Affine quantization of the Brans-Dicke theory: Smooth bouncing and the equivalence between the Einstein and Jordan frames

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(Received 8 October 2018; published 24 January 2019)

In this work, we present a complete analysis of the quantization of the classical Brans-Dicke theory using the method of affine quantization in the Hamiltonian description of the theory. The affine quantization method is based on the symmetry of the phase space of the system, in this case the (positive) half-plane, which is identified with the affine group. We consider a Friedmann-Lemaître-Robertson-Walker type spacetime, and since the scale factor is always positive, the affine method seems to be more suited than the canonical quantization for our quantum cosmology. We find the wave function of the Brans-Dicke universe and its energy spectrum. A smooth bounce is expected at the semiclassical level in the quantum phase-space portrait. We also address the problem of equivalence between the Jordan and Einstein frames.

DOI: 10.1103/PhysRevD.99.023524

I. INTRODUCTION

After the formulation of general relativity (GR), some modified theories arose in an attempt to explain open problems in cosmology, such as inflation and the observed accelerated expansion. One of the oldest modifications of GR is the Brans-Dicke theory (BDT), proposed in the early 1960s by Carl H. Brans and Robert H. Dicke [1], in which there is a nonminimal time-dependent coupling of the long-range scalar field with geometry, that is, with gravity. The BDT also introduces an adimensional constant ω such that, for a constant gravitational coupling, GR is recovered at the limit $\omega \to \infty$, if the trace of the energy-momentum tensor is not null [2–4]. Today, it is well known that, classically, the BDT is practically indistinguishable from GR, with the constant ω estimated to be over 40 000 [5,6]. Interestingly, the Brans-Dicke scalar field arises naturally in superstring cosmology, associated with the dilaton, which couples directly with the matter field [7]. The dilaton is equivalent to the graviton for a theory with dynamical gravitational coupling. In spite of the fact that the BDT is classically no different from GR, the quantum treatment can reveal new dynamics for the primordial Universe. There are also claims that the BDT cannot reproduce GR for scale-invariant matter content. In fact, in this case, it has been shown that ω can display various effects depending on its value, such as a symmetry breaking resulting in a binary phase structure. However, for a strong coupling $\omega \to \infty$, the BDT reproduces GR only in the quantised version [8].

With the assumption that quantum effects cannot be ignored at early stages of the Universe, the quantization of the classical BDT in its Hamiltonian description is relevant to better understand this era. We will assume a minisuperspace, a configuration with reduced degrees of freedom for homogeneous cosmologies, which can be understood as a projection of the whole superspace, containing only the largest wavelength modes of the size of the Universe [9]. Minisuperspaces are considered to be toy models, since they reduce the superspace, that is, the observable universe on the largest scales, which have infinite degrees of freedom. However, it is still a fairly good approximation of the superspace to study certain properties. This allows one to target specific behaviors such as the dynamics of the volume of the Universe or to investigate the nature of the initial singularity and the inflationary phase.

We choose to explore the quantization of the BDT with the affine quantization instead of the canonical one, since the domain of the variables involved (scale factor and scalar field) is the real half-line, and its phase space can be identified with the affine group. With this, we also avoid the operator-ordering problem arising in the case of canonical quantization (see, e.g., discussions in Refs. [10,11]). The affine quantization is also equipped with a "dequantization"

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map that allows us to obtain classical expressions from quantum operators. In the canonical quantization, the classical measurements are obtained by the expectation values of the classical observables, but in the affine quantization, the classical system is recovered with possible corrections through this dequantization map, called quantum corrections or lower symbols. While being a fairly recent subject of interest in cosmology, the affine quantization points to interesting applications, such as the removal of divergences in nonrenormalizable theories [12,13] or the nonsingular expanding (and possibly cyclic) universes [14].

This work, in which we will investigate the quantization of the BDT applying the affine method, is a continuation of the analysis initiated in Ref. [15]. We present the wave function for a Brans-Dicke universe, and we draw its quantum phase space. Then, it is shown that a bounce is expected, avoiding the initial singularity. We also raise the question about the equivalence between the Jordan and Einstein frames. This paper is organized as follows. In Sec. II, we review the classical derivation of the BDT with a perfect fluid introduced via the Schutz formalism. In Sec. III, we introduce the affine quantization method as well as a more direct way to obtain classical estimates: the quantum phase-space portraits. In Sec. IV, we apply the affine quantization to the BDT to obtain the Wheeler-DeWitt equation in the Jordan frame and in the Einstein frame. Finally, we derive the semiclassical version of the Hamiltonian constraint in both frames. In the last section, we present our results and discuss the dependence of the parameters on the solutions.

II. BRANS-DICKE THEORY WITH A PERFECT FLUID

The Brans-Dicke theory is characterized by the introduction of a scalar field nonminimally coupled to gravity, and it is described by the gravitational Lagrangian

$$\mathcal{L}_G = \sqrt{-g}\left\{\varphi R - \omega \frac{\varphi_{;\rho}\varphi^{;\rho}}{\varphi}\right\}. \tag{1}$$

The Brans-Dicke coupling parameter ω is chosen to be a constant in this work. Let us consider a homogeneous and isotropic universe,

$$ds^2 = N^2(t)dt^2 - a^2(t)[dx^2 + dy^2 + dz^2], \tag{2}$$

where N and a are, respectively, the lapse function and the scale factor. Then, the Lagrangian (1) becomes

$$\mathcal{L}_G = \frac{1}{N}\left\{6[\varphi a\dot{a}^2 + a^2\dot{a}\,\dot{\varphi}] - \omega a^3\frac{\dot{\varphi}^2}{\varphi}\right\}, \tag{3}$$

where we have already discarded the surface terms. The Lagrangian of the system is completed with a matter component, which we will consider to be a radiative perfect fluid, defined by the equation of state $p = \rho/3$.

Let us use the Schutz formalism to introduce the perfect fluid [16], in which the 4-velocity of a baryonic perfect fluid is described by four potentials,[1] the specific enthalpy μ and the entropy s of the fluid and another two with no clear physical meaning; let us call them ϵ and θ. After some considerations [17,18], the matter Lagrangian becomes

$$\mathcal{L}_M = -\frac{1}{3}\left(\frac{3}{4}\right)^4\frac{a^3}{N^3}(\dot{\epsilon} + \theta\dot{s})^4 e^{-3s}. \tag{4}$$

Since we are interested in the quantum corrections of this system, we must describe the theory with the Hamiltonian formalism. To do so, let us write the Lagrangians above as functions of the conjugate momenta, defined by

$$p_q = \frac{\partial\mathcal{L}}{\partial\dot{q}}. \tag{5}$$

With this, from (4), we obtain the matter super-Hamiltonian [19][2]

$$\mathcal{H}_M = -p_\epsilon^{\frac{4}{3}}a^{-1}e^s, \tag{6}$$

where $p_\epsilon = -N\rho_0 U^0 a^3$, with ρ_0 the rest mass density of the fluid and U the 4-velocity. Let us introduce the following canonical transformations [20]:

$$T = -p_s e^{-s} p_\epsilon^{-\frac{4}{3}}; \qquad p_T = p_\epsilon^{\frac{4}{3}} e^s;$$
$$\bar{\epsilon} = \epsilon - \frac{4}{3}\frac{p_s}{p_\epsilon}; \qquad \bar{p}_\epsilon = p_\epsilon. \tag{7}$$

Then, the super-Hamiltonian for the matter component becomes

$$\mathcal{H}_M = -\frac{N}{a}p_T. \tag{8}$$

The Hamiltonian for the gravitational part is given by the Legendre transformation of $\mathcal{L}_G$,

$$\mathcal{H}_G = \dot{a}p_a + \dot{\varphi}p_\varphi - \mathcal{L}_G, \tag{9}$$

where the conjugate momenta are

$$p_a = \frac{6}{N}(2\varphi a\dot{a} + a^2\dot{\varphi}); \qquad p_\varphi = \frac{6}{N}a^2\dot{a} - 2\frac{\omega}{N}a^3\frac{\dot{\varphi}}{\varphi}. \tag{10}$$

Expressing the generalized velocities in terms of the momenta, we obtain

[1] There are six potentials in total, but they reduce to four in a homogeneous and isotropic medium.

[2] The Hamiltonian defined on the minisuperspace, where the spacelike metric and nongravitational fields belong to a finite set and their conjugate momentum is identically zero.

$$\dot{a} = \frac{\omega N}{(3+2\omega)\varphi a}\left(\frac{p_a}{6} + \frac{\varphi p_\varphi}{2\omega a}\right);$$
$$\dot{\varphi} = \frac{N\varphi}{2(3+2\omega)a^3}\left(\frac{a p_a}{\varphi} - 2p_\varphi\right), \tag{11}$$

which, after some algebra, gives us

$$\mathcal{H}_G = \frac{N}{3+2\omega}\left[\frac{\omega}{12\varphi a}p_a^2 + \frac{1}{2a^2}p_a p_\varphi - \frac{\varphi}{2a^3}p_\varphi^2\right]. \tag{12}$$

Therefore, the Hamiltonian of the BDT is given by

$$\mathcal{H} = N\left\{\frac{1}{(3+2\omega)}\left[\frac{\omega}{12\varphi a}p_a^2 + \frac{1}{2a^2}p_a p_\varphi - \frac{\varphi}{2a^3}p_\varphi^2\right] - \frac{1}{a}p_T\right\}, \tag{13}$$

where p_T, p_a, and p_φ are the conjugate momenta associated with the matter component, the scale factor a, and the field φ, respectively.

The classical Hamiltonian constraint $\mathcal{H} \approx 0$ still holds (notice that here $\approx$ means "weakly equal," so that $\mathcal{H}$ is a first class constraint; i.e., its Poisson brackets with other constraints are vanishing on the constrained space) for the BDT with a perfect fluid [21,22]. Thus, we have

$$\frac{\omega}{12\varphi}p_a^2 + \frac{1}{2a}p_a p_\varphi - \frac{\varphi}{2a^2}p_\varphi^2 = (3+2\omega)p_T. \tag{14}$$

The quantization of this constraint results in the Wheeler-DeWitt equation. We can interpret it as a Schrödinger-like equation and, from it, obtain the cosmological scenarios at a quantum level [23]. Now, instead of the canonical quantization used in Ref. [23], we will introduce another quantization method, based on the symmetry group of the system's phase space. This kind of quantization is completed with a quantum phase-space portrait, which accounts for a quantum correction to the classical trajectories of the theory, that we will use to analyze the BDT at early cosmological times.

III. AFFINE QUANTIZATION

A. Mathematical background

First, let us introduce the affine quantization method mentioned earlier. The model requires the scale factor and the scalar field, our two dynamical variables, to be positive, with the zero value being a geometrical singularity. Thus, the phase space is a four-dimensional space which is the Cartesian product of two half-planes,[3]

$$\Pi_+^2 \coloneqq \{(a, p_a)\times(\varphi, p_\varphi) | a > 0, \varphi > 0, p_a, p_\varphi \in \mathbb{R}\}. \tag{15}$$

Since it is a Cartesian product, we can analyze each half-plane separately. Thus, we will present the theory behind this method of quantization for a generic phase space and then apply it to our specific case (for a more detailed presentation, see e.g., Refs. [24,25]).

The half-plane $\Pi_+ \coloneqq \{(q,p) | q > 0, p \in \mathbb{R}\}$ with a multiplication operation defined by

$$(q,p)(q_0,p_0) = \left(qq_0, \frac{p_0}{q} + p\right); \quad q \in \mathbb{R}_+^*, \quad p \in \mathbb{R} \tag{16}$$

is identified with the affine group $\mathrm{Aff}_+(\mathbb{R})$ of the real line. The group acts on $\mathbb{R}$ as follows:

$$(q,p)\cdot x = \frac{x}{q} + p, \quad \forall x \in \mathbb{R}. \tag{17}$$

On a physical level, one can interpret (17) as a contraction/dilation (depending on if $q > 1$ or $q < 1$) of space plus a translation. We shall equip the half-plane with the measure $dqdp$, which is invariant under the left action of the affine group on itself [26].

Rigorously, the affine quantization is a covariant integral method, that combines the properties of symmetry from the affine group with all the resources of integral calculus. This method makes use of *coherent states* [27] to construct the quantization map, the definition of which is connected with the symmetry of the phase space, as we will see. First, let us explain the integral quantization method. Given a group G and a unitary irreducible representation (UIR) of it, the quantization map transforms a classical function (or distribution) into an operator using a bounded square-integrable operator M and a measure $d\nu$, such as

$$\int_G M(g)d\nu(g) = I, \tag{18}$$

where $g \in G$, $M(g) = U(g)MU^{-1}(g)$. This is the resolution of the identity for the operator M. With this, from a classical observable $f(g)$, we obtain the corresponding operator

$$A_f = \int_G M(g)f(g)d\nu(g). \tag{19}$$

For the affine group, that is $G = \mathrm{Aff}_+(\mathbb{R})$, we have two nonequivalent UIR $U_\pm$, plus a trivial one U_0 [28,29]. Let us choose $U = U_+$, which acts on the Hilbert space $L^2(R_+^*, dx/x^{\alpha+1})$ as

$$(U(q,p)\psi)(x) = \frac{e^{ipx}}{\sqrt{q^{-\alpha}}}\psi\left(\frac{x}{q}\right). \tag{20}$$

We choose the operator M such as

[3] In the case of radiative matter, at least [23].

$$M = |\psi\rangle\langle\psi|; \qquad \psi \in L^2\left(R_+^*, \frac{dx}{x^{\alpha+1}}\right) \cap L^2\left(R_+^*, \frac{dx}{x^{\alpha+2}}\right). \tag{21}$$

The normalized vectors ψ are arbitrarily chosen providing the square-integrability condition (21), and they are known as *fiducial vectors*. For simplicity, we will consider only real fiducial vectors and will choose $\alpha = -1$. The action (20) of the UIR of U over fiducial vectors produces the quantum states

$$|q, p\rangle \coloneqq U(q, p)|\psi\rangle. \tag{22}$$

These states are called *affine coherent states* or *wavelets*. It is easy to show that

$$\int_{\Pi_+} |q, p\rangle\langle q, p| \frac{dqdp}{2\pi c_{-1}} = I, \tag{23}$$

where the constant c_{-1} depends on the choice of ψ and is defined as

$$c_\gamma = c_\gamma(\psi) \coloneqq \int_0^\infty |\psi(x)|^2 \frac{dx}{x^{2+\gamma}}. \tag{24}$$

Hence, the quantization maps (19) become

$$f(q, p) \mapsto A_f = \int_{\Pi_+} f(q, p)|q, p\rangle\langle q, p| \frac{dqdp}{2\pi c_{-1}}. \tag{25}$$

With this, one can easily verify that the quantization of the elementary functions position q^β (for any β), momentum p, and kinetic energy[4] p^2 yields

$$A_{q^\beta} = \frac{c_{\beta-1}}{c_{-1}} \hat{Q}^\beta; \qquad A_p = -i\frac{\partial}{\partial x} = \hat{P};$$
$$A_{p^2} = \hat{P}^2 + \frac{c_{-3}^{(1)}}{c_{-1}} \hat{Q}^{-2}, \tag{26}$$

with $\hat{Q}$ being the position operator defined by $\hat{Q}f(x) = xf(x)$ and the constant $c_{-3}^{(1)}$ defined as

$$c_\gamma^{(\beta)}(\psi) \coloneqq \int_0^\infty |\psi^{(\beta)}(x)|^2 \frac{dx}{x^{2+\gamma}}. \tag{27}$$

Notice that, in this affine quantization method, the only dependence on the fiducial vector ψ is in the constant coefficients of the quantum operators. Thus, the arbitrariness of ψ does not play a fundamental role in the quantization. This is an advantage to be explored. For example, we can adjust the fiducial vectors to regain the self-adjoint character of the operator p^2 [26]. Choosing ψ such that $4c_{-3}^{(1)} \geq 3c_{-1}$, the kinetic operator becomes essentially self-adjoint [30], which is a desired characteristic since a Hermitian operator must be self-adjoint. A Hermitian operator can be obtained by imposing boundary conditions. However, there is a continuous infinity of possible boundaries, and thus the choice of a representation is arbitrary (this is the *operator-ordering* problem of the canonical quantization). In the affine quantization, the choice of a fiducial vector can naturally result in an essentially self-adjoint operator, which means there is only one possible extension of it and, therefore, no need to impose boundary conditions. We stress, however, that choosing fiducial vectors is not the same as choosing boundary conditions. Self-adjoint-ness is a well-known problem in the canonical quantization of this theory, and it has been studied extensively in Ref. [23]. However, with the affine quantization, we naturally recover the quantum symmetrization of the classical product momentum position

$$qp \mapsto A_{qp} = \frac{c_0}{c_{-1}} \frac{\hat{Q}\hat{P} + \hat{P}\hat{Q}}{2}, \tag{28}$$

up to a constant that once again depends on the choice of the fiducial vector.

[4]Up to a factor.

B. Quantum phase-space portraits

The construction of the affine quantization method presented in the previous section using coherent states allows us to define a dequantization map, named the quantum phase-space portrait, in a very obvious way: by calculating the expectation value of an operator with respect to the coherent states. That is, given a quantum operator A_f, we obtain a classical function $\check{f}$ such that

$$\check{f}(q, p) = \langle q, p|A_f|q, p\rangle. \tag{29}$$

If the operator is obtained from a classical function f, as suggested in the notation, then $\check{f}$ is a quantum correction or lower symbol of the original f [31]. It corresponds to the average value of $f(q, p)$ with respect to the probability density distribution

$$\rho_\phi(q, p) = \frac{1}{2\pi c_{-1}} |\langle q, p|\phi\rangle|^2, \tag{30}$$

with $|\phi\rangle = |q', p'\rangle$. We can also define the time evolution of the distribution (30) with respect to time through a Hamiltonian operator $\hat{H} = A_H$, using the time evolution operator $e^{-i\hat{H}t}$. Then,

$$\rho_\phi(q, p, t) \coloneqq \frac{1}{2\pi c_{-1}} |\langle q, p|e^{-i\hat{H}t}|\phi\rangle|^2. \tag{31}$$

Thus, if you consider the operator $M = \rho$, the lower symbol of A_f becomes [24]

$$\check{f}(z) = \int tr(\rho(z)\rho(z'))f(z')\frac{d^2z'}{\pi}, \tag{32}$$

with tr the trace. From the resolution of the identity (18), one finds $tr(\rho(z)\rho(z'))$ is a probability distribution of the phase space, and $\check{f}$ is indeed an average measurement of the classical f.

From Eq. (29), using (25), the quantum correction $\check{f}$ of a classical function f is then

$$\check{f}(q,p)=\frac{1}{2\pi c_{-1}}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{dq'dp'}{qq'}\int_0^{\infty}\int_0^{\infty}dxdx'f(q',p')\times\left[e^{ip(x'-x)}e^{-ip'(x'-x)}\psi\left(\frac{x}{q}\right)\psi\left(\frac{x'}{q}\right)\psi\left(\frac{x}{q'}\right)\psi\left(\frac{x'}{q'}\right)\right]. \tag{33}$$

Thus, it is not necessary to find the operator A_f of a classical function f to obtain its lower symbol. One can use the above formula (33) to do so. For example, the quantum correction of the classical functions q^β, p, and p^2 are given by

$$\check{q^\beta} = \frac{c_{\beta-1}c_{-\beta-2}}{c_{-1}}q^\beta; \qquad \check{p} = p;$$
$$\check{p^2} = p^2 + \left(c_{-2}^{(1)} + \frac{c_0 c_{-3}^{(1)}}{c_{-1}}\right)\frac{1}{q^2}, \tag{34}$$

with the constants c_γ and $c_\gamma^{(\beta)}$ defined in (24) and (27), respectively. Notice that the corrections also depend on the choice of specific fiducial vectors to determine these constants.

IV. AFFINE QUANTIZATION OF THE BDT

A. Quantization in the Jordan frame

Now that we have introduced the affine quantization method and the quantum phase-space portrait coming from it, we can apply the method to the BDT presented in Sec. II, since the variables a and φ are both positively defined. However, the Schutz variable associated to the fluid has the whole real line as its domain, and therefore we cannot apply the affine method in it. Nevertheless, we can use another integral quantization method based on the Weyl-Heisenberg group, which acts on the real line [32]. Here, we could also use the canonical quantization for this variable, since it works just fine for parameters in the whole line, a domain that does not have any singularity and, therefore, no problems of self-adjoint-ness.[5] In both cases, we have

$$p_T \mapsto \hat{P}_T = -i\frac{\partial}{\partial T}; \qquad p_T \mapsto \check{p}_T = p_T = E. \tag{35}$$

To build the coherent states of the variables a and φ, let us name the respective fiducial vectors as ψ_a and ψ_φ, which are *a priori* not the same. Then, the coherent states are given by

$$|a,p_a\rangle = U_a|\psi_a\rangle \Rightarrow \langle x|a,p_a\rangle = \frac{e^{ip_a x}}{\sqrt{a}}\psi_a\left(\frac{x}{a}\right) \tag{36}$$

$$|\varphi,p_\varphi\rangle = U_\varphi|\psi_\varphi\rangle \Rightarrow \langle y|\varphi,p_\varphi\rangle = \frac{e^{ip_\varphi y}}{\sqrt{\varphi}}\psi_\varphi\left(\frac{y}{\varphi}\right). \tag{37}$$

With this, the quantization of Eq. (14) results in the following Wheeler-DeWitt equation,

$$\left\{-\omega\lambda_1\frac{1}{\varphi}\partial_a^2+(\omega\lambda_2-\lambda_3)\frac{1}{\varphi a^2}-\lambda_4\frac{1}{a}\partial_a\partial_\varphi+\lambda_5\frac{\varphi}{a^2}\partial_\varphi^2+\lambda_6\frac{1}{a^2}\partial_\varphi\right\}\Psi(a,\varphi,T)=-i(3+2\omega)\partial_T\Psi(a,\varphi,T), \tag{38}$$

where $\Psi(a,\varphi,T)$ is the wave function. The constants λ_i are given by

$$\lambda_1 = \frac{1}{12c_{-1}(\varphi)}; \qquad \lambda_2 = \frac{1}{12c_{-1}(\varphi)}\frac{c_{-3}^{(1)}(a)}{c_{-1}(a)};$$
$$\lambda_3 = \frac{1}{2}\frac{c_{-3}(a)}{c_{-1}(a)}\frac{c_{-2}^{(1)}(\varphi)}{c_{-1}(\varphi)}; \qquad \lambda_4 = \frac{1}{2c_{-1}(a)};$$
$$\lambda_5 = \frac{1}{2}\frac{c_{-3}(a)}{c_{-1}(a)}\frac{c_0(\varphi)}{c_{-1}(\varphi)}; \qquad \lambda_6 = \frac{1}{2}\frac{c_{-3}(a)}{c_{-1}(a)}\frac{c_0(\varphi)}{c_{-1}(\varphi)}+\frac{1}{4c_{-1}(a)}, \tag{39}$$

and we defined

$$c_\gamma^{(j)}(a) = \int_0^\infty [\psi_a^{(j)}(x)]^2\frac{dx}{x^{2+\gamma}};$$
$$c_\gamma^{(j)}(\varphi) = \int_0^\infty [\psi_\varphi^{(j)}(x)]^2\frac{dx}{x^{2+\gamma}}. \tag{40}$$

If we choose $\psi_a = \psi_\varphi$, then $c_\gamma^{(j)}(a) = c_\gamma^{(j)}(\varphi) = c_\gamma^{(j)}$. So, let us choose a fiducial vector such that

$$\psi_a = \psi_\varphi = \frac{9}{\sqrt{6}}x^{\frac{3}{2}}e^{-\frac{3x}{2}}. \tag{41}$$

With these vectors, we have $c_{-2} = c_{-1} = 1$, and $c_{-3}^{(1)} = 3/4$, which, as mentioned before, is a necessary condition for the quantized kinetic energy to be an essentially self-adjoint operator [30]. In turn, this gives the us the Wheeler-DeWitt equation in the Jordan frame

[5] Using the Weyl-Heisenberg method can give us the advantage of introducing another constant that depends on the fiducial vector chosen in the quantization. This can be an asset used to adjust energy levels, for example.

$$\left\{-\frac{\omega}{12}\frac{1}{\varphi}\partial_a^2+\left(\frac{\omega}{16}-\frac{3}{4}\right)\frac{1}{\varphi a^2}-\frac{1}{2a}\partial_a\partial_\varphi+\frac{\varphi}{a^2}\partial_\varphi^2+\frac{5}{4a^2}\partial_\varphi\right\}\Psi$$
$$=-i(3+2\omega)\partial_T\Psi. \tag{42}$$

From this equation, absorbing the constant $12(3+2\omega)\omega^{-1}$ into the temporal parameter, that is, accounting it as energy, we find the Hamiltonian for the BDT in the Jordan frame to be

$$H_J=\frac{1}{\varphi}\partial_a^2-\frac{12}{\omega}\left(\frac{\omega}{16}-\frac{3}{4}\right)\frac{1}{\varphi a^2}+\frac{6}{\omega a}\partial_a\partial_\varphi$$
$$-\frac{12}{\omega}\frac{\varphi}{a^2}\partial_\varphi^2-\frac{15}{\omega a^2}\partial_\varphi. \tag{43}$$

It is easy to see that the Hamiltonian (43) is essentially self-adjoint for the usual measure $dad\varphi$ on the Hilbert space, as expected. One can notice that Eq. (42) is not separable. We can work around this problem by considering the Einstein frame instead.

B. Conformal transformation of affine operators

The Jordan and Einstein frames are related to each other by a conformal transformation given by $g_{\mu\nu}=\phi^{-1}\tilde{g}_{\mu\nu}$, where $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ represent the metric tensors in each frame, respectively. Thus, before analyzing the equivalence between these frames, let us first comment on how affine operators change with a conformal transformation.

As opposed to what happens in the canonical quantization (see Ref. [15]), the affine operators are uniquely defined by Eq. (25). Also, if A_f is the operator obtained from a classical function $f(q,p)$, with q being a positive-defined variable and p its associated momentum, then for a general conformal scaling factor $\Omega(q)$ on the domain, we have

$$\Omega^2(q)A_f\neq A_{\Omega^2(q)f}. \tag{44}$$

Therefore, we need to be careful when we quantize models related by conformal transformations. Even if the constraint obtained from a Hamiltonian is classical, we cannot cancel overall coefficients [for instance, the factor $1/b$ in Eq. (50)]. To illustrate this, let us give an example. Consider $\Omega^2(q)=q$ and $f(q,p)=p$. The operator $A_{\Omega^2 f}$ is given by (28), and then

$$A_{\Omega^2 f}=A_{qp}=\frac{c_0}{c_{-1}}\frac{\hat{Q}\hat{P}+\hat{P}\hat{Q}}{2}\neq\hat{Q}\hat{P}=qA_p=\Omega^2(q)A_f. \tag{45}$$

This means that, classically, it is always possible to cancel non-null coefficients; however, quantizing the constraint in different frames can result in very different scenarios, because of (44). In conclusion, we cannot cancel out non-null functions before quantizing to compare the quantization of two different frames connected by a transformation of coordinates.

C. Quantization in the Einstein frame

Since the seminal paper of Brans and Dicke [1], we know that two formulations of the theory (and, in fact, for every scalar-tensor theory) are possible. These formulations, related by a conformal transformation, are the target of a long debate on which of these frames is physically relevant. Some authors claim they are equivalent classically but should be different at the quantum level [33,34], while others claim that both are equivalent at classical and quantum levels [35–38]. Some also claim that the equivalence is broken by off-shell one-loop quantum corrections, but recovered on shell [39]. Since theoretical predictions depend entirely on the conformal frame we are working on, a natural question that arises is if there is a preferred frame or not, and which one is the most suitable to observations. In the Jordan frame, we found the differential equation governing the wave function evolution (42); however, as a crossed term appeared in the partial derivatives, finding a solution can be difficult. Let us now analyze the problem in the Einstein frame instead.

The Brans-Dicke Lagrangian, with a nonminimally coupled scalar field, is given by (1), and by using the conformal transformation, $g_{\mu\nu}=\varphi^{-1}\tilde{g}_{\mu\nu}$, where $g_{\mu\nu}$ is the metric in the nonminimal coupling frame, the Lagrangian reads as

$$\mathcal{L}_G=\sqrt{-\tilde{g}}\left[\tilde{R}-\left(\omega+\frac{3}{2}\right)\frac{\varphi_{;\rho}\varphi^{;\rho}}{\varphi^2}\right], \tag{46}$$

which is the Lagrangian for general relativity with a minimally coupled scalar field. The Lagrangian (1) is written in the Jordan frame, and (46) is written in the Einstein frame. The conformal transformation is given by the change of coordinates

$$N'=\varphi^{\frac{1}{2}}N;\qquad b=\varphi^{\frac{1}{2}}a;\qquad \varphi'=\varphi, \tag{47}$$

and applying these to (3), we obtain

$$\mathcal{L}_G=\frac{1}{N'}\left[6b\dot{b}^2-\left(\omega+\frac{3}{2}\right)b^3\left(\frac{\dot{\varphi}'}{\varphi'}\right)^2\right]. \tag{48}$$

The total Hamiltonian is thus

$$H_T=N'\left(\frac{p_b^2}{24b}-\frac{\varphi'^2}{2(3+2\omega)b^3}p_{\varphi'}^2-\frac{p_T}{b}\right), \tag{49}$$

and the constraint $H_T = 0$ gives us[6]

$$\frac{p_b^2}{24b} - \frac{\varphi'^2}{2(3+2\omega)b^3} p_{\varphi'}^2 = \frac{p_T}{b}. \tag{50}$$

In order to quantize Eq. (50), it is necessary to know the Hilbert space in the Einstein frame. From the change of variables (7), the measure becomes

$$dad\varphi = \varphi'^{-\frac{1}{2}} dbd\varphi'. \tag{51}$$

Thus, the Hilbert space for the coordinates (b, φ') is $L^2(\mathbb{R}_+^* \times \mathbb{R}_+^*, \varphi'^{-\frac{1}{2}} dbd\varphi')$. Then, according to definition (21), the fiducial vectors $\psi_{\varphi'}$ are defined on another Hilbert space:

$$\psi_{\varphi'} \in L^2\left(R_+^*, \frac{dx}{x^{\frac{1}{2}}}\right) \cap L^2\left(R_+^*, \frac{dx}{x^{\frac{3}{2}}}\right). \tag{52}$$

With this measure, the operator associated with the kinetic energy is given by

$$A_{p^2} = -\partial_{\varphi'}^2 + \frac{1}{2\varphi'}\partial_{\varphi'} + \left(\frac{c_{-5/2}^{(1)}(\varphi')}{c_{-1/2}(\varphi')} - \frac{3}{8}\right)\frac{1}{\varphi'^2}, \tag{53}$$

which is already self-adjoint.

Now, for the coordinate b, using (25), we obtain

$$A_{b^{-1}p_b^2} = -\frac{1}{c_{-1}(b)}\frac{1}{b}\partial_b^2 + \frac{1}{c_{-1}(b)}\frac{1}{b^2}\partial_b - \left(\frac{1 - c_{-4}^{(1)}(b)}{c_{-1}(b)}\right)\frac{1}{b^3}. \tag{54}$$

For the coordinate φ', we get

$$A_{\varphi'^2 p_{\varphi'}^2} = -\frac{11}{8}\frac{c_{3/2}}{c_{-1/2}} + \frac{c_{-1/2}^{(1)}}{c_{-1/2}} - \frac{3}{2}\frac{c_{3/2}}{c_{-1/2}}\varphi'\partial_{\varphi'} - \frac{c_{3/2}}{c_{-1/2}}\varphi'^2\partial_{\varphi'}^2. \tag{55}$$

Then, the quantization of Eq. (50) results in

$$\left\{-\varpi\partial_b^2 + \frac{\varpi}{b}\partial_b + (\tilde{\lambda}_1\varpi + \tilde{\lambda}_2)\frac{1}{b^2} + \frac{\tilde{\lambda}_3}{b^2}\left(\varphi'^2\partial_{\varphi'}^2 + \frac{3}{2}\varphi'\partial_{\varphi'}\right)\right\}\Psi = -24\varpi i\partial_T\Psi, \tag{56}$$

with $\varpi = \omega + \frac{3}{2}$, and $\tilde{\lambda}_i$ are given by

$$\tilde{\lambda}_1 = c_{-4}^{(1)}(b) - 1;$$
$$\tilde{\lambda}_2 = \frac{3}{4}\frac{c_{-4}(b)}{c_{-1/2}(\varphi')}\left(\frac{11}{8}c_{3/2}(\varphi') - c_{-1/2}^{(1)}(\varphi')\right);$$
$$\tilde{\lambda}_3 = \frac{c_{-4}(b)c_{3/2}(\varphi')}{c_{-1/2}(\varphi')}. \tag{57}$$

On the other hand, one can change variables as in (47) directly on (38). This yields

$$\left[-\left(\omega\lambda_1 + \frac{\lambda_4}{2} - \frac{\lambda_5}{4}\right)\partial_b^2 + \frac{\lambda_5 - \lambda_4}{4}\frac{1}{b}\partial_b + (\omega\lambda_2 - \lambda_3)\frac{1}{b^2} + \left(\frac{\lambda_5}{2} - \lambda_4\right)\frac{\varphi'}{b}\partial_b\partial_{\varphi'} + \lambda_5\frac{\varphi'^2}{b^2}\partial_{\varphi'}^2 + \lambda_6\frac{\varphi'}{b^2}\partial_{\varphi'}\right]\Psi = -i(3+2\omega)\partial_T\Psi, \tag{58}$$

with λ_i given in (39). Notice that the coefficients λ_i are in terms of $c_\lambda^{(i)}(a)$ and $c_\lambda^{(i)}(\varphi)$, while the coefficients in Eq. (56) are in terms of $c_\lambda^{(i)}(b)$ and $c_\lambda^{(i)}(\varphi')$. Considering the freedom in the choice of the fiducial vectors,[7] and comparing Eqs. (56) and (58), we conclude that there is equivalence between the Einstein and Jordan frames only if

$$\frac{\lambda_5}{2} - \lambda_4 = 0 \Rightarrow c_{-3}(a) = 2\frac{c_{-1}(\varphi)}{c_0(\varphi)}. \tag{59}$$

In a way, this result is similar to the one found in Ref. [26], in which it is concluded that the equivalence depends on the choice of *ordering factors* for the canonical quantization, which are related to the coefficients of the Hamiltonian operator. In our case, the unitary equivalence is then obtained if we impose some constraints on the fiducial vectors:

$$4c_{-3}^{(1)} \geq 3c_{-1} \quad \text{for } \psi_a, \psi_b, \psi_\varphi; \quad \text{and} \quad c_{-3}(a) = \frac{2c_{-1}(\varphi)}{c_0(\varphi)}. \tag{60}$$

Let us solve, without loss of generality, Eq. (56). We suppose the following separation of variables: $\Psi(b, \varphi, t) \coloneqq X(b)Y(\varphi)P(T)$. We obtain, for the function of time

$$P(T) = A\exp\left[i\frac{ET}{24}\right], \tag{61}$$

where $E/24$ is the energy constant. This results in the following system of partial differential equations,

[6]We keep the 1/b factor in order to avoid inconsistences in the quantization (see the discussion in Sec. IV. B).

[7]The quantization is not determined by this choice, although there is an inequality constraint ($4c_{-3}^{(1)} \geq 3c_{-1}$) in order to obtain a Hermitian operator (see the discussion at the end of Sec. III. A).

$$\left\{-\partial_b^2 + \frac{1}{b}\partial_b + \frac{1}{\varpi}[\tilde{\lambda}_1\varpi + \tilde{\lambda}_2 - \tilde{\lambda}_3 k^2]\frac{1}{b^2}\right\}X(b) = EX(b);$$
$$\left\{\varphi^2\partial_\varphi^2 + \frac{3}{2}\varphi\partial_\varphi\right\}Y(\varphi) = -k^2 Y(\varphi), \tag{62}$$

with k^2 being a separation constant. The general solutions are given by

$$X(b) = C_1 b J_\nu(\sqrt{E}b) + C_2 b Y_\nu(\sqrt{E}b), \tag{63}$$

$$Y(\varphi) = D_1\varphi^{-\frac{1}{4}(\sqrt{1-16k^2}+1)} + D_2\varphi^{\frac{1}{4}(\sqrt{1-16k^2}-1)}, \tag{64}$$

with J_ν and Y_ν the Bessel functions of first and second kinds, respectively; $C_{1,2}$ and $D_{1,2}$ are integration constants,

$$\nu = \sqrt{\frac{(1+\tilde{\lambda}_1)\varpi + (\tilde{\lambda}_2 - \tilde{\lambda}_3 k^2)}{\varpi}}; \tag{65}$$

and $k^2 < 1/16$. The wave function of the Universe $\Psi_T(b,\varphi) = X(b)Y(\varphi)$ must be square integrable. This is the reason for the choice of the limit set for the separation constant. Equation (62) is known as Euler equation, and the solution (64) corresponds to said limit of k^2. The solution for $k^2 = 1/16$ gives similar results; however, $k^2 > 1/16$ results in a non-square-integrable wave function. This is also the reason why we choose a negative sign for the separation constant. Also, since Y_n blows up at the origin, we must take $C_2 = 0$. Now, let us consider the following transformation for the variable φ:

$$\sigma = \ln\varphi \Rightarrow d\sigma = \frac{1}{\varphi}d\varphi. \tag{66}$$

With this, the solution (64) becomes[8]

$$Y(\sigma) = D_1 e^{-\frac{\sigma}{4}(\sqrt{1-16k^2}+1)} + D_2 e^{\frac{\sigma}{4}(\sqrt{1-16k^2}-1)}. \tag{67}$$

For the sake of simplicity, let us consider $D_2 = 0$. We construct the wave packet as

$$\Psi = N\int_{-\frac{1}{4}}^{\frac{1}{4}} dk b J_\nu(\sqrt{E}b) e^{-\frac{\sigma}{4}(\sqrt{1-16k^2}+1)} e^{i\frac{E}{24}T}, \tag{68}$$

where N is a normalization constant. Therefore, the norm of the wave packets is

[8]With this, it becomes more evident why it is only square integrable for $k^2 < 1/16$.

$$\langle\Psi|\Psi\rangle = N^2\int_0^{b_0}\int_0^\infty \varphi^{-\frac{1}{2}} db d\varphi \int_{-\frac{1}{4}}^{\frac{1}{4}}\int_{-\frac{1}{4}}^{\frac{1}{4}} dk dk' b^2 e^{-\frac{\sigma}{2}}$$
$$\times J_\nu(\sqrt{E}b) J_{\nu'}(\sqrt{E}b) e^{i(\frac{1}{4}\sqrt{|1-16k'^2|} - \frac{1}{4}\sqrt{|1-16k^2|})\sigma}, \tag{69}$$

or, writing only in terms of σ,

$$\langle\Psi|\Psi\rangle = N^2\int_0^{b_0}\int_{-\infty}^\infty db d\sigma \int_{-\frac{1}{4}}^{\frac{1}{4}}\int_{-\frac{1}{4}}^{\frac{1}{4}} dk dk' b^2 J_\nu(\sqrt{E}b)$$
$$\times J_{\nu'}(\sqrt{E}b) e^{i(\frac{1}{4}\sqrt{|1-16k'^2|} - \frac{1}{4}\sqrt{|1-16k^2|})\sigma}, \tag{70}$$

where the prime on the ν indicates $\nu(k')$ and we can take $b_0 = 1$ as the value of the scale factor today. Performing the integrals over σ and k' gives

$$\langle\Psi|\Psi\rangle = 8\pi N^2 \int_0^{b_0}\int_{-\frac{1}{4}}^{\frac{1}{4}} db dk b^2 J_\nu(\sqrt{E}b) J_\nu(\sqrt{E}b). \tag{71}$$

Now, we shall consider an approximation for the limit $\omega \gg k^2$. This approximation is relevant due to our understanding of today's estimate of the Brans-Dicke constant ω. Notice that, in this limit, the Bessel index (65) becomes $\nu = \sqrt{1+\tilde{\lambda}_1}$, and then (71) becomes

$$\langle\Psi|\Psi\rangle = 8\pi N^2 \int_0^{b_0} db b^2 J_{\nu=\sqrt{1+\tilde{\lambda}_1}}(\sqrt{E}b) J_{\nu=\sqrt{1+\tilde{\lambda}_1}}(\sqrt{E}b). \tag{72}$$

The solution is given in terms of the regularized generalized hypergeometric function ${}_2\tilde{F}_3$ [40] as

$$\langle\Psi|\Psi\rangle = 4\sqrt{\pi}N^2 b_0^3 \Gamma\left(\nu + \frac{1}{2}\right)\Gamma\left(\nu + \frac{3}{2}\right)(b_0\sqrt{E})^{2\nu}$$
$$\times {}_2\tilde{F}_3\left(\nu + \frac{1}{2}, \nu + \frac{3}{2}; \nu + 1, \nu + \frac{5}{2}, 2\nu + 1; -b_0^2 E\right). \tag{73}$$

The regularized generalized hypergeometric functions are defined as the power series

$${}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) := \frac{1}{\Gamma(b_1)\ldots\Gamma(b_q)}\sum_{n=0}^\infty \frac{(a_1)_n\ldots(a_p)_n}{(b_1)_n\ldots(b_q)_n}\frac{z^n}{n!}, \tag{74}$$

with the recurrence relations

$$(a_j)_0 = 1; \text{ and}$$
$$(a_j)_n = a_j(a_j+1)(a_j+2)\ldots(a_j+n-1), \text{ for } n \geq 1. \tag{75}$$

The norm of the wave packet becomes

$$\langle\Psi|\Psi\rangle = A(b_0\sqrt{E})^{2\nu}\sum_{n=0}^{\infty}\frac{(\nu+\frac{1}{2})_n}{(\nu+1)_n(2\nu+1)_n}\frac{(-b_0^2E)^n}{n!}, \quad (76)$$

where

$$A = 4\sqrt{\pi}N^2\frac{b_0^3}{(\nu+1)}\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+2)\Gamma(2\nu+1)}. \quad (77)$$

Then, Eq. (76) suggests that the energy spectrum is discrete. This means we can write $\langle\Psi|\Psi\rangle = \sum_n\langle\Psi_n|\Psi_n\rangle$, and the energy levels satisfy the equations

$$\langle\Psi_0|\Psi_0\rangle = A(b_0\sqrt{E})^{2\nu}, \quad (78)$$

and for a general $n \geq 1$,

$$\langle\Psi_n|\Psi_n\rangle = A(b_0\sqrt{E})^{2\nu}\sum_{n=0}^{\infty}\frac{(\nu+\frac{1}{2})_n}{(\nu+1)_n(2\nu+1)_n}\frac{(-b_0^2E)^n}{n!}. \quad (79)$$

D. Quantum phase-space portrait of the BDT

Let us consider the formalism introduced in Sec. III. B. The constraint (14), $\mathcal{H}_T = 0$, can be rewritten in its semiclassical version using (33) to calculate each term. For the sake of simplicity, we will keep the same letter for the energy constant, so $\check{p}_T = E$, and hence

$$\frac{\omega}{12}\frac{1}{\varphi}p_a^2+(\omega\kappa_1-\kappa_2)\frac{1}{a^2\varphi}+\frac{1}{2a}p_ap_\varphi-\kappa_3\frac{\varphi}{a^2}p_\varphi^2=(3+2\omega)E, \quad (80)$$

with the constants κ_i being

$$\kappa_1 = \frac{1}{12}\left(\frac{c_0(a)c_{-3}^{(1)}(a)}{c_{-1}(a)}+c_{-2}^{(1)}(a)\right);$$
$$\kappa_2 = \frac{1}{2}\frac{c_0(a)c_{-3}(a)}{c_{-1}(a)}\left(\frac{c_0(\varphi)c_{-3}^{(1)}(\varphi)}{c_{-1}(\varphi)}+c_{-2}^{(1)}(\varphi)\right);$$
$$\kappa_3 = \frac{1}{2}\frac{c_0(a)c_{-3}(a)}{c_{-1}(a)}\frac{c_0(\varphi)c_{-3}(\varphi)}{c_{-1}(\varphi)},$$

where $c_\gamma^{(j)}(a)$ and $c_\gamma^{(j)}(\varphi)$ are

$$c_\gamma^{(j)}(a) = \int_0^\infty[\psi_a^{(j)}(x)]^2\frac{dx}{x^{2+\gamma}};$$
$$c_\gamma^{(j)}(\varphi) = \int_0^\infty[\psi_\varphi^{(j)}(x)]^2\frac{dx}{x^{2+\gamma}}. \quad (81)$$

If we choose $\psi_a = \psi_\varphi$, then $c_\gamma^{(j)}(a) = c_\gamma^{(j)}(\varphi) = c_\gamma^{(j)}$. With this in mind, let us choose a fiducial vector such that

$$\psi_a = \psi_\varphi = \frac{9}{\sqrt{6}}x^{\frac{3}{2}}e^{-\frac{3x}{2}}. \quad (82)$$

With these vectors, we have $c_{-2}=c_{-1}=1$, and $c_{-3}^{(1)} = 3/4$, the latter being a necessary condition for the quantized Hamiltonian to be an essentially self-adjoint operator [30]. We want this condition to hold even if we are not doing the quantization explicitly, since the semiclassical trajectories are probabilistic along the path that a quantum state evolves. Then, Eq. (80) becomes

$$\frac{\omega}{12}\frac{1}{\varphi}p_a^2+\frac{9}{8}(\omega-2)\frac{1}{a^2\varphi}+\frac{1}{2a}p_ap_\varphi-2\frac{\varphi}{a^2}p_\varphi^2=(3+2\omega)E. \quad (83)$$

The expression (83) allows us to analyze the expected behavior of the scale factor a for the early universe, for a given initial value of the scalar field $\varphi(t_0) = \varphi_0$ and its momentum at this instant $p_\varphi(t_0) = p_{\varphi 0}$.

Notice that Eq. (14) is the classical Hamiltonian constraint in the Jordan frame. To compare the expected behavior of the scale factor in the Jordan frame with that in the Einstein frame, let us calculate the quantum phase-space portrait of Eq. (50), the Hamiltonian constraint in the Einstein frame. We have

$$(b^{-1}p_b^2)\check{} = \frac{p_b^2}{b}+\frac{c_{-1}^{(1)}(b)+c_1(b)c_{-4}^{(1)}(b)-c_1(b)}{c_{-1}(b)}\frac{1}{b^3}; \quad (84)$$

$$(\varphi'^2p_{\varphi'}^2)\check{} = \frac{c_{3/2}(\varphi')c_{-7/2}(\varphi')}{c_{-1/2}(\varphi')}\varphi'^2p_{\varphi'}^2(\varphi') + \left(\frac{c_{3/2}(\varphi')c_{-7/2}^{(1)}(\varphi')}{c_{-1/2}(\varphi')}+\frac{c_{-3/2}(\varphi')c_{-1/2}^{(1)}(\varphi')}{c_{-1/2}(\varphi')} - \frac{11}{8}\frac{c_{3/2}(\varphi')c_{-3/2}(\varphi')}{c_{-1/2}(\varphi')}\right). \quad (85)$$

Then, the quantum correction of (50) becomes

$$\frac{3+2\omega}{24}\left[p_b^2+\kappa_4\frac{1}{b^2}\right]-\frac{1}{b^2}[\kappa_5\varphi'^2p_{\varphi'}^2+\kappa_6] = (3+2\omega)E', \quad (86)$$

with E' the energy and the constants

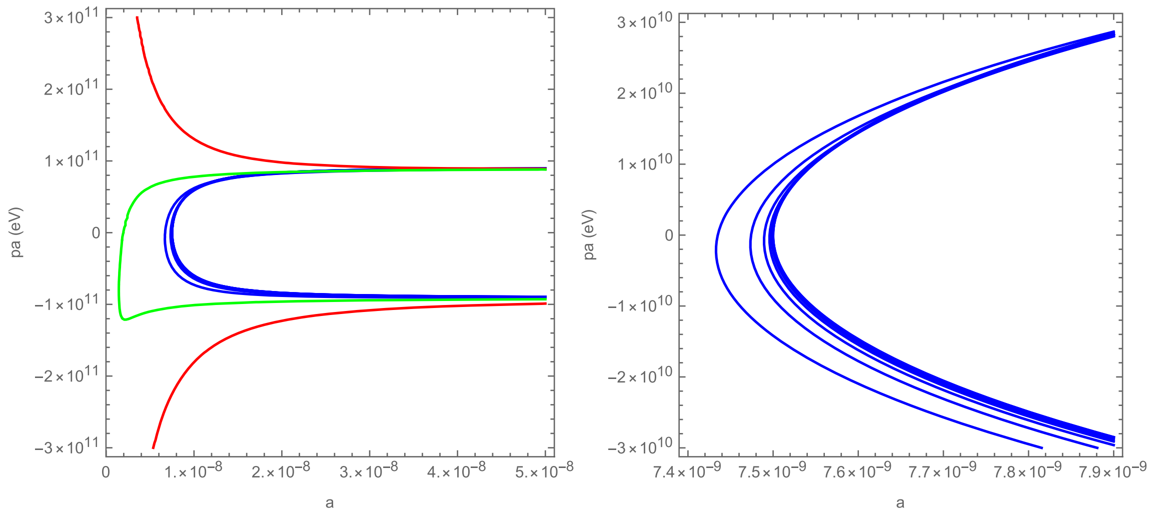

FIG. 1. Quantum phase space of the scalar field in the Jordan frame, using $\omega = 410\,000$ and $E_0 = 10^{16}$. The left figure is for a range $1 \le p_\varphi \le 10^3$, while for the right figure, the range is smaller $1 \le p_\varphi \le 10^2$.

$$\kappa_4 = \frac{c_{-1}^{(1)}(b) + c_1(b)c_{-4}^{(1)}(b) - c_1(b)}{c_{-1}(b)};$$

$$\kappa_5 = \frac{1}{2}\frac{c_{-4}(b)c_1(b)}{c_{-1}(b)}\frac{c_{3/2}(\varphi')c_{-7/2}(\varphi')}{c_{-1/2}(\varphi')};$$

$$\kappa_6 = \frac{1}{2}\frac{c_{-4}(b)c_1(b)}{c_{-1}(b)}\frac{c_{3/2}(\varphi')}{c_{-1/2}(\varphi')}\left(c_{-7/2}^{(1)}(\varphi') + \frac{c_{-3/2}(\varphi')c_{-1/2}^{(1)}(\varphi')}{c_{3/2}(\varphi')} - \frac{11}{8}c_{-3/2}(\varphi')\right).$$

By choosing the fiducial vectors as before, we find

$$\frac{3+2\omega}{24}p_b^2 + \left[\frac{1296 - 1500\sqrt{3\pi} + 864\omega}{64}\right]\frac{1}{b^2} - \frac{525\sqrt{3\pi}}{16b^2}\varphi'^2 p_{\varphi'}^2 = (3+2\omega)E'. \quad (87)$$

Equations (83) and (87) are the quantum corrections of the classical Brans-Dicke theory described in the Jordan and Einstein frames, respectively. To understand the consequences of these corrections, let us build the quantum phase space of the BDT in both these frames.

V. PHASE-SPACE PORTRAITS

As mentioned before, in this section, we present the quantum phase-space portraits coming from Eqs. (83) and (87). The aim is to understand the behavior of the scale factor a, which is connected to the volume of the Universe, so the phase spaces shown here are with reference to this variable. Notice, however, that there are still other free parameters: the scalar field φ, the energy E, and the Brans-Dicke constant ω. These parameters will be varied for the sake of understanding their influence on the issue. Without loss of generality, let us consider the initial state of the scalar field to be $\varphi_0 = 1$.

A. Jordan frame

For the Jordan frame, let us set the energy at E_0 and construct the phase space for a range of values of p_φ (Fig. 1). Each curve represents a value for the velocity (momentum) of the scalar field. In each plot, we have a total of ten curves. For each curve, the lower the minimum of the scale factor is, the higher p_φ is. Notice that, up until an upper value for p_φ, the curves are of a smooth bouncing for the universe, including solutions with a possible inflationary phase. Above a certain value of p_φ, divergent curves appear. If one assumes that this type of divergence does not describe a physical reality (favoring smoothness), then the scalar field must have a limit in momentum. Otherwise, this model predicts a singularity formed by an accelerated contraction of a prior universe, reaching null volume as the (modulus of the) momentum goes to infinity, followed by a decelerated inflation.[9]

Now, we study the effect of the Brans-Dicke parameter ω (Fig. 2). In the left figure, we take $\omega = 41\,000$ and see there are more divergent lines than in the generic case considered in Fig. 1. In the right figure, we increased ω to $4\,100\,000$. Notice that it requires a much greater initial momentum for the scale factor to obtain divergent solutions. Therefore, a larger ω seems to lead to a more well-behaved theory. This is a result of interest, since the larger ω is, the greater the coupling between matter and the scalar field, that is, the smaller the effects of the scalar field are. This would

[9]Notice that we are reading the graphics in the clockwise direction.

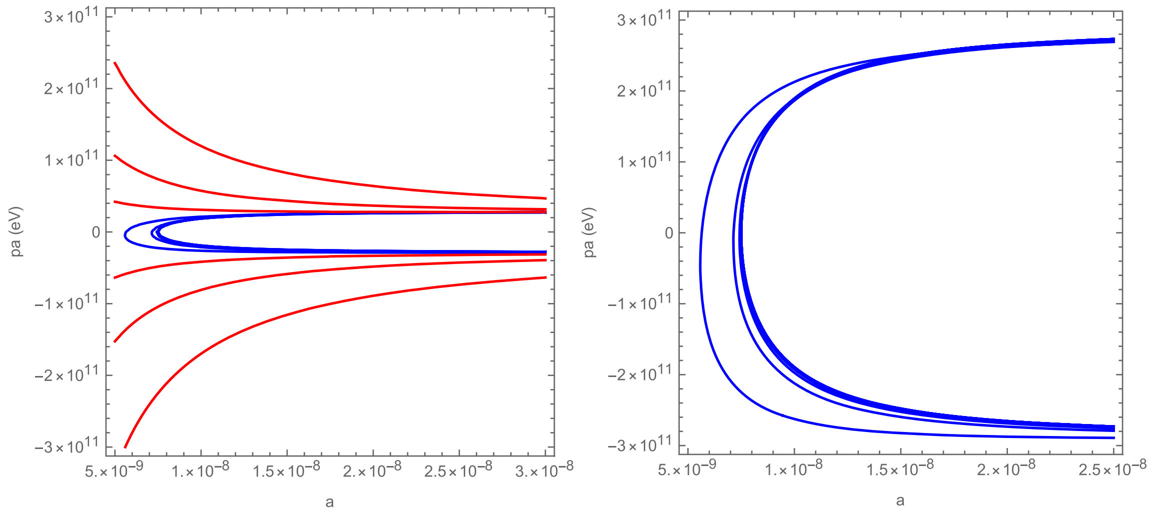

FIG. 2. The effect of the Brans-Dicke constant in the scalar field phase space. Once again, we use $E_0 = 10^{16}$ and consider the range $1 \leq p_\varphi \leq 10^3$. The left figure is for $\omega = 41\,000$, and the right figure is for $\omega = 4\,100\,000$.

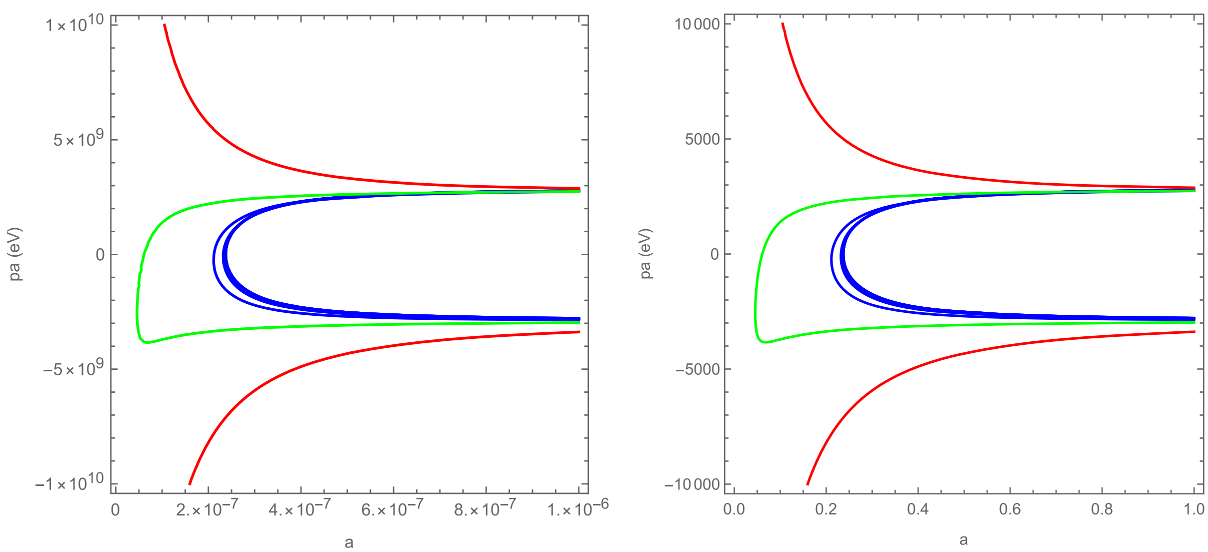

FIG. 3. The change in the energy of the system results in a change of scale for the solutions. In the left figure, we take $E = 10^{13}$, and in the right figure, we take $E = 10$. The same values were used as before for p_φ and ω: $1 \leq p_\varphi \leq 10^3$ and $\omega = 410\,000$.

correspond to the weak-field limit we observe today. Actually, for a perfect fluid (as in our case), we recover GR in this limit [41].

The variation of the energy parameter does not change the behavior of the solutions, but it results in a change of scale in the phase space (Fig. 3). So, the energy can determine the scale with which inflation happens.

Up until now, we have considered the initial value of the scalar field to be $\varphi_0 = 1$, but we also want to understand the effects of the initial condition on the behavior of the solutions (Fig. 4). Thus, in Fig. 4, we show the direct influence of changing the value for the scalar field on the solutions. The top row shows greater values for φ_0, from 10 to 10^4 (left to right). We notice that the greater φ_0 is, the more singularities we obtain. Conversely, in the second row, we lower it from 0.1 to 10^{-4}. The solutions tend to bounce instead of singularities. As expected, the results are consistent with the study on ω.

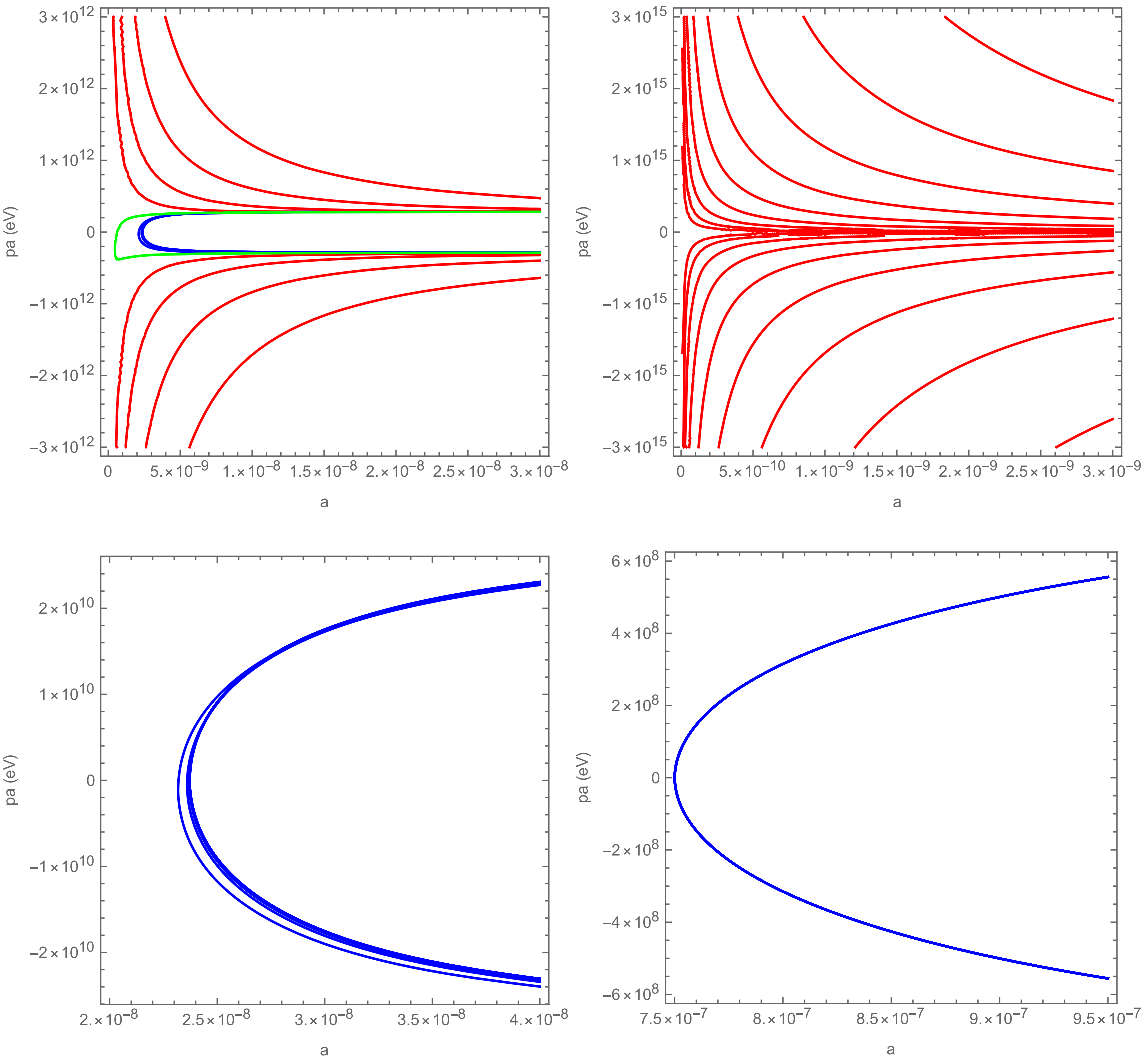

FIG. 4. The top row shows the solutions for high values of φ_0: top left is $\varphi_0 = 10$, and top right is $\varphi_0 = 10^4$. The bottom row is for low values of φ_0: bottom left is $\varphi_0 = 10^{-1}$, and bottom right is $\varphi_0 = 10^{-4}$. For these, we are considering $\omega = 410\,000$, $E = 10^{16}$, and $1 \leq p_\varphi \leq 10^3$.

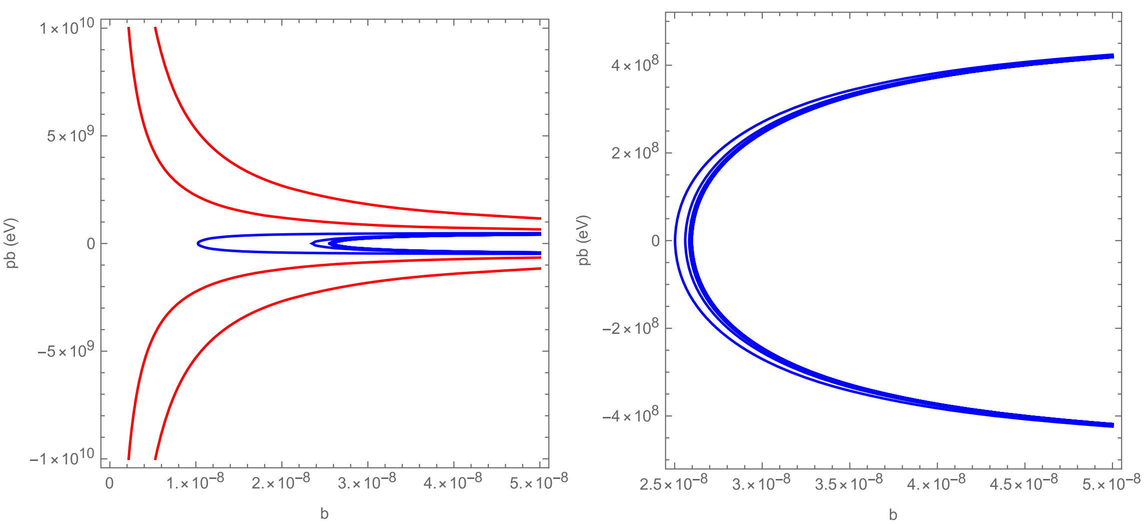

FIG. 5. Quantum phase space of the scalar field in the Einstein frame, using $\omega = 410\,000$ and $E_0 = 10^{16}$. The left figure is for a range $1 \leq p_\varphi \leq 10^3$, while for the right figure, the range is smaller $1 \leq p_\varphi \leq 10^2$.

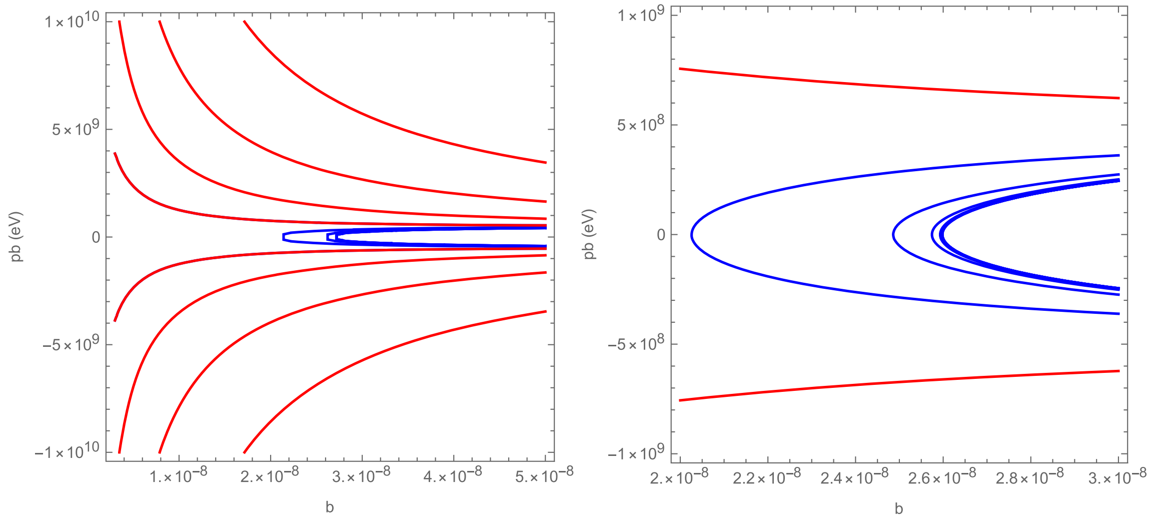

FIG. 6. The effect of the Brans-Dicke constant in the scalar field phase space. Once again, we take $E_0 = 10^{16}$ and consider the range $1 \leq p_\varphi \leq 10^3$. The left figure is for $\omega = 41,000$, and the right one is for $\omega = 4\,100\,000$.

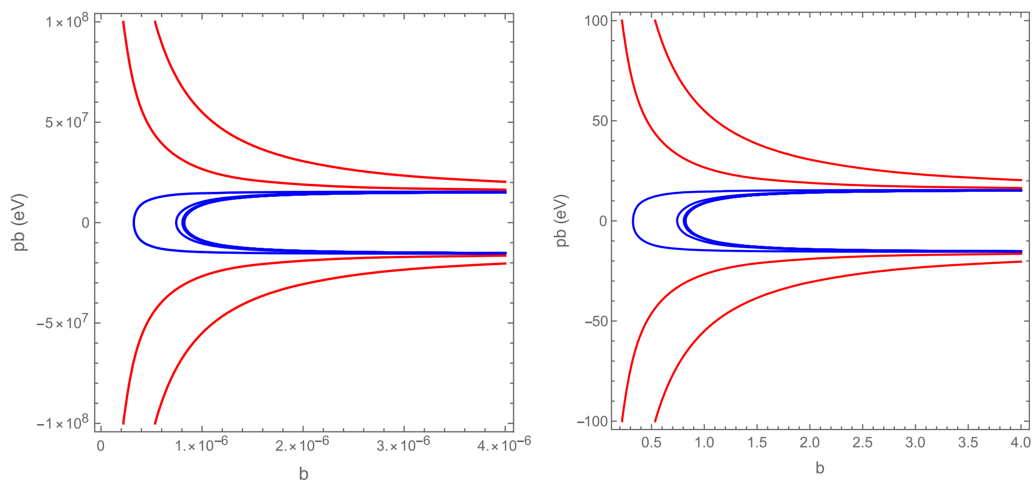

FIG. 7. The change in the energy of the system results in a change of scale for the solutions. In the left figure, we take $E = 10^{13}$, and in the right figure, we take $E = 10$. The same values were used as before for p_φ and ω: $1 \leq p_\varphi \leq 10^3$ and $\omega = 410\,000$.

B. Einstein frame

In the Einstein frame, we have symmetric bounces without any inflationary epoch,[10] as we see in Fig. 5. By varying once again ω (Fig. 6) and the energy (Fig. 7), we arrive at the same conclusions as in the Jordan frame, i.e., that the larger ω is, the less divergent the curves we obtain are, and varying the energy induces a scaling in the phase space. We also show the effect of the scalar field in Fig. 8.

Notice that these results are consistent with what was found in the Jordan frame, which provides further evidence that the frames are equivalent. Remember, though, that, in spite of choosing specific fiducial vectors, this analysis is still qualitative, since one can always choose different wavelets and also restore the unities (we chose $c = \hbar = 1$). For our purpose, this qualitative analysis is enough.

[10]Inflation may be interpreted as a "stretching" of the solutions induced by the conformal transformation by going from the Einstein frame to the Jordan frame.

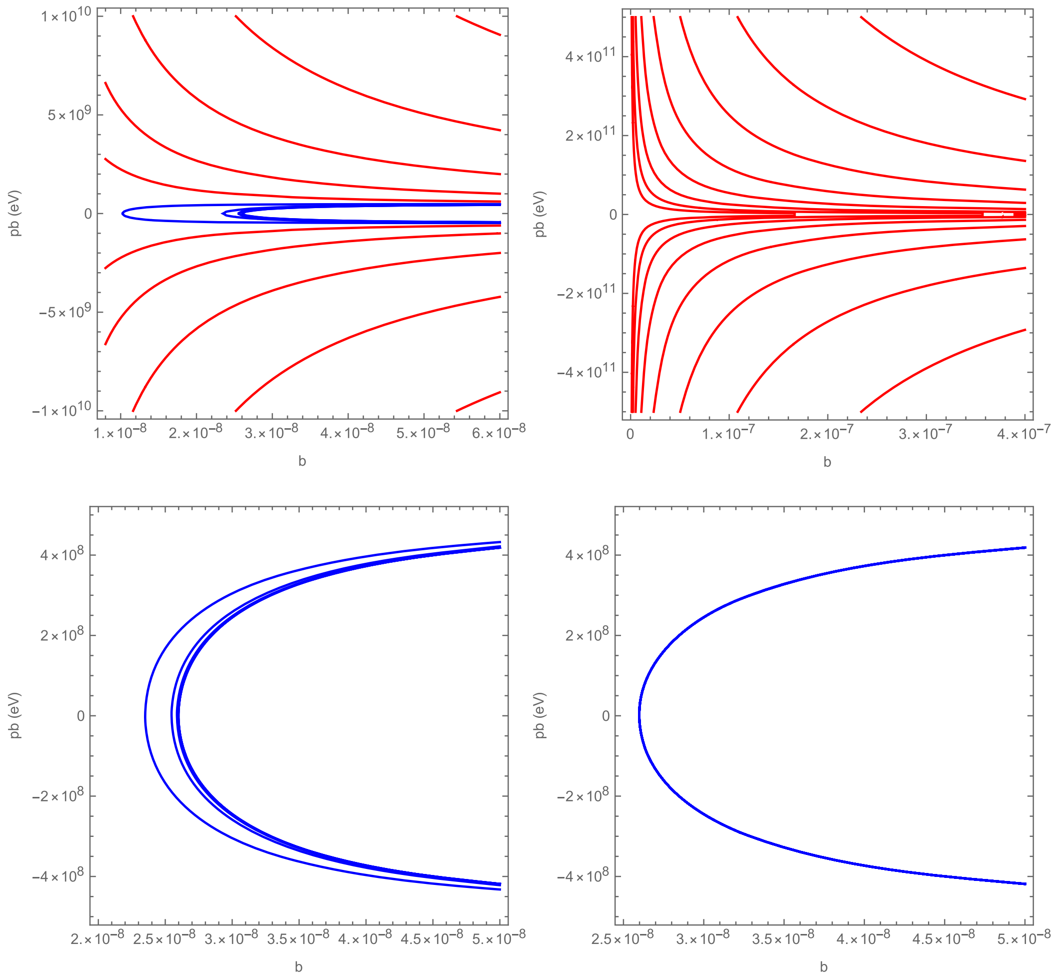

FIG. 8. The top row shows the solutions for high values of φ_0: top left is $\varphi_0 = 10$, and top right is $\varphi_0 = 10^4$. The bottom row is for low values of φ: bottom left is $\varphi_0 = 10^{-1}$, and bottom right is $\varphi_0 = 10^{-4}$. For these, we are considering $\omega = 410\,000$, $E = 10^{16}$, and $1 \leq p_\varphi \leq 10^3$.

VI. CONCLUSIONS

In this work, we presented the quantization of the Brans-Dicke theory using the affine covariant integral method and the cosmological scenarios arising from it. We introduced the classical Hamiltonian formalism of the BDT and the mathematical foundations of this quantization method, in order to familiarize the reader with the concepts used later on. Our model is completed with the addition of a radiative matter component in form of a perfect fluid, introduced via the Schutz formalism, which we adopted as the clock. The affine quantization is based on the symmetry of the phase space of the system, and we can choose the free parameters, namely the fiducial vectors, in a way to build an essentially self-adjoint Hamiltonian operator. The quantization of the Hamiltonian constraint results in the Wheeler-DeWitt equation, from which we obtain a Schrödinger-like equation (38), with the radiative matter providing the time parameter. One expected setback of this quantization is that it results in a nonseparable partial differential equation. We can work around this problem by changing frames, making a conformal transformation of the coordinates.

The BDT is described in the Jordan frame, and a conformal change of coordinates transforms the BDT into GR with a scalar field, i.e., the Einstein frame. The equivalence between these frames is still debatable (see, e.g., Refs. [33–37]), and our results may contribute to this debate. In the Einstein frame, the Schrödinger-like equation is separable and becomes easier to deal with. We presented the classical GR with a scalar-field model corresponding to the BDT in the Einstein frame and quantized it using the affine method. We also performed a change of coordinates in the already quantized Schrödinger-like equation in the Jordan frame. Considering the freedom in the choice of the fiducial vectors, we found an equivalent equation. However, we conclude that the Hamiltonian operator in the Einstein frame is only essentially self-adjoint if we consider different fiducial vectors while quantizing the theory in each frame, or if we change the domains (i.e., the measure) of the operators in the respective Hilbert space. In any case, one may argue that, because of this, there is no equivalence between the frames. However, the role of the fiducial vectors during the quantization is precisely to open up opportunities for adjustment, since it is based on a statistical method ($|\langle q, p|\phi\rangle|^2$ is interpreted as the probability density distribution of the function ϕ; see, e.g., Ref. [27]). Thus, considering different fiducial vectors in different frames should not invalidate the equivalence between them. We choose to solve the Wheeler-deWitt equation obtained from the classical BDT in the Einstein frame, in order to do a qualitative analysis, since this equation has a relatively simple solution. From it, we were able to conclude that the energy spectrum of the Hamiltonian operator in the Einstein frame is discrete.

The affine quantization method is completed with a dequantization, known as the quantum phase-space portrait or lower symbol, that transforms the quantized operator into a classical function, by means of their fiducial vectors expectation values. This dequantization provides a quantum correction for classical observables, from which we can analyze the behavior of these observables in semiclassical environments. Even if we cannot find the wave function of the universe in the Jordan frame, we can use the quantum phase space to compare the results with the ones from the quantum phase space in the Einstein frame. Thus, we find quantum corrections for the Hamiltonian constraint in both frames in Sec. IV. D and compare the results in Sec. V, drawing the phase-space portrait for the scale factor, to better understand the behavior of the (volume of the) universe in earlier stages.

We obtained two types of solutions in both frames: bounces and singularities. For both types, we predict a prior universe. For the singular cases in the Jordan frame, there is an accelerated contraction, with a singular point where the volume of the Universe becomes null, followed by a decelerated inflationary era. However, if we limit the momentum of the scalar field, we obtain only bouncing solutions. Thus, we may argue that the scalar field should have a limited velocity, since this discards the singular solutions. We also analyzed the influence of other parameters in the solutions. In the limit $\omega \to \infty$, in which we expect to reproduce GR (for our model, at least), bounces become more expected. It is interesting to see that an inflationary stage also appears for bounces in this frame. In the Einstein frame, however, we do not have any inflationary era, but similar conclusions can be drawn, with the exception that both singular and bouncing solutions are symmetric.

The use of affine quantization in cosmology is a nascent subject, with the desirable feature of providing solutions without singularities for a natural range of parameters. This is in adequation with other results on various cosmological scenarios (see Refs. [42–46]) as well as a result on the quantum Belinski-Khalatnikov-Lifshitz scenario using the affine coherent states quantization for a Bianchi IX universe [47], in which they suggest that quantizing GR should also lead to bouncing solutions. We intend to continue to explore this line of research in future works.

ACKNOWLEDGMENTS

E. F. was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior—Brasil—Finance Code 001 and by the Institute of Cosmology and Gravitation. E. F. and C. R. A. thank immensely Professor Jean-Pierre Gazeau for his support, the referee for his useful comments, and also Chris Pattison for his diligent proofreading of this work.

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