The abelian gauge-Yukawa β-functions at large N_f

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ABSTRACT: We study the impact of the Yukawa interaction in the large-N_f limit to the abelian gauge theory. We compute the coupled β-functions for the system in a closed form at $\mathcal{O}(1/N_f)$.

Contents

1 Introduction

A comprehensive understanding of the UV behaviour of gauge-Yukawa theories has become of key importance with the growing interest in the asymptotic-safety paradigm [1–4]. Prime candidates for these considerations are gauge-Yukawa models with a large number of fermion flavours, N_f. Computing the leading large-N_f contribution to the β-functions was pioneered by evaluating the $\mathcal{O}(1/N_f)$ gauge β-functions [5–7] for N_f fermions charged under the gauge group; see also Refs [8, 9].

We recently computed the $\mathcal{O}(1/N_f)$ β-function for Yukawa-theory [10] inspired by the earlier works [11, 12]. The Yukawa-theory is closely related to the Gross–Neveu model, which has been extensively studied in the past using a different approach; see e.g. Refs [13–16]. For Gross–Neveu–Yukawa model the behaviour near the fixed point in terms of critical exponents is known up to $\mathcal{O}(1/N_f^2)$ [17, 18]. However, the strength of our analysis is that we readily achieved a closed form expression of the β-function, and as shown in the present work, the procedure is straighforwardly generalisable to include gauge interactions.

In this paper, we compute the leading $1/N_f$ contributions to the β-functions of the gauge-Yukawa system in a closed form. This result is new and sheds light to the impact of the Yukawa interaction to the gauge theory in the large-N_f limit.

The gauge contribution to the Yukawa β-function was computed in the abelian case in Ref. [11] and later generalised to non-abelian and semi-simple gauge groups in Ref. [12] assuming that only one flavour of fermions couples to the scalar via Yukawa interaction. We relax this assumption and show that it is possible to get a closed form expressions also in the general case. The current result provides a groundwork for several interesting extensions including e.g. non-abelian gauge groups and chiral fermions.

The paper is organized as follows: In Sec. 2 we introduce the framework and notations and in Sec. 3 compute the new contributions to the renormalization constants and β-functions. In Sec. 4 we collect the results and comment on the structure of the coupled system, and in Sec. 5 we conclude.

2 The framework

We consider the massless U(1) gauge theory with N_f fermion flavours (QED) with a gauge-singlet real scalar field coupling to the fermionic multiplet, ψ, via Yukawa interaction:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + i\bar{\psi}\not{D}\psi + y\bar{\psi}\psi\phi. \tag{2.1}$$

We define the rescaled gauge and Yukawa couplings,

$$E \equiv \frac{e^2}{4\pi^2}N_f, \text{ and } K \equiv \frac{y^2}{4\pi^2}N_f, \tag{2.2}$$

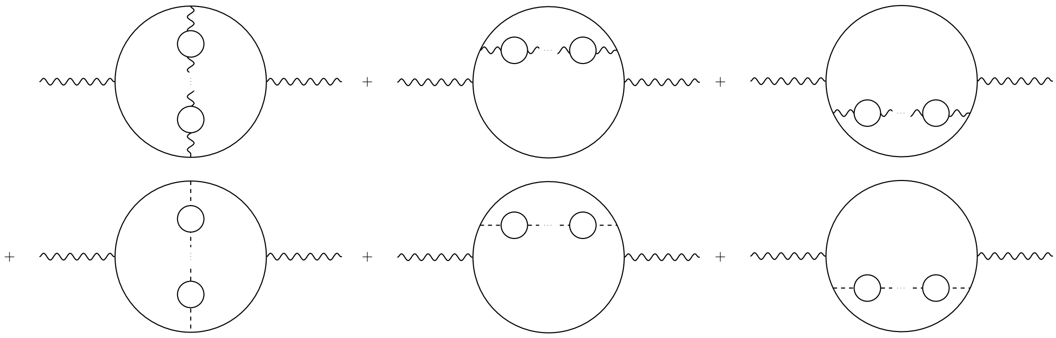


Figure 1: Photon self-energy corrections.

which are kept constant in the limit $N_f \to \infty$. The purpose of this paper is to derive the coupled system of β-functions for E and K at the $1/N_f$ level:

$$\beta_E \equiv \frac{\mathrm{d}E}{\mathrm{d}\ln\mu} = E\left(K\frac{\partial}{\partial K} + E\frac{\partial}{\partial E}\right) G_1(K,E), \tag{2.3}$$

$$\beta_K \equiv \frac{\mathrm{d}K}{\mathrm{d}\ln\mu} = K\left(K\frac{\partial}{\partial K} + E\frac{\partial}{\partial E}\right) H_1(K,E), \tag{2.4}$$

where G_1 and H_1 are defined by

$$\log Z_E \equiv \log Z_3^{-1} = \sum_{n=1}^{\infty} \frac{G_n(K,E)}{\epsilon^n}, \tag{2.5}$$

$$\log Z_K \equiv \log(Z_S^{-1} Z_F^{-2} Z_V^2) = \sum_{n=1}^{\infty} \frac{H_n(K,E)}{\epsilon^n}, \tag{2.6}$$

and Z_3, Z_S, Z_F, and Z_V are the renormalization constants for the photon, the scalar, and the fermion wave function, and the 1PI vertex, respectively.

The photon wave function renormalization constant, Z_3, is given by

$$Z_3 = 1 - \mathrm{div}\left\{Z_3 \Pi_0(p^2, Z_K K, Z_E E, \epsilon)\right\}, \tag{2.7}$$

where Π_0 is the self-energy divided by the external momentum squared, p^2, and we denote the poles of X in ϵ by $\mathrm{div}X$. The self-energy can be written as

$$\begin{aligned} &\Pi_0(p^2, K_0, E_0, \epsilon) \\ &\quad = E_0\, \Pi_E^{(1)}(p^2,\epsilon) + \frac{1}{N_f}\sum_{n=2}^{\infty}\left(E_0^n \Pi_E^{(n)}(p^2,\epsilon) + E_0 K_0^{n-1} \Pi_K^{(n)}(p^2,\epsilon)\right) + \mathcal{O}(1/N_f^2), \end{aligned} \tag{2.8}$$

where $\Pi_E^{(1)}$ is the one-loop contribution, and $\Pi_E^{(n)}$ and $\Pi_K^{(n)}$ contain the n-loop part consisting of $n-2$ fermion bubbles in the gauge and Yukawa chains summing over the topologies given in Fig. 1.

The scalar wave function renormalization constant, Z_S, is determined via

$$Z_S = 1 - \mathrm{div}\left\{Z_S S_0(p^2, Z_K K, Z_E E, \epsilon)\right\}, \tag{2.9}$$

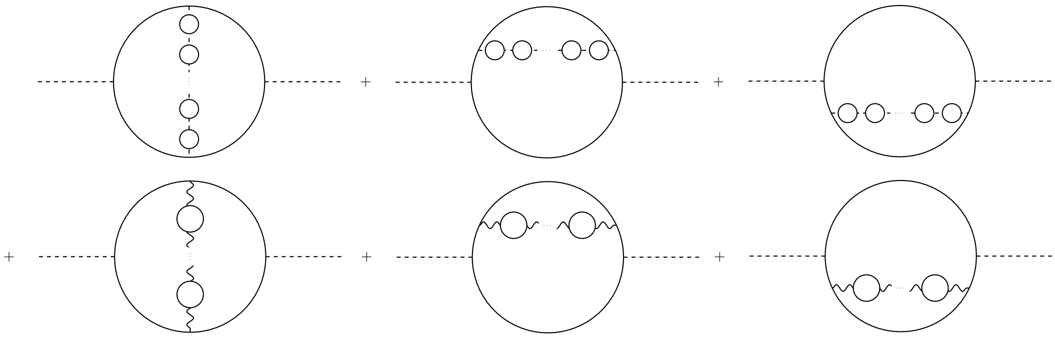


Figure 2: Scalar self-energy corrections.

with the scalar self-energy given by

$$
\begin{aligned}
&S_0(p^2,K_0,E_0,\epsilon)\\
&\quad = K_0 S_K^{(1)}(p^2,\epsilon) + \frac{1}{N_f}\sum_{n=2}^{\infty}\Big(K_0^n S_K^{(n)}(p^2,\epsilon) + K_0 E_0^{n-1} S_E^{(n)}(p^2,\epsilon)\Big) + \mathcal{O}(1/N_f^2),
\end{aligned} \tag{2.10}
$$

where $S_K^{(1)}$ is the one-loop result, and $S_K^{(n)}$ and $S_E^{(n)}$ the n-loop terms consisting of $n-2$ fermion bubbles in the Yukawa and gauge chains summing over the topologies shown in Fig. 2.

For the fermion self-energy and vertex renormalization constants, the lowest non-trivial contributions are already $\mathcal{O}(1/N_f)$, and we have

$$
Z_f = 1 - \mathrm{div}\left\{\Sigma_0(p^2, Z_K K, Z_E E, \epsilon)\right\}, \tag{2.11}
$$

$$
\Sigma_0(p^2,K_0,E_0,\epsilon) = 1 + \frac{1}{N_f}\sum_{n=1}^{\infty}\Big(K_0^n \Sigma_K^{(n)}(p^2,\epsilon) + E_0^n \Sigma_E^{(n)}(p^2,\epsilon)\Big) + \mathcal{O}(1/N_f^2), \tag{2.12}
$$

where $\Sigma_K^{(n)}$ and $\Sigma_E^{(n)}$ are depicted in Fig. 3a with $n-1$ fermion bubbles. Similarly,

$$
Z_V = 1 - \mathrm{div}\left\{V_0(p^2, Z_K K, Z_E E, \epsilon)\right\}, \tag{2.13}
$$

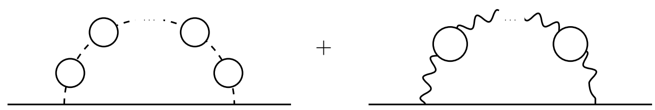
$$
V_0(p^2,K_0,E_0,\epsilon) = 1 + \frac{1}{N_f}\sum_{n=1}^{\infty}\Big(K_0^n V_K^{(n)}(p^2,\epsilon) + E_0^n V_E^{(n)}(p^2,\epsilon)\Big) + \mathcal{O}(1/N_f^2), \tag{2.14}
$$

where $V_K^{(n)}$ and $V_E^{(n)}$ contain $n-1$ fermion bubbles and are shown diagrammatically in Fig. 3b.

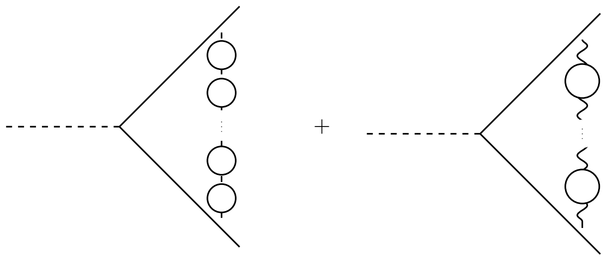
The term corresponding to pure QED, $\Pi_E^{(n)}$, was computed in Ref. [6], and the pure-Yukawa contributions, $S_K^{(n)}$, $\Sigma_K^{(n)}$ and $V_K^{(n)}$, in Ref. [10]. Their contribution to the β-functions, Eqs (2.3) and (2.4), is

$$
\beta_E(K=0) = E^2\left[\frac{2}{3} + \frac{1}{4N_f}\int_0^{2/3E}\pi_E(t)\mathrm{d}t\right] + \mathcal{O}(1/N_f^2), \tag{2.15}
$$

$$
\beta_K(E=0) = K^2\left[1 + \frac{1}{N_f}\left(\frac{3}{2} + \int_0^{K}\xi_K(t)\mathrm{d}t\right)\right] + \mathcal{O}(1/N_f^2), \tag{2.16}
$$



(a) Fermion self-energy corrections.



(b) Vertex corrections.

Figure 3: Gauge and Yukawa contributions to fermion self-energy and the vertex corrections due to a chain of fermion bubbles.

where

$$\pi_E(t) = \frac{\Gamma(4-t)(1-t)\left(1-\frac{t}{3}\right)\left(1+\frac{t}{2}\right)}{\Gamma\left(2-\frac{t}{2}\right)^2\Gamma\left(3-\frac{t}{2}\right)\Gamma\left(1+\frac{t}{2}\right)}, \tag{2.17}$$

$$\xi_K(t) = -\frac{\Gamma(4-t)}{\Gamma(2-\frac{t}{2})\Gamma(3-\frac{t}{2})\pi t}\sin\left(\frac{\pi t}{2}\right) \tag{2.18}$$

The impact of the mixed contributions, namely $\Pi_K^{(n)}$, and $S_E^{(n)}$, $\Sigma_E^{(n)}$, $V_E^{(n)}$, is evaluated in the next section.

3 Mixed contributions

In this section we derive the mixed contributions to the renormalization constants for the photon self-energy, the fermion self-energy, the Yukawa vertex, and the scalar self-energy, and eventually compute the coupled β-functions.

3.1 The Yukawa contribution to the QED β-function

The Yukawa contribution to the photon self-energy (depicted in the second row of Fig. 1), is obtained by substituting Eq. (2.8) in Eq. (2.7). We get

$$Z_3(K) = -\frac{E}{N_f}\text{div}\left\{\sum_{n=1}^{\infty}(Z_K K)^n \Pi_K^{(n+1)}(p^2,\epsilon)\right\}. \tag{3.1}$$

Notice that the diagrams involving a horizontal bubble chain differ from the corresponding ones for the scalar self-energy in Fig. 2 just by an overall factor $(2-d)$ coming from the algebra of the γ-matrices. Altogether, we find

$$\Pi_K^{(n)}(p^2,\epsilon) = (-1)^{n-1}\frac{3}{4(d-1)n\epsilon^{n-1}}\pi_K(p^2,\epsilon,n), \tag{3.2}$$

where $\pi_K(p^2,\epsilon,n)$ can be expanded as

$$\pi_K(p^2,\epsilon,n) = \sum_{j=0}^{\infty}\pi_K^{(j)}(p^2,\epsilon)(n\epsilon)^j, \tag{3.3}$$

with $\pi_K^{(j)}(p^2,\epsilon)$ regular for $\epsilon \to 0$. Recalling that $Z_K = \left(1-\frac{1}{\epsilon}K\right)^{-1} + \mathcal{O}(1/N_f)$, we can evaluate $Z_3(K)$ from Eq. (3.1):

$$\begin{aligned} Z_3(K) &= -\frac{E}{N_f}\mathrm{div}\left\{\sum_{n=1}^{\infty}K^n\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{1}{\epsilon^i}\Pi_K^{(n-i+1)}(p^2,\epsilon)\right\} \\ &= -\frac{3E}{4N_f}\mathrm{div}\left\{\sum_{n=1}^{\infty}\frac{(-K)^n}{(d-1)\epsilon^n}\sum_{j=0}^{n-1}\pi_K^{(j)}(p^2,\epsilon)\epsilon^j\right. \\ &\qquad\left.\times\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i(n-i+1)^{j-1}\right\} \\ &= -\frac{3E}{4N_f}\mathrm{div}\left\{\sum_{n=1}^{\infty}\frac{(-K)^n}{(d-1)\epsilon^n}\pi_K^{(0)}(\epsilon)\frac{(-1)^{n+1}}{n(n+1)}\right\} \\ &= -\frac{3E}{4N_f}\frac{1}{\epsilon}\int_0^K\frac{\pi_K^{(0)}(t)}{t-3}\left(1-\frac{t}{K}\right)\mathrm{d}t, \end{aligned} \tag{3.4}$$

where we used

$$\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i(n-i+1)^{j-1} = \frac{(-1)^{n+1}}{n(n+1)}\delta_{j,0}, \quad j=0,\dots,n-1 \tag{3.5}$$

and restricted ourselves to the $1/\epsilon$ pole. The function $\pi_K^{(0)}$ is independent of p^2, as it should, and reads

$$\pi_K^{(0)}(t) = \frac{(t-2)(t-1)\Gamma(5-t)}{6\Gamma(3-\frac{t}{2})^2\pi t}\sin\left(\frac{\pi t}{2}\right). \tag{3.6}$$

The contribution of $Z_3(K)$ to β_E, Eq. (2.3), is found to be

$$\beta_E(K\neq 0) = E^2\frac{3}{4N_f}\int_0^K\pi_K(t)\mathrm{d}t. \tag{3.7}$$

where we have defined

$$\pi_K(t) \equiv \frac{\pi_K^{(0)}(t)}{t-3}. \tag{3.8}$$

We show the function $\pi_K(t)$ in Fig. 4. Since $\pi_K(t)$ has a first order pole at $t=3$, the first singularity of $\beta_E(K\neq 0)$ occurs at $K=3$ and is a logarithmic one. The next singularity of $\pi_K(t)$ is found at $t=5$ (first order) and would result in a logarithmic singularity of $\beta_E(K\neq 0)$ at $K=5$.

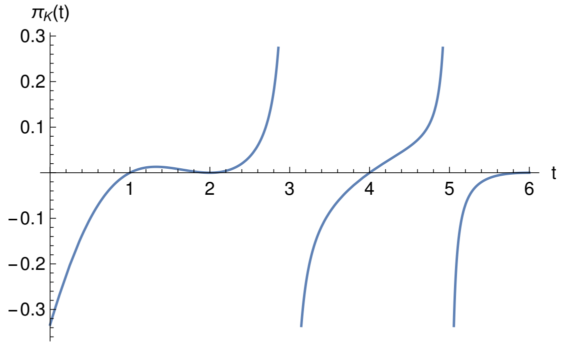


Figure 4: The function $\pi_K(t)$ defined in Eq. (3.8).

3.2 The QED contribution to the Yukawa β-function

The QED contribution to the fermion self-energy and to the Yukawa vertex is closely related to the pure-Yukawa case. This is because the gauge chain is equivalent to the Yukawa chain besides an overall factor. In fact, $\Sigma_E^{(n)}$ and $V_E^{(n)}$ are related to $\Sigma_K^{(n)}$ and $V_K^{(n)}$ as

$$\Sigma_E^{(n)}(\not{p}) =(d-2)\left(\frac{d-2}{d-1}\right)^{n-1}\Sigma_K^{(n)}(\not{p}), \tag{3.9}$$

$$V_E^{(n)}(p^2) = -\,d\left(\frac{d-2}{d-1}\right)^{n-1}V_K^{(n)}(p^2). \tag{3.10}$$

The factors $(d-2)$ and $-d$ come from the algebra of the γ-matrices, while $\left(\frac{d-2}{d-1}\right)^{n-1}$ takes into account the difference in replacing $\Pi_E^{(1)}$ with $S_K^{(1)}$. Notice that $g_{\mu\nu}$ is the only relevant Lorentz structure in the photon propagator, since the $k_\mu k_\nu$ term do not contribute to the β-function.

Making use the relations Eqs (3.9) and (3.10), $\Sigma_E^{(n)}$ and $V_E^{(n)}$ are expanded as

$$\Sigma_E^{(n)}(\not{p}) = (-1)^{n-1}\left(\frac{2}{3}\right)^n\frac{3}{4n\epsilon^n}\sigma_E(p^2,\epsilon,n), \tag{3.11}$$

$$\sigma_E(p^2,\epsilon,n) = \sum_{j=0}^{\infty}\sigma_E^{(j)}(p^2,\epsilon)(n\epsilon)^j, \tag{3.12}$$

and

$$V_E^{(n)}(p^2) = (-1)^{n-1}\frac{3}{n\epsilon^n}\left(\frac{2}{3}\right)^n v_E(p^2,\epsilon,n), \tag{3.13}$$

$$v_E(p^2,\epsilon,n) = \sum_{j=0}^{\infty}v_E^{(j)}(p^2,\epsilon)(n\epsilon)^j. \tag{3.14}$$

Using the one-loop result $Z_E = \left(1-\frac{2}{3}E\right)^{-1} + \mathcal{O}(1/N_f)$, and applying the same summation procedure as in Ref. [10] for the fermion self-energy and the vertex, Eqs (2.11) and (2.13) yield

$$Z_f(E) = -\frac{1}{N_f}\sum_{n=1}^{\infty}\text{div}\left\{(Z_E E)^n \Sigma_E^{(n)}(p^2,\epsilon)\right\} = -\frac{1}{N_f}\frac{3}{4\epsilon}\int_0^{\frac{2}{3}E}\sigma_E^{(0)}(t)\text{d}t, \tag{3.15}$$

$$Z_V(E) = -\frac{1}{N_f}\sum_{n=1}^{\infty}\text{div}\left\{(Z_E E)^n V_E^{(n)}(p^2,\epsilon)\right\} = -\frac{1}{N_f}\frac{3}{\epsilon}\int_0^{\frac{2}{3}E}v_E^{(0)}(t)\text{d}t, \tag{3.16}$$

where we kept only the $1/\epsilon$ pole. The functions $\sigma_E^{(0)}$ and $v_E^{(0)}$ are independent of p^2, and are given by

$$\sigma_E^{(0)}(t) = \frac{2\Gamma(4-t)}{3\pi\Gamma\left(1-\frac{t}{2}\right)\Gamma\left(3-\frac{t}{2}\right)t}\sin\left(\frac{\pi t}{2}\right), \tag{3.17}$$

$$v_E^{(0)}(t) = \left(\frac{1-\frac{t}{4}}{1-\frac{t}{2}}\right)^2\sigma_E^{(0)}(t). \tag{3.18}$$

The QED contribution to the scalar self-energy is shown in the second row of Fig. 2. The diagrams involving a horizontal gauge chain are related to the ones in the pure-Yukawa case analogoursly to Eq. (3.9). Altogether, we find

$$S_E^{(n)}(p^2,\epsilon) = (-1)^n\left(\frac{2}{3}\right)^n\frac{27}{4n(n-1)\epsilon^n}s_E(p^2,\epsilon,n), \tag{3.19}$$

$$s_E(p^2,\epsilon,n) = \sum_{j=0}^{\infty}s_E^{(j)}(p^2,\epsilon)(n\epsilon)^j. \tag{3.20}$$

The QED contribution in Eq. (2.9) is singled out as follows:

$$Z_S(E) = -K\text{div}\left\{Z_f(E)^{-2}Z_V(E)^2 S_K^{(1)}(p^2,\epsilon) + \frac{1}{N_f}\sum_{n=1}^{\infty}(Z_E E)^n S_E^{(n+1)}(p^2,\epsilon)\right\}. \tag{3.21}$$

To evaluate the right-hand side of Eq. (3.21), we closely follow the procedure in Ref. [10] for the scalar self-energy:

$$\begin{aligned}
Z_S(E) &= -\frac{K}{N_f}\sum_{n=1}^{\infty}E^n\text{div}\left\{\left(1-\frac{2}{3}\frac{E}{\epsilon}\right)^{-n}\left[2S_F^{(1)}\left(\Sigma_E^{(n)}-V_E^{(n)}\right)+S_E^{(n+1)}\right]\right\}\\
&= -\frac{K}{N_f}\sum_{n=1}^{\infty}E^n\text{div}\left\{\sum_{i=0}^{n-1}\binom{n-1}{i}\left(\frac{2}{3}\right)^i\frac{1}{\epsilon^i}\right.\\
&\qquad\qquad\left.\times\left[2S_F^{(1)}\left(\Sigma_E^{(n-i)}-V_E^{(n-i)}\right)+S_E^{(n-i+1)}\right]\right\}\\
&= -3\frac{K}{N_f}\sum_{n=1}^{\infty}\left(-\frac{2}{3}E\right)^n\text{div}\left\{\frac{1}{\epsilon^n}\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i\frac{\xi_E(p^2,\epsilon,n-i)}{(n-i)(n-i+1)\epsilon^{n+1}}\right\},
\end{aligned} \tag{3.22}$$

where we defined

$$\xi_E(p^2,\epsilon,n) = \epsilon(n+1)\,2\,S_F^{(1)}\left(v_E(p^2,\epsilon,n)-\frac{1}{4}\sigma_E(p^2,\epsilon,n)\right)-\frac{3}{2}s_E(p^2,\epsilon,n+1), \tag{3.23}$$

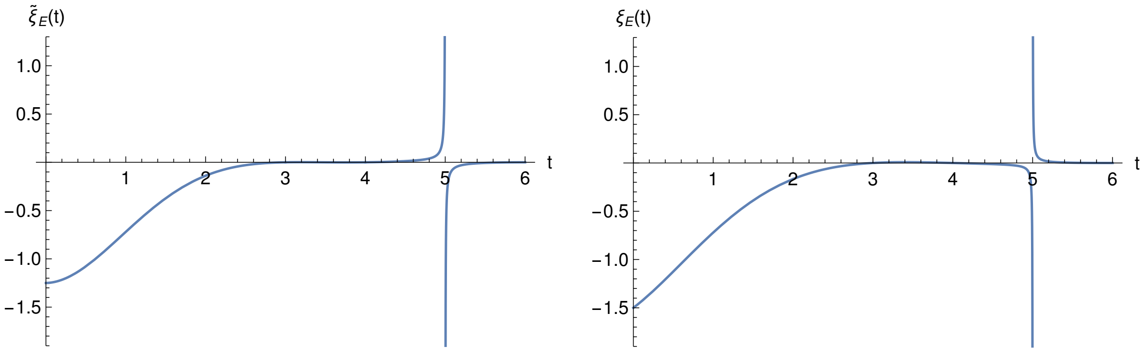


Figure 5: The functions $\tilde{\xi}_E(t)$ (left panel) and $\xi_E(t)$ (right panel) defined in Eqs (3.31) and (3.28), respectively.

and $S_F^{(1)}$ is the finite part of the one-loop bubble $S_K^{(1)}$. Then, by expanding

$$\xi_E(p^2, \epsilon, n-i) = \sum_{j=0}^{\infty} \epsilon^j (n-i+1)^j \xi_E^{(j)}(p^2, \epsilon), \tag{3.24}$$

and using

$$\sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i \frac{(n-i+1)^{j-1}}{n-i} = \begin{cases} \frac{(-1)^{n+1}}{n+1} & j=0 \\ \frac{(-1)^{n+1}}{n} & j=1,..,n \end{cases}, \tag{3.25}$$

we can further simplify the expression to

$$\begin{aligned} Z_S(E) =& 3\frac{K}{N_f} \sum_{n=1}^{\infty} \left(\frac{2}{3}E\right)^n \operatorname{div}\left\{ \frac{1}{(n+1)\epsilon^{n+1}} \xi_E^{(0)}(p^2,\epsilon) + \frac{1}{n\epsilon^{n+1}} \sum_{j=1}^{\infty} \xi_E^{(j)}(p^2,\epsilon)\epsilon^j \right\} \\ =& \frac{9K}{2EN_f} \sum_{n=2}^{\infty} \left(\frac{2}{3}E\right)^n \operatorname{div}\left\{ \frac{1}{\epsilon^n} \left(\frac{\xi_E^{(0)}(p^2,\epsilon)}{n} + \frac{\xi_E(p^2,\epsilon,0) - \xi_E^{(0)}(p^2,\epsilon)}{n-1} \right) \right\} \\ =& \frac{9K}{2EN_f} \frac{1}{\epsilon} \int_0^{\frac{2}{3}E} \left(\xi_E^{(0)}(t) - \xi_E^{(0)}(0) + \frac{2}{3} \frac{\xi_E(p^2,t,0) - \xi_E^{(0)}(t)}{t} E \right) \mathrm{d}t, \end{aligned} \tag{3.26}$$

where we kept the $1/\epsilon$ pole only. The function $\xi_E(p^2, t, 0) = \lim_{n\to 0} \xi_E(p^2, t, n)$ has to be independent of p^2 for the consistency of the computation. This is indeed the case: we checked that

$$\frac{3}{2} s_E(p^2, t, 1) = 2\left(1 + t\, S_F^{(1)}(p^2, t)\right) \left(v_E^{(0)}(t) - \frac{1}{4}\sigma_E^{(0)}(t) \right), \tag{3.27}$$

and therefore

$$\xi_E(p^2, t, 0) = -2 v_E^{(0)}(t) + \frac{1}{2}\sigma_E^{(0)}(t) \equiv \xi_E(t). \tag{3.28}$$

Finally, we find:

$$Z_S(E) = \frac{3K}{\epsilon N_f} \left\{ \frac{3}{2E} \int_0^{\frac{2}{3}E} \left(\xi_E^{(0)}(t) - \xi_E^{(0)}(0) \right) \mathrm{d}t + \int_0^{\frac{2}{3}E} \frac{\xi_E(t) - \xi_E^{(0)}(t)}{t} \mathrm{d}t \right\}. \tag{3.29}$$

With Eqs (3.15), (3.16) and (3.29) at hand, we can compute the QED contribution to the Yukawa β-function:

$$\beta_K(E \neq 0) = -\frac{3K^2}{N_f}\left\{\int_0^{\frac{2}{3}E} \tilde{\xi}_E(t)\mathrm{d}t + \frac{3}{2} + \left(1 - \frac{2E}{3K}\right)\xi_E\left(\frac{2}{3}E\right)\right\}. \tag{3.30}$$

where we have defined

$$\tilde{\xi}_E(t) \equiv \frac{\xi_E(t) - \xi_E^{(0)}(t)}{t}. \tag{3.31}$$

The functions $\xi_E(t)$ and $\tilde{\xi}_E(t)$ are explicitly given by

$$\xi_E(t) = -\frac{2(t-3)^2\Gamma(2-t)}{3\Gamma\left(2-\frac{t}{2}\right)\Gamma\left(3-\frac{t}{2}\right)\pi t}\sin\left(\frac{\pi t}{2}\right), \tag{3.32}$$

$$\tilde{\xi}_E(t) = \frac{(15+t-5t^2+t^3)\Gamma(4-t)}{9(t-2)\Gamma\left(2-\frac{t}{2}\right)\Gamma(3-\frac{t}{2})\pi t}\sin\left(\frac{\pi t}{2}\right). \tag{3.33}$$

We plot the functions $\tilde{\xi}_E(t)$ and $\xi_E(t)$ in Fig. 5. The first singularity of $\beta_K(E \neq 0)$ is at $E = 15/2$ and consists of a first-order pole coming from $\xi_E(t)$ plus a logarithmic singularity arising from the integration of $\tilde{\xi}_E(t)$, both at $t = 5$.

4 The coupled system

Here we summarize and discuss our results for the coupled system. Combining Eqs (2.15) and (2.16) with the new results in Eqs (3.7) and (3.30), we obtain

$$\frac{\beta_K}{K^2} = 1 - \frac{3}{N_f}\left\{1 - \frac{1}{3}\int_0^K \xi_K(t)\mathrm{d}t + \int_0^{\frac{2}{3}E}\tilde{\xi}_E(t)\mathrm{d}t + \left(1-\frac{2E}{3K}\right)\xi_E\left(\frac{2}{3}E\right)\right\}, \tag{4.1}$$

$$\frac{\beta_E}{E^2} = \frac{2}{3} + \frac{1}{4N_f}\left\{\int_0^{\frac{2}{3}E}\pi_E(t)\mathrm{d}t + 3\int_0^K \pi_K(t)\mathrm{d}t\right\}. \tag{4.2}$$

Near the Gaussian fixed point, these can be expanded as

$$\begin{aligned}\beta_E =& \frac{2}{3}E^2 + \frac{1}{2N_f}E^3 - \frac{1}{4N_f}E^2K - \frac{11}{72N_f}E^4 + \frac{7}{32N_f}E^2K^2 \\ &- \frac{77}{1944N_f}E^5 - \frac{3}{64N_f}E^2K^3 + \dots\end{aligned} \tag{4.3}$$

$$\begin{aligned}\beta_K =& \left(1 + \frac{3}{2N_f}\right)K^2 - \frac{3}{N_f}EK - \frac{3}{2N_f}K^3 + \frac{5}{4N_f}EK^2 + \frac{5}{6N_f}E^2K \\ &+ \frac{7}{16N_f}K^4 - \frac{1}{2N_f}E^2K^2 + \frac{35}{108N_f}E^3K \\ &+ \frac{11}{96N_f}K^5 + \frac{1}{3888N_f}\left(-1625 + 1296\zeta_3\right)E^3K^2 + \frac{1}{648N_f}(83 - 144\zeta_3)E^4K\dots\end{aligned} \tag{4.4}$$

We have checked that the expansions agree with the known four-loop results [19–23] in the leading order in N_f. Furthermore, the $-\frac{2E}{3K}\xi_E\left(\frac{2}{3}E\right)$ part in the last term of Eq. (4.1)

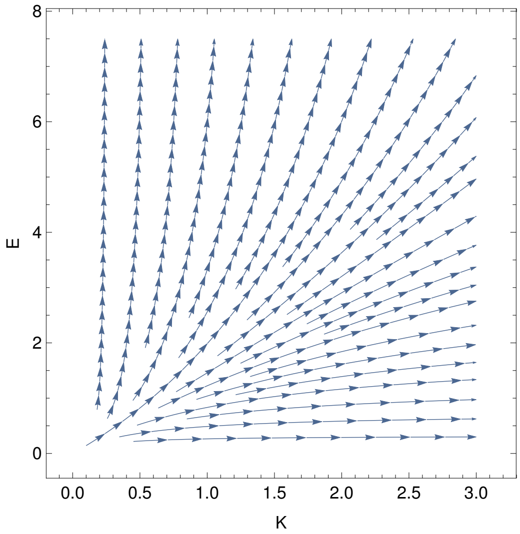


Figure 6: The flow diagram for the coupled system with $N_f = 30$. The arrows point towards UV.

corresponds to the result of Refs [11, 12], and we have checked that our result agrees with those.

The first singularity of the pure-QED β-function is located at $E = 15/2$, whereas for the pure-Yukawa case it occurs at $K = 5$. These known singularities are now accompanied by the ones from the mixed contributions, Eqs (3.7) and (3.30). As we noticed in Section 3, $\beta_E(K \neq 0)$ has the first singularity at $K = 3$, while $\beta_K(E \neq 0)$ at $E = 15/2$. The former, similarly to the pure gauge and Yukawa cases, is a logarithmic singularity, whereas the latter is a pole of first order.

The $\mathcal{O}(1/N_f)$ coupled system has only the three already known fixed points: the Gaussian fixed point, and the pure-QED (near $E = 15/2$) and pure-Yukawa (near $K = 3$) fixed points.

We show the flow diagram for $N_f = 30$ outside the vicinity of the singularities in Fig. 6. Near $K = 3$, the logarithmic singularity in β_E arising from $\pi_K(t)$ dominates making the gauge coupling to increase and approach the value $E = 15/2$. Near $E = 15/2$, however, β_K has a pole arising from $\xi_E(t)$ eventually dominating the flow, and driving the Yukawa coupling to zero near $E = 15/2$. The flow may be extended setting $K \equiv 0$ and switching to pure-QED, so that the gauge coupling reaches the fixed point as $E \to 15/2$ in the UV.

5 Conclusions

We have computed the leading $1/N_f$ mixed contributions for the β-functions for abelian gauge-Yukawa theory with N_f fermion flavours coupling to a gauge-singlet real scalar. Together with the known results for the pure-QED and pure-Yukawa cases, this allows the study of the abelian gauge-Yukawa system.

The flow in the interacting theory leads to the vanishing Yukawa coupling near the gauge coupling value $E = 15/2$ due to the peculiar interplay of the singularities. However, the gauge β-function is still positive around $(K, E) = (0, 15/2)$, and E keeps growing before eventually reaching the fixed point due to the known a logarithmic singluarity near $E = 15/2$.

Our work extends the previous results towards a more complete picture of gauge-Yukawa theories in the large-N_f limit.

Acknowledgments

We thank John Gracey for valuable comments.

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