big]$ in the BRST cohomology, see (I.4.23), suggests the following speculation: if the loop momentum changed as $\ell^m \to \ell^m - 2\pi i k^m_1$ at the same time as z_1 goes around the B-cycle, then the integrand (3.1) would be single valued as a function of both z_1 and ℓ^m.

As it stands the above speculation is not compelling enough as we did not consider how the Koba–Nielsen factor (2.3) behaves under these changes. Luckily, the quasi-periodicity $\theta_1(z{+}\tau,\tau) = -e^{-i\pi\tau-2\pi i z}\theta_1(z,\tau)$ of the odd Jacobi theta function (2.4) implies that the absolute value of the Koba–Nielsen factor is *invariant* under the simultaneous transformation of $z_1 \to z_1{+}\tau$ and $\ell^m \to \ell^m - 2\pi i k^m_1$,

$$\big|\mathcal{I}_n(\ell - 2\pi i k_1)\big|_{z_1\to z_1+\tau} = \big|\mathcal{I}_n(\ell)\big| \,. \tag{3.2}$$

Hence, the loop-integrated open- and closed-string expressions $\int d^D\ell\, |\mathcal{I}_n(\ell)|\langle\mathcal{K}_5(\ell)\rangle$ and $\int d^D\ell\, |\mathcal{I}_n(\ell)|^2\langle\mathcal{K}_5(\ell)\rangle\langle\tilde{\mathcal{K}}_5(-\ell)\rangle$ will still lead to single-valued functions of the punctures in the conventional sense of [10]. But the above reasoning suggests that one can even talk about single-valued chirally-split superstring integrands by also letting the loop momentum change along the B-cycle. Furthermore, the same analysis can be performed for shifts along the A-cycle (without any modification of the loop momentum as $z_1 \to z_1 + 1$), motivating the following definition:

Definition 1 (GEI). *A generalized elliptic integrand (GEI) is a single-valued function $f(z_i,\ell,\tau,k_j)$ of the lattice coordinates z_j, $j = 1,\ldots,n$, the loop momentum ℓ^m, the modular parameter τ and the external momenta k^m_j such that*

$$f(z'_j,\ell',\tau,k_j) = f(z_j,\ell,\tau,k_j) \tag{3.3}$$

as z_j and ℓ^m go around the A and B cycles

$$\begin{aligned} &A\text{-cycle}: \quad (z'_j,\ell') = (z_j+1,\ell)\,, \\ &B\text{-cycle}: \quad (z'_j,\ell') = (z_j+\tau,\ell-2\pi i k_j)\,. \end{aligned} \tag{3.4}$$

By their dependence on ℓ^m and k_j^m, GEIs may have free vector indices $f^{m_1 m_2\cdots}(z_j,\ell,\tau,k_j)$.

As the absolute value of the Koba–Nielsen factor is by itself a GEI, the five-point example (3.1) suggests that superstring correlators are given by GEIs in the above sense,

$$\mathcal{K}_n(\ell-2\pi i k_j)\big|_{z_j\to z_j+\tau}=\mathcal{K}_n(\ell)\,. \tag{3.5}$$

We will see that this observation harbors valuable constructive input to the derivation of correlators from first principles. Furthermore, the argument above suggests a deeper connection between BRST invariance of pure-spinor superspace expressions and GEIs. As we will see in the following sections, this synergy is quite powerful and leads to many interesting results.

Integrands depending on ℓ,k,z and τ satisfying the key property (3.3) were used for the first time in [4], where the acronym GEI was coined. As detailed in section 7, integrating the GEIs in n-point closed-string integrands over $\int d^D\ell\,|\mathcal{I}_n(\ell)|^2$ yields *modular forms* of weight $(n{-}4,n{-}4)$ and leads to modular invariant closed-string amplitudes (2.2).

3.2. The linearized-monodromy operator

Given a monomial in $g^{(n)}_{ij}$, the monodromies as $z_j\to z_j+\tau$ are polynomials in $2\pi i$ by (2.12). We will be interested in combinations of $g^{(n)}_{ij}$ and the loop momentum such that the monodromies are compensated by shifts $\ell\to\ell-2\pi i k_j$ and the defining property (3.5) of GEIs is attained. In order to efficiently identify GEIs, we formally truncate the combined transformations of $g^{(n)}_{ij}$ and ℓ to the linear order in $2\pi i$ and study the operator

$$\delta_j\ell=-2\pi i k_j\,,\quad \delta_j g^{(n)}_{jm}=-2\pi i g^{(n-1)}_{jm}\,,\quad n\geq 1\,, \tag{3.6}$$

where $\delta_j g^{(0)}_{jm}=0$ and $\delta_j g^{(n)}_{im}=0$ for all $i,m\neq j$. This operator probes the *linearized monodromy* w.r.t. a given puncture $\delta_j:\ z_j\to z_j+\tau$ with the accompanying shift $\ell\to\ell-2\pi i k_j$. Accordingly, it is understood to obey a Leibniz property

$$\delta_j\big(f_1(\ell,z_j)f_2(\ell,z_j)\big)=f_1(\ell,z_j)\big(\delta_j f_2(\ell,z_j)\big)+f_2(\ell,z_j)\big(\delta_j f_1(\ell,z_j)\big) \tag{3.7}$$

for arbitrary functions f_i of the loop momentum and the punctures. It is convenient to assemble the linearized monodromies w.r.t. all of $z_1,z_2,\ldots,z_n$ into a single operator as

$$D=-\frac{1}{2\pi i}\sum_{j=1}^{n}\Omega_j\delta_j\,, \tag{3.8}$$

where we have introduced formal variables Ω_j to track the contribution of the $j^{\rm th}$ puncture. Then, (3.6) and the shorthand notation $\Omega_{ij} \equiv \Omega_i - \Omega_j$ give rise to

$$Dg^{(n)}_{ij} = \Omega_{ij} g^{(n-1)}_{ij} \, , \qquad D\ell^m = \sum_{j=1}^n \Omega_j k_j^m = \sum_{j=2}^n \Omega_{j1} k_j^m \, , \tag{3.9}$$

where momentum conservation $k_1^m = -k_2^m - \cdots - k_n^m$ has been used in the last relation. For example,

$$Dg^{(1)}_{12} = \Omega_{12} \, , \qquad Dg^{(2)}_{12} = \Omega_{12} g^{(1)}_{12} \, , \qquad D\big(g^{(1)}_{12} \ell^m\big) = \Omega_{12}\ell^m + g^{(1)}_{12} \sum_{j=2}^n \Omega_{j1} k_j^m \, . \tag{3.10}$$

Note that D will be later on argued to play a role similar to the BRST operator Q of the pure-spinor formalism. One can enforce that D shares the nilpotency $Q^2 = 0$ by defining the formal variables Ω_j to be fermionic[5]. However, the choice of statistics for the Ω_j won't affect any calculation done in this work, so we defer this decision to follow-up research.

Since the linearized monodromy operator D only picks the terms linear in $2\pi i$ that arise from the transformation $z_j \to z_j + \tau$ and $\ell \to \ell - 2\pi i k_j$, invariance $DE = 0$ is only a *necessary* condition for E to be a GEI. It remains to check if the higher orders in $2\pi i$ also drop out from the image of E under the above shift of z_j and ℓ. For all solutions to $DE = 0$ studied in this work, we have checked that they constitute a GEI on a case-by-case basis, and it would be interesting to find a general argument. In many cases, single-valuedness can be seen from the generating-function techniques in later sections.

4. Bootstrapping shuffle-symmetric worldsheet functions

In this section we will construct a system of worldsheet functions $\mathcal{Z}$ for superstring correlators on a genus-one Riemann surface by analogies with kinematic factors. When the latter are organized in terms of Berends–Giele superfields as detailed in part I, their variation under the pure-spinor BRST operator will be used as a prototype to prescribe monodromy variations for the $\mathcal{Z}$-functions. As a consequence, the combinatorics of BRST-invariant

[5] The conditional nilpotency of D for fermionic formal variables Ω_j follows from the fact that linearized monodromies (3.6) w.r.t. different punctures commute, $\delta_i \delta_j = \delta_j \delta_i$. This commutativity property follows from (3.6) and (3.7).

kinematic factors can be borrowed to anticipate D-invariant combinations of $\mathcal{Z}$-functions, i.e. GEIs.

The correspondence between the pure-spinor BRST charge Q acting on superfields and the monodromy operator D acting on functions is the first facet of a duality between kinematics and worldsheet functions. Further aspects of the duality will be presented in section 5 that lead to a variety of applications. In particular, the duality between kinematics and worldsheet functions implies a double-copy structure of open-superstring one-loop amplitudes discussed in [4] and expanded in part III.

4.1. Shuffle-symmetric worldsheet functions

In the computation of tree-level correlators for n-point open-string amplitudes [19], the nested OPE singularities were captured by worldsheet functions of the following form[6]

$$\mathcal{Z}^{\rm tree}_{123\ldots p} \equiv \frac{1}{z_{12}z_{23}\ldots z_{p-1,p}} \, . \tag{4.1}$$

It follows from partial-fraction relations such as $(z_{12}z_{23})^{-1} + {\rm cyc}(1,2,3) = 0$ that the tree-level functions satisfy *shuffle symmetries*[7] (e.g. $\mathcal{Z}^{\rm tree}_{1\shuffle 23} = \mathcal{Z}^{\rm tree}_{123} + \mathcal{Z}^{\rm tree}_{213} + \mathcal{Z}^{\rm tree}_{231} = 0$) [22]

$$\mathcal{Z}^{\rm tree}_{A\shuffle B} = 0, \quad \forall\, A, B \neq \emptyset \, . \tag{4.2}$$

Since the appearance of shuffle-symmetric worldsheet functions (4.1) at tree level can be traced back to the short-distance behavior of vertex operators, the same structure must persist at higher genus. Therefore we assume that the short-distance singularities at one loop arise from analogous chains built from functions $g^{(1)}(z,\tau) = \frac{1}{z} + \mathcal{O}(z)$

$$g^{(1)}_{12}\, g^{(1)}_{23} \ldots g^{(1)}_{p-1,p} \, . \tag{4.3}$$

As a fundamental starting point in obtaining one-loop n-point correlators of the open superstring, the worldsheet functions associated with nested OPE singularities will be required to obey shuffle symmetries like their tree-level counterparts, i.e.,

$$\mathcal{Z}^{1-{\rm loop}}_{\ldots, A\shuffle B, \ldots} = 0 \, , \qquad \forall\, A, B \neq \emptyset \, . \tag{4.4}$$

---

[6] Note that the worldsheet functions $(z_{12}z_{23}\ldots z_{p-1,p})^{-1}$ at tree level arise from cyclic Parke–Taylor factors $(z_{12}z_{23}\ldots z_{p-1,p}z_{p,n}z_{n,1})^{-1}$ in an SL_2-frame where $z_n \to \infty$.

[7] The *shuffle product* of words A and B of length n and m generates all $\frac{(n+m)!}{n!m!}$ possible ways to interleave the letters of A and B without changing their orderings within A and B, see (I.3.2) for a recursive definition. A more elaborate account on the combinatorics on words can be found in section I.3.1, based on the mathematics literature [20,21].

At multiplicity $p=2$, antisymmetry of $g^{(1)}_{12}=-g^{(1)}_{21}$ suffices to make it shuffle-symmetric. However, for the tentative one-loop counterpart $g^{(1)}_{12}g^{(1)}_{23}$ of $\mathcal{Z}^{\rm tree}_{123}$ it is easy to see that the Fay identity (2.17) prevents the shuffle relation $\mathcal{Z}^{\rm tree}_{1\shuffle 23}=0$ from generalizing. Luckily, the same Fay identity also suggests how to restore the shuffle symmetry without altering the pole at $z_i\rightarrow z_j$ by adding non-singular $g^{(2)}_{ij}$-functions. One can check via (2.17) that both of

$$\mathcal{Z}^{(\rm i)}_{123}\equiv g^{(1)}_{12}g^{(1)}_{23}+\frac{1}{2}(g^{(2)}_{12}+g^{(2)}_{23})\,,\qquad \mathcal{Z}^{(\rm ii)}_{123}\equiv g^{(1)}_{12}g^{(1)}_{23}+g^{(2)}_{12}+g^{(2)}_{23}-g^{(2)}_{13} \tag{4.5}$$

share the desired shuffle symmetry of $\mathcal{Z}^{\rm tree}_{123}$. Also at higher multiplicity, the non-singular functions $g^{(n)}_{ij}$ with $n\geq 2$ admit various shuffle symmetric completions of $g^{(1)}_{12}g^{(1)}_{23}\ldots g^{(1)}_{p-1,p}$ which reproduce the singularity structure of (4.1) and qualify as one-loop counterparts of $\mathcal{Z}^{\rm tree}_{123\ldots p}$. From the availability of two shuffle-symmetric multiplicity-three candidates in (4.5), one can anticipate that many more options arise at higher multiplicities. In the next subsection we will identify a guiding principle to prefer $\mathcal{Z}^{(\rm ii)}_{123}$ over $\mathcal{Z}^{(\rm i)}_{123}$ in our representations of one-loop correlators and to select higher-multiplicity generalizations.

4.2. Duality between monodromy and BRST variations

We will now prescribe the monodromy variation $D\mathcal{Z}_{12\ldots p}$ of shuffle-symmetric worldsheet functions by analogies with Berends–Giele superfields $M_{12\ldots p}$ that share the shuffle symmetry and are reviewed in section I.5. The idea is to impose the combinatorics of the BRST variation $QM_{12\ldots p}$ to carry over to the worldsheet functions, $\mathcal{Z}_{12\ldots p}\leftrightarrow M_{12\ldots p}$. This relationship is at the heart of an emerging proposal for a *duality between worldsheet functions kinematics* – monodromy variations are taken to be dual to BRST variations.

4.2.1. Scalar monodromy variations

The Berends–Giele superfields at one loop have multiple slots, starting with the scalar kinematics $M_AM_{B,C,D}$ of section I.5.1. Accordingly, the simplest one-loop worldsheet functions should inherit the slot structure $\mathcal{Z}_{A,B,C,D}$ with shuffle symmetries in all of A,B,C,D. Throughout this work, whenever multiparticle labels $A,B,\ldots$ in a subscript are separated by a comma rather than a vertical bar, then they are understood to be freely interchangeable, $\mathcal{Z}_{A,B,\ldots}=\mathcal{Z}_{B,A,\ldots}$.

The BRST variation (I.5.18) of $M_{B,C,D}$ can be written as linear combinations of the BRST invariants $C_{i|P,Q,R}$ [23] reviewed in section I.5.2. Accordingly, the corresponding

D-variations of $\mathcal{Z}_{A,B,C,D}$ should be written in terms of GEIs $E_{i|A,B,C,D}$, i.e. D-invariant combinations of simpler $\mathcal{Z}$-functions. More explicitly, the parallel is taken to be

$$QM_{A,B,C} = C_{a_1|a_2\ldots a_{|A|},B,C} - C_{a_{|A|}|a_1\ldots a_{|A|-1},B,C} + (A\leftrightarrow B,C)\,, \tag{4.6}$$

$$D\mathcal{Z}_{A,B,C,D} = \Omega_{a_1} E_{a_1|a_2\ldots a_{|A|},B,C,D} - \Omega_{a_{|A|}} E_{a_{|A|}|a_1\ldots a_{|A|-1},B,C,D} + (A\leftrightarrow B,C,D)\,, \tag{4.7}$$

where the length of the word $A = a_1a_2\ldots a_{|A|}$ is denoted by $|A|$, and the bookkeeping variables Ω_j of (3.8) always follow the special label of $E_{j|\ldots}$, e.g.

$$\begin{aligned} QM_{1,2,3} &= 0\,, \qquad & QM_{12,3,4} &= C_{1|2,3,4} - C_{2|1,3,4}\,,\\ D\mathcal{Z}_{1,2,3,4} &= 0\,, \qquad & D\mathcal{Z}_{12,3,4,5} &= \Omega_1 E_{1|2,3,4,5} - \Omega_2 E_{2|1,3,4,5}\,. \end{aligned} \tag{4.8}$$

The $E_{i|\ldots}$ on the right-hand sides will be defined in analogy[8] with $C_{i|\ldots}$, and this analogy will be reflected by the notation: The duality between superfields and $\mathcal{Z}_{A,B,C,D}$ as well as the resulting correspondence between Q and D imply that BRST invariants $C_{i|A,B,C}$ should be dualized to GEIs. By the vertical-bar notation, the symmetries $C_{i|A,B,\ldots} = C_{i|B,A,\ldots}$ $E_{i|A,B,\ldots} = E_{i|B,A,\ldots}$ do not extend to the external-state label i in the first entry.

At this point, we can identify a preferred choice among the two multiplicity-three candidates (4.5). Based on (3.10), we have

$$D\mathcal{Z}^{(\mathrm{i})}_{123} = \Omega_1(g^{(1)}_{23} + \frac{1}{2}g^{(1)}_{12}) + \frac{1}{2}\Omega_2 g^{(1)}_{12} + (1\leftrightarrow 3)\,, \qquad D\mathcal{Z}^{(\mathrm{ii})}_{123} = \Omega_{13}(g^{(1)}_{12}+g^{(1)}_{23}+g^{(1)}_{31})\,, \tag{4.9}$$

where the second variation will later be shown to be equal to $\Omega_{13}E_{1|23,4,5}$. Since it is easy to see that $D\mathcal{Z}^{(\mathrm{i})}_{123}$ is not single-valued, only the second option has the required structure (4.7) on the right-hand side. In order to reconcile the expression for $\mathcal{Z}^{(\mathrm{ii})}_{123}$ with the slot structure of scalar worldsheet functions $\mathcal{Z}_{A,B,C,D}$ in (4.7), from now on we use the notation $\mathcal{Z}_{123,4,5,6} = \mathcal{Z}^{(\mathrm{ii})}_{123}$, also see section 4.4.3.

[8] We note the mismatch between the slots of $E_{i|A,B,C}$ defined in analogy with the BRST invariants and the slots of $E_{i|A,B,C,D}$ appearing in the right-hand side of (4.7). This difference is inconsequential for functions up to multiplicity nine and can be bypassed by defining the *extension* of scalar GEIs by $E_{i|A,B,C,D} \equiv E_{i|A,B,C}$ and by adding extra permutations to the tensorial GEIs.

4.2.2. Tensorial monodromy variations

The same ideas can be reused at higher tensor ranks r to infer tensorial worldsheet functions $\mathcal{Z}^{m_1\ldots m_r}_{A,B,C,..}$ involving loop momenta, external momenta and $g^{(n)}_{ij}$, with shuffle symmetries in multiple slots $A,B,\ldots$. These tensorial functions will be constructed by imposing their linearized monodromies to follow the BRST variation of tensorial kinematic building blocks $M^{m_1\ldots m_r}_{A,B,C,\ldots}$ in pure-spinor superspace. Explicitly, the map is

$$QM^{m_1\ldots m_r}_{A,B,C,\ldots} \longleftrightarrow D\mathcal{Z}^{m_1\ldots m_r}_{A,B,C,\ldots}\,, \tag{4.10}$$

and we will use the following results for the left-hand side [23],

$$\begin{aligned} QM^{m_1m_2\ldots m_r}_{A,B,C,\ldots} &= \delta^{(m_1m_2}\mathcal{Y}^{m_3\ldots m_r)}_{A,B,C,\ldots} \\ &\quad + C^{m_1m_2\ldots m_r}_{a_1|a_2\ldots a_{|A|},B,C,\ldots} - C^{m_1m_2\ldots m_r}_{a_{|A|}|a_1\ldots a_{|A|-1},B,C,\ldots} + (A\leftrightarrow B,C,\ldots) \\ &\quad + \delta_{|A|,1}rk^{(m_1}_{a_1}C^{m_2\ldots m_r)}_{a_1|B,C,\ldots} + (A\leftrightarrow B,C,\ldots)\,, \end{aligned} \tag{4.11}$$

with tensorial anomaly superfields $\mathcal{Y}^{m_1m_2\ldots}_{A,B,\ldots}$ and (pseudo-)invariants[9] $C^{m_1m_2\ldots}_{1|A,B,\ldots}$ [23], e.g.

$$\begin{aligned} QM^m_{1,2,3,4} &= k^m_1C_{1|2,3,4} + (1\leftrightarrow 2,3,4) \\ QM^m_{12,3,4,5} &= C^m_{1|2,3,4,5} - C^m_{2|1,3,4,5} + \big[k^m_3C_{3|12,4,5} + (3\leftrightarrow 4,5)\big]\,. \end{aligned} \tag{4.12}$$

Here and in the following, Lorentz indices are (anti)symmetrized such that each inequivalent term has unit coefficient, e.g. $k_1^{(m_1}k_2^{m_2}\ldots k_r^{m_r)} \equiv k_1^{m_1}k_2^{m_2}\ldots k_r^{m_r} + \mathrm{perm}(m_1,\ldots,m_r)$, for a total of $r!$ terms. In case of symmetric tensors, imposing unit coefficients leads to fewer terms such as $\delta^{(mn}k^{p)} \equiv \delta^{mn}k^p + \delta^{mp}k^n + \delta^{np}k^m$, and expanding the symmetrization of $\delta^{(m_1m_2}\mathcal{Y}^{m_3\ldots m_r)}_{A,B,C,\ldots}$ in (4.11) yields $\binom{r}{2}$ terms.

The vectorial BRST invariants $C^m_{i|A,\ldots}$ on the right-hand sides of (4.11) and (4.12) are composed of $M_{A,B,C}$, $M^m_{A,B,C,D}$ and external momenta. Similarly, we will later on obtain vectorial GEIs $E^m_{i|A,B,\ldots}$ involving ℓ^m, k^m_j and $g^{(n)}_{ij}$ by following the same composition rules. According to the duality (4.10), the defining property of $\mathcal{Z}^{m_1\ldots m_r}_{A,B,C,..}$ is

$$\begin{aligned} D\mathcal{Z}^{m_1m_2\ldots m_r}_{A,B,C,\ldots} = \Big[&\Omega_{a_1}E^{m_1m_2\ldots m_r}_{a_1|a_2\ldots a_{|A|},B,C,\ldots} - \Omega_{a_{|A|}}E^{m_1m_2\ldots m_r}_{a_{|A|}|a_1\ldots a_{|A|-1},B,C,\ldots} \\ &+ \delta_{|A|,1}\Omega_{a_1}k^{(m_1}_{a_1}E^{m_2\ldots m_r)}_{a_1|B,C,\ldots} + (A\leftrightarrow B,C,\ldots)\Big]\,, \end{aligned} \tag{4.13}$$

[9] The defining property of pseudo-invariants is that their BRST variation is entirely expressible in terms of anomaly superfields [23].

where the anomalous superfield in the first line of (4.11) does not have any worldsheet counterpart. The examples in (4.12) then translate into

$$
\begin{aligned}
D\mathcal{Z}^m_{1,2,3,4,5} &= \Omega_1 k_1^m E_{1|2,3,4,5} + (1 \leftrightarrow 2,3,4,5) \\
D\mathcal{Z}^m_{12,3,4,5,6} &= \Omega_1 E^m_{1|2,3,4,5,6} - \Omega_2 E^m_{2|1,3,4,5,6} + \big[\Omega_3 k_3^m E_{3|12,4,5,6} + (3 \leftrightarrow 4,5,6)\big] \, .
\end{aligned}
\tag{4.14}
$$

4.2.3. Refined bootstrap equations

The duality between superspace kinematics and worldsheet functions suggests to introduce a notion of refined $\mathcal{Z}$-functions defined via monodromies

$$
Q\mathcal{J}^{m_1\ldots m_r}_{A_1,\ldots,A_d|B_1,B_2,\ldots} \longleftrightarrow D\mathcal{Z}^{m_1\ldots m_r}_{A_1,\ldots,A_d|B_1,B_2,\ldots} \, , \tag{4.15}
$$

where the Berends–Giele superfields $\mathcal{J}$ are derived from the refined building blocks of section I.4.4. The number $d \geq 1$ of slots on the left of the vertical bar is referred to as the degree of refinement. The left-hand side of (4.15) is given in terms of refined anomaly superfields $\mathcal{Y}^{m_1 m_2\ldots}_{A_1,\ldots,A_d|B,\ldots}$ and (pseudo-)invariants $P^{m_1\ldots m_r}_{1|A_1,\ldots,A_d|B,\ldots}$ [23],

$$
\begin{aligned}
&Q\mathcal{J}^{m_1\ldots m_r}_{A_1,\ldots,A_d|B_1,B_2,\ldots} = \delta^{(m_1 m_2}\mathcal{Y}^{m_3\ldots m_r)}_{A_1,\ldots,A_d|B_1,\ldots,} \\
&+ \big[\mathcal{Y}^{m_1\ldots m_r}_{A_2,\ldots,A_d|A_1,B_1,\ldots,} + (A_1 \leftrightarrow A_2,\ldots,A_d)\big] \\
&+ \big[\delta_{|A_1|,1} k^p_{a_1} P^{pm_1\ldots m_r}_{a_1|A_2,\ldots,A_d|B_1,\ldots} + (A_1 \leftrightarrow A_2,\ldots,A_d)\big] \\
&+ \big[\delta_{|B_1|,1} k^{(m_1}_{b_1} P^{m_2\ldots m_r)}_{b_1|A_1,\ldots,A_d|B_2,\ldots} + (B_1 \leftrightarrow B_2,\ldots)\big] \\
&+ \big[P^{m_1\ldots m_r}_{a_1|a_2\ldots a_{|A_1|},A_2,\ldots,A_d|B_1,\ldots} - P^{m_1\ldots m_r}_{a_{|A_1|}|a_1\ldots a_{|A_1|-1},A_2,\ldots,A_d|B_1,\ldots} + (A_1 \leftrightarrow A_2,\ldots,A_d)\big] \\
&+ \big[P^{m_1\ldots m_r}_{b_1|A_1,\ldots,A_d|b_2\ldots b_{|B_1|},B_2,\ldots} - P^{m_1\ldots m_r}_{b_{|B_1|}|A_1,\ldots,A_d|b_1\ldots b_{|B_1|-1},B_2,\ldots} + (B_1 \leftrightarrow B_2,\ldots)\big] \, ,
\end{aligned}
\tag{4.16}
$$

for instance,

$$
\begin{aligned}
Q\mathcal{J}^m_{1|23,4,5,6,7} &= \mathcal{Y}^m_{1,23,4,5,6,7} + k_1^p C^{mp}_{1|23,4,5,6,7} \\
&\quad + \big[k_4^m P_{4|1|23,5,6,7} + (4 \leftrightarrow 5,6,7)\big] + P^m_{2|1|3,4,5,6,7} - P^m_{3|1|2,4,5,6,7} \, .
\end{aligned}
\tag{4.17}
$$

Accordingly, the refined versions of the worldsheet functions comprising ℓ, k_j^m and $g^{(n)}_{ij}$ are characterized by the following monodromies

$$
\begin{aligned}
&D\mathcal{Z}^{m_1\ldots m_r}_{A_1,\ldots,A_d|B_1,B_2,\ldots} = \big[\delta_{|A_1|,1}\Omega_{a_1} k^p_{a_1} E^{pm_1\ldots m_r}_{a_1|A_2,\ldots,A_d|B_1,\ldots} + (A_1 \leftrightarrow A_2,\ldots,A_d)\big] \\
&+ \big[\delta_{|B_1|,1}\Omega_{b_1} k^{(m_1}_{b_1} E^{m_2\ldots m_r)}_{b_1|A_1,\ldots,A_d|B_2,\ldots} + (B_1 \leftrightarrow B_2,\ldots)\big] \\
&+ \big[\Omega_{a_1} E^{m_1\ldots m_r}_{a_1|a_2\ldots a_{|A_1|},A_2,\ldots,A_d|B_1,\ldots} - \Omega_{a_{|A_1|}} E^{m_1\ldots m_r}_{a_{|A_1|}|a_1\ldots a_{|A_1|-1},A_2,\ldots,A_d|B_1,\ldots} + (A_1 \leftrightarrow A_2,\ldots,A_d)\big] \\
&+ \big[\Omega_{b_1} E^{m_1\ldots m_r}_{b_1|A_1,\ldots,A_d|b_2\ldots b_{|B_1|},B_2,\ldots} - \Omega_{b_{|B_1|}} E^{m_1\ldots m_r}_{b_{|B_1|}|A_1,\ldots,A_d|b_1\ldots b_{|B_1|-1},B_2,\ldots} + (B_1 \leftrightarrow B_2,\ldots)\big] \, ,
\end{aligned}
\tag{4.18}
$$

where the anomalous superfields in the first line of (4.16) do not have any worldsheet counterpart. The right-hand side of (4.18) features refined GEIs $E^{m_1\ldots}_{i|A_1,\ldots,A_d|B_1,\ldots}$ which will enter the correlators discussed in part III as the coefficients of refined superfields. For example, the monodromy variation dual to (4.17) reads

$$
\begin{aligned}
D\mathcal{Z}^m_{1|23,4,\ldots,8} &= \Omega_1 k_1^p E^{mp}_{1|23,4,\ldots,8} \\
&\quad + \big[k_4^m \Omega_4 E_{4|1|23,5,\ldots,8} + (4 \leftrightarrow 5,\ldots,8)\big] + E^m_{2|1|3,4,\ldots,8} - E^m_{3|1|2,4,\ldots,8}\,.
\end{aligned}
\tag{4.19}
$$

The above patterns were discovered upon studying correlators previously obtained by various other considerations at multiplicities four, five and six. At higher multiplicities, the existence of worldsheet functions subject to (4.13) and (4.18) is a working hypothesis—so far confirmed by explicit construction up to and including eight points.

4.2.4. An ambiguity caused by Eisenstein series

Given a solution $\mathcal{Z}_{A,B,\ldots}$ to monodromy-variation equations, it is always possible to deform it by an arbitrary GEI. A partial resolution to this ambiguity is quite natural in view of the defining properties of one-loop correlators: We require the words $A, B, \ldots$ of $\mathcal{Z}_{A,B,\ldots}$ to reflect tree-level-like singularities $(z_{a_1a_2}z_{a_2a_3}\ldots z_{a_{|A|-1}a_{|A|}})^{-1}(z_{b_1b_2}z_{b_2b_3}\ldots z_{b_{|B|-1}b_{|B|}})^{-1}$, cf. (4.1). This requirement fixes the most singular term to be $(g^{(1)}_{a_1a_2}g^{(1)}_{a_2a_3}\ldots g^{(1)}_{a_{|A|-1}a_{|A|}})$ and should prevent the addition of non-constant functions with vanishing monodromies, as they would necessarily modify this singularity structure. At the level of unrefined scalar GEIs, this follows from the fact that non-constant elliptic functions always involve singularities as $z_i \to z_j$, and we expect this property to carry over to tensorial and refined GEIs.

However, this requirement cannot determine the presence (or absence) of terms proportional to a holomorphic Eisenstein series G_n, for they are monodromy invariant ($D\mathrm{G}_n = 0$) as well as constant functions on the worldsheet ($\frac{\partial \mathrm{G}_n}{\partial z_j} = 0$). The construction of $\mathcal{Z}_{A,B,\ldots}$ and GEIs from $g^{(n)}_{ij}$ automatically qualifies holomorphic Eisenstein series $\mathrm{G}_n = -g^{(n)}_{ii}$ as possible constituents. Moreover, G_n are known to arise in ($n \geq 8$)-point one-loop correlators from the spin sums in the RNS formalism [13,14].

By the weight counting of section 2.1.2, the first instance where the above ambiguity may affect the expressions for shuffle-symmetric functions happens at eight points. And indeed, we will see in section III.3.5 that the eight-point correlator is plagued by unwanted appearances of G_4 whose kinematic coefficient remains undetermined in this work.

4.2.5. Lie-symmetric worldsheet functions

From the discussion in section I.5.1, Berends–Giele superfields $M_{A,B,C}$ subject to shuffle symmetries can be translated to local building blocks $T_{A,B,C}$ that satisfy Lie symmetries (cf. section I.3.4). The dictionary in (I.5.8) boils down to the KLT-matrix $S(\cdot|\cdot)_i$ [24] (also known as the *momentum kernel* [25]) that cancels the kinematic poles of the Berends–Giele currents and is recursively defined by

$$S(P,j|Q,j,R)_i = (k_{iQ}\cdot k_j)S(P|Q,R)_i, \qquad S(\emptyset|\emptyset)_i = 1\,, \tag{4.20}$$

for instance

$$S(2|2)_1 = (k_1\cdot k_2), \quad S(23|23)_1 = (k_{12}\cdot k_3)(k_1\cdot k_2), \quad S(23|32)_1 = (k_1\cdot k_3)(k_1\cdot k_2)\,. \tag{4.21}$$

In analogous fashion, one can also define worldsheet functions that satisfy Lie symmetries. To this effect we define, in analogy with (I.5.8),

$$Z^{(s)m_1\ldots}_{aA,bB,\ldots} \equiv \sum_{A',B',\ldots} S(A|A')_a S(B|B')_b \cdots \mathcal{Z}^{m_1\ldots}_{aA',bB',\ldots}\,, \tag{4.22}$$

$$E^{(s)m_1\ldots}_{1|aA,bB,\ldots} \equiv \sum_{A',B',\ldots} S(A|A')_a S(B|B')_b \cdots E^{m_1\ldots}_{1|aA',bB',\ldots}\,, \tag{4.23}$$

where the matrix $S(A|A')_a$ defined in (4.20) contributes $|A|$ powers of $s_{ij} = k_i\cdot k_j$. For example,

$$\begin{aligned} &Z^{(s)}_{1,2,3,4} = \mathcal{Z}_{1,2,3,4}, \quad Z^{(s)}_{12,3,4,5} = s_{12}\mathcal{Z}_{12,3,4,5} \\ &Z^{(s)}_{12,34,5,6} = s_{12}s_{34}\mathcal{Z}_{12,34,5,6}, \quad Z^{(s)}_{123,4,5,6} = (s_{13}+s_{23})s_{12}\mathcal{Z}_{123,4,5,6} + s_{13}s_{12}\mathcal{Z}_{132,4,5,6}\,. \end{aligned} \tag{4.24}$$

One can explicitly check that $Z^{(s)}_{123,4,5,6}$ indeed obeys the Lie symmetries in $A = 123$; $Z^{(s)}_{123,4,5,6} + Z^{(s)}_{213,4,5,6} = 0$, and $Z^{(s)}_{123,4,5,6} + Z^{(s)}_{231,4,5,6} + Z^{(s)}_{321,4,5,6} = 0$. Similarly, one may verify the Lie symmetries at higher multiplicities. The superscript in $Z^{(s)}$ reminds of the presence of monomials in s_{ij}.

4.3. Worldsheet dual expansions of BRST pseudo-invariants

In this section we will see the first non-trivial consequence of the conjectural duality between worldsheet functions and kinematics: the systematic construction of GEIs. This is done by exploiting the analogy between monodromy variations of $\mathcal{Z}$-functions and the BRST variations of Berends–Giele currents put forward in section 4.2. The tentative idea is to assemble GEIs or "worldsheet invariants" following the same combinatorics used in building kinematic BRST invariants $C_{1|A,B,C}$ and $C^m_{1|A,B,C,D}$ in (I.5.20) and (I.5.21) from Berends–Giele currents. It turns out that the worldsheet invariants constructed in this way give rise to GEIs as defined in section 3, i.e., their monodromy variations vanish.

At four and five points, the expressions for $C_{1|2,3,4}, C_{1|23,4,5}$ and $C^m_{1|2,3,4,5}$ in (I.5.20) and (I.5.21) translate into

$$\begin{aligned} E_{1|2,3,4} &= \mathcal{Z}_{1,2,3,4}\,, \\ E_{1|23,4,5} &= \mathcal{Z}_{1,23,4,5} + \mathcal{Z}_{12,3,4,5} - \mathcal{Z}_{13,2,4,5}\,, \\ E^m_{1|2,3,4,5} &= \mathcal{Z}^m_{1,2,3,4,5} + \big[k^m_2 \mathcal{Z}_{12,3,4,5} + (2 \leftrightarrow 3,4,5)\big]\,, \end{aligned} \tag{4.25}$$

while at six points we have the unrefined GEIs,

$$\begin{aligned} E_{1|234,5,6} &= \mathcal{Z}_{1,234,5,6} + \mathcal{Z}_{12,34,5,6} + \mathcal{Z}_{123,4,5,6} + \mathcal{Z}_{412,3,5,6} - \mathcal{Z}_{14,23,5,6} + \mathcal{Z}_{143,2,5,6}\,, \\ E_{1|23,45,6} &= \mathcal{Z}_{1,23,45,6} + \mathcal{Z}_{12,45,3,6} - \mathcal{Z}_{13,45,2,6} + \mathcal{Z}_{14,23,5,6} - \mathcal{Z}_{15,23,4,6} \\ &\quad - \mathcal{Z}_{412,3,5,6} + \mathcal{Z}_{314,2,5,6} + \mathcal{Z}_{215,3,4,6} - \mathcal{Z}_{315,2,4,6}\,, \\ E^m_{1|23,4,5,6} &= \mathcal{Z}^m_{1,23,4,5,6} + \mathcal{Z}^m_{12,3,4,5,6} - \mathcal{Z}^m_{13,2,4,5,6} + k^m_3 \mathcal{Z}_{123,4,5,6} - k^m_2 \mathcal{Z}_{132,4,5,6} \\ &\quad + \big[k^m_4 \mathcal{Z}_{14,23,5,6} - k^m_4 \mathcal{Z}_{214,3,5,6} + k^m_4 \mathcal{Z}_{314,2,5,6} + (4 \leftrightarrow 5,6)\big]\,, \\ E^{mn}_{1|2,3,4,5,6} &= \mathcal{Z}^{mn}_{1,2,3,4,5,6} + \big[k^m_2 \mathcal{Z}^n_{12,3,4,5,6} + k^n_2 \mathcal{Z}^m_{12,3,4,5,6} + (2 \leftrightarrow 3,4,5,6)\big] \\ &\quad - \big[(k^m_2 k^n_3 + k^n_2 k^m_3)\mathcal{Z}_{213,4,5,6} + (2,3|2,3,4,5,6)\big]\,, \end{aligned} \tag{4.26}$$

see (I.5.22) for the superspace counterpart of the tensor. Moreover, six points admit one instance of a refined GEI dual to the $P_{1|2|3,4,5,6}$ superfield (I.5.24),

$$E_{1|2|3,4,5,6} = \mathcal{Z}_{2|1,3,4,5,6} + k^m_2 \mathcal{Z}^m_{12,3,4,5,6} + \big[s_{23}\mathcal{Z}_{123,4,5,6} + (3 \leftrightarrow 4,5,6)\big]\,. \tag{4.27}$$

The analogous seven-point expansions are displayed in Appendix A.2.3.

Based on the D-variations from section 4.2, it is straightforward to verify that all of (4.25) and (4.26) are indeed GEIs upon using momentum conservation[10]. As we will see in the next section, the above GEIs have obvious extensions by one extra word (slot) to match the slot structure on the right-hand side of the above D-variations. For instance, $E_{1|23,4,5,6} \equiv E_{1|23,4,5}$ will be needed for $D\mathcal{Z}_{123,4,5,6} = \Omega_1 E_{1|23,4,5,6} - \Omega_3 E_{3|12,4,5,6}$.

[10] In order to see that (4.27) defines a GEI as well, one can either employ the explicit representation assembled in (4.37) or insert the integration-by-parts identity (5.1) among GEIs into the D variation obtained from (4.18).

4.4. The bootstrap

At first glance, the discussion in sections 4.2 and 4.3 seems to suffer from a *chicken-and-egg* dilemma; in section 4.2, to obtain the monodromy variations of the shuffle symmetric functions one needs the associated GEIs from section 4.3, while the expressions of the GEIs require the shuffle-symmetric functions from section 4.2.

The way out of this conundrum is to note that this self-recursive structure can be exploited to *bootstrap* the shuffle-symmetric $\mathcal{Z}$-functions order by order in multiplicity, starting with the four-point solution which is taken to be a constant. We will see how this works in practice in the following subsections.

Note that the functions obtained below will be used inside one-loop correlators of the open superstring, and as such, are considered to be multiplied by the overall Koba–Nielsen factor (2.3). Therefore functions that differ by derivatives of the Koba–Nielsen factor given in (2.22) and (2.23) are considered equivalent as will be indicated by the symbol $\cong$.

4.4.1. Four-point worldsheet functions

From the computation of the four-point correlator in [26,3], it follows that the four-point shuffle-symmetric worldsheet function is a constant. Similarly, the expansion (4.25) implies that also its corresponding GEI is a constant. Both are normalized to one,

$$\mathcal{Z}_{1,2,3,4} \equiv 1\,, \quad E_{1|2,3,4} \equiv 1\,. \tag{4.28}$$

To proceed to the next level we define the slot extension of (4.28) as $E_{1|2,3,4,5} \equiv 1$.

4.4.2. Five-point worldsheet functions

According to (4.7) and (4.13), the monodromy variations of the shuffle-symmetric functions $\mathcal{Z}_{12,3,4,5}$ and $\mathcal{Z}^m_{1,2,3,4,5}$ at five points are given by

$$\begin{aligned} D\mathcal{Z}_{12,3,4,5} &= \Omega_1 E_{1|2,3,4,5} - \Omega_2 E_{2|1,3,4,5} = \Omega_{12} \\ D\mathcal{Z}^m_{1,2,3,4,5} &= \Omega_1 k_1^m E_{1|2,3,4,5} + (1 \leftrightarrow 2,3,4,5) = \sum_{j=1}^{5} \Omega_j k_j^m\,. \end{aligned} \tag{4.29}$$

A closer inspection of the linearized monodromies (3.9) naturally leads to the following solutions of (4.29),

$$\mathcal{Z}_{12,3,4,5} = g^{(1)}_{12}\,, \qquad \mathcal{Z}^m_{1,2,3,4,5} = \ell^m\,. \tag{4.30}$$

These expressions reproduce the desired singularity structure $\mathcal{Z}_{12,3,4,5} = z_{12}^{-1} + \mathcal{O}(z_{12})$ and regularity of $\mathcal{Z}^m_{1,2,3,4,5}$, cf. section 4.2.4.

4.4.2.1. Assembling five-point GEIs

We can now assemble associated GEIs from the expansions in (4.25),

$$\begin{aligned} E_{1|23,4,5} &= \mathcal{Z}_{1,23,4,5} + \mathcal{Z}_{12,3,4,5} - \mathcal{Z}_{13,2,4,5} \\ &= g^{(1)}_{23} + g^{(1)}_{12} - g^{(1)}_{13} \,, \end{aligned} \tag{4.31}$$

$$\begin{aligned} E^m_{1|2,3,4,5} &= \mathcal{Z}^m_{1,2,3,4,5} + \big[k^m_2 \mathcal{Z}_{12,3,4,5} + (2 \leftrightarrow 3,4,5)\big] \\ &= \ell^m + \big[k^m_2 g^{(1)}_{12} + (2 \leftrightarrow 3,4,5)\big] \,. \end{aligned} \tag{4.32}$$

It is easy to check that (4.31) and (4.32) are indeed invariant under monodromy variations (using momentum conservation in the latter case). Before proceeding to the next multiplicity, we define the slot extension of (4.31) and (4.32),

$$E_{1|23,4,5,6} \equiv E_{1|23,4,5} \,, \qquad E^m_{1|2,3,4,5,6} \equiv \ell^m + \big[k^m_2 g^{(1)}_{12} + (2 \leftrightarrow 3,4,5,6)\big] \,, \tag{4.33}$$

including an extra permutation $2 \leftrightarrow 6$ in the vector GEI. These extensions are natural from the generating functions for GEIs to be given in a later work and they will be used on the right-hand sides of the monodromy variations of six-point $\mathcal{Z}$-functions below.

4.4.3. Six-point worldsheet functions

According to (4.13) and (4.18), the six-point shuffle-symmetric worldsheet functions satisfy the following monodromy variations:

$$\begin{aligned} D\mathcal{Z}_{123,4,5,6} &= \Omega_1 E_{1|23,4,5,6} - \Omega_3 E_{3|12,4,5,6} \,, \\ D\mathcal{Z}_{12,34,5,6} &= \Omega_1 E_{1|2,34,5,6} - \Omega_2 E_{2|1,34,5,6} + \Omega_3 E_{3|12,4,5,6} - \Omega_4 E_{4|12,3,5,6} \,, \\ D\mathcal{Z}^m_{12,3,4,5,6} &= \Omega_1 E^m_{1|2,3,4,5,6} - \Omega_2 E^m_{2|1,3,4,5,6} + \big[k^m_3 \Omega_3 E_{3|12,4,5,6} + (3 \leftrightarrow 4,5,6)\big] \,, \\ D\mathcal{Z}^{mn}_{1,2,3,4,5,6} &= k^m_1 \Omega_1 E^n_{1|2,3,4,5,6} + k^n_1 \Omega_1 E^m_{1|2,3,4,5,6} + (1 \leftrightarrow 2,3,4,5,6) \,, \\ D\mathcal{Z}_{2|1,3,4,5,6} &= \Omega_2 k^m_2 E^m_{2|1,3,4,5,6} \,. \end{aligned} \tag{4.34}$$

In the appendix A.1 we will obtain the following solutions,

$$\begin{aligned} \mathcal{Z}_{123,4,5,6} &= g^{(1)}_{12} g^{(1)}_{23} + g^{(2)}_{12} + g^{(2)}_{23} - g^{(2)}_{13} \,, \\ \mathcal{Z}_{12,34,5,6} &= g^{(1)}_{12} g^{(1)}_{34} + g^{(2)}_{13} + g^{(2)}_{24} - g^{(2)}_{14} - g^{(2)}_{23} \,, \\ \mathcal{Z}^m_{12,3,4,5,6} &= \ell^m g^{(1)}_{12} + (k^m_2 - k^m_1) g^{(2)}_{12} + \big[k^m_3 (g^{(2)}_{13} - g^{(2)}_{23}) + (3 \leftrightarrow 4,5,6)\big] \,, \\ \mathcal{Z}^{mn}_{1,2,3,4,5,6} &= \ell^m \ell^n + \big[(k^m_1 k^n_2 + k^n_1 k^m_2) g^{(2)}_{12} + (1,2|1,2,3,4,5,6)\big] \,, \\ \mathcal{Z}_{2|1,3,4,5,6} &= 0 \,. \end{aligned} \tag{4.35}$$

In accordance with the discussion in section 4.1, their behavior as the vertex insertions collide corresponds to their tree-level counterparts. For instance, the short-distance behavior $\mathcal{Z}_{123,4,5,6} \to (z_{12} z_{23})^{-1}$ is the same as that of $\mathcal{Z}^{\rm tree}_{123}$.

4.4.3.1. Assembling six-point GEIs

Plugging the above solutions into the expansions (4.26) of six-point GEIs leads to

$$\begin{aligned} E_{1|234,5,6} &= g_{12}^{(1)} g_{23}^{(1)} - g_{12}^{(1)} g_{24}^{(1)} + g_{12}^{(1)} g_{34}^{(1)} - g_{14}^{(1)} g_{23}^{(1)} + g_{14}^{(1)} g_{24}^{(1)} - g_{14}^{(1)} g_{34}^{(1)} \\ &\quad + g_{23}^{(1)} g_{34}^{(1)} + g_{23}^{(2)} - g_{24}^{(2)} + g_{34}^{(2)} \, , \\ E_{1|23,45,6} &= \big(g_{12}^{(1)} + g_{23}^{(1)} + g_{31}^{(1)}\big)\big(g_{14}^{(1)} + g_{45}^{(1)} + g_{51}^{(1)}\big) \, , \\ E^m_{1|23,4,5,6} &= \big(\ell^m + k_4^m g_{14}^{(1)} + k_5^m g_{15}^{(1)} + k_6^m g_{16}^{(1)}\big)\big(g_{12}^{(1)} + g_{23}^{(1)} + g_{31}^{(1)}\big) \\ &\quad + \big[k_2^m (g_{13}^{(1)} g_{23}^{(1)} + g_{12}^{(2)} - g_{13}^{(2)} - g_{23}^{(2)}) - (2 \leftrightarrow 3)\big] \, , \\ E^{mn}_{1|2,3,4,5,6} &= \ell^m \ell^n + \big[k_2^{(m} k_3^{n)} g_{12}^{(1)} g_{13}^{(1)} + (2,3|2,3,4,5,6)\big] \\ &\quad + \big[\ell^{(m} k_2^{n)} g_{12}^{(1)} + 2 k_2^m k_2^n g_{12}^{(2)} + (2 \leftrightarrow 3,4,5,6)\big] \, , \end{aligned} \tag{4.36}$$

as well as

$$\begin{aligned} E_{1|2|3,4,5,6} &= -2 s_{12} g_{12}^{(2)} + g_{12}^{(1)} (\ell \cdot k_2 + s_{23} g_{23}^{(1)} + s_{24} g_{24}^{(1)} + s_{25} g_{25}^{(1)} + s_{26} g_{26}^{(1)}) \\ &\cong \partial g_{12}^{(1)} + s_{12} (g_{12}^{(1)})^2 - 2 s_{12} g_{12}^{(2)} \, . \end{aligned} \tag{4.37}$$

The second line follows from the first one via integration by parts according to (2.20).

The slot-extensions of the above GEIs are given by

$$\begin{aligned} E_{1|234,5,6,7} &\equiv E_{1|234,5,6} \, , \quad E_{1|23,45,6,7} \equiv E_{1|23,45,6} \, , \\ E^m_{1|23,4,5,6,7} &\equiv E^m_{1|23,4,5,6} + k_7^m g_{17}^{(1)} \big(g_{12}^{(1)} + g_{23}^{(1)} + g_{31}^{(1)}\big) \, , \\ E^{mn}_{1|2,3,4,5,6,7} &\equiv \ell^m \ell^n + \big[k_2^{(m} k_3^{n)} g_{12}^{(1)} g_{13}^{(1)} + (2,3|2,\ldots,7)\big] \\ &\quad + \big[\ell^{(m} k_2^{n)} g_{12}^{(1)} + 2 k_2^m k_2^n g_{12}^{(2)} + (2 \leftrightarrow 3,4,5,6,7)\big] \, , \\ E_{1|2|3,4,5,6,7} &\equiv -2 s_{12} g_{12}^{(2)} + g_{12}^{(1)} (\ell \cdot k_2 + s_{23} g_{23}^{(1)} + s_{24} g_{24}^{(1)} + \ldots + s_{27} g_{27}^{(1)}) \\ &\cong \partial g_{12}^{(1)} + s_{12} (g_{12}^{(1)})^2 - 2 s_{12} g_{12}^{(2)} \quad , \end{aligned} \tag{4.38}$$

and they will be used to bootstrap the shuffle-symmetric functions at seven points.

4.4.4. Seven-point worldsheet functions

At seven points, the monodromy variations for the scalar shuffle-symmetric functions following from (4.13) and (4.38) are given by

$$\begin{aligned} D\mathcal{Z}_{1234,5,6,7} &= \Omega_1 E_{1|234,5,6,7} - \Omega_4 E_{4|123,5,6,7} \, , \\ D\mathcal{Z}_{123,45,6,7} &= \Omega_1 E_{1|23,45,6,7} - \Omega_3 E_{3|12,45,6,7} + \Omega_4 E_{4|123,5,6,7} - \Omega_5 E_{5|123,4,6,7} \, , \\ D\mathcal{Z}_{12,34,56,7} &= \Omega_1 E_{1|2,34,56,7} - \Omega_2 E_{2|1,34,56,7} + (12 \leftrightarrow 34, 56) \, , \end{aligned} \tag{4.39}$$

and admit the following solutions:

$$\begin{aligned}\mathcal{Z}_{1234,5,6,7} &= g^{(1)}_{12}g^{(1)}_{23}g^{(1)}_{34} + g^{(3)}_{12} + g^{(3)}_{23} + g^{(3)}_{34} - 2g^{(3)}_{41} \\ &\quad + g^{(1)}_{12}\big(g^{(2)}_{23} + g^{(2)}_{34} - g^{(2)}_{41}\big) + g^{(1)}_{23}\big(g^{(2)}_{12} + g^{(2)}_{34} - g^{(2)}_{41}\big) + g^{(1)}_{34}\big(g^{(2)}_{12} + g^{(2)}_{23} - g^{(2)}_{41}\big)\,, \\ \mathcal{Z}_{123,45,6,7} &= g^{(1)}_{12}g^{(1)}_{23}g^{(1)}_{45} + g^{(1)}_{45}\big(g^{(2)}_{12} + g^{(2)}_{23} - g^{(2)}_{13}\big) \\ &\quad + \big(g^{(1)}_{12} + g^{(1)}_{23} + g^{(1)}_{31}\big)\big(g^{(2)}_{14} - g^{(2)}_{15} + g^{(2)}_{35} - g^{(2)}_{34}\big)\,, \\ \mathcal{Z}_{12,34,56,7} &= g^{(1)}_{12}g^{(1)}_{34}g^{(1)}_{56} + g^{(1)}_{12}\big(g^{(2)}_{35} - g^{(2)}_{36} - g^{(2)}_{45} + g^{(2)}_{46}\big) \\ &\quad + g^{(1)}_{34}\big(g^{(2)}_{15} - g^{(2)}_{16} - g^{(2)}_{25} + g^{(2)}_{26}\big) + g^{(1)}_{56}\big(g^{(2)}_{13} - g^{(2)}_{14} - g^{(2)}_{23} + g^{(2)}_{24}\big) \\ &\quad + g^{(1)}_{15}\big(g^{(2)}_{13} - g^{(2)}_{14} - g^{(2)}_{35} + g^{(2)}_{45}\big) + g^{(1)}_{16}\big(g^{(2)}_{14} - g^{(2)}_{13} + g^{(2)}_{36} - g^{(2)}_{46}\big) \\ &\quad + g^{(1)}_{25}\big(g^{(2)}_{24} - g^{(2)}_{23} - g^{(2)}_{45} + g^{(2)}_{35}\big) + g^{(1)}_{26}\big(g^{(2)}_{23} - g^{(2)}_{24} - g^{(2)}_{36} + g^{(2)}_{46}\big)\,,\end{aligned} \tag{4.40}$$

The solutions for the tensorial functions will be presented in Appendix A.2, see in particular (A.26), (A.29) and (A.30).

In addition to the above unrefined solutions, the monodromy variations of the three seven-point topologies of refined worldsheet functions following from (4.18) read

$$\begin{aligned} D\mathcal{Z}_{12|3,4,5,6,7} &= \Omega_1 E_{1|2|3,4,5,6,7} - \Omega_2 E_{2|1|3,4,5,6,7}\,, \\ D\mathcal{Z}_{1|23,4,5,6,7} &= \Omega_1 k^p_1 E^p_{1|23,4,5,6,7} + \Omega_2 E_{2|1|3,4,5,6,7} - \Omega_3 E_{3|1|2,4,5,6,7}\,, \\ D\mathcal{Z}^m_{1|2,3,4,5,6,7} &= \Omega_1 k^p_1 E^{pm}_{1|2,3,4,5,6,7} + \big[\Omega_2 k^m_2 E_{2|1|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\big]\,,\end{aligned} \tag{4.41}$$

where the extended GEI above were defined in (4.38), with solutions

$$\begin{aligned} \mathcal{Z}_{12|3,4,5,6,7} &= \partial g^{(2)}_{12} + s_{12}g^{(1)}_{12}g^{(2)}_{12} - 3s_{12}g^{(3)}_{12}\,, \\ \mathcal{Z}_{1|23,4,5,6,7} &= \mathcal{Z}_{13|2,4,5,6,7} - \mathcal{Z}_{12|3,4,5,6,7}\,, \\ \mathcal{Z}^m_{1|2,3,4,5,6,7} &= -\big[k^m_2 \mathcal{Z}_{12|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\big]\,.\end{aligned} \tag{4.42}$$

Although not manifest, the worldsheet singularities of the above functions are the ones expected from their labeling according to the discussion in section 4.1. For instance, the function $\mathcal{Z}_{12,34,56,7}$ can only have singularities as $z_1 \to z_2$ (corresponding to the word 12) and similarly for 34 and 56. However, its expansion contains certain factors of $g^{(1)}_{ij}$ that suggest the presence of "forbidden" singularities; like $g^{(1)}_{15}\big(g^{(2)}_{13} - g^{(2)}_{14} - g^{(2)}_{35} + g^{(2)}_{45}\big)$ as $z_1 \to z_5$. But a careful analysis using the Laurent expansions (B.6) shows that it is in fact non-singular as $z_1 \to z_5$ (similar conclusions apply for the other terms). Note that functions which involve Mandelstam variables such as $s_{12}g^{(1)}_{12}$ are considered non-singular as they don't generate kinematic poles when integrated along with the Koba–Nielsen factor. Therefore all functions in (4.42) are in fact non-singular upon integration over z_j.

Having the shuffle-symmetric worldsheet functions we can now assemble seven-point GEIs as discussed in the previous section. The results are displayed in Appendix A, see in particular (A.31) to (A.34). Also, the building blocks of section 6 turn out to admit the compact representations (6.22) or (6.23).

4.4.5. Eight-point shuffle-symmetric worldsheet functions

The system of monodromy variations can be solved explicitly at eight points following the bootstrap approach. This will be done in the appendix A.3.

5. Duality between worldsheet functions and kinematics

In this section, we will illustrate various further facets of the duality between worldsheet functions and kinematics. It will be exemplified that GEIs $E^{\cdots}_{1|\ldots}$ share the relations and symmetries of the kinematic factors $C^{\cdots}_{1|\ldots}$ and $P^{\cdots}_{1|\ldots}$ discussed in part I. Some of these relations will be shown to have an echo at the level of the $\mathcal{Z}$-functions. We spell out the concrete evidence for the duality and formulate conjectures for the all-multiplicity patterns. If these conjectures are correct, the kinematic and worldsheet ingredients of the open-string correlators $\mathcal{K}_n$ in (C.1) to (C.4) enter on completely symmetric footing. Like this, we support the double-copy structure of one-loop open-string amplitudes [4] up to and including seven points. At eight points we will sometimes encounter terms proportional to the holomorphic Eisenstein series G_4 that do not have a corresponding kinematic companion. Accommodating these terms with the duality between worldsheet functions and kinematics is left for a future work.

5.1. The GEI dual to BRST-cohomology identities

The appearance of the correlators $\mathcal{K}_n(\ell)$ in open- and closed-string amplitudes is insensitive to BRST-exact terms. This has been exploited in [23] to derive so-called Jacobi identities in the BRST cohomology that relate momentum contractions $k_1^m C^{m\cdots}_{1|A,B,\ldots}$ and $k_A^m C^{m\cdots}_{1|A,B,\ldots}$ to (pseudo-)invariants of lower tensor rank, see section I.5.4. We will now exemplify that GEIs $E^{m\cdots}_{1|A,B,\ldots}$ obey the same Jacobi identities between different tensor rank and degree of refinement, where BRST-exact terms translate into total derivatives.

5.1.1. Five points

Based on momentum conservation, one can show that the following combinations of five-point GEIs (4.31) and (4.32) conspire to total Koba–Nielsen derivatives (2.20)

$$\begin{aligned} k_1^m E^m_{1|2,3,4,5} &= \partial_1 \log \mathcal{I}_5 \\ k_2^m E^m_{1|2,3,4,5} + \big[s_{23} E_{1|23,4,5} + (3 \leftrightarrow 4,5) \big] &= \partial_2 \log \mathcal{I}_5 \end{aligned} \tag{5.1}$$

and can therefore be dropped from open- and closed-string amplitudes. The first relation is in one-to-one correspondence with the cohomology identity $Q\mathcal{J}_{1|2,3,4,5} = k_1^m C^m_{1|2,3,4,5} + \Delta_{1|2,3,4,5}$ after dropping the BRST-exact anomaly factor $\Delta_{1|2,3,4,5}$ (cf. section I.5.3). Similarly, the second line of (5.1) has the kinematic counterpart (I.5.41) involving $k_2^m C^m_{1|2,3,4,5}$.

5.1.2. Six points

Similarly, at six points we find Jacobi relations among the GEIs in (4.36) and (4.37) which exactly match the kinematic identities listed in (I.5.42) (cf. section 10 of [23]),

$$\begin{aligned} k_4^m E^m_{1|23,4,5,6} &\cong -s_{24} E_{1|324,5,6} + s_{34} E_{1|234,5,6} - s_{45} E_{1|23,45,6} - s_{46} E_{1|23,46,5} \,, \\ k_{23}^m E^m_{1|23,4,5,6} &\cong \big[s_{24} E_{1|324,5,6} - s_{34} E_{1|234,5,6} + (4 \leftrightarrow 5,6) \big] + E_{1|2|3,4,5,6} - E_{1|3|2,4,5,6} \,, \\ k_1^m E^m_{1|23,4,5,6} &\cong E_{1|3|2,4,5,6} - E_{1|2|3,4,5,6} \,, \\ k_2^m E^{mn}_{1|2,3,4,5,6} &\cong k_2^n E_{1|2|3,4,5,6} - \big[s_{23} E^n_{1|23,4,5,6} + (3 \leftrightarrow 4,5,6) \big] \,, \\ k_1^m E^{mn}_{1|2,3,4,5,6} &\cong - \big[k_2^n E_{1|2|3,4,5,6} + (2 \leftrightarrow 3,4,5,6) \big] \,. \end{aligned} \tag{5.2}$$

As in (2.22), the $\cong$ notation is a reminder that z_j-derivatives have been discarded in passing to the right-hand side. Note that momentum conservation reduces the identities for contraction with k_1 to combinations of the remaining ones involving $k_A^m E^{m\ldots}_{1|A,\ldots}$.

5.1.3. Higher multiplicity

More generally, the elliptic identities that are dual to the BRST-cohomology identities in section 9 of [23] can be written as

$$\begin{aligned} k^p_{A_1} E^p_{1|A_1,\ldots,A_4} &= -\big[E_{1|S[A_1,A_2],A_3,\ldots,A_4} + (A_2 \leftrightarrow A_3,\ldots,A_4) \big] \\ &\quad + \sum_{XY=A_1} \big[E_{1|X|Y,A_2,\ldots,A_4} - (X \leftrightarrow Y) \big] \,, \\ k^p_{A_1} E^{pm}_{1|A_1,\ldots,A_5} &= k^m_{A_1} E_{1|A_1|A_2,\ldots,A_5} - \big[E^m_{1|S[A_1,A_2],A_3,\ldots,A_5} + (A_2 \leftrightarrow A_3,\ldots,A_5) \big] \\ &\quad + \sum_{XY=A_1} \big[E^m_{1|X|Y,A_2,\ldots,A_5} - (X \leftrightarrow Y) \big] \,, \end{aligned} \tag{5.3}$$

where the $S[A,B]$ map is defined in section I.5.1.1 and yields $E_{1|S[2,3],4,5} = s_{23}E_{1|23,4,5}$ as well as $E_{1|S[23,4],5,6} = s_{34}E_{1|234,5,6} - s_{24}E_{1|324,5,6}$ in the simplest cases. Given a word $A = a_1a_2\ldots a_{|A|}$, the sum over deconcatenations $XY = A$ is understood to comprise all non-empty $X = a_1a_2\ldots a_j$ and $Y = a_{j+1}\ldots a_{|A|}$ with $j = 1,2,\ldots,|A|-1$. We have verified all of (5.3) up to and including eight points, and their higher-point generalizations are plausible by the dual kinematic identities given in (I.5.43) and [23]. Note the absence of elliptic-function duals to the BRST-exact anomaly terms $\Delta^{m_1\ldots}_{1|A_1,\ldots}$ without refined slots.

Following the worldsheet duals of the higher-rank identities in section 9 of [23], one arrives at

$$\begin{aligned}
k^p_{A_1}E^{pmn}_{1|A_1,\ldots,A_6} &= k^{(m}_{A_1}E^{n)}_{1|A_1|A_2,\ldots,A_6} - \big[E^{mn}_{1|S[A_1,A_2],A_3,\ldots,A_6} + (A_2 \leftrightarrow A_3,\ldots,A_6)\big] \\
&\quad + \sum_{XY=A_1}\big[E^{mn}_{1|X|Y,A_2,\ldots,A_6} - (X\leftrightarrow Y)\big] + \delta^{mn}G_{1|A_1|A_2,\ldots,A_6} \qquad (5.4)\\
k^p_{A_1}E^{pm_1\ldots m_r}_{1|A_1,\ldots,A_{r+4}} &= k^{(m_1}_{A_1}E^{m_2\ldots m_r)}_{1|A_1|A_2,\ldots,A_{r+4}} - \big[E^{m_1\ldots m_r}_{1|S[A_1,A_2],A_3,\ldots,A_{r+4}} + (A_2 \leftrightarrow A_3,\ldots,A_{r+4})\big] \\
&\quad + \sum_{XY=A_1}\big[E^{m_1\ldots m_r}_{1|X|Y,A_2,\ldots,A_{r+4}} - (X\leftrightarrow Y)\big] + \delta^{(m_1m_2}G^{m_3\ldots m_r)}_{1|A_1|A_2,\ldots,A_{r+4}}\,,
\end{aligned}$$

for some a priori undetermined GEIs $G_{1|\ldots}$ in the trace component. The latter can be thought of as a tentative GEI dual of the refined anomaly superfields $\Delta^{m_1\ldots}_{1|A_1,\ldots,A_d|B,\ldots}$ that are no longer BRST-exact if $d \geq 1$, see section I.5.3. The representations of GEIs up to and including eight points given in this work yield $G^{m_1\ldots}_{1|A_1|A_2,\ldots} = 0$, e.g. the seven-point GEIs in (A.31) to (A.34) can be checked to obey

$$k^p_2E^{mnp}_{1|2,3,\ldots,7} = k^{(m}_2E^{n)}_{1|2|3,\ldots,7} - \big[s_{23}E^{mn}_{1|23,4,\ldots,7} + (3\leftrightarrow 4,\ldots,7)\big] \qquad (5.5)$$

which amounts to $G_{1|2|3,4,5,6,7} = 0$ in (5.4). Still, it is worthwhile to keep in mind that non-zero choices of $G^{m_1\ldots}_{1|A_1|A_2,\ldots}$ are still compatible with the duality between kinematics and worldsheet functions.

Finally, the above identities generalize straightforwardly to slot-extensions of GEIs such as $E_{1|A,B,C} \to E_{1|A,B,C,D}$ and its generalizations in the $D\mathcal{Z}$-variations, namely

$$\begin{aligned}
k^p_{A_1}E^{pm_1\ldots m_r}_{1|A_1,\ldots,A_{r+5}} &= k^{(m_1}_{A_1}E^{m_2\ldots m_r)}_{1|A_1|A_2,\ldots,A_{r+5}} - \big[E^{m_1\ldots m_r}_{1|S[A_1,A_2],A_3,\ldots,A_{r+5}} + (A_2 \leftrightarrow A_3,\ldots,A_{r+5})\big] \\
&\quad + \sum_{XY=A_1}\big[E^{m_1\ldots m_r}_{1|X|Y,A_2,\ldots,A_{r+5}} - (X\leftrightarrow Y)\big] + \delta^{(m_1m_2}G^{m_3\ldots m_r)}_{1|A_1|A_2,\ldots,A_{r+5}}\,. \qquad (5.6)
\end{aligned}$$

At higher degree of refinement, appropriate choices of GEIs should obey the dual of the most general Jacobi identity (I.5.45) on the kinematic side

$$\begin{aligned}0 &= \big[G^{m_1\ldots m_r}_{1|A_2,\ldots,A_d|A_1,B_1,\ldots,B_{r+d+2}} + (A_1 \leftrightarrow A_2,\ldots,A_d)\big] + \delta^{(m_1 m_2} G^{m_3\ldots m_r)}_{1|A_1,\ldots,A_d|B_1,\ldots,B_{r+d+2}} \\ &+ \big[k^p_{A_1} E^{pm_1\ldots m_r}_{1|A_2,\ldots,A_d|A_1,B_1,\ldots,B_{r+d+2}} + (A_1 \leftrightarrow A_2,\ldots,A_d)\big] - k^{(m_1}_{A_1A_2\ldots A_d} E^{m_2\ldots m_r)}_{1|A_1,\ldots,A_d|B_1,\ldots,B_{r+d+2}} \\ &+ \Big(\big[E^{m_1\ldots m_r}_{1|A_2,\ldots,A_d|S[A_1,B_1],B_2,\ldots,B_{r+d+2}} + (B_1 \leftrightarrow B_2,\ldots,B_{r+d+2})\big] \\ &- \sum_{XY=A_1} (E^{m_1\ldots m_r}_{1|X,A_2,\ldots,A_d|Y,B_1,\ldots,B_{r+d+2}} - E^{m_1\ldots m_r}_{1|Y,A_2,\ldots,A_d|X,B_1,\ldots,B_{r+d+2}}) + (A_1 \leftrightarrow A_2,\ldots,A_d)\Big)\,,\end{aligned} \tag{5.7}$$

that are checked up to and including eight points. These proposals will serve as a key input for the all-multiplicity construction of GEIs from generating series. Note that the first term $G^{m_1\ldots m_r}_{1|A_2,\ldots,A_d|A_1,B_1,\ldots,B_{r+d+2}}$ does not have any refined slots at $d=1$ and should vanish by the duality with the BRST-exact unrefined anomaly superfields. In fact, we even observe stronger identities among seven- and eight-point GEIs such as

$$0 \cong k^m_3 E^m_{1|2|3,4,5,6,7} - s_{23}E_{1|23|4,5,6,7} + \big[s_{34}E_{1|2|34,5,6,7} + (4 \leftrightarrow 5,6,7)\big] \tag{5.8}$$

with a single momentum contraction, which implies (5.7) upon symmetrization in $2 \leftrightarrow 3$. The kinematic dual of (5.8) involving $k^m_3 P^m_{1|2|3,4,5,6,7} + \Delta_{1|3|2,4,5,6,7}$ can be found in (I.5.44). As detailed in section III.4.4.4, identities like (5.8) that involve just a single momentum contraction $k^p_{A_1} E^{pm_1\ldots}_{1|A_2,\ldots,A_d|A_1,B_1,\ldots}$ play a key role for the path towards local and BRST-invariant n-point correlators in future work.

5.2. The GEI dual to BRST change-of-basis identities

In section 11 of [23] several identities among (pseudo-)invariants were derived using BRST-cohomology manipulations that implement a *change of basis*[11]. The simplest examples are

$$\begin{aligned}C_{3|12,4,5} &= C_{1|23,4,5} + Q(\ldots)\,, \\ C_{2|1,34,5} &= C_{1|2,34,5} + C_{1|23,4,5} - C_{1|24,3,5} + Q(\ldots)\,, \\ C^m_{2|1,3,4,5} &= C^m_{1|2,3,4,5} + \big[k^m_3 C_{1|23,4,5} + (3 \leftrightarrow 4,5)\big] + Q(\ldots)\,, \\ P_{2|1|3,4,5,6} &= P_{1|2|3,4,5,6} + \mathcal{Y}_{12,3,4,5,6} + Q(\ldots)\,,\end{aligned} \tag{5.9}$$

where the right-hand side is written in terms of the canonical basis of $C_{1|A,B,C}$ and $P_{1|A|B,C,D,E}$ with leg 1 in the first position of the subscript. The BRST-exact terms in the ellipses are spelled out in [23]. Naturally, these identities have an elliptic dual under $C \to E$ as well as its "refined" version[12] $P \to E$.

[11] They were referred to as "BRST-canonicalization" identities in [23].

[12] The pseudo-invariant $P_{i|A|B,\ldots}$ should really be denoted $C_{i|A|B,\ldots}$, as it would unify this and countless other formulas.

5.2.1. Five points

It is straightforward to show that the five-point GEIs in (6.16) obey change-of-basis identities dual to (5.9),

$$
\begin{aligned}
E_{3|12,4,5} &= E_{1|23,4,5}\,, \\
E_{2|1,34,5} &= E_{1|2,34,5} + E_{1|23,4,5} - E_{1|24,3,5}\,, \\
E^m_{2|1,3,4,5} &= E^m_{1|2,3,4,5} + \big[k_3^m E_{1|23,4,5} + (3 \leftrightarrow 4,5)\big]\,.
\end{aligned}
\tag{5.10}
$$

As detailed in appendix A, similar change-of-basis identities involving GEIs play a major role in the solution of the monodromy-variation equations. On the right-hand sides of the monodromy variations $D\mathcal{Z}$ in section 4.2, however, the GEIs are "extended" to have one additional word. While the scalar identities in (5.10) hold in identical form for $E_{1|23,4,5,6} = E_{1|23,4,5}$, the vector identity is extended by an obvious extra permutation involving leg 6, i.e. $E^m_{2|1,3,4,5,6} = E^m_{1|2,3,4,5,6} + \big[k_3^m E_{1|23,4,5,6} + (3 \leftrightarrow 4,5,6)\big]$.

5.2.2. Six points

Change-of-basis identities among six-point GEIs take the identical form as compared to the relations among (pseudo-)invariants in section 11 and appendix F of [23],

$$
\begin{aligned}
E_{2|134,5,6} &= E_{1|342,5,6} \\
E_{2|13,45,6} &= E_{1|32,45,6} + E_{1|324,5,6} - E_{1|325,4,6} \\
E_{2|1,345,6} &= E_{1|2,345,6} + E_{1|234,5,6} + E_{1|254,3,6} + E_{1|325,4,6} + E_{1|23,45,6} + E_{1|25,43,6} \\
E_{2|1,34,56} &= E_{1|2,34,56} + E_{1|23,56,4} - E_{1|24,56,3} + E_{1|25,34,6} - E_{1|26,34,5} \\
&\quad - E_{1|325,6,4} + E_{1|326,5,4} + E_{1|425,6,3} - E_{1|426,5,3} \\
E^m_{2|13,4,5,6} &= E^m_{1|32,4,5,6} + \big[k_4^m E_{1|324,5,6} + (4 \leftrightarrow 5,6)\big] \\
E^m_{2|1,34,5,6} &= E^m_{1|2,34,5,6} + E^m_{1|23,4,5,6} - E^m_{1|24,3,5,6} + k_4^m E_{1|234,5,6} - k_3^m E_{1|243,5,6} \\
&\quad + \big[k_5^m (E_{1|25,34,6} - E_{1|325,4,6} + E_{1|425,3,6}) + (5 \leftrightarrow 6)\big]\,.
\end{aligned}
\tag{5.11}
$$

Similar to the translation of BRST variations to $D\mathcal{Z}$-variations in the previous section, the $\mathcal{Y}$-superfield in the pseudo-invariant identity of (5.9) has no GEI analogue,

$$
\begin{aligned}
E_{2|1|3,4,5,6} &= E_{1|2|3,4,5,6} \\
E_{2|3|1,4,5,6} &= E_{1|3|2,4,5,6} + k_3^m E^m_{1|23,4,5,6} + \big[s_{34} E_{1|234,5,6} + (4 \leftrightarrow 5,6)\big] \\
E^{mn}_{2|1,3,4,5,6} &= E^{mn}_{1|2,3,4,5,6} + \big[k_3^{(m} E^{n)}_{1|23,4,5,6} + (3 \leftrightarrow 4,5,6)\big] \\
&\quad - \big[k_3^{(m} k_4^{n)} E_{1|324,5,6} + (3,4|3,4,5,6)\big]\,.
\end{aligned}
\tag{5.12}
$$

More general cases such as expanding $E_{2|13,45,67,89}$ in terms of $E_{1|\ldots}$ have no matching analogous identities in terms of $C_{1|A,B,C}$, so the required change-of-basis identities are not readily available from [23]. These identities can, however, be generated using the general algorithm described in the appendix I.A.3.

5.3. The worldsheet analogue of kinematic trace relations

We have seen in section I.4.4.4 that the kinematic building blocks satisfy certain identities that relate traces of tensorial building blocks at refinement d to sums of building blocks[13] of refinement $d{+}1$. For instance, (I.4.47) at the level of Berends–Giele currents reads [23]

$$\frac{1}{2}\delta_{np}\mathcal{J}^{npm_1\ldots m_r}_{A_1,\ldots,A_d|B_1,\ldots,B_{d+r+5}} = \mathcal{J}^{m_1\ldots m_r}_{A_1,\ldots,A_d,B_1|B_2,\ldots,B_{d+r+5}} + (B_1\leftrightarrow B_2,\ldots,B_{d+r+5})\,, \quad (5.13)$$

and it is natural to ask what is the corresponding statement in terms of worldsheet functions. Given that this identity relates BRST-covariant Berends–Giele superfields rather than (pseudo-)invariants, their worldsheet analogues should concern the $\mathcal{Z}$-functions subject to non-vanishing D-variations. Note that the worldsheet functions depend on one additional word when compared to their kinematic counterpart ($\mathcal{Z}_{A,B,C,D} \leftrightarrow M_{A,B,C}$), therefore their trace relations will also have one extra permutation.

5.3.1. Six points

At six points one can show from the explicit solutions (4.35) for the $\mathcal{Z}$-functions that the following trace relation is satisfied up to a total derivative (2.23) in τ:

$$\frac{1}{2}\delta_{mn}\mathcal{Z}^{mn}_{1,2,3,4,5,6} \cong \mathcal{Z}_{1|2,3,4,5,6} + (1\leftrightarrow 2,3,4,5,6)\,. \quad (5.14)$$

In order so see this, we note that the functions $\mathcal{Z}_{1|2,3,4,5,6}$ on the right-hand side vanish (see (4.35) and appendix A.1), and the trace of the tensor in (4.35) yields the τ-derivative (2.21) of the Koba–Nielsen factor,

$$\frac{1}{2}\delta_{mn}\mathcal{Z}^{mn}_{1,2,3,4,5,6} = \frac{1}{2}\ell^2 + \big[s_{12}g^{(2)}_{12} + (1,2|1,2,3,4,5,6)\big] = 2\pi i\frac{\partial}{\partial\tau}\log\mathcal{I}_6(\ell)\,. \quad (5.15)$$

5.3.2. Seven points

Similarly, the solutions of the seven-point monodromy variations in section 4.4.4 satisfy

$$\begin{aligned}
&\frac{1}{2}\mathcal{Z}^{mpp}_{1,2,3,4,5,6,7} - \big[\mathcal{Z}^{m}_{2|1,3,4,5,6,7} + (2\leftrightarrow 3,\ldots,7)\big] \cong \mathcal{Z}^{m}_{1|2,3,4,5,6,7}\,, \quad (5.16)\\
&\frac{1}{2}\mathcal{Z}^{pp}_{12,3,4,5,6,7} - \big[\mathcal{Z}_{3|12,4,5,6,7} + (3\leftrightarrow 4,5,6,7)\big] \cong \mathcal{Z}_{12|3,4,5,6,7}\,,
\end{aligned}$$

in accordance with the expectation from the analogy with kinematic building blocks (5.13). Note that τ-derivatives acting on both the Koba–Nielsen factor and ℓ^m or $g^{(1)}_{12}$ have been discarded in (5.16), using the mixed heat equation (2.9) for the latter.

[13] Note that the building blocks with $d=0$ are denoted by M rather than $\mathcal{J}$.

5.3.3. Eight points

From the discussion in section 4.2.4 we know that the solutions of the eight-point monodromy variations are slightly ambiguous due to the Eisenstein series G_4. This freedom can be exploited to yield two sets of solutions differing by terms proportional to G_4, depending on whether they satisfy the trace relations or not. On the one hand, the *naive* solutions to the monodromy equations in the appendix A.3 fail to satisfy all but one of the dual trace relations,

$$\begin{aligned}
&\frac{1}{2}\mathcal{Z}^{pp}_{12,34,5,\ldots,8} - \big[\mathcal{Z}_{12|34,5,\ldots,8} + (12 \leftrightarrow 34,5,\ldots,8)\big] \cong -R_{12,34,5,6,7,8}\,, \\
&\frac{1}{2}\mathcal{Z}^{pp}_{123,4,5,6,7,8} - \big[\mathcal{Z}_{123|4,5,6,\ldots,8} + (123 \leftrightarrow 4,\ldots,8)\big] \cong -R_{123,4,5,6,7,8}\,, \\
&\frac{1}{2}\mathcal{Z}^{mpp}_{12,3,4,5,6,7,8} - \big[\mathcal{Z}^{m}_{12|3,4,5,6,\ldots,8} + (12 \leftrightarrow 3,\ldots,8)\big] \cong -R^{m}_{12,3,4,5,6,7,8}\,, \\
&\frac{1}{2}\mathcal{Z}^{mnpp}_{1,2,3,4,5,6,7,8} - \big[\mathcal{Z}^{mn}_{1|2,4,5,6,7,8} + (1 \leftrightarrow 2,3,\ldots,8)\big] \cong -R^{mn}_{1,2,3,4,5,6,7,8}\,, \\
&\frac{1}{2}\mathcal{Z}^{pp}_{1|2,3,4,5,6,7,8} - \big[\mathcal{Z}_{1,2|3,4,5,6,7,8} + (2 \leftrightarrow 3,\ldots,8)\big] \cong 0\,,
\end{aligned} \tag{5.17}$$

where

$$\begin{aligned}
R_{12,34,5,6,7,8} &= 3\mathrm{G}_4\big(s_{13} - s_{14} - s_{23} + s_{24}\big)\,, \\
R_{123,4,5,6,7,8} &= 3\mathrm{G}_4\big(s_{12} - 2s_{13} + s_{23}\big)\,, \\
R^{m}_{12,3,4,5,6,7,8} &= 3\mathrm{G}_4\big(s_{12}(k_2^m - k_1^m) + \big[k_3^m(s_{13} - s_{23}) + (3 \leftrightarrow 4,5,6,7,8)\big]\big)\,, \\
R^{mn}_{1,2,3,4,5,6,7,8} &= 3\mathrm{G}_4 k_1^{(m} k_2^{n)} s_{12} + (1,2|1,2,\ldots,8)\,.
\end{aligned} \tag{5.18}$$

But note that these failed trace relations are a peculiarity of certain eight-point $\mathcal{Z}$-functions that will be used in the eight-point correlator in section III.3.5. Since these functions will be multiplying local kinematic building blocks, one may exploit the kinematic trace relations reviewed in section I.4.4.4 to add *deformations*

$$\hat{\mathcal{Z}} \equiv \mathcal{Z} + \delta\mathcal{Z}\,, \tag{5.19}$$

while keeping the overall eight-point correlator unchanged. Starting from the naive solutions $\mathcal{Z}$ of the monodromy variations in the appendix A.3, the deformed functions $\hat{\mathcal{Z}}$ in (5.19) can be made to satisfy all trace relations by adding[14]

$$\begin{aligned}
&\delta\mathcal{Z}^{mn}_{12,34,5,6,7,8} = -\delta^{mn} R_{12,34,5,6,7,8}\,, \qquad \delta\mathcal{Z}^{mn}_{123,4,5,6,7,8} = -\delta^{mn} R_{123,4,5,6,7,8}\,, \\
&\delta\mathcal{Z}^{mnp}_{12,3,4,5,6,7,8} = -\delta^{(mn} R^{p)}_{12,3,4,5,6,7,8}\,, \\
&\delta\mathcal{Z}^{mnpq}_{1,2,3,4,5,6,7,8} = -\delta^{(mn} R^{pq)}_{1,2,3,4,5,6,7,8} + \frac{1}{4}\delta^{(mn}\delta^{pq)} R^{aa}_{1,2,3,4,5,6,7,8}\,,
\end{aligned} \tag{5.20}$$

[14] Beware of the definition (I.2.3), in particular, $\delta^{(mn}\delta^{pq)} = \delta^{mn}\delta^{pq} + \delta^{mp}\delta^{nq} + \delta^{mq}\delta^{pn}$.

in the unrefined cases, while the deformations of the refined functions read,

$$\delta\mathcal{Z}_{123|4,5,6,7,8} = -R_{123,4,5,6,7,8}\,, \qquad \delta\mathcal{Z}_{1|23,45,6,7,8} = -R_{23,45,1,6,7,8}\,,$$
$$\delta\mathcal{Z}^m_{1|23,4,5,6,7,8} = -R^m_{23,1,4,5,6,7,8}\,, \qquad \delta\mathcal{Z}_{1|234,5,6,7,8} = -R_{234,1,5,6,7,8}\,,$$
$$\delta\mathcal{Z}^m_{12|3,4,5,6,7,8} = -R^m_{12,3,4,5,6,7,8}\,, \qquad \delta\mathcal{Z}_{12|34,5,6,7,8} = -R_{12,34,5,6,7,8}\,,$$
$$\delta\mathcal{Z}^{mn}_{1|2,3,4,5,6,7,8} = -R^{mn}_{1,2,3,4,5,6,7,8} - \frac{1}{4}\delta^{mn}R^{aa}_{1,2,3,4,5,6,7,8}\,. \tag{5.21}$$

In addition, in order to preserve the last trace relation of (5.17), we have

$$\delta\mathcal{Z}_{1,2|3,4,5,6,7,8} = -\frac{1}{4}R^{aa}_{1,2,3,4,5,6,7,8}\,, \tag{5.22}$$

where the shorthands R proportional to G_4 were defined in (5.18). Once we present the eight-point correlator in section III.3.5, it will be straightforward to verify that the above deformations (5.19) keep it invariant.

5.4. The worldsheet analogue of kinematic anomaly invariants

The vanishing of the six-point function $\mathcal{Z}_{1|2,3,4,5,6}$ can be understood as a correspondence between refined worldsheet functions at multiplicity n and unrefined $\mathcal{Y}$ superfields at multiplicity $n-1$. More precisely, the BRST-exact linear combinations $\Delta_{1|\ldots}$ of unrefined anomaly superfields [23] reviewed in section I.5.3 are observed to match the vanishing of the corresponding linear combinations of refined worldsheet functions under the map

$$\mathcal{Y}^{m\ldots}_{1A,B_1,\ldots} \leftrightarrow \mathcal{Z}^{m\ldots}_{1A|B_1,\ldots}\,. \tag{5.23}$$

In the following we will use the notation $\mathcal{Z}^{\Delta}_{1|A,B,C,D,E}$ to denote the worldsheet counterpart of $\Delta_{1|A,B,C,D,E}$ that follows the same combinatorics (with obvious generalizations to tensors and refined cases). We will see that the six- and seven-point $\mathcal{Z}^{\Delta}$ vanish up to total derivatives (confirming the suggested duality) whereas subtle contributions $\sim \mathrm{G}_4$ may arise at eight points.

5.4.1. Six points

At six points, the vanishing of the components $\langle\Delta_{1|2,3,4,5}\rangle = \langle\mathcal{Y}_{1,2,3,4,5}\rangle$ suggests that its worldsheet analogue under the map (5.23) also vanishes. Indeed, as anticipated in (4.35) and detailed in appendix A.1,

$$\mathcal{Z}^{\Delta}_{1|2,3,4,5,6} \equiv \mathcal{Z}_{1|2,3,4,5,6} \cong 0\,. \tag{5.24}$$

5.4.2. Seven points

The natural next step is to check whether the seven-point refined functions following from the combinatorics of the six-point BRST-exact superfields [23],

$$\begin{aligned}\Delta_{1|23,4,5,6} &= \mathcal{Y}_{1,23,4,5,6} + \mathcal{Y}_{12,3,4,5,6} - \mathcal{Y}_{13,2,4,5,6}\,, \\ \Delta^m_{1|2,3,4,5,6} &= \mathcal{Y}^m_{1,2,3,4,5,6} + \big[k_2^m \mathcal{Y}_{12,3,4,5,6} + (2\leftrightarrow 3,\ldots,6)\big]\,,\end{aligned} \tag{5.25}$$

also vanish. This is indeed the case, as the solutions (4.42) for the refined $\mathcal{Z}$-functions imply the vanishing of

$$\begin{aligned}\mathcal{Z}^{\Delta}_{1|23,4,5,6,7} &\equiv \mathcal{Z}_{1|23,4,5,6,7} + \mathcal{Z}_{12|3,4,5,6,7} - \mathcal{Z}_{13|2,4,5,6,7} \cong 0\,, \\ \mathcal{Z}^{\Delta,m}_{1|2,3,4,5,6,7} &\equiv \mathcal{Z}^m_{1|2,3,4,5,6,7} + \big[k_2^m \mathcal{Z}_{12|3,4,5,6,7} + (2\leftrightarrow 3,4,\ldots,7)\big] \cong 0\,.\end{aligned} \tag{5.26}$$

Therefore, the pattern established in the six-point vanishing of $\mathcal{Z}_{1|2,3,4,5,6}$ in (5.24) extends to seven points; worldsheet functions that correspond to BRST-exact superfields $\Delta_{1|\ldots}$ vanish up to total derivatives.

5.4.3. Eight points

However, at eight points something peculiar happens. From the superfield expansions of the BRST-exact anomaly building blocks, the map (5.23) leads to

$$\begin{aligned}\mathcal{Z}^{\Delta}_{1|234,5,6,7,8} &= \mathcal{Z}_{1|234,5,6,7,8} + \mathcal{Z}_{12|34,5,6,7,8} + \mathcal{Z}_{123|4,5,6,7,8} - \mathcal{Z}_{124|3,5,6,7,8} \\ &\quad - \mathcal{Z}_{14|23,5,6,7,8} - \mathcal{Z}_{142|3,5,6,7,8} + \mathcal{Z}_{143|2,5,6,7,8}\,, \\ \mathcal{Z}^{\Delta}_{1|23,45,6,7,8} &= \mathcal{Z}_{1|23,45,6,7,8} + \mathcal{Z}_{12|45,3,6,7,8} - \mathcal{Z}_{13|45,2,6,7,8} + \mathcal{Z}_{14|23,5,6,7,8} - \mathcal{Z}_{15|23,4,6,7,8} \\ &\quad - \mathcal{Z}_{412|3,5,6,7,8} + \mathcal{Z}_{314|2,5,6,7,8} + \mathcal{Z}_{215|3,4,6,7,8} - \mathcal{Z}_{315|2,4,6,7,8}\,, \\ \mathcal{Z}^{\Delta,m}_{1|23,4,5,6,7,8} &= \mathcal{Z}^m_{1|23,4,\ldots,8} + \mathcal{Z}^m_{12|3,\ldots,8} - \mathcal{Z}^m_{13|2,4,\ldots,8} + k_3^m \mathcal{Z}_{123|4,5,6,7,8} - k_2^m \mathcal{Z}_{132|4,5,6,7,8} \\ &\quad + \big[k_4^m \mathcal{Z}_{14|23,5,6,7,8} - k_4^m \mathcal{Z}_{214|3,5,6,7,8} + k_4^m \mathcal{Z}_{314|2,5,6,7,8} + (4\leftrightarrow 5,6,7,8)\big]\,, \\ \mathcal{Z}^{\Delta,mn}_{1|2,3,4,5,6,7,8} &= \mathcal{Z}^{mn}_{1|2,3,4,5,6,7,8} + \big[k_2^m \mathcal{Z}^n_{12|3,4,5,6,7,8} + k_2^n \mathcal{Z}^m_{12|3,4,5,6,7,8} + (2\leftrightarrow 3,\ldots,8)\big] \\ &\quad - \big[(k_2^m k_3^n + k_2^n k_3^m)\mathcal{Z}_{213|4,5,6,7,8} + (2,3|2,\ldots,8)\big]\,.\end{aligned} \tag{5.27}$$

In addition, the worldsheet analogue of the non-BRST-exact building block $\Delta_{1|2|3,4,5,6,7}$ in (I.5.35) gives rise to

$$\mathcal{Z}^{\Delta}_{1|2|3,\ldots,8} = \mathcal{Z}_{1,2|3,\ldots,8} + k_2^m \mathcal{Z}^m_{12|3,\ldots,8} + \big[s_{23}\mathcal{Z}_{123|4,\ldots,8} + (3\leftrightarrow 4,5,\ldots,8)\big]\,. \tag{5.28}$$

Given that the monodromy variations used to obtain the eight-point functions $\mathcal{Z}$ cannot detect explicit appearances of the modular form G_4, we have two possible scenarios:

i) use $\mathcal{Z}$-functions without G_4 corrections that do not satisfy the trace relations;
ii) use $\hat{\mathcal{Z}}$-functions in (5.19) with G_4 corrections that satisfy the trace relations.

It turns out that the functions from option *i*) lead to vanishing $\mathcal{Z}^{\Delta}$, including (5.28):

$$\begin{aligned} &\mathcal{Z}^{\Delta}_{1|234,5,6,7,8} \cong 0\,, \qquad \mathcal{Z}^{\Delta\, m}_{1|23,4,5,6,7,8} \cong 0\,, \qquad \mathcal{Z}^{\Delta}_{1|2|3,\ldots,8} \cong 0\,, \\ &\mathcal{Z}^{\Delta}_{1|23,45,6,7,8} \cong 0\,, \qquad \mathcal{Z}^{\Delta\, mn}_{1|2,3,4,5,6,7,8} \cong 0\,. \end{aligned} \tag{5.29}$$

The trace-satisfying functions $\hat{\mathcal{Z}}$ from option *ii*), however, lead to non-vanishing analogues $\hat{\mathcal{Z}}^{\Delta}$ that are defined by replying $\mathcal{Z} \to \hat{\mathcal{Z}}$ in (5.27) and (5.28),

$$\begin{aligned} \hat{\mathcal{Z}}^{\Delta}_{1|234,5,6,7,8} &= 3\mathrm{G}_4\big(2s_{13} - s_{12} - s_{14} + 2s_{24} - s_{23} - s_{34}\big)\,, \\ \hat{\mathcal{Z}}^{\Delta}_{1|23,45,6,7,8} &= 3\mathrm{G}_4\big(s_{25} + s_{34} - s_{24} - s_{35}\big)\,, \\ \hat{\mathcal{Z}}^{\Delta,m}_{1|23,4,5,6,7,8} &= 3\mathrm{G}_4\Big[s_{23}k_2^m - s_{12}(2k_2^m + k_3^m) - \big[k_4^m s_{24} + (4 \leftrightarrow 5,6,7,8)\big] - (2 \leftrightarrow 3)\Big]\,, \\ \hat{\mathcal{Z}}^{\Delta,mn}_{1|2,3,4,5,6,7,8} &= 3\mathrm{G}_4 s_{23}\Big(k_2^{(m}k_2^{n)} + k_3^{(m}k_3^{n)} - k_2^{(m}k_3^{n)} + \frac{1}{2}\delta^{mn}(s_{12} + s_{13} - s_{23})\Big) + (2,3|2,\ldots,8)\,, \\ \hat{\mathcal{Z}}^{\Delta}_{1|2|3,4,5,6,7,8} &= 3\mathrm{G}_4\big(3s_{23}s_{24} + s_{13}s_{14} - s_{34}(s_{23} + s_{24} + \frac{1}{2}s_{34} + \frac{1}{2}s_{12})\big) + (3,4|3,\ldots,8)\,. \end{aligned} \tag{5.30}$$

As will become clear in the discussion of the eight-point correlator in section III.3.5, the subtleties associated to the presence or absence of G_4 terms are responsible for the difficulties in obtaining a BRST-closed eight-point correlator.

5.5. The GEI dual to trace relations

Also the trace relations among pseudo-invariants such as $\frac{1}{2}\delta_{mn}C^{mn}_{1|2,3,4,5,6} = P_{1|2|3,\ldots,6}$ $+(2 \leftrightarrow 3,4,5,6)$ and its generalizations in (I.5.29) have an echo at the level of GEIs.

5.5.1. Six points

At six points, the GEIs (4.36) and (4.37) are related by

$$\begin{aligned} \frac{1}{2}\delta_{mn}E^{mn}_{1|2,3,4,5,6} &= \frac{1}{2}\ell^2 + \big[s_{12}g^{(2)}_{12} + (1,2|1,2,3,4,5,6)\big] + \big[E_{1|2|3,4,5,6} + (2 \leftrightarrow 3,4,5,6)\big] \\ &= \big[E_{1|2|3,4,5,6} + (2 \leftrightarrow 3,4,5,6)\big] + 2\pi i \frac{\partial}{\partial\tau}\log \mathcal{I}_6(\ell)\,, \end{aligned} \tag{5.31}$$

where we have used (2.21) to identify $\frac{1}{2}\ell^2 + \sum_{i<j} s_{ij}g^{(2)}_{ij}$ as a τ-derivative of the Koba–Nielsen factor. Note that this trace relation has a $\mathcal{Z}$-function counterpart given in (5.15).

5.5.2. Seven points

Similarly, we have checked that the seven-point tensor traces of GEIs obey relations analogous to the dual (pseudo-)invariants,

$$\begin{aligned}
\frac{1}{2}\delta_{mn}E^{mn}_{1|23,4,5,6,7} &\cong E_{1|23|4,5,6,7} + \big[E_{1|4|23,5,6,7} + (4 \leftrightarrow 5,6,7)\big] \\
\frac{1}{2}\delta_{np}E^{mnp}_{1|2,3,4,5,6,7} &\cong \big[E^{m}_{1|2|3,4,5,6,7} + (2 \leftrightarrow 3,\ldots,7)\big]\,.
\end{aligned} \tag{5.32}$$

Similar to the $\mathcal{Z}$-function counterparts (5.16), the equivalence $\cong$ refers to τ-derivatives that have been discarded.

5.5.3. Eight points

At eight points, however, the GEI-duals of the kinematic trace relations (I.5.29) exhibit deviations proportional to G_4. After expanding the GEIs in terms of $\hat{\mathcal{Z}}$-functions (obtained from the Berends–Giele expansion of their corresponding pseudo BRST invariants, see appendix A) one can show that

$$\begin{aligned}
\frac{1}{2}\delta_{mn}E^{mn}_{1|234,5,6,7,8} - \big[E_{1|234|5,6,7,8} + (234 \leftrightarrow 5,6,7,8)\big] &\cong \hat{\mathcal{Z}}^{\Delta}_{1|234,5,6,7,8} \\
\frac{1}{2}\delta_{mn}E^{mn}_{1|23,45,6,7,8} - \big[E_{1|23|45,6,7,8} + (23 \leftrightarrow 45,6,7,8)\big] &\cong \hat{\mathcal{Z}}^{\Delta}_{1|23,45,6,7,8} \\
\frac{1}{2}\delta_{np}E^{mnp}_{1|23,4,5,6,7,8} - \big[E^{m}_{1|23|4,5,6,7,8} + (23 \leftrightarrow 4,5,6,7,8)\big] &\cong \hat{\mathcal{Z}}^{\Delta,m}_{1|23,4,5,6,7,8} \\
\frac{1}{2}\delta_{pq}E^{mnpq}_{1|2,3,\ldots,8} - \big[E^{mn}_{1|2|3,4,\ldots,8} + (2 \leftrightarrow 3,\ldots,8)\big] &\cong \hat{\mathcal{Z}}^{\Delta,mn}_{1|2,3,4,5,6,7,8} \\
\frac{1}{2}\delta_{mn}E^{mn}_{1|2|3,\ldots,8} - \big[E_{1|2,3|4,\ldots,8} + (3 \leftrightarrow 4,\ldots,8)\big] &\cong \hat{\mathcal{Z}}^{\Delta}_{1|2|3,4,5,6,7,8}\,,
\end{aligned} \tag{5.33}$$

where the various functions $\hat{\mathcal{Z}}^{\Delta}$ are described in section 5.4.3 and defined in (5.30). The above results were obtained using the trace-satisfying representation $\hat{\mathcal{Z}}$ in the expansions of the GEIs. We know from (5.29) that all $\mathcal{Z}^{\Delta}$-functions vanish if we use the representation of shuffle-symmetric functions that do not satisfy the trace relations, so one could wonder if the above elliptic traces would vanish in that case. Unfortunately, this does not happen. In fact, the first three relations of (5.33) are independent on the choice of $\mathcal{Z}$ or $\hat{\mathcal{Z}}$, while the other two change (but do not vanish in either case).

5.5.4. Higher multiplicities

At higher multiplicity, suitable choices of the GEIs are expected to admit the dual of the kinematic relation (I.5.29),

$$\delta_{np}\widehat{E}^{npm_1\ldots m_r}_{1|B_1,\ldots,B_{r+5}} = 2\widehat{E}^{m_1\ldots m_r}_{1|B_1|B_2,\ldots,B_{r+5}} + (B_1 \leftrightarrow B_2,\ldots,B_{r+5})\,, \tag{5.34}$$

or more generally, the dual of the higher-refinement relation (I.5.30),

$$\delta_{np}\widehat{E}^{npm_1\ldots m_r}_{1|A_1,\ldots,A_d|B_1,\ldots,B_{d+r+5}} = 2\widehat{E}^{m_1\ldots m_r}_{1|A_1,\ldots,A_d,B_1|B_2,\ldots,B_{d+r+5}} + (B_1 \leftrightarrow B_2,\ldots,B_{d+r+5})\,. \tag{5.35}$$

The hat notation in (5.34) and (5.35) is used to indicate that, beyond seven points, the expressions for E presented in this work do not necessarily match the trace-satisfying GEIs $\widehat{E}$. We leave it to the future to identify the missing redefinitions by $\mathrm{G}_{k\geq 4}$ relating the GEIs E of this work to the trace-satisfying GEIs $\widehat{E}$ in (5.34) and (5.35).

6. Simplified representations of GEIs

In this section, we review and extend the construction of elliptic functions from the Kronecker–Eisenstein series [27,28] and identify ubiquitous building blocks for GEIs. These building blocks turn out to yield compact expressions for the GEIs in section 4.4 and will be used to present explicit all-multiplicity formulae for unrefined GEI of tensor rank $r \leq 2$.

6.1. Elliptic functions and their extensions

One can show via (2.11) that the cyclic product $F(z_{12},\alpha)F(z_{23},\alpha)\ldots F(z_{n-1,n},\alpha)F(z_{n,1},\alpha)$ of Kronecker–Eisenstein series (2.5) is an elliptic function of the punctures $z_1, z_2, \ldots, z_n$ [27],

$$F(z_{12},\alpha)F(z_{23},\alpha)\ldots F(z_{n-1,n},\alpha)F(z_{n,1},\alpha) = \sum_{w=0}^{\infty} \alpha^{-n+w} V_w(1,2,\ldots,n)\,, \tag{6.1}$$

where the dependence on τ is kept implicit for ease of notation. Since this property is independent on α, each term on the right-hand side of (6.1) is an elliptic function V_w in n punctures $z_1, z_2, \ldots, z_n$ by itself. At the level of linearized monodromies (3.8), we have $DF(z_{ij},\alpha) = \alpha\Omega_{ij}F(z_{ij},\alpha)$ and therefore

$$DV_w(1,2,\ldots,n) = 0\,. \tag{6.2}$$

The simplest examples of the elliptic functions V_w in (6.1) are $V_0(1,2,\ldots,n)=1$ and

$$V_1(1,2,\ldots,n)=\sum_{j=1}^{n} g^{(1)}_{j,j+1}\,, \qquad V_2(1,2,\ldots,n)=\sum_{j=1}^{n} g^{(2)}_{j,j+1}+\sum_{1\leq i<j}^{n} g^{(1)}_{i,i+1}g^{(1)}_{j,j+1}\,, \tag{6.3}$$

subject to cyclic identification $z_{n+1}\equiv z_1$. Their generating series in (6.1) and the reflection properties

$$F(-z,-\alpha,\tau)=-F(z,\alpha,\tau)\,, \qquad g^{(n)}(-z,\tau)=(-1)^n g^{(n)}(z,\tau) \tag{6.4}$$

imply cyclicity and reflection (anti-)symmetry for the functions V_w,

$$V_w(1,2,3,\ldots,n)=V_w(2,3,\ldots,n,1)=(-1)^w V_w(1,n,\ldots,3,2)\,. \tag{6.5}$$

Moreover, one can show via Fay relations (2.15) or (2.16) that the functions $V_w(1,2,\ldots,n)$ with $w=n-2$ obey the shuffle symmetry

$$V_{n-2}(1,(2,3,\ldots,j)\sqcup\!\sqcup(j+1,\ldots,n))=0\,, \qquad j=2,3,\ldots,n-1\,. \tag{6.6}$$

Given that shuffle symmetry is shared by Berends–Giele currents and (pseudo-)invariant kinematic factors, the $V_w(1,2,\ldots,n)$ with $w=n-2$ will play a key role for the duality between worldsheet functions and kinematics.

6.1.1. Derivative extension of elliptic functions

Compact representations of vectorial and tensorial GEIs will require extensions of the set of V_w-functions (6.1) that are covariant rather than invariant under linearized monodromies. Functions with these properties can be constructed by inserting a derivative with respect to the bookkeeping variable α into their generating series:

$$F(z_{12},\alpha)F(z_{23},\alpha)\ldots F(z_{n-1,n},\alpha)\partial_\alpha F(z_{n,1},\alpha)\equiv\sum_{w=-1}^{\infty}\alpha^{-n+w}\partial V_w(1,2,\ldots,n)\,. \tag{6.7}$$

The notation ∂V_w for the functions on the right-hand side reminds of the α-derivative on the left-hand side and should not be confused with $\frac{\partial}{\partial z_j}$. Based on $D\partial_\alpha F(z_{ij},\alpha)=\Omega_{ij}\big[\alpha\partial_\alpha F(z_{ij},\alpha)+F(z_{ij},\alpha)\big]$, the monodromy variations of the ∂V_w-functions in (6.7) can be written as

$$D\partial V_w(1,2,\ldots,n)=\Omega_{n1}V_w(1,2,\ldots,n)\,. \tag{6.8}$$

Given that their D-variation is expressible in terms of the elliptic V_w-functions of (6.1), the ∂V_w-functions are said to be *monodromy-covariant*.

The desired expressions for the $\mathcal{Z}$-functions and GEIs turn out to only involve $\partial V_w(1,2,\ldots,n)$ with $w=n-2$. The simplest examples admit the following expansions

$$\begin{aligned}
\partial V_0(1,2) &= g^{(1)}_{21} \qquad (6.9)\\
\partial V_1(1,2,3) &= g^{(2)}_{31} - g^{(1)}_{12}g^{(1)}_{23} - g^{(2)}_{12} - g^{(2)}_{23} = g^{(1)}_{31}\left(g^{(1)}_{12}+g^{(1)}_{23}\right) + 2g^{(2)}_{31}\\
\partial V_2(1,2,3,4) &= g^{(1)}_{41}\left(g^{(1)}_{12}g^{(1)}_{23} + g^{(1)}_{12}g^{(1)}_{34} + g^{(1)}_{23}g^{(1)}_{34} + g^{(2)}_{12} + g^{(2)}_{23} + g^{(2)}_{34}\right)\\
&\quad + 2g^{(2)}_{41}\left(g^{(1)}_{12}+g^{(1)}_{23}+g^{(1)}_{34}\right) + 3g^{(3)}_{41}\,,\\
\partial V_3(1,2,3,4,5) &= g^{(1)}_{51}\Big(g^{(1)}_{12}g^{(1)}_{23}g^{(1)}_{34} + g^{(1)}_{12}g^{(1)}_{23}g^{(1)}_{45} + g^{(1)}_{12}g^{(1)}_{34}g^{(1)}_{45} + g^{(1)}_{23}g^{(1)}_{34}g^{(1)}_{45} + g^{(1)}_{12}g^{(2)}_{23}\\
&\qquad + g^{(1)}_{12}g^{(2)}_{34} + g^{(1)}_{12}g^{(2)}_{45} + g^{(1)}_{23}g^{(2)}_{12} + g^{(1)}_{23}g^{(2)}_{34} + g^{(1)}_{23}g^{(2)}_{45} + g^{(1)}_{34}g^{(2)}_{12} + g^{(1)}_{34}g^{(2)}_{23}\\
&\qquad + g^{(1)}_{34}g^{(2)}_{45} + g^{(1)}_{45}g^{(2)}_{12} + g^{(1)}_{45}g^{(2)}_{23} + g^{(1)}_{45}g^{(2)}_{34} + g^{(3)}_{12} + g^{(3)}_{23} + g^{(3)}_{34} + g^{(3)}_{45}\Big)\\
&\quad + 2g^{(2)}_{51}\Big(g^{(1)}_{12}g^{(1)}_{23} + g^{(1)}_{12}g^{(1)}_{34} + g^{(1)}_{12}g^{(1)}_{45} + g^{(1)}_{23}g^{(1)}_{34} + g^{(1)}_{23}g^{(1)}_{45} + g^{(1)}_{34}g^{(1)}_{45}\\
&\qquad + g^{(2)}_{12} + g^{(2)}_{23} + g^{(2)}_{34} + g^{(2)}_{45}\Big)\\
&\quad + 3g^{(3)}_{51}\left(g^{(1)}_{12}+g^{(1)}_{23}+g^{(1)}_{34}+g^{(1)}_{45}\right) + 4g^{(4)}_{51}\,,
\end{aligned}$$

as one can check via $\partial_\alpha F(z,\alpha) = -\frac{1}{\alpha^2} + \sum_{n=1}^{\infty} n\alpha^{n-1}g^{(n+1)}(z)$. Alternatively, the expansions (6.9) can be written using the definition $V_p(\not{1},2,3,\ldots,\not{q}) \equiv V_p(1,2,3,\ldots,q)\big|_{g^{(k)}_{1q}\to 0}$ as $\partial V_w(1,\ldots,n) = \sum_{p=1}^{w+1} p g^{(p)}_{1n} V_{w+1-p}(\not{1},2,3,\ldots,\not{n})$ when $w=n-2$.

Note that the cyclicity of V_w does not extend to the ∂V_w, but the shuffle symmetry (6.6) at $w=n-2$ reappears in a modified form:

$$\partial V_{n-2}((1,2,\ldots,j)\sqcup(j{+}1,\ldots,n)) = 0\,, \qquad j=1,2,\ldots,n{-}1\,. \tag{6.10}$$

Also, the generating series (6.7) immediately implies the reflection property (valid for general w and n)

$$\partial V_w(1,2,\ldots,n) = (-1)^{w+1}\partial V_w(n,\ldots,2,1)\,. \tag{6.11}$$

6.1.2. Higher-derivative extension of elliptic functions

By extending (6.7) to involve higher derivatives in α, we are led to monodromy covariant functions $\partial^M V_w$ in

$$F(z_{12},\alpha)F(z_{23},\alpha)\ldots F(z_{n-1,n},\alpha)\partial^M_\alpha F(z_{n,1},\alpha) \equiv \sum_{w=-M}^{\infty} \alpha^{-n+w}\partial^M V_w(1,2,\ldots,n)\,, \tag{6.12}$$

where $D\partial_\alpha^M F(z_{ij},\alpha) = \Omega_{ij}[\alpha\partial_\alpha^M F(z_{ij},\alpha) + M\partial_\alpha^{M-1}F(z_{ij},\alpha)]$ implies that

$$D\partial^M V_w(1,2,\ldots,n) = M\Omega_{n1}\partial^{M-1}V_w(1,2,\ldots,n)\,. \tag{6.13}$$

The expansion of $\partial_\alpha^M F(z_{n,1},\alpha)$ in terms of $g_{ij}^{(n)}$ gives rise to expressions such as

$$\begin{aligned}
\partial^M V_0(1,2) &= M!\, g_{21}^{(M)} \\
\partial^2 V_1(1,2,3) &= 2g_{31}^{(2)}\big(g_{12}^{(1)} + g_{23}^{(1)}\big) + 6g_{31}^{(3)} \\
\partial^2 V_2(1,2,3,4) &= 2g_{41}^{(2)}\big(g_{12}^{(1)}g_{23}^{(1)} + g_{12}^{(1)}g_{34}^{(1)} + g_{23}^{(1)}g_{34}^{(1)} + g_{12}^{(2)} + g_{23}^{(2)} + g_{34}^{(2)}\big) \\
&\quad + 6g_{41}^{(3)}\big(g_{12}^{(1)} + g_{23}^{(1)} + g_{34}^{(1)}\big) + 12g_{41}^{(4)}\,.
\end{aligned} \tag{6.14}$$

Again, the cyclic symmetry of V_w is lost for $\partial^M V_w$ with $M \geq 1$, and there is no analogue of the shuffle symmetries (6.6) and (6.10) at $M \geq 2$. Still, the reflection property in (6.5) generalizes to

$$\partial^M V_w(1,2,\ldots,n) = (-1)^{w+M}\partial^M V_w(n,\ldots,2,1)\,. \tag{6.15}$$

6.2. Explicit examples of GEIs

In this section, we apply the elliptic functions V_w and their derivative-extensions $\partial^M V_w$ to cast the GEIs from the bootstrap procedure into compact form. Given the trivial GEI $E_{1|2,3,4} = 1$ at four points, the simplest example of the V_w-functions occurs at five points, where the GEIs (4.31) and (4.32) can be rewritten as

$$E_{1|23,4,\ldots} = V_1(1,2,3)\,, \qquad E^m_{1|2,3,4,\ldots} = \ell^m + \sum_{j\geq 2} k_j^m g_{1j}^{(1)}\,, \tag{6.16}$$

see (6.3) for V_1. Here and in the following, the number of slots (i.e. the upper bound on the summation range for $j \geq 2$) is kept unspecified in order to account for the extensions as in (4.33).

6.2.1. Six points

At six points, the definitions in (6.3) and (6.9) can be used to condense the scalars and the vector GEI in (4.36) to

$$\begin{aligned}
E_{1|234,5,\ldots} &= V_2(1,2,3,4)\,, \\
E_{1|23,45,\ldots} &= V_1(1,2,3)V_1(1,4,5)\,, \\
E^m_{1|23,4,5,\ldots} &= \Big(\ell^m + \sum_{j\geq 4} k_j^m g_{1j}^{(1)}\Big)V_1(1,2,3) + k_2^m\partial V_1(2,3,1) - k_3^m\partial V_1(3,2,1)\,, \\
E^{mn}_{1|2,3,4,5,\ldots} &= \ell^m\ell^n + \sum_{j\geq 2}\ell^{(m}k_j^{n)}g_{1j}^{(1)} + 2\sum_{j\geq 2}k_j^m k_j^n g_{1j}^{(2)} + \sum_{2\leq i<j}k_i^{(m}k_j^{n)}g_{1i}^{(1)}g_{1j}^{(1)}\,,
\end{aligned} \tag{6.17}$$

where the unspecified summation range automatically accounts for the extensions in (4.38). With the covariant monodromy variation (6.8) of ∂V_1 at hand, it is easy to verify that $DE^m_{1|23,4,5,\ldots} = 0$. One may identify the above $g^{(1)}_{1j}$ and $g^{(2)}_{1j}$ as $-\partial V_0(1,j)$ and $\frac{1}{2}\partial^2 V_0(1,j)$, respectively, to arrive at a uniform presentation for the coefficients of k^m_j.

In view of $\partial_1 g^{(1)}_{12} = V_2(1,2) - \mathrm{G}_2$, the refined GEI (4.37) is also expressible in terms of elliptic functions

$$\begin{aligned} E_{1|2|3,4,5,\ldots} &= (1-s_{12})V_2(1,2) - \mathrm{G}_2 \\ &\cong -2s_{12}g^{(2)}_{12} + g^{(1)}_{12}\Big(\ell\cdot k_2 + \sum_{j\geq 3} s_{2j}g^{(1)}_{2j}\Big)\,, \end{aligned} \tag{6.18}$$

where the last line again follows from integration by parts.

6.2.2. Seven points

At seven points, the scalar GEIs in (A.31) can be compactly written as

$$\begin{aligned} E_{1|2345,6,7,\ldots} &= V_3(1,2,3,4,5)\,, \\ E_{1|234,56,7,\ldots} &= V_2(1,2,3,4)V_1(1,5,6)\,, \\ E_{1|23,45,67,\ldots} &= V_1(1,2,3)V_1(1,4,5)V_1(1,6,7)\,, \end{aligned} \tag{6.19}$$

and the vectors (A.32) simplify as well when expressed in terms of V_w- and ∂V_w-functions,

$$\begin{aligned} E^m_{1|234,5,6,\ldots} &= \Big(\ell^m + \sum_{j\geq 5} g^{(1)}_{1j}k^m_j\Big)V_2(1,2,3,4) + k^m_2\partial V_2(2,3,4,1) \\ &\quad + k^m_4\partial V_2(4,3,2,1) - k^m_3\big[\partial V_2(3,2,4,1) + \partial V_2(3,4,2,1)\big]\,, \\ E^m_{1|23,45,6,\ldots} &= \Big(\ell^m + \sum_{j\geq 6} g^{(1)}_{1j}k^m_j\Big)V_1(1,2,3)V_1(1,4,5) \\ &\quad + V_1(1,4,5)\big[k^m_2\partial V_1(2,3,1) - k^m_3\partial V_1(3,2,1)\big] \\ &\quad + V_1(1,2,3)\big[k^m_4\partial V_1(4,5,1) - k^m_5\partial V_1(5,4,1)\big]\,. \end{aligned} \tag{6.20}$$

Similarly, the $\partial^2 V_w$-functions in (6.12) allow for compact representations of the two- and three-tensors in (A.33)[15],

$$E^{mn}_{1|23,4,5,\ldots} = \Big(\ell^m\ell^n + \sum_{j\geq 4} g^{(1)}_{1j}\ell^{(m}k^{n)}_j\Big)V_1(1,2,3) \tag{6.21}$$

[15] Note that our conventions lead to $\ell^{(m}\ell^n k^{p)}_j = \ell^m\ell^n k^p_j + \ell^m\ell^p k^n_j + \ell^n\ell^p k^m_j$.

$$
\begin{aligned}
&+ \partial V_1(2,3,1)\Big(\ell^{(m}k_2^{n)} + \sum_{j\geq 4} k_2^{(m}k_j^{n)} g_{1j}^{(1)}\Big) + \partial^2 V_1(2,3,1) k_2^m k_2^n \\
&- \partial V_1(3,2,1)\Big(\ell^{(m}k_3^{n)} + \sum_{j\geq 4} k_3^{(m}k_j^{n)} g_{1j}^{(1)}\Big) - \partial^2 V_1(3,2,1) k_3^m k_3^n \\
&+ 2V_1(1,2,3)\sum_{j\geq 4} k_j^m k_j^n g_{1j}^{(2)} + V_1(1,2,3)\sum_{4\leq i<j} k_i^{(m}k_j^{n)} g_{1i}^{(1)} g_{1j}^{(1)} \\
&+ \frac{1}{2}k_2^{(m}k_3^{n)}\big[\partial^2 V_1(1,2,3) - \partial^2 V_1(1,3,2) + \partial^2 V_1(2,1,3)\big]\,,
\end{aligned}
$$

$$
\begin{aligned}
E^{mnp}_{1|2,3,4,5,\ldots} &= \ell^m \ell^n \ell^p + \sum_{j\geq 2} \ell^{(m}\ell^n k_j^{p)} g_{1j}^{(1)} + \sum_{2\leq i<j} \ell^{(m}k_i^n k_j^{p)} g_{1i}^{(1)} g_{1j}^{(1)} \\
&+ 2\sum_{j\geq 2} \ell^{(m}k_j^n k_j^{p)} g_{1j}^{(2)} + \sum_{2\leq i<j<l} k_i^{(m}k_j^n k_l^{p)} g_{1i}^{(1)} g_{1j}^{(1)} g_{1l}^{(1)} \\
&+ 6\sum_{j\geq 2} k_j^m k_j^n k_j^p g_{1j}^{(3)} + 2\sum_{2\leq i<j} \big[k_i^{(m}k_i^n k_j^{p)} g_{1i}^{(2)} g_{1j}^{(1)} + (i\leftrightarrow j)\big]\,.
\end{aligned}
$$

Using the monodromy variation (6.13) of $\partial^M V_w$ and the shuffle symmetries (6.6) and (6.10) of V_{n-2} and ∂V_{n-2}, all the above $E^{\cdots}_{1|\ldots}$ can be verified to be GEIs with pen-and-paper effort. Similarly, one can show that the refined GEIs (A.34) whose combinatorics mimic the Berends–Giele expansion of the refined superfields P from [23] can be rewritten more compactly as

$$
\begin{aligned}
E_{1|23|4,5,\ldots} &= -s_{123}V_3(1,2,3) + (g_{12}^{(1)} + g_{31}^{(1)})\partial g_{23}^{(1)} + \partial g_{23}^{(2)} \\
E_{1|4|23,5,\ldots} &= \big[\partial g_{14}^{(1)} - s_{14}V_2(1,4)\big]V_1(1,2,3) - s_{24}V_3(1,2,4) + s_{34}V_3(1,3,4) \\
E^m_{1|2|3,4,5,\ldots} &= \big[\partial g_{12}^{(1)} - s_{12}V_2(1,2)\big]\Big(\ell^m + \sum_{j\geq 3} k_j^m g_{1j}^{(1)}\Big) + \sum_{j\geq 3} k_j^m s_{2j} V_3(1,2,j) \\
&\quad + k_2^m\big[\partial g_{12}^{(2)} + s_{12}(g_{12}^{(1)}g_{12}^{(2)} - 3g_{12}^{(3)})\big]\,.
\end{aligned}
\tag{6.22}
$$

Alternatively, using integration-by-parts identities leads to

$$
\begin{aligned}
E_{1|23|4,5,\ldots} &= \Big(g_{12}^{(1)}g_{23}^{(1)} + \tfrac{1}{2}(g_{12}^{(2)} + g_{23}^{(2)})\Big)\Big(\ell\cdot k_3 + \sum_{j\geq 4} s_{3j} g_{3j}^{(1)}\Big) \\
&- \Big(g_{13}^{(1)}g_{32}^{(1)} + \tfrac{1}{2}(g_{13}^{(2)} + g_{32}^{(2)})\Big)\Big(\ell\cdot k_2 + \sum_{j\geq 4} s_{2j} g_{2j}^{(1)}\Big) \\
&- \Big(s_{23}\big[3g_{23}^{(3)} + 2g_{23}^{(2)}(g_{12}^{(1)} + g_{31}^{(1)}) + \tfrac{1}{2}g_{23}^{(1)}(g_{12}^{(2)} + g_{13}^{(2)})\big] + \mathrm{cyc}(1,2,3)\Big)\,, \\
E_{1|4|23,5,\ldots} &= V_1(1,2,3)\Big[g_{14}^{(1)}\big(\ell\cdot k_4 - s_{24}g_{24}^{(1)} - s_{34}g_{34}^{(1)} + \sum_{j\geq 5} s_{4j} g_{4j}^{(1)}\big) - 2s_{14}g_{14}^{(2)}\Big] \\
&- s_{24}V_3(1,2,4) + s_{34}V_3(1,3,4)\,,
\end{aligned}
\tag{6.23}
$$

$$E^m_{1|2|3,4,5,\ldots} = \Big(\ell^m + \sum_{j\geq 3} k_j^m g^{(1)}_{1j}\Big)\Big[g^{(1)}_{12}\big(k_2\cdot \ell + \sum_{l\geq 3} s_{2l} g^{(1)}_{2l}\big) - 2s_{12}g^{(2)}_{12}\Big]$$
$$+ \sum_{j\geq 3} k_j^m s_{2j} V_3(1,2,j) + k_2^m \Big[g^{(2)}_{12}\big(k_2\cdot \ell + \sum_{l\geq 3} s_{2l} g^{(1)}_{2l}\big) - 3s_{12}g^{(3)}_{12}\Big]\,.$$

Instead of (6.22), one can also use $\partial_1 g^{(1)}_{12} = V_2(1,2) - \mathrm{G}_2$ and $\partial_1 g^{(2)}_{12} = 3g^{(3)}_{12} - g^{(1)}_{12}g^{(2)}_{12} - \mathrm{G}_2 g^{(1)}_{12}$ to write

$$E_{1|23|4,5,\ldots} = (1-s_{123})V_3(1,2,3) - \mathrm{G}_2 V_1(1,2,3)\,, \tag{6.24}$$

and the analogous identities for z_j-derivatives of general $g^{(n)}_{ij}$-functions read[16]

$$\partial_z g^{(n)}(z,\tau) = (n{+}1)g^{(n+1)}(z,\tau) - g^{(1)}(z,\tau)g^{(n)}(z,\tau) - \sum_{k=2}^{n+1} \mathrm{G}_k\, g^{(n+1-k)}(z,\tau)\,. \tag{6.25}$$

6.3. Closed all-multiplicity formulae for GEIs

6.3.1. Scalars at all multiplicities

The above examples of scalar GEIs in (6.16), (6.17) and (6.19) line up with

$$E_{1|A,B,C} = V_{|A|-1}(1,A)V_{|B|-1}(1,B)V_{|C|-1}(1,C)\,. \tag{6.26}$$

Given that all the $V_w(1,2,\ldots,n)$-functions on the right-hand side have $w = n{-}2$, the GEIs in (6.26) exhibit the desired shuffle symmetry in each slot by (6.6). Although only the functions (6.26) with three multiparticle slots enter open-string amplitudes, the later discussion will benefit from an extension to unspecified numbers of slots,

$$E_{1|A_1,A_2,\ldots} = \prod_{j\geq 1} V_{|A_j|-1}(1,A_j)\,. \tag{6.27}$$

6.3.2. Closed formulae for vectors and two-tensors

The above examples of vector and two-tensor GEIs can be lined up with the closed formulae

$$\begin{aligned} E^m_{1|A,B,C,\ldots} &= \ell^m V_{|A|-1}(1,A)V_{|B|-1}(1,B)V_{|C|-1}(1,C)\ldots \\ &+ \Big[\sum_{j=1}^{|A|}(-1)^{j-1}k^m_{a_j}\partial V_{|A|-1}(a_j,(a_{j-1}\ldots a_2a_1 \sqcup\!\sqcup a_{j+1}\ldots a_{|A|}),1) \\ &\quad\times V_{|B|-1}(1,B)V_{|C|-1}(1,C)\ldots + (A\leftrightarrow B,C,\ldots)\Big] \end{aligned} \tag{6.28}$$

[16] This follows from the expansion of $(\partial_z - \partial_\alpha)F(z,\alpha,\tau) = (g^{(1)}(\alpha,\tau) - g^{(1)}(z,\tau))F(z,\alpha,\tau)$.

as well as

$$
\begin{aligned}
E^{mn}_{1|A,B,C,\ldots} &= \ell^m \ell^n V_{|A|-1}(1,A) V_{|B|-1}(1,B) V_{|C|-1}(1,C)\ldots \\
&+ \Big[\sum_{j=1}^{|A|} (-1)^{j-1} \ell^{(m} k^{n)}_{a_j} \partial V_{|A|-1}(a_j, (a_{j-1}\ldots a_2 a_1 \sqcup a_{j+1}\ldots a_{|A|}), 1) \\
&\quad \times V_{|B|-1}(1,B) V_{|C|-1}(1,C)\ldots + (A \leftrightarrow B, C, \ldots) \Big] \\
&+ \Big[\sum_{i=1}^{|A|} \sum_{j=1}^{|B|} (-1)^{i+j} k^{(m}_{a_i} k^{n)}_{b_j} \partial V_{|A|-1}(a_i, (a_{i-1}\ldots a_2 a_1 \sqcup a_{i+1}\ldots a_{|A|}), 1) \\
&\quad \times \partial V_{|B|-1}(b_j, (b_{j-1}\ldots b_1 \sqcup b_{j+1}\ldots b_{|B|}), 1) \\
&\quad \times V_{|C|-1}(1,C) V_{|D|-1}(1,D)\ldots + (A,B|A,B,C,D\ldots) \Big] \qquad (6.29) \\
&+ \frac{1}{2} \Big[k^{(m}_A \sum_{j=1}^{|A|} k^{n)}_{a_j} (-1)^{j-1} \partial^2 V_{|A|-1}(a_j, (a_{j-1}\ldots a_2 a_1 \sqcup a_{j+1}\ldots a_{|A|}), 1) \\
&\quad \times V_{|B|-1}(1,B) V_{|C|-1}(1,C)\ldots + (A \leftrightarrow B, C, \ldots) \Big] \\
&- \frac{1}{2} \Big[V_{|B|-1}(1,B) V_{|C|-1}(1,C)\ldots \sum_{1=i<j}^{|A|} (-1)^{i+j+|A|} k^{(m}_{a_i} k^{n)}_{a_j} \\
&\quad \times \partial^2 V_{|A|-1}(a_i, (a_{i-1}\ldots a_2 a_1 1 a_{|A|}\ldots a_{j+1} \sqcup a_{i+1} a_{i+2}\ldots a_{j-1}), a_j) + (A \leftrightarrow B, C, \ldots) \Big] .
\end{aligned}
$$

Up to multiplicity seven, the complete set of unrefined GEIs is accessible from the above closed formulae and (6.21). At higher tensor rank, the system of $\partial^M V_w(1,2,\ldots,n)$-functions in (6.12) is no longer sufficient to represent the coefficients of $k_i^{(m} k_j^n k_l^{p)}$ and higher-rank terms. This shortcoming motivates the development of more powerful tools for all-multiplicity and all-rank constructions of GEIs, which we leave for a future work.

7. Integrating the loop momentum and modular invariance

The purpose of this section is to set the stage for integrating the one-loop correlators of part III over the loop momentum. We will see below that loop-integrated GEIs yield manifestly single-valued worldsheet functions that largely conspire to modular weight $(n{-}4,0)$. The loop integrals of individual GEIs at $(n{\geq}6)$ points also feature terms of different modular weights that (as will be shown in part III) cancel from the amplitude by kinematic identities among their coefficients. Such modular anomalies will be illustrated to follow the patterns of BRST anomalies of pseudo-invariants. Like this, we extend the duality between worldsheet functions and kinematics to anomalies.

7.1. The non-holomorphic Kronecker–Eisenstein series

As detailed in section 2.1, the meromorphic constituents $g^{(n)}(z,\tau)$ of the chirally-split open-string correlators $\mathcal{K}_n(\ell)$ descend from the Kronecker–Eisenstein series (2.5). The doubly-periodic counterparts of $g^{(n)}(z,\tau)$ that will result from loop integration can be generated from the non-holomorphic completion [5],

$$\Omega(z,\alpha,\tau) \equiv e^{2\pi i\alpha \frac{\operatorname{Im} z}{\operatorname{Im} \tau}} F(z,\alpha,\tau) \equiv \sum_{n=0}^{\infty} \alpha^{n-1} f^{(n)}(z,\tau)\,, \tag{7.1}$$

where the exponential factor is tailored to cancel the B-cycle monodromies (2.11),

$$\begin{aligned} f^{(n)}(z,\tau) &= f^{(n)}(z+1,\tau) = f^{(n)}(z+\tau,\tau) \\ \Omega(z,\alpha,\tau) &= \Omega(z+1,\alpha,\tau) = \Omega(z+\tau,\alpha,\tau)\,. \end{aligned} \tag{7.2}$$

The doubly-periodic but non-holomorphic functions $f^{(n)}$ in (7.1) are related to the holomorphic $g^{(n)}$ with B-cycle monodromies (2.12) via [14]

$$f^{(n)}(z,\tau) \equiv \sum_{k=0}^{n} \frac{\nu^k}{k!} g^{(n-k)}(z,\tau)\,, \quad \nu \equiv 2\pi i \frac{\operatorname{Im} z}{\operatorname{Im} \tau}\,, \tag{7.3}$$

where the simplest examples are $f^{(0)} = 1$ and

$$f^{(1)}(z,\tau) = g^{(1)}(z,\tau) + \nu\,, \quad f^{(2)}(z,\tau) = g^{(2)}(z,\tau) + \nu g^{(1)}(z,\tau) + \frac{1}{2}\nu^2\,. \tag{7.4}$$

Apart from double-periodicity, the non-holomorphic Kronecker–Eisenstein series and the functions $f^{(n)}$ exhibit covariant modular transformations with holomorphic weights $(1,0)$ and $(n,0)$, respectively, [7]

$$\begin{aligned} \Omega\left(\frac{z}{c\tau+d}, \frac{\alpha}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) &= (c\tau+d)\,\Omega(z,\alpha,\tau)\,, \\ f^{(n)}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) &= (c\tau+d)^n\, f^{(n)}(z,\tau)\,, \end{aligned} \tag{7.5}$$

where a,b,c,d form an $\mathrm{SL}_2(\mathbb{Z})$ matrix. Similarly, each holomorphic derivative in z adds holomorphic weight $(1,0)$ to the $f^{(n)}$. However, meromorphicity of the $g^{(n)}$ is replaced by the condition

$$\Big(\frac{\partial}{\partial \overline{\tau}} + \frac{\operatorname{Im} z}{\operatorname{Im} \tau} \frac{\partial}{\partial \overline{z}}\Big) f^{(n)}(z,\tau) = 0 \tag{7.6}$$

following from

$$\frac{\partial}{\partial \overline{z}} f^{(n)}(z,\tau) = -\frac{\pi}{\operatorname{Im}\tau} f^{(n-1)}(z,\tau)\,, \qquad \frac{\partial}{\partial \overline{\tau}} f^{(n)}(z,\tau) = \frac{\pi \operatorname{Im} z}{(\operatorname{Im}\tau)^2} f^{(n-1)}(z,\tau)\,. \tag{7.7}$$

It will be convenient to extend the shorthand notation (2.14) for $g_{ij}^{(n)}$ to their doubly-periodic counterparts,

$$f_{ij}^{(n)} \equiv f^{(n)}(z_i - z_j, \tau)\,, \tag{7.8}$$

which we will use from now on. The Fay identity (2.15) of the Kronecker–Eisenstein series is unchanged when replacing $F(\ldots) \rightarrow \Omega(\ldots)$. Accordingly, the relations (2.16) to rearrange products $g_{12}^{(n)} g_{23}^{(m)}$ also hold when globally trading $g_{ij}^{(n)} \rightarrow f_{ij}^{(n)}$. For instance, the simplest examples (7.4) of $f^{(n)}$ satisfy the analogue $f_{12}^{(1)} f_{23}^{(1)} + f_{12}^{(2)} + \operatorname{cyc}(1,2,3) = 0$ of (2.17).

7.2. Integrating out the loop momentum

In this section, we set the stage for loop integrals over both the Koba–Nielsen factor

$$\begin{aligned} \big|\mathcal{I}_n(\ell)\big|^2 = \exp\Big(\sum_{i<j}^n s_{ij} \Big\{ \log\big|\theta_1(z_{ij},\tau)\big|^2 - \frac{i\pi}{\tau-\overline{\tau}} \Big[\sum_{j=1}^n k_j (z_j - \overline{z}_j) \Big]^2 \\ + \frac{\tau-\overline{\tau}}{4\pi i} \Big[\ell + 2\pi i \sum_{j=1}^n k_j \frac{z_j - \overline{z}_j}{\tau-\overline{\tau}} \Big]^2 \Big\} \Big) \end{aligned} \tag{7.9}$$

and ℓ-dependent open- and closed-string correlators in the amplitudes (2.1) and (2.2). For closed-string correlators independent on ℓ, the result of the Gaussian loop integral

$$\hat{\mathcal{I}}_n \equiv \int d^D\ell\ \big|\mathcal{I}_n(\ell)\big|^2 = \frac{(2\pi i)^D}{(2\operatorname{Im}\tau)^{\frac{D}{2}}} \exp\Big(\sum_{i<j}^n s_{ij} \Big[\log\big|\theta_1(z_{ij},\tau)\big|^2 - \frac{2\pi}{\operatorname{Im}\tau} (\operatorname{Im} z_{ij})^2 \Big] \Big) \tag{7.10}$$

has already been spelled out in (I.2.26). Zero-mode integration at $n \geq 5$ points, however, requires generalizations of (7.10) to additional polynomials $p(\ell)$ in the loop momentum besides $|\mathcal{I}_n(\ell)|^2$. We will use the square-bracket notation

$$\int d^D\ell\ \big|\mathcal{I}_n(\ell)\big|^2 p(\ell) = \hat{\mathcal{I}}_n\,[[p(\ell)]] \tag{7.11}$$

to compactly address the net effect $[[p(\ell)]]$ of the shifts in the Gaussian integration variable in (7.9). The right-hand side of (7.11) is normalized to $[[1]] = 1$, and the loop integrals over polynomials in ℓ are most conveniently written in terms of the shorthands

$$\nu_{ij} \equiv 2\pi i \frac{\operatorname{Im} z_{ij}}{\operatorname{Im}\tau}\,, \quad L_0^m \equiv -\sum_{j=1}^n k_j^m \nu_j = \sum_{j=2}^n k_j^m \nu_{1j}\,, \tag{7.12}$$

where momentum conservation has been used to eliminate $k_1^m = -k_2^m - \cdots - k_n^m$ from the definition of L_0^m. As a result of straightforward Gaussian integration, we have (recall the convention (I.2.3) where all terms generated by (anti)symmetrization of indices have unit coefficient, e.g., $\delta^{(mn}k^{p)} \equiv \delta^{mn}k^p + \delta^{mp}k^n + \delta^{np}k^m$)

$$
\begin{aligned}
[[\ell^m]] &= L_0^m \,, \\
[[\ell^m\ell^n]] &= L_0^m L_0^n - \frac{\pi}{\mathrm{Im}\,\tau}\delta^{mn} \,, \\
[[\ell^m\ell^n\ell^p]] &= L_0^m L_0^n L_0^p - \frac{\pi}{\mathrm{Im}\,\tau}\delta^{(mn}L_0^{p)} \,, \\
[[\ell^m\ell^n\ell^p\ell^q]] &= L_0^m L_0^n L_0^p L_0^q - \frac{\pi}{\mathrm{Im}\,\tau}\delta^{(mn}L_0^p L_0^{q)} + \Big(\frac{\pi}{\mathrm{Im}\,\tau}\Big)^2 \delta^{m(n}\delta^{pq)} \,,
\end{aligned}
\tag{7.13}
$$

which are sufficient to integrate open-string correlators at $n \leq 8$ points and closed-string correlators at $n \leq 6$ points. In general, following standard Gaussian integration rules, one has to sum over all possibilities to perform pairwise contractions $\ell^m\ell^n \to -\frac{\pi}{\mathrm{Im}\,\tau}\delta^{mn}$ on a subset of the loop momenta in the integrand while setting the others to $\ell^m \to L_0^m$.

The open-string analogue of (7.11) reads

$$
\int d^D\ell \, \big|\mathcal{I}_n(\ell)\big| \, p(\ell) = \hat{\mathcal{I}}_n^{\rm open} \, [[p(\ell)]] \,, \tag{7.14}
$$

where $\hat{\mathcal{I}}_n^{\rm open}$ is defined in (I.2.27), and one can take advantage of the same expressions (7.13) for $[[p(\ell)]]$ that apply to the closed string. The imaginary parts in (7.12) then ensure that the results (7.13) can be specialized to all the open-string topologies by suitable choices of the integration domains for z_j and τ.

In summary, (7.11) and (7.14) are tailored to express the open- and closed-string amplitudes (2.1) and (2.2) in the following form

$$
\begin{aligned}
\mathcal{A}_n &= \sum_{\rm top} C_{\rm top} \int_{D_{\rm top}} d\tau \, dz_2 \, dz_3 \, \ldots \, dz_n \, \hat{\mathcal{I}}_n^{\rm open} \, [[\, \langle \mathcal{K}_n(\ell) \rangle \,]] \,, \\
\mathcal{M}_n &= \int_{\mathcal{F}} d^2\tau \, d^2 z_2 \, d^2 z_3 \, \ldots \, d^2 z_n \, \hat{\mathcal{I}}_n \, [[\, \langle \mathcal{K}_n(\ell) \rangle \, \langle \tilde{\mathcal{K}}_n(-\ell) \rangle \,]] \,,
\end{aligned}
\tag{7.15}
$$

where all the remnants of the loop momenta in the correlators are captured by the Gaussian brackets $[[\ldots]]$ exemplified in (7.13). In the remainder of this section, we will evaluate $[[E_{1|\ldots}]]$ for various GEIs and elaborate on the modified integration-by-parts rules adapted to (7.10) instead of $|\mathcal{I}_n(\ell)|^2$. This will be applied in section III.4 to provide manifestly single-valued expressions for open- and closed-string correlators $[[\mathcal{K}_n(\ell)]]$ and $[[\mathcal{K}_n(\ell)\tilde{\mathcal{K}}_n(-\ell)]]$.

7.2.1. Integrating unrefined GEIs

In section 3, GEIs $E^{\cdots}_{1|\ldots}$ have been introduced as meromorphic functions that are doubly-periodic up to shifts of the loop momentum. Hence, upon integration over ℓ, GEIs are guaranteed to become doubly-periodic, and the functions $f^{(n)}$ in (7.1) turn out to be the natural framework to represent the dependence of $[[E^{\cdots}_{1|\ldots}]]$ on z_j.

Unrefined scalar GEIs $E_{1|A,B,C}$ were found to be elliptic functions in the conventional sense and expressible in terms of the V_w-functions of (6.1), see e.g. (6.16) and (6.17). Given that the generating series (6.1) of V_w are unchanged when the Kronecker–Eisenstein series are replaced by their doubly-periodic completions (7.1), one can globally replace $g^{(n)} \to f^{(n)}$ in any V_w, and in fact, in any $E_{1|A,B,C}$. For instance, all the imaginary parts ν_{ij} of (7.12) cancel out from

$$V_1(1,2,\ldots,n) = \sum_{j=1}^{n} f^{(1)}_{j,j+1}\,, \qquad V_2(1,2,\ldots,n) = \sum_{j=1}^{n} f^{(2)}_{j,j+1} + \sum_{1\leq i<j}^{n} f^{(1)}_{i,i+1} f^{(1)}_{j,j+1}\,, \tag{7.16}$$

which gives rise to $[[E_{1|2,3,4}]] = 1$ and

$$\begin{aligned}
[[E_{1|23,4,5}]] &= V_1(1,2,3) = f^{(1)}_{12} + f^{(1)}_{23} + f^{(1)}_{31} \qquad (7.17)\\
[[E_{1|234,5,6}]] &= V_2(1,2,3,4) = f^{(1)}_{12} f^{(1)}_{34} + f^{(1)}_{23} f^{(1)}_{41} + \big[f^{(1)}_{12} f^{(1)}_{23} + f^{(2)}_{12} + \mathrm{cyc}(1,2,3,4)\big]\\
[[E_{1|23,45,6}]] &= V_1(1,2,3)V_1(1,4,5) = (f^{(1)}_{12} + f^{(1)}_{23} + f^{(1)}_{31})(f^{(1)}_{14} + f^{(1)}_{45} + f^{(1)}_{51})\,.
\end{aligned}$$

Similarly, the all-multiplicity formula (6.26) for unrefined scalar GEIs generalizes to

$$[[E_{1|A,B,C}]] = V_{|A|-1}(1,A) V_{|B|-1}(1,B) V_{|C|-1}(1,C)\,. \tag{7.18}$$

The $[[\ldots]]$ have no effect on these ℓ-independent functions but have been included into (7.17) and (7.18) to harmonize with the examples below.

For vectorial and tensorial GEIs, the loop momenta integrate to polynomials in ν_{ij} as a result of the Gaussian brackets in (7.13). In order to manifest the double-periodicity of $[[E^{m_1\ldots}_{1|A,B,\ldots}]]$, these factors of ν_{ij} can be combined with the meromorphic functions $g^{(n)}_{ij}$ to obtain their doubly-periodic completion $f^{(n)}_{ij}$. Based on the conversion (7.3) between $g^{(n)}_{ij}$ and $f^{(n)}_{ij}$ as well as the expressions for the GEIs in (4.32) and (4.36), we find

$$\begin{aligned}
[[E^m_{1|2,3,4,5}]] &= k^m_2 f^{(1)}_{12} + (2 \leftrightarrow 3,4,5)\,, \qquad (7.19)\\
[[E^m_{1|23,4,5,6}]] &= k^m_3 f^{(1)}_{12} f^{(1)}_{23} + k^m_2 f^{(1)}_{13} f^{(1)}_{23} + \big[k^m_4 f^{(1)}_{14}(f^{(1)}_{23} + f^{(1)}_{12} + f^{(1)}_{31}) + (4 \leftrightarrow 5,6)\big]\\
&\quad + k^m_{23}(f^{(2)}_{12} - f^{(2)}_{13}) + (k^m_3 - k^m_2) f^{(2)}_{23}\,,\\
[[E^{mn}_{1|2,3,4,5,6}]] &= -\frac{\pi}{\mathrm{Im}\,\tau}\delta^{mn} + 2\big[k^m_2 k^n_2 f^{(2)}_{12} + (2 \leftrightarrow 3,4,5,6)\big]\\
&\quad + \big[(k^m_2 k^n_3 + k^m_2 k^n_3) f^{(1)}_{12} f^{(1)}_{13} + (2,3|2,3,4,5,6)\big]\,,
\end{aligned}$$

and higher-multiplicity results will be given below.

7.2.2. Modular anomalies

By the modular weight $(w,0)$ of $f^{(w)}_{ij}$, see (7.5), almost all of the examples (7.17) to (7.19) of integrated n-point GEIs are modular forms of weight $(n{-}4,0)$. The only exception is the first term $-\frac{\pi}{\operatorname{Im}\tau}\delta^{mn}$ of modular weight $(1,1)$ in the expression (7.19) for the tensor $[[E^{mn}_{1|2,3,4,5,6}]]$ whose remaining terms $f^{(2)}_{ij}$ and $f^{(1)}_{ij}f^{(1)}_{kl}$ carry weight $(2,0)$. Accordingly, contributions to $[[E^{m_1m_2\ldots}_{1|A,B,\ldots}]]$ at n points that depart from modular weight $(n{-}4,0)$ are referred to as a *modular anomalies*, the simplest example being the above $-\frac{\pi}{\operatorname{Im}\tau}\delta^{mn}$.

For unrefined GEIs, modular anomalies can be conveniently traced back to contractions $\ell^m\ell^n\to-\frac{\pi}{\operatorname{Im}\tau}\delta^{mn}$, so they only arise at tensor rank $r{\geq}2$ (the situation for refined GEIs is different, see section 7.2.4). Scalar GEIs $[[E_{1|A,B,C}]]=E_{1|A,B,C}$ reduce to elliptic V_w-functions of weight $(w,0)$, and the integral $[[E^m_{1|A,B,C,D}]]$ over vector GEIs follows from setting $\ell^m\to 0$ and $g^{(n)}_{ij}\to f^{(n)}_{ij}$, see e.g. (7.19). The modular anomalies of the tensorial seven-points GEIs (6.21) are the contributions $\sim\frac{\pi}{\operatorname{Im}\tau}$ in

$$
\begin{aligned}
[[E^{mn}_{1|23,4,5,6,7}]] &= -\frac{\pi}{\operatorname{Im}\tau}\delta^{mn}V_1(1,2,3)+2V_1(1,2,3)\big[k_4^mk_4^nf^{(2)}_{14}+(4\leftrightarrow 5,6,7)\big] \\
&\quad + V_1(1,2,3)\big[k_4^{(m}k_5^{n)}f^{(1)}_{14}f^{(1)}_{15}+(4,5|4,5,6,7)\big] \\
&\quad + \Big(\big[k_2^{(m}k_4^{n)}f^{(1)}_{14}+(4\leftrightarrow 5,6,7)\big]\big[2f^{(2)}_{12}+f^{(1)}_{12}(f^{(1)}_{23}{+}f^{(1)}_{31})\big]-(2\leftrightarrow 3)\Big) \\
&\quad + \Big(k_2^mk_2^n\big[6f^{(3)}_{12}+2f^{(2)}_{12}(f^{(1)}_{23}{+}f^{(1)}_{31})\big]-(2\leftrightarrow 3)\Big) \\
&\quad + k_2^{(m}k_3^{n)}\big[2f^{(3)}_{12}+2f^{(3)}_{31}-f^{(3)}_{23}+f^{(1)}_{23}(f^{(2)}_{12}{+}f^{(2)}_{13})\big]\ . \\
[[E^{mnp}_{1|2,3,4,\ldots,7}]] &= -\frac{\pi}{\operatorname{Im}\tau}\delta^{(mn}\big[k_2^{p)}f^{(1)}_{12}+(2\leftrightarrow 3,4,\ldots,7)\big]+6\big[k_2^mk_2^nk_2^pf^{(3)}_{12}+(2\leftrightarrow 3,\ldots,7)\big] \\
&\quad + 2\big[k_2^{(m}k_2^nk_3^{p)}f^{(2)}_{12}f^{(1)}_{13}+k_2^{(m}k_3^nk_3^{p)}f^{(1)}_{12}f^{(2)}_{13}+(2,3|2,3,4,\ldots)\big] \\
&\quad + \big[k_2^{(m}k_3^nk_4^{p)}f^{(1)}_{12}f^{(1)}_{13}f^{(1)}_{14}+(2,3,4|2,3,\ldots,7)\big]\,.
\end{aligned}
\tag{7.20}
$$

Before pointing out analogous modular anomalies in the loop integrals of refined GEIs, we shall elaborate on the integration-by-parts relations relevant to the results for $[[E^{m_1\ldots}_{1|A|B,\ldots}]]$.

7.2.3. Integration by parts

The integration-by-parts relations of meromorphic correlators $\mathcal{K}_n(\ell)$ were governed by the derivatives of the ℓ-dependent Koba–Nielsen factor $\mathcal{I}_n(\ell)$, see section 2.3. Accordingly, the loop-integrated Koba–Nielsen factor $\hat{\mathcal{I}}_n$ in (7.10) gives rise to a modified set of integration-by-parts relations. The z_j-derivatives (2.22) straightforwardly generalize to

$$
\frac{\partial}{\partial z_i}\hat{\mathcal{I}}_n=\Big(\sum_{j\neq i}^n s_{ij}f^{(1)}_{ij}\Big)\hat{\mathcal{I}}_n\,, \tag{7.21}
$$

while the τ-derivative (2.23) requires more adjustments after integration over ℓ. After momentum conservation, the Koba–Nielsen exponent in (7.10) has the following τ-derivative

$$\frac{\partial}{\partial \tau} \sum_{i<j}^{n} s_{ij} \Big[\log \big|\theta_1(z_{ij},\tau)\big|^2 - \frac{2\pi}{\mathrm{Im}\,\tau} (\mathrm{Im}\, z_{ij})^2 \Big] = \sum_{i<j}^{n} s_{ij} \Big[\frac{1}{2\pi i} f^{(2)}_{ij} - \frac{\mathrm{Im}\, z_{ij}}{\mathrm{Im}\,\tau} f^{(1)}_{ij} \Big] \, , \tag{7.22}$$

where the admixtures of $f^{(1)}_{ij}$ cancel from the action of the differential operator

$$\nabla_\tau \equiv \frac{\partial}{\partial \tau} + \sum_{j=2}^{n} \frac{\mathrm{Im}\, z_{j1}}{\mathrm{Im}\,\tau} \frac{\partial}{\partial z_j} \tag{7.23}$$

depending on n punctures z_j. The operator ∇_τ obeys the usual Leibniz property and appears naturally in the following generalization of the mixed heat equation (2.9),

$$\nabla_\tau f^{(w)}_{ij} = \frac{w}{2\pi i} \partial f^{(w+1)}_{ij} - \frac{w}{2i \ \mathrm{Im}\,\tau} f^{(w)}_{ij} \, . \tag{7.24}$$

Then, after taking the prefactor of $\hat{\mathcal{I}}_n \sim (\mathrm{Im}\,\tau)^{-D/2}$ in (7.10) into account, a convenient analogue of the τ-derivative (2.23) after loop integration reads

$$\nabla_\tau \hat{\mathcal{I}}_n = \hat{\mathcal{I}}_n \Big\{ \frac{1}{2\pi i} \sum_{i<j}^{n} s_{ij} f^{(2)}_{ij} + \frac{iD}{4\,\mathrm{Im}\,\tau} \Big\} \, , \tag{7.25}$$

where we will set the number of spacetime dimensions to $D = 10$ henceforth. The operator (7.23) can be aligned into the following boundary term

$$\begin{aligned} &\frac{\partial}{\partial \tau} \big(h(z,\tau) \hat{\mathcal{I}}_n \big) + \sum_{p=2}^{n} \frac{\partial}{\partial z_p} \Big(\frac{\mathrm{Im}\, z_{p1}}{\mathrm{Im}\,\tau} h(z,\tau) \hat{\mathcal{I}}_n \Big) \\ &\quad = h(z,\tau) \hat{\mathcal{I}}_n \Big\{ \frac{1}{2\pi i} \sum_{i<j}^{n} s_{ij} f^{(2)}_{ij} + \frac{n-6}{2i \ \mathrm{Im}\,\tau} \Big\} + \hat{\mathcal{I}}_n \nabla_\tau h(z,\tau) \, , \end{aligned} \tag{7.26}$$

with $h(z,\tau)$ denoting an arbitrary function on the worldsheet. Since both of (7.21) and (7.26) integrate to zero within string amplitudes, we conclude the following equivalence classes of integrated correlators $[[\ldots]]$,

$$\Big(\sum_{j\neq i}^{n} s_{ij} f^{(1)}_{ij} \Big) h(z,\tau) + \frac{\partial h(z,\tau)}{\partial z_i} \cong 0 \, , \qquad \forall\, h(z,\tau) \, , \tag{7.27}$$

$$\Big(\sum_{i<j}^{n} s_{ij} f^{(2)}_{ij} \Big) h(z,\tau) + 2\pi i \Big(\frac{n-6}{2i \ \mathrm{Im}\,\tau} + \nabla_\tau \Big) h(z,\tau) \cong 0 \, , \qquad \forall\, h(z,\tau) \, , \tag{7.28}$$

see (2.22) and (2.23) for their chirally-split analogues. The simplest example of (7.28) with $h(z,\tau)=1$ has been used in [29] to identify the BRST variation of the ($n=6$)-point closed-string amplitude as a boundary term.

Note that the holomorphic derivative $\frac{\partial}{\partial z_i}$ in (7.27) acts non-trivially on the contributions $\overline{f}^{(w)}_{ij}$ from the opposite chiral half in closed-string amplitudes. This follows from the complex conjugate

$$\frac{\partial}{\partial z}\overline{f}^{(n)}(z,\tau) = -\frac{\pi}{\operatorname{Im}\tau}\overline{f}^{(n-1)}(z,\tau)\,, \qquad \frac{\partial}{\partial \tau}\overline{f}^{(n)}(z,\tau) = \frac{\pi \operatorname{Im} z}{(\operatorname{Im}\tau)^2}\overline{f}^{(n-1)}(z,\tau) \tag{7.29}$$

of (7.7) and gives rise to examples such as [30,31]

$$f^{(1)}_{12}\overline{f}^{(1)}_{23} \cong \frac{1}{s_{12}}\Big(\overline{f}^{(1)}_{23}\sum_{j=3}^{n} s_{2j}f^{(1)}_{2j} - \frac{\pi}{\operatorname{Im}\tau}\Big)\,. \tag{7.30}$$

The differential operator (7.23) in turn annihilates undifferentiated $\overline{f}^{(w)}_{ij}$ and only acts on $\overline{z}$-derivatives of the $\overline{f}^{(w)}_{ij}$ from the opposite chiral half in closed-string amplitudes

$$\nabla_\tau \overline{f}^{(w)}_{ij} = 0\,, \qquad \nabla_\tau\Big(\frac{\partial \overline{f}^{(w)}_{ij}}{\partial \overline{z}}\Big) = -\frac{\pi \overline{f}^{(w-1)}_{ij}}{2i\,(\operatorname{Im}\tau)^2}\,. \tag{7.31}$$

On these grounds, the analysis of boundary terms in τ is facilitated when loop-integrated GEIs $[[E^{m_1\cdots}_{1|A,\ldots}]]$ are expressed in terms of undifferentiated $f^{(w)}_{ij}$.

7.2.4. Integrating refined GEIs

After loop integration, the integration-by-parts equivalent representations of the simplest refined GEI $E_{1|2|3,4,5,6}$ in (4.37) translate into

$$\begin{aligned}
[[E_{1|2|3,4,5,6}]] &= -\frac{\pi}{\operatorname{Im}\tau} + \partial f^{(1)}_{12} + s_{12}(f^{(1)}_{12})^2 - 2s_{12}f^{(2)}_{12} \\
&\cong -\frac{\pi}{\operatorname{Im}\tau} - 2s_{12}f^{(2)}_{12} + f^{(1)}_{12}(s_{23}f^{(1)}_{23} + s_{24}f^{(1)}_{24} + s_{25}f^{(1)}_{25} + s_{26}f^{(1)}_{26}) \\
&\cong -2s_{12}f^{(2)}_{12} + f^{(1)}_{12}\big[s_{23}f^{(1)}_{23} + (3\leftrightarrow 4,5,6)\big] + \nu_{12}\big[s_{12}f^{(1)}_{12} + (1\leftrightarrow 3,4,5,6)\big]\,.
\end{aligned} \tag{7.32}$$

The first line follows from inserting $\partial g^{(1)}_{12} = \partial f^{(1)}_{12} - \frac{\pi}{\operatorname{Im}\tau}$ into (4.37), and the second and third line result from the integration-by-parts relation (7.27) after discarding $\partial_2(f^{(1)}_{12}\hat{\mathcal{I}}_6)$ and $\partial_2(\nu_{12}\hat{\mathcal{I}}_6)$, respectively. One can also arrive at last line by inserting $[[\ell^m]] = \sum_{j=2}^{6} k^m_j \nu_{1j}$ into the first line of (4.37) and expressing all the $g^{(n)}_{ij}$ in terms of $f^{(n)}_{ij}$ and ν_{ij}.

At seven points, the refined GEIs (6.22) integrate to

$$
\begin{aligned}
[[E_{1|23|4,5,6,7}]] &= -\frac{\pi}{\operatorname{Im}\tau} V_1(1,2,3) - s_{123} V_3(1,2,3) + (f_{12}^{(1)} + f_{31}^{(1)})\partial f_{23}^{(1)} + \partial f_{23}^{(2)} \qquad (7.33)\\
[[E_{1|4|23,5,6,7}]] &= -\frac{\pi}{\operatorname{Im}\tau} V_1(1,2,3) + \big[\partial f_{14}^{(1)} - s_{14} V_2(1,4)\big] V_1(1,2,3) - s_{24} V_3(1,2,4) + s_{34} V_3(1,3,4)\\
[[E^m_{1|2|3,4,5,6,7}]] &= -\frac{\pi}{\operatorname{Im}\tau}\big[k_2^m f_{12}^{(1)} + (2\leftrightarrow 3,\ldots,7)\big] + k_2^m\big[\partial f_{12}^{(2)} + s_{12}(f_{12}^{(1)} f_{12}^{(2)} - 3 f_{12}^{(3)})\big]\\
&\quad + \big[\partial f_{12}^{(1)} - s_{12} V_2(1,2)\big]\big[k_3^m f_{13}^{(1)} + (3\leftrightarrow 4,\ldots,7)\big] + \big[k_3^m s_{23} V_3(1,2,3) + (3\leftrightarrow 4,\ldots,7)\big]\,,
\end{aligned}
$$

where we reiterate that the elliptic V_w-functions are unchanged under the global replacement of $g_{ij}^{(n)} \rightarrow f_{ij}^{(n)}$. One can perform integrations by parts (7.27) similar to (7.32) to avoid the appearance of $\partial f_{ij}^{(n)}$ on the right-hand side. Similar to (7.19) and (7.20), the factors of $\frac{\pi}{\operatorname{Im}\tau}$ on the right-hand sides of (7.32) and (7.33) signal a modular anomaly: They depart from the purely holomorphic modular weights $(n,0)$ and $(n{+}1,0)$ of the $f_{ij}^{(n)}$ and $\partial f_{ij}^{(n)}$.

Note that the trace relations (5.31) and (5.32) of six- and seven-point GEIs can be verified at the level of the above expressions for the $[[E_{1|\ldots}]]$: While

$$
\begin{aligned}
&\frac{1}{2}\delta_{mn}[[E^{mn}_{1|2,3,4,5,6}]]\hat{\mathcal{I}}_6 + \big([[E_{1|2|3,4,5,6}]] + (2\leftrightarrow 3,4,5,6)\big)\hat{\mathcal{I}}_6\\
&\qquad = 2\pi i \frac{\partial}{\partial\tau}\hat{\mathcal{I}}_6 + 2\pi i \sum_{p=2}^{6} \frac{\partial}{\partial z_p}\Big(\frac{\operatorname{Im} z_{p1}}{\operatorname{Im}\tau}\hat{\mathcal{I}}_6\Big) \qquad (7.34)
\end{aligned}
$$

is a consequence of (7.26) at $n=6$ and $h(z,\tau)=1$, the seven-point analogues require a specialization of (7.28) to[17]

$$
\Big(\frac{n-6}{2i\operatorname{Im}\tau} + \nabla_\tau\Big) f_{ij}^{(1)}\Big|_{n=7} = \frac{\partial f_{ij}^{(2)}}{2\pi i}\,, \qquad (7.35)
$$

see (7.24) for the action of ∇_τ on $f_{ij}^{(w)}$.

[17] More generally, the choices of $h(z,\tau)$ in (7.28) relevant to integrated n-point closed-string correlators $[[\mathcal{K}_n(\ell)\tilde{\mathcal{K}}_n(-\ell)]]$ have the form $\frac{1}{(\operatorname{Im}\tau)^m}\prod_k f_{a_k b_k}^{(w_k)} \overline{f}_{c_k d_k}^{(\overline{w}_k)}$ with $m + \sum_k w_k = n-6$. In these cases, the second line of the following equivalence relation (7.28) vanishes (we are suppressing the $\overline{f}_{c_k d_k}^{(\overline{w}_k)}$ they are annihilated by ∇_τ),

$$
\begin{aligned}
0 \cong\; &\Big(\frac{1}{(\operatorname{Im}\tau)^m}\prod_k f_{a_k b_k}^{(w_k)}\Big(\sum_{i<j}^{n} s_{ij} f_{ij}^{(2)}\Big) + \sum_r \frac{w_r \partial f_{a_r b_r}^{(w_r+1)}}{(\operatorname{Im}\tau)^m}\prod_{k\neq r} f_{a_k b_k}^{(w_k)}\Big)\\
&+ 2\pi i\Big(\frac{n-6-m-w_1-w_2-\ldots}{2i\operatorname{Im}\tau}\Big)\frac{1}{(\operatorname{Im}\tau)^m}\prod_k f_{a_k b_k}^{(w_k)}\,.
\end{aligned}
$$

7.2.5. Modular anomalies versus BRST anomalies

The above instances of modular anomalies furnish another incarnation of the duality between kinematics and worldsheet functions. Modular anomalies are proposed to be the worldsheet counterpart of anomalous BRST variations such as

$$\begin{aligned} QC^{mn}_{1|2,3,4,5,6} &= -\delta^{mn}\Gamma_{1|2,3,4,5,6}\,, \qquad QP_{1|2|3,4,5,6} = -\Gamma_{1|2,3,4,5,6} \\ QC^{mn}_{1|23,4,\ldots,7} &= -\delta^{mn}\Gamma_{1|23,4,\ldots,7}\,, \qquad QP_{1|23|4,5,6,7} = QP_{1|4|23,5,6,7} = -\Gamma_{1|23,4,5,6,7} \\ QC^{mnp}_{1|2,3,\ldots,7} &= -\delta^{(mn}\Gamma^{p)}_{1|2,3,\ldots,7}\,, \qquad QP^{m}_{1|2|3,\ldots,7} = -\Gamma^{m}_{1|2,3,\ldots,7}\,, \end{aligned} \tag{7.36}$$

where the anomaly invariants $\Gamma_{1|\ldots}$ are defined in section I.5.2.3, and generalizations of (7.36) can be found in (I.5.28). The idea is to associate the anomaly invariants with the slot extensions $[[E_{1|2,3,4,5,6}]] = 1$ and

$$[[E_{1|23,4,5,6,7}]] = V_1(1,2,3)\,, \qquad [[E^{m}_{1|2,3,\ldots,7}]] = k_2^m f^{(1)}_{12} + (2\leftrightarrow 3,\ldots,7) \tag{7.37}$$

of earlier results according to the general dictionary

$$\Gamma^{m_1\ldots m_r}_{1|A_1,\ldots,A_d|B_1,\ldots,B_{d+r+5}} \leftrightarrow \frac{\pi}{\mathrm{Im}\,\tau}[[E^{m_1\ldots m_r}_{1|A_1,\ldots,A_d|B_1,\ldots,B_{d+r+5}}]]\,. \tag{7.38}$$

Under these identifications, the combinatorics of (7.36) literally translates into the following modular anomalies at six points

$$\begin{aligned} [[E^{mn}_{1|2,3,4,5,6}]] &= -\frac{\pi}{\mathrm{Im}\,\tau}\delta^{mn} + \text{modular weight }(2,0) \\ [[E_{1|2|3,4,5,6}]] &= -\frac{\pi}{\mathrm{Im}\,\tau} + \text{modular weight }(2,0) \end{aligned} \tag{7.39}$$

and at seven points

$$\begin{aligned} [[E^{mn}_{1|23,4,5,6,7}]] &= -\frac{\pi}{\mathrm{Im}\,\tau}\delta^{mn}V_1(1,2,3) + \text{modular weight }(3,0) \\ [[E^{mnp}_{1|2,3,4,5,6,7}]] &= -\frac{\pi}{\mathrm{Im}\,\tau}\delta^{(mn}\big[k_2^{p)}f^{(1)}_{12} + (2\leftrightarrow 3,\ldots,7)\big] + \text{modular weight }(3,0) \\ [[E^{m}_{1|2|3,4,5,6,7}]] &= -\frac{\pi}{\mathrm{Im}\,\tau}\big[k_2^{m}f^{(1)}_{12} + (2\leftrightarrow 3,\ldots,7)\big] + \text{modular weight }(3,0) \\ [[E_{1|23|4,5,6,7}]] &= -\frac{\pi}{\mathrm{Im}\,\tau}V_1(1,2,3) + \text{modular weight }(3,0) \\ [[E_{1|4|23,5,6,7}]] &= -\frac{\pi}{\mathrm{Im}\,\tau}V_1(1,2,3) + \text{modular weight }(3,0)\,, \end{aligned} \tag{7.40}$$

where the weight-$(n-4,0)$ parts can be found in (7.19), (7.20), (7.32) and (7.33). As we will see in section III.4.2, the above instances of modular anomalies drop out from the integrated six-point correlator $[[\mathcal{K}_6(\ell)]]$. The cancellation of modular anomalies will be shown to furnish a dual to the localization of BRST anomalies $Q\mathcal{K}_n(\ell)$ on the boundary of moduli space.

While the dictionary (7.38) is expected to extend to higher multiplicity, it is not clear whether it applies to higher powers $(\frac{\pi}{\operatorname{Im}\tau})^m$ with $m \geq 2$. It remains to clarify whether the absence of tensor structures $\delta^{m(n}\delta^{pq)}$ in $QC^{mnpq}_{1|2,3,\ldots,8} = -\delta^{(mn}\Gamma^{pq)}_{1|2,\ldots,8}$ can be reconciled with the contribution $[[\ell^m\ell^n\ell^p\ell^q]] = (\frac{\pi}{\operatorname{Im}\tau})^2\delta^{m(n}\delta^{pq)} + \ldots$ to $[[E^{mnpq}_{1|2,\ldots,8}]]$.

8. Conclusions

In this paper we continued setting up the ingredients that will be needed to build up one-loop correlators for massless open- and closed-string amplitudes in the pure-spinor formalism. We have introduced two classes of worldsheet functions that will manifest different aspects of the correlators to be assembled in part III. Both of them are constructed from loop momenta and combinations of Jacobi theta functions $g^{(n)}_{ij} = g^{(n)}(z_i - z_j, \tau)$ that are the coefficients in the Laurent expansion of the Kronecker–Eisenstein series [5].

The first class of worldsheet functions, denoted by $\mathcal{Z}$, is designed to capture the worldsheet singularities arising when the vertex operators approach each other on a genus-one surface. These singularities are straightforward to handle via an OPE analysis, and their behavior when the vertices are close together is the same as products of $1/z_{ij} = 1/(z_i - z_j)$ functions well-known from the tree-level correlators.

However, the OPE analysis is not enough to completely determine the one-loop $\mathcal{Z}$-functions as there can be non-singular pieces that do not vanish on a genus-one surface[18]. Instead, our starting point to constrain the non-singular pieces is the following observation on tree-level correlators: The products of singular functions $1/z_{ij}$ at genus zero end up assembling chains $1/(z_{12}z_{23}\ldots z_{p-1,p})$ [19] that obey shuffle symmetries among their labels $1, 2, \ldots, p$. By imposing the same shuffle symmetries among the labels of their one-loop counterparts $\mathcal{Z}$ and using Fay identities one proves the existence of non-singular pieces in the one-loop worldsheet functions.

The algorithmic determination of these non-singular pieces follows from another surprising feature of these functions; their properties mimic those of superfield building blocks

[18] These non-singular parts are absent at tree level where the knowledge of the singular behavior is enough to fix the whole function.

discussed in part I. More precisely, the role of the pure-spinor BRST charge acting on the superfields is replaced by a monodromy operator acting on the genus-one functions and the loop momentum. This observation, among others along the same lines, has been interpreted as a duality between worldsheet functions and kinematics.

The second class of worldsheet functions discussed in this paper concerns the generalized elliptic integrands (GEIs) briefly introduced in [4]. GEIs are monodromy-invariant combinations of $\mathcal{Z}$-functions, and already their very construction is driven by the duality between worldsheet functions and kinematics: GEIs can be assembled from the monodromy-covariant functions $\mathcal{Z}$ in exactly the same combinatorial manner as kinematic BRST invariants are assembled from Berends–Giele superfield building blocks (reviewed in part I). These definitions lead to a plethora of relations that apply in similar if not identical form to the superfield building blocks, manifesting various further incarnations of the duality between worldsheet functions and kinematics.

A multitude of identities among $\mathcal{Z}$-functions and GEIs has been discussed in this paper that support their duality connection with superfield building blocks. However, we observed that holomorphic Eisenstein series lead to departures from a strict duality between functions and kinematics starting at eight points. The solution to this puzzling behavior, for instance through systematic redefinitions of $\mathcal{Z}$-functions and GEIs via Eisenstein series, will be left for the future. Furthermore, a preliminary analysis indicates that the functions considered in this paper admit compact generating-series representations whose detailed presentation we also leave for future work.

The relevance of both the $\mathcal{Z}$-functions as well as GEIs for the assembly of one-loop correlators will become apparent in the sequel part III of this series of papers.

Acknowledgements: We are indebted to the IAS Princeton and to Nima Arkani-Hamed for kind hospitality during an inspiring visit which initiated this project. This research was supported by the Munich Institute for Astro- and Particle Physics (MIAPP) of the DFG cluster of excellence "Origin and Structure of the Universe", and we are grateful to the organizers for creating a stimulating atmosphere. CRM is supported by a University Research Fellowship from the Royal Society. The research of OS was supported in part by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Research, Innovation and Science.

Appendix A. Bootstrapping the shuffle-symmetric worldsheet functions

This appendix complements the results of the bootstrap techniques for $\mathcal{Z}$-functions outlined in section 4.4 with derivations based on the system of monodromy variations. The key steps will be presented in detail for six points and for some selected seven- and eight-point functions; the results from the omitted derivations can be obtained with reasonable effort [32] and do not require any new methods.

In the derivations below we will use the representations of GEIs obtained in section 6 as they lead to considerably shorter results; in some cases, they even suggest pattern-driven general closed formulæ.

A.1. Six points

The starting point at six points is given by the extended GEIs (4.33) from the five-point results (4.31) and (4.32), namely

$$E_{1|23,4,5,6} = V_1(1,2,3)\,, \qquad E^m_{1|2,3,4,5,6} = \ell^m - \big[\partial V_0(1,2) + (2 \leftrightarrow 3,4,5,6)\big]\,. \tag{A.1}$$

They are written in terms of the V_w- and ∂V_w-functions with generating series in (6.1), (6.7) and (6.12) for convenience. As we have seen in section 4.4, the monodromy variations of the six-point shuffle-symmetric worldsheet functions are given by

$$\begin{aligned}
D\mathcal{Z}_{123,4,5,6} &= \Omega_1 E_{1|23,4,5,6} - \Omega_3 E_{3|12,4,5,6}\,, \\
D\mathcal{Z}_{12,34,5,6} &= \Omega_1 E_{1|2,34,5,6} - \Omega_2 E_{2|1,34,5,6} + \Omega_3 E_{3|12,4,5,6} - \Omega_4 E_{4|12,3,5,6}\,, \\
D\mathcal{Z}^m_{12,3,4,5,6} &= \Omega_1 E^m_{1|2,3,4,5,6} - \Omega_2 E^m_{2|1,3,4,5,6} + \big[k^m_3 \Omega_3 E_{3|12,4,5,6} + (3 \leftrightarrow 4,5,6)\big]\,, \\
D\mathcal{Z}^{mn}_{1,2,3,4,5,6} &= k^m_1 \Omega_1 E^n_{1|2,3,4,5,6} + k^n_1 \Omega_1 E^m_{1|2,3,4,5,6} + (1 \leftrightarrow 2,3,4,5,6)\,, \\
D\mathcal{Z}_{2|1,3,4,5,6} &= \Omega_2 k^m_2 E^m_{2|1,3,4,5,6}\,.
\end{aligned} \tag{A.2}$$

To solve these equations using the generating-series techniques of section 6 it will be convenient to rewrite the above GEIs in a basis where leg 1 is in the special slot. This can be done by exploiting the duality with the BRST invariants and using the identities of section 5.2.2. In this new basis we have:

$$D\mathcal{Z}_{123,4,5,6} = \Omega_{13} E_{1|23,4,5,6}\,, \tag{A.3}$$

$$D\mathcal{Z}_{12,34,5,6} = \Omega_{12} E_{1|2,34,5,6} + \Omega_{32} E_{1|23,4,5,6} + \Omega_{24} E_{1|24,3,5,6}\,, \tag{A.4}$$

$$D\mathcal{Z}^m_{12,3,4,5,6} = \Omega_{12} E^m_{1|2,3,4,5,6} - \big[\Omega_{23} k^m_3 E_{1|23,4,5,6} + (3 \leftrightarrow 4,5,6)\big]\,, \tag{A.5}$$

$$\begin{aligned}
D\mathcal{Z}^{mn}_{1,2,3,4,5,6} &= \big[\Omega_{21}(k^m_2 E^n_{1|2,3,4,5,6} + k^n_2 E^m_{1|2,3,4,5,6}) + (2 \leftrightarrow 3,4,5,6)\big] \\
&\quad + \big[(k^m_2 k^n_3 + k^n_2 k^m_3)\Omega_{23} E_{1|23,4,5,6} + (2,3|2,3,4,5,6)\big]\,,
\end{aligned} \tag{A.6}$$

$$D\mathcal{Z}_{2|1,3,4,5,6} = \Omega_2 k^m_2 \big(E^m_{1|2,3,4,5,6} + \big[k^m_3 E_{1|23,4,5,6} + (3 \leftrightarrow 4,5,6)\big]\,. \tag{A.7}$$

The scalar equations are easily solved using cyclic symmetry of $V_w(1,2,\ldots,n)$ and the monodromy variations $D\partial V_w(1,2,\ldots,n) = -\Omega_{1n}V_w(1,2,\ldots,n)$. We get,

$$\begin{aligned}
\mathcal{Z}_{123,4,5,6} &= -\partial V_1(1,2,3)\,, \\
\mathcal{Z}_{12,34,5,6} &= -\partial V_0(1,2)V_1(1,3,4) + \partial V_1(4,1,2) - \partial V_1(3,1,2)\,,
\end{aligned} \tag{A.8}$$

whose equivalence with the solutions presented in (4.35) is easily established using Fay identities. Let us now solve the monodromy variation (A.5) of the vectorial function

$$D\mathcal{Z}^m_{12,3,4,5,6} = \Omega_{12}E^m_{1|2,3,4,5,6} - \big[\Omega_{23}k_3^m E_{1|23,4,5,6} + (3\leftrightarrow 4,5,6)\big] \tag{A.9}$$

$$= \Omega_{12}\ell^m + \Omega_{12}k_2^m g^{(1)}_{12} + \big[k_3^m(\Omega_{12}g^{(1)}_{13} - \Omega_{23}V_1(1,2,3)) + (3\leftrightarrow 4,5,6)\big]\,, \tag{A.10}$$

where the second line follows from (A.1). Noting that $D(g^{(1)}_{12}\ell^m) = \Omega_{12}\ell^m - g^{(1)}_{12}\sum_{j=2}^6\Omega_{1j}k_j^m$ one can rewrite (A.10) as follows

$$\begin{aligned}
D\mathcal{Z}^m_{12,3,4,5,6} &= \big(\Omega_{12}\ell^m - g^{(1)}_{12}\sum_{j=2}^{6}\Omega_{1j}k_j^m\big) + 2k_2^m\Omega_{12}g^{(1)}_{12} \\
&\quad + \big[k_3^m(\Omega_{12}g^{(1)}_{13} + \Omega_{13}g^{(1)}_{12} - \Omega_{23}V_1(1,2,3)) + (3\leftrightarrow 4,5,6)\big]\,.
\end{aligned} \tag{A.11}$$

The solution to (A.11) can be obtained by inspection and is given by

$$\begin{aligned}
\mathcal{Z}^m_{12,3,4,5,6} &= g^{(1)}_{12}\ell^m + 2k_2^m g^{(2)}_{12} + \big[k_3^m\big(g^{(1)}_{12}g^{(1)}_{13} - \partial V_1(3,1,2)\big) + (3\leftrightarrow 4,5,6)\big] \\
&= \ell^m g^{(1)}_{12} + (k_2^m - k_1^m)g^{(2)}_{12} + \big[k_3^m(g^{(2)}_{13} - g^{(2)}_{23}) + (3\leftrightarrow 4,5,6)\big]\,,
\end{aligned} \tag{A.12}$$

see (4.35). The equality in the last line follows from momentum conservation and $g^{(1)}_{12}g^{(1)}_{13} - \partial V_1(3,1,2) - g^{(2)}_{12} = g^{(2)}_{13} - g^{(2)}_{23}$, which can be shown using Fay identities. As a side remark, note that one can arrive at (A.12) from (A.10) using an effective "integration" rule $\int\Omega_{ij}g^{(n)}_{ij} = (n{+}1)g^{(n+1)}_{ij}$, $\forall\, n\in\mathbb{N}$ to "invert" the D operator.

The solution to the tensorial monodromy variation (A.6),

$$\begin{aligned}
D\mathcal{Z}^{mn}_{1,2,3,4,5,6} &= \big[\Omega_{21}k_2^{(m}E^{n)}_{1|2,3,4,5,6} + (2\leftrightarrow 3,4,5,6)\big] \\
&\quad + \big[k_2^{(m}k_3^{n)}\Omega_{23}E_{1|23,4,5,6} + (2,3|2,3,4,5,6)\big]
\end{aligned}$$

can be found similarly. First one plugs in the vectorial extended GEI from (A.1) to obtain

$$\begin{aligned}
D\mathcal{Z}^{mn}_{1,2,3,4,5,6} &= \big[\Omega_{21}k_2^{(m}\ell^{n)} + 2k_2^m k_2^n\Omega_{21}g^{(1)}_{12} + (2\leftrightarrow 3,4,5,6)\big] \\
&\quad + \big[k_2^{(m}k_3^{n)}\big(\Omega_{21}g^{(1)}_{13} + \Omega_{31}g^{(1)}_{12} - \Omega_{32}V_1(3,1,2)\big) + (2,3|2,3,4,5,6)\big]\,,
\end{aligned} \tag{A.13}$$

whose solution is easily found after noticing that $D(\ell^m \ell^n) = \Omega_{21} k_2^{(m} \ell^{n)} + (2 \leftrightarrow 3,4,5,6)$,

$$\begin{aligned} \mathcal{Z}^{mn}_{1,2,3,4,5,6} &= \ell^m \ell^n - 2\big[k_2^m k_2^n g^{(2)}_{12} + (2 \leftrightarrow 3,4,5,6)\big] \\ &\quad - \big[k_2^{(m} k_3^{n)} \big(g^{(1)}_{12} g^{(1)}_{13} - \partial V_1(3,1,2)\big) + (2,3|2,3,4,5,6)\big] . \end{aligned} \tag{A.14}$$

Fay identities imply that (A.14) is equivalent to the expression given in (4.35),

$$\mathcal{Z}^{mn}_{1,2,3,4,5,6} = \ell^m \ell^n + \big[k_1^{(m} k_2^{n)} g^{(2)}_{12} + (1,2|1,2,3,4,5,6)\big] . \tag{A.15}$$

The solution to the refined worldsheet function can be easily found. After plugging in the extended GEIs on the right-hand side of (A.7), straightforward algebra leads to

$$D\mathcal{Z}_{2|1,3,4,5,6} = \Omega_2\big((\ell \cdot k_2) + s_{21} g^{(1)}_{21} + s_{23} g^{(1)}_{23} + s_{24} g^{(1)}_{24} + s_{25} g^{(1)}_{25} + s_{26} g^{(1)}_{26}\big) \cong 0 , \tag{A.16}$$

which vanishes in view of the total-derivative relation (2.20). Therefore, one can choose

$$\mathcal{Z}_{2|1,3,4,5,6} = 0 , \tag{A.17}$$

see (4.35). As mentioned in section 4.4.3, this vanishing is compatible with a duality between refined worldsheet functions and BRST-exact superfields, see (5.24).

Plugging the results above in the expressions (4.26) and (4.27) leads to the expressions (4.36), (4.37) and (6.17) for GEIs. Their seven-point extensions (4.38) will be used in the next step of the bootstrap procedure.

A.2. Seven points

The solution to the scalar monodromy variations

$$\begin{aligned} D\mathcal{Z}_{1234,5,6,7} &= \Omega_1 E_{1|234,5,6,7} - \Omega_4 E_{4|123,5,6,7} , \\ D\mathcal{Z}_{123,45,6,7} &= \Omega_1 E_{1|23,45,6,7} - \Omega_3 E_{3|12,45,6,7} , + \Omega_4 E_{4|123,5,6,7} - \Omega_5 E_{5|123,4,6,7} \\ D\mathcal{Z}_{12,34,56,7} &= \Omega_1 E_{1|2,34,56,7} - \Omega_2 E_{2|1,34,56,7} + (12 \leftrightarrow 34, 56) , \end{aligned} \tag{A.18}$$

is easily obtained after rewriting the GEIs in the canonical basis and using (6.17),

$$\begin{aligned} D\mathcal{Z}_{1234,5,6,7} &= \Omega_{14} V_2(1,2,3,4) \\ D\mathcal{Z}_{123,45,6,7} &= \Omega_{13} V_1(1,2,3) V_1(1,4,5) - \Omega_{34} V_2(1,2,3,4) + \Omega_{35} V_2(1,2,3,5) \\ D\mathcal{Z}_{12,34,56,7} &= \Omega_{12} V_1(1,3,4) V_1(1,5,6) \\ &\quad + \big[\Omega_{23}\big(V_2(1,2,3,6) - V_1(1,2,3)V_1(1,5,6) - V_2(1,2,3,5)\big) - (3 \leftrightarrow 4)\big] \\ &\quad + \big[\Omega_{25}\big(V_2(1,2,5,4) - V_1(1,2,5)V_1(1,3,4) - V_2(1,2,5,3)\big) - (5 \leftrightarrow 6)\big] . \end{aligned} \tag{A.19}$$

Noting the fundamental equation (6.8) and cyclicity of $V_w(1,\dots,n)$ we arrive at the following solutions

$$\begin{aligned}
\mathcal{Z}_{1234,5,6,7} &= -\partial V_2(1,2,3,4)\,, \\
\mathcal{Z}_{123,45,6,7} &= -\partial V_1(1,2,3)V_1(1,4,5) + \partial V_2(5,1,2,3) - \partial V_2(4,1,2,3) \\
\mathcal{Z}_{12,34,56,7} &= -\partial V_0(1,2)V_1(1,3,4)V_1(1,5,6) + \partial V_1(4,1,2)V_1(1,5,6) \\
&\quad - \partial V_1(5,1,2)V_1(1,3,4) - \partial V_1(3,1,2)V_1(1,5,6) + \partial V_1(6,1,2)V_1(1,3,4) \\
&\quad + \partial V_2(3,6,1,2) - \partial V_2(3,5,1,2) + \partial V_2(4,5,1,2) - \partial V_2(4,6,1,2) \\
&\quad + \partial V_2(6,3,1,2) - \partial V_2(5,3,1,2) + \partial V_2(5,4,1,2) - \partial V_2(6,4,1,2)\,.
\end{aligned} \tag{A.20}$$

A long but straightforward application of Fay identities demonstrates the equivalence between the above solutions and the ones presented in the main text, (4.40). While the above form of the functions is easy to derive from the monodromy variations, it does not expose the singularity structure as the vertex positions approach each other. This constitutes a drawback of the representation in (A.20) and motivates the rewriting in (4.40).

A.2.1. Vectorial seven-point functions

The monodromy variation (4.13) of the vectorial seven-point function $\mathcal{Z}^m_{123,4,5,6,7}$ can be written in a basis of GEIs as

$$\begin{aligned}
D\mathcal{Z}^m_{123,4,5,6,7} &= \Omega_{13}E^m_{1|23,4,5,6,7} + \big[k_4^m\Omega_{43}E_{1|234,5,6,7} + (4\leftrightarrow 5,6,7)\big] \\
&= \Omega_{13}V_1(1,2,3)\ell^m + k_2^m\Omega_{13}\partial V_1(2,3,1) - k_3^m\Omega_{13}\partial V_1(3,2,1) \\
&\quad + \big[k_4^m\big(\Omega_{13}g^{(1)}_{14}V_1(1,2,3) - \Omega_{34}V_2(1,2,3,4)\big) + (4\leftrightarrow 5,6,7)\big]\,.
\end{aligned} \tag{A.21}$$

Similarly as before, in order to integrate the term containing ℓ^m in the above variation, we add and subtract $\partial V_1(1,2,3)\sum_{j=2}^7\Omega_{1j}k_j^m$ to obtain

$$\begin{aligned}
D\mathcal{Z}^m_{123,4,5,6,7} &= \Omega_{13}V_1(1,2,3)\ell^m + \partial V_1(1,2,3)\big[\Omega_{12}k_2^m + (2\leftrightarrow 3,4,5,6,7)\big] \\
&\quad + k_2^m\big(\Omega_{13}\partial V_1(2,3,1) - \Omega_{12}\partial V_1(1,2,3)\big) - k_3^m\big(\Omega_{13}\partial V_1(3,2,1) + \Omega_{13}\partial V_1(1,2,3)\big) \\
&\quad + \big[k_4^m\big(\Omega_{13}g^{(1)}_{14}V_1(1,2,3) - \Omega_{14}\partial V_1(1,2,3) - \Omega_{34}V_2(1,2,3,4)\big) + (4\leftrightarrow 5,6,7)\big]\,.
\end{aligned} \tag{A.22}$$

One can then show that (A.22) integrates to

$$\begin{aligned}
\mathcal{Z}^m_{123,4,5,6,7} &= -\ell^m\partial V_1(1,2,3) + k_3^m\partial^2 V_1(1,2,3) \\
&\quad + \frac{1}{2}k_2^m\big[\partial^2 V_1(1,2,3) + \partial^2 V_1(2,3,1) + \partial^2 V_1(2,1,3)\big] \\
&\quad - \big[k_4^m\big(\partial V_2(4,1,2,3) + g^{(1)}_{14}\partial V_1(1,2,3)\big) + (4\leftrightarrow 5,6,7)\big]\,,
\end{aligned} \tag{A.23}$$

which can be rewritten as

$$\begin{aligned}\mathcal{Z}^m_{123,4,5,6,7} &= -\ell^m \partial V_1(1,2,3) - \frac{1}{2}k_1^m \partial^2 V_1(1,2,3) + \frac{1}{2}k_3^m \partial^2 V_1(1,2,3) \\ &+ \frac{1}{2}k_2^m \big[\partial^2 V_1(2,3,1) + \partial^2 V_1(2,1,3)\big] \\ &+ \big[k_4^m V_1(1,2,3)(g^{(2)}_{14} - g^{(2)}_{34}) + (4 \leftrightarrow 5,6,7)\big] \,.\end{aligned} \tag{A.24}$$

To see this one uses momentum conservation and the identity

$$\partial V_2(4,1,2,3) + g^{(1)}_{14}\partial V_1(1,2,3) = -V_1(1,2,3)(g^{(2)}_{14} - g^{(2)}_{34}) - \frac{1}{2}\partial^2 V_1(1,2,3) \,. \tag{A.25}$$

Alternatively, the expression (A.24) can be rewritten in terms of $g^{(n)}_{ij}$-functions in order to make its singularity structure more evident,

$$\begin{aligned}\mathcal{Z}^m_{123,4,5,6,7} &= \ell^m \big[g^{(1)}_{12} g^{(1)}_{23} + g^{(2)}_{12} - g^{(2)}_{13} + g^{(2)}_{23}\big] + (k_3^m - k_1^m)\big(g^{(1)}_{12} g^{(2)}_{13} + g^{(1)}_{23} g^{(2)}_{13} - 3g^{(3)}_{13}\big) \\ &+ k_2^m \big(g^{(1)}_{13}(g^{(2)}_{12} - g^{(2)}_{23}) + g^{(2)}_{13}(g^{(1)}_{23} - g^{(1)}_{12})\big) \\ &+ \big[k_4^m \big(g^{(1)}_{12} + g^{(1)}_{23} + g^{(1)}_{31}\big)\big(g^{(2)}_{14} - g^{(2)}_{34}\big) + (4 \leftrightarrow 5,6,7)\big] \,.\end{aligned} \tag{A.26}$$

Similarly, the monodromy variation of $\mathcal{Z}^m_{12,34,5,6,7}$,

$$\begin{aligned}D\mathcal{Z}^m_{12,34,5,6,7} &= \Omega_{12} E^m_{1|2,34,5,6,7} - \Omega_{23} E^m_{1|23,4,5,6,7} + \Omega_{24} E^m_{1|24,3,5,6,7} \\ &+ k_3^m \Omega_{24} E_{1|243,5,6,7} - k_4^m \Omega_{23} E_{1|234,5,6,7} \\ &+ \Big[k_5^m \big(\Omega_{25} E_{1|254,3,6,7} - \Omega_{25} E_{1|253,4,6,7} - \Omega_{25} E_{1|25,34,6,7} \\ &\quad + \Omega_{24} E_{1|245,3,6,7} - \Omega_{23} E_{1|235,4,6,7} + (5 \leftrightarrow 6,7\Big] \,,\end{aligned} \tag{A.27}$$

is readily integrated and yields, after using identities similar to (A.25) and momentum conservation, the following result:

$$\begin{aligned}\mathcal{Z}^m_{12,34,5,6,7} &= \ell^m \Big[g^{(1)}_{12} V_1(1,3,4) - \partial V_1(3,1,2) + \partial V_1(4,1,2)\Big] \\ &- \Big[k_1^m \big(g^{(2)}_{12} V_1(2,3,4) + \frac{1}{2}\partial^2 V_1(1,2,4) - \frac{1}{2}\partial^2 V_1(1,2,3)\big) - (1 \leftrightarrow 2)\Big] \\ &- \Big[k_3^m \big(g^{(2)}_{34} V_1(4,1,2) + \frac{1}{2}\partial^2 V_1(3,4,2) - \frac{1}{2}\partial^2 V_1(3,4,1)\big) - (3 \leftrightarrow 4)\Big] \\ &+ \Big[k_5^m \big(g^{(2)}_{51} V_1(1,3,4) - g^{(2)}_{52} V_1(2,3,4) + g^{(2)}_{53} V_1(3,1,2) - g^{(2)}_{54} V_1(4,1,2)\big) + (5 \leftrightarrow 6,7)\Big] \,.\end{aligned} \tag{A.28}$$

Its expansion in terms of $g_{ij}^{(n)}$-functions can be shown to read,

$$\begin{aligned}\mathcal{Z}^m_{12,34,5,6,7} &= \ell^m \big(g_{12}^{(1)} g_{34}^{(1)} + g_{13}^{(2)} - g_{14}^{(2)} + g_{24}^{(2)} - g_{23}^{(2)}\big) + (k_{12}^m - k_{34}^m)\big(g_{14}^{(3)} - g_{13}^{(3)} + g_{23}^{(3)} - g_{24}^{(3)}\big) \\ &\quad + \big[g_{34}^{(1)} g_{12}^{(2)}(k_2^m - k_1^m) + k_1^m g_{12}^{(1)}\big(g_{23}^{(2)} - g_{24}^{(2)}\big) + k_2^m g_{12}^{(1)}\big(g_{13}^{(2)} - g_{14}^{(2)}\big) + (12 \leftrightarrow 34)\big] \\ &\quad + \Big\{k_5^m \big[g_{15}^{(1)}(g_{13}^{(2)} - g_{14}^{(2)} + g_{45}^{(2)} - g_{35}^{(2)}) + g_{25}^{(1)}(g_{24}^{(2)} - g_{23}^{(2)} + g_{35}^{(2)} - g_{45}^{(2)}) \\ &\qquad + g_{12}^{(1)}(g_{35}^{(2)} - g_{45}^{(2)}) + g_{34}^{(1)}(g_{15}^{(2)} - g_{25}^{(2)}) + g_{14}^{(3)} - g_{13}^{(3)} + g_{23}^{(3)} - g_{24}^{(3)}\big] + (5 \leftrightarrow 6,7)\Big\} .\end{aligned} \tag{A.29}$$

This completes the bootstrapping of the vectorial shuffle-symmetric functions for seven points.

A.2.2. Tensorial functions

An analogous procedure can be used for solving the tensorial seven-point functions starting from their monodromy variations given in (4.13). The outcome can be written as

$$\begin{aligned}\mathcal{Z}^{mn}_{12,3,4,5,6,7} &= \ell^m \ell^n g_{12}^{(1)} + \big[\ell^{(m} k_3^{n)}(g_{13}^{(2)} - g_{23}^{(2)}) + 2k_3^m k_3^n (g_{13}^{(3)} - g_{23}^{(3)}) + (3 \leftrightarrow 4,5,6,7)\big] \\ &\quad + g_{12}^{(2)}(\ell^{(m} k_2^{n)} - \ell^{(m} k_1^{n)}) + g_{12}^{(3)}(2k_1^m k_1^n + 2k_2^m k_2^n - k_1^m k_2^n - k_2^m k_1^n) \\ &\quad + \big[k_3^{(m} k_1^{n)}(g_{12}^{(1)} g_{23}^{(2)} - g_{13}^{(3)} + g_{23}^{(3)}) + k_3^{(m} k_2^{n)}(g_{12}^{(1)} g_{31}^{(2)} - g_{13}^{(3)} + g_{23}^{(3)}) + (3 \leftrightarrow 4,5,6,7)\big] \\ &\quad + \big[k_3^{(m} k_4^{n)}(g_{12}^{(1)} g_{34}^{(2)} + g_{34}^{(1)}(g_{13}^{(2)} - g_{23}^{(2)} - g_{14}^{(2)} + g_{24}^{(2)}) \\ &\qquad + g_{13}^{(3)} - g_{23}^{(3)} + g_{14}^{(3)} - g_{24}^{(3)}) + (3,4|3,4,5,6,7)\big] , \\ \mathcal{Z}^{mnp}_{1,2,3,4,5,6,7} &= \ell^m \ell^n \ell^p + \big[k_1^{(m} k_2^n \ell^{p)} g_{12}^{(2)} - k_1^{(m}(k_1^n - k_2^n) k_2^{p)} g_{12}^{(3)} + (1,2|1,2,3,4,5,6,7)\big] \\ &\quad + \big[k_1^{(m} k_2^n k_3^{p)}\big(g_{23}^{(1)}(g_{12}^{(2)} - g_{13}^{(2)}) + g_{12}^{(3)} + g_{13}^{(3)}\big) + (1,2,3|1,2,3,4,5,6,7)\big] .\end{aligned} \tag{A.30}$$

Note that the coefficient of $k_1^m k_2^n k_3^p$ in the last line is totally symmetric in $1,2,3$. Again, their singularity structure within a given word is the same as in their tree-level counterparts, see section 4.1.

A.2.3. Assembling seven-point GEIs

Now that the shuffle-symmetric $\mathcal{Z}$-functions at seven points are known, one can assemble the GEIs as described in section 4.3. The scalar GEIs follow from the replacement rule

$M_AM_{B,C,D} \to \mathcal{Z}_{A,B,C,D}$ applied to the Berends–Giele expansion of the BRST invariants $C_{1|2345,6,7}$, $C_{1|234,56,7}$ and $C_{1|23,45,67}$ from [23],

$$
\begin{aligned}
E_{1|2345,6,7} &= \mathcal{Z}_{1,2345,6,7} + \mathcal{Z}_{512,34,6,7} + \mathcal{Z}_{12,345,6,7} + \mathcal{Z}_{123,45,6,7} + \mathcal{Z}_{1234,5,6,7}\\
&\quad + \mathcal{Z}_{5123,4,6,7} + \mathcal{Z}_{51,234,6,7} + \mathcal{Z}_{451,23,6,7} + \mathcal{Z}_{3451,2,6,7} + \mathcal{Z}_{4512,3,6,7}\,,\\
E_{1|234,56,7} &= \mathcal{Z}_{1,234,56,7} + \mathcal{Z}_{214,3,56,7} + \mathcal{Z}_{15,234,6,7} - \mathcal{Z}_{16,234,5,7} + \mathcal{Z}_{12,34,56,7}\\
&\quad + \mathcal{Z}_{123,4,56,7} + \mathcal{Z}_{14,32,56,7} + \mathcal{Z}_{143,2,56,7} + \mathcal{Z}_{612,34,5,7} + \mathcal{Z}_{6123,4,5,7}\\
&\quad + \mathcal{Z}_{5124,3,6,7} + \mathcal{Z}_{614,32,5,7} + \mathcal{Z}_{6143,2,5,7} + \mathcal{Z}_{5142,3,6,7} - \mathcal{Z}_{512,34,6,7}\\
&\quad - \mathcal{Z}_{5123,4,6,7} - \mathcal{Z}_{6124,3,5,7} - \mathcal{Z}_{514,32,6,7} - \mathcal{Z}_{5143,2,6,7} - \mathcal{Z}_{6142,3,5,7}\,,\\
E_{1|23,45,67} &= \mathcal{Z}_{1,23,45,67} + \mathcal{Z}_{12,3,45,67} - \mathcal{Z}_{13,2,45,67} + \mathcal{Z}_{14,5,23,67} - \mathcal{Z}_{15,4,23,67}\\
&\quad + \mathcal{Z}_{16,7,23,45} - \mathcal{Z}_{17,6,23,45} + \mathcal{Z}_{217,3,45,6} - \mathcal{Z}_{317,2,45,6} - \mathcal{Z}_{216,3,45,7}\\
&\quad + \mathcal{Z}_{316,2,45,7} + \mathcal{Z}_{413,5,67,2} - \mathcal{Z}_{513,4,67,2} - \mathcal{Z}_{412,5,67,3} + \mathcal{Z}_{512,4,67,3}\\
&\quad + \mathcal{Z}_{615,7,23,4} - \mathcal{Z}_{715,6,23,4} - \mathcal{Z}_{614,7,23,5} + \mathcal{Z}_{714,6,23,5} + \mathcal{Z}_{7135,2,4,6}\\
&\quad + \mathcal{Z}_{7153,2,4,6} - \mathcal{Z}_{7125,3,4,6} - \mathcal{Z}_{7152,3,4,6} - \mathcal{Z}_{7134,2,5,6} - \mathcal{Z}_{7143,2,5,6}\\
&\quad + \mathcal{Z}_{7124,3,5,6} + \mathcal{Z}_{7142,3,5,6} - \mathcal{Z}_{6135,2,4,7} - \mathcal{Z}_{6153,2,4,7} + \mathcal{Z}_{6125,3,4,7}\\
&\quad + \mathcal{Z}_{6152,3,4,7} + \mathcal{Z}_{6134,2,5,7} + \mathcal{Z}_{6143,2,5,7} - \mathcal{Z}_{6124,3,5,7} - \mathcal{Z}_{6142,3,5,7}\,,
\end{aligned}
\tag{A.31}
$$

and read as in (6.19) after the solutions for $\mathcal{Z}$ obtained above are plugged in. Similarly, the lengthy expansions of the vectorial GEIs

$$
\begin{aligned}
E^m_{1|234,5,6,7} &= \mathcal{Z}^m_{1,234,5,6,7} + \mathcal{Z}^m_{123,4,5,6,7} + \mathcal{Z}^m_{412,3,5,6,7} + \mathcal{Z}^m_{341,2,5,6,7}\\
&\quad + \mathcal{Z}^m_{12,34,5,6,7} + \mathcal{Z}^m_{41,23,5,6,7} + k^m_2 \mathcal{Z}_{1432,5,6,7} + k^m_4 \mathcal{Z}_{1234,5,6,7}\\
&\quad - k^m_3(\mathcal{Z}_{1423,5,6,7} + \mathcal{Z}_{1243,5,6,7}) - \big[k^m_5(\mathcal{Z}_{51,234,6,7} + \mathcal{Z}_{512,34,6,7} + \mathcal{Z}_{514,32,6,7}\\
&\qquad + \mathcal{Z}_{5123,4,6,7} + \mathcal{Z}_{5143,2,6,7} - \mathcal{Z}_{5124,3,6,7} + \mathcal{Z}_{5142,3,6,7}) + (5 \leftrightarrow 6,7)\big]\,, \qquad \text{(A.32)}\\
E^m_{1|23,45,6,7} &= \mathcal{Z}^m_{1,23,45,6,7} + \mathcal{Z}^m_{12,3,45,6,7} - \mathcal{Z}^m_{13,2,45,6,7} + \mathcal{Z}^m_{14,23,5,6,7} - \mathcal{Z}^m_{15,23,4,6,7}\\
&\quad + \mathcal{Z}^m_{413,2,5,6,7} + \mathcal{Z}^m_{512,3,4,6,7} - \mathcal{Z}^m_{412,3,5,6,7} - \mathcal{Z}^m_{513,2,4,6,7}\\
&\quad + \big[k^m_3(\mathcal{Z}_{123,45,6,7} - \mathcal{Z}_{4123,5,6,7} + \mathcal{Z}_{5123,4,6,7}) - (2 \leftrightarrow 3)\big]\\
&\quad + \big[k^m_5(\mathcal{Z}_{145,23,6,7} - \mathcal{Z}_{2145,3,6,7} + \mathcal{Z}_{3145,2,6,7}) - (4 \leftrightarrow 5)\big]\\
&\quad - \big[k^m_6(\mathcal{Z}_{61,23,45,7} + \mathcal{Z}_{612,3,45,7} - \mathcal{Z}_{613,2,45,7} + \mathcal{Z}_{614,23,5,7} - \mathcal{Z}_{615,23,4,7}\\
&\qquad - (\mathcal{Z}_{6134,2,5,7} + \mathcal{Z}_{6143,2,5,7}) - (\mathcal{Z}_{6125,3,4,7} + \mathcal{Z}_{6152,3,4,7})\\
&\qquad + (\mathcal{Z}_{6135,2,4,7} + \mathcal{Z}_{6153,2,4,7}) + (\mathcal{Z}_{6124,3,5,7} + \mathcal{Z}_{6142,3,5,7})) + (6 \leftrightarrow 7)\big]\,,
\end{aligned}
$$

collapse to a few terms (6.20) when rewritten in terms of V_w- and ∂V_w-functions. Similarly, the tensorial GEIs

$$\begin{aligned}
E^{mn}_{1|23,4,5,6,7} &= \mathcal{Z}^{mn}_{1,23,4,5,6,7} + \mathcal{Z}^{mn}_{12,3,4,5,6,7} - \mathcal{Z}^{mn}_{13,2,4,5,6,7} + k_3^{(m}\mathcal{Z}^{n)}_{123,4,5,6,7} - k_2^{(m}\mathcal{Z}^{n)}_{132,4,5,6,7} \\
&\quad + \big[k_4^{(m}k_5^{n)}\big\{ - \mathcal{Z}_{514,23,6,7} + (\mathcal{Z}_{1245,3,6,7} + \text{symm}(2,4,5)) \\
&\qquad - (\mathcal{Z}_{1345,2,6,7} + \text{symm}(3,4,5))\big\} + (4,5|4,5,6,7)\big] \\
&\quad + \big[k_4^{(m}\big\{\mathcal{Z}^{n)}_{14,23,5,6,7} - \mathcal{Z}^{n)}_{214,3,5,6,7} + \mathcal{Z}^{n)}_{314,2,5,6,7} \qquad\qquad (\text{A.33}) \\
&\qquad + k_2^{n)}\mathcal{Z}_{4132,5,6,7} - k_3^{n)}\mathcal{Z}_{4123,5,6,7}\big\} + (4 \leftrightarrow 5,6,7)\big]\,, \\
E^{mnp}_{1|2,3,4,5,6,7} &= \mathcal{Z}^{mnp}_{1,2,3,4,5,6,7} + \big[k_2^{(m}\mathcal{Z}^{np)}_{12,3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\big] \\
&\quad - \big[k_2^{(m}k_3^{n}\mathcal{Z}^{p)}_{213,4,5,6,7} + (2,3|2,3,4,5,6,7)\big] \\
&\quad + \big[k_2^{(m}k_3^{n}k_4^{p)}(\mathcal{Z}_{1234,5,6,7} + \text{symm}(2,3,4)) + (2,3,4|2,3,4,5,6,7)\big]\ ,
\end{aligned}$$

become as compact as (6.21).

Moreover, there are three topologies of refined GEIs at seven points,

$$\begin{aligned}
E_{1|23|4,5,6,7} &= \mathcal{Z}_{23|1,4,5,6,7} + \mathcal{Z}_{3|12,4,5,6,7} - \mathcal{Z}_{2|13,4,5,6,7} + k_3^m\mathcal{Z}^m_{123,4,5,6,7} \\
&\quad - k_2^m\mathcal{Z}^m_{132,4,5,6,7} + \big[(s_{34}\mathcal{Z}_{1234,5,6,7} - s_{24}\mathcal{Z}_{1324,5,6,7}) + (4 \leftrightarrow 5,6,7)\big] \\
E_{1|2|34,5,6,7} &= \mathcal{Z}_{2|1,34,5,6,7} + \mathcal{Z}_{2|13,4,5,6,7} - \mathcal{Z}_{2|14,3,5,6,7} \\
&\quad - s_{23}(\mathcal{Z}_{1243,5,6,7} + \mathcal{Z}_{1423,5,6,7}) + s_{24}(\mathcal{Z}_{1234,5,6,7} + \mathcal{Z}_{1324,5,6,7}) \\
&\quad + k_2^m(\mathcal{Z}^m_{12,34,5,6,7} - \mathcal{Z}^m_{213,4,5,6,7} + \mathcal{Z}^m_{214,3,5,6,7}) \qquad\qquad (\text{A.34}) \\
&\quad + \big[s_{25}(\mathcal{Z}_{125,34,6,7} - \mathcal{Z}_{3125,4,6,7} + \mathcal{Z}_{4125,3,6,7}) + (5 \leftrightarrow 6,7)\big]\ , \\
E^m_{1|2|3,4,5,6,7} &= \mathcal{Z}^m_{2|1,3,4,5,6,7} + \big[k_3^m\big\{\mathcal{Z}_{2|13,4,5,6,7} - k_2^p\mathcal{Z}^p_{213,4,5,6,7}\big\} + (3 \leftrightarrow 4,5,6,7)\big] \\
&\quad + k_2^p\mathcal{Z}^{pm}_{12,3,4,5,6,7} + \big[s_{23}\big\{\mathcal{Z}^m_{123,4,5,6,7} - k_4^m\mathcal{Z}_{4123,5,6,7} - k_5^m\mathcal{Z}_{5123,4,6,7} \\
&\qquad - k_6^m\mathcal{Z}_{6123,4,5,7} - k_7^m\mathcal{Z}_{7123,4,5,6}\big\} + (3 \leftrightarrow 4,5,6,7)\big]\,,
\end{aligned}$$

and integration-by-parts identities lead to the compact representations (6.22) or (6.23).

The above GEIs, in turn, will be used as input in the monodromy-variation equations to bootstrap the eight-point shuffle-symmetric functions.

A.3. Eight points

In a similar fashion, it is possible to find all the solutions to the scalar shuffle-symmetric functions from the monodromy variations (4.13) using various change-of-basis identities such as (I.A.21). A long but straightforward analysis leads to,

$$\begin{aligned}
\mathcal{Z}_{12345,6,7,8} &= -\partial V_3(1,2,3,4,5)\,, \qquad\qquad (A.35)\\
\mathcal{Z}_{123,456,7,8} &= -\partial V_1(1,2,3)V_2(1,4,5,6) - \partial V_2(4,1,2,3)V_1(1,5,6)\\
&\quad + \partial V_2(6,1,2,3)V_1(1,4,5) - \partial V_3(4,5,1,2,3) + \partial V_3(4,6,1,2,3)\\
&\quad + \partial V_3(6,4,1,2,3) - \partial V_3(6,5,1,2,3)\,,\\
\mathcal{Z}_{1234,56,7,8} &= -\partial V_2(1,2,3,4)V_1(1,5,6) + \partial V_3(6,1,2,3,4) - \partial V_3(5,1,2,3,4)\,,\\
\mathcal{Z}_{123,45,67,8} &= -\partial V_1(1,2,3)V_1(1,4,5)V_1(1,6,7) + \partial V_2(5,1,2,3)V_1(1,6,7)\\
&\quad - \partial V_2(6,1,2,3)V_1(1,4,5) - \partial V_2(4,1,2,3)V_1(1,6,7) + \partial V_2(7,1,2,3)V_1(1,4,5)\\
&\quad + \partial V_3(4,7,1,2,3) - \partial V_3(4,6,1,2,3) + \partial V_3(5,6,1,2,3) - \partial V_3(5,7,1,2,3)\\
&\quad + \partial V_3(7,4,1,2,3) - \partial V_3(6,4,1,2,3) + \partial V_3(6,5,1,2,3) - \partial V_3(7,5,1,2,3)\,,\\
\mathcal{Z}_{12,34,56,78} &= g^{(1)}_{12} V_1(1,3,4)V_1(1,5,6)V_1(1,7,8)\\
&\quad - V_1(1,3,4)V_1(1,5,6)\partial V_1(7,1,2) + V_1(1,3,4)V_1(1,5,6)\partial V_1(8,1,2)\\
&\quad - V_1(1,3,4)V_1(1,7,8)\partial V_1(5,1,2) - V_1(1,5,6)V_1(1,7,8)\partial V_1(3,1,2)\\
&\quad + V_1(1,5,6)V_1(1,7,8)\partial V_1(4,1,2) + V_1(1,3,4)V_1(1,7,8)\partial V_1(6,1,2)\\
&\quad - V_1(1,3,4)\partial V_2(5,7,1,2) + V_1(1,3,4)\partial V_2(5,8,1,2) + V_1(1,3,4)\partial V_2(6,7,1,2)\\
&\quad - V_1(1,3,4)\partial V_2(6,8,1,2) - V_1(1,3,4)\partial V_2(7,5,1,2) + V_1(1,3,4)\partial V_2(7,6,1,2)\\
&\quad + V_1(1,3,4)\partial V_2(8,5,1,2) - V_1(1,3,4)\partial V_2(8,6,1,2) - V_1(1,5,6)\partial V_2(3,7,1,2)\\
&\quad + V_1(1,5,6)\partial V_2(3,8,1,2) + V_1(1,5,6)\partial V_2(4,7,1,2) - V_1(1,5,6)\partial V_2(4,8,1,2)\\
&\quad - V_1(1,5,6)\partial V_2(7,3,1,2) + V_1(1,5,6)\partial V_2(7,4,1,2) + V_1(1,5,6)\partial V_2(8,3,1,2)\\
&\quad - V_1(1,5,6)\partial V_2(8,4,1,2) - V_1(1,7,8)\partial V_2(3,5,1,2) + V_1(1,7,8)\partial V_2(3,6,1,2)\\
&\quad + V_1(1,7,8)\partial V_2(4,5,1,2) - V_1(1,7,8)\partial V_2(4,6,1,2) - V_1(1,7,8)\partial V_2(5,3,1,2)\\
&\quad + V_1(1,7,8)\partial V_2(5,4,1,2) + V_1(1,7,8)\partial V_2(6,3,1,2) - V_1(1,7,8)\partial V_2(6,4,1,2)\\
&\quad - \partial V_3(3,5,7,1,2) + \partial V_3(3,5,8,1,2) + \partial V_3(3,6,7,1,2) - \partial V_3(3,6,8,1,2)\\
&\quad - \partial V_3(3,7,5,1,2) + \partial V_3(3,7,6,1,2) + \partial V_3(3,8,5,1,2) - \partial V_3(3,8,6,1,2)\\
&\quad + \partial V_3(4,5,7,1,2) - \partial V_3(4,5,8,1,2) - \partial V_3(4,6,7,1,2) + \partial V_3(4,6,8,1,2)
\end{aligned}$$

$$
\begin{aligned}
&+\partial V_3(4,7,5,1,2)-\partial V_3(4,7,6,1,2)-\partial V_3(4,8,5,1,2)+\partial V_3(4,8,6,1,2)\\
&-\partial V_3(5,3,7,1,2)+\partial V_3(5,3,8,1,2)+\partial V_3(5,4,7,1,2)-\partial V_3(5,4,8,1,2)\\
&-\partial V_3(5,7,3,1,2)+\partial V_3(5,7,4,1,2)+\partial V_3(5,8,3,1,2)-\partial V_3(5,8,4,1,2)\\
&+\partial V_3(6,3,7,1,2)-\partial V_3(6,3,8,1,2)-\partial V_3(6,4,7,1,2)+\partial V_3(6,4,8,1,2)\\
&+\partial V_3(6,7,3,1,2)-\partial V_3(6,7,4,1,2)-\partial V_3(6,8,3,1,2)+\partial V_3(6,8,4,1,2)\\
&-\partial V_3(7,3,5,1,2)+\partial V_3(7,3,6,1,2)+\partial V_3(7,4,5,1,2)-\partial V_3(7,4,6,1,2)\\
&-\partial V_3(7,5,3,1,2)+\partial V_3(7,5,4,1,2)+\partial V_3(7,6,3,1,2)-\partial V_3(7,6,4,1,2)\\
&+\partial V_3(8,3,5,1,2)-\partial V_3(8,3,6,1,2)-\partial V_3(8,4,5,1,2)+\partial V_3(8,4,6,1,2)\\
&+\partial V_3(8,5,3,1,2)-\partial V_3(8,5,4,1,2)-\partial V_3(8,6,3,1,2)+\partial V_3(8,6,4,1,2)\,.
\end{aligned}
$$

The sheer size of the solution for $\mathcal{Z}_{12,34,56,78}$ can be traced back to the length of the intermediate identity (I.A.21) used in its derivation. Fortunately, the combinatorics of such solutions can be understood in terms of the multi-word rhomap (I.A.3), and an efficient algorithm to generate them at arbitrary multiplicity will be provided below.

A.3.1. Vectorial shuffle-symmetric functions

The monodromy variation of $\mathcal{Z}^m_{1234,5,6,7,8}$ can be written in the canonical basis of GEIs as

$$
\begin{aligned}
D\mathcal{Z}^m_{1234,5,6,7,8} &= \Omega_{14}E^m_{1|234,5,6,7,8} - \Big[k_5^m\Omega_{45}E_{1|2345,6,7,8} + (5\leftrightarrow 6,7,8)\Big]\\
&= \Omega_{14}V_2(1,2,3,4)\ell^m + k_2^m\Omega_{14}\partial V_2(2,3,4,1) \qquad\qquad (A.36)\\
&\quad - k_3^m\Omega_{14}\big(\partial V_2(3,2,4,1)+\partial V_2(3,4,2,1)\big) + k_4^m\Omega_{14}\partial V_2(4,3,2,1)\\
&\quad + \Big[k_5^m\big(\Omega_{14}V_2(1,2,3,4)g^{(1)}_{15} - \Omega_{45}V_3(1,2,3,4,5)\big) + (5\leftrightarrow 6,7,8)\Big]\,.
\end{aligned}
$$

Note that there is no linear combination of $\partial^2 V_2(i,j,k,l)$-functions that integrates to $\Omega_{14}\partial V_2(2,3,4,1)$ as can be checked using $D\partial^2V_2(i,j,k,l) = 2\Omega_{li}\partial V_2(i,j,k,l)$. However, the integration of $\Omega_{14}V_2(1,2,3,4)\ell^m$ produces correction terms since

$$
D(-\partial V_2(1,2,3,4)\ell^m) = \Omega_{14}V_2(1,2,3,4)\ell^m + \partial V_2(1,2,3,4)\sum_{j=2}^{8}\Omega_{1j}k_j^m. \qquad (A.37)
$$

By adding and subtracting the sum on the right-hand side prior to integration produces corrections to the other terms $\sim k_i^m$. For example, the new k_2^m terms can be "integrated" as

$$
\begin{aligned}
\int\Big(\Omega_{14}\partial V_2(2,3,4,1) - \Omega_{12}\partial V_2(1,2,3,4)\Big) &= \frac{1}{2}\Big(\partial^2V_2(1,4,3,2)+\partial^2V_2(2,1,3,4) \qquad (A.38)\\
&\quad + \partial^2V_2(2,3,1,4)+\partial^2V_2(1,2,3,4)\Big)\,.
\end{aligned}
$$

Repeating the same steps as in the previous analyses yields,

$$\begin{aligned}\mathcal{Z}^m_{1234,5,6,7,8} &= -\ell^m \partial V_2(1,2,3,4) + k_4^m \partial^2 V_2(1,2,3,4) \\ &+ \frac{1}{2} k_2^m \big[\partial^2 V_2(1,4,3,2) + \partial^2 V_2(2,1,3,4) + \partial^2 V_2(2,3,1,4) + \partial^2 V_2(1,2,3,4)\big] \\ &- \frac{1}{2} k_3^m \big[\partial^2 V_2(1,4,2,3) + \partial^2 V_2(3,4,2,1) + \partial^2 V_2(4,1,2,3) - \partial^2 V_2(1,2,3,4)\big] \\ &- \Big[k_5^m \big(\partial V_2(1,2,3,4) g^{(1)}_{15} + \partial V_3(5,1,2,3,4)\big) + (5 \leftrightarrow 6,7,8)\Big] ,\end{aligned} \tag{A.39}$$

which can be rewritten as,

$$\begin{aligned}\mathcal{Z}^m_{1234,5,6,7,8} &= -\ell^m \partial V_2(1,2,3,4) - \frac{1}{2} k_1^m \partial^2 V_2(1,2,3,4) + \frac{1}{2} k_4^m \partial^2 V_2(1,2,3,4) \\ &+ \frac{1}{2} k_2^m \big[\partial^2 V_2(1,4,3,2) + \partial^2 V_2(2,1,3,4) + \partial^2 V_2(2,3,1,4)\big] \\ &- \frac{1}{2} k_3^m \big[\partial^2 V_2(1,4,2,3) + \partial^2 V_2(3,4,2,1) + \partial^2 V_2(4,1,2,3)\big] \\ &+ \Big[k_5^m V_2(1,2,3,4)(g^{(2)}_{15} - g^{(2)}_{45}) + (5 \leftrightarrow 6,7,8)\Big]\end{aligned} \tag{A.40}$$

after using the weight-four version of (A.25),

$$\partial V_2(1,2,3,4) g^{(1)}_{15} + \partial V_3(5,1,2,3,4) = -V_2(1,2,3,4)(g^{(2)}_{15} - g^{(2)}_{45}) - \frac{1}{2} \partial^2 V_2(1,2,3,4) . \tag{A.41}$$

The expression for $\mathcal{Z}^m_{123,45,6,7,8}$ can be obtained similarly and a long analysis leads to

$$\begin{aligned}\mathcal{Z}^m_{123,45,6,7,8} &= -\ell^m \big(V_1(1,4,5)\partial V_1(1,2,3) + \partial V_2(4,1,2,3) - \partial V_2(5,1,2,3)\big) \\ &+ \frac{1}{2}\big[k_1^m \big(-V_1(3,4,5)\partial^2 V_1(1,2,3) + \partial^2 V_2(1,2,3,4) - \partial^2 V_2(1,2,3,5)\big) + (1 \leftrightarrow 3)\big] \\ &+ \frac{1}{2} k_2^m \big(V_1(3,4,5)\partial^2 V_1(2,1,3) - \partial^2 V_2(2,1,3,4) + \partial^2 V_2(2,1,3,5) \\ &\quad + V_1(1,4,5)\partial^2 V_1(2,3,1) - \partial^2 V_2(2,3,1,4) + \partial^2 V_2(2,3,1,5)\big) \\ &+ \big[k_4^m \big(g^{(1)}_{14} \partial V_2(5,1,2,3) - \partial V_1(1,2,3)\partial V_1(1,5,4) + \partial V_3(5,4,1,2,3) \\ &\quad - \frac{1}{2}\big(V_1(3,4,5)\partial^2 V_1(1,2,3) + \partial^2 V_2(1,2,3,5) + \partial^2 V_2(1,2,4,3) \\ &\quad + \partial^2 V_2(1,4,2,3) + \partial^2 V_2(3,2,1,4)\big)\big) - (4 \leftrightarrow 5)\big] \\ &+ \big[k_6^m \big(g^{(2)}_{46} V_2(1,2,3,4) - g^{(2)}_{16} V_1(1,3,2)V_1(1,4,5) \\ &\quad - g^{(2)}_{36} V_1(1,2,3)V_1(3,4,5) - g^{(2)}_{56} V_2(1,2,3,5)\big) + (6 \leftrightarrow 7,8)\big] .\end{aligned} \tag{A.42}$$

Solving the monodromy variation of $\mathcal{Z}^m_{12,34,56,7,8}$ along similar lines yields a long formula which we suppress (it can be downloaded in [33]). This completes the bootstrap procedure for the vectorial shuffle-symmetric eight-point functions. From the above solutions we can derive the eight-point vectorial GEIs which in turn allow to bootstrap the vectorial $\mathcal{Z}$-functions at nine points.

A.3.2. Tensorial and refined functions

Given their sizes, the tensorial and refined shuffle-symmetric functions will be omitted. Their explicit expansions are available to download as plain text files in [33], and are ready to be used with `FORM` [32].

A.4. A closed formula for scalar shuffle-symmetric functions

The solutions for the scalar shuffle-symmetric functions can be generated via a conjectural closed formula. This empirical observation is based on the word-invariant map $\mathcal{I}(\ldots)$ defined in the appendix I.A and is given by

$$\mathcal{Z}_{A,B,C,D} = -A \odot \mathcal{I}(\emptyset|B,C,D)\,, \tag{A.43}$$

where the $\odot$ operation is defined by

$$aA \odot (B,C,D,E) \equiv (BaA|aC,aD,aE)_V\,, \tag{A.44}$$

$$(A|B,C,D)_V \equiv \partial V_{|A|-2}(A)V_{|B|-2}(B)V_{|C|-2}(C)V_{|D|-2}(D)\,, \tag{A.45}$$

with the understanding that $V_0(i,j) \equiv 1$ and $\partial V_{-1}(i) \equiv -1$.

For example, let us consider $\mathcal{Z}_{123,45,67,8} = -123 \odot \mathcal{I}(\emptyset|45,67,8)$. A straightforward application of the recursions in appendix I.A leads to,

$$\begin{aligned}-\mathcal{I}(\emptyset|45,67,8) = &-(\emptyset|67,45,8)-(6|45,7,8)-(64|5,7,8)+(65|4,7,8)\\ &+(7|45,6,8)+(74|5,6,8)-(75|4,6,8)-(4|67,5,8)\\ &-(46|7,5,8)+(47|6,5,8)+(5|67,4,8)+(56|7,4,8)-(57|6,4,8)\,.\end{aligned} \tag{A.46}$$

Now, using the definition (A.44) yields

$$\begin{aligned}-123\odot\mathcal{I}(\emptyset|45,67,8) = &-(123|167,145,18)_V-(6123|145,17,18)_V-(64123|15,17,18)_V\\ &+(65123|14,17,18)_V+(7123|145,16,18)_V+(74123|15,16,18)_V\\ &-(75123|14,16,18)_V-(4123|167,15,18)_V-(46123|17,15,18)_V\\ &+(47123|16,15,18)_V+(5123|167,14,18)_V+(56123|17,14,18)_V\\ &-(57123|16,14,18)_V\,.\end{aligned} \tag{A.47}$$

Finally, the definition (A.45) leads to the correct expression for $\mathcal{Z}_{123,45,67,8}$ from (A.35).

Two comments are in order. First, one may notice that the combinatorics of the permutations in (A.47) is closely related to the change-of-basis identity expressing $-C_{3|12,45,67,8}$ in a basis of $C_{1|A,B,C}$ originally derived in [23]. For instance,

$$\begin{aligned}-C_{3|12,45,67} = &-C_{1|23,45,67} - C_{1|236,7,45} - C_{1|2364,5,7} \\ &+ C_{1|2365,4,7} + C_{1|237,6,45} + C_{1|2374,5,6} \\ &- C_{1|2375,4,6} - C_{1|234,5,67} - C_{1|2346,7,5} \\ &+ C_{1|2347,6,5} + C_{1|235,4,67} + C_{1|2356,7,4} \\ &- C_{1|2357,6,4} \,.\end{aligned} \tag{A.48}$$

Comparing the expressions (A.47) and (A.48) we see that $C_{1|23A,B,C} \to (A123, 1B, 1C, 18)$ maps one expression into the other.

Second, the right-hand side of the algorithm $\mathcal{Z}_{A,B,C,D} = -A \odot \mathcal{I}(\emptyset|B,C,D)$ is not manifestly symmetric under exchange of the words $A \leftrightarrow B, C, D$. Therefore, it must generate identities among the various functions $V_n(\ldots)$ and their generalizations $\partial V_n(\ldots)$. Consider, for example, the five-point function $\mathcal{Z}_{1,23,4,5}$ and evaluate it in the two inequivalent orderings using the algorithm (A.43); we see that $-23 \odot \mathcal{I}(\emptyset|1,4,5)$ and $-1 \odot \mathcal{I}(\emptyset|23,4,5)$ give rise to

$$\mathcal{Z}_{23,1,4,5} = -\partial V_0(2,3)\,, \qquad \mathcal{Z}_{1,23,4,5} = V_1(1,2,3) - \partial V_0(1,2) + \partial V_0(1,3)\,, \tag{A.49}$$

which are, of course, equal. A bit less obvious is the equality of both $\mathcal{Z}_{1,234,5,6}$ and $\mathcal{Z}_{234,1,5,6}$ under the algorithm (A.43), as this implies

$$\begin{aligned}V_2(1,2,3,4) = &V_1(1,2,3)\partial V_0(1,4) - V_1(1,3,4)\partial V_0(1,2) - \partial V_1(1,2,4) + \partial V_1(1,3,2) \\ &+ \partial V_1(1,3,4) - \partial V_1(1,4,2) - \partial V_1(2,3,4)\,.\end{aligned}$$

These observations imply that it is advantageous to write the expansion of the scalar GEIs in a certain "canonical" order such as: $E_{1|234,5,6} = \mathcal{Z}_{1,234,5,6} + \mathcal{Z}_{12,34,5,6} + \mathcal{Z}_{123,4,5,6} + \mathcal{Z}_{412,3,5,6} - \mathcal{Z}_{14,23,5,6} + \mathcal{Z}_{143,2,5,6}$ since it takes the shortest form $E_{1|234,5,6} = V_2(1,2,3,4)$.

Appendix B. The Jacobi theta expansion of $g^{(n)}(z,\tau)$

In this appendix we will list, for convenience, the explicit expansions of the coefficients of the Kronecker–Eisenstein series in terms of Jacobi theta functions for the first few cases.

Recall that the functions $g^{(n)}(z,\tau)$ admit the following recursive expansion [14]

$$g^{(n)}(z,\tau)=\frac{1}{n}\sum_{j=1}^{n}\mathcal{E}_j(z,\tau)g^{(n-j)}(z,\tau)\,,\quad g^{(1)}(z,\tau)\equiv E_1(z,\tau)\,,\quad g^{(0)}(z,\tau)\equiv 1\,,\qquad \text{(B.1)}$$

where $\mathcal{E}_j(z,\tau)\equiv(-1)^j\big(\mathrm{G}_j(\tau)-E_j(z,\tau)\big)$ and $E_{n+1}(z,\tau)=(-1)^n\frac{1}{n!}\partial^{n+1}\log\theta_1(z,\tau)$ with $\mathrm{G}_j(\tau)$ denoting the Eisenstein series (2.7). It is a matter of tedious algebra to get:

$$\begin{aligned}
g^{(1)}(z,\tau)&=\frac{\theta_1^{(1)}(z,\tau)}{\theta_1(z,\tau)} &\text{(B.2)}\\
g^{(2)}(z,\tau)&=\frac{1}{2!}\frac{\theta_1^{(2)}(z,\tau)}{\theta_1(z,\tau)}-\frac{1}{3!}\frac{\theta_1^{(3)}(0,\tau)}{\theta_1^{(1)}(0,\tau)}\\
g^{(3)}(z,\tau)&=\frac{1}{3!}\frac{\theta_1^{(3)}(z,\tau)}{\theta_1(z,\tau)}-\frac{1}{3!}\frac{\theta_1^{(3)}(0,\tau)}{\theta_1^{(1)}(0,\tau)}g^{(1)}(z,\tau)\\
g^{(4)}(z,\tau)&=\frac{1}{4!}\frac{\theta_1^{(4)}(z,\tau)}{\theta_1(z,\tau)}-\frac{1}{3!}\frac{\theta_1^{(3)}(0,\tau)}{\theta_1^{(1)}(0,\tau)}g^{(2)}(z,\tau)-\frac{1}{5!}\frac{\theta_1^{(5)}(0,\tau)}{\theta_1^{(1)}(0,\tau)}\\
g^{(5)}(z,\tau)&=\frac{1}{5!}\frac{\theta_1^{(5)}(z,\tau)}{\theta_1(z,\tau)}-\frac{1}{3!}\frac{\theta_1^{(3)}(0,\tau)}{\theta_1^{(1)}(0,\tau)}g^{(3)}(z,\tau)-\frac{1}{5!}\frac{\theta_1^{(5)}(0,\tau)}{\theta_1^{(1)}(0,\tau)}g^{(1)}(z,\tau)\,.
\end{aligned}$$

The surprisingly simple pattern above arises from non-trivial cancellations such as

$$\mathrm{G}_4-\frac{1}{2}\mathrm{G}_2^2=-\frac{1}{30}\frac{\theta_1^{(5)}(0,\tau)}{\theta_1^{(1)}(0,\tau)}\,,\qquad 8\mathrm{G}_6-6\mathrm{G}_2\mathrm{G}_4+\mathrm{G}_2^3=-\frac{1}{105}\frac{\theta_1^{(7)}(0,\tau)}{\theta_1^{(1)}(0,\tau)}\,,\qquad \text{(B.3)}$$

where the expansion of the Eisenstein series in terms of Jacobi theta functions reads [34]

$$\begin{aligned}
\mathrm{G}_2(\tau)&=-\frac{1}{3}\frac{\theta_1^{(3)}(0,\tau)}{\theta_1^{(1)}(0,\tau)}\,,\qquad \mathrm{G}_4(\tau)=-\frac{1}{30}\frac{\theta_1^{(5)}(0,\tau)}{\theta_1^{(1)}(0,\tau)}+\frac{1}{18}\left(\frac{\theta_1^{(3)}(0,\tau)}{\theta_1^{(1)}(0,\tau)}\right)^2 &\text{(B.4)}\\
\mathrm{G}_6(\tau)&=-\frac{1}{840}\frac{\theta_1^{(7)}(0,\tau)}{\theta_1^{(1)}(0,\tau)}+\frac{1}{120}\frac{\theta_1^{(5)}(0,\tau)}{\theta_1^{(1)}(0,\tau)}\frac{\theta_1^{(3)}(0,\tau)}{\theta_1^{(1)}(0,\tau)}-\frac{1}{108}\left(\frac{\theta_1^{(3)}(0,\tau)}{\theta_1^{(1)}(0,\tau)}\right)^3.
\end{aligned}$$

B.1. Laurent series expansion of the $g^{(n)}(z,\tau)$-functions

The Laurent expansion of $g^{(n)}(z,\tau)$ follows from (B.1) and [35] (note $\binom{p}{0}\equiv 1$)

$$E_n(z,\tau)=\frac{1}{z^n}+(-1)^n\sum_{m=1}^{\infty}\binom{2m-1}{n-1}\mathrm{G}_{2m}(\tau)z^{2m-1}\,.\qquad \text{(B.5)}$$

More explicitly,

$$
\begin{aligned}
g^{(1)}(z,\tau) &= \frac{1}{z} - \mathrm{G}_2 z - \mathrm{G}_4 z^3 - \mathrm{G}_6 z^5 + \mathcal{O}(z^7) \\
g^{(2)}(z,\tau) &= -\mathrm{G}_2 + \frac{1}{2}\big(\mathrm{G}_2^2 - 5\mathrm{G}_4\big) z^2 + \big(\mathrm{G}_2\mathrm{G}_4 - \frac{7}{2}\mathrm{G}_6\big) z^4 + \mathcal{O}(z^6) \\
g^{(3)}(z,\tau) &= \frac{1}{2}\big(\mathrm{G}_2^2 - 5\mathrm{G}_4\big) z + \frac{1}{2}\big(5\mathrm{G}_2\mathrm{G}_4 - \frac{35}{3}\mathrm{G}_6 - \frac{1}{3}\mathrm{G}_2^3\big) z^3 + \mathcal{O}(z^5) \\
g^{(4)}(z,\tau) &= -\mathrm{G}_4 + \frac{1}{2}\big(5\mathrm{G}_2\mathrm{G}_4 - \frac{35}{3}\mathrm{G}_6 - \frac{1}{3}\mathrm{G}_2^3\big) z^2 + \mathcal{O}(z^4)\,.
\end{aligned}
\tag{B.6}
$$

Appendix C. String correlators and GEIs

For convenience, in this appendix we quote from part III of this series of papers a representation of the one-loop correlators utilizing GEIs for $n = 4,5,6,7$, as they are frequently referred to in this work.

$$
\mathcal{K}_4(\ell) = C_{1|2,3,4} E_{1|2,3,4} \tag{C.1}
$$

$$
\mathcal{K}_5(\ell) = C^m_{1|2,3,4,5} E^m_{1|2,3,4,5} + \big[C_{1|23,4,5} s_{23} E_{1|23,4,5} + (2,3|2,3,4,5)\big]\,, \tag{C.2}
$$

$$
\begin{aligned}
\mathcal{K}_6(\ell) &= \frac{1}{2} C^{mn}_{1|2,3,4,5,6} E^{mn}_{1|2,3,4,5,6} - \big[P_{1|2|3,4,5,6} E_{1|2|3,4,5,6} + (2 \leftrightarrow 3,\ldots,6)\big] \\
&\quad + \big[s_{23} C^m_{1|23,4,5,6} E^m_{1|23,4,5,6} + (2,3|2,3,\ldots,6)\big] \\
&\quad + \Big(\big[s_{23}s_{45} C_{1|23,45,6} E_{1|23,45,6} + \mathrm{cyc}(3,4,5)\big] + (6 \leftrightarrow 5,4,3,2)\Big) \\
&\quad + \Big(\big[s_{23}s_{34} C_{1|234,5,6} E_{1|234,5,6} + \mathrm{cyc}(2,3,4)\big] + (2,3,4|2,3,\ldots,6)\Big)
\end{aligned}
\tag{C.3}
$$

$$
\begin{aligned}
\mathcal{K}_7(\ell) &= \frac{1}{6} C^{mnp}_{1|2,3,4,5,6,7} E^{(s)mnp}_{1|2,3,4,5,6,7} \\
&\quad + \frac{1}{2} C^{mn}_{1|23,4,5,6,7} E^{(s)mn}_{1|23,4,5,6,7} + (2,3|2,3,4,5,6,7) \\
&\quad + \big[C^m_{1|234,5,6,7} E^{(s)m}_{1|234,5,6,7} + C^m_{1|243,5,6,7} E^{(s)m}_{1|243,5,6,7}\big] + (2,3,4|2,3,4,5,6,7) \\
&\quad + \big[C^m_{1|23,45,6,7} E^{(s)m}_{1|23,45,6,7} + \mathrm{cyc}(2,3,4)\big] + (6,7|2,3,4,5,6,7) \\
&\quad + \big[C_{1|2345,6,7} E^{(s)}_{1|2345,6,7} + \mathrm{perm}(3,4,5)\big] + (2,3,4,5|2,3,4,5,6,7) \\
&\quad + \big[C_{1|234,56,7} E^{(s)}_{1|234,56,7} + C_{1|243,56,7} E^{(s)}_{1|243,56,7} + \mathrm{cyc}(5,6,7)\big] + (2,3,4|2,3,4,5,6,7) \\
&\quad + \big[C_{1|23,45,67} E^{(s)}_{1|23,45,67} + \mathrm{cyc}(4,5,6)\big] + (3 \leftrightarrow 4,5,6,7) \\
&\quad - P^m_{1|2|3,\ldots,7} E^{(s)m}_{1|2|3,\ldots,7} + (2 \leftrightarrow 3,4,5,6,7) \\
&\quad - P_{1|23|4,\ldots,7} E^{(s)}_{1|23|4,\ldots,7} + (2,3|2,3,4,5,6,7) \\
&\quad - \big[P_{1|2|34,5,6,7} E^{(s)}_{1|2|34,5,6,7} + \mathrm{cyc}(2,3,4)\big] + (2,3,4|2,3,4,5,6,7)\,.
\end{aligned}
\tag{C.4}
$$

By the symmetric role of GEIs and BRST (pseudo-)invariants, these representations manifest the double-copy structure of one-loop open-superstring amplitudes [4]. Other properties of one-loop correlators including locality are manifest in various alternative representations given in part III.

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