Towards the n-point one-loop superstring amplitude III: One-loop correlators and their double-copy structure

Carlos R. Mafra[†] and Oliver Schlotterer[‡,*]

[†] *Mathematical Sciences and STAG Research Centre, University of Southampton, Highfield, Southampton, SO17 1BJ, UK*

[‡] *Max–Planck–Institut für Gravitationsphysik Albert–Einstein–Institut, 14476 Potsdam, Germany*

[*] *Perimeter Institute for Theoretical Physics, Waterloo, ON N2L 2Y5, Canada*

In this final part of a series of three papers, we will assemble supersymmetric expressions for one-loop correlators in pure-spinor superspace that are BRST invariant, local, and single valued. A key driving force in this construction is the generalization of a so far unnoticed property at tree-level; the correlators have the symmetry structure akin to *Lie polynomials*. One-loop correlators up to seven points are presented in a variety of representations manifesting different subsets of their defining properties. These expressions are related via identities obeyed by the kinematic superfields and worldsheet functions spelled out in the first two parts of this series and reflecting a duality between the two kinds of ingredients.

Interestingly, the expression for the eight-point correlator following from our method seems to capture correctly all the dependence on the worldsheet punctures but leaves undetermined the coefficient of the holomorphic Eisenstein series G_4. By virtue of chiral splitting, closed-string correlators follow from the double copy of the open-string results.

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[†] email: c.r.mafra@soton.ac.uk

[‡] email: olivers@aei.mpg.de

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1. Introduction

This is the third part of a series of papers [1] towards the derivation of one-loop correlators of massless open- and closed-superstring states using techniques from the pure-spinor formalism [2,3]. We often refer to section and equation numbers from part I & part II and then prefix these numbers by the corresponding roman numerals I and II. The main result of this paper is the assembly of local one-loop correlators in pure-spinor superspace [4] up to eight points. This will be done by combining two main ingredients:

1. local kinematic building blocks introduced in part I that capture the essentials of the pure-spinor zero-mode saturation rules and transform covariantly under the BRST charge
2. worldsheet functions introduced in part II capturing the singularities generated by OPE contractions among vertex operators. In particular, their monodromies as the vertex positions are moved around the genus-one cycles also follow a notion of "covariance". More precisely, the monodromies are described by a system of equations that share the same properties of the so-called BRST invariants and naturally lead to a *duality* between kinematics and worldsheet functions.

The fundamental guiding principle that will act as the recipe to combine these two ingredients will correspond to the one-loop generalization of a symmetry property obeyed by the analogous tree-level correlators derived in [5] and reviewed in section 2.1 below. More precisely, the tree-level correlators are composed from products of Lie-symmetric kinematic building blocks $V_{123\ldots p}$ and shuffle-symmetric worldsheet functions $\mathcal{Z}_{123\ldots p} = (z_{12}z_{23}\ldots z_{p-1,p})^{-1}$. Given the similar structure between these symmetries and the composing elements in a theorem of Ree concerning Lie polynomials [6], we dubbed the correlators obtained in this way as having a *Lie-polynomial form*. We will see that this line of reasoning leads to a key assumption of this paper, that the local n-point one-loop correlators of the open superstring can be written as

$$\mathcal{K}_n(\ell) = \sum_{r=0}^{n-4} \frac{1}{r!}\Big(V_{A_1} T^{m_1\ldots m_r}_{A_2,\ldots,A_{r+4}} \mathcal{Z}^{m_1\ldots m_r}_{A_1,\ldots,A_{r+4}} + \big[12\ldots n|A_1,\ldots,A_{r+4}\big]\Big) + \text{ corrections}\,. \tag{1.1}$$

Definitions of the kinematic building blocks $T^{m_1\ldots m_r}_{A_2,\ldots,A_{r+4}}$ and the worldsheet functions $\mathcal{Z}^{m_1\ldots m_r}_{A_1,\ldots,A_{r+4}}$ can be found in part I and part II, respectively[1]. The notation for the permutations in terms of partitions of words addresses the kind of permutations resulting from

[1] The worldsheet functions can also be downloaded from [7] as text files in `FORM` [8] format.

the interplay between shuffles and Lie symmetries, and is explained in the appendix A. As discussed in part II, a beneficial side effect of requiring shuffle symmetry for the worldsheet functions is that the resulting functions automatically contain non-singular pieces that are invisible to an OPE analysis. Lie symmetries in turn refer to the generalized Jacobi identities satisfied by the kinematic building blocks, in lines with the Bern–Carrasco–Johansson duality between color and kinematics [9].

The notions of locality, BRST invariance and single-valuedness will then lead to a discussion for why the "+ corrections" are needed starting at $n \geq 7$ points. In section 3, a multitude of representations for the correlators with $n = 4, 5, 6, 7$ (including the "+ corrections" at $n = 7$) will be given that expose different subsets of their properties. While the $n = 8$ correlator following from the proposal (1.1) satisfies many non-trivial constraints, it fails to be BRST invariant by terms proportional to the holomorphic Eisenstein series G_4. In the future, we expect to address this challenging leftover problem in order to extend our results to arbitrary numbers of points.

Section 4 is dedicated to manifesting the modular properties of the open- and closed-string correlators by integrating out the loop momentum. We will relate a double-copy structure of the open-string correlators [10] to the low-energy limit of the closed-string amplitudes. This incarnation of the duality between kinematics and worldsheet functions is checked in detail up to multiplicity seven, and we describe the problems and perspectives in the quest for an n-point generalization at the end of section 4.

2. One-loop correlators of the open superstring: general structure

In this section, we set the stage for assembling one-loop correlators $\mathcal{K}_n(\ell)$ from the system of kinematic building blocks and worldsheet functions introduced in part I and II. By their definition in section I.2.2, correlators $\mathcal{K}_n(\ell)$ carry the kinematic dependence of one-loop open-string amplitudes among n massless states

$$\mathcal{A}_n = \sum_{\mathrm{top}} C_{\mathrm{top}} \int_{D_{\mathrm{top}}} d\tau \, dz_2 \, dz_3 \, \ldots \, dz_n \int d^D \ell \; |\mathcal{I}_n(\ell)| \, \langle \mathcal{K}_n(\ell) \rangle \, , \tag{2.1}$$

with $\langle \ldots \rangle$ denoting the zero-mode integration prescription of the pure-spinor formalism [2]. The integration domains D_{top} for the modular parameter τ and vertex positions z_j are tailored to the topologies of a cylinder or a Möbius strip with associated color factors

C_{top}, see [11] for details. The integration over loop momenta ℓ is an integral part of the chiral-splitting method [12,13,14], which allows to derive massless closed-string one-loop amplitudes from an integrand of double-copy form

$$\mathcal{M}_n = \int_{\mathcal{F}} d^2\tau \, d^2 z_2 \, d^2 z_3 \, \ldots \, d^2 z_n \int d^D \ell \; |\mathcal{I}_n(\ell)|^2 \, \langle \mathcal{K}_n(\ell) \rangle \, \langle \tilde{\mathcal{K}}_n(-\ell) \rangle \,, \tag{2.2}$$

with $\mathcal{F}$ denoting the fundamental domain for inequivalent tori w.r.t. the modular group. Both of (2.1) and (2.2) involve the universal one-loop Koba–Nielsen factor

$$\mathcal{I}_n(\ell) \equiv \exp\Big(\sum_{i<j}^{n} s_{ij} \log \theta_1(z_{ij}, \tau) + \sum_{j=1}^{n} z_j (\ell \cdot k_j) + \frac{\tau}{4\pi i} \ell^2 \Big) \,, \tag{2.3}$$

with lightlike external momenta k_j, where we use the shorthands

$$s_{ij} \equiv k_i \cdot k_j \,, \qquad z_{ij} \equiv z_i - z_j \tag{2.4}$$

and conventions where $2\alpha' = 1$ for open strings and $\frac{\alpha'}{2} = 1$ for closed ones.

In trying to calculate multiparticle one-loop amplitudes using the pure-spinor prescription (I.2.4), one soon realizes that most efforts tend to be hampered by the complicated nature of the b-ghost (I.2.5). This difficulty, however, motivates a less direct approach which illuminates the structure of the answer in a somewhat unexpected way; the organizing principle will be drawn from the tree-level correlators of [5].

2.1. Lessons from tree-level correlators

Recall that n-point open-string tree amplitudes in the pure-spinor formulation require the evaluation of the following n-point correlation function [2],

$$\langle\!\langle V_1(z_1) \prod_{j=2}^{n-2} U_j(z_j) V_{n-1}(z_{n-1}) V_n(\infty) \rangle\!\rangle_{\text{tree}} \equiv \langle \mathcal{K}_n^{\text{tree}} \rangle \prod_{i<j}^{n} |z_{ij}|^{s_{ij}} \,, \tag{2.5}$$

see (I.2.6) and (I.2.7) for the vertex operators V_i and U_j. The definition of the tree-level *correlators* $\mathcal{K}_n^{\text{tree}}$ on the right-hand side is analogous to that of one-loop correlators $\mathcal{K}_n(\ell)$, cf. (I.2.28). The idea is to strip off the universal factors of $|z_{ij}|^{s_{ij}}$ from the path integral, i.e. the tree-level analogue of the one-loop Koba–Nielsen factor (2.3). The computation of the correlators $\mathcal{K}_n^{\text{tree}}$ boils down to using the CFT rules of the pure-spinor formalism to perform OPE contractions among the vertex operators in (2.5).

One of the crucial steps in the calculation of [5] was showing that the multiparticle vertex operators V_P [15] could be used as the fundamental building blocks of the correlator; for example, in terms of the function $\mathcal{Z}_P^{\rm tree}$ defined by

$$\mathcal{Z}^{\rm tree}_{123\ldots p} \equiv \frac{1}{z_{12}z_{23}\ldots z_{p-1,p}}\,, \tag{2.6}$$

we have

$$\begin{aligned} V_1(z_1)U_2(z_2) &\cong V_{12}\mathcal{Z}^{\rm tree}_{12} \\ V_1(z_1)U_2(z_2)U_3(z_3) &\cong V_{123}\mathcal{Z}^{\rm tree}_{123} + V_{132}\mathcal{Z}^{\rm tree}_{132}\,, \end{aligned} \tag{2.7}$$

where the symbol $\cong$ is a reminder that the above relations are valid up to total derivatives and BRST-exact quantities. As reviewed in section II.4.1, the accompanying functions exhibit shuffle symmetry such as $\mathcal{Z}^{\rm tree}_{12}+\mathcal{Z}^{\rm tree}_{21}=0$, $\mathcal{Z}^{\rm tree}_{123}-\mathcal{Z}^{\rm tree}_{321}=0$ and $\mathcal{Z}^{\rm tree}_{123}+\mathcal{Z}^{\rm tree}_{213}+\mathcal{Z}^{\rm tree}_{231}=0$ dual to the Lie symmetries $V_{12}=-V_{21}$, $V_{123}=-V_{213}$ and $V_{123}+V_{231}+V_{312}=0$, cf. (I.3.25). When combined with (2.7), these symmetries lead to the following generalization:

$$V_1(z_1)\prod_{i=1}^n U_{a_i}(z_{a_i}) \cong \sum_{|A|=n} V_{1A}\mathcal{Z}^{\rm tree}_{1A}\,, \qquad \mathcal{Z}^{\rm tree}_{A\sqcup B}=0\,, \quad \forall A,B\neq\emptyset\,, \tag{2.8}$$

which eventually gives rise to the solution found in [5]. As detailed in section I.3.1, the summation range $|A|=n$ in (2.8) refers to the $n!$ words A formed by permutations of $a_1a_2\ldots a_{|A|}$ with $|A|=n$.

At this point one may realize that the right-hand side of (2.8) has the structure of a *Lie polynomial* [16,6] and that the expressions for the n-point correlators at tree level obtained in [5] can be written in terms of their products. More precisely, $\mathcal{K}_n^{\rm tree}$ is given by two copies of (2.8) with $n{-}2$ deconcatenations $AB=23\ldots n{-}2$ and an overall permutation over $(n{-}3)!$ letters for a total of $(n{-}2)!$ terms:

$$\mathcal{K}_n^{\rm tree} = \sum_{AB=23\ldots n-2}\big(V_{1A}\mathcal{Z}^{\rm tree}_{1A}\big)\big(V_{n-1,B}\mathcal{Z}^{\rm tree}_{n-1,B}\big)V_n + {\rm perm}(23\ldots n{-}2)\,. \tag{2.9}$$

For example ($\mathcal{Z}_i^{\rm tree}\equiv 1$),

$$\begin{aligned} \mathcal{K}_3^{\rm tree} &= V_1V_2V_3\,, \qquad \mathcal{K}_4^{\rm tree} = V_{12}\mathcal{Z}^{\rm tree}_{12}V_3V_4 + V_1V_{32}\mathcal{Z}^{\rm tree}_{32}V_4\,, \\ \mathcal{K}_5^{\rm tree} &= \big(V_{123}\mathcal{Z}^{\rm tree}_{123}+V_{132}\mathcal{Z}^{\rm tree}_{132}\big)V_4V_5 + V_1\big(V_{423}\mathcal{Z}^{\rm tree}_{423}+V_{432}\mathcal{Z}^{\rm tree}_{432}\big)V_5 \\ &\quad + \big(V_{12}\mathcal{Z}^{\rm tree}_{12}\big)\big(V_{43}\mathcal{Z}^{\rm tree}_{43}\big)V_5 + \big(V_{13}\mathcal{Z}^{\rm tree}_{13}\big)\big(V_{42}\mathcal{Z}^{\rm tree}_{42}\big)V_5\,. \end{aligned} \tag{2.10}$$

Note that $V_{123}\mathcal{Z}^{\rm tree}_{123} + V_{132}\mathcal{Z}^{\rm tree}_{132}$ in (2.10) is symmetric in $1,2,3$ even though only two out of 3! permutations are spelled out. This is a consequence of the Lie-polynomial structure of the correlator[2]; the right-hand side of (2.8) is permutation symmetric in $1, a_1, a_2, \ldots, a_{|A|}$ even though only the weaker symmetry in $a_1, a_2, \ldots, a_{|A|}$ is manifest.

The Lie-polynomial structure of the building blocks in the tree-level correlator (2.9) motivates us to search for a similar organization of the one-loop correlators.

2.2. Assembling one-loop correlators

Let us summarize what we have seen in part I and II in order to better understand the motivation behind the general form of the one-loop correlators $\mathcal{K}_n(\ell)$ to be proposed shortly.

- In section I.3, we reviewed the definition of local superfields that satisfy generalized Jacobi identities and, in section I.4, we showed how they can be assembled in several classes of local building blocks.
- In section II.4, we constructed functions composed of the expansion coefficients of the Kronecker–Eisenstein series that obey shuffle symmetries when the vertex insertion points are permuted.

Let us thread the above points together in view of the tree-level structure discussed above. Firstly, since the short-distance singularities within the correlator are independent on the global properties of the Riemann surface, the shuffle symmetries of the worldsheet functions should also be a property of the worldsheet functions at one loop. And secondly, the shuffle symmetry obeyed by the functions are the driving force in the Lie-polynomial organization of the tree-level correlators with local kinematic building blocks. When taken together these points suggest that the *superfields and worldsheet functions of one-loop correlators have the same symmetry structure of Lie polynomials.* This realization will lead to a beautiful organization of superstring one-loop correlators.

2.3. The Lie-polynomial structure of one-loop correlators

The additional zero modes at genus one, in particular the availability of loop momenta, allow for a significantly richer system of kinematic building blocks as compared to the tree-level kinematics $V_{1A}V_{n-1,B}V_n$ in (2.9). Also their accompanying worldsheet functions must accommodate the different OPE singularities and powers of loop momentum characteristic

[2] This follows from the identity $\sum_A \frac{1}{|A|}\mathcal{Z}_A V_A = \sum_B \mathcal{Z}_{iB}V_{iB}$ [6].

to each zero-mode saturation pattern, see e.g. (I.3.23) and (I.3.24). The corresponding Lie polynomials will therefore differ with respect to these features but will preserve their mathematical characterization as sums over products of shuffle- and Lie-symmetric objects.

To exemplify and to give a preview of what is ahead, the four-, five-, and six-point correlators at one loop will be written as products of local kinematic building blocks $T^{m_1\ldots}_{A,B,\ldots}$ (cf. sections I.4.1 to I.4.3) and worldsheet functions (cf. section II.4.4) as follows,

$$
\begin{aligned}
\mathcal{K}_4(\ell) &= V_1 T_{2,3,4} \mathcal{Z}_{1,2,3,4}\,, \\
\mathcal{K}_5(\ell) &= V_1 T^m_{2,3,4,5} \mathcal{Z}^m_{1,2,3,4,5} + \big[V_{12} T_{3,4,5} \mathcal{Z}_{12,3,4,5} + (2 \leftrightarrow 3,4,5)\big] \\
&\quad + \big[V_1 T_{23,4,5} \mathcal{Z}_{1,23,4,5} + (2,3|2,3,4,5)\big]\,, \\
\mathcal{K}_6(\ell) &= \frac{1}{2} V_1 T^{mn}_{2,3,4,5,6} \mathcal{Z}^{mn}_{1,2,3,4,5,6} \\
&\quad + \big[V_{12} T^m_{3,4,5,6} \mathcal{Z}^m_{12,3,4,5,6} + (2 \leftrightarrow 3,4,5,6)\big] \\
&\quad + \big[V_1 T^m_{23,4,5,6} \mathcal{Z}^m_{1,23,4,5,6} + (2,3|2,3,4,5,6)\big] \\
&\quad + \big[V_{123} T_{4,5,6} \mathcal{Z}_{123,4,5,6} + V_{132} T_{4,5,6} \mathcal{Z}_{132,4,5,6} + (2,3|2,3,4,5,6)\big] \\
&\quad + \big[(V_{12} T_{34,5,6} \mathcal{Z}_{12,34,5,6} + \mathrm{cyc}(2,3,4)) + (2,3,4|2,3,4,5,6)\big] \\
&\quad + \big[(V_1 T_{2,34,56} \mathcal{Z}_{1,2,34,56} + \mathrm{cyc}(3,4,5)) + (2 \leftrightarrow 3,4,5,6)\big] \\
&\quad + \big[V_1 T_{234,5,6} \mathcal{Z}_{1,234,5,6} + V_1 T_{243,5,6} \mathcal{Z}_{1,243,5,6} + (2,3,4|2,3,4,5,6)\big]\,,
\end{aligned}
\tag{2.11}
$$

where $m,n,p,\ldots = 0,1,\ldots,9$ denote Lorentz-vector indices. As in part I and II, the separation of words $A,B,\ldots$ through a comma in a subscript indicates that the parental object is symmetric under $A \leftrightarrow B$, e.g. $T_{A,B,C} = T_{B,A,C} = T_{A,C,B}$. The generalized Jacobi symmetries of V_P then apply to all of $A,B,\ldots$, e.g. $T_{234,5,6} + \mathrm{cyc}(2,3,4) = 0$. Moreover, $+(a_1,\ldots,a_p|a_1,\ldots,a_{p+q})$ refers to summing over all the $\binom{p+q}{p}$ subsets of $a_1,\ldots,a_{p+q}$ involving p elements a_i in the place of $a_1,\ldots,a_p$.

The Lie-polynomial form of the correlator (2.11) is also convenient for obtaining different representations. For example, after rewriting $\sum_A V_{iA}\mathcal{Z}_{iA} = \sum_{A,B} V_{iA}\delta_{A,B}\mathcal{Z}_{iB}$ one can use (I.5.2) to obtain the *trading identity*,

$$
\sum_A V_{iA} \mathcal{Z}_{iA} = \sum_A M_{iA} Z^{(s)}_{iA}\ . \tag{2.12}
$$

Shuffle symmetric Berends–Giele currents M_B and Lie-symmetric worldsheet functions $Z^{(s)}_B$ are defined in (I.5.1) and (II.4.22), respectively, and (2.12) can be easily generalized to any number of words. This kind of manipulation played a key role in expressing the tree-level correlator in terms of an $(n-3)!$ basis of worldsheet functions in [5].

We will be concerned with the particulars of the expressions (2.11) in section 3; for the moment we note that their growing number of terms calls for a more convenient notation. In the subsequent discussion we will distill the combinatorial properties of these permutation sums and propose an intuitive notation for them.

2.3.1. Stirling cycle permutation sums

In order to grasp the combinatorics of (2.11), note that the symmetries of the Lie polynomial $\sum_A V_A \mathcal{Z}_A$ imply that only $(p-1)!$ permutations are independent for words of length p. This is true for each word A_i in terms such as $V_{A_1} T_{A_2,A_3,A_4} \mathcal{Z}_{A_1,A_2,A_3,A_4}$. For an n-point correlator these words A_i must encompass all particle labels, that is $|A_1|+|A_2|+|A_3|+|A_4| = n$. Therefore the sums of $V_{A_1} T_{A_2,A_3,A_4} \mathcal{Z}_{A_1,A_2,A_3,A_4}$ in the correlators of (2.11) can be interpreted as being all the permutations of n labels that are composed of 4 cycles, or p cycles in the general case of tensorial $V_{A_1} T^{m_1 m_2 \ldots}_{A_2,A_3,\ldots,A_p}$. This is the characterization of the *Stirling cycle numbers*[3] $\left[{n \atop p}\right]$.

Using the above interpretation, the scalar building blocks in (2.11) are generated by the following combinatorial notation

$$\mathcal{K}_n(\ell)\,\big|_{V_A T_{B,C,D}} = V_A T_{B,C,D} \mathcal{Z}_{A,B,C,D} + \big[12\ldots n|A,B,C,D\big]\,, \tag{2.13}$$

where $+\big[12\ldots n|A,B,C,D\big]$ indicates a sum over the *Stirling cycle permutations* of the set $\{1,2,\ldots,n\}$, defined in the appendix A. As a consequence of this definition, each term of (2.13) has leg one as the first letter of A, cf. (A.3).

Similarly, the vector contribution to $\mathcal{K}_5(\ell)$ and $\mathcal{K}_6(\ell)$ in (2.11) follows the same combinatorial pattern as the scalars and its contribution is captured by extending the Stirling cycle permutations to five slots in a similar manner,

$$\mathcal{K}_n(\ell)\,\big|_{V_A T^m_{B,C,D,E}} = V_A T^m_{B,C,D,E} \mathcal{Z}^m_{A,B,C,D,E} + \big[12\ldots n|A,B,C,D,E\big]\,. \tag{2.14}$$

The generalization of the above sums to more slots is straightforward.

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[3] We are following the terminology and notation proposed in [17]; they are commonly known as the *Stirling numbers of the first kind*.

2.3.2. Unrefined Lie polynomials

The Stirling cycle permutations allow for a straightforward generalization of the correlators in (2.11) to multiplicities $n \geq 4$,

$$\mathcal{K}_n^{(0)}(\ell) = \sum_{r=0}^{n-4} \frac{1}{r!} \Big(V_{A_1} T^{m_1 \ldots m_r}_{A_2,\ldots,A_{r+4}} \mathcal{Z}^{m_1 \ldots m_r}_{A_1,\ldots,A_{r+4}} + \big[12\ldots n|A_1,\ldots,A_{r+4}\big] \Big) , \tag{2.15}$$

where the summand with $r = 0$ and $r = 1$ reproduces (2.13) and (2.14), respectively. The reason for the superscript in $\mathcal{K}_n^{(0)}(\ell)$ will become clear below, and this is related to the corrections in (1.1). Expanding the sum yields,

$$\begin{aligned} \mathcal{K}_n^{(0)}(\ell) = {} & V_{A_1} T_{A_2,A_3,A_4} \mathcal{Z}_{A_1,A_2,A_3,A_4} + \big[12\ldots n|A_1,\ldots,A_4\big] \\ & + \frac{1}{1!} V_{A_1} T^{m_1}_{A_2,\ldots,A_5} \mathcal{Z}^{m_1}_{A_1,A_2,\ldots,A_5} + \big[12\ldots n|A_1,\ldots,A_5\big] \\ & + \frac{1}{2!} V_{A_1} T^{m_1 m_2}_{A_2,\ldots,A_6} \mathcal{Z}^{m_1 m_2}_{A_1,A_2,\ldots,A_6} + \big[12\ldots n|A_1,\ldots,A_6\big] \\ & \qquad \vdots \\ & + \frac{1}{(n-4)!} V_{A_1} T^{m_1 \ldots m_{n-4}}_{A_2,\ldots,A_n} \mathcal{Z}^{m_1 \ldots m_{n-4}}_{A_1,\ldots,A_n} + \big[12\ldots n|A_1,A_2,\ldots,A_n\big] . \end{aligned} \tag{2.16}$$

We will see in section 3 that (2.15) gives the correct form of the one-loop correlators up to and including six points, i.e., $\mathcal{K}_n(\ell) = \mathcal{K}_n^{(0)}(\ell)$ for $n \leq 6$. By "correct" we mean that the resulting correlators satisfy a number of requirements detailed in section 2.4, the most stringent ones being BRST invariance and single-valuedness.

So the question to consider is whether the expression (2.16) provides the complete answer for correlators with seven or more external states. Unfortunately this is not the case; the explicit construction of the seven-point correlator indicates that the proposal (2.16) needs to be amended by terms involving superfields with higher degrees of refinement defined in section I.4.4. This will be done below and leads to an expression for $\mathcal{K}_7$ that passes all consistency checks. At eight points and beyond, however, the appearance of Eisenstein series in the correlators cannot be determined by the methods in this work. Hence, we will only propose an expression for $\mathcal{K}_8$ up to an unknown kinematic factor multiplying G_4 while completely fixing its dependence on the z_j.

2.3.3. Including refined building blocks

The reason why $\mathcal{K}_n^{(0)}(\ell)$ in (2.16) cannot be the full expression for the one-loop correlator for $n \geq 7$ is related to BRST invariance; it is not difficult to show that the seven-point instance is not BRST invariant using the worldsheet functions discussed in part II. However, the desired invariance can still be achieved by adding corrections containing *refined* superfields $J_{A|B,C,D,E}$ and their tensorial generalizations, cf. section I.4.4. The patterns encountered at multiplicities seven and eight suggest the following organization; the n-point correlator contains contributions with varying degree d of refinement according to,

$$\mathcal{K}_n^{\text{Lie}}(\ell) \equiv \sum_{d=0}^{\lfloor \frac{n-4}{2} \rfloor} (-1)^d \mathcal{K}_n^{(d)}(\ell)\,. \tag{2.17}$$

The alternating minus sign in (2.17) is chosen for later convenience. The $d=0$ contribution is given by (2.15) for $n \geq 4$, while the first instance of refined corrections with $d=1$ could already appear in the six-point correlator,

$$\mathcal{K}_6^{(1)}(\ell) = V_1 J_{2|3,4,5,6} \mathcal{Z}_{2|1,3,4,5,6} + (2 \leftrightarrow 3,4,5,6)\,. \tag{2.18}$$

However, as detailed in section II.4.4.3, the accompanying functions $\mathcal{Z}_{2|1,3,4,5,6}$ can be chosen to vanish, i.e. $\mathcal{K}_6^{(1)}(\ell) = 0$. Therefore, the first non-vanishing contribution to (2.17) with $d=1$ occurs at seven points,

$$\begin{aligned}\mathcal{K}_7^{(1)}(\ell) &= V_1 J^m_{2|3,4,5,6,7} \mathcal{Z}^m_{2|1,3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7) \\ &\quad + \big[V_{12} J_{3|4,5,6,7} \mathcal{Z}_{3|12,4,5,6,7} + V_{13} J_{2|4,5,6,7} \mathcal{Z}_{2|13,4,5,6,7} + (2,3|2,3,\ldots,7)\big] \\ &\quad + \big[V_1 J_{23|4,5,6,7} \mathcal{Z}_{23|1,4,5,6,7} + (2,3|2,3,\ldots,7)\big] \\ &\quad + \big[(V_1 J_{2|34,5,6,7} \mathcal{Z}_{2|1,34,5,6,7} + \text{cyc}(2,3,4)) + (2,3,4|2,3,\ldots,7)\big]\,,\end{aligned} \tag{2.19}$$

see (II.4.42) for the refined worldsheet functions $\mathcal{Z}_{A|B,\ldots}$. Similarly, eight points give rise to the first non-vanishing instance of $d=2$,

$$\mathcal{K}_8^{(2)}(\ell) = V_1 J_{2,3|4,5,6,7,8}\, \mathcal{Z}_{2,3|1,4,5,6,7,8} + (2,3|2,3,4,5,6,7,8)\,. \tag{2.20}$$

These expressions generalize to $n \geq 7$ points at generic degree d of refinement as

$$\begin{aligned}\mathcal{K}_n^{(d)}(\ell) = \sum_{r=0}^{n-4-2d} \frac{1}{r!} \Big(&\big(V_{A_1} J^{m_1\ldots m_r}_{A_2,\ldots,A_{d+1}|A_{d+2},\ldots,A_{r+4+2d}} \mathcal{Z}^{m_1\ldots m_r}_{A_2,\ldots,A_{d+1}|A_1,A_{d+2},\ldots,A_{r+4+2d}} \\ &+ (A_2,\ldots,A_{d+1}|A_2,\ldots,A_{r+4+2d})\big) + \big[12\ldots n|A_1,\ldots,A_{r+4+2d}\big]\Big)\,.\end{aligned} \tag{2.21}$$

More explicitly, expanding the sum in (2.21) for the case $d=1$ yields,

$$\begin{aligned}
\mathcal{K}^{(1)}_n(\ell) &= \big(V_{A_1} J_{A_2|A_3,\ldots,A_6} \mathcal{Z}_{A_2|A_1,A_3,\ldots,A_6} + (A_2{\leftrightarrow}A_3,\ldots,A_6)\big) + \big[1\ldots n|A_1,\ldots,A_6\big] \qquad (2.22)\\
&+ \frac{1}{1!}\big(V_{A_1} J^m_{A_2|A_3,\ldots,A_7} \mathcal{Z}^m_{A_2|A_1,A_3,\ldots,A_7} + (A_2{\leftrightarrow}A_3,\ldots,A_7)\big) + \big[1\ldots n|A_1,\ldots,A_7\big]\\
&+ \frac{1}{2!}\big(V_{A_1} J^{mn}_{A_2|A_3,\ldots,A_8} \mathcal{Z}^{mn}_{A_2|A_1,A_3,\ldots,A_8} + (A_2{\leftrightarrow}A_3,\ldots,A_8)\big) + \big[1\ldots n|A_1,\ldots,A_8\big]\\
&\quad\vdots\\
&+ \frac{1}{(n{-}6)!}\big(V_{A_1} J^{m_1\ldots m_{n-6}}_{A_2|A_3,\ldots,A_n} \mathcal{Z}^{m_1\ldots m_{n-6}}_{A_2|A_1,A_3,\ldots,A_n} + (A_2{\leftrightarrow}A_3,\ldots,A_n)\big) + \big[1\ldots n|A_1,\ldots,A_n\big]\,.
\end{aligned}$$

The collection of $\mathcal{K}^{(d)}_n(\ell)$ with $d=0,1,\ldots,\lfloor\frac{n-4}{2}\rfloor$ summarized by (2.17) makes up the bulk of the open-string one-loop correlators and will be referred to as its Lie-series part. The expressions (2.21) for $\mathcal{K}^{(d)}_n(\ell)$ with $d\geq 1$ furnish a large class of the corrections in (1.1). We will see that up to and including eight points, the BRST variation of (2.17) is purely anomalous (it is written in terms of the anomalous superfields Y, see section I.4.3.1) and it is natural to conjecture that this behavior is valid for arbitrary n.

2.3.4. BRST variation of the Lie-polynomial correlator

By BRST covariance of their kinematic building blocks in section I.4, the Q variations of the above $\mathcal{K}^{\rm Lie}_n(\ell)$ boil down to ghost-number four superfields $V_{A_1}V_{A_2}T^{m_1\ldots}_{A_3,A_4,\ldots}$ and $V_{A_1}Y^{m_1\ldots}_{A_2,A_3,\ldots}$. As will be detailed in the next section, the coefficients of these ghost-number four combinations read as follows in the simplest non-vanishing variations,

$$\begin{aligned}
-Q\mathcal{K}^{\rm Lie}_5(\ell)\big|_{V_1V_2T_{3,4,5}} &= k_2^m\mathcal{Z}^m_{1,2,3,4,5} + s_{21}\mathcal{Z}_{21,3,4,5} + s_{23}\mathcal{Z}_{1,23,4,5} + s_{24}\mathcal{Z}_{1,24,3,5} + s_{25}\mathcal{Z}_{1,25,3,4}\\
-Q\mathcal{K}^{\rm Lie}_6(\ell)\big|_{V_1V_2T^m_{3,4,5,6}} &= k_2^n\mathcal{Z}^{mn}_{1,2,3,4,5,6} - k_2^m\mathcal{Z}_{2|1,3,4,5,6} + \big[s_{21}\mathcal{Z}^m_{21,3,4,5,6} + (1\leftrightarrow 3,4,5,6)\big]\\
-Q\mathcal{K}^{\rm Lie}_6(\ell)\big|_{V_{12}V_3T_{4,5,6}} &= k_3^m\mathcal{Z}^m_{12,3,4,5,6} + s_{31}\mathcal{Z}_{312,4,5,6} - s_{32}\mathcal{Z}_{321,4,5,6} + \big[s_{34}\mathcal{Z}_{12,34,5,6} + (4{\leftrightarrow}5,6)\big]\\
-Q\mathcal{K}^{\rm Lie}_6(\ell)\big|_{V_1V_2T_{34,5,6}} &= k_2^m\mathcal{Z}^m_{1,2,34,5,6} + s_{23}\mathcal{Z}_{1,234,5,6} - s_{24}\mathcal{Z}_{1,243,5,6} + \big[s_{21}\mathcal{Z}_{21,34,5,6} + (1{\leftrightarrow}5,6)\big]\\
-Q\mathcal{K}^{\rm Lie}_6(\ell)\big|_{V_1V_{23}T_{4,5,6}} &= k_{23}^m\mathcal{Z}^m_{1,23,4,5,6} - \mathcal{Z}_{2|1,3,4,5,6} + \mathcal{Z}_{3|1,2,4,5,6} \qquad (2.23)\\
&\quad + \big[s_{31}\mathcal{Z}_{231,4,5,6} - s_{21}\mathcal{Z}_{321,4,5,6} + (1\leftrightarrow 4,5,6)\big]\,,
\end{aligned}$$

as well as

$$-Q\mathcal{K}^{\rm Lie}_6(\ell)\big|_{V_1Y_{2,3,4,5,6}} = \frac{1}{2}\mathcal{Z}^{mm}_{1,2,3,4,5,6} - \big[\mathcal{Z}_{2|1,3,4,5,6} + (2\leftrightarrow 3,4,5,6)\big]\,, \qquad (2.24)$$

where $\mathcal{K}^{\rm Lie}_{n=5,6}(\ell)$ already furnish the complete correlators $\mathcal{K}_n(\ell)$. Note that we have disregarded the vanishing of $\mathcal{Z}_{2|1,3,4,5,6}$ for later convenience, and one can compactly absorb the Mandelstam invariants in (2.23) into the $S[A,B]$ map defined in (I.5.13), e.g. $s_{31}\mathcal{Z}_{312,4,5,6} - s_{32}\mathcal{Z}_{321,4,5,6} = \mathcal{Z}_{S[3,12],4,5,6}$. Based on these examples and analogous observations on $Q\mathcal{K}^{\rm Lie}_n(\ell)$ for higher values of n, it is possible infer a general pattern and propose closed formulae. We organize the general conjecture on the BRST variation of the correlator (2.17) into the following Stirling permutation sums,

$$
\begin{aligned}
Q\mathcal{K}^{\rm Lie}_n(\ell) = &-\sum_{r=0}^{n-5}\frac{1}{r!}T^{(0,r)}_{A_1|A_2,\ldots,A_{r+5}} + \big[12\ldots n|A_1,\ldots,A_{r+5}\big] \\
&+\sum_{r=0}^{n-6}\frac{1}{r!}Y^{(0,r)}_{A_1|A_2,\ldots,A_{r+6}} + \big[12\ldots n|A_1,\ldots,A_{r+6}\big] \\
&-\sum_{r=0}^{n-7}\frac{1}{r!}T^{(1,r)}_{A_1|A_2,\ldots,A_{r+7}} + \big[12\ldots n|A_1,\ldots,A_{r+7}\big] \\
&+\sum_{r=0}^{n-8}\frac{1}{r!}Y^{(1,r)}_{A_1|A_2,\ldots,A_{r+8}} + \big[12\ldots n|A_1,\ldots,A_{r+8}\big] \\
&+\ \cdots\,,
\end{aligned}
\tag{2.25}
$$

where the suppressed terms $T^{(d,r)}$ and $Y^{(d,r)}$ in $\ldots$ refer to higher degree of refinement $d\geq 2$ and start to contribute at $n=9$. The case $r=0$ is understood as containing no vector indices in the superfields, and a upper negative integer in the sum must be discarded; $\sum_{r=0}^{-i}(\ldots)\to 0$. The shorthands $T^{(d,r)}$ contain T-like[4] building blocks, and their definitions at refinement $d=0,1$

$$
\begin{aligned}
T^{(0,r)}_{A_1|A_2,\ldots,A_{r+5}} &\equiv V_{A_1}V_{A_2}T^{m_1\ldots m_r}_{A_3,\ldots,A_{r+5}}\Theta^{(0)\,m_1\ldots m_r}_{A_2|A_1,A_3,\ldots,A_{r+5}} + (A_2\leftrightarrow A_3,\ldots,A_{r+5})\,, \\
T^{(1,r)}_{A_1|A_2,\ldots,A_{r+7}} &\equiv \Big(\big[V_{A_1}V_{A_2}J^{m_1\ldots m_r}_{A_3|A_4,\ldots,A_{r+7}}\Theta^{(1)\,m_1\ldots m_r}_{A_2|A_3|A_1,A_4,\ldots,A_{r+7}} + (A_3\leftrightarrow A_4,\ldots,A_{r+7})\big] \\
&\qquad + (A_2\leftrightarrow A_3,\ldots,A_{r+7})\Big)
\end{aligned}
\tag{2.26}
$$

admit an obvious generalization to higher values of d. Similarly, the shorthands $Y^{(d,r)}$ contain anomalous superfields Y with degree of refinement d, see equations (I.4.18) and (I.4.32), and their definitions at $d=0,1$,

$$
\begin{aligned}
Y^{(0,r)}_{A_1|A_2,\ldots,A_{r+6}} &\equiv V_{A_1}Y^{m_1\ldots m_r}_{A_2,A_3,\ldots,A_{r+6}}\Xi^{(0)\,m_1\ldots m_r}_{A_1|A_2,\ldots,A_{r+6}}\,, \\
Y^{(1,r)}_{A_1|A_2,\ldots,A_{r+8}} &\equiv V_{A_1}\big[Y^{m_1\ldots m_r}_{A_2|A_3,\ldots,A_{r+8}}\Xi^{(1)\,m_1\ldots m_r}_{A_1|A_2|A_3,\ldots,A_{r+8}} + (A_2\leftrightarrow A_3,\ldots,A_{r+8})\big]\,,
\end{aligned}
\tag{2.27}
$$

[4] Recall that the $J_{A|\ldots}$ building block is naturally identified as a $d=1$ refined version of T.

suggest their analogues at $d \geq 2$. Finally, the shorthands $\Theta^{(d)}$ and $\Xi^{(d)}$ stand for the following linear combinations of worldsheet functions with degree d of refinement that capture the right-hand sides of (2.23),

$$\begin{aligned}
\Theta^{(0)\, m_1 m_2 \ldots m_r}_{A|B_1,B_2,\ldots,B_{r+4}} &\equiv k_A^p \mathcal{Z}^{p m_1 m_2 \ldots m_r}_{A,B_1,B_2,\ldots,B_{r+4}} + \big[\mathcal{Z}^{m_1 m_2 \ldots m_r}_{S[A,B_1],B_2,\ldots,B_{r+4}} + (B_1 \leftrightarrow B_2, \ldots, B_{r+4})\big] \\
&\quad - k_A^{(m_1} \mathcal{Z}^{m_2 \ldots m_r)}_{A|B_1,\ldots,B_{r+4}} - \sum_{A=XY} \big(\mathcal{Z}^{m_1 m_2 \ldots m_r}_{X|Y,B_1,\ldots,B_{r+4}} - (X \leftrightarrow Y)\big) , \qquad (2.28) \\
\Theta^{(1)\, m_1 m_2 \ldots m_r}_{A|B|B_1,B_2,\ldots,B_{r+5}} &\equiv -k_A^p \mathcal{Z}^{p m_1 \ldots m_r}_{B|A,B_1,\ldots,B_{r+5}} - \mathcal{Z}^{m_1 \ldots m_r}_{S[A,B]|B_1,\ldots,B_{r+5}} \\
&\quad - \big[\mathcal{Z}^{m_1 \ldots m_r}_{B|S[A,B_1],\ldots,B_{r+5}} + (B_1 \leftrightarrow B_2, \ldots, B_{r+5})\big] \\
&\quad + k_A^{(m_1} \mathcal{Z}^{m_2 \ldots m_r)}_{A,B|B_1,\ldots,B_{r+5}} + \sum_{A=XY} \big(\mathcal{Z}^{m_1 m_2 \ldots m_r}_{X,B|Y,B_1,\ldots,B_{r+5}} - (X \leftrightarrow Y)\big) ,
\end{aligned}$$

(recall that $S[A,B]$ denotes the S-map defined in (I.5.13)), and

$$\begin{aligned}
\Xi^{(0)\, m_1 m_2 \ldots m_r}_{A_1|B_1,\ldots,B_{r+5}} &\equiv -\frac{1}{2} \mathcal{Z}^{pp m_1 \ldots m_r}_{A_1,B_1,\ldots,B_{r+5}} + \big[\mathcal{Z}^{m_1 \ldots m_r}_{B_1|A_1,B_2,\ldots,B_{r+5}} + (B_1 \leftrightarrow B_2, \ldots B_{r+5})\big] , \qquad (2.29) \\
\Xi^{(1)\, m_1 m_2 \ldots m_r}_{A_1|A_2|B_1,\ldots,B_{r+6}} &\equiv \frac{1}{2} \mathcal{Z}^{pp m_1 \ldots m_r}_{A_2|A_1,B_1,\ldots,B_{r+6}} - \big[\mathcal{Z}^{m_1 \ldots m_r}_{A_2,B_1|A_1,B_2,\ldots,B_{r+6}} + (B_1 \leftrightarrow B_2, \ldots B_{r+6})\big] .
\end{aligned}$$

Hence, after modding out by Lie symmetries of the superfields, combining (2.25) and (2.26) identifies $\Theta^{(d)\, m_1 m_2 \ldots m_r}_{A_2|B_1,\ldots,B_d|A_1,A_3,\ldots}$ to be the coefficient of $V_{A_1} V_{A_2} J^{m_1 \ldots m_r}_{B_1,\ldots,B_d|A_3,\ldots}$ in $Q\mathcal{K}^{\text{Lie}}_n$. Similarly, $\Xi^{(d)\, m_1 m_2 \ldots m_r}_{A_1|B_1,\ldots,B_d|A_2,\ldots}$ turns out to be the coefficient of $V_{A_1} Y^{m_1 \ldots m_r}_{B_1,\ldots,B_d|A_2,\ldots}$ by (2.25) and (2.27).

Two comments are in order here. First, notice that the presentation of the BRST variation as a Stirling permutation sum (with the conventions of the appendix A) is essential to fix the ambiguity of $V_A V_B = -V_B V_A$ in matching the $V_A V_B$ products in (2.26) to $\Theta^{(d)}_{B|A,\ldots}$. For example, the conventions of the appendix A fix the relative ordering between the cycles (1)(234) in the permutation sum such that we get $V_1 V_{234} T_{5,6,7}\, \Theta^{(0)}_{234|1,5,6,7}$ rather than $V_{234} V_1 T_{5,6,7}\, \Theta^{(0)}_{1|234,5,6,7}$. And second, although a bit surprising, the BRST variation leads to crossing-symmetric definitions such as $\Theta^{(0)}_{A|B,C,D,E}$ in B, C, D and E; in other words, the worldsheet functions multiplying $V_{1234} V_5 T_{6,7,8}$ are related by a relabeling of those that multiply $V_1 V_2 T_{3456,7,8}$.

The monodromy variations of section II.4.2 and the elliptic identities of section II.5 imply that the definitions (2.28) are generalized elliptic integrands (GEIs); $D\Theta^{(d)} = 0$. Moreover, by inserting the solutions of the bootstrap procedure in section II.4.4 up to $n = 8$ points, the GEIs $\Theta^{(d)}$ are in fact found to vanish up to total derivatives. The coefficients $\Xi^{(d)}$ of the anomalous terms, however, turn out to be non-zero. Instead, the trace relations among worldsheet functions discussed in section II.5.3 simplify the explicit expressions of $\Xi^{(d)}$ from (2.29) at $n \leq 7$ points to a single term. In summary, we obtain

$$\Theta^{(0)\, m_1 m_2 \ldots m_r}_{A|B_1,B_2,\ldots,B_{r+4}} \cong 0\,, \qquad \Theta^{(1)\, m_1 m_2 \ldots m_r}_{A|B|B_1,B_2,\ldots,B_{r+4}} \cong 0\,, \qquad n \leq 8\,, \tag{2.30}$$

and

$$\Xi^{(0)\, m_1 m_2 \ldots m_r}_{A_1|B_1,\ldots,B_{r+6}} \cong -\mathcal{Z}_{A_1|B_1,\ldots,B_{r+6}}\,, \qquad \Xi^{(1)\, m_1 m_2 \ldots m_r}_{A_1|A_2|B_1,\ldots,B_{r+6}} \cong \mathcal{Z}_{A_1,A_2|B_1,\ldots,B_{r+6}}\,, \qquad n \leq 7\,, \tag{2.31}$$

see section 3.5.1 for the eight-point examples of $\Xi^{(d)}$.

The simplest examples of (2.28) are given by (see (2.23) for the former two),

$$\begin{aligned}
\Theta^{(0)}_{1|2,3,4,5} &= k_1^p \mathcal{Z}^p_{1,2,3,4,5} + \big[s_{12}\mathcal{Z}_{12,3,4,5} + (2 \leftrightarrow 3,4,5)\big]\,, \\
\Theta^{(0)}_{12|3,4,5,6} &= k_{12}^p \mathcal{Z}^p_{12,3,4,5,6} + \big[s_{23}\mathcal{Z}_{123,4,5,6} - s_{13}\mathcal{Z}_{213,4,5,6} + (3 \leftrightarrow 4,5,6)\big] \\
&\quad - \mathcal{Z}_{1|2,3,4,5,6} + \mathcal{Z}_{2|1,3,4,5,6}\,, \\
\Theta^{(1)}_{1|2|3,4,5,6,7} &= -k_1^p \mathcal{Z}^p_{2|1,3,4,5,6,7} - s_{12}\mathcal{Z}_{12|3,4,5,6,7} - \big[s_{13}\mathcal{Z}_{2|13,4,5,6,7} + (3 \leftrightarrow 4,5,6,7)\big]\,,
\end{aligned} \tag{2.32}$$

and one can verify from the expressions for $\mathcal{Z}$ in section II.4.4 that these linear combinations indeed yield total derivatives. For more examples, see the appendix C.

2.3.5. Anomalous Lie polynomials

Given the non-vanishing expressions for $\Xi^{(d)}$ in (2.31), the Lie-series part $\mathcal{K}_n^{\rm Lie}(\ell)$ of the correlators from (2.17) is not BRST invariant for $n \geq 7$. More precisely, we have $T^{(d,r)} = 0$ but $Y^{(d,r)} \neq 0$ in (2.25). Fortunately, these non-vanishing terms are purely anomalous and suggest to add corrections containing exclusively anomalous superfields of the form $Y^{m_1 \ldots}_{A_1,\ldots,A_d|B_1,\ldots}$, see (I.4.18), (I.4.19) and (I.4.42). Therefore our proposal for the one-loop correlator becomes,

$$\mathcal{K}_n(\ell) = \mathcal{K}_n^{\rm Lie}(\ell) + \mathcal{K}_n^{Y}(\ell)\,, \tag{2.33}$$

for some $\mathcal{K}_n^{Y}(\ell)$ to be determined. Such an anomaly sector $\mathcal{K}_n^{Y}(\ell)$ is plausible by the kinematic identities of section I.5.4, as they mix anomalous and non-anomalous terms. Up to and including six points, we have

$$\mathcal{K}_n^{Y}(\ell) = 0\,, \quad \text{for } n \leq 6\,. \tag{2.34}$$

From multiplicity seven on, we need to find an expression for the anomaly sector $\mathcal{K}_n^{Y}(\ell)$ such that the full correlator satisfies the criteria summarized below. Even though we will find the proper $\mathcal{K}_n^{Y}(\ell)$ in the seven-point example of section 3, this is done case-by-case, so it would be desirable to understand the general pattern behind them.

2.4. Final assembly of one-loop correlators

The general form of the one-loop correlators (2.33) was suggested by analogy with the Lie-polynomial structure observed at tree level [5]. The one-loop correlators $\mathcal{K}_n(\ell)$ are expressions in the cohomology [18] of pure-spinor superspace that depend on the loop momentum ℓ^m and the zero modes of the pure spinor λ^α and of the superspace coordinate θ^α. Moreover, they are also expanded in terms of worldsheet functions that have to be integrated over the vertex operator insertions points as well as over the moduli space that parametrize the different genus-one surfaces. Given this setting, the final assembly of one-loop correlators $\mathcal{K}_n(\ell)$ as defined in (2.1) must satisfy the following fundamental requirements:

1. The correlator must be in the cohomology of the BRST operator;
2. The correlator must be a single-valued function with respect to both z_i and ℓ^m;
3. The correlator must admit a local representation;
4. The correlator must be manifestly[5] symmetric in the labels $(2, 3, \ldots, n)$.

These conditions arise from general CFT considerations applied to the pure-spinor amplitude prescription (I.2.4), and they are compatible with the tree-level arguments that led to the Lie-polynomial proposal (2.33). The notion of single-valuedness in 2. is defined in (II.3.3), and 3. refers to the absence of kinematic poles s_P^{-1} in a local representation of $\mathcal{K}_n$. The combination of 1. and 3. turns out to be particularly constraining: Any BRST-invariant linear combination of the building blocks of section I.4 has been checked to vanish in the cohomology at $5 \leq n \leq 8$ points (see appendix I.B for further details). Therefore there is no freedom of adding BRST-invariant local terms multiplying single-valued functions at these multiplicities.

In the next section we write down explicit examples of one-loop correlators fulfilling the above criteria up to seven points. Moreover, we propose an expression at eight points with mild violations of 1. and 3.: Its BRST variation vanishes only up to local terms proportional to the Eisenstein series of modular weight four, G_4, and certain terms in the anomaly sector $\mathcal{K}_8^Y$ violate locality. We expect that the eight-point proposal to be given in section 3.5 differs from the correct correlator $\mathcal{K}_8$ by G_4 multiplying an unknown kinematic factor, i.e. it correctly captures all dependence on the z_i.

---

[5] The symmetry with respect to leg 1 is not manifest in the prescription (I.2.4) and therefore can be verified only up to total τ derivatives originating from BRST integration by parts [3].

3. One-loop correlators of the open superstring: examples

We will now apply all the techniques developed in the previous sections to obtain explicit expressions for the one-loop correlators of the open superstring in a manifestly supersymmetric fashion. The correlators at four, five, six and seven points meet all the requirements described in section 2.4, and we will elaborate on the aforementioned issues with the eight-point correlator below.

3.1. Four points

The four-point correlator is uniquely determined by the zero-mode integration over the pure-spinor variables and it was firstly computed by Berkovits in [3]. Using the definition (I.4.1) its correlator can be written as the manifestly local pure-spinor superspace expression

$$\mathcal{K}_4(\ell) = V_1 T_{2,3,4} \, . \tag{3.1}$$

Note that there are no worldsheet singularities among the vertex positions nor an explicit dependence on the loop momentum ℓ^m. This is in accordance with the general discussion in section I.2.1.3 that a n-point correlator $\mathcal{K}_n(\ell)$ is a polynomial in loop momenta of degree $n{-}4$ and that the maximum number of OPE contractions is also $n{-}4$. It has been shown in [19] using BRST cohomology identities in pure-spinor superspace that the one-loop correlator (3.1) is proportional to its tree-level counterpart (2.10),

$$\langle V_1 T_{2,3,4} \rangle = s_{12} s_{23} A^{\rm SYM}(1,2,3,4) \, . \tag{3.2}$$

Therefore it reproduces the well-known [11,20] supersymmetric completion of $t_8 F^4$ and the one-loop amplitudes of Brink, Green and Schwarz with bosonic external states [21].

3.2. Five points

The reasoning behind the derivation of the five-point correlator will be presented in detail as it constitutes the prototype for similar derivations at higher points. Not surprisingly, the outcome of the following analysis is in accordance with the general features of one-loop correlators summarized in section 2.3.

As discussed in section I.2.1.3, the pure-spinor prescription [3] implies that the five-point correlator $\mathcal{K}_5(\ell)$ is a polynomial of degree one in ℓ with at most one OPE singularity. Therefore the correlator is composed of two classes of terms containing: (i) one OPE contraction, (ii) one loop momentum. Let us consider them in turn.

3.2.1. The OPEs

The two inequivalent OPEs $V_1(z_1)U_2(z_2)$ and $U_2(z_2)U_3(z_3)$ can be derived from (I.2.11) and give rise to two-particle vertex operators (I.3.16) and (I.3.19),

$$V_1(z_1)U_2(z_2) \to g^{(1)}_{12} V_{12}(z_2)\,, \qquad U_2(z_2)U_3(z_3) \to g^{(1)}_{23} U_{23}(z_3)\,, \tag{3.3}$$

where $g^{(w)}_{ij} \equiv g^{(w)}(z_i - z_j, \tau)$ refer to expansion coefficients of the Kronecker–Eisenstein series, see (II.2.5), with $g^{(1)}(z,\tau) = \partial_z \log \theta_1(z,\tau)$. In both cases the zero-mode integration for d_α and N_{mn} only admits the b-ghost sector $b^{(4)}$ defined in section I.2.1.3 and yields $T_{A,B,C}$ according to the multiplicity-agnostic rule (I.3.23). In assembling all the ten OPE channels we obtain

$$\mathcal{K}_5(\ell)\big|_{\rm OPE} = \big[g^{(1)}_{12} V_{12} T_{3,4,5} + (2 \leftrightarrow 3,4,5)\big] + \big[g^{(1)}_{23} V_1 T_{23,4,5} + (2,3|2,3,4,5)\big]\,. \tag{3.4}$$

3.2.2. Adjoining the loop momentum

Five points is the first instance where a loop momentum can be extracted from the external vertices or the b-ghost. According to the discussion of section I.4.2, the relevant b-ghost sectors are $b^{(4)}$ and $b^{(2)}$, and they give rise to the schematic contributions $\ell_m V_1 A^m_2 T_{3,4,5}$ and $\ell_m V_1 W^m_{2,3,4,5}$, respectively. BRST covariance fixes their relative coefficients to

$$\mathcal{K}_5(\ell)\big|_\ell = \ell_m V_1 T^m_{2,3,4,5}\,, \tag{3.5}$$

see (I.4.6). By adjoining the contribution (3.4) from OPEs, one arrives at the following final expression for the five-point correlator anticipated in section II.3.1:

$$\begin{aligned}\mathcal{K}_5(\ell) &= \mathcal{K}_5(\ell)\big|_\ell + \mathcal{K}_5(\ell)\big|_{\rm OPE} \\ &= \ell_m V_1 T^m_{2,3,4,5} + \big[g^{(1)}_{12} V_{12} T_{3,4,5} + (2 \leftrightarrow 3,4,5)\big] + \big[g^{(1)}_{23} V_1 T_{23,4,5} + (2,3|2,3,4,5)\big]\,.\end{aligned} \tag{3.6}$$

It will be rewarding to rewrite the correlator (3.6) in a slightly more abstract manner, since the higher-point generalization will become more natural in this way. The correlator lines up with the Lie-polynomial structure of (2.15),

$$\begin{aligned}\mathcal{K}_5(\ell) &= V_{A_1} T^m_{A_2,\ldots,A_5} \mathcal{Z}^m_{A_1,\ldots,A_5} + \big[12345|A_1,\ldots,A_5\big] \\ &\quad + V_{A_1} T_{A_2,\ldots,A_4} \mathcal{Z}_{A_1,\ldots,A_4} + \big[12345|A_1,\ldots,A_4\big]\,,\end{aligned} \tag{3.7}$$

where the notation for the permutations is explained after (2.13) and in the appendix A. Expanding the above permutations leads to the following $\left[{5 \atop 5}\right] + \left[{5 \atop 4}\right] = 1 + 10 = 11$ terms,

$$\begin{aligned} \mathcal{K}_5(\ell) = V_1 T^m_{2,3,4,5} \mathcal{Z}^m_{1,2,3,4,5} \\ + V_{12} T_{3,4,5} \mathcal{Z}_{12,3,4,5} + (2 \leftrightarrow 3,4,5) \\ + V_1 T_{23,4,5} \mathcal{Z}_{1,23,4,5} + (2,3|2,3,4,5)\,. \end{aligned} \tag{3.8}$$

In comparing (3.8) with (3.6) we can read off the following $\mathcal{Z}$-functions,

$$\mathcal{Z}_{12,3,4,5} = g^{(1)}_{12}\,, \qquad \mathcal{Z}^m_{1,2,3,4,5} = \ell^m\,, \tag{3.9}$$

which correspond to the functions (II.4.30) studied in section II.4.4.2. As this example demonstrates, the presentation of the correlator as the Lie-polynomial (3.7) organizes the worldsheet functions in a way that manifests the parallels with the kinematic building blocks as highlighted in section II.4.

In summary, the five-point one-loop correlator (3.8) is a manifestly local expression of superfields that was obtained using general arguments based on the amplitude prescription of the pure-spinor formalism. If we want to argue that it is also the *correct* correlator, it must be BRST invariant and single-valued as well.

3.2.3. BRST invariance

It is straightforward to use the BRST variations of the local building blocks – (I.3.32), (I.4.3) and (I.4.8) – to obtain the $n = 5$ instance of the general BRST variation (2.25). Indeed, a short calculation yields

$$\begin{aligned} Q\mathcal{K}_5(\ell) &= -V_1 V_2 T_{3,4,5} \Big[k^m_2 \mathcal{Z}^m_{1,2,3,4,5} + \big[s_{21} \mathcal{Z}_{21,3,4,5} + (1 \leftrightarrow 3,4,5) \big] \Big] + (2 \leftrightarrow 3,4,5) \\ &= -V_1 V_2 T_{3,4,5} \Theta^{(0)}_{2|1,3,4,5} + (2 \leftrightarrow 3,4,5)\,, \end{aligned} \tag{3.10}$$

where in the second line we used the shorthand defined in (2.28). At first sight (3.10) appears to be different than zero, but luckily this particular arrangement of integrands turns out to be a total worldsheet derivative,

$$k^m_2 \mathcal{Z}^m_{1,2,3,4,5} + \big[s_{21} \mathcal{Z}_{21,3,4,5} + (1 \leftrightarrow 3,4,5) \big] = (\ell \cdot k_2) + \big[s_{21} g^{(1)}_{21} + (1 \leftrightarrow 3,4,5) \big] \cong 0\,, \tag{3.11}$$

where we used the expansions (3.9) and the identity (II.2.22). Therefore the five-point correlator (3.8) is BRST invariant.

3.2.4. Single-valuedness

From the discussion in section II.2.1.1, it follows that the monodromies around the A-cycle vanish for any combination of ℓ^m and $g_{ij}^{(n)}$, so the correlator (3.8) will be single valued if its monodromies around the B-cycle also vanish. In this case, the variations (II.3.9) yield

$$\begin{aligned} D\mathcal{K}_5(\ell) &= \Omega_1\Big(k_1^m V_1 T^m_{2,3,4,5} + \big[V_{12}T_{3,4,5} + 2 \leftrightarrow 3,4,5\big]\Big) \\ &\quad + \Omega_2\Big(k_2^m V_1 T^m_{2,3,4,5} + V_{21}T_{3,4,5} + \big[V_1 T_{23,4,5} + 3 \leftrightarrow 4,5\big]\Big) + (2 \leftrightarrow 3,4,5)\,, \end{aligned} \tag{3.12}$$

see section II.3.2 for the linearized-monodromy operator D. Note that the superspace expressions that multiply the formal variables Ω_i for $i = 1, \ldots, 5$ in the definition (II.3.8) of D are BRST-closed and local. However, as discussed in the appendix I.B, the BRST cohomology is empty for local superspace expressions and therefore the above combinations must be BRST-exact. In fact, one can show via (I.4.23) and (I.5.41) that

$$D\mathcal{K}_5(\ell) = \Omega_1 Q J_{1|2,3,4,5} + \big[\Omega_2(QD_{1|2|3,4,5} - \Delta_{1|2,3,4,5}) + (2 \leftrightarrow 3,4,5)\big] \cong 0\,. \tag{3.13}$$

Since the anomalous superfield $\Delta_{1|2,3,4,5}$ was shown to be BRST-exact in [22], the monodromy variation (3.13) vanishes in the cohomology of the pure-spinor superspace (indicated by $\cong 0$), and the correlator (3.8) is therefore single-valued.

3.2.5. Duality between worldsheet functions and BRST invariants

The vanishing of (3.11) is a clear indication of the duality between worldsheet functions and BRST invariants discussed in section II.4 and pointed out in [10]; it corresponds to the BRST-exact linear combination of superfields in (3.12) under the replacement (II.4.10),

$$\begin{aligned} & 0 \cong k_2^m V_1 T^m_{2,3,4,5} + V_{21}T_{3,4,5} + V_1 T_{23,4,5} + V_1 T_{24,3,5} + V_1 T_{25,3,4} \\ \Longleftrightarrow\quad & 0 \cong k_2^m Z^{(s)m}_{1,2,3,4,5} + Z^{(s)}_{21,3,4,5} + Z^{(s)}_{23,1,4,5} + Z^{(s)}_{24,1,3,5} + Z^{(s)}_{25,1,3,4}\,, \end{aligned} \tag{3.14}$$

where the Lie-symmetric worldsheet functions $Z^{(s)}$ have been introduced in section II.4.2.5. The superspace expression in the first line of (3.14) vanishes because it is BRST exact, see (I.5.41), while the integrand in the second line vanishes because it is a total worldsheet derivative, see (II.2.22). This correspondence between BRST invariance and monodromy invariance is a central example of the *duality* between pure-spinor-superspace expressions and one-loop worldsheet functions. In fact, further investigation of such relations led to the discussions presented in section II.4.

3.2.6. Different representations of the five-point correlator

Since the correlator (3.8) is local, single-valued and BRST invariant, it meets the criteria of section 2.4 to be the open-superstring five-point correlator. We will now exploit the properties of both the superspace expressions and the worldsheet functions to rewrite it in various ways that manifest different subsets of these fundamental properties.

3.2.6.1. The $C \cdot \mathcal{Z}$ representation: manifesting BRST invariance

Integration-by-parts identities can be used to yield a manifestly BRST closed representation of the correlator: first rewrite (3.8) in terms of Berends–Giele currents M_A and $M_{B,C,D}$ associated with V_A and $T_{B,C,D}$ using the trading identity (2.12) of the Lie polynomial as

$$\begin{aligned}\mathcal{K}_5(\ell) = M_1 M^m_{2,3,4,5} \mathcal{Z}^m_{1,2,3,4,5} + \big[M_{12} M_{3,4,5} s_{12} \mathcal{Z}_{12,3,4,5} + (2 \leftrightarrow 3,4,5)\big] \\ + M_1 M_{23,4,5} s_{23} \mathcal{Z}_{1,23,4,5} + (2,3|2,3,4,5)\,.\end{aligned} \tag{3.15}$$

Next, the integration-by-parts identity (3.11) can be used to eliminate all functions of the form $\mathcal{Z}_{1i,A,B,C}$ with $i \neq \emptyset$ (i.e. all of $g^{(1)}_{12}, g^{(1)}_{13}, g^{(1)}_{14}, g^{(1)}_{15}$). Doing this leads to

$$\begin{aligned}\mathcal{K}_5(\ell) = \mathcal{Z}^m_{1,2,3,4,5}\Big(M_1 M^m_{2,3,4,5} + \big[k^m_2 M_{12} M_{3,4,5} + (2 \leftrightarrow 3,4,5)\big]\Big) \\ + \big[s_{23} \mathcal{Z}_{1,23,4,5}(M_1 M_{23,4,5} + M_{12} M_{3,4,5} - M_{13} M_{2,4,5}) + (2,3|2,3,4,5)\big]\,.\end{aligned} \tag{3.16}$$

In this way, the terms inside the round brackets build up the Berends–Giele expansions of the BRST invariants from (I.5.20) and (I.5.21) such that (3.16) becomes

$$\mathcal{K}_5(\ell) = C^m_{1|2,3,4,5} \mathcal{Z}^m_{1,2,3,4,5} + \big[C_{1|23,4,5}\, s_{23} \mathcal{Z}_{1,23,4,5} + (2,3|2,3,4,5)\big]\,. \tag{3.17}$$

Since $C^m_{1|A,B,C,D}$ and $C_{1|A,B,C}$ are both BRST closed, (3.17) constitutes a manifestly BRST invariant representation of the local correlator (3.8).

3.2.6.2. The $T \cdot E$ representation: manifesting single-valuedness

Since the five-point correlator (3.8) is single valued, it is worthwhile to spell out a representation that manifests this property. To do this, we rewrite the terms containing a factor of V_{1A} with non-empty A using the BRST cohomology identity

$$V_{12} T_{3,4,5} \cong k^m_2 V_1 T^m_{2,3,4,5} + \big[V_1 T_{23,4,5} + (3 \leftrightarrow 4,5)\big]\,, \tag{3.18}$$

which follows from the (I.5.41) and the BRST-exactness of $\Delta_{1|2,3,4,5}$. Doing this replacement in the correlator (3.8) and collecting terms leads to

$$\begin{aligned}\mathcal{K}_5(\ell) &= V_1 T^m_{2,3,4,5}\Big(\mathcal{Z}^m_{1,2,3,4,5} + \big[k_2^m \mathcal{Z}_{12,3,4,5} + (2 \leftrightarrow 3,4,5)\big]\Big) \\ &\quad + \big[V_1 T_{23,4,5}\big(\mathcal{Z}_{1,23,4,5} + \mathcal{Z}_{12,3,4,5} - \mathcal{Z}_{13,2,4,5}\big) + (2,3|2,3,4,5)\big]\,.\end{aligned} \tag{3.19}$$

The combinations of $\mathcal{Z}$-functions the round brackets can be identified with the GEIs $E^m_{1|2,3,4,5}$ and $E_{1|23,4,5}$ from (II.4.31) and (II.4.32), respectively. Using these functions, the correlator (3.19) takes the manifestly single-valued form:

$$\mathcal{K}_5(\ell) = V_1 T^m_{2,3,4,5} E^m_{1|2,3,4,5} + \big[V_1 T_{23,4,5} E_{1|23,4,5} + (2,3|2,3,4,5)\big]\,. \tag{3.20}$$

This representation reverses the roles of worldsheet functions and kinematic factors in comparison to (3.17)[6]: Manifest BRST invariance is traded for manifest monodromy invariance.

3.2.6.3. The $C \cdot E$ representation: manifesting BRST invariance & single-valuedness

The five-point correlator can also be rewritten such as to manifest both BRST invariance and single-valuedness. To this effect we eliminate $\mathcal{Z}^m_{1,2,3,4,5} = E^m_{1|2,3,4,5} - [k_2^m \mathcal{Z}_{12,3,4,5} + (2 \leftrightarrow 3,4,5)]$ as well as $\mathcal{Z}_{1,23,4,5} = E_{1|23,4,5} - \mathcal{Z}_{12,3,4,5} + \mathcal{Z}_{13,2,4,5}$ from (3.17) and use the BRST cohomology identity (I.5.41) to obtain

$$\mathcal{K}_5(\ell) = C^m_{1|2,3,4,5} E^m_{1|2,3,4,5} + [C_{1|23,4,5} s_{23} E_{1|23,4,5} + (2,3|2,3,4,5)]\,, \tag{3.21}$$

which reproduces the double-copy expression for the five-point correlator proposed in [10] and manifests both BRST invariance and single-valuedness.

3.2.7. Summary of representations

As shown above, there are multiple Lie-polynomial representations of the five-point correlator according to which features are chosen to be manifested:

$$\begin{aligned}\mathcal{K}_5(\ell) &= V_1 T^m_{2,3,4,5} \mathcal{Z}^m_{1,2,3,4,5} + \big[V_{12} T_{3,4,5} \mathcal{Z}_{12,3,4,5} + (2 \leftrightarrow 3,4,5)\big] \\ &\quad + \big[V_1 T_{23,4,5} \mathcal{Z}_{1,23,4,5} + (2,3|2,3,4,5)\big]\,, \\ \mathcal{K}_5(\ell) &= V_1 T^m_{2,3,4,5} E^m_{1|2,3,4,5} + \big[V_1 T_{23,4,5} E_{1|23,4,5} + (2,3|2,3,4,5)\big]\,, \\ \mathcal{K}_5(\ell) &= C^m_{1|2,3,4,5} \mathcal{Z}^m_{1,2,3,4,5} + \big[C_{1|23,4,5} s_{23} \mathcal{Z}_{1,23,4,5} + (2,3|2,3,4,5)\big]\,, \\ \mathcal{K}_5(\ell) &= C^m_{1|2,3,4,5} E^m_{1|2,3,4,5} + [C_{1|23,4,5} s_{23} E_{1|23,4,5} + (2,3|2,3,4,5)]\,.\end{aligned} \tag{3.22}$$

[6] This becomes particularly transparent by introducing $Z^{(s)m}_{1,2,3,4,5} = \mathcal{Z}^m_{1,2,3,4,5}$ and $Z^{(s)}_{1,23,4,5} = s_{23}\mathcal{Z}_{1,23,4,5}$ in (3.17).

In addition to the above, the single-valued representation of the five-point correlator obtained by explicit integration over the loop momentum will be presented in section 4.

We remark that the one-loop five-point amplitude in the open superstring has been computed with the RNS and GS formalisms for states in the Neveu-Schwarz sector [23,24,25] and in the Ramond sector [26,27]. Manifestly supersymmetric expressions were obtained in [28] using the non-minimal pure-spinor formalism [29] and later in [30] using the minimal pure-spinor formalism.

3.3. Six points

We will now show that the general formulas summarized in section 2.3 give rise to the correct six-point one-loop correlator. The Lie-polynomial form of the six-point correlator is given by

$$\begin{aligned}\mathcal{K}_6(\ell) = \frac{1}{2} V_{A_1} T^{mn}_{A_2,\ldots,A_6} \mathcal{Z}^{mn}_{A_1,\ldots,A_6} + \big[123456|A_1,\ldots,A_6\big] \\ + V_{A_1} T^{m}_{A_2,\ldots,A_5} \mathcal{Z}^{m}_{A_1,\ldots,A_5} + \big[123456|A_1,\ldots,A_5\big] \\ + V_{A_1} T_{A_2,\ldots,A_4} \mathcal{Z}_{A_1,\ldots,A_4} + \big[123456|A_1,\ldots,A_4\big]\,,\end{aligned} \tag{3.23}$$

with the following worldsheet functions as derived in section II.4.4.3,

$$\begin{aligned}\mathcal{Z}_{123,4,5,6} &= g^{(1)}_{12} g^{(1)}_{23} + g^{(2)}_{12} + g^{(2)}_{23} - g^{(2)}_{13}\,, \\ \mathcal{Z}_{12,34,5,6} &= g^{(1)}_{12} g^{(1)}_{34} + g^{(2)}_{13} + g^{(2)}_{24} - g^{(2)}_{14} - g^{(2)}_{23}\,, \\ \mathcal{Z}^{m}_{12,3,4,5,6} &= \ell^m g^{(1)}_{12} + (k^m_2 - k^m_1) g^{(2)}_{12} + \big[k^m_3 (g^{(2)}_{13} - g^{(2)}_{23}) + (3 \leftrightarrow 4,5,6)\big]\,, \\ \mathcal{Z}^{mn}_{1,2,3,4,5,6} &= \ell^m \ell^n + \big[(k^m_1 k^n_2 + k^n_1 k^m_2) g^{(2)}_{12} + (1,2|1,2,3,4,5,6)\big]\,.\end{aligned} \tag{3.24}$$

Note that a possible contribution of a $d=1$ refined sector according to (2.21) is suppressed since the monodromy variations (II.4.34) are compatible with $\mathcal{Z}_{1|2,3,4,5,6} = 0$. The explicit expansion of the Stirling cycle permutations in (3.23) generates a total of $\left[{6 \atop 6}\right] + \left[{6 \atop 5}\right] + \left[{6 \atop 4}\right] = 1 + 15 + 85 = 101$ terms,

$$\begin{aligned}\mathcal{K}_6(\ell) = \frac{1}{2} V_1 T^{mn}_{2,3,4,5,6} \mathcal{Z}^{mn}_{1,2,3,4,5,6} \\ + V_{12} T^{m}_{3,4,5,6} \mathcal{Z}^{m}_{12,3,4,5,6} + (2 \leftrightarrow 3,4,5,6) \\ + V_1 T^{m}_{23,4,5,6} \mathcal{Z}^{m}_{1,23,4,5,6} + (2,3|2,3,4,5,6) \\ + V_{123} T_{4,5,6} \mathcal{Z}_{123,4,5,6} + V_{132} T_{4,5,6} \mathcal{Z}_{132,4,5,6} + (2,3|2,3,4,5,6) \\ + V_1 T_{234,5,6} \mathcal{Z}_{1,234,5,6} + V_1 T_{243,5,6} \mathcal{Z}_{1,243,5,6} + (2,3,4|2,3,4,5,6) \\ + \big[\big(V_{12} T_{34,5,6} \mathcal{Z}_{12,34,5,6} + \mathrm{cyc}(2,3,4)\big) + (2,3,4|2,3,4,5,6)\big] \\ + \big[\big(V_1 T_{2,34,56} \mathcal{Z}_{1,2,34,56} + \mathrm{cyc}(3,4,5)\big) + (2 \leftrightarrow 3,4,5,6)\big]\,,\end{aligned} \tag{3.25}$$

where the indicated permutations defined in (I.3.13) act on a line-by-line basis.

We will now prove that the six-point correlator (3.23) is both single-valued and BRST invariant, in accordance with the expectations outlined in section 2.4.

3.3.1. BRST invariance

As anticipated in section 2.3.4, the BRST algebra of the building blocks leads to the following Q-variation of the correlator (3.23),

$$
\begin{aligned}
Q\mathcal{K}_6(\ell) = &-\frac{1}{2}V_1 Y_{2,3,4,5,6}\mathcal{Z}^{mm}_{1,2,3,4,5,6} \\
&- V_1 V_2 T^m_{3,4,5,6}\Theta^{(0)\,m}_{2|1,3,4,5,6} - V_{12}V_3 T_{4,5,6}\Theta^{(0)}_{3|12,4,5,6} + (2\leftrightarrow 3,4,5,6) \\
&- \big[V_1 V_2 T_{34,5,6}\Theta^{(0)}_{2|1,34,5,6} + (3,4|3,4,5,6)\big] + (2\leftrightarrow 3,4,5,6) \\
&- V_1 V_{23} T_{4,5,6}\Theta^{(0)}_{23|1,4,5,6} + (2,3|2,3,4,5,6)\,,
\end{aligned} \tag{3.26}
$$

where the shorthands $\Theta^{(0)\,m\ldots}_{1|A,\ldots}$ were defined in (2.28). After discarding the vanishing refined $\mathcal{Z}$-function, they are given by

$$
\begin{aligned}
\Theta^{(0)\,m}_{2|1,3,4,5,6} &= k_2^n \mathcal{Z}^{mn}_{1,2,3,4,5,6} + \big[s_{21}\mathcal{Z}^m_{21,3,4,5,6} + (1\leftrightarrow 3,4,5,6)\big] \cong 0\,, \\
\Theta^{(0)}_{3|12,4,5,6} &= k_3^m \mathcal{Z}^m_{12,3,4,5,6} + s_{31}\mathcal{Z}_{312,4,5,6} - s_{32}\mathcal{Z}_{321,4,5,6} + \big[s_{34}\mathcal{Z}_{12,34,5,6} + (4\leftrightarrow 5,6)\big] \cong 0\,, \\
\Theta^{(0)}_{2|1,34,5,6} &= k_2^m \mathcal{Z}^m_{1,2,34,5,6} + s_{23}\mathcal{Z}_{1,234,5,6} - s_{24}\mathcal{Z}_{1,243,5,6} + \big[s_{21}\mathcal{Z}_{21,34,5,6} + (1\leftrightarrow 5,6)\big] \cong 0\,, \\
\Theta^{(0)}_{23|1,4,5,6} &= k_{23}^m \mathcal{Z}^m_{1,23,4,5,6} + \big[s_{31}\mathcal{Z}_{231,4,5,6} - s_{21}\mathcal{Z}_{321,4,5,6} + (1\leftrightarrow 4,5,6)\big] \cong 0\,,
\end{aligned} \tag{3.27}
$$

and conspire to total derivatives in z_j. Therefore the BRST variation is proportional to the trace $\mathcal{Z}^{mm}_{1,2,3,4,5,6}$,

$$
Q\mathcal{K}_6(\ell) = -\frac{1}{2}V_1 Y_{2,3,4,5,6}\mathcal{Z}^{mm}_{1,2,3,4,5,6} = -2\pi i\, V_1 Y_{2,3,4,5,6}\frac{\partial}{\partial\tau}\log\mathcal{I}_6(\ell) \cong 0\,, \tag{3.28}
$$

where the total τ derivative of the Koba–Nielsen factor has been identified in (II.5.15). Thus, the BRST variation is a boundary term in moduli space [31], and the usual mechanism of anomaly cancellation [32] implies that the amplitudes computed from the correlator (3.23) are BRST invariant.

3.3.1.1. The $C \cdot \mathcal{Z}$ representation: manifesting BRST invariance

Now that BRST invariance of the six-point correlator is proven, let us rewrite it using the BRST invariants from section I.5.2, in a similar spirit as done with the five-point correlator in the previous section. There are different ways to achieve this, one uses the trading identity (2.12) to rewrite the Lie polynomial (3.23) as,

$$\begin{aligned}\mathcal{K}_6(\ell) = \frac{1}{2} M_{A_1} M^{mn}_{A_2,\ldots,A_6} Z^{(s),mn}_{A_1,\ldots,A_6} + \big[123456|A_1,\ldots,A_6\big] \\ + M_{A_1} M^{m}_{A_2,\ldots,A_5} Z^{(s),m}_{A_1,\ldots,A_5} + \big[123456|A_1,\ldots,A_5\big] \\ + M_{A_1} M_{A_2,\ldots,A_4} Z^{(s)}_{A_1,\ldots,A_4} + \big[123456|A_1,\ldots,A_4\big]\,,\end{aligned} \tag{3.29}$$

where $Z^{(s)}$ is defined in (II.4.22). The idea now is to exploit the fact that terms of the form $M_1 M^{m\cdots}_{A,B,\ldots}$, which feature the single-particle Berends–Giele current M_1, are the leading terms in the expansion of the BRST (pseudo-)invariants from section I.5.2.1. Therefore they can be rewritten as

$$M_1 M^{m\cdots}_{A,B,\ldots} = C^{m\cdots}_{1|A,B,\ldots} + \cdots\,, \tag{3.30}$$

where the terms in the ellipsis on the right-hand side are linear combinations of $M_{1A} M^{m\cdots}_{B,\ldots}$ with $A \neq \emptyset$ that uniquely follow from the definition of the BRST pseudo-invariants in (I.5.20) to (I.5.22). Plugging in the above expressions into the correlator (3.29) yields

$$\begin{aligned}\mathcal{K}_6(\ell) = \frac{1}{2} C^{mn}_{1|A_1,\ldots,A_5} Z^{(s)mn}_{1,A_1,\ldots,A_5} + \big[23456|A_1,\ldots,A_5\big] \\ + C^{m}_{1|A_1,\ldots,A_4} Z^{(s)m}_{1,A_1,\ldots,A_4} + \big[23456|A_1,\ldots,A_4\big] \\ + C_{1|A_1,\ldots,A_3} Z^{(s)}_{1,A_1,\ldots,A_3} + \big[23456|A_1,\ldots,A_3\big]\,.\end{aligned} \tag{3.31}$$

To arrive at (3.31) the following three topologies of terms (and their permutations) were discarded as they are total derivatives:

$$\begin{aligned} s_{34} M_{12} M_{34,5,6} \Theta^{(0)}_{2|1,34,5,6} \cong 0\,, \qquad M_{12} M^{m}_{3,4,5,6} \Theta^{(0)\,m}_{2|1,3,4,5,6} \cong 0\,, \\ M_{123} M_{4,5,6} \Big(k_3^m \Theta^{(0)\,m}_{2|1,3,4,5,6} + s_{12} \Theta^{(0)}_{3|12,4,5,6} + \big[s_{34} \Theta^{(0)}_{2|1,34,5,6} + (4 \leftrightarrow 5,6)\big]\Big) \cong 0\,.\end{aligned} \tag{3.32}$$

Expanding the Stirling cycle permutations in (3.31) yields the following $\left[{5 \atop 5}\right] + \left[{5 \atop 4}\right] + \left[{5 \atop 3}\right] = 1 + 10 + 35 = 46$ terms,

$$\begin{aligned}\mathcal{K}_6(\ell) = \frac{1}{2} C^{mn}_{1|2,3,4,5,6} Z^{(s)mn}_{1,2,3,4,5,6} + \big[C^{m}_{1|23,4,5,6} Z^{(s)m}_{1,23,4,5,6} + (2,3|2,3,4,5,6)\big] \\ + \big[C_{1|234,5,6} Z^{(s)}_{1,234,5,6} + C_{1|243,5,6} Z^{(s)}_{1,243,5,6} + (2,3,4|2,3,4,5,6)\big] \\ + \big[C_{1|23,45,6} Z^{(s)}_{1,23,45,6} + C_{1|24,35,6} Z^{(s)}_{1,24,35,6} + C_{1|25,34,6} Z^{(s)}_{1,25,34,6} + (6 \leftrightarrow 2,3,4,5)\big]\,.\end{aligned} \tag{3.33}$$

Note one important difference between the expansion above and an earlier representation; unlike the local Lie polynomial (3.23) in which six labels are distributed among the available slots, in the non-local representation (3.31) only five labels participate in the Stirling permutations. Like this, the initially 101 terms in (3.25) conspire to the considerably smaller number of 46 terms in (3.33).

As a consistency check, we note that the scalar $C_{1|A,B,C}$ and vectorial $C^m_{1|A,B,C,D}$ are manifestly BRST closed while the BRST variation of the two-tensor $C^{mn}_{1|A,B,C,D,E}$ is proportional to δ^{mn} [22]. Hence, we arrive at the same conclusion as in (3.28)

$$Q\mathcal{K}_6(\ell) = -\frac{1}{2} V_1 Y_{2,3,4,5,6} \mathcal{Z}^{mm}_{1,2,3,4,5,6} = -2\pi i \, V_1 Y_{2,3,4,5,6} \frac{\partial}{\partial\tau} \log \mathcal{I}_6(\ell) \cong 0 \,, \tag{3.34}$$

since $Z^{(s)\,mn}_{1,2,3,4,5,6} = \mathcal{Z}^{mn}_{1,2,3,4,5,6}$ follows from (II.4.22).

3.3.2. Single-valuedness

To prove that the correlator (3.23) is single-valued it is sufficient to show that its integration-by-parts-equivalent representation (3.33) is single-valued. After a tedious calculation using the monodromy variations (II.4.34) one gets

$$D\mathcal{K}_6(\ell) = \Omega_1 \delta\mathcal{K}^{(1)}_6 + \Omega_2 \delta\mathcal{K}^{(2)}_6 + \cdots + \Omega_6 \delta\mathcal{K}^{(6)}_6 \,, \tag{3.35}$$

where

$$\delta\mathcal{K}^{(1)}_6 = k^n_1 C^{mn}_{1|2,3,4,5,6} E^m_{1|2,3,4,5,6} + \big[k^m_1 s_{23} C^m_{1|23,4,5,6} E_{1|23,4,5,6} + (2,3|2,3,4,5,6) \big] \tag{3.36}$$

$$\begin{aligned} \delta\mathcal{K}^{(2)}_6 &= k^m_2 C^{mn}_{1|2,3,4,5,6} E^n_{2|1,3,4,5,6} + \big[s_{23} C^m_{1|23,4,5,6} E^m_{2|1,3,4,5,6} + (3 \leftrightarrow 4,5,6) \big] \\ &\quad + \big[k^m_2 s_{34} C^m_{1|2,34,5,6} E_{2|1,34,5,6} + (3,4|3,4,5,6) \big] \\ &\quad + \big[\big(s_{23} s_{45} C_{1|23,45,6} E_{2|1,3,45,6} + s_{23} s_{34} C_{1|234,5,6} E_{2|1,34,5,6} + \mathrm{cyc}(3,4,5) \big) + (3 \leftrightarrow 4,5,6) \big] \,, \end{aligned}$$

and the other $\delta\mathcal{K}^{(i)}_6$ for $i = 3,4,5,6$ are obtained from relabeling of $\delta\mathcal{K}^{(2)}_6$ under $(2 \leftrightarrow 3)$, $(2 \leftrightarrow 4)$ and so forth. The structural difference between $\delta\mathcal{K}^{(1)}_6$ and $\delta\mathcal{K}^{(j)}_6$ for $j \neq 1$ arises from the choice of basis for the BRST pseudo-invariants which singles out leg number 1 in $C^{m\ldots}_{1|A,\ldots}$. To expose this, one uses the kinematic change-of-basis identities dual to (II.5.11) and (II.5.12) [22] (also see section II.5.2 and the appendix I.A.3) to rewrite $\delta\mathcal{K}^{(2)}_6$ in a basis of $C^{m\ldots}_{2|A,\ldots}$ to obtain[7]

$$\delta\mathcal{K}^{(2)}_6 = k^m_2 C^{mn}_{2|1,3,4,5,6} E^n_{2|1,3,4,5,6} + \big[k^m_2 s_{13} C^m_{2|13,4,5,6} E_{2|13,4,5,6} + (1,3|1,3,4,5,6) \big] \,, \tag{3.37}$$

[7] The anomalous term in the change-of-basis identity $C^{mn}_{1|2,3,4,5,6} = \delta^{mn} \mathcal{Y}_{21,3,4,5,6} + C^{mn}_{2|1,3,4,5,6} + \ldots$ [22] has already been discarded from (3.37) since the accompanying GEI $k^m_2 E^n_{2|1,3,4,5,6}$ vanishes upon contraction with δ^{mn}.

which is clearly the relabeling of $\delta\mathcal{K}_6^{(1)}$ under $1 \leftrightarrow 2$. Therefore, it suffices to demonstrate the vanishing of $\delta\mathcal{K}_6^{(1)}$ to prove that the correlator (3.23) is single-valued.

To show that $\delta\mathcal{K}_6^{(1)}$ vanishes, we use the kinematic BRST cohomology identities [22]

$$
\begin{aligned}
k_1^m C^m_{1|23,4,5,6} &\cong -P_{1|2|3,4,5,6} + P_{1|3|2,4,5,6} - \Delta_{1|23,4,5,6} \\
k_1^n C^{mn}_{1|2,3,4,5,6} &\cong -\big[k_2^m P_{1|2|3,4,5,6} + (2 \leftrightarrow 3,4,5,6)\big] - \Delta^m_{1|2,3,4,5,6}\,,
\end{aligned}
\tag{3.38}
$$

that follow from (I.5.42) and can be used to bring (3.36) into the following form

$$
\begin{aligned}
\delta\mathcal{K}_6^{(1)} \cong{}& -\Delta^m_{1|2,3,4,5,6} E^m_{1|2,3,4,5,6} - \big[\Delta_{1|23,4,5,6} s_{23} E_{1|23,4,5,6} + (2,3|2,3,4,5,6)\big] \\
&- \Big\{P_{1|2|3,4,5,6}\big(k_2^m E^m_{1|2,3,4,5,6} + \big[s_{23}E_{1|23,4,5,6} + (3 \leftrightarrow 4,5,6)\big]\big) + (2 \leftrightarrow 3,4,5,6)\Big\}\,.
\end{aligned}
\tag{3.39}
$$

The coefficients of $P_{1|2|3,4,5,6}$ in the second line in turn conspire to total derivatives,

$$
k_2^m E^m_{1|2,3,4,5,6} + \big[s_{23}E_{1|23,4,5,6} + (3 \leftrightarrow 4,5,6)\big] \cong 0\,,
\tag{3.40}
$$

see section II.5.1. Finally, combining the relabelings of the first line of (3.39), we arrive at

$$
\begin{aligned}
-D\mathcal{K}_6(\ell) \cong{}& \Omega_1\Big(\Delta^m_{1|2,3,4,5,6}E^m_{1|2,3,4,5,6} + \big[\Delta_{1|23,4,5,6}s_{23}E_{1|23,4,5,6} + (2,3|2,3,4,5,6)\big]\Big) \\
&+ (1 \leftrightarrow 2,3,4,5,6)\,.
\end{aligned}
\tag{3.41}
$$

As reviewed in section I.5.3, the unrefined anomalous building blocks $\Delta^{m_1\ldots}_{1|A_1,\ldots}$ are BRST exact, so the monodromy variation (3.41) vanishes in the cohomology of the pure-spinor BRST charge, finishing the proof that the six-point correlator (3.23) is single valued. By the interplay of the cohomology identity (3.38) and the GEI relation (3.40), our proof of $D\mathcal{K}_6(\ell) \cong 0$ constitutes an illuminating showcase of the duality between kinematics and worldsheet functions.

We note that there are other ways to prove the single-valuedness of the six-point correlator (3.23). One such proof, given in section 4, follows by explicitly integrating the loop momentum from the correlator while verifying that only the single-valued functions $f_{ij}^{(n)}$ in (II.7.1) build up in the outcome. Another proof, presented in the appendix B, uses manipulations involving the manifestly-local representation $T\cdot\mathcal{Z}$. However, in exploiting the BRST invariance of the correlator in its $C\cdot\mathcal{Z}$ representation the proof above is considerably simpler than the others.

3.3.2.1. The $C \cdot E$ representation: manifesting BRST invariance & single-valuedness

As another application of the duality between kinematics and worldsheet functions, we shall now derive a manifestly BRST-invariant and single-valued representation of the six-point correlator. The idea is to start from the $C \cdot \mathcal{Z}$ representation (3.33) and to exploit the dual

$$\mathcal{Z}^{m\ldots}_{1,A,B,\ldots} = E^{m\ldots}_{1|A,B,\ldots} + \cdots \tag{3.42}$$

of (3.30): Each $\mathcal{Z}$-function with leg one in a single-particle slot is taken as a leading term of a GEI, see (II.4.26), and the additional terms in the ellipsis of (3.42) are of the form $\mathcal{Z}^{m\ldots}_{1C,D,\ldots}$ with $C \neq \emptyset$. In this way, a long sequence of BRST cohomology identities given in section I.5.4 leads to the following manifestly BRST-invariant and single-valued Lie-polynomial form of (3.33),

$$\begin{aligned}\mathcal{K}_6(\ell) = {}& \frac{1}{2} C^{mn}_{1|A_1,\ldots,A_5} E^{(s),mn}_{1|A_1,\ldots,A_5} + \big[23456|A_1,\ldots,A_5\big] \\ &+ C^{m}_{1|A_1,\ldots,A_4} E^{(s)m}_{1|A_1,\ldots,A_4} + \big[23456|A_1,\ldots,A_4\big] \\ &+ C_{1|A_1,\ldots,A_3} E^{(s)}_{1|A_1,\ldots,A_3} + \big[23456|A_1,\ldots,A_3\big] \\ &- \big[P_{1|A_1|A_2,\ldots,A_5} E^{(s)}_{1|A_1|A_2,\ldots,A_5} + (A_1 \leftrightarrow A_2,\ldots,A_5)\big] + \big[23456|A_1,\ldots,A_5\big] \,.\end{aligned} \tag{3.43}$$

The GEIs have been expressed in terms of the Lie symmetric $E^{(s)m\ldots}_{1|A,\ldots}$ defined in (II.4.23), and similar to (3.31), only five legs participate in the Stirling permutations. More explicitly, expanding the above sums over Stirling cycle permutations yields

$$\begin{aligned}\mathcal{K}_6(\ell) = {}& \frac{1}{2} C^{mn}_{1|2,3,4,5,6} E^{mn}_{1|2,3,4,5,6} - \big[P_{1|2|3,4,5,6} E_{1|2|3,4,5,6} + (2 \leftrightarrow 3,4,5,6)\big] \\ &+ \big[C^{m}_{1|23,4,5,6}\, s_{23} E^{m}_{1|23,4,5,6} + (2,3|2,3,4,5,6)\big] \\ &+ \big[\big(C_{1|23,45,6}\, s_{23} s_{45} E_{1|23,45,6} + \mathrm{cyc}(3,4,5)\big) + (6 \leftrightarrow 5,4,3,2)\big] \\ &+ \big[\big(C_{1|234,5,6}\, s_{23} s_{34} E_{1|234,5,6} + \mathrm{cyc}(2,3,4)\big) + (2,3,4|2,3,4,5,6)\big] \,,\end{aligned} \tag{3.44}$$

and reproduces the double-copy expression for the six-point correlator proposed in [10]. The refined GEI $E_{1|2|3,4,5,6}$ arises from its expansion (II.4.27) in terms of $\mathcal{Z}$-functions and boils down to the $g^{(n)}_{ij}$ in (II.4.37). By the vanishing of $\mathcal{Z}_{2|1,3,4,5,6}$, the $C \cdot \mathcal{Z}$-representation (3.33) of the six-point correlator does not feature any analogue of the terms $P_{1|2|3,4,5,6} E_{1|2|3,4,5,6}$ in the first line of (3.44).

Furthermore, from the trace relation (II.5.31) among GEIs,

$$\frac{1}{2} \delta_{mn} E^{mn}_{1|2,3,4,5,6} = \big[E_{1|2|3,4,5,6} + (2 \leftrightarrow 3,4,5,6)\big] + 2\pi i \frac{\partial}{\partial \tau} \log \mathcal{I}_6(\ell) \,, \tag{3.45}$$

one concludes that the BRST variation of (3.44) is a boundary term [10]

$$\begin{aligned} Q\mathcal{K}_6(\ell) &= -V_1 Y_{2,3,4,5,6}\Big(\frac{1}{2}E^{mm}_{1|2,3,4,5,6} - \big[E_{1|2|3,4,5,6} + (2\leftrightarrow 3,4,5,6)\big]\Big) \\ &= -2\pi i\, V_1 Y_{2,3,4,5,6}\frac{\partial}{\partial\tau}\log\mathcal{I}_6(\ell) \cong 0\,, \end{aligned} \tag{3.46}$$

as required by the anomaly cancellation condition.

3.3.2.2. The $T\cdot E$ representation: manifesting locality & single-valuedness

The $C\cdot E$ representation (3.43) is not manifestly local, but it is written in terms of GEIs manifesting monodromy invariance. However, by construction, we know that (3.43) is equivalent to the local representation (3.23), so all the non-localities within the pseudo-invariants C and P must be spurious. In the following discussions we exploit this reasoning to find a new representation that is both manifestly local and monodromy invariant.

We can do this starting from (3.44), plugging in the Berends–Giele expansion of the pseudo-invariants and separating terms according to their kinematic poles. The non-local terms turn out to vanish (as will be exemplified below) while the local terms conspire to produce the full correlator $\mathcal{K}_6(\ell)$. After going through the algebra we obtain the following manifestly local and monodromy-invariant form of the six-point correlator $\mathcal{K}_6(\ell)$,

$$\begin{aligned} \mathcal{K}_6(\ell) = {}& \frac{1}{2}V_1 T^{mn}_{2,3,4,5,6}E^{mn}_{1|2,3,4,5,6} - \big[V_1 J_{2|3,4,5,6}E_{1|2|3,4,5,6} + (2\leftrightarrow 3,4,5,6)\big]\,, \\ &+ \big[V_1 T^m_{23,4,5,6}E^m_{1|23,4,5,6} + (2,3|2,3,4,5,6)\big] \\ &+ \big[V_1 T_{234,5,6}E_{1|234,5,6} + V_1 T_{243,5,6}E_{1|243,5,6} + (2,3,4|2,3,4,5,6)\big] \\ &+ \big[\big(V_1 T_{2,34,56}E_{1|2,34,56} + \mathrm{cyc}(3,4,5)\big) + (2\leftrightarrow 3,4,5,6)\big]\,. \end{aligned} \tag{3.47}$$

The non-local terms from the kinematic side turn out to vanish due to identities obeyed by their accompanying worldsheet functions. For instance, one such class of terms (featuring an uncancelled s_{12} pole) is given by,

$$M_{12}T_{34,5,6}\big(k_2^m E^m_{1|2,34,5,6} + s_{23}E_{1|234,5,6} - s_{24}E_{1|243,5,6} + s_{25}E_{1|25,34,6} + s_{26}E_{1|26,34,5}\big) \cong 0\,, \tag{3.48}$$

whose vanishing follows from one of the GEI relations (II.5.2). Similarly, one can check that all the other classes of non-local terms vanish as well. In summary, the expressions (3.25), (3.33), (3.44) and (3.47) for the six-point correlator generalize the four representations of the five-point correlator in (3.22).

Note that the $T\cdot E$ representation (3.47) is related to the $C\cdot\mathcal{Z}$ representation (3.33) through the duality between kinematics and worldsheet functions: In order to see this, one needs to adjoin the vanishing terms $-\big[\mathcal{Z}_{2|1,3,4,5,6}P_{1|2|3,4,5,6} + (2\leftrightarrow 3,4,5,6)\big]$ to the latter.

3.3.3. Comparison with older results

To conclude the discussion of the six-point correlator, we make contact between the above representations and a manifestly BRST invariant expression for the six-point amplitude that has been presented in [33]. Starting from the $C \cdot \mathcal{Z}$-representation (3.33), expanding the worldsheet functions and collecting terms yields,

$$\begin{aligned}\mathcal{K}_6(\ell) &= \frac{1}{2}\ell_m \ell_n C^{mn}_{1|2,3,4,5,6} + \ell_m \big[s_{23} g^{(1)}_{23} C^m_{1|23,4,5,6} + (2,3|2,\ldots,6)\big] \\ &+ \big[\big(s_{23}s_{34} g^{(1)}_{23} g^{(1)}_{34} C_{1|234,5,6} + \mathrm{cyc}(2,3,4)\big) + (2,3,4|2,3,4,5,6)\big] \\ &+ \big[\big(s_{23}s_{45} g^{(1)}_{23} g^{(1)}_{45} C_{1|23,45,6} + \mathrm{cyc}(3,4,5)\big) + (6 \leftrightarrow 5,4,3,2)\big] \\ &+ \big[g^{(2)}_{12} C_{1|2|3,4,5,6} + (2 \leftrightarrow 3,4,5,6)\big] + \big[g^{(2)}_{23} C_{1|(23)|4,5,6} + (2,3|2,3,4,5,6)\big]\end{aligned} \tag{3.49}$$

where we defined the following shorthands for the coefficients of $g^{(2)}_{12}$ and $g^{(2)}_{23}$,

$$\begin{aligned}C_{1|2|3,4,5,6} &\equiv k_1^m k_2^n C^{mn}_{1|2,3,4,5,6} + \big[s_{23} k_1^m C^m_{1|23,4,5,6} + (3 \leftrightarrow 4,5,6)\big]\,, \\ C_{1|(23)|4,5,6} &\equiv k_2^m k_3^n C^{mn}_{1|2,3,4,5,6} + s_{23}(k_3^m - k_2^m) C^m_{1|23,4,5,6} \\ &\quad + \big[s_{24} k_3^m C^m_{1|24,3,5,6} + s_{34} k_2^m C^m_{1|34,2,5,6} + (4 \leftrightarrow 5,6)\big] \\ &\quad + \big[s_{34}s_{23} C_{1|234,5,6} + s_{23}s_{24} C_{1|324,5,6} - s_{24}s_{34} C_{1|243,5,6} + (4 \leftrightarrow 5,6)\big] \\ &\quad + \Big[s_{24}s_{35} C_{1|24,35,6} + s_{25}s_{34} C_{1|25,34,6} + (4,5|4,5,6)\Big]\,.\end{aligned} \tag{3.50}$$

These combinations are easily seen to satisfy

$$QC_{1|2|3,4,5,6} = -s_{12} V_1 Y_{2,3,4,5,6}\,, \qquad QC_{1|(23)|4,5,6} = -s_{23} V_1 Y_{2,3,4,5,6}\,, \tag{3.51}$$

so the BRST variation of (3.49) reproduces the desired Koba–Nielsen derivative in τ. Using BRST cohomology identities one can show that

$$\begin{aligned}C_{1|2|3,4,5,6} &\cong s_{12} P_{1|2|3,4,5,6}\,, \\ C_{1|(23)|4,5,6} &\cong \frac{1}{2} s_{23}\Big(P_{1|2|3,4,5,6} + P_{1|3|2,4,5,6} + (k_3^m - k_2^m) C^m_{1|23,4,5,6} \\ &\quad + \big[s_{34} C_{1|234,5,6} + s_{24} C_{1|324,5,6} + (4 \leftrightarrow 5,6)\big]\Big)\,,\end{aligned} \tag{3.52}$$

which will imply, after integration over the loop momentum in section 4, that (3.49) gives rise to an equivalent version of the six-point pure-spinor correlator expression of [33][8].

The bosonic six-point one-loop amplitude of the open superstring was computed in the RNS formalism, see [23,24] for the parity even part and [34] for the parity odd part.

[8] To see the equivalence we note, in particular, equation (3.15) of [33].

3.4. Seven points

Following the general structure of the one-loop correlator presented in (2.33), the local seven-point correlator is proposed to be

$$\mathcal{K}_7(\ell) = \mathcal{K}_7^{\rm Lie}(\ell) + \mathcal{K}_7^{\rm Y}(\ell)\,, \tag{3.53}$$

where $\mathcal{K}_n^{\rm Lie}(\ell)$ is defined in (2.17) and the anomaly sector $\mathcal{K}_7^{\rm Y}(\ell)$ will be determined below. The unrefined contribution to $\mathcal{K}_n^{\rm Lie}(\ell) = \mathcal{K}_7^{(0)}(\ell) - \mathcal{K}_7^{(1)}(\ell)$ follows the pattern of (2.16),

$$\begin{aligned}\mathcal{K}_7^{(0)}(\ell) &= \frac{1}{3!} V_{A_1} T^{mnp}_{A_2,\ldots,A_7} \mathcal{Z}^{mnp}_{A_1,\ldots,A_7} + \big[1234567|A_1,\ldots,A_7\big]\\ &+ \frac{1}{2!} V_{A_1} T^{mn}_{A_2,\ldots,A_6} \mathcal{Z}^{mn}_{A_1,\ldots,A_6} + \big[1234567|A_1,\ldots,A_6\big]\\ &+ V_{A_1} T^{m}_{A_2,\ldots,A_5} \mathcal{Z}^{m}_{A_1,\ldots,A_5} + \big[1234567|A_1,\ldots,A_5\big]\\ &+ V_{A_1} T_{A_2,\ldots,A_4} \mathcal{Z}_{A_1,\ldots,A_4} + \big[1234567|A_1,\ldots,A_4\big]\,,\end{aligned} \tag{3.54}$$

with a total number of terms given by $\left[{7 \atop 7}\right] + \left[{7 \atop 6}\right] + \left[{7 \atop 5}\right] + \left[{7 \atop 4}\right] = 1+21+175+735 = 932$ (see its explicit expansion in (A.6)). The worldsheet functions entering (3.54) and the subsequent equations are determined from their monodromy variations. The solutions for the three topologies of scalar $\mathcal{Z}$-functions, the two topologies of vectorial ones and the tensorial ones can be found in (II.4.40), (II.A.26), (II.A.29) and (II.A.30), respectively.

The above $\mathcal{K}_7^{(0)}(\ell)$ alone is not BRST invariant, and this fact motivates the introduction of refined contributions $\mathcal{K}_7^{(1)}(\ell)$ to (2.17). In fact, the general discussion of refined correlators $\mathcal{K}_n^{(d)}(\ell)$ in section 2.3.2 originated from the explicit findings of this example. The seven-point expression

$$\begin{aligned}\mathcal{K}_7^{(1)}(\ell) &= \big[V_{A_1} J^m_{A_2|A_3,\ldots,A_7} \mathcal{Z}^m_{A_2|A_1,A_3,\ldots,A_7} + (A_2 \leftrightarrow A_3,\ldots,A_7)\big] + \big[1234567|A_1,\ldots,A_7\big]\\ &\quad + \big[V_{A_1} J_{A_2|A_3,\ldots,A_6} \mathcal{Z}_{A_2|A_1,A_3,\ldots,A_6} + (A_2 \leftrightarrow A_3,\ldots,A_6)\big] + \big[1234567|A_1,\ldots,A_6\big]\\ &= V_1 J^m_{2|3,4,5,6,7} \mathcal{Z}^m_{2|1,3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\\ &\quad + \big[V_{12} J_{3|4,5,6,7} \mathcal{Z}_{3|12,4,5,6,7} + V_{13} J_{2|4,5,6,7} \mathcal{Z}_{2|13,4,5,6,7} + (2,3|2,3,\ldots,7)\big]\\ &\quad + \big[V_1 J_{23|4,5,6,7} \mathcal{Z}_{23|1,4,5,6,7} + (2,3|2,3,\ldots,7)\big]\\ &\quad + \big[(V_1 J_{2|34,5,6,7} \mathcal{Z}_{2|1,34,5,6,7} + {\rm cyc}(2,3,4)) + (2,3,4|2,3,\ldots,7)\big]\end{aligned} \tag{3.55}$$

with $5\left[{7 \atop 6}\right] + 6\left[{7 \atop 7}\right] = 105+6 = 111$ terms in total lines up with the general proposal (2.22) at refinement $d=1$. We have seen in (II.4.42) that the three topologies of refined functions appearing in (3.55) are simple combinations of

$$\begin{aligned}\mathcal{Z}_{12|3,4,5,6,7} &= \partial g^{(2)}_{12} + s_{12} g^{(1)}_{12} g^{(2)}_{12} - 3 s_{12} g^{(3)}_{12}\\ &\cong -3 s_{12} g^{(3)}_{12} + g^{(2)}_{12}(\ell\cdot k_2 + s_{23} g^{(1)}_{23} + s_{24} g^{(1)}_{24} + \ldots + s_{27} g^{(1)}_{27})\,.\end{aligned} \tag{3.56}$$

However, the sum of the $d = 0$ and $d = 1$ correlators in (3.54) and (3.55) is still not enough to yield a BRST-invariant seven-point correlator, see the discussion of $Q\mathcal{K}_n^{\rm Lie}(\ell)$ in section 2.3.4. This necessitates the additional purely anomalous contribution to (3.53) given by

$$\mathcal{K}_7^Y(\ell) = -\Delta_{1|2|3,4,5,6,7}\mathcal{Z}_{12|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\,, \tag{3.57}$$

where the anomalous superfield $\Delta_{1|2|3,\ldots,7}$ is defined in (I.5.35). By the arguments of section I.5.3.1, the components $\langle\Delta_{1|2|3,\ldots,7}\rangle$ cannot have any kinematic pole, so addition of (3.57) does not spoil the locality of the seven-point correlator (3.53).

Note that the representation of $\mathcal{Z}_{12|3,4,5,6,7}$ in the second line of (3.56) manifests that $\mathcal{K}_7(\ell)$ can be written without any derivatives $\partial g_{ij}^{(m)}$ or products $g_{ij}^{(1)} g_{ij}^{(2)}$ with coinciding arguments. This observation should play an important role for the transcendentality properties upon integration over z_j.

3.4.1. BRST invariance

In order to show that the full correlator (3.54) is BRST invariant, let us first consider its non-anomalous part, $Q\mathcal{K}_7^{\rm Lie}(\ell)$. This computation can be organized according to the ghost-number four products of superfields it generates; this general structure was anticipated in section 2.3.4 but it is instructive to see it again in this particular case:

$$\begin{aligned} Q\mathcal{K}_7^{\rm Lie} &= -\frac{1}{2}T^{(0,2)}_{1|2,3,4,5,6,7} - T^{(1,0)}_{1|2,3,4,5,6,7} \\ &\quad - \big[T^{(0,1)}_{12|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\big] - \big[T^{(0,1)}_{1|23,4,5,6,7} + (2,3|2,3,4,5,6,7)\big] \\ &\quad - \big[T^{(0,0)}_{123|4,5,6,7} + T^{(0,0)}_{132|4,5,6,7} + (2,3|2,3,4,5,6,7)\big] \\ &\quad - \big[T^{(0,0)}_{1|234,5,6,7} + T^{(0,0)}_{1|243,5,6,7} + (2,3,4|2,3,4,5,6,7)\big] \\ &\quad - \big[T^{(0,0)}_{12|34,5,6,7} + T^{(0,0)}_{13|24,5,6,7} + T^{(0,0)}_{14|23,5,6,7} + (2,3,4|2,3,4,5,6,7)\big] \\ &\quad - \big[T^{(0,0)}_{1|23,45,6,7} + T^{(0,0)}_{1|24,53,6,7} + T^{(0,0)}_{1|25,34,6,7} + (6,7|2,3,4,5,6,7)\big] \\ &\quad + Y^{(0,1)}_{1|2,3,4,5,6,7} + \big[Y^{(0,0)}_{12|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\big] \\ &\quad + \big[Y^{(0,0)}_{1|23,4,5,6,7} + (2,3|2,3,4,5,6,7)\big]\,, \end{aligned} \tag{3.58}$$

where, following (2.26), the non-anomalous building blocks are contained in

$$\begin{aligned} T^{(0,2)}_{1|2,3,4,5,6,7} &= V_1 V_2 T^{mn}_{3,\ldots,7}\Theta^{(0)\,mn}_{2|1,3,\ldots,7} + (2 \leftrightarrow 3,4,5,6,7)\,, \\ T^{(0,1)}_{12|3,4,5,6,7} &= V_{12} V_3 T^{m}_{4,\ldots,7}\Theta^{(0)\,m}_{3|12,4,\ldots,7} + (3 \leftrightarrow 4,5,6,7)\,, \end{aligned} \tag{3.59}$$

$$
\begin{aligned}
T^{(0,1)}_{1|23,4,5,6,7} &= V_1 V_{23} T^m_{4,\ldots,7} \Theta^{(0)\,m}_{23|1,4,\ldots,7} + (23 \leftrightarrow 4,5,6,7)\,,\\
T^{(0,0)}_{123|4,5,6,7} &= V_{123} V_4 T_{5,\ldots,7} \Theta^{(0)}_{4|123,5,\ldots,7} + (4 \leftrightarrow 5,6,7)\,,\\
T^{(0,0)}_{12|34,5,6,7} &= V_{12} V_{34} T_{5,\ldots,7} \Theta^{(0)}_{34|12,5,\ldots,7} + (34 \leftrightarrow 5,6,7)\,,\\
T^{(0,0)}_{1|234,5,6,7} &= V_1 V_{234} T_{5,\ldots,7} \Theta^{(0)}_{234|1,5,\ldots,7} + (234 \leftrightarrow 5,6,7)\,,\\
T^{(0,0)}_{1|23,45,6,7} &= V_1 V_{23} T_{45,6,7} \Theta^{(0)}_{23|1,45,6,7} + (23 \leftrightarrow 45,6,7)\,,\\
T^{(1,0)}_{1|2,3,4,5,6,7} &= V_1 V_2 \big[J_{3|4,\ldots,7} \Theta^{(1)}_{2|3|1,4,\ldots,7} + (3 \leftrightarrow 4,5,6,7) \big] + (2 \leftrightarrow 3,4,5,6,7)\,,
\end{aligned}
$$

while the anomalous building blocks are contained in $Y^{(d,r)}$ given by (2.27),

$$
\begin{aligned}
Y^{(0,1)}_{1|2,3,4,5,6,7} &= V_1 Y^m_{2,3,4,5,6,7}\, \Xi^{(0)\,m}_{1|2,3,\ldots,7} \\
Y^{(0,0)}_{12|3,4,5,6,7} &= V_{12} Y_{3,4,5,6,7}\, \Xi^{(0)}_{12|3,\ldots,7}\,,\\
Y^{(0,0)}_{1|23,4,5,6,7} &= V_1 Y_{23,4,5,6,7}\, \Xi^{(0)}_{1|23,4,\ldots,7}\,.
\end{aligned}
\tag{3.60}
$$

It is evident from the above permutations that the general compact expression (2.25) leads to involved combinatorics resulting in many terms present in the seven-point BRST variation (3.58), even when written using the shorthands $\Theta^{(d)}$ and $\Xi^{(d)}$ defined in (2.28) and (2.29). Fortunately, the analysis of the outcome is also greatly simplified by this very same organization, as it suffices to check only a handful of different *topologies* of $\Theta^{(d)}$ and $\Xi^{(d)}$ rather than all their permutations. In fact, it is straightforward to check that all $T^{(d,r)}$ terms above vanish due to

$$
\begin{aligned}
V_1 V_2 T^{mn}_{3,4,5,6,7} \Theta^{(0)\,mn}_{2|1,3,4,5,6,7} &\cong 0, & V_1 V_2 T^m_{34,5,6,7} \Theta^{(0)\,m}_{2|1,34,5,6,7} &\cong 0,\\
V_1 V_{23} T^m_{4,5,6,7} \Theta^{(0)\,m}_{23|1,4,5,6,7} &\cong 0, & V_1 V_2 T_{34,56,7} \Theta^{(0)}_{2|1,34,56,7} &\cong 0,\\
V_1 V_2 T_{345,6,7} \Theta^{(0)}_{2|1,345,6,7} &\cong 0, & V_1 V_{23} T_{45,6,7} \Theta^{(0)}_{23|1,45,6,7} &\cong 0,\\
V_1 V_{234} T_{5,6,7} \Theta^{(0)}_{234|1,5,6,7} &\cong 0, & V_1 V_2 J_{3|4,5,6,7} \Theta^{(1)}_{2|3|1,4,5,6,7} &\cong 0\,,
\end{aligned}
\tag{3.61}
$$

whose explicit expansions in terms of shuffle-symmetric functions $\mathcal{Z}$ can be found in the appendix C. The coefficients of V_{1A} with $A \neq \emptyset$ are just relabellings of the $\Theta^{(d)}$ in (3.61) and therefore vanish as well.

Using the results above the BRST variation of (3.54) is purely anomalous

$$
\begin{aligned}
Q\mathcal{K}^{\text{Lie}}_7(\ell) = {}& V_1 Y^m_{2,3,4,5,6,7}\, \Xi^{(0)\,m}_{1|2,3,4,5,6,7} \\
&+ V_{12} Y_{3,4,5,6,7}\, \Xi^{(0)}_{12|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\\
&+ V_1 Y_{23,4,5,6,7}\, \Xi^{(0)}_{1|23,4,5,6,7} + (2,3|2,3,4,5,6,7)\,,
\end{aligned}
\tag{3.62}
$$

and is written entirely using the linear combinations $\Xi^{(0)}$ of (2.29),

$$\begin{aligned}
\Xi^{(0)\,m}_{1|2,3,4,5,6,7} &= -\frac{1}{2}\mathcal{Z}^{mpp}_{1,2,3,4,5,6,7} + \big[\mathcal{Z}^{m}_{2|1,3,4,5,6,7} + (2\leftrightarrow 3,\ldots,7)\big] \cong -\mathcal{Z}^{m}_{1|2,3,4,5,6,7} \\
\Xi^{(0)}_{12|3,4,5,6,7} &= -\frac{1}{2}\mathcal{Z}^{pp}_{12,3,4,5,6,7} + \big[\mathcal{Z}_{3|12,4,5,6,7} + (3\leftrightarrow 4,5,6,7)\big] \cong -\mathcal{Z}_{12|3,4,5,6,7} \\
\Xi^{(0)}_{1|23,4,5,6,7} &= -\frac{1}{2}\mathcal{Z}^{pp}_{1,23,4,5,6,7} + \big[\mathcal{Z}_{23|1,4,5,6,7} + (23\leftrightarrow 4,5,6,7)\big] \cong -\mathcal{Z}_{1|23,4,5,6,7}\,.
\end{aligned} \tag{3.63}$$

Similar to (3.28), the $\cong$ symbol indicates that boundary terms w.r.t. τ have been discarded in the second step of each line. The rearrangements of the above sums have the same structure as the trace relations among non-refined and refined building blocks, see section I.4.4.4. As discussed in section II.5.3, the worldsheet functions found via the bootstrap method of section II.4.4.4 satisfy the *dual* trace relations exploited in (3.63), and (3.62) becomes,

$$\begin{aligned}
Q\mathcal{K}^{\rm Lie}_7(\ell) = &-V_1 Y^m_{2,3,4,5,6,7}\,\mathcal{Z}^m_{1|2,3,4,5,6,7} \\
&- V_{12}Y_{3,4,5,6,7}\,\mathcal{Z}_{12|3,4,5,6,7} + (2\leftrightarrow 3,4,5,6,7) \\
&- V_1 Y_{23,4,5,6,7}\,\mathcal{Z}_{1|23,4,5,6,7} + (2,3|2,3,4,5,6,7)\,.
\end{aligned} \tag{3.64}$$

By the relations (II.4.42) between the three topologies of refined $\mathcal{Z}$-functions, the BRST variation (3.64) can then be written as,

$$\begin{aligned}
Q\mathcal{K}^{\rm Lie}_7(\ell) = &\Big(k^m_2 V_1 Y^m_{2,3,4,5,6,7} + V_{21}Y_{3,4,5,6,7} + \big[V_1 Y_{23,4,5,6,7} + (3\leftrightarrow 4,5,6,7)\big]\Big)\mathcal{Z}_{12|3,4,5,6,7} \\
&+ (2\leftrightarrow 3,4,5,6,7)\,.
\end{aligned} \tag{3.65}$$

From (I.5.36) we recognize the terms inside the parenthesis in (3.65) as the BRST variation of $\Delta_{1|2|3,4,5,6,7}$, that is, the expression for $\mathcal{K}^Y_7(\ell)$ in (3.57) is tailored to cancel

$$Q\mathcal{K}^{\rm Lie}_7(\ell) = Q\Delta_{1|2|3,4,5,6,7}\mathcal{Z}_{12|3,4,5,6,7} + (2\leftrightarrow 3,4,5,6,7) = -Q\mathcal{K}^Y_7(\ell)\,. \tag{3.66}$$

Therefore the full correlator (3.54) is BRST invariant up to total derivatives,

$$Q\big(\mathcal{K}^{\rm Lie}_7(\ell) + \mathcal{K}^Y_7(\ell)\big) = Q\mathcal{K}_7(\ell) \cong 0\,. \tag{3.67}$$

Before showing that (3.54) is also monodromy invariant, it will be convenient to rewrite it using the pseudo-invariants of section I.5.2, as that will simplify the proof considerably.

3.4.1.1. The $C \cdot \mathcal{Z}$ representation: manifesting BRST invariance

Given that the correlator (3.54) is BRST invariant, it is rewarding to rewrite it in terms of BRST pseudo-invariants. This can be done following the same procedure applied in detail for the six-point correlator in subsection 3.3.1.1, so it will only be sketched here again; we rewrite $M_1 M^{m\ldots}_{A,B,\ldots} = C^{m\ldots}_{1|A,B,\ldots} + \cdots$ and $M_1 \mathcal{J}^{m\ldots}_{A,B,\ldots} = P^{m\ldots}_{1|A,B,\ldots} + \cdots$ and collect the terms containing a factor of M_{1P} with $P \neq \emptyset$. A long but straightforward analysis using integration-by-parts relations (2.30) for the $\mathcal{Z}$-functions shows that all terms proportional to M_{1P} vanish and we arrive at

$$
\begin{aligned}
\mathcal{K}_7(\ell) = {} & \frac{1}{6} C^{mnp}_{1|A_1,\ldots,A_6} Z^{(s),mnp}_{1,A_1,\ldots,A_6} + \big[234567|A_1,\ldots,A_6\big] \\
& + \frac{1}{2} C^{mn}_{1|A_1,\ldots,A_5} Z^{(s),mn}_{1,A_1,\ldots,A_5} + \big[234567|A_1,\ldots,A_5\big] \\
& + \ C^{m}_{1|A_1,\ldots,A_4} Z^{(s)m}_{1,A_1,\ldots,A_4} + \big[234567|A_1,\ldots,A_4\big] \\
& + \ C_{1|A_1,\ldots,A_3} Z^{(s)}_{1,A_1,\ldots,A_3} + \big[234567|A_1,\ldots,A_3\big] \\
& - \big[P^{m}_{1|A_1|A_2,\ldots,A_6} Z^{(s)\,m}_{A_1|1,A_2,\ldots,A_6} + (A_1 \leftrightarrow A_2,\ldots,A_6)\big] + \big[234567|A_1,\ldots,A_6\big] \\
& - \big[P_{1|A_1|A_2,\ldots,A_5} Z^{(s)}_{A_1|1,A_2,\ldots,A_5} + (A_1 \leftrightarrow A_2,\ldots,A_5)\big] + \big[234567|A_1,\ldots,A_5\big] \\
& - \Delta_{1|2|3,4,5,6,7} \mathcal{Z}_{12|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\,.
\end{aligned}
\tag{3.68}
$$

Note that only six legs participate in the Stirling permutations, and $Z^{(s)}_{\ldots}$ are defined in (II.4.22). To compute the BRST variation of (3.68) it will be convenient to recall that [22]

$$
\begin{aligned}
QP^{m}_{1|2|3,4,5,6,7} &= -\Gamma^{m}_{1|2,3,4,5,6,7}\,, \qquad & QC^{mnp}_{1|2,3,4,5,6,7} &= -\delta^{(mn}\Gamma^{p)}_{1|2,3,4,5,6,7} \\
QP_{1|23|4,5,6,7} &= -\Gamma_{1|23,4,5,6,7}\,, & QC^{mn}_{1|23,4,5,6,7} &= -\delta^{mn}\Gamma_{1|23,4,5,6,7} \\
QP_{1|2|34,5,6,7} &= -\Gamma_{1|2,34,5,6,7}\,, & QC^{m}_{1|A,B,C,D} &= QC_{1|A,B,C} = 0 \\
Q\Delta_{1|2|3,4,5,6,7} &= k_2^m \Gamma^{m}_{1|2,3,4,5,6,7} + \big[s_{23}\Gamma_{1|23,4,5,6,7} + (3 \leftrightarrow 4,5,6,7)\big]\,, &&
\end{aligned}
\tag{3.69}
$$

see (I.5.27) for the anomaly invariants $\Gamma_{1|\ldots}$. After straightforward algebra and using the trace relations (3.63) we obtain,

$$
\begin{aligned}
Q\mathcal{K}_7(\ell) = {} & \Gamma^{m}_{1|2,3,4,5,6,7}\big(\mathcal{Z}^{m}_{1|2,3,4,5,6,7} + \big[k_2^m \mathcal{Z}_{12|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\big]\big) \\
& - s_{23}\Gamma_{1|23,4,5,6,7}(\mathcal{Z}_{1|23,4,5,6,7} + \mathcal{Z}_{12|3,4,5,6,7} - \mathcal{Z}_{13|2,4,5,6,7}) + (2,3|2,3,4,5,6,7) \\
\cong {} & 0\,.
\end{aligned}
\tag{3.70}
$$

The linear combinations of worldsheet functions in (3.70) correspond to the BRST-exact anomalous kinematic factors displayed in section II.5.4 and, as we have seen in (II.5.26), they vanish up to total derivatives. Therefore, BRST invariance of the representation (3.68) is indeed confirmed.

3.4.2. Single-valuedness

We will take the manifestly BRST-invariant representation (3.68) of the seven-point correlator as a starting point to verify monodromy invariance. Using the monodromy variations of the seven-point $\mathcal{Z}$-functions discussed in section II.4.4.4 and in the appendix II.A, a long but straightforward calculation implies,

$$D\mathcal{K}_7(\ell) = \Omega_1 \delta\mathcal{K}_7^{(1)} + \cdots + \Omega_7 \delta\mathcal{K}_7^{(7)} \,, \tag{3.71}$$

where ($E^{(s)m\ldots}_{1|A,\ldots}$ is defined in (II.4.23))

$$\begin{aligned}\delta\mathcal{K}_7^{(1)} = {} & \frac{1}{2} k_1^m C^{mnp}_{1|A_2,\ldots,A_7} E^{(s)np}_{1|A_2,\ldots,A_7} + \big[234567|A_2,\ldots,A_7\big] \\ & + k_1^m C^{mn}_{1|A_2,\ldots,A_6} E^{(s)n}_{1|A_2,\ldots,A_6} + \big[234567|A_2,\ldots,A_6\big] \\ & + k_1^m C^{m}_{1|A_2,\ldots,A_5} E^{(s)}_{1|A_2,\ldots,A_5} + \big[234567|A_2,\ldots,A_5\big] \\ & - \big[k_1^m P^m_{1|2|3,4,5,6,7} + \Delta_{1|2|3,4,5,6,7}\big] E^{(s)}_{1|2|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\,.\end{aligned} \tag{3.72}$$

The other $\delta\mathcal{K}_7^{(j)}$ for $j = 2,\ldots,7$ can be obtained from $\delta\mathcal{K}_7^{(1)}$ by relabeling of $1 \leftrightarrow j$ in both the kinematics and GEIs of (3.72). To verify this last statement one uses the change-of-basis identities for pseudo-invariants derived in [22]. This is because the relabeling of $\delta\mathcal{K}_7^{(j)}$ for $j = 2,\ldots,7$ involves pseudo-invariants outside of the *canonical* basis $C^{m\ldots}_{1|\ldots}$ (i.e. $C^{m\ldots}_{j|\ldots}$ with $j \neq 1$), whereas the monodromy variation of (3.71) obviously contains only elements in the canonical basis. See the analogous six-point analysis described in section 3.3.2 for more details.

The appearance of momentum contractions in (3.72) signals the need to use the BRST cohomology identities derived in [22] and reviewed in section I.5.4. In addition, one also needs their elliptic dual identities involving momentum contractions of $k_1^m E^{m\ldots}$ (cf. section II.5.1) and the trace identity

$$\frac{1}{2} E^{mm}_{1|2|3,4,5,6,7} = \big[E_{1|2|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\big] + 2\pi i \frac{\partial}{\partial\tau} \log \mathcal{I}_7(\ell)\,. \tag{3.73}$$

After long but straightforward manipulations one finally concludes that the monodromy variation (3.72) is BRST-exact and given by

$$\begin{aligned}\delta\mathcal{K}_7^{(1)} = {} & \frac{1}{2} \Delta^{mn}_{1|A_2,\ldots,A_7} E^{(s)mn}_{1|A_2,\ldots,A_6} + \big[234567|A_2,\ldots,A_7\big] \\ & + \Delta^{m}_{1|A_2,\ldots,A_6} E^{(s)m}_{1|A_2,\ldots,A_6} + \big[234567|A_2,\ldots,A_6\big] \\ & + \Delta_{1|A_2,\ldots,A_5} E^{(s)}_{1|A_2,\ldots,A_5} + \big[234567|A_2,\ldots,A_5\big] \cong 0\,.\end{aligned} \tag{3.74}$$

It is crucial to note that only *unrefined* building blocks $\Delta_{1|\ldots}$ arise, whose BRST exactness is discussed in section I.5.3. Since the other $\delta\mathcal{K}_7^{(j)}$ are relabellings of (3.74), it follows that the complete monodromy variation $D\mathcal{K}_7(\ell)$ in (3.71) is BRST-exact and therefore vanishes in the cohomology; $D\mathcal{K}_7(\ell) \cong 0$.

3.4.2.1. The $C \cdot E$ representation: manifesting BRST invariance & single-valuedness

Having derived the $C \cdot \mathcal{Z}$ representation and shown that it is single valued, we can re-express it to manifest both BRST and monodromy invariance. We proceed similarly as in the six-point case by inserting $\mathcal{Z}^{m\ldots}_{1,A,B,\ldots} = E^{m\ldots}_{1|A,B,\ldots} + \cdots$ into the $C \cdot \mathcal{Z}$ representation (3.68) and using a long sequence of BRST cohomology identities described in [22]. Doing this leads to a manifestly BRST-invariant and single-valued expression neatly summarized by the following Stirling permutation sums

$$
\begin{aligned}
\mathcal{K}_7(\ell) = &\sum_{r=0}^{3} C^{m_1\ldots m_r}_{1|A_1,\ldots,A_{r+3}} E^{(s)\,m_1\ldots m_r}_{1|A_1,\ldots,A_{r+3}} + \big[234567|A_1,\ldots,A_{r+3}\big] \\
&- \sum_{r=0}^{1} \big[P^{m_1\ldots m_r}_{1|A_1|A_2,\ldots,A_{r+5}} E^{(s)\,m_1\ldots m_r}_{1|A_1|A_2\ldots,A_{r+5}} + (A_1 \leftrightarrow A_2,\ldots,A_{r+5})\big] + \big[2\ldots7|A_1,\ldots,A_{r+5}\big]\,,
\end{aligned}
\tag{3.75}
$$

where $E^{(s)m\ldots}_{1|A,\ldots}$ is defined in (II.4.23), and the terms proportional to $\Delta_{1|2|3,4,5,6,7}$ drop by the trace relations (3.63). Expanding the Stirling permutation sums in (3.75) yields

$$
\begin{aligned}
\mathcal{K}_7(\ell) = &\frac{1}{6} C^{mnp}_{1|2,3,4,5,6,7} E^{(s)mnp}_{1|2,3,4,5,6,7} \\
&+ \frac{1}{2} C^{mn}_{1|23,4,5,6,7} E^{(s)mn}_{1|23,4,5,6,7} + (2,3|2,3,4,5,6,7) \\
&+ \big[C^{m}_{1|234,5,6,7} E^{(s)m}_{1|234,5,6,7} + C^{m}_{1|243,5,6,7} E^{(s)m}_{1|243,5,6,7}\big] + (2,3,4|2,3,4,5,6,7) \\
&+ \big[C^{m}_{1|23,45,6,7} E^{(s)m}_{1|23,45,6,7} + \mathrm{cyc}(2,3,4)\big] + (6,7|2,3,4,5,6,7) \\
&+ \big[C_{1|2345,6,7} E^{(s)}_{1|2345,6,7} + \mathrm{perm}(3,4,5)\big] + (2,3,4,5|2,3,4,5,6,7) \\
&+ \big[C_{1|234,56,7} E^{(s)}_{1|234,56,7} + C_{1|243,56,7} E^{(s)}_{1|243,56,7} + \mathrm{cyc}(5,6,7)\big] + (2,3,4|2,3,4,5,6,7) \\
&+ \big[C_{1|23,45,67} E^{(s)}_{1|23,45,67} + \mathrm{cyc}(4,5,6)\big] + (3 \leftrightarrow 4,5,6,7) \\
&- P^{m}_{1|2|3,4,5,6,7} E^{(s)m}_{1|2|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7) \\
&- P_{1|23|4,5,6,7} E^{(s)}_{1|23|4,5,6,7} + (2,3|2,3,4,5,6,7) \\
&- \big[P_{1|2|34,5,6,7} E^{(s)}_{1|2|34,5,6,7} + \mathrm{cyc}(2,3,4)\big] + (2,3,4|2,3,4,5,6,7)\,,
\end{aligned}
\tag{3.76}
$$

for a total number of 326 terms with pseudo-invariants C and 81 terms with P. This is the double-copy expression for the seven-point correlator implicitly proposed in [10]. Similar to (3.68), only six legs participate in the Stirling permutations, but there is no analogue of the terms $\Delta_{1|2|3,\ldots,7}\mathcal{Z}_{12|3,\ldots,7}$ in the last line of the $C \cdot \mathcal{Z}$ representation.

3.4.2.2. The $T \cdot E$ representation: manifesting locality & single-valuedness

From the $C \cdot E$ representation (3.76) one can derive a manifestly local and single-valued representation following the same ideas as explained for the six-point case in section 3.3.2.2. The end result is given by,

$$\begin{aligned}
\mathcal{K}_7(\ell) &= \frac{1}{6} V_1 T^{mnp}_{2,3,\ldots,7} E^{mnp}_{1|2,3,\ldots,7} \qquad (3.77)\\
&+ \frac{1}{2} V_1 T^{mn}_{23,4,5,6,7} E^{mn}_{1|23,4,5,6,7} + (2,3|2,3,4,5,6,7)\\
&+ \big[V_1 T^m_{234,5,6,7} E^m_{1|234,5,6,7} + V_1 T^m_{243,5,6,7} E^m_{1|243,5,6,7} \big] + (2,3,4|2,3,4,5,6,7)\\
&+ \big[V_1 T^m_{23,45,6,7} E^m_{1|23,45,6,7} + \mathrm{cyc}(2,3,4) \big] + (6,7|2,3,4,5,6,7)\\
&+ \big[V_1 T_{2345,6,7} E_{1|2345,6,7} + \mathrm{perm}(3,4,5) \big] + (2,3,4,5|2,3,4,5)\\
&+ \big[V_1 T_{234,56,7} E_{1|234,56,7} + V_1 T_{243,56,7} E_{1|243,56,7} + \mathrm{cyc}(5,6,7) \big] + (2,3,4|2,3,4,5,6,7)\\
&+ \big[V_1 T_{23,45,67} E_{1|23,45,67} + \mathrm{cyc}(4,5,6) \big] + (3 \leftrightarrow 4,5,6,7)\\
&- V_1 J^m_{2|3,4,5,6,7} E^m_{1|2|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\\
&- V_1 J_{23|4,5,6,7} E_{1|23|4,5,6,7} + (2,3|2,3,4,5,6,7)\\
&- \big[V_1 J_{2|34,5,6,7} E_{1|2|34,5,6,7} + \mathrm{cyc}(2,3,4) \big] + (2,3,4|2,3,4,5,6,7)\,.
\end{aligned}$$

Similar to the six-point case (3.47), this $T \cdot E$ representation is related to the $C \cdot \mathcal{Z}$ representation through the duality between kinematics and worldsheet functions, up to the fact that (3.77) does not exhibit any dual of the terms $\Delta_{1|2|3,\ldots,7} \mathcal{Z}_{12|3,\ldots,7}$ in (3.68). Moreover, the combinatorial structure of (3.77) is identical to that of the $C \cdot E$ representation (3.76). In addition, proving BRST invariance of the representation (3.77) requires the same elliptic worldsheet identities used to generate (3.77) from (3.76).

3.5. Eight points

Following the general structure of one-loop correlators presented in (2.17) and (2.33), the manifestly local Lie-series part of the eight-point correlator is proposed to be

$$\mathcal{K}_8^{\mathrm{Lie}}(\ell) \equiv \mathcal{K}_8^{(0)}(\ell) - \mathcal{K}_8^{(1)}(\ell) + \mathcal{K}_8^{(2)}(\ell)\,, \qquad (3.78)$$

which will later receive a purely anomalous correction $\mathcal{K}_8^Y(\ell)$. The unrefined part with $d = 0$ follows the general pattern indicated in (2.16),

$$\mathcal{K}_8^{(0)}(\ell) = \frac{1}{4!} V_{A_1} T^{mnpq}_{A_2,\ldots,A_8} \mathcal{Z}^{mnpq}_{A_1,\ldots,A_8} + \big[12345678|A_1,\ldots,A_8 \big] \qquad (3.79)$$

$$
\begin{aligned}
&+\frac{1}{3!}V_{A_1}T^{mnp}_{A_2,\ldots,A_7}\mathcal{Z}^{mnp}_{A_1,\ldots,A_7} + \big[12345678|A_1,\ldots,A_7\big]\\
&+\frac{1}{2!}V_{A_1}T^{mn}_{A_2,\ldots,A_6}\mathcal{Z}^{mn}_{A_1,\ldots,A_6} + \big[12345678|A_1,\ldots,A_6\big]\\
&+\ \ V_{A_1}T^{m}_{A_2,\ldots,A_5}\mathcal{Z}^{m}_{A_1,\ldots,A_5} + \big[12345678|A_1,\ldots,A_5\big]\\
&+\ \ V_{A_1}T_{A_2,\ldots,A_4}\mathcal{Z}_{A_1,\ldots,A_4} + \big[12345678|A_1,\ldots,A_4\big]\,,
\end{aligned}
$$

and contains $\left[{8 \atop 8}\right]+\left[{8 \atop 7}\right]+\left[{8 \atop 6}\right]+\left[{8 \atop 5}\right]+\left[{8 \atop 4}\right] = 1+28+322+1960+6769 = 9080$ terms, where we recall that $\left[{n \atop p}\right]$ denotes the Stirling cycle number. The correlator (3.78) also contains $7\left[{8 \atop 8}\right]+6\left[{8 \atop 7}\right]+5\left[{8 \atop 6}\right] = 7+168+1610 = 1785$ terms with refinement $d=1$,

$$
\begin{aligned}
\mathcal{K}^{(1)}_8(\ell) = \frac{1}{2!}&\big[V_{A_1}J^{mn}_{A_2|A_3,\ldots,A_8}\mathcal{Z}^{mn}_{A_2|A_1,A_3,\ldots,A_8} + (A_2 \leftrightarrow A_3,\ldots,A_8)\big] + [1\ldots 8|A_1,\ldots,A_8]\\
&+\big[V_{A_1}J^{m}_{A_2|A_3,\ldots,A_7}\mathcal{Z}^{m}_{A_2|A_1,A_3,\ldots,A_7} + (A_2 \leftrightarrow A_3,\ldots,A_7)\big] + [1\ldots 8|A_1,\ldots,A_7]\\
&+\big[V_{A_1}J_{A_2|A_3,\ldots,A_6}\mathcal{Z}_{A_2|A_1,A_3,\ldots,A_6} + (A_2 \leftrightarrow A_3,\ldots,A_6)\big] + [1\ldots 8|A_1,\ldots,A_6]\,,
\end{aligned}
\tag{3.80}
$$

and $\binom{7}{2} = 21$ terms with refinement $d=2$,

$$
\mathcal{K}^{(2)}_8(\ell) = V_1 J_{2,3|4,5,6,7,8}\mathcal{Z}_{2,3|1,4,5,6,7,8} + (2,3|2,3,4,5,6,7,8)\,. \tag{3.81}
$$

The worldsheet functions appearing in the expansions above can be obtained solving the system of monodromy variations described in section II.4.4, and their explicit expressions can be found in the appendix II.A.

One can also show using the trace relations among local building blocks that the overall correlator (3.78) is unchanged when using trace-satisfying worldsheet functions $\hat{\mathcal{Z}}$ defined in (II.5.19) instead of the naive ones from the solutions to the monodromy variations,

$$
\hat{\mathcal{K}}^{\mathrm{Lie}}_8(\ell) = \mathcal{K}^{\mathrm{Lie}}_8(\ell)\,. \tag{3.82}
$$

3.5.1. BRST variation

The computation of $Q\mathcal{K}^{\mathrm{Lie}}_8$ can be performed in a straightforward fashion using the variations of the local superfields given in section I.4 and is given by the general identity (2.25) with $n=8$ (see its $n=7$ instance in (3.58)). To check whether the correlator is BRST invariant, it suffices to analyze a few distinct linear combinations of worldsheet functions encompassed in the definitions of $\Theta^{(d)}$ and $\Xi^{(d)}$ in (2.28) and (2.29).

One can show that the eight-point $\mathcal{Z}$-functions derived via the bootstrap approach (cf. appendix II.A.3) imply the vanishing of all $\Theta^{(d)}$ topologies of worldsheet functions, see (2.30). For some of these topologies, more than ten $\mathcal{Z}$-functions conspire in a highly non-trivial way to yield total Koba–Nielsen derivatives that integrate to zero. The full list of inequivalent topologies can be found in the appendix C.

However, the combinations $\Xi^{(d)}$ defined in (2.29) do not vanish when the solutions to the monodromy equations are plugged in. For instance, the contributions to $Q\mathcal{K}_8^{\rm Lie}$ proportional to $V_1 Y_{23,45,6,7,8}$ are given by

$$\begin{aligned}\Xi^{(0)}_{23|1,45,6,7,8} &\equiv -\frac{1}{2}\mathcal{Z}^{pp}_{23,1,45,6,7,8} + \big[\mathcal{Z}_{23|1,45,6,7,8} + (23 \leftrightarrow 45,6,7,8)\big] \\ &= -\mathcal{Z}_{1|23,45,6,7,8} + R_{1,23,45,6,7,8} \\ &= -\hat{\mathcal{Z}}_{1|23,45,6,7,8}\,,\end{aligned} \tag{3.83}$$

where the R-functions where defined in (II.5.18) are proportional to G_4 – they will be written down below in (3.88) for convenience – and we used the definition (II.5.19) of $\hat{\mathcal{Z}}$ in passing to the last line. The analysis for the other eight-point building blocks is similar,

$$\Xi^{(0)\,m_1 m_2\ldots m_r}_{A_1|B_1,\ldots,B_{r+6}} \cong -\hat{\mathcal{Z}}_{A_1|B_1,\ldots,B_{r+6}}\,, \qquad \Xi^{(1)\,m_1 m_2\ldots m_r}_{A_1|A_2|B_1,\ldots,B_{r+6}} \cong \hat{\mathcal{Z}}_{A_1,A_2|B_1,\ldots,B_{r+6}}\,, \qquad n=8\,, \tag{3.84}$$

and the BRST variation of (3.78) becomes

$$\begin{aligned}Q\mathcal{K}_8^{\rm Lie}(\ell) &= -\frac{1}{2}V_{A_1}Y^{mn}_{A_2,\ldots,A_8}\,\hat{\mathcal{Z}}^{mn}_{A_1|A_2,\ldots,A_8} + [12\ldots8|A_1,\ldots,A_8] \\ &\quad - V_{A_1}Y^{m}_{A_2,\ldots,A_7}\,\hat{\mathcal{Z}}^{m}_{A_1|A_2,\ldots,A_7} + [12\ldots8|A_1,\ldots,A_7] \\ &\quad - V_{A_1}Y_{A_2,\ldots,A_6}\,\hat{\mathcal{Z}}_{A_1|A_2,\ldots,A_6} + [12\ldots8|A_1,\ldots,A_6] \\ &\quad + \big[V_1 Y_{2|3,4,5,6,7,8}\,\hat{\mathcal{Z}}_{1,2|3,4,5,6,7,8} + (2\leftrightarrow 3,4,5,6,7,8)\big]\,,\end{aligned} \tag{3.85}$$

which can be written more explicitly as

$$\begin{aligned}Q\mathcal{K}_8^{\rm Lie}(\ell) = &-\frac{1}{2}V_1 Y^{mn}_{2,3,4,5,6,7,8}\,\hat{\mathcal{Z}}^{mn}_{1|2,3,4,5,6,7,8} \\ &- \big[V_1 Y^m_{23,4,5,6,7,8}\,\hat{\mathcal{Z}}^m_{1|23,4,5,6,7,8} + (2,3|2,3,4,5,6,7,8)\big] \\ &- \big[V_{12} Y^m_{3,4,5,6,7,8}\,\hat{\mathcal{Z}}^m_{12|3,4,5,6,7,8} + (2\leftrightarrow 3,4,5,6,7,8)\big] \\ &- \big[V_{123} Y_{4,5,6,7,8}\,\hat{\mathcal{Z}}_{123|4,5,6,7,8} + V_{132} Y_{4,5,6,7,8}\,\hat{\mathcal{Z}}_{132|4,5,6,7,8} + (2,3|2,3,4,5,6,7,8)\big] \\ &- \big[V_1 Y_{234,5,6,7,8}\,\hat{\mathcal{Z}}_{1|234,5,6,7,8} + V_1 Y_{243,5,6,7,8}\,\hat{\mathcal{Z}}_{1|243,5,6,7,8} + (2,3,4|2,3,4,5,6,7,8)\big] \\ &- \big[\big(V_{12} Y_{34,5,6,7,8}\,\hat{\mathcal{Z}}_{12|34,5,6,7,8} + \mathrm{cyc}(2,3,4)\big) + (2,3,4|2,3,4,5,6,7,8)\big] \\ &- \big[\big(V_1 Y_{2,3,4,56,78}\,\hat{\mathcal{Z}}_{1|2,3,4,56,78} + \mathrm{cyc}(5,6,7)\big) + (2,3,4|2,3,4,5,6,7,8)\big] \\ &+ \big[V_1 Y_{2|3,4,5,6,7,8}\,\hat{\mathcal{Z}}_{1,2|3,4,5,6,7,8} + (2\leftrightarrow 3,4,5,6,7,8)\big]\,.\end{aligned} \tag{3.86}$$

In terms of the undeformed functions, the BRST variation is given by

$$
\begin{aligned}
Q\mathcal{K}_8^{\rm Lie}(\ell) = &-\frac{1}{2}V_{A_1}Y^{mn}_{A_2,\ldots,A_8}\left(\mathcal{Z}^{mn}_{A_1|A_2,\ldots,A_8} - R^{mn}_{A_1,\ldots,A_8}\right) + [12\ldots 8|A_1,\ldots,A_8] \\
&- V_{A_1}Y^{m}_{A_2,\ldots,A_7}\left(\mathcal{Z}^{m}_{A_1|A_2,\ldots,A_7} - R^{m}_{A_1,\ldots,A_7}\right) + [12\ldots 8|A_1,\ldots,A_7] \\
&- V_{A_1}Y_{A_2,\ldots,A_6}\left(\mathcal{Z}_{A_1|A_2,\ldots,A_6} - R_{A_1,\ldots,A_6}\right) + [12\ldots 8|A_1,\ldots,A_6] \\
&+ \left[V_1 Y_{2|3,4,5,6,7,8}\,\mathcal{Z}_{1,2|3,4,5,6,7,8} + (2\leftrightarrow 3,4,5,6,7,8)\right],
\end{aligned} \tag{3.87}
$$

where the R-functions are all proportional to G_4,

$$
\begin{aligned}
R_{12,34,5,6,7,8} &= 3\mathrm{G}_4\big(s_{13}-s_{14}-s_{23}+s_{24}\big)\,, \\
R_{123,4,5,6,7,8} &= 3\mathrm{G}_4\big(s_{12}-2s_{13}+s_{23}\big)\,, \\
R^m_{12,3,4,5,6,7,8} &= 3\mathrm{G}_4\big(s_{12}(k_2^m-k_1^m) + \left[k_3^m(s_{13}-s_{23}) + (3\leftrightarrow 4,5,6,7,8)\right]\big)\,, \\
R^{mn}_{1,2,3,4,5,6,7,8} &= 3\mathrm{G}_4 k_1^{(m}k_2^{n)}s_{12} + (1,2|1,2,\ldots,8)\,.
\end{aligned} \tag{3.88}
$$

Note that the trace relation $Y^{mm}_{2,3,\ldots,8} = 2Y_{2|3,\ldots,8} + (2\leftrightarrow 3,\ldots,8)$ implies that the contributions of $R^{aa}_{1,2,\ldots,8}$ in (II.5.21) and (II.5.22) cancel. The remaining task is to compensate the leftover variation (3.87) by adding an anomaly sector $\mathcal{K}_8^Y(\ell)$ to the eight-point correlator.

3.5.2. Purely anomalous sector

The strategy to cancel the terms (3.86) in a bid to achieve BRST invariance is similar to the seven-point case; we propose to add a purely anomalous contribution to the eight-point correlator (3.78),

$$
\mathcal{K}_8(\ell) = \mathcal{K}_8^{\rm Lie}(\ell) + \mathcal{K}_8^Y(\ell)\,. \tag{3.89}
$$

By analogy with the expression (3.57) for $\mathcal{K}_7^Y(\ell)$, we start from an ansatz comprising anomalous Δ superfields of (I.C.1) and some unknown worldsheet functions U,

$$
\begin{aligned}
\mathcal{K}_8^Y(\ell) = &\left[\Delta^m_{1|2|3,4,\ldots,8}\,U^m_{1|2|3,4,\ldots,8} + (2\leftrightarrow 3,4,\ldots,8)\right] \\
&+ \left[\Delta_{1|23|4,\ldots,8}\,s_{23}U_{1|23|4,\ldots,8} + (2,3|2,3,4,\ldots,8)\right] \\
&+ \left[\big(\Delta_{1|2|34,\ldots,8}\,s_{34}U_{1|2|34,\ldots,8} + \mathrm{cyc}(2,3,4)\big) + (2,3,4|2,3,4,5,6,7)\right].
\end{aligned} \tag{3.90}
$$

In fact, (3.90) is the most general linear combination of anomalous building blocks such that their BRST variations are expressible in terms of $V_{1A}Y^{m_1\cdots}_{\cdots}$ rather than $V_B Y^{m_1\cdots}_{1\ldots}$ with $1\notin B$. Any other combination of $Y^{m_1\cdots}_{\cdots}$ in (3.90) would lead to terms $V_B,\ 1\notin B$ in $Q\mathcal{K}_8^Y(\ell)$ that cannot be cancelled by (3.87). In contrast to their seven-point counterpart

$\Delta_{1|2|3,4,5,6,7}$, the eight-point instances of the Δ superfields exhibit kinematic poles (cf. appendix I.C), so (3.90) amounts to a mild violating of manifest locality.

In order to determine the U-functions in (3.90) we start by noting that $Q^2\mathcal{K}_8^{\rm Lie}(\ell)=0$ implies that $Q\mathcal{K}_8^{\rm Lie}(\ell)$ is BRST closed. Therefore all the ghost-number-four superfields $V_A Y^{m_1m_2\ldots}_{B_1,B_2,\ldots}$ in (3.87) must combine to ghost-number four BRST invariants given by Γ defined in [22] (also see the alternative algorithm in appendix I.A.2 for the unrefined cases). This can be seen by rewriting the local superfields in (3.86) in terms of Berends–Giele currents using (I.5.1) followed by $M_A\mathcal{Y}^{m_1m_2\ldots}_{B_1,B_2,\ldots}\to\delta_{|A|,1}\Gamma^{m_1m_2\ldots}_{A|B_1,B_2,\ldots}$ where

$$
\begin{aligned}
\Gamma^{mn}_{1|2,3,4,5,6,7,8} &= M_1\mathcal{Y}^{mn}_{2,\ldots,8} + \big[k_2^m M_{12}\mathcal{Y}^n_{3,4,5,6,7,8} + k_2^n M_{12}\mathcal{Y}^m_{3,4,5,6,7,8} + (2\leftrightarrow 3,\ldots,8)\big] \\
&\quad - \big[(k_2^m k_3^n + k_2^n k_3^m)M_{312}\mathcal{Y}_{4,5,6,7,8} + (2,3|2,3,4,5,6,7,8)\big] \qquad (3.91)\\
\Gamma^m_{1|23,4,5,6,7,8} &= M_1\mathcal{Y}^m_{23,4,5,6,7,8} + \big[M_{12}\mathcal{Y}^m_{3,4,5,6,7,8} + M_{123}\mathcal{Y}_{4,5,6,7,8}k_3^m - (2\leftrightarrow 3)\big] \\
&\quad + \big[k_4^m\big(M_{14}\mathcal{Y}_{23,5,6,7,8} - M_{214}\mathcal{Y}_{3,5,6,7,8} + M_{314}\mathcal{Y}_{2,5,6,7,8}\big) + (4\leftrightarrow 5,6,7,8)\big] \\
\Gamma_{1|234,5,6,7,8} &= M_1\mathcal{Y}_{234,5,6,7,8} + M_{12}\mathcal{Y}_{34,5,6,7,8} + M_{123}\mathcal{Y}_{4,5,6,7,8} \\
&\quad + M_{214}\mathcal{Y}_{3,5,6,7,8} - M_{14}\mathcal{Y}_{23,5,6,7,8} + M_{143}\mathcal{Y}_{2,5,6,7,8} \\
\Gamma_{1|23,45,6,7,8} &= M_1\mathcal{Y}_{23,45,6,7,8} + \big[M_{12}\mathcal{Y}_{45,3,6,7,8} - (2\leftrightarrow 3)\big] \\
&\quad + \big[M_{14}\mathcal{Y}_{23,5,6,7,8} + M_{215}\mathcal{Y}_{3,4,6,7,8} - M_{315}\mathcal{Y}_{2,4,6,7,8} - (4\leftrightarrow 5)\big] \\
\Gamma_{1|2|3,4,5,6,7,8} &= M_1\mathcal{Y}_{2|3,4,\ldots,8} + M_{12}k_2^m\mathcal{Y}^m_{3,4,\ldots,8} + \big[s_{23}M_{123}\mathcal{Y}_{4,\ldots,8} + (3\leftrightarrow 4,\ldots,8)\big]\,.
\end{aligned}
$$

Under these transformations, it is possible to verify that (3.86) is identical to

$$
\begin{aligned}
Q\mathcal{K}_8^{\rm Lie}(\ell) = &-\frac{1}{2}\Gamma^{mn}_{A_1|A_2,\ldots,A_8}\,Z^{(s),mn}_{A_1|A_2,\ldots,A_8} + [12\ldots8|A_1,\ldots,A_8] \qquad (3.92)\\
&-\Gamma^m_{A_1|A_2,\ldots,A_7}\,Z^{(s),m}_{A_1|A_2,\ldots,A_7} + [12\ldots8|A_1,\ldots,A_7] \\
&-\Gamma_{A_1|A_2,\ldots,A_6}\,Z^{(s)}_{A_1|A_2,\ldots,A_6} + [12\ldots8|A_1,\ldots,A_6] \\
&+\big[\Gamma_{1|2|3,4,5,6,7,8}\,Z^{(s)}_{1,2|3,4,5,6,7,8} + (2\leftrightarrow 3,4,5,6,7,8)\big] \\
&+\frac{1}{2}V_{A_1}Y^{mn}_{A_2,\ldots,A_8}\,R^{mn}_{A_1,\ldots,A_8} + [12\ldots8|A_1,\ldots,A_8] \\
&+V_{A_1}Y^m_{A_2,\ldots,A_7}\,R^m_{A_1,\ldots,A_7} + [12\ldots8|A_1,\ldots,A_7] \\
&+V_{A_1}Y_{A_2,\ldots,A_6}\,R_{A_1,\ldots,A_6} + [12\ldots8|A_1,\ldots,A_6]\,,
\end{aligned}
$$

where $Z^{(s)}$ is defined in (II.4.22). It is beneficial to rewrite (3.86) in this way because the Q-variation of the anomalous correlator (3.90) takes the the same form once we insert the expressions for $Q\Delta$ in (I.5.39):

$$
\begin{aligned}
Q\mathcal{K}_8^Y(\ell) &= \Gamma^{mn}_{1|2,3,4,5,6,7,8}\left[k_2^m U^n_{1|2|3,4,5,6,7,8} + (2\leftrightarrow 3,4,5,6,7,8)\right] \\
&\quad + \big[s_{23}\Gamma^m_{1|23,4,\ldots,8}\big(U^m_{1|2|3,\ldots,8} - U^m_{1|3|2,\ldots,8} \\
&\qquad + \left[k_{23}^m U_{1|23|4,\ldots,8} + (23\leftrightarrow 4,5,6,7,8)\right]\big) + (2,3|2,3,4,5,6,7,8)\big] \\
&\quad + \big[s_{23}s_{45}\Gamma_{1|23,45,6,7,8}\big(U_{1|2|3,45,6,7,8} - U_{1|3|2,45,6,7,8} \\
&\qquad + U_{1|4|5,23,6,7,8} - U_{1|5|4,23,6,7,8}\big) + (2,3|4,5|2,3,4,5,6,7,8)\big] \\
&\quad + \Big[\Gamma_{1|234,5,6,7,8}\big[s_{23}s_{24}\big(U_{1|23|4,\ldots,8} - U_{1|4|23,\ldots,8} - U_{1|3|24,\ldots,8} + U_{1|24|3,\ldots,8}\big) \\
&\qquad\qquad + s_{23}s_{34}\big(U_{1|23|4,\ldots,8} - U_{1|4|23,\ldots,8} + U_{1|2|34,\ldots,8} - U_{1|34|2,\ldots,8}\big)\big] \\
&\qquad + \Gamma_{1|243,5,6,7,8}\big[s_{23}s_{24}\big(U_{1|24|3,\ldots,8} - U_{1|3|24,\ldots,8} - U_{1|4|23,\ldots,8} + U_{1|23|4,\ldots,8}\big) \\
&\qquad\qquad + s_{24}s_{34}\big(U_{1|24|3,\ldots,8} - U_{1|3|24,\ldots,8} + U_{1|2|43,\ldots,8} - U_{1|43|2,\ldots,8}\big)\big] \\
&\qquad + (2,3,4|2,3,4,5,6,7,8)\Big] \\
&\quad - \big[\Gamma_{1|2|3,4,5,6,7,8}\big(k_2^m U^m_{1|2|3,4,5,6,7,8} + s_{23}U_{1|23|4,5,6,7,8} + s_{24}U_{1|24|3,5,6,7,8} \\
&\qquad + \cdots + s_{28}U_{1|28|3,4,5,6,7}\big) + (2\leftrightarrow 3,4,5,6,7,8)\big]\,.
\end{aligned}
\tag{3.93}
$$

As we will see, the functions U in the anomalous correlator (3.90) can be chosen such as to cancel all Γ terms from (3.92). This can be achieved provided the following equations hold:

$$
0 \cong \Big(\mathcal{Z}^{mn}_{1|2,3,4,5,6,7,8} - \left[k_2^{(m}U^{n)}_{1|2|3,4,5,6,7,8} + (2\leftrightarrow 3,\ldots,8)\right]\Big)\Gamma^{mn}_{1|2,3,4,5,6,7,8} \tag{3.94}
$$

$$
0 \cong \Big(\mathcal{Z}^m_{1|23,4,\ldots,8} - U^m_{1|2|3,\ldots,8} + U^m_{1|3|2,\ldots,8} - \left[k_{23}^m U_{1|23|4,\ldots,8} + (23\leftrightarrow 4,\ldots,8)\right]\Big)\Gamma^m_{1|23,4,\ldots,8} \tag{3.95}
$$

$$
0 \cong \Big(\mathcal{Z}_{1|23,45,6,7,8} - \left[U_{1|4|23,5,6,7,8} + U_{1|5|23,4,6,7,8} - (23\leftrightarrow 45)\right]\Big)\Gamma_{1|23,45,6,7,8} \tag{3.96}
$$

$$
\begin{aligned}
0 \cong \Big(&s_{23}(s_{24}+s_{34})\mathcal{Z}_{1|234,5,6,7,8} + s_{23}s_{24}\mathcal{Z}_{1|243,5,6,7,8} \\
&- s_{23}s_{24}\big(U_{1|23|4,\ldots,8} + U_{1|24|3,\ldots,8} - U_{1|3|24,\ldots,8} - U_{1|4|23,\ldots,8}\big) \\
&- s_{23}s_{34}\big(U_{1|2|34,\ldots,8} + U_{1|23|4,\ldots,8} - U_{1|34|2,\ldots,8} - U_{1|4|23,\ldots,8}\big)\Big)\Gamma_{1|234,5,6,7,8}
\end{aligned}
\tag{3.97}
$$

$$
0 \cong \Big(\mathcal{Z}_{1,2|3,\ldots,8} - k_2^m U^m_{1|2|3,\ldots,8} - \left[s_{23}U_{1|23|4,\ldots,8} + (3\leftrightarrow 4,\ldots,8)\right]\Big)\Gamma_{1|2|3,4,5,6,7,8}\,. \tag{3.98}
$$

To solve these equations it will be convenient to exploit the vanishing of $\mathcal{Z}^\Delta$ according to (II.5.29). For instance, $0 \cong \mathcal{Z}^{\Delta,mn}_{1|2,3,4,5,6,7,8}$ can be used to rewrite

$$
\mathcal{Z}^{mn}_{1|2,3,4,5,6,7,8} = -\left[k_2^{(m}\mathcal{Z}^{n)}_{12|3,4,5,6,7,8} + (2\leftrightarrow 3,\ldots,8)\right] + \left[k_2^{(m}k_3^{n)}\mathcal{Z}_{213|4,5,6,7,8} + (2,3|2,\ldots,8)\right] \tag{3.99}
$$

allowing (3.94) to be solved for U^m. In turn, plugging U^m into (3.95) and using the vanishing of $\mathcal{Z}^{\Delta}_{1|23,4,5,6,7,8}$ from (II.5.27) allows the determination of the other two topologies of U in (3.90). The resulting expressions

$$\begin{aligned} U^m_{1|2|3,4,5,6,7,8} &= -\mathcal{Z}^m_{12|3,\ldots,8} + \frac{1}{2}\big[k^m_3 \mathcal{Z}_{213|4,\ldots,8} + (3 \leftrightarrow 4,\ldots,8)\big]\,, \\ U_{1|23|4,5,6,7,8} &= \frac{1}{2}\big(\mathcal{Z}_{132|4,5,6,7,8} - \mathcal{Z}_{123|4,5,6,7,8}\big)\,, \\ U_{1|2|34,5,6,7,8} &= \frac{1}{2}\big(\mathcal{Z}_{312|4,5,6,7,8} - \mathcal{Z}_{412|3,5,6,7,8}\big)\,, \end{aligned} \tag{3.100}$$

are consistent with the remaining equations (3.96) to (3.98). One could wonder if the trace relation $\Gamma^{mn}_{1|2,3,\ldots,8} = 2\Gamma_{1|2|3,\ldots,8} + (2 \leftrightarrow 3,\ldots,8)$ among the anomaly invariants might generate corrections to the last equation (3.98) from tensor traces in (3.94). This is not the case because the chosen representation of $\mathcal{Z}^{mn}$ in the tensorial equation (3.94) does not feature any $\delta^{mn}\mathrm{G}_4$ deformations.

Given that the expressions (3.100) for the U-functions in $\mathcal{K}^Y_8(\ell)$ solve all of (3.94) to (3.98), the BRST variation of the overall correlator (3.89) reduces to

$$\begin{aligned} Q(\mathcal{K}^{\mathrm{Lie}}_8(\ell) + \mathcal{K}^Y_8(\ell)) = \frac{1}{2} &V_{A_1} Y^{mn}_{A_2,\ldots,A_8}\, R^{mn}_{A_1,\ldots,A_8} + [12\ldots 8|A_1,\ldots,A_8] \\ &+ V_{A_1} Y^m_{A_2,\ldots,A_7}\, R^m_{A_1,\ldots,A_7} + [12\ldots 8|A_1,\ldots,A_7] \\ &+ V_{A_1} Y_{A_2,\ldots,A_6}\, R_{A_1,\ldots,A_6} + [12\ldots 8|A_1,\ldots,A_6]\,. \end{aligned} \tag{3.101}$$

The R-functions from (3.88) are all proportional to the holomorphic Eisenstein series G_4, i.e. any dependence of the BRST variation (3.101) on ℓ or the z_j has cancelled.

Unfortunately, we explicitly checked [8] that there is no manifestly local deformation of the correlator that can be used to cancel the remaining terms in (3.101). Therefore, even though the BRST variation of $\mathcal{K}^{\mathrm{Lie}}_8(\ell) + \mathcal{K}^Y_8(\ell)$ turns out to be a local expression, its component expansion is non-local, see appendix I.C for the kinematic poles of the $\Delta_{1|\ldots}$ superfields in (3.90). This suggests that there may be another non-local sector whose BRST variation cancels (3.101), although we have not been able to pinpoint it yet. We leave the quest for finding such a completion to future investigations.

4. Modular forms: Integrating out the loop momentum

This section is dedicated to the integration over the loop momentum which will lead to manifestly single-valued one-loop correlators. In this way, the correlators acquire well-defined weights under modular transformations, namely holomorphic weight $n-4$ for the loop integral of $\mathcal{K}_n(\ell)$.

At the same time, closed-string correlators are no longer chirally split after integration over the loop momenta [12,13,14]. We will describe the systematics of the interactions between left- and right movers that arises from loop integration of the holomorphic squares $|\mathcal{K}_n(\ell)|^2$. The setup of $\mathcal{Z}$-functions and GEIs turns out to provide an efficient handle on the vector contractions between left- and right movers and the loss of meromorphicity of the open-string contributions after integration over ℓ.

Let us briefly summarize the notation of part I & II. As detailed in sections I.2.2 and II.7.2, the net effect of loop integration on the Koba–Nielsen factor (2.3) is captured by

$$\hat{\mathcal{I}}_n^{\rm open} = \frac{(2\pi i)^D}{(\mathrm{Im}\,\tau)^{\frac{D}{2}}} \exp\Big(\sum_{i<j}^n s_{ij} \Big[\log \big|\theta_1(z_{ij},\tau)\big| - \frac{\pi (\mathrm{Im}\, z_{ij})^2}{\mathrm{Im}\,\tau} \Big] \Big) \, , \tag{4.1}$$

$$\hat{\mathcal{I}}_n = \frac{(2\pi i)^D}{(2\,\mathrm{Im}\,\tau)^{\frac{D}{2}}} \exp\Big(\sum_{i<j}^n s_{ij} \Big[\log \big|\theta_1(z_{ij},\tau)\big|^2 - \frac{2\pi}{\mathrm{Im}\,\tau} (\mathrm{Im}\, z_{ij})^2 \Big] \Big) \, .$$

After factorizing these universal quantities in the worldsheet integrand of open- and closed-string amplitudes (2.1) and (2.2),

$$\mathcal{A}_n = \sum_{\rm top} C_{\rm top} \int_{D_{\rm top}} d\tau\, dz_2\, dz_3\, \ldots\, dz_n\, \hat{\mathcal{I}}_n^{\rm open}\, [[\, \langle \mathcal{K}_n(\ell) \rangle\,]] \, , \tag{4.2}$$

$$\mathcal{M}_n = \int_{\mathcal{F}} d^2\tau\, d^2 z_2\, d^2 z_3\, \ldots\, d^2 z_n\, \hat{\mathcal{I}}_n\, [[\, \langle \mathcal{K}_n(\ell) \rangle\, \langle \tilde{\mathcal{K}}_n(-\ell) \rangle\,]] \, ,$$

the leftover integrand w.r.t. the punctures z_j and modular parameters τ is furnished by "loop-integrated" correlators $[[\mathcal{K}_n(\ell)]]$ and $[[\mathcal{K}_n(\ell)\tilde{\mathcal{K}}_n(-\ell)]]$ in combination with the zero-mode prescription $\langle \ldots \rangle$ of the pure-spinor formalism [2]. Hence, the notation $[[\ldots]]$ in (4.2) addresses the net effect of shifting the loop momentum as a Gaussian integration variable, cf. section II.7.2. The normalization is chosen as $[[1]] = 1$, and the simplest non-trivial examples $[[\ell^m]] = L_0^m$ and $[[\ell^m \ell^n]] = L_0^m L_0^n - \frac{\pi}{\mathrm{Im}\,\tau}\delta^{mn}$ are most compactly written in terms of the non-meromorphic quantity $L_0^m = \sum_{j=2}^n k_j^m \nu_{1j}$ with $\nu_{ij} \equiv 2\pi i \frac{\mathrm{Im}\, z_{ij}}{\mathrm{Im}\,\tau}$, see (II.7.13) for integration over higher powers of ℓ. The contribution $-\frac{\pi}{\mathrm{Im}\,\tau}\delta^{mn}$ to $[[\ell^m \ell^n]]$ is the first instance of the aforementioned interactions between left- and right movers.

4.1. Five-point open-string correlators

Starting from this section, we apply the techniques of integrating the loop momentum to the correlators $\mathcal{K}_n(\ell)$ of section 3. We will complement the direct integration of GEIs with a study of the $T\cdot\mathcal{Z}$ and $C\cdot\mathcal{Z}$ representations where the origin of the kinematic factors from the OPEs is more transparent.

4.1.1. The $T\cdot\mathcal{F}$ representation: manifesting locality & single-valuedness

As discussed in [14], integration over the loop momentum leads to manifestly single-valued representations of chirally-split correlators. We therefore integrate out the loop momentum from the representation (3.8) using (II.7.13) to obtain

$$\begin{aligned}[[\mathcal{K}_5(\ell)]] &= \big[k_2^m V_1 T^m_{2,3,4,5}\nu_{12} + V_{12}T_{3,4,5}g^{(1)}_{12} + (2\leftrightarrow 3,4,5)\big] + \big[V_1T_{23,4,5}g^{(1)}_{23} + (2,3|2,3,4,5)\big]\\ &= \big[V_{12}T_{3,4,5}(g^{(1)}_{12}+\nu_{12})+(2\leftrightarrow 3,4,5)\big]+\big[V_1T_{23,4,5}(g^{(1)}_{23}+\nu_{23})+(2,3|2,3,4,5)\big]\,, \end{aligned}\tag{4.3}$$

where to arrive in the second line we used the cohomology identity (3.18) as $k_2^m V_1 T^m_{2,3,4,5} \cong V_{12}T_{3,4,5} - \big[V_1T_{23,4,5} + (3\leftrightarrow 4,5)\big]$ and $\nu_{13}-\nu_{12} = \nu_{23}$. So we see that the single-valued functions $f^{(1)}_{ij} = g^{(1)}_{ij} + \nu_{ij}$ are constructively obtained and we get the following correlator

$$\begin{aligned}[[\mathcal{K}_5(\ell)]] &= V_{12}T_{3,4,5}\,\mathcal{F}_{12,3,4,5} + (2\leftrightarrow 3,4,5)\\ &\quad + V_1T_{23,4,5}\mathcal{F}_{1,23,4,5} + (2,3|2,3,4,5)\,,\end{aligned}\tag{4.4}$$

in terms of manifestly single-valued functions $\mathcal{F}_{12,3,4,5} \equiv f^{(1)}_{12}$. Given that the functions $f^{(w)}_{ij}$ defined by (II.7.1) carry w units of holomorphic modular weight, see (II.7.5), the correlator (4.4) is a modular form of weight one.

4.1.2. The $C\cdot\mathcal{F}$ representation: manifesting BRST invariance & single-valuedness

It is also possible to obtain a representation without the loop momentum which manifests both BRST invariance and single-valuedness. This can be achieved in at least two ways: integrating out the loop momentum from the $C\cdot\mathcal{Z}$ representation (3.17) or using integration-by-parts identities to eliminate all $f^{(1)}_{1j}$ integrands with $j = 2,3,4,5$ from (4.4).

First, integrating out the loop momentum from (3.17) using the identity,

$$L_0^m C^m_{1|2,3,4,5} = -\sum_{j=1}^{5}\nu_j k_j^m C^m_{1|2,3,4,5} = \big[s_{23}\nu_{23}C_{1|23,4,5} + (2,3|2,3,4,5)\big]\,,\tag{4.5}$$

leads to the manifestly single-valued and BRST-invariant form of the five-point correlator

$$[[\mathcal{K}_5(\ell)]] = C_{1|23,4,5} s_{23} f_{23}^{(1)} + (2,3|2,3,4,5) \, . \tag{4.6}$$

This form reproduces the five-point one-loop correlator proposed in [28] and rederived in [30,10]. Alternatively, one can arrive at the representation (4.6) using integration-by-parts identities (II.7.27) in the local and single-valued representation (4.4). In fact, this is how (4.6) was originally derived in [30]. The derivations of this paper are based on single-valuedness and BRST-invariance constraints, and one obtains a much richer perspective on the correlators. In summary, the integration over the loop momentum yields two additional representations of the five-point correlator from (3.22),

$$\begin{aligned}
[[\mathcal{K}_5(\ell)]] &= \big[V_{12} T_{3,4,5} \mathcal{F}_{12,3,4,5} + (2 \leftrightarrow 3,4,5)\big] + \big[V_1 T_{23,4,5} \mathcal{F}_{1,23,4,5} + (2,3|2,3,4,5)\big] \, , \\
[[\mathcal{K}_5(\ell)]] &= C_{1|23,4,5} F^{(s)}_{1,23,4,5} + (2,3|2,3,4,5) \, ,
\end{aligned} \tag{4.7}$$

where we used the shorthand $F^{(s)}_{1,23,4,5} = s_{23} \mathcal{F}_{1,23,4,5} = s_{23} f_{23}^{(1)}$ in the second line. More generally, by analogy with the $Z^{(s)}$ in (II.4.22), we define the following Lie-symmetry satisfying analogues of the shuffle-symmetric $\mathcal{F}^{m_1 m_2 \ldots}_{A,B,\ldots}$-functions,

$$F^{(s) m_1 \ldots}_{aA,bB,\ldots} \equiv \sum_{A',B',\ldots} S(A|A')_a S(B|B')_b \cdots \mathcal{F}^{m_1 \ldots}_{aA',bB',\ldots} \, , \tag{4.8}$$

which will be tensorial at higher multiplicity. We see that integrating out the loop momentum from the functions $\mathcal{Z}$ in (3.8) has the same effect as sending $\ell \to 0$ and $g^{(1)}_{ij} \to f^{(1)}_{ij}$. However, these replacement rules are tied to the present open-string context and no longer apply to the closed-string five-point correlators of section (4.20).

4.1.3. Lessons from the $T \cdot E$ and $C \cdot E$ representations

As an alternative to the earlier computations, one can start from the representations (3.20) or (3.21) of $\mathcal{K}_5(\ell)$ in terms of GEIs and insert the results (II.7.17) and (II.7.19) for their loop integral. The manifestly local $T \cdot E$ representation (3.20) yields

$$\begin{aligned}
[[\mathcal{K}_5(\ell)]] &= V_1 T_{23,4,5} f_{23}^{(1)} + (2,3|2,3,4,5) \\
&\quad + V_1 (k_2^m T^m_{2,3,4,5} + T_{23,4,5} + T_{24,3,5} + T_{25,3,4}) f_{12}^{(1)} + (2 \leftrightarrow 3,4,5)
\end{aligned} \tag{4.9}$$

after reorganizing terms which agrees with the $T \cdot \mathcal{F}$ representation (4.4) up to the cohomology identity (3.18). Similarly, the manifestly BRST-invariant $C \cdot E$ representation (3.21) yields

$$\begin{aligned}
[[\mathcal{K}_5(\ell)]] &= s_{23} C_{1|23,4,5} f_{23}^{(1)} + (2,3|2,3,4,5) \\
&\quad + (k_2^m C^m_{1|2,3,4,5} + s_{23} C_{1|23,4,5} + s_{24} C_{1|24,3,5} + s_{25} C_{1|25,3,4}) f_{12}^{(1)} + (2 \leftrightarrow 3,4,5) \, ,
\end{aligned} \tag{4.9}$$

after reorganizing terms. This in turn matches the $C \cdot \mathcal{F}$ representation (4.6), because the second line is BRST exact by (I.5.41).

4.2. Six-point open-string correlators

4.2.1. The $T \cdot \mathcal{F}$ representation: manifesting locality & single-valuedness

We already know that the six-point correlator (3.23) is single-valued, and in this section this will be manifested by integrating out the loop momentum and checking that the generated variables ν_{ij} combine into single-valued functions $f_{ij}^{(n)}$ according to (II.7.3)

$$g_{ij}^{(1)} + \nu_{ij} = f_{ij}^{(1)}, \qquad g_{ij}^{(2)} + \nu_{ij} g_{ij}^{(1)} + \frac{1}{2}\nu_{ij}^2 = f_{ij}^{(2)} . \tag{4.11}$$

Indeed, integrating out the loop momentum in the representation (3.23) using (II.7.3) yields

$$\begin{aligned} [[\mathcal{K}_6(\ell)]] = & \frac{1}{2} V_{A_1} T^{mn}_{A_2,\ldots,A_6} \mathcal{F}^{mn}_{A_1,\ldots,A_6} + [123456|A_1,\ldots,A_6] \\ & + V_{A_1} T^{m}_{A_2,\ldots,A_5} \mathcal{F}^{m}_{A_1,\ldots,A_5} + [123456|A_1,\ldots,A_5] \\ & + V_{A_1} T_{A_2,\ldots,A_4} \mathcal{F}_{A_1,\ldots,A_4} + [123456|A_1,\ldots,A_4] , \end{aligned} \tag{4.12}$$

with manifestly single-valued worldsheet functions given by

$$\begin{aligned} \mathcal{F}_{123,4,5,6} &\equiv f_{12}^{(1)} f_{23}^{(1)} + f_{12}^{(2)} + f_{23}^{(2)} - f_{13}^{(2)} , \\ \mathcal{F}_{12,34,5,6} &\equiv f_{12}^{(1)} f_{34}^{(1)} + f_{13}^{(2)} + f_{24}^{(2)} - f_{14}^{(2)} - f_{23}^{(2)} , \\ \mathcal{F}^m_{12,3,4,5,6} &\equiv (k_2^m - k_1^m) f_{12}^{(2)} + \big[k_3^m (f_{13}^{(2)} - f_{23}^{(2)}) + (3 \leftrightarrow 4,5,6) \big] , \\ \mathcal{F}^{mn}_{1,2,3,4,5,6} &\equiv \big[(k_1^m k_2^n + k_1^n k_2^m) f_{12}^{(2)} + (1,2|1,2,3,4,5,6) \big] . \end{aligned} \tag{4.13}$$

To see this, we use the integration-by-parts identity (II.7.27) obtained from $\partial_2(\nu_2 \hat{\mathcal{I}}_6) = 0$,

$$\nu_2 f_{12}^{(1)} \cong \frac{1}{s_{12}} \Big(\frac{\pi}{\operatorname{Im} \tau} + \big[s_{23} \nu_2 f_{23}^{(1)} + (3 \leftrightarrow 4,5,6) \big] \Big) , \tag{4.14}$$

and drop the BRST-exact linear combinations given by (I.B.2) and (I.B.14). Given that an additional summand of $\frac{\pi}{\operatorname{Im}\tau}$ arises from the loop integral over $\frac{1}{2}\ell_m \ell_n V_1 T^{mn}_{2,3,4,5,6}$, the coefficient of the modular anomaly cancels by the building-block trace relation (I.4.45). Similarly as in the five-point open-string calculations, the functions $\mathcal{F}$ in (4.13) are related to $\mathcal{Z}$ from (3.24) by $\ell \to 0$ and $g_{ij}^{(n)} \to f_{ij}^{(n)}$. Furthermore, we see from (II.7.5) that the non-holomorphic six-point correlator (4.12) is a modular form of weight two.

4.2.2. The $C\cdot\mathcal{F}$ representation: manifesting BRST invariance & single-valuedness

There are several alternatives to deriving a manifestly BRST-invariant form of the correlator without the loop momentum. The most straightforward way is to use integration-by-parts identities (II.7.27) in the representation (4.12). A long calculation very similar to the derivation of (3.49) in section 3.3.3 leads to

$$\begin{aligned}[[\mathcal{K}_6(\ell)]] &= \big[\big(s_{23}s_{34}f^{(1)}_{23}f^{(1)}_{34}C_{1|234,5,6}+\text{cyc}(2,3,4)\big)+(2,3,4|2,3,4,5,6)\big] \\ &\quad+\big[\big(s_{23}s_{45}f^{(1)}_{23}f^{(1)}_{45}C_{1|23,45,6}+\text{cyc}(3,4,5)\big)+(6\leftrightarrow 5,4,3,2)\big] \\ &\quad+\big[f^{(2)}_{12}C_{1|2|3,4,5,6}+(2\leftrightarrow 3,4,5,6)\big]+\big[f^{(2)}_{23}C_{1|(23)|4,5,6}+(2,3|2,3,4,5,6)\big]\,,\end{aligned}\tag{4.15}$$

see (3.50) or (3.52) for the pseudo-invariant kinematic factors $C_{1|2|3,4,5,6}$ and $C_{1|(23)|4,5,6}$. This representation reproduces the correlator proposed in [33] based on BRST cohomology properties together with an anomaly-cancellation analysis. At the same time, (4.15) can be easily checked to be equivalent to

$$\begin{aligned}[[\mathcal{K}_6(\ell)]] &= \frac{1}{2}C^{mn}_{1|2,3,4,5,6}F^{(s)mn}_{1,2,3,4,5,6}+\big[C^{m}_{1|23,4,5,6}F^{(s)m}_{1,23,4,5,6}+(2,3|2,3,4,5,6)\big] \\ &\quad+\big[\big(C_{1|234,5,6}F^{(s)}_{1,234,5,6}+C_{1|243,5,6}F^{(s)}_{1,243,5,6}\big)+(2,3,4|2,3,4,5,6)\big] \\ &\quad+\big[\big(C_{1|23,45,6}F^{(s)}_{1,23,45,6}+\text{cyc}(3,4,5)\big)+(6\leftrightarrow 5,4,3,2)\big]\,,\end{aligned}\tag{4.16}$$

using the Lie-symmetric version (4.8) of the functions $\mathcal{F}^{m\cdots}_{A,B,\ldots}$ in (4.13).

4.2.3. Lessons from the $T\cdot E$ and $C\cdot E$ representations

Again, one can combine the above results for the loop integrals over six-point GEIs with the $T\cdot E$ and $C\cdot E$ representations of $\mathcal{K}_6(\ell)$ in (3.47) and (3.44), respectively. Based on the loop integration $[[E^{mn}_{1|2,\ldots}]]=-\frac{\pi}{\text{Im}\,\tau}\delta^{mn}+\ldots$ and $[[E_{1|2|3,\ldots}]]=-\frac{\pi}{\text{Im}\,\tau}+\ldots$ in (II.7.19) and (II.7.32), the cancellation of the modular anomalies is transparent in both representations: Either by the trace relation (I.4.45) among local building blocks or by the trace relation (I.5.29) among pseudo-invariants.

In particular, the terms $\mathcal{K}_6(\ell)=\frac{1}{2}C^{mn}_{1|2,\ldots}E^{mn}_{1|2,\ldots}-\big[P_{1|2|3,\ldots}E_{1|2|3,\ldots}+(2\leftrightarrow 3,\ldots)\big]+\ldots$ in the $C\cdot E$ representations (3.44) illustrate the duality between BRST anomalies and modular anomalies: In the same way as the modular anomaly of $[[\mathcal{K}_6(\ell)]]$ cancels by the trace relation (I.5.29) between $C^{mn}_{1|2,\ldots}$ and $P_{1|2|3,\ldots}$, the BRST anomaly localizes to a boundary term in moduli space since the GEIs $E^{mn}_{1|2,\ldots}$ and $E_{1|2|3,\ldots}$ satisfy the dual trace relation (II.5.31) (or (II.7.34) after integration over ℓ).

Also, note that the $C\cdot\mathcal{F}$ representation (4.16) results from straightforward regroupings of terms in the integrated $C\cdot E$ representations (3.44): There is no need to perform integration by parts on the $f^{(n)}_{ij}$, and the coefficients of $f^{(1)}_{1j}$, $j=2,3,4,5,6$ are easily seen to vanish after using Fay relations and cohomology identities of section I.5.4.

4.3. Closed-string correlators

One of the major motivations for chiral splitting is that closed-string correlators are literally the square of open-string correlators before integration over ℓ, cf. (2.2). Performing the loop integral reveals modular invariance of the closed-string amplitude representation (4.2), at the expense of introducing interactions between left- and right-movers. We will now illustrate these interactions based on examples up to six points.

Most obviously, the expressions of section II.7.2 for integrated GEIs are augmented by additional terms involving $\frac{\pi}{\operatorname{Im}\tau}$ when the opposite-chirality sector contributes additional loop momenta, e.g.

$$
\begin{aligned}
[[\ell^n E^m_{1|2,3,4,5}]] &= -\frac{\pi}{\operatorname{Im}\tau}\delta^{mn} + L_0^n\big[k_2^m f_{12}^{(1)} + (2\leftrightarrow 3,4,5)\big]\,, \qquad (4.17)\\
[[\ell^n E^m_{1|23,4,5,6}]] &= -\frac{\pi}{\operatorname{Im}\tau}\delta^{mn}V_1(1,2,3) + L_0^n\Big(k_3^m f_{12}^{(1)}f_{23}^{(1)} + k_2^m f_{13}^{(1)}f_{23}^{(1)} + k_{23}^m(f_{12}^{(2)} - f_{13}^{(2)})\\
&\quad + (k_3^m - k_2^m)f_{23}^{(2)} + \big[k_4^m f_{14}^{(1)}V_1(1,2,3) + (4\leftrightarrow 5,6)\big]\Big)\,,\\
[[\ell^p\ell^q E^{mn}_{1|2,3,4,5,6}]] &= \Big(\frac{\pi}{\operatorname{Im}\tau}\Big)^2\delta^{m(n}\delta^{pq)} - \frac{\pi}{\operatorname{Im}\tau}L_0^{(p}\delta^{q)(m}\big[k_2^{n)}f_{12}^{(1)} + (2\leftrightarrow 3,4,5,6)\big]\\
&\quad - \frac{\pi}{\operatorname{Im}\tau}\delta^{mn}L_0^pL_0^q + 2\Big(L_0^pL_0^q - \frac{\pi}{\operatorname{Im}\tau}\delta^{pq}\Big)\big[k_2^mk_2^n f_{12}^{(2)} + (2\leftrightarrow 3,4,5,6)\big]\\
&\quad + \Big(L_0^pL_0^q - \frac{\pi}{\operatorname{Im}\tau}\delta^{pq}\Big)\big[(k_2^mk_3^n + k_2^mk_3^n)f_{12}^{(1)}f_{13}^{(1)} + (2,3|2,3,4,5,6)\big]\,,\\
[[\ell^m E_{1|2|3,4,5,6}]] &= -\frac{\pi}{\operatorname{Im}\tau}f_{12}^{(1)}k_2^m + L_0^m\Big(-\frac{\pi}{\operatorname{Im}\tau} - 2s_{12}f_{12}^{(2)} + f_{12}^{(1)}\big[s_{23}f_{23}^{(1)} + (3\leftrightarrow 4,5,6)\big]\Big)\\
&= L_0^m\Big(-\frac{\pi}{\operatorname{Im}\tau} + \partial f_{12}^{(1)} + s_{12}(f_{12}^{(1)})^2 - 2s_{12}f_{12}^{(2)}\Big)\,.
\end{aligned}
$$

Once these additional loop momenta are regrouped into complex conjugate GEIs, the net effect of the additional L_0^m is to recombine the $\overline{g}^{(n)}$ functions to

$$
\overline{f}^{(n)}(z,\tau) \equiv \sum_{k=0}^{n}\frac{(-\nu)^k}{k!}\overline{g}^{(n-k)}(z,\tau)\,. \qquad (4.18)
$$

The minus signs relative to (II.7.3) are due to $\nu_{ij}\to-\nu_{ij}$ under complex conjugation. Likewise, our normalization conventions for the loop momentum transforms $\ell\to-\ell$ in passing from GEIs to their complex conjugates, as reflected in the notation $\tilde{\mathcal{K}}_n(-\ell)$ for right-moving correlators in (4.2). For instance, the vectorial GEI in $\tilde{\mathcal{K}}_5(-\ell) = \overline{E}^m_{1|2,3,4,5}\tilde{C}^m_{1|2,3,4,5} + \ldots$ reads $\overline{E}^m_{1|2,3,4,5} = -\ell + \big[k_2^m\overline{g}_{12}^{(1)} + (2\leftrightarrow 3,4,5)\big]$, and the loop integral of its holomorphic square can be performed via (4.17),

$$
[[E^m_{1|2,3,4,5}\overline{E}^n_{1|2,3,4,5}]] = \frac{\pi}{\operatorname{Im}\tau}\delta^{mn} + \big[k_2^m f_{12}^{(1)} + (2\leftrightarrow 3,4,5)\big]\big[k_2^n\overline{f}_{12}^{(1)} + (2\leftrightarrow 3,4,5)\big]\,. \qquad (4.19)
$$

The first term exemplifies that factors of $\frac{\pi}{\operatorname{Im}\tau}$ are not necessarily associated with modular anomalies in a closed-string setup: Both $\frac{\pi}{\operatorname{Im}\tau}$ and the remaining terms $f^{(1)}_{ij}\overline{f}^{(1)}_{kl}$ in (4.19) have modular weights $(1,1)$, in lines with modular invariance of the five-point amplitude (4.2). In fact, the cancellation of modular anomalies in integrated open-string six-point correlators applies separately to both chiral halves of the closed-string calculation.

4.3.1. Five points

Starting from the $T\cdot\mathcal{Z}$ representation (3.8) of the open-string five-point correlator, loop integration over its holomorphic square yields

$$\begin{aligned}[[\mathcal{K}_5(\ell)\tilde{\mathcal{K}}_5(-\ell)]] = \Big|\big[V_{12}T_{3,4,5}\mathcal{F}_{12,3,4,5}+(2\leftrightarrow 3,4,5)\big] + \big[V_1T_{23,4,5}\mathcal{F}_{1,23,4,5}+(2,3|2,3,4,5)\big]\Big|^2 \\ + \frac{\pi}{\operatorname{Im}\tau}V_1T^m_{2,3,4,5}\tilde{V}_1\tilde{T}^m_{2,3,4,5}\,. \qquad (4.20)\end{aligned}$$

The second line augments the square of the integrated open-string correlator in its $T\cdot\mathcal{F}$ representation (4.4) by a left-right contraction. The recombination of $g^{(1)}_{ij}+\nu_{ij}=f^{(1)}_{ij}$ and $\overline{g}^{(1)}_{ij}-\nu_{ij}=\overline{f}^{(1)}_{ij}$ follows the mechanism of the open-string context, see (4.3).

The local form (4.20) of the five-point closed-string correlator has been spelled out in [35]. As already emphasized in the reference, integrations by parts (II.7.27) are more subtle in presence of both $f^{(n)}_{ij}$ and $\overline{f}^{(n)}_{ij}$: Additional terms $\frac{\pi}{\operatorname{Im}\tau}$ may arise in trading $s_{12}f^{(1)}_{12}$ for $s_{23}f^{(1)}_{23}+(3\leftrightarrow 4,5)$ on the left-moving side, depending on the labels of the accompanying right-moving $\overline{f}^{(1)}_{ij}$, see e.g. (II.7.30). Hence, one cannot just replace the left-moving terms in the first line of (4.20) by their manifestly BRST-invariant counterparts (4.6) without inspecting the respective right-movers and altering the coefficient of $\frac{\pi}{\operatorname{Im}\tau}$.

Instead, a manifestly BRST-invariant rewriting of (4.20) can be conveniently found by integrating the $C\cdot\mathcal{Z}$ representation of $\mathcal{K}_5(\ell)\tilde{\mathcal{K}}_5(-\ell)$,

$$[[\mathcal{K}_5(\ell)\tilde{\mathcal{K}}_5(-\ell)]] = \big|s_{23}f^{(1)}_{23}C_{1|23,4,5}+(2,3|2,3,4,5)\big|^2 + \frac{\pi}{\operatorname{Im}\tau}C^m_{1|2,3,4,5}\tilde{C}^m_{1|2,3,4,5}\,. \quad (4.21)$$

This representation has been firstly given in [33], based on a long sequence of integration-by-parts identities in (4.20) and carefully tracking all $\partial_i\overline{f}^{(1)}_{ij}=\overline{\partial}_if^{(1)}_{ij}=-\frac{\pi}{\operatorname{Im}\tau}$ [35].

4.3.2. Six points

A manifestly BRST invariant closed-string six-point correlator has been proposed in [33]

$$
\begin{aligned}
[[\mathcal{K}_6(\ell)\tilde{\mathcal{K}}_6(-\ell)]] = \mathcal{K}_6^{\rm open}\tilde{\mathcal{K}}_6^{\rm open} + \frac{\pi}{{\rm Im}\,\tau}\big| s_{23} f_{23}^{(1)} C^m_{1|23,4,5,6} + (2,3|2,3,4,5,6)\big|^2 \\
+ \Big(\frac{\pi}{{\rm Im}\,\tau}\Big)^2 \Big(\frac{1}{2} C^{mn}_{1|2,3,4,5,6}\tilde{C}^{mn}_{1|2,3,4,5,6} - \big[|P_{1|2|3,4,5,6}|^2 + (2\leftrightarrow 3,4,5,6)\big]\Big)\,,
\end{aligned}
\tag{4.22}
$$

where $\mathcal{K}_6^{\rm open}$ is essentially the representation of $[[\mathcal{K}_6(\ell)]]$ given in (4.15),

$$
\begin{aligned}
\mathcal{K}_6^{\rm open} &= \big[\big(s_{23}s_{34} f_{23}^{(1)} f_{34}^{(1)} C_{1|234,5,6} + {\rm cyc}(2,3,4)\big) + (2,3,4|2,3,4,5,6)\big] \\
&+ \big[\big(s_{23}s_{45} f_{23}^{(1)} f_{45}^{(1)} C_{1|23,45,6} + {\rm cyc}(3,4,5)\big) + (6\leftrightarrow 5,4,3,2)\big] \\
&+ \big[f_{12}^{(2)} C_{1|2|3,4,5,6} + (2\leftrightarrow 3,4,5,6)\big] + \big[f_{23}^{(2)} C_{1|(23)|4,5,6} + (2,3|2,3,4,5,6)\big]\,,
\end{aligned}
\tag{4.23}
$$

up to Koba–Nielsen derivatives to be detailed below. The pseudo-invariants $C_{1|2|3,4,5,6}$ and $C_{1|(23)|4,5,6}$ have been defined in (3.50). The second line of (4.22) has not yet been derived from first principles but was inferred by indirect arguments including properties of the low-energy limit [33]. In appendix D, we will demonstrate the terms $|P_{1|2|3,4,5,6}|^2$ in (4.22) to follow from a careful analysis of integration-by-parts identities.

Our derivation of (4.22) starts from the $C \cdot E$ representation (3.44) of $\mathcal{K}_6(\ell)$ and a convenient organization of the loop integrals in the closed-string case according to the number of contractions $\ell^m \ell^n \to -\frac{\pi}{{\rm Im}\,\tau}$ between left- and right-movers

$$
\begin{aligned}
[[\mathcal{K}_6(\ell)\tilde{\mathcal{K}}_6(-\ell)]] &= [[\mathcal{K}_6(\ell)]] \cdot [[\tilde{\mathcal{K}}_6(-\ell)]] \\
&+ \frac{\pi}{{\rm Im}\,\tau}\Big[\Big[\frac{\partial \mathcal{K}_6(\ell)}{\partial \ell_m}\Big]\Big]\delta_{mn}\Big[\Big[\frac{\partial \tilde{\mathcal{K}}_6(-\ell)}{\partial(-\ell_n)}\Big]\Big] \\
&+ \frac{1}{2}\Big(\frac{\pi}{{\rm Im}\,\tau}\Big)^2 C^{mn}_{1|2,3,4,5,6}\tilde{C}^{mn}_{1|2,3,4,5,6}\,.
\end{aligned}
\tag{4.24}
$$

The double contractions between left- and right movers are sensitive to no contribution to other than $\mathcal{K}_6(\ell) = \frac{1}{2}\ell_m \ell_n C^{mn}_{1|2,3,4,5,6} + \ldots$ and lead to the last line. For the vectorial open-string constituents of (4.24), the representation (3.44) gives rise to

$$
\begin{aligned}
\Big[\Big[\frac{\partial \mathcal{K}_6(\ell)}{\partial \ell_m}\Big]\Big] &= \big[C^m_{1|23,4,5,6} s_{23} f_{23}^{(1)} + (2,3|2,3,4,5,6)\big] \\
&+ \big[P_{1|2|3,4,5,6} k_2^m \nu_{12} + (2\leftrightarrow 3,4,5,6)\big]\,,
\end{aligned}
\tag{4.25}
$$

see appendix D for intermediate steps. Finally, the scalar contributions to (4.24)

$$
[[\mathcal{K}_6(\ell)]] = \mathcal{K}_6^{\rm open} - \big[N_{1|2|3,4,5,6} P_{1|2|3,4,5,6} + (2\leftrightarrow 3,4,5,6)\big]
\tag{4.26}
$$

augment (4.23) by total derivatives

$$N_{1|2|3,4,5,6}\hat{\mathcal{I}}_6 = \Big(\frac{\pi}{\operatorname{Im}\tau} + \nu_{12}(s_{12}f^{(1)}_{12} - \big[s_{23}f^{(1)}_{23} + (3\leftrightarrow 4,5,6)\big])\Big)\hat{\mathcal{I}}_6 = -\frac{\partial}{\partial z_2}(\nu_{12}\hat{\mathcal{I}}_6)$$

$$\tilde{N}_{1|2|3,4,5,6}\hat{\mathcal{I}}_6 = \Big(\frac{\pi}{\operatorname{Im}\tau} - \nu_{12}(s_{12}\overline{f}^{(1)}_{12} - \big[s_{23}\overline{f}^{(1)}_{23} + (3\leftrightarrow 4,5,6)\big])\Big)\hat{\mathcal{I}}_6 = \frac{\partial}{\partial \overline{z}_2}(\nu_{12}\hat{\mathcal{I}}_6) \quad (4.27)$$

that have been dropped in the open-string context of (4.15), see (II.7.32). In the present closed-string context, however, the quantity $\mathcal{K}^{\rm open}_6$ in (4.23) cannot be replaced by integration-by-parts equivalent representations of $[[\mathcal{K}_6(\ell)]]$, say the local expression in (4.12). The factors of $\overline{f}^{(w)}_{ij}$ in the accompanying $\tilde{\mathcal{K}}^{\rm open}_6$ in (4.22) are affected by holomorphic total derivatives via (II.7.29).

Upon insertion into (4.24), the first line of (4.25) and the holomorphic square of $\mathcal{K}^{\rm open}_6$ in (4.26) explain the first line of the final result (4.22). The second line of (4.22), however, arises from the factors of ν_{1j} in (4.25) and (4.27) through a sequence of integrations by parts, see appendix D for details. Note that the modular anomalies of (4.24) cancel separately in both $[[\mathcal{K}_6(\ell)]]$ and $[[\tilde{\mathcal{K}}_6(-\ell)]]$, following the mechanisms of section 4.2.3. The BRST anomaly of (4.22) was shown in [33] to yield a boundary term in τ, based on a special case of (II.7.28).

4.3.3. Higher multiplicity

The organization of the closed-string loop integration in the six-point example (4.24) readily generalizes to higher multiplicity. The left-right contractions in the seven-point correlator can be captured via

$$\begin{aligned}
[[\mathcal{K}_7(\ell)\tilde{\mathcal{K}}_7(-\ell)]] &= [[\mathcal{K}_7(\ell)]]\cdot[[\tilde{\mathcal{K}}_7(-\ell)]] + \frac{\pi}{\operatorname{Im}\tau}\Big[\Big[\frac{\partial\mathcal{K}_7(\ell)}{\partial\ell_m}\Big]\Big]\delta^{mn}\Big[\Big[\frac{\partial\tilde{\mathcal{K}}_7(-\ell)}{\partial(-\ell_n)}\Big]\Big] \quad (4.28)\\
&+\frac{1}{2!}\Big(\frac{\pi}{\operatorname{Im}\tau}\Big)^2\Big[\Big[\frac{\partial^2\mathcal{K}_7(\ell)}{\partial\ell_m\partial\ell_n}\Big]\Big]\delta^{mp}\delta^{nq}\Big[\Big[\frac{\partial^2\tilde{\mathcal{K}}_7(-\ell)}{\partial(-\ell_p)\partial(-\ell_q)}\Big]\Big]\\
&+\frac{1}{3!}\Big(\frac{\pi}{\operatorname{Im}\tau}\Big)^3\Big[\Big[\frac{\partial^3\mathcal{K}_7(\ell)}{\partial\ell_m\partial\ell_n\partial\ell_p}\Big]\Big]\delta^{mq}\delta^{nr}\delta^{ps}\Big[\Big[\frac{\partial^3\tilde{\mathcal{K}}_7(-\ell)}{\partial(-\ell_q)\partial(-\ell_r)\partial(-\ell_s)}\Big]\Big]\,,
\end{aligned}$$

where the last line evaluates to $\frac{1}{3!}(\frac{\pi}{\operatorname{Im}\tau})^3 C^{mnp}_{1|2,\ldots}\tilde{C}^{mn}_{1|2,\ldots}$ when the $C\cdot E$ representation (3.76) of $\mathcal{K}_7(\ell)$ is used. By adapting the techniques of appendix D, it should be possible to bring (4.28) into a form similar to (4.22) where all the $g^{(n)}_{ij}$ and $\overline{g}^{(n)}_{ij}$ are completed to $f^{(n)}_{ij}$ and $\overline{f}^{(n)}_{ij}$ through the loop integration. The most laborious part of this calculation might be to identify the seven-point generalization of the terms like $-(\frac{\pi}{\operatorname{Im}\tau})^2|P_{1|2|3,\ldots}|^2$ in (4.22).

Note that the all-multiplicity generalization of (4.28) reads

$$[[\mathcal{K}_n(\ell)\tilde{\mathcal{K}}_n(-\ell)]] = \sum_{r=0}^{n-4} \frac{1}{r!}\Big(\frac{\pi}{\mathrm{Im}\,\tau}\Big)^r \Big[\Big[\frac{\partial^r \mathcal{K}_r(\ell)}{\partial\ell_{m_1}\partial\ell_{m_2}\ldots\partial\ell_{m_r}}\Big]\Big] \\ \times \delta^{m_1p_1}\delta^{m_2p_2}\ldots\delta^{m_rp_r}\Big[\Big[\frac{\partial^r\tilde{\mathcal{K}}_n(-\ell)}{\partial(-\ell_{p_1})\partial(-\ell_{p_2})\ldots\partial(-\ell_{p_r})}\Big]\Big]\,. \tag{4.29}$$

4.4. Closed-string low-energy limits versus open-string correlators

The one-loop low-energy effective action[9] of type-IIB and type-IIA superstrings features a supersymmetrized higher-curvature operator[10] R^4 at its leading order in α' [21]. Hence, the low-energy limit of one-loop closed-string amplitudes yields matrix elements with a single insertion of a supersymmetrized R^4 operator. By inspection of their ($n \leq 7$)-point examples, these matrix elements will be shown to relate to open-string correlators by the duality between pseudo-invariants and GEIs [10].

4.4.1. Up to six points

Once the z_j-dependence of closed-string correlators $[[\mathcal{K}_n(\ell)\tilde{\mathcal{K}}_n(-\ell)]]$ is expressed in terms of $f^{(n)}_{ij}$ and $\overline{f}^{(n)}_{ij}$, their low-energy limit can be conveniently extracted through the techniques of [25,35]. The idea is to perform the α'-expansion of the integrals in (4.2) over the punctures while keeping τ finite[11] in this process. Then, the leftover τ-integration at the leading order in α' straightforwardly yields the volume $\frac{\pi}{3}$ of moduli space.

[9] See [36] for the exact coefficient of the R^4 operator in the type-IIB effective action, including all perturbative and non-perturbative contributions.

[10] While the R^4 operators in the tree-level effective action of the type-IIA and type-IIB theories are identical, at one loop they differ by a contribution proportional to $\epsilon_{10}\epsilon_{10}R^4$ [37,38]. As detailed in [35], the type-IIB matrix elements of this section are proportional to the $\alpha'^3\zeta_3$-order of the respective tree amplitudes, where the proportionality constant depends on the R-symmetry charge of the components (say gravitons or dilatons).

[11] This approach yields a power series in α' that is tailored to infer the one-loop low-energy effective action. The branch cuts of the overall amplitude due to the $\tau \to i\infty$ limit of the moduli-space integral are disentangled when integrating over the z_j at fixed values τ. Still, in analyzing effective interactions beyond the low-energy limit, a subtle interplay between the branch cuts and the power-series part has to be taken into account [39,40].

In this setup, the representations (4.21) and (4.22) of the closed-string correlators are tailored to extract the following low-energy limits [33]

$$\begin{aligned}
\mathcal{M}_4^{R^4} &= C_{1|2,3,4}\tilde{C}_{1|2,3,4} \\
\mathcal{M}_5^{R^4} &= C^m_{1|2,3,4,5}\tilde{C}^m_{1|2,3,4,5} + \big[C_{1|23,4,5}\, s_{23}\tilde{C}_{1|23,4,5} + (2,3|2,3,4,5)\big] \\
\mathcal{M}_6^{R^4} &= \frac{1}{2}C^{mn}_{1|2,3,4,5,6}\tilde{C}^{mn}_{1|2,3,4,5,6} - \big[P_{1|2|3,4,5,6}\tilde{P}_{1|2|3,4,5,6} + (2 \leftrightarrow 3,4,5,6)\big] \\
&\quad + \big[C^m_{1|23,4,5,6}\, s_{23}\tilde{C}^m_{1|23,4,5,6} + (2,3|2,3,4,5,6)\big] \\
&\quad + \big[\big(C_{1|23,45,6}\, s_{23}s_{45}\tilde{C}_{1|23,45,6} + \mathrm{cyc}(3,4,5)\big) + (6 \leftrightarrow 5,4,3,2)\big] \\
&\quad + \big[\big(C_{1|234,5,6}\, s_{23}s_{34}\tilde{C}_{1|234,5,6} + \mathrm{cyc}(2,3,4)\big) + (2,3,4|2,3,4,5,6)\big]\,,
\end{aligned} \tag{4.30}$$

also see [21] and [25,35] for earlier discussions of the four- and five-point examples.

As highlighted in [10], the expressions (4.30) for the matrix elements of R^4 are related to open-string correlators $\mathcal{K}_n(\ell)$ by the duality between kinematics and worldsheet functions. More precisely, the above $\mathcal{M}_n^{R^4}$ can be mapped to the $C \cdot E$ representations of the open-string correlators by trading the right-moving pseudo-invariants for the GEIs with the same slot structure [10],

$$\mathcal{K}_n(\ell) = \mathcal{M}_n^{R^4}\,\big|_{\tilde{C},\tilde{P}\to E}\,. \tag{4.31}$$

This can be checked from the formal rewriting $\mathcal{K}_4(\ell) = C_{1|2,3,4}E_{1|2,3,4}$ of (3.1) and as well as the expressions (3.21) and (3.44) for $\mathcal{K}_5(\ell)$ and $\mathcal{K}_6(\ell)$, respectively.

In the same way as open-string correlators admit a variety of representations, one can rewrite the matrix elements (4.30) such as to manifest their locality properties. The idea is to aim for a kinematic analogue of the $T \cdot E$ representations $\mathcal{K}_4(\ell) = V_1T_{2,3,4}E_{1|2,3,4}$ as well as (3.20) and (3.47) of the open-string correlators. The duality between pseudo-invariants and GEIs translates these manifestly local representations of $\mathcal{K}_n(\ell)$ into

$$\begin{aligned}
\mathcal{M}_4^{R^4} &= V_1T_{2,3,4}\tilde{C}_{1|2,3,4} \\
\mathcal{M}_5^{R^4} &= V_1T^m_{2,3,4,5}\tilde{C}^m_{1|2,3,4,5} + V_1\big[T_{23,4,5}\,\tilde{C}_{1|23,4,5} + (2,3|2,3,4,5)\big] \\
\mathcal{M}_6^{R^4} &= \frac{1}{2}V_1T^{mn}_{2,3,4,5,6}\tilde{C}^{mn}_{1|2,3,4,5,6} - V_1\big[J_{2|3,4,5,6}\tilde{P}_{1|2|3,4,5,6} + (2 \leftrightarrow 3,4,5,6)\big] \\
&\quad + V_1\big[T^m_{23,4,5,6}\,\tilde{C}^m_{1|23,4,5,6} + (2,3|2,3,4,5,6)\big] \\
&\quad + V_1\big[\big(T_{23,45,6}\,\tilde{C}_{1|23,45,6} + \mathrm{cyc}(3,4,5)\big) + (6 \leftrightarrow 5,4,3,2)\big] \\
&\quad + V_1\big[T_{234,5,6}\,\tilde{C}_{1|234,5,6} + T_{243,5,6}\,\tilde{C}_{1|243,5,6} + (2,3,4|2,3,4,5,6)\big]\,.
\end{aligned} \tag{4.32}$$

These representations of the matrix elements are tailored to connect with the Feynman diagrams of the effective action proportional to $R + R^4$: All the propagators stem from the right-moving pseudo-invariants whose expansion in terms of Berends–Giele currents is reviewed in section I.5.2. These Berends–Giele constituents manifest that each term in (4.32) has at most $n-4$ propagators, reflecting at least one vertex of valence ≥ 4 in each diagram.

The equivalence of (4.30) and (4.32) can be checked without any further calculation by exploiting the duality between kinematics and worldsheet functions: Given that $\mathcal{K}_n(\ell)$ and $\mathcal{M}_4^{R^4}$ are related by exchange of pseudo-invariants and GEIs, the manipulations that connect the $T \cdot E$ and $C \cdot E$ representations of the correlators apply in identical form to the matrix elements of R^4. This follows from the observations of section II.5.1 that all the integration-by-parts identities among GEIs at $n \leq 6$ points have a counterpart in the BRST cohomology, relating pseudo-invariants of different tensor rank.

In summary, the low-energy limit of closed-string one-loop amplitudes results in supersymmetrized matrix elements of R^4 that share the structure of open-string correlators, cf. (4.31). Like this, the duality between kinematics and worldsheet functions connects the representations (4.30) and (4.32) and implies that the matrix elements are both local and BRST invariant.

4.4.2. Seven points

As explained in section 4.3.3, the low-energy limit of the closed-string seven-point amplitude may not be readily available from the expression (4.28) for the loop-integrated correlator. Still, it is tempting to invoke the connection between matrix elements of R^4 and open-string correlators to propose a candidate expression on the basis of the $C \cdot E$ representation (3.76) of $\mathcal{K}_7(\ell)$:

$$\begin{aligned}\mathcal{M}_7^{R^4} = {} & \frac{1}{6} C^{mnp}_{1|2,3,4,5,6,7} \tilde{C}^{(s)mnp}_{1|2,3,4,5,6,7} \\ & + \frac{1}{2} C^{mn}_{1|23,4,5,6,7} \tilde{C}^{(s)mn}_{1|23,4,5,6,7} + (2,3|2,3,4,5,6,7) \\ & + \big[C^{m}_{1|234,5,6,7} \tilde{C}^{(s)m}_{1|234,5,6,7} + C^{m}_{1|243,5,6,7} \tilde{C}^{(s)m}_{1|243,5,6,7} \big] + (2,3,4|2,3,4,5,6,7) \\ & + \big[C^{m}_{1|23,45,6,7} \tilde{C}^{(s)m}_{1|23,45,6,7} + \mathrm{cyc}(2,3,4) \big] + (6,7|2,3,4,5,6,7) \\ & + \big[C_{1|2345,6,7} \tilde{C}^{(s)}_{1|2345,6,7} + \mathrm{perm}(3,4,5) \big] + (2,3,4,5|2,3,4,5,6,7) \\ & + \big[C_{1|234,56,7} \tilde{C}^{(s)}_{1|234,56,7} + C_{1|243,56,7} \tilde{C}^{(s)}_{1|243,56,7} + \mathrm{cyc}(5,6,7) \big] + (2,3,4|2,3,4,5,6,7)\end{aligned} \tag{4.33}$$

$$
\begin{aligned}
&+ \left[C_{1|23,45,67}\tilde{C}^{(s)}_{1|23,45,67} + \text{cyc}(4,5,6)\right] + (3 \leftrightarrow 4,5,6,7)\\
&- P^m_{1|2|3,4,5,6,7}\tilde{P}^{(s)m}_{1|2|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\\
&- P_{1|23|4,5,6,7}\tilde{P}^{(s)}_{1|23|4,5,6,7} + (2,3|2,3,4,5,6,7)\\
&- \left[P_{1|2|34,5,6,7}\tilde{P}^{(s)}_{1|2|34,5,6,7} + \text{cyc}(2,3,4)\right] + (2,3,4|2,3,4,5,6,7)\,.
\end{aligned}
$$

The superscripts $^{(s)}$ of the right-moving pseudo-invariants instruct to perform the matrix multiplications with $S(A|A')_a$ as in the definitions (II.4.22) and (II.4.23) of $Z^{(s)}$ and $E^{(s)}$, e.g. $\tilde{C}^{(s)}_{1|23,45,67} = s_{23}s_{45}s_{67}\tilde{C}_{1|23,45,67}$.

In order to validate the proposal (4.33), we shall verify that the BRST invariant expression is at the same time compatible with the locality properties of an R^4 matrix element. Since the seven-point pseudo-invariants obey the relations of the dual GEIs up to the anomalous $\Delta^{\cdots}_{1|\ldots}$ superfields, cf. section II.5.1.3, we can apply the manipulations of the correlator $\mathcal{K}_7(\ell)$ to the above expression for $\mathcal{M}^{R^4}_7$. In the same way as integration-by-parts relations among GEIs yield the $T \cdot E$ representation (3.77) for $\mathcal{K}_7(\ell)$, (4.33) must be equivalent to

$$
\begin{aligned}
\mathcal{M}^{R^4}_7 &= \frac{1}{6}V_1 T^{mnp}_{2,3,4,5,6,7}\tilde{C}^{mnp}_{1|2,3,4,5,6,7} \qquad (4.34)\\
&+ \frac{1}{2}V_1 T^{mn}_{23,4,5,6,7}\tilde{C}^{mn}_{1|23,4,5,6,7} + (2,3|2,3,4,5,6,7)\\
&+ \left[V_1 T^m_{234,5,6,7}\tilde{C}^m_{1|234,5,6,7} + V_1 T^m_{243,5,6,7}\tilde{C}^m_{1|243,5,6,7}\right] + (2,3,4|2,3,4,5,6,7)\\
&+ \left[V_1 T^m_{23,45,6,7}\tilde{C}^m_{1|23,45,6,7} + \text{cyc}(2,3,4)\right] + (6,7|2,3,4,5,6,7)\\
&+ \left[V_1 T_{2345,6,7}\tilde{C}_{1|2345,6,7} + \text{perm}(3,4,5)\right] + (2,3,4,5|2,3,4,5)\\
&+ \left[V_1 T_{234,56,7}\tilde{C}_{1|234,56,7} + V_1 T_{243,56,7}\tilde{C}_{1|243,56,7} + \text{cyc}(5,6,7)\right] + (2,3,4|2,3,4,5,6,7)\\
&+ \left[V_1 T_{23,45,67}\tilde{C}_{1|23,45,67} + \text{cyc}(4,5,6)\right] + (3 \leftrightarrow 4,5,6,7)\\
&- V_1 J^m_{2|3,4,5,6,7}\tilde{P}^m_{1|2|3,4,5,6,7} + (2 \leftrightarrow 3,4,5,6,7)\\
&- V_1 J_{23|4,5,6,7}\tilde{P}_{1|23|4,5,6,7} + (2,3|2,3,4,5,6,7)\\
&- \left[V_1 J_{2|34,5,6,7}\tilde{P}_{1|2|34,5,6,7} + \text{cyc}(2,3,4)\right] + (2,3,4|2,3,4,5,6,7)\,,
\end{aligned}
$$

and we have made a separate check that the BRST non-exact $\Delta_{1|2|3,\ldots}$ are absent. Given that the candidate expression (4.33) is both BRST invariant and local, we expect it to match with the seven-point matrix element of R^4. This corroborates the correspondence (4.31) up to multiplicity seven [10].

4.4.3. Eight points

At eight points, the analysis of section 3.5 led to obstacles in constructing a BRST-invariant and local open-string correlator from the methods of this work. A closely related problem is the availability of the holomorphic Eisenstein series G_4 as a deformation of eight-point GEIs. Any addition of G_4 is compatible with the defining property (II.3.3) of GEIs and the desired modular weight four upon loop integration. While the construction of eight-point GEIs subject to trace relation is left for the future, we shall propose an eight-point candidate for $\mathcal{M}_8^{R^4}$,

$$\begin{aligned}\mathcal{M}_8^{R^4} &= \sum_{r=0}^{4} \frac{1}{r!} C^{m_1\ldots m_r}_{1|A_1,\ldots,A_{r+3}} \tilde{C}^{(s)\, m_1\ldots m_r}_{1|A_1,\ldots,A_{r+3}} + \big[2345678|A_1,\ldots,A_{r+3}\big] \qquad (4.35)\\ &- \sum_{r=0}^{2} \frac{1}{r!} \big[P^{m_1\ldots m_r}_{1|A_1|A_2,\ldots,A_{r+5}} \tilde{P}^{(s)\, m_1\ldots m_r}_{1|A_1|A_2\ldots,A_{r+5}} + (A_1 \leftrightarrow A_2,\ldots,A_{r+5})\big] + \big[23\ldots8|A_1,\ldots,A_{r+5}\big]\\ &+ \big[P_{1|2,3|4,5,6,7,8} \tilde{P}_{1|2,3|4,5,6,7,8} + (2,3|2,3,4,5,6,7,8)\big]\,,\end{aligned}$$

see section 2.3 and appendix A for the notation $[2345678|A_1,\ldots,A_j]$. The BRST variation of (4.35) vanishes by the trace relations (I.5.29) and (I.5.30) among the pseudo-invariants, and we expect it to be equivalent to the following local representation,

$$\begin{aligned}\mathcal{M}_8^{R^4} &= \sum_{r=0}^{4} \frac{1}{r!} V_1 T^{m_1\ldots m_r}_{A_1,\ldots,A_{r+3}} \tilde{C}^{m_1\ldots m_r}_{1|A_1,\ldots,A_{r+3}} + \big[2345678|A_1,\ldots,A_{r+3}\big] \qquad (4.36)\\ &- \sum_{r=0}^{2} \frac{1}{r!} V_1 \big[J^{m_1\ldots m_r}_{A_1|A_2,\ldots,A_{r+5}} \tilde{P}^{m_1\ldots m_r}_{1|A_1|A_2\ldots,A_{r+5}} + (A_1 \leftrightarrow A_2,\ldots,A_{r+5})\big] + \big[23\ldots8|A_1,\ldots,A_{r+5}\big]\\ &+ V_1 \big[J_{2,3|4,5,6,7,8} \tilde{P}_{1|2,3|4,5,6,7,8} + (2,3|2,3,4,5,6,7,8)\big]\,.\end{aligned}$$

By the cohomology identities and trace relations of the right-moving pseudo-invariants, also the local representation (4.36) is BRST invariant, see the detailed argument below. And since all the left-moving superspace building blocks of (4.36) appear in (4.35) with the same right-moving coefficient, the two expressions should be equivalent.

Since the trace relations (II.5.33) among eight-point GEIs exhibit the inhomogeneities proportional to G_4, the duality between kinematics and worldsheet functions does not generate any BRST-invariant candidate correlators from (4.36). But it is encouraging to see that it is only the constant G_4 and none of the vast set of z_j- and ℓ-dependent eight-point $\mathcal{Z}$-functions or GEIs that obstructs the construction of local and BRST-invariant correlators.

4.4.4. Higher multiplicity

In order to obtain a more general perspective on its BRST invariance, we note that (4.36) is closely related to the Lie-series contributions $\mathcal{K}_8^{\rm Lie}(\ell)$ of (2.17): By replacing the $\mathcal{Z}$-functions in (2.21) according to

$$\begin{aligned}\mathcal{Z}^{m_1\ldots m_r}_{1A,B_1,B_2,\ldots,B_{r+3}} &\rightarrow \delta_{A,\emptyset}\tilde{C}^{m_1\ldots m_r}_{1|B_1,B_2,\ldots,B_{r+3}} \\ \mathcal{Z}^{m_1\ldots m_r}_{B_1,\ldots,B_d|1A,B_{d+1},\ldots,B_{r+d+3}} &\rightarrow \delta_{A,\emptyset}\tilde{P}^{m_1\ldots m_r}_{1|B_1,\ldots,B_d|B_{d+1},\ldots,B_{r+d+3}}\end{aligned} \tag{4.37}$$

such that all of the accompanying V_{1A} with $A \neq \emptyset$ are set to zero, we recover (4.36) from $\mathcal{K}_8^{\rm Lie}(\ell)$. Then, the coefficients $\Theta^{(d)}$ and $\Xi^{(d)}$ in (2.28) and (2.29) of the ghost-number four superfields in $Q\mathcal{K}_8^{\rm Lie}(\ell)$ are mapped to

$$\begin{aligned}\Theta^{(0)\,m_1m_2\ldots m_r}_{A|1,B_1,\ldots,B_{r+3}} &\rightarrow k^p_A\tilde{C}^{pm_1m_2\ldots m_r}_{1|A,B_1,B_2,\ldots,B_{r+3}} + \big[\tilde{C}^{m_1m_2\ldots m_r}_{1|S[A,B_1],B_2,\ldots,B_{r+3}} + (B_1\leftrightarrow B_2,\ldots,B_{r+3})\big] \\ &\quad - k_A^{(m_1}\tilde{P}^{m_2\ldots m_r)}_{1|A|B_1,\ldots,B_{r+3}} - \sum_{A=XY}\big(\tilde{P}^{m_1m_2\ldots m_r}_{1|X|Y,B_1,\ldots,B_{r+3}} - (X\leftrightarrow Y)\big)\,, \\ \Theta^{(1)\,m_1m_2\ldots m_r}_{A|B|1,B_1,\ldots,B_{r+4}} &\rightarrow -k^p_A\tilde{P}^{pm_1\ldots m_r}_{1|B|A,B_1,\ldots,B_{r+4}} - \tilde{P}^{m_1\ldots m_r}_{1|S[A,B]|B_1,\ldots,B_{r+4}} \\ &\quad - \big[\tilde{P}^{m_1\ldots m_r}_{1|B|S[A,B_1],B_2,\ldots,B_{r+4}} + (B_1\leftrightarrow B_2,\ldots,B_{r+4})\big] \\ &\quad + k_A^{(m_1}\tilde{P}^{m_2\ldots m_r)}_{1|A,B|B_1,\ldots,B_{r+4}} + \sum_{A=XY}\big(\tilde{P}^{m_1m_2\ldots m_r}_{1|X,B|Y,B_1,\ldots,B_{r+4}} - (X\leftrightarrow Y)\big)\,,\end{aligned} \tag{4.38}$$

as well as

$$\begin{aligned}\Xi^{(0)\,m_1m_2\ldots m_r}_{1|B_1,\ldots,B_{r+5}} &\rightarrow -\frac{1}{2}\tilde{C}^{ppm_1\ldots m_r}_{1|B_1,\ldots,B_{r+5}} + \big[\tilde{P}^{m_1\ldots m_r}_{1|B_1|B_2,\ldots,B_{r+5}} + (B_1\leftrightarrow B_2,\ldots B_{r+5})\big]\,, \\ \Xi^{(1)\,m_1m_2\ldots m_r}_{1|A|B_1,\ldots,B_{r+6}} &\rightarrow \frac{1}{2}\tilde{P}^{ppm_1\ldots m_r}_{1|A|B_1,\ldots,B_{r+6}} - \big[\tilde{P}^{m_1\ldots m_r}_{1|A,B_1|B_2,\ldots,B_{r+6}} + (B_1\leftrightarrow B_2,\ldots B_{r+6})\big]\,.\end{aligned} \tag{4.39}$$

At multiplicity eight, (4.39) vanishes, and all instances of (4.38) boil down to the anomalous $\Delta_{1|\ldots}$ superfields by the results of section I.5.4. The BRST non-exact $\Delta_{1|\ldots}$ in turn drop out from $Q\mathcal{M}_8^{R^4}$ by the trace relations of the local superfields at ghost-number four, confirming BRST invariance of (4.36). Moreover, all of $\Xi^{(0)}, \Xi^{(1)}$ and $\Theta^{(0)}$ at arbitrary higher multiplicity are mapped to zero or $\Delta_{1|\ldots}$ under (4.37) – see e.g. (I.5.43).

For the images of $\Theta^{(1)}$, by contrast, only their ($n \leq 8$)-point instances are known to reduce to $\Delta_{1|\ldots}$ in the BRST cohomology, see (I.5.44). It is an open question whether the same is true at $n \geq 9$ points and for generalizations $\Theta^{(d)}$ with higher refinement $d \geq 2$.

Note, however, that the vanishing kinematic factors on the right-hand side of (4.39) are the result of translating $\mathcal{Z}$-functions to kinematic factors via (4.37). At the level of open-string correlators, i.e. before applying the (non-invertible) map (4.37), the $\Xi^{(d)}$ are generically non-zero, cf. (3.84).

Up to these open questions on the pseudo-invariants, it appears likely to arrive at BRST-invariant and local expressions for n-point matrix elements of R^4 by applying the map (4.37) to $\mathcal{K}_n^{\rm Lie}(\ell)$. Then, the leftover task to generate BRST-invariant and local correlators in a $T\cdot E$ representation would be to identify a suitable system of GEIs: Such $T\cdot E$ representations of $\mathcal{K}_n(\ell)$ would follow from $\mathcal{M}_n^{R^4}$ through the duality between pseudo-invariants and GEIs if the latter can be made to

- satisfy all the trace relations dual to those of the pseudo-invariants
- obey the analogue of the condition $\Theta^{(d)}=0$ (possibly up to analogues of the BRST non-exact anomaly superfields $\Delta_{1|\ldots}$, cf. the objects $G_{1|\ldots}$ in (II.5.6) and (II.5.7)),

$$
\begin{aligned}
0 \cong -k_A^p E^{pm_1\ldots m_r}_{1|B_1,\ldots,B_d|A,C_1,\ldots,C_{r+d+3}} &- \big[E^{m_1\ldots m_r}_{1|S[A,B_1],B_2,\ldots,B_d|C_1,\ldots,C_{r+d+3}} + (B_1\leftrightarrow B_2,\ldots,B_d)\big]\\
&- \big[E^{m_1\ldots m_r}_{1|B_1,\ldots,B_d|S[A,C_1],C_2,\ldots,C_{r+d+3}} + (C_1\leftrightarrow C_2,\ldots,C_{r+d+3})\big] \qquad (4.40)\\
&+ k_A^{(m_1}E^{m_2\ldots m_r)}_{1|A,B_1,\ldots,B_d|C_1,\ldots,C_{r+d+3}} + \sum_{A=XY}\big(E^{m_1m_2\ldots m_r}_{1|X,B_1,\ldots,B_d|Y,C_1,\ldots,C_{r+d+3}} - (X\leftrightarrow Y)\big)\,.
\end{aligned}
$$

In case one succeeds in generating local and BRST invariant $T\cdot E$ representations for $\mathcal{K}_n(\ell)$ in this way, one would still have to find and inspect the corresponding $T\cdot\mathcal{Z}$ form: For consistency with the OPEs among vertex operators, the correlators must admit a representation, where the slots of the multiparticle superfields in $V_A, T^{m_1,\ldots}_{B,C,\ldots}$ and $J^{m_1,\ldots}_{B_1,\ldots,B_d|C,\ldots}$ line up with the singularity structure of the accompanying $g^{(1)}_{ij}$. As a drawback of the GEIs in $T\cdot E$ representations, their slot structure does not expose the singularities of the $g^{(1)}_{ij}$.

In spite of the large list of open questions, we are optimistic that the above ideas will on the long run guide a path towards an explicit n-point open-string correlator.

5. Conclusions

It is appropriate to summarize the achievements and future directions arising from this series of three papers [1]. We have presented a method to determine manifestly local one-loop correlators of the pure-spinor superstring. Their dependence on the external polarizations is organized in terms of BRST-covariant building blocks discussed in part I.

A bootstrap procedure is introduced to assemble the accompanying worldsheet functions from loop momenta and coefficients of the Kronecker–Eisenstein series. As a key input of the bootstrap, the monodromies of the worldsheet function around the B-cycle are taken to mirror the BRST variations of the associated kinematic factors. This is a first example for a multifaceted *duality* between kinematics and worldsheet functions described in part II.

The bootstrap approach results in shuffle-symmetric worldsheet functions that conspire with the Lie symmetries of the kinematic factors: The two kinds of ingredients combine into a Lie-polynomial structure which leads to a natural ansatz for manifestly local n-point correlators. Up to six points, the Lie polynomials are BRST-invariant by themselves and reproduce the non-local correlators known from earlier work [33]. At seven points, the Lie-polynomial ansatz exhibits a simple BRST variation which can be cancelled by adding a local collection of certain *anomalous* superfields to the full correlator. Starting from eight points, however, an anomalous BRST variation along with the holomorphic Eisenstein series G_4 remains uncancelled. Like this, we can only give an incomplete proposal for the eight-point correlator, leaving a single kinematic factor along with G_4 undetermined. We leave it as an open problem for the future to understand the systematics of Eisenstein series G_k in $(n{\geq}8)$-point correlators.

Further aspects of the duality between kinematics and worldsheet functions concern the BRST-(pseudo-)invariants obtained from certain non-local combinations of kinematic building blocks [22]. By exporting their underlying combinatorial pattern to the shuffle-symmetric worldsheet functions, one is led to the notion of *generalized elliptic integrands* (GEIs) whose B-cycle monodromies cancel upon integration over the loop momentum. GEIs are observed to share the relations of the dual kinematic factors up to seven points, but the preliminary definition of eight-point GEIs are found to violate certain trace relations by a factor of G_4. Hence, it remains to incorporate G_k into $(n{\geq}8)$-point GEIs in order to realize the duality between kinematics and worldsheet functions at all multiplicities.

We rewrite the $(n{\leq}7)$-point correlators in terms of (pseudo-)invariants and/or GEIs such as to manifest the respective kinds of invariances. When both of BRST-invariance and monodromy invariance are manifested, the (pseudo-)invariants and GEIs are found to enter on completely symmetric footing. This kind of exchange symmetry between kinematics and worldsheet functions is reminiscent of the disk amplitudes of [5,41], where gauge-theory trees and Parke–Taylor integrands are freely interchangeable. Hence, the observed duality between kinematics and worldsheet functions up to and including seven points induces a double-copy structure in one-loop open-superstring amplitudes [10]. In the same way as disk amplitudes are dual to supergravity trees when replacing worldsheet integrals by kinematics, the duality maps one-loop open-superstring amplitudes to matrix elements of the supersymmetrized higher-curvature operator R^4.

The results of this work result suggest a variety of follow-up directions.

Higher genus: Most obviously, the systems of BRST-covariant kinematic building blocks and shuffle-symmetric worldsheet functions call for an extension to higher genus. First instances of BRST-covariant vectorial superfields have been studied in the low-energy regime of two-loop five-point [42,43] and three-loop four-point amplitudes [44,45]. The principle of BRST-covariance should guide their systematic generalizations to higher tensor ranks as well as analogues of the refined and anomalous building blocks of this work.

As to the worldsheet functions, one would need to identify a higher-genus generalization of the Kronecker–Eisenstein series and its expansion coefficients, where the elliptic functions of [46] may play a role. It would be interesting to extend the duality between worldsheet functions and kinematics – in particular between monodromy and BRST variations – to the multiloop level.

Gravitational operators versus open-string correlators: There is an intuitive reason to find the matrix elements of R^4 and no other gravitational operator as the kinematic duals to one-loop open-string correlators: The supersymmetrized higher-curvature operator R^4 governs the low-energy limit of the corresponding closed-string amplitudes. Accordingly, the supersymmetrized matrix elements of[12] D^4R^4 and D^6R^4 are likely to imprint their double-copy structure on two-loop and three-loop open-string correlators.

In one-loop string amplitudes with reduced supersymmetry in turn, the closed-string low-energy limit results in matrix elements of R^2 [47]. Hence, the open-string one-loop correlators with half- and quarter-maximal supersymmetry of [48] should share the double-copy structure of R^2 involving GEIs similar to the ones in this work.

Field-theory limits and ambitwistors: The framework of chiral splitting is a natural starting point to determine loop integrands of super-Yang–Mills and supergravity in momentum space from the field-theory limit. We leave it to follow-up work to investigate the $\tau \to i\infty$ degeneration of GEIs relevant to field-theory amplitudes and the emergence of new color-kinematics dual representations.

Moreover, the superstring correlators of this work can be exported to the one-loop amplitudes of the ambitwistor string [49,50]. It will then be interesting to explore the

[12] The shorthands D^4R^4 and D^6R^4 are understood to comprise the companion terms $D^2R^5+R^6$ and $D^4R^5 + D^2R^6 + R^7$ of the same mass dimension determined by non-linear supersymmetry.

interplay of GEIs with the color-kinematics dual field-theory amplitudes obtained from the methods of [51,52]. The same questions will arise at higher genus [53,54].

GEIs and scalar amplitudes: The double-copy structure of open-string tree amplitudes [5,41] motivated the interpretation of Parke–Taylor-type disk integrals as scattering amplitudes in effective field theories of scalars. Indeed, the low-energy limit of disk integrals reproduces the tree amplitudes of bi-adjoint scalars with a ϕ^3 interaction [55] and Goldstone bosons [56,57]. Similarly, higher orders in their α'-expansion suggest higher-mass-dimension deformations of the respective Lagrangians collectively referred to as *Z-theory* [56,58,57].

In one-loop string amplitudes, GEIs are found to play a role similar to the Parke–Taylor factors at tree level. Hence, it is tempting to compare the moduli-space integrals of GEIs with loop integrands in scalar field theories – for worldsheets of both toroidal and cylinder topology. Also, it will be interesting to compare such integrated GEIs with the forward limits of Z-theory amplitudes.

Connections with combinatorics: After observing that several patterns and identities obeyed by the BRST pseudo-invariants are also satisfied by the GEIs, one is left wondering if these *kinematic* and *worldsheet-function* invariants could be a manifestation of a more fundamental mathematical property of objects constructed from building blocks subject to the shuffle symmetries. After all, the combinatorics of these "invariants" can be generated by linear maps acting on words that also feature prominently in the free-Lie-algebra literature. We suspect that many combinatorial algorithms on words have direct relevance to the study of scattering amplitudes and in particular string-theory correlators, and that many relations among amplitudes can be understood in terms of free-Lie-algebra structures.

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Appendix A. Stirling cycle permutation sums

In order to explain the Stirling cycle permutation sums used throughout this work it is convenient to start by briefly recalling the definition, using the notation and terminology proposed in [17], of the *Stirling cycle numbers* $\left[{n \atop p}\right]$ and *Stirling set numbers* $\left\{{n \atop p}\right\}$.

The Stirling set number $\left\{{n \atop p}\right\}$ represents the number of ways to partition a set of n elements into p non-empty sets [59]. For example, $\left\{{4 \atop 2}\right\} = 7$ because there are seven ways to split the set $\{1, 2, 3, 4\}$ into two non-empty subsets:

$$\begin{aligned}&\{1,2,3\}\cup\{4\}, \quad \{1,2,4\}\cup\{3\}, \quad \{1,3,4\}\cup\{2\}, \quad \{2,3,4\}\cup\{1\}, \\ &\{1,2\}\cup\{3,4\}, \quad \{1,3\}\cup\{2,4\}, \quad \{1,4\}\cup\{2,3\}.\end{aligned} \tag{A.1}$$

The Stirling cycle number $\left[{n \atop p}\right]$ is closely related and represents the number of ways to split n objects into p cycles[13]. It is easy to write down the different arrangements of cycles once the Stirling set partitions have been worked out: simply convert a given k-element subset into its $(k-1)!$ distinct cycles as $\{1, 2, \ldots, k\} \to (123\ldots k) + \mathrm{perm}(2, 3, \ldots, k)$. For example, using the above subset decomposition of $\left\{{4 \atop 2}\right\}$ we obtain $\left[{4 \atop 2}\right] = 11$:

$$\begin{aligned}&(123)(4), \quad (132)(4), \quad (124)(3), \quad (142)(3), \quad (134)(2), \quad (143)(2), \\ &(1)(234), \quad (1)(243), \quad (12)(34), \quad (13)(24), \quad (14)(23)\,.\end{aligned} \tag{A.2}$$

Since there is no unique way of representing a product of disjoint cycles we fix this ambiguity by ordering the cycles as follows:

i. each cycle is written with its smallest element first, (A.3)

ii. the cycles are written in increasing order of its smallest element.

For example, (65)(471)(23) becomes (147)(23)(56). Given the above conventions we can now define:

Definition 2. *The Stirling cycle permutation sum of a generic object* $S_{A_1,A_2,\ldots,A_p}$ *with* p *slots is denoted by*

$$S_{A_1,A_2,\ldots,A_p} + [1, 2, \ldots, n|A_1, A_2, \ldots, A_p]\,, \tag{A.4}$$

and it represents the sum over all $\left[{n \atop p}\right]$ *ways to partition the set* $\{1, 2, \ldots, n\}$ *into* p *cycles, ordered according to (A.3), and that are distributed to* $S_{A_1,\ldots,A_p}$ *as follows,*

$$(a_1 \ldots a_{n_a})(b_1 \ldots b_{n_b}) \cdots (p_1 \ldots p_{n_p}) \to S_{a_1\ldots a_{n_a},b_1\ldots b_{n_b},\ldots,p_1\ldots p_{n_p}}\,.$$

[13] A cycle is defined up to cyclic rearrangements; $(12\ldots k) = (23\ldots k1)$.

To illustrate the above definition, let us consider $C_{1|A,B,C} + \big[2,3,4,5,6|A,B,C\big]$. In this case, all the 35 partitions of $\{2,3,4,5,6\}$ into 3 cycles ordered according to the above convention are given by

$$
\begin{aligned}
&(2)(3)(456),\ (2)(3)(465),\ (2)(356)(4),\ (2)(365)(4),\ (2)(346)(5),\\
&(2)(364)(5),\ (2)(345)(6),\ (2)(354)(6),\ (256)(3)(4),\ (265)(3)(4),\\
&(246)(3)(5),\ (264)(3)(5),\ (245)(3)(6),\ (254)(3)(6),\ (236)(4)(5),\\
&(263)(4)(5),\ (235)(4)(6),\ (253)(4)(6),\ (234)(5)(6),\ (243)(5)(6),\\
&(2)(34)(56),\ (2)(35)(46),\ (2)(36)(45),\ (24)(3)(56),\ (25)(3)(46),\\
&(26)(3)(45),\ (23)(4)(56),\ (25)(36)(4),\ (26)(35)(4),\ (23)(46)(5),\\
&(24)(36)(5),\ (26)(34)(5),\ (23)(45)(6),\ (24)(35)(6),\ (25)(34)(6)\,.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
C_{1|A,B,C} + \big[2,3,4,5,6|A,B,C\big] = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (A.5)\\
C_{1|2,3,456} + C_{1|2,3,465} + C_{1|2,356,4} + C_{1|2,365,4} + C_{1|2,346,5} + C_{1|2,364,5} + C_{1|2,345,6}\\
+C_{1|2,354,6} + C_{1|256,3,4} + C_{1|265,3,4} + C_{1|246,3,5} + C_{1|264,3,5} + C_{1|245,3,6} + C_{1|254,3,6}\\
+C_{1|236,4,5} + C_{1|263,4,5} + C_{1|235,4,6} + C_{1|253,4,6} + C_{1|234,5,6} + C_{1|243,5,6} + C_{1|2,34,56}\\
+C_{1|2,35,46} + C_{1|2,36,45} + C_{1|24,3,56} + C_{1|25,3,46} + C_{1|26,3,45} + C_{1|23,4,56} + C_{1|25,36,4}\\
+C_{1|26,35,4} + C_{1|23,46,5} + C_{1|24,36,5} + C_{1|26,34,5} + C_{1|23,45,6} + C_{1|24,35,6} + C_{1|25,34,6}\,.
\end{aligned}
$$

For some typical numbers appearing in this work, we note that the total number of terms in the local representation (2.15) of $\mathcal{K}^{(0)}_n(\ell)$ is given by $T_n \equiv \big[{n \atop 4}\big] + \big[{n \atop 5}\big] + \cdots + \big[{n \atop n}\big]$, while in the corresponding manifestly BRST-invariant representation they become $C_n \equiv \big[{n-1 \atop 3}\big] + \big[{n-1 \atop 4}\big] + \cdots + \big[{n-1 \atop n-1}\big]$. For example, $T_n = 1, 11, 101, 932, 9080, 94852, 1066644$ and $C_n = 1, 7, 46, 326, 2556, 22212, 212976$ for $n = 4,5,6,7,8,9,10$.

A.0.1. Stirling cycle permutations of the seven-point $d=0$ correlator

For convenience and to provide yet another explicit example of a Stirling cycle permutation sum, we write down the complete expansion of the unrefined Lie polynomials in (3.54),

$$
\begin{aligned}
\mathcal{K}^{(0)}_7(\ell) &= \frac{1}{6} V_1 T^{mnp}_{2,3,4,5,6,7} \mathcal{Z}^{mnp}_{1,2,3,4,5,6,7} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (A.6)\\
&\quad + \frac{1}{2}\big[V_{12} T^{mn}_{3,4,5,6,7} \mathcal{Z}^{mn}_{12,3,4,5,6,7} + (2 \leftrightarrow 3,\ldots,7)\big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{2}\big[V_1 T^{mn}_{23,4,5,6,7}\mathcal{Z}^{mn}_{1,23,4,5,6,7} + (2,3|2,3,\ldots,7)\big]\\
&+ \big[(V_{123}T^m_{4,5,6,7}\mathcal{Z}^m_{123,4,5,6} + V_{132}T^m_{4,5,6,7}\mathcal{Z}^m_{132,4,5,6}) + (2,3|2,3,\ldots,7)\big]\\
&+ \big[\big(V_1 T^m_{234,5,6,7}\mathcal{Z}^m_{1,234,5,6,7} + V_1 T^m_{243,5,6,7}\mathcal{Z}^m_{1,243,5,6,7}\big) + (2,3,4|2,3,\ldots,7)\big]\\
&+ \big[\big(V_{12}T^m_{34,5,6,7}\mathcal{Z}^m_{12,34,5,6,7} + (3,4|3,4,\ldots,7)\big) + (2\leftrightarrow 3,\ldots,7)\big]\\
&+ \big[\big(V_1 T^m_{23,45,6,7}\mathcal{Z}^m_{1,23,45,6,7} + \mathrm{cyc}(2,3,4)\big) + (6,7|2,3,\ldots,7)\big]\\
&+ \big[\big(V_{1234}T_{5,6,7}\mathcal{Z}_{1234,5,6,7} + \mathrm{perm}(2,3,4)\big) + (2,3,4|2,3,\ldots,7)\big]\\
&+ \big[\big(V_{123}T_{45,6,7}\mathcal{Z}_{123,45,6,7} + V_{132}T_{45,6,7}\mathcal{Z}_{132,45,6,7} + (4,5|4,5,6,7)\big) + (2,3|2,\ldots,7)\big]\\
&+ \big[\big(V_{12}T_{345,6,7}\mathcal{Z}_{12,345,6,7} + V_{12}T_{354,6,7}\mathcal{Z}_{12,354,6,7} + \mathrm{cyc}(2,3,4,5)\big) + (6,7|2,\ldots,7)\big]\\
&+ \big[\big(V_{12}T_{3,45,67}\mathcal{Z}_{12,3,45,67} + V_{13}T_{2,45,67}\mathcal{Z}_{13,2,45,67} + \mathrm{cyc}(4,5,6)\big) + (2,3|2,\ldots,7)\big]\\
&+ \big[\big(V_1 T_{2345,6,7}\mathcal{Z}_{1,2345,6,7} + \mathrm{perm}(3,4,5)\big) + (2,3,4,5|2,3,\ldots,7)\big]\\
&+ \big[\big(V_1 T_{23,4,567}\mathcal{Z}_{1,23,4,567} + V_1 T_{23,4,576}\mathcal{Z}_{1,23,4,576} + \mathrm{cyc}(2,3,4)\big) + (2,3,4|2,\ldots,7)\big]\\
&+ \big[\big(V_1 T_{23,45,67}\mathcal{Z}_{1,23,45,67} + \mathrm{cyc}(4,5,6)\big) + (3\leftrightarrow 4,5,6,7)\big]\,.
\end{aligned}
$$

It is straightforward but tedious to see that there are 932 terms above, reproducing the number $T_7 = 932$ discussed above.

A.1. Lie polynomials

There are several characterizations of a Lie polynomial in the mathematics literature, see for example [16]. For our purposes, Lie polynomials are composed by linear combinations of nested commutators in a given set of non-commutative indeterminates. For example, if $t^{a_1}, t^{a_2}, t^{a_3}$ are non-commutative, $P = [t^{a_1}, [t^{a_2}, t^{a_3}]] + 3[[t^{a_1}, t^{a_2}], t^{a_3}]$ is a Lie polynomial while $N = t^{a_1}t^{a_2}t^{a_3}$ is not.

The identification of a Lie-polynomial structure within the correlators of this work stems from a theorem proved by Ree [6]. Using the notation of section I.3.1, the theorem states that if M_A satisfies shuffle symmetries (i.e., $M_{R\sqcup\!\sqcup S} = 0$, for any $R, S \neq \emptyset$) and t^{a_i} are non-commutative indeterminates with $t^A \equiv t^{a_1}t^{a_2}\ldots t^{a_p}$, a sum over all words A of length p of the form

$$
P = \sum_A M_A t^A \tag{A.7}
$$

gives rise to a Lie polynomial of degree p. For example, at degree two the shuffle symmetry on $M_{a_1a_2}$ implies that $M_{a_2a_1} = -M_{a_1a_2}$ and the sum (A.7) becomes $P = M_{a_1a_2}t^{a_1}t^{a_2} + M_{a_2a_1}t^{a_2}t^{a_1} = M_{a_1a_2}[t^{a_1}, t^{a_2}]$. Hence P is a Lie polynomial.

In general, the sum in (A.7) can be rewritten as a sum proportional to $\sum M_A t^{\ell(A)}$ where $\ell(A)$ is the Dynkin map defined in (I.3.8). Thus, the Lie polynomial arising from (A.7) has the form of a sum over products of objects satisfying shuffle symmetries and objects satisfying generalized Jacobi symmetries. Schematically we have $P = \sum(\text{shuffle})(\text{Lie})$. This is precisely the structure within each word (slot) in the local form of the one-loop correlators found in this work, see for example (2.15).

Appendix B. Monodromy invariance of the six-point correlator

In this appendix we demonstrate the monodromy invariance of the six-point correlator in its local representation (3.23) and thereby provide an alternative to the proof in section 3.3.2 with manifest BRST invariance. It will be convenient to define the following shorthands:

$$\begin{aligned}
X^{(a)}_{1|23,4,5,6} &\equiv V_1 k_1^p T^p_{23,4,5,6} - V_{231}T_{4,5,6} - V_{41}T_{23,5,6} - V_{51}T_{23,4,6} - V_{61}T_{23,4,5} \qquad \text{(B.1)}\\
X^{(a)m}_{1|2,3,4,5,6} &\equiv V_1 k_1^n T^{mn}_{2,3,4,5,6} - V_{21}T^m_{3,4,5,6} - V_{31}T^m_{2,4,5,6} - V_{41}T^m_{2,3,5,6} - V_{51}T^m_{2,3,4,6} - V_{61}T^m_{2,3,4,5}\\
X^{(b)}_{13|2|4,5,6} &\equiv V_{13} k_2^p T^p_{2,4,5,6} - V_{132}T_{4,5,6} - V_{13}T_{42,5,6} - V_{13}T_{52,4,6} - V_{13}T_{62,4,5}\\
X^{(b)}_{1|2|34,5,6} &\equiv V_1 k_2^p T^p_{2,34,5,6} - V_{12}T_{34,5,6} - V_1T_{342,5,6} - V_1T_{34,52,6} - V_1T_{34,62,5}\\
X^{(b)m}_{1|2|3,4,5,6} &\equiv V_1 k_2^n T^{mn}_{2,3,4,5,6} - V_{12}T^m_{3,4,5,6} - V_1T^m_{32,4,5,6} - V_1T^m_{42,3,5,6} - V_1T^m_{52,3,4,6} - V_1T^m_{62,3,4,5}\,.
\end{aligned}$$

Using the monodromy variations of the six-point functions (II.A.2) we get the following variation of the correlator (3.23),

$$D\mathcal{K}_6(\ell) = \Omega_1\delta\mathcal{K}^{(1)}_6 + \Omega_2\delta\mathcal{K}^{(2)}_6 + \cdots + \Omega_6\delta\mathcal{K}^{(6)}_6\,, \qquad \text{(B.2)}$$

where

$$\delta\mathcal{K}^{(1)}_6 = E^m_{1|2,3,4,5,6}X^{(a)m}_{1|2,3,4,5,6} + \big[E_{1|23,4,5,6}X^{(a)}_{1|23,4,5,6} + (2,3|2,3,4,5,6)\big] \qquad \text{(B.3)}$$

$$\begin{aligned}
\delta\mathcal{K}^{(2)}_6 &= E^m_{2|1,3,4,5,6}X^{(b)m}_{1|2|3,4,5,6} + \big[E_{2|13,4,5,6}X^{(b)}_{13|2|4,5,6} + (3\leftrightarrow 4,5,6)\big] \qquad \text{(B.4)}\\
&\quad + \big[E_{2|34,1,5,6}\,X^{(b)}_{1|2|34,5,6} + (3,4|3,4,5,6)\big]\,,
\end{aligned}$$

and the other $\delta\mathcal{K}^{(i)}_6$ for $i=3,4,5,6$ are obtained by relabeling of $\delta\mathcal{K}^{(2)}_6$ in (B.4). Since the bookkeeping variables Ω_i are independent, all the $\delta\mathcal{K}^{(i)}_6$ must vanish separately.

For the Ω_1 terms in (B.3), after using the BRST identities

$$\begin{aligned}
QJ^m_{1|2,3,4,5,6} &= X^{(a)m}_{1|2,3,4,5,6} + \Delta^m_{1,2,3,4,5,6} + \big[k_2^m(V_2J_{1|3,4,5,6} - \mathcal{Y}_{12,3,4,5,6}) + (2\leftrightarrow 3,4,5,6)\big]\\
QJ_{1|23,4,5,6} &= X^{(a)}_{1|23,4,5,6} + s_{23}(V_2J_{1|3,4,5,6} - V_3J_{1|2,4,5,6}) + Y_{1,23,4,5,6}
\end{aligned}$$

together with their elliptic counterpart $k_2^m E^m_{1|2,3,4,5,6} = -\big[s_{23}E_{1|23,4,5,6} + (3 \leftrightarrow 4,5,6)\big]$, one arrives at a BRST-exact variation since the unrefined $\Delta^{m\cdots}_{A|B,C,\ldots}$ are BRST exact [22],

$$\delta K_6^{(1)} = E^m_{1|2,3,4,5,6}\Big(QJ^m_{1|2,3,4,5,6} - \Delta^m_{1|2,3,4,5,6}\Big) + \Big[E_{1|23,4,5,6}\big(QJ_{1|23,4,5,6} - s_{23}\Delta_{1|23,4,5,6}\big) + (2,3|2,3,4,5,6)\Big]. \tag{B.5}$$

For the Ω_2 terms in (B.4), upon using BRST identities (see section 8 of [22] for the $D_{\ldots}$)

$$QD^m_{1|2|3,4,5,6} = X^{(b)m}_{1|2|3,4,5,6} - k_2^m V_1 J_{2|3,4,5,6} + \Delta^m_{1|2,3,4,5,6} + \Big[\frac{k_3^m}{s_{13}}X^{(b)}_{13|2|4,5,6} + (3 \leftrightarrow 4,5,6)\Big]$$
$$s_{34}QD_{1|2|34,5,6} = X^{(b)}_{1|2|34,5,6} + \frac{s_{34}}{s_{13}}X^{(b)}_{13|2|4,5,6} - \frac{s_{34}}{s_{14}}X^{(b)}_{14|2|3,5,6} + s_{34}\Delta_{1|34,2,5,6} \tag{B.6}$$

and relabellings of $k_2^m E^m_{1|2,3,4,5,6} = -\big[s_{23}E_{1|23,4,5,6} + (3 \leftrightarrow 4,5,6)\big]$, one gets

$$\delta K_6^{(2)} = E^m_{2|1,3,4,5,6}(QD^m_{1|2|3,4,5,6} - \Delta^m_{1|2,3,4,5,6}) + \big[E_{2|34,1,5,6}(s_{34}QD_{1|2|34,5,6} - s_{34}\Delta_{1|34,2,5,6}) + (3,4|3,4,5,6)\big] \tag{B.7}$$

which vanishes in the cohomology for the same reason as above. Therefore, the six-point correlator (3.23) is confirmed to be single valued.

The above proof can be extended to higher-point correlators, but since it is simpler to prove monodromy invariance using a non-local representation with manifest BRST invariance (see section 3.3.2), we will omit further discussions.

Appendix C. Vanishing linear combinations of worldsheet functions

In this appendix we write down a few explicit expansions of the vanishing linear combinations of worldsheet functions given by $\Theta^{(d)}$ from (2.28).

At six points, the three topologies of worldsheet functions were expanded in (3.27) and are easily checked to be zero.

C.1. Seven points

At seven points, the inequivalent topologies of $\Theta^{(0)}$ are given by

$$\Theta^{(0)}_{2|1,34,56,7} = k_2^m \mathcal{Z}^m_{1,2,34,56,7} - s_{12}\mathcal{Z}_{12,34,56,7} + s_{27}\mathcal{Z}_{1,27,34,56} + s_{23}\mathcal{Z}_{1,234,56,7} - s_{24}\mathcal{Z}_{1,243,56,7} + s_{25}\mathcal{Z}_{1,256,34,7} - s_{26}\mathcal{Z}_{1,265,34,7} \tag{C.1}$$

$$
\begin{aligned}
\Theta^{(0)}_{23|1,45,6,7} &= k^m_{23}\mathcal{Z}^m_{1,23,45,6,7} - s_{12}\mathcal{Z}_{123,45,6,7} + s_{13}\mathcal{Z}_{132,45,6,7} \\
&\quad - s_{24}\mathcal{Z}_{1,3245,6,7} + s_{34}\mathcal{Z}_{1,2345,6,7} + s_{25}\mathcal{Z}_{1,3254,6,7} - s_{35}\mathcal{Z}_{1,2354,6,7} \\
&\quad + \big[s_{36}\mathcal{Z}_{1,236,45,7} - s_{26}\mathcal{Z}_{1,326,45,7} + (6\leftrightarrow 7)\big] \\
&\quad - \mathcal{Z}_{2|3,1,45,6,7} + \mathcal{Z}_{3|2,1,45,6,7} \\
\Theta^{(0)}_{234|1,5,6,7} &= k^m_{234}\mathcal{Z}^m_{1,234,5,6,7} + \big[s_{13}(\mathcal{Z}_{1324,5,6,7} + \mathcal{Z}_{1342,5,6,7}) \\
&\qquad - s_{12}\mathcal{Z}_{1234,5,6,7} - s_{14}\mathcal{Z}_{1432,5,6,7} + (1\leftrightarrow 5,6,7)\big] \\
&\quad - \mathcal{Z}_{2|34,1,5,6,7} - \mathcal{Z}_{23|4,1,5,6,7} + \mathcal{Z}_{34|2,1,5,6,7} + \mathcal{Z}_{4|23,1,5,6,7} \\
\Theta^{(0)}_{4|123,5,6,7} &= k^m_4\mathcal{Z}^m_{123,4,5,6,7} + s_{34}\mathcal{Z}_{4321,5,6,7} + s_{14}\mathcal{Z}_{4123,5,6,7} \\
&\quad - s_{24}(\mathcal{Z}_{4213,5,6,7} + \mathcal{Z}_{4231,5,6,7}) + \big[s_{45}\mathcal{Z}_{123,45,6,7} + (5\leftrightarrow 6,7)\big] \\
\Theta^{(0)\,m}_{2|1,34,5,6,7} &= k^n_2\mathcal{Z}^{mn}_{1,2,34,5,6,7} - s_{12}\mathcal{Z}^m_{12,34,5,6,7} + s_{23}\mathcal{Z}^m_{1,234,5,6,7} - s_{24}\mathcal{Z}^m_{1,243,5,6,7} \\
&\quad + \big[s_{25}\mathcal{Z}^m_{1,25,34,6,7} + (5\leftrightarrow 6,7)\big] - k^m_2\mathcal{Z}_{2|1,34,5,6,7} \\
\Theta^{(0)\,m}_{23|1,4,5,6,7} &= k^n_{23}\mathcal{Z}^{mn}_{1,23,4,5,6,7} - s_{12}\mathcal{Z}^m_{123,4,5,6,7} + s_{13}\mathcal{Z}^m_{132,4,5,6,7} \\
&\quad + \big[s_{34}\mathcal{Z}^m_{1,234,5,6,7} - s_{24}\mathcal{Z}^m_{1,324,5,6,7} + (4\leftrightarrow 5,6,7)\big] \\
&\quad - k^m_{23}\mathcal{Z}_{23|1,4,5,6,7} - \mathcal{Z}^m_{2|3,1,4,5,6,7} + \mathcal{Z}^m_{3|2,1,4,5,6,7} \\
\Theta^{(0)\,mn}_{2|1,3,4,5,6,7} &= k^p_2\mathcal{Z}^{mnp}_{1,2,3,4,5,6,7} - s_{12}\mathcal{Z}^{mn}_{12,3,4,5,6,7} + \big[s_{23}\mathcal{Z}^{mn}_{1,23,4,5,6,7} + (3\leftrightarrow 4,5,6,7)\big] \\
&\quad - k^{(m}_2\mathcal{Z}^{n)}_{2|1,3,4,5,6,7} \\
\Theta^{(1)}_{2|3|1,4,5,6,7} &= -k^p_2\mathcal{Z}_{3|2,1,4,5,6,7} - s_{23}\mathcal{Z}_{23|1,4,5,6,7} - \big[s_{21}\mathcal{Z}_{3|21,4,5,6,7} + (1\leftrightarrow 4,5,6,7)\big]
\end{aligned}
$$

and can also be verified to be zero up to total derivatives.

C.2. Eight points

At eight points, the following topologies can be shown to vanish up to total derivatives:

$$
\begin{aligned}
V_1V_2T^{mnp}_{3,4,5,6,7,8}\Theta^{(0)\,mnp}_{2|1,3,4,5,6,7,8} &\cong 0, & V_1V_2T^{mn}_{34,5,6,7,8}\Theta^{(0)\,mn}_{2|1,34,5,6,7,8} &\cong 0 \qquad \text{(C.2)}\\
V_1V_{23}T^{mn}_{4,5,6,7,8}\Theta^{(0)\,mn}_{23|1,4,5,6,7,8} &\cong 0, & V_1V_2T^{m}_{34,56,7,8}\Theta^{(0)\,m}_{2|1,34,56,7,8} &\cong 0 \\
V_1V_2T^{m}_{345,6,7,8}\Theta^{(0)\,m}_{2|1,345,6,7,8} &\cong 0, & V_1V_{23}T^{m}_{45,6,7,8}\Theta^{(0)\,m}_{23|1,45,6,7,8} &\cong 0 \\
V_1V_{234}T^{m}_{5,6,7,8}\Theta^{(0)\,m}_{234|1,5,6,7,8} &\cong 0, & V_1V_{23}T_{45,67,8}\Theta^{(0)}_{23|1,45,67,8} &\cong 0 \\
V_1V_{234}T_{56,7,8}\Theta^{(0)}_{234|1,56,7,8} &\cong 0, & V_1V_{2345}T_{6,7,8}\Theta^{(0)}_{2345|1,6,7,8} &\cong 0 \\
V_1V_2T_{34,56,78}\Theta^{(0)}_{2|1,34,56,78} &\cong 0, & V_1V_2T_{345,67,8}\Theta^{(0)}_{2|1,345,67,8} &\cong 0 \\
V_1V_2T_{3456,7,8}\Theta^{(0)}_{2|1,3456,7,8} &\cong 0, & V_1V_{23}T_{456,7,8}\Theta^{(0)}_{23|1,456,7,8} &\cong 0
\end{aligned}
$$

as well as

$$
\begin{aligned}
V_1V_2J_{34|5,6,7,8}\Theta^{(1)}_{2|34|1,5,6,7,8} &\cong 0, \qquad V_1V_2J_{3|45,6,7,8}\Theta^{(1)}_{2|3|1,45,6,7,8} \cong 0 \\
V_1V_2J^m_{3|4,5,6,7,8}\Theta^{(1)\,m}_{2|3|1,4,5,6,7,8} &\cong 0, \qquad V_1V_{23}J_{4|5,6,7,8}\Theta^{(1)}_{23|4|1,5,6,7,8} \cong 0\,.
\end{aligned}
\tag{C.3}
$$

The coefficients of V_{1A} with $A \neq \emptyset$ are just relabellings of the $\Theta^{(d)}$ in (C.2) and (C.3) and therefore vanish as well.

The explicit expansion of all eight-point topologies from (C.2) and (C.3) is somewhat lengthy, so let us display just a couple of examples. It is not hard to be convinced that their vanishing, up to total derivatives, is a non-trivial statement:

$$
\begin{aligned}
\Theta^{(0)}_{23|1,45,67,8} &= k^m_{23}\mathcal{Z}^m_{1,23,45,67,8} + \big[s_{38}\mathcal{Z}_{1,238,45,67} + s_{13}\mathcal{Z}_{132,45,67,8} - (2\leftrightarrow 3)\big] \\
&\quad + \big[(s_{24}+s_{34})\mathcal{Z}_{1,2345,67,8} - (4\leftrightarrow 5)\big] + \big[(s_{26}+s_{36})\mathcal{Z}_{1,2367,45,8} - (6\leftrightarrow 7)\big] \\
&\quad + \big[s_{24}(\mathcal{Z}_{1,2435,67,8} + \mathcal{Z}_{1,2453,67,8}) - (4\leftrightarrow 5)\big] \\
&\quad + \big[s_{26}(\mathcal{Z}_{1,2637,45,8} + \mathcal{Z}_{1,2673,45,8}) - (6\leftrightarrow 7)\big] \\
&\quad - \mathcal{Z}_{2|1,3,45,67,8} + \mathcal{Z}_{3|1,2,45,67,8} \cong 0\,, \\
\Theta^{(0)}_{2|1,345,67,8} &= k^m_2\mathcal{Z}_{1,2,345,67,8} + s_{23}\mathcal{Z}_{1,2345,67,8} - s_{24}(\mathcal{Z}_{1,2435,67,8} + \mathcal{Z}_{1,2453,67,8}) \\
&\quad + s_{25}\mathcal{Z}_{1,2543,67,8} + \big[s_{26}\mathcal{Z}_{1,267,345,8} - (6\leftrightarrow 7)\big] \\
&\quad + s_{28}\mathcal{Z}_{1,28,345,67} - s_{12}\mathcal{Z}_{12,345,67,8} \cong 0\,, \\
\Theta^{(0)\,m}_{234|1,5,6,7,8} &= k^p_{234}\mathcal{Z}^{pm}_{234,1,5,6,7,8} - k^m_{234}\mathcal{Z}_{234|1,5,6,7,8} \\
&\quad + \big[s_{14}\mathcal{Z}^m_{2341,5,6,7,8} - s_{13}(\mathcal{Z}^m_{2431,5,6,7,8} + \mathcal{Z}^m_{4231,5,6,7,8}) + s_{12}\mathcal{Z}^m_{4321,5,6,7,8} + (1\leftrightarrow 5,6,7,8)\big] \\
&\quad - \mathcal{Z}^m_{2|34,1,5,6,7,8} - \mathcal{Z}^m_{23|4,1,5,6,7,8} + \mathcal{Z}^m_{34|2,1,5,6,7,8} + \mathcal{Z}^m_{4|23,1,5,6,7,8} \cong 0\,, \\
\Theta^{(1)}_{3|4|12,5,6,7,8} &= -k^p_3\mathcal{Z}^p_{4|12,3,5,6,7,8} - s_{34}\mathcal{Z}_{34|12,5,6,7,8} - s_{31}\mathcal{Z}_{4|312,5,6,7,8} \\
&\quad + s_{32}\mathcal{Z}_{4|321,5,6,7,8} - \big[s_{35}\mathcal{Z}_{4|35,12,6,7,8} + (5\leftrightarrow 6,7,8)\big] \cong 0\ .
\end{aligned}
\tag{C.4}
$$

Appendix D. Proof of (4.22)

The purpose of this appendix is to deliver intermediate steps in deriving the manifestly BRST-invariant representation (4.22) of the six-point closed-string correlator that has been proposed in [33].

D.1. Single contraction between left and right movers

The open-string contribution (4.25) involving a single vector contraction between left- and right-movers stems from the derivative

$$\begin{aligned}\frac{\partial \mathcal{K}_6(\ell)}{\partial \ell_m} &= C^{mn}_{1|2,3,4,5,6}\big(\ell^n + \big[g^{(1)}_{12} k_2^n + (2\leftrightarrow 3,4,5,6)\big]\big) \\ &\quad + \big[C^m_{1|23,4,5,6} s_{23} V_1(1,2,3) + (2,3|2,3,4,5,6)\big] \\ &\quad - \big[P_{1|2|3,4,5,6} k_2^m g^{(1)}_{12} + (2\leftrightarrow 3,4,5,6)\big]\end{aligned} \tag{D.1}$$

of the $C\cdot E$ representation (3.44) (using the loop-momentum-dependent form of $E_{1|2|3,4,5,6}$ in the first line of (II.4.37)). Upon integration over ℓ, we obtain

$$\begin{aligned}\Big[\Big[\frac{\partial \mathcal{K}_6(\ell)}{\partial \ell_m}\Big]\Big] &= C^{mn}_{1|2,3,4,5,6}\big[f^{(1)}_{12} k_2^n + (2\leftrightarrow 3,4,5,6)\big]\big) \\ &\quad + \big[C^m_{1|23,4,5,6} s_{23} V_1(1,2,3) + (2,3|2,3,4,5,6)\big] \\ &\quad + \big[P_{1|2|3,4,5,6} k_2^m (\nu_{12} - f^{(1)}_{12}) + (2\leftrightarrow 3,4,5,6)\big] \\ &\cong \big[C^m_{1|23,4,5,6} s_{23} f^{(1)}_{23} + (2,3|2,3,4,5,6)\big] \\ &\quad + \big[P_{1|2|3,4,5,6} k_2^m \nu_{12} + (2\leftrightarrow 3,4,5,6)\big] \\ &\quad + \big[f^{(1)}_{12}\Big(k_2^n C^{mn}_{1|2,3,\ldots,6} - k_2^m P_{1|2|3,4,5,6} + s_{23} C^m_{1|23,4,5,6} \\ &\qquad + s_{24} C^m_{1|24,3,5,6} + s_{25} C^m_{1|25,3,4,6} + s_{26} C^m_{1|26,3,4,5}\Big) + (2\leftrightarrow 3,4,5,6)\big]\,,\end{aligned} \tag{D.2}$$

and BRST-exactness of the coefficient of $f^{(1)}_{12}$ in the last two lines leads to (4.25). As we will see, the ν_{1j}-dependent terms in the second line of (4.24),

$$\begin{aligned}&\big[C^m_{1|23,4,5,6} s_{23} f^{(1)}_{23} + (2,3|2,3,\ldots,6)\big]\big[\tilde C^m_{1|23,4,5,6} s_{23} \overline{f}^{(1)}_{23} + (2,3|2,3,\ldots,6)\big] \\ &+ \big[P_{1|2|3,4,5,6} k_2^m \nu_{12} + (2\leftrightarrow 3,4,5,6)\big]\big[\tilde C^m_{1|23,4,5,6} s_{23} \overline{f}^{(1)}_{23} + (2,3|2,3,\ldots,6)\big] \\ &- \big[C^m_{1|23,4,5,6} s_{23} f^{(1)}_{23} + (2,3|2,3,\ldots,6)\big]\big[\tilde P_{1|2|3,4,5,6} k_2^m \nu_{12} + (2\leftrightarrow 3,4,5,6)\big] \\ &- \big[P_{1|2|3,4,5,6} k_2^m \nu_{12} + (2\leftrightarrow 3,4,5,6)\big]\big[\tilde P_{1|2|3,4,5,6} k_2^m \nu_{12} + (2\leftrightarrow 3,4,5,6)\big]\,,\end{aligned} \tag{D.3}$$

will cancel in the end (where all of (D.3) is accompanied by a factor of $\frac{\pi}{\mathrm{Im}\,\tau}$).

D.2. Contributions from two (anti-)holomorphic derivatives

We shall now elaborate on the contributions of the (anti-)holomorphic derivatives $N_{1|2|3,\ldots}$ and $\tilde{N}_{1|2|3,\ldots}$ (4.27) that arise in the expression (4.26) for $[[\mathcal{K}_6(\ell)]]$. Combinations of both $N_{1|2|3,\ldots}$ and $\tilde{N}_{1|2|3,\ldots}$ can lead to the following two inequivalent situations,

$$\begin{aligned} N_{1|2|3,4,5,6}\tilde{N}_{1|3|2,4,5,6} &= \nu_{12}\frac{\partial}{\partial z_2}\Big(\frac{\pi}{\operatorname{Im}\tau} - \nu_{13}(s_{13}\overline{f}^{(1)}_{13} + s_{23}\overline{f}^{(1)}_{23} - \big[s_{34}\overline{f}^{(1)}_{34} + (4\leftrightarrow 5,6)\big])\Big) \\ &= \frac{\pi}{\operatorname{Im}\tau}\nu_{12}\nu_{13}s_{23} \end{aligned} \tag{D.4}$$

$$\begin{aligned} N_{1|2|3,4,5,6}\tilde{N}_{1|2|3,4,5,6} &= \nu_{12}\frac{\partial}{\partial z_2}\Big(\frac{\pi}{\operatorname{Im}\tau} - \nu_{12}(s_{12}\overline{f}^{(1)}_{12} - \big[s_{23}\overline{f}^{(1)}_{23} + (3\leftrightarrow 4,5,6)\big])\Big) \\ &= -\frac{\pi\nu_{12}}{\operatorname{Im}\tau}\Big(s_{12}\overline{f}^{(1)}_{21} + \big[s_{23}\overline{f}^{(1)}_{23} + (3\leftrightarrow 4,5,6)\big] + \nu_{12}(s_{12}+s_{23}+\ldots+s_{26})\Big) \\ &= -\frac{\pi\nu_{12}}{\operatorname{Im}\tau}\frac{\partial}{\partial\overline{z}_2}\log\hat{\mathcal{I}}_6 \cong \frac{\pi}{\operatorname{Im}\tau}\frac{\partial}{\partial\overline{z}_2}\nu_{12} = \Big(\frac{\pi}{\operatorname{Im}\tau}\Big)^2, \end{aligned} \tag{D.5}$$

using $\frac{\partial}{\partial\overline{z}_2}\overline{f}^{(1)}_{2j} = -\frac{\pi}{\operatorname{Im}\tau}$ and momentum conservation. Hence, the part of $[[\mathcal{K}_6(\ell)]]\cdot[[\tilde{\mathcal{K}}_6(-\ell)]]$ with two factors of $N_{1|2|3,\ldots}$ and $\tilde{N}_{1|2|3,\ldots}$ adds up to

$$\begin{aligned} &\big[N_{1|2|3,4,5,6}P_{1|2|3,4,5,6} + (2\leftrightarrow 3,4,5,6)\big]\big[\tilde{N}_{1|2|3,4,5,6}\tilde{P}_{1|2|3,4,5,6} + (2\leftrightarrow 3,4,5,6)\big] \\ &\quad= \frac{\pi}{\operatorname{Im}\tau}\big[P_{1|2|3,4,5,6}k_2^m\nu_{12} + (2\leftrightarrow 3,4,5,6)\big]\big[\tilde{P}_{1|2|3,4,5,6}k_2^m\nu_{12} + (2\leftrightarrow 3,4,5,6)\big] \\ &\qquad+ \Big(\frac{\pi}{\operatorname{Im}\tau}\Big)^2\big[|P_{1|2|3,4,5,6}|^2 + (2\leftrightarrow 3,4,5,6)\big]\,. \end{aligned} \tag{D.6}$$

Note that the first line of the right-hand side cancels the last line in (D.3), and the last term of (D.6) will interfere with the crossterms to be discussed next.

D.3. Contributions from one (anti-)holomorphic derivative

Finally, there are crossterm contributions to $[[\mathcal{K}_6(\ell)]]\cdot[[\tilde{\mathcal{K}}_6(-\ell)]]$

$$\begin{aligned} N_{1|2|3,4,5,6}\tilde{\mathcal{K}}_6^{\rm open} &= \nu_{12}\frac{\partial}{\partial z_2}\Big\{\frac{1}{2}\tilde{C}^{mn}_{1|2,\ldots,6}[[\overline{E}^{mn}_{1|2,\ldots,6}]] + \big(s_{23}\tilde{C}^{m}_{1|23,4,5,6}[[\overline{E}^{mn}_{1|23,4,5,6}]] + (2,3|2,\ldots,6)\big) \\ &+ \Big[\tilde{P}_{1|2|3,4,5,6}\Big(\frac{\pi}{\operatorname{Im}\tau} + 2s_{12}\overline{f}^{(2)}_{12} - \overline{f}^{(1)}_{12}\big[s_{23}\overline{f}^{(1)}_{23} + (3\leftrightarrow 4,5,6)\big]\Big) + (2\leftrightarrow 3,4,5,6)\Big]\Big\} \end{aligned} \tag{D.7}$$

where $\tilde{\mathcal{K}}_6^{\rm open}$ given by (4.23) is obtained from $C\cdot E$ representation (3.44) of $\tilde{\mathcal{K}}_6(-\ell)$. The coefficient of $\tilde{P}_{1|2|3,4,5,6}$ is most subtle to evaluate since the z_2-derivative of the $\overline{f}^{(n)}_{ij}$ functions generates a Koba–Nielsen derivative w.r.t. $\overline{z}_2$:

$$\begin{aligned} &\nu_{12}\frac{\partial}{\partial z_2}\Big(\frac{\pi}{\operatorname{Im}\tau} + 2s_{12}\overline{f}^{(2)}_{12} - \overline{f}^{(1)}_{12}\big[s_{23}\overline{f}^{(1)}_{23} + (3\leftrightarrow 4,5,6)\big]\Big) \\ &\quad= \frac{\pi\nu_{12}}{\operatorname{Im}\tau}\Big(s_{12}\overline{f}^{(1)}_{12} - \big[s_{23}\overline{f}^{(1)}_{23} + (3\leftrightarrow 4,5,6)\big]\Big) \\ &\quad= -\frac{\pi\nu_{12}}{\operatorname{Im}\tau}\frac{\partial}{\partial\overline{z}_2}\log\hat{\mathcal{I}}_6 = \frac{\pi}{\operatorname{Im}\tau}\frac{\partial}{\partial\overline{z}_2}\nu_{12} = \big(\frac{\pi}{\operatorname{Im}\tau}\big)^2 \end{aligned} \tag{D.8}$$

Given that the only contributions to $\frac{\partial}{\partial z_2}[[\overline{E}^{\cdots}_{1|\ldots}]]$ arise from the quantity L_0^m in (II.7.12), the remaining terms in (D.7) reduce to vector contractions of $\frac{\partial L_0^m}{\partial z_2} = -\frac{\pi}{\mathrm{Im}\,\tau} k_2^m$,

$$\begin{aligned}
N_{1|2|3,4,5,6}\tilde{\mathcal{K}}_6^{\rm open} &= \Big(\frac{\pi}{\mathrm{Im}\,\tau}\Big)^2 \tilde{P}_{1|2|3,4,5,6} + \frac{\pi \nu_{12}}{\mathrm{Im}\,\tau}\Big\{ k_2^m \tilde{C}^{mn}_{1|2,3,4,5,6}\big[k_2^n \overline{f}^{(1)}_{12} + (2\leftrightarrow 3,4,5,6)\big] \\
&\quad + k_2^m \big[s_{23}\tilde{C}^m_{1|23,4,5,6}(\overline{f}^{(1)}_{12} + \overline{f}^{(1)}_{23} + \overline{f}^{(1)}_{31}) + (2,3|2,3,4,5,6)\big] \\
&\quad - \big[s_{23}\tilde{P}_{1|3|2,4,5,6}\overline{f}^{(1)}_{13} + (3\leftrightarrow 4,5,6)\big]\Big\} \qquad\qquad \text{(D.9)}\\
&= \Big(\frac{\pi}{\mathrm{Im}\,\tau}\Big)^2 \tilde{P}_{1|2|3,4,5,6} + \frac{\pi \nu_{12}}{\mathrm{Im}\,\tau} k_2^m \big[s_{23}\tilde{C}^m_{1|23,4,5,6}\overline{f}^{(1)}_{23} + (2,3|2,3,4,5,6)\big]\,,
\end{aligned}$$

where we have repeated the simplifications of (D.2) in the last step. By adjoining permutations and the complex conjugate of (D.9), one arrives at

$$\begin{aligned}
&-\big[N_{1|2|3,4,5,6}P_{1|2|3,4,5,6} + (2\leftrightarrow 3,4,5,6)\big]\tilde{\mathcal{K}}_6^{\rm open} - \big[\tilde{N}_{1|2|3,4,5,6}\tilde{P}_{1|2|3,4,5,6} + (2\leftrightarrow 3,4,5,6)\big]\mathcal{K}_6^{\rm open} \\
&= -\frac{\pi}{\mathrm{Im}\,\tau}\big[P_{1|2|3,4,5,6}k_2^m \nu_{12} + (2\leftrightarrow 3,4,5,6)\big]\big[s_{23}\tilde{C}^m_{1|23,4,5,6}\overline{f}^{(1)}_{23} + (2,3|2,3,4,5,6)\big] \\
&\quad + \frac{\pi}{\mathrm{Im}\,\tau}\big[s_{23}C^m_{1|23,4,5,6}f^{(1)}_{23} + (2,3|2,3,4,5,6)\big]\big[\tilde{P}_{1|2|3,4,5,6}k_2^m \nu_{12} + (2\leftrightarrow 3,4,5,6)\big] \\
&\quad - 2\Big(\frac{\pi}{\mathrm{Im}\,\tau}\Big)^2 \big[|P_{1|2|3,4,5,6}|^2 + (2\leftrightarrow 3,4,5,6)\big]\,, \qquad\qquad \text{(D.10)}
\end{aligned}$$

where the last term interferes with (D.6), and the remaining terms on the right-hand side cancel the crossterms in the second and third line of (D.3). In summary, by combining (D.3), (D.6) and (D.10), one arrives at the most subtle contributions $\sim C^m_{1|23,\ldots}\tilde{C}^m_{1|ij,\ldots}$ and $|P_{1|i|j,\ldots}|^2$ in (4.22), concluding the derivation of the proposal in [33].

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