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The β-function for Yukawa theory at large N_f

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Abstract: We compute the β-function for a massless Yukawa theory in a closed form at the order $\mathcal{O}(1/N_f)$ in the spirit of the expansion in a large number of flavours N_f. We find an analytic expression with a finite radius of convergence, and the first singularity occurs at the coupling value $K = 5$.

Contents

1 Introduction

The success of the Standard Model in describing the electroweak scale phenomena notwithstanding the apparent problems with the high-energy behaviour have lead to revival of interest in better understanding the UV properties of general gauge-Yukawa theories, see e.g. Refs [1–3]. In particular, gauge-Yukawa theories with a large number of fermion flavours, N_f, provide interesting candidates within the asymptotic-safety framework as opposed to the traditional asymptotic-freedom paradigm [4, 5].

The groundwork for these considerations was laid few decades ago with the computation of the leading large-N_f behaviour of the gauge β-functions [6–8] for N_f fermion charged under the gauge group; see also Refs [9, 10]. The leading $1/N_f$ contribution to the β-function is obtained by resumming the gauge self-energy diagrams with ever increasing chain of fermion bubbles constituting a power series in $K = \alpha N_f/\pi$. It was noticed that this series has a finite radius of convergence; in the case of U(1) gauge group $K = 15/2$. Furthermore, the leading $1/N_f$ contribution to the U(1) β-function has a negative pole at $K = 15/2$, thereby suggesting that this behaviour could cure the Landau-pole behaviour of the SM U(1) coupling, see e.g. Refs [9, 11, 12].

Recently, a further step towards a more complete understanding of these models was achieved by working out the leading $1/N_f$ contribution from the gauge sector to a Yukawa coupling [13]; an extension to semi-simple gauge groups was discussed in Ref. [14]. However, only a single fermion flavour was assumed to couple to the scalar, and the scalar self-energy remained uneffected by the N_f fermion bubbles. Our work is the first step to bridge this remaining gap: we provide the leading $1/N_f$ β-function for pure Yukawa theory, where N_f flavours of fermions couple to the scalar field via Yukawa interaction. We leave the more detailed study within a general gauge-Yukawa framework for future work. Interestingly, the pure Yukawa model is closely related to the Gross–Neveu–Yukawa model, whose critical exponents have been recently computed up to $1/N_f^2$ [15, 16]; see also the earlier studies on the Gross–Neveu model e.g. Refs [17, 18].

The paper is organized as follows: In Sec. 2 we introduce the framework and notations and in Sec. 3 give the expressions for the renormalization constants. In Sec. 4 we perform the resummations of the bubble chains and give closed form expressions for the renormalization constants. In Sec. 5 we collect the results, and write down the final expression for the β-function, and in Sec. 6 we conclude. Explicit formulas for the loop integrals are given in Appendix A.

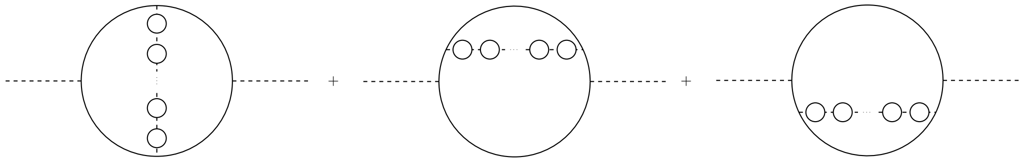
2 The framework and definitions

We consider the massless Yukawa theory for a real scalar field, ϕ, and a fermionic multiplet, ψ, consisting of N_f flavours interacting through the usual Yukawa interaction:

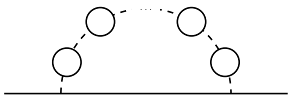
$$\mathcal{L}_{\text{Yuk}} = g\bar{\psi}\psi\phi. \tag{2.1}$$

We define the rescaled coupling,

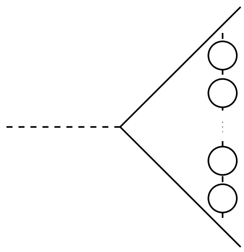
$$K \equiv \frac{g^2}{4\pi^2} N_f, \tag{2.2}$$



(a) Scalar self-energy corrections.



(b) Fermion self-energy correction.



(c) Vertex correction.

Figure 1: Scalar self-energy, fermion self-energy, and vertex corrections due to a chain of fermion bubbles.

which is kept constant in the limit $N_f \to \infty$. The β-function of the rescaled coupling, K, can then be expanded in powers of $1/N_f$ as

$$\beta(K) \equiv \frac{\mathrm{d}K}{\mathrm{d}\ln\mu} = K^2\left[F_0 + \frac{1}{N_f}F_1(K)\right] + \mathcal{O}\left(1/N_f^2\right). \tag{2.3}$$

The purpose of this paper is to compute F_0 and $F_1(K)$. The former is entirely fixed at the one-loop level and can be derived just by rescaling the well-known result for the β-function at that order, while the evaluation of $F_1(K)$ requires the resummation of diagrams in Fig. 1 involving all-order fermion-bubble chains.

The β-function can be obtained from

$$\beta = K^2\frac{\partial G_1(K)}{\partial K}, \tag{2.4}$$

where G_1 is defined by

$$\ln Z_K \equiv \ln\left(Z_S^{-1}Z_F^{-2}Z_V^2\right) = \sum_{n=1}^{\infty}\frac{G_n(K)}{\epsilon^n}, \tag{2.5}$$

and Z_S, Z_F, and Z_V are the renormalization constants for the scalar wave function, the fermion wave function, and the 1PI vertex, respectively. The scalar wave function renormalization constant is determined via

$$Z_S = 1 - \mathrm{div}\{Z_S\Pi_0(p^2, Z_K K, \epsilon)\}, \tag{2.6}$$

where $\Pi_0(p^2, K_0, \epsilon)$ is the scalar self-energy divided by p^2, where p is the external momentum. Here and in the following, $\mathrm{div}X$ denotes the poles of X in ϵ. The self-energy can be written as

$$\Pi_0(p^2, K_0, \epsilon) = K_0 \Pi^{(1)}(p^2, \epsilon) + \frac{1}{N_f} \sum_{n=2}^{\infty} K_0^n \Pi^{(n)}(p^2, \epsilon), \tag{2.7}$$

where $\Pi^{(1)}$ gives the one-loop result, and $\Pi^{(n)}$ the n-loop part containing $n-2$ fermion bubbles in the chain, and summing over the topologies given in Fig. 1a. Other contributions are of higher order in $1/N_f$ and are thus omitted.

For the fermion self-energy and vertex renormalization constants, the lowest non-trivial contributions are already $\mathcal{O}(1/N_f)$, and we, therefore, have

$$Z_F = 1 - \text{div}\left\{\Sigma_0(p^2, Z_K K, \epsilon)\right\}, \tag{2.8}$$

$$\Sigma_0(p^2, K_0, \epsilon) = \frac{1}{N_f} \sum_{n=1}^{\infty} K_0^n \Sigma^{(n)}(p^2, \epsilon), \tag{2.9}$$

where $\Sigma^{(n)}$ is depicted in Fig. 1b with $n-1$ fermion bubbles. Similarly,

$$Z_V = 1 - \text{div}\left\{V_0(p^2, Z_K K, \epsilon)\right\}, \tag{2.10}$$

$$V_0(p^2, K_0, \epsilon) = \frac{1}{N_f} \sum_{n=1}^{\infty} K_0^n V^{(n)}(p^2, \epsilon), \tag{2.11}$$

where $V^{(n)}$ again contains $n-1$ fermion bubbles and is shown diagrammatically in Fig 1c.

Finally, we briefly comment on the scalar three-point and four-point functions, assuming that they are generated via fermion loops: the former exactly vanishes for massless fermions, while the latter is found to be already $\mathcal{O}(1/N_f)$ at the lowest order. Therefore, they can be neglected for the purpose of our analysis.

3 Renormalization constants

In this section our goal is to extract the contributions to the renormalization constants that are $\mathcal{O}(1/N_F)$ and relevant for the computation of the β-function.

Our starting point for Z_S is Eq. (2.6). Using the expansion of the scalar self-energy, Eq. (2.7), we obtain

$$Z_S = 1 - \text{div}\left\{ Z_S Z_K K \Pi^{(1)}(p^2, \epsilon) + \frac{1}{N_f} \sum_{n=2}^{\infty} Z_S (Z_K K)^n \Pi^{(n)}(p^2, \epsilon) \right\}. \tag{3.1}$$

Recalling that $Z_K \equiv Z_S^{-1} Z_F^{-2} Z_V^2$ and substituting Eqs (2.8) and (2.10), the first term between brackets can be written as

$$\begin{aligned} &\text{div}\left\{ Z_S Z_K \Pi^{(1)}(p^2, \epsilon) K \right\} \\ &= K \text{div}\left\{ \Pi^{(1)} \right\} + \frac{1}{N_f} \text{div}\left\{ 2K \,\text{div}\left\{ \Sigma_0(p^2, Z_K K, \epsilon) - V_0(p^2, Z_K K, \epsilon) \right\} \Pi^{(1)}(p^2, \epsilon) \right\}. \end{aligned} \tag{3.2}$$

The $\Pi^{(1)}$ part corresponds to the one-loop diagram and is given by

$$\begin{aligned}\Pi^{(1)}(p^2,\epsilon) \equiv &\text{div}\left\{\Pi^{(1)}\right\} + \Pi_{\text{F}}^{(1)}(p^2,\epsilon) = \frac{1}{(4\pi)^{d/2-2}}\frac{G(1,1)}{2}(-p^2)^{d/2-2} \\ =&\frac{1}{\epsilon} + \Pi_{\text{F}}^{(1)}(p^2,\epsilon),\end{aligned} \tag{3.3}$$

where $d = 4 - \epsilon$, the loop function, $G(1,1)$, is defined in Eq. (A.2) in Appendix A.1, and we have introduced the notation $\Pi_{\text{F}}^{(1)}$ to indicate the finite part of $\Pi^{(1)}$. Then,

$$\begin{aligned}&\text{div}\left\{Z_S Z_K \Pi^{(1)}(p^2,\epsilon)K\right\} \\ &= \frac{K}{\epsilon} + \frac{1}{N_f}\text{div}\Big\{2K\,\text{div}\left\{\Sigma_0(p^2, Z_K K,\epsilon) - V_0(p^2, Z_K K,\epsilon)\right\} \\ &\qquad\qquad \times \left(\text{div}\left\{\Pi^{(1)}\right\} + \Pi_{\text{F}}^{(1)}(p^2,\epsilon)\right)\Big\} \\ &= \frac{K}{\epsilon} + \frac{1}{N_f}\text{div}\left\{2K\Pi_{\text{F}}^{(1)}(p^2,\epsilon)\left[\Sigma_0(p^2, Z_K K,\epsilon) - V_0(p^2, Z_K K,\epsilon)\right]\right\} \\ &\quad + \frac{1}{N_f} \times \text{ higher poles},\end{aligned} \tag{3.4}$$

where the higher poles, i.e., higher than $1/\epsilon$, arise from the product of two divergent parts and will be omitted because they play no role in what follows. Then, at the lowest order in $1/N_f$,

$$Z_S = 1 - \frac{K}{\epsilon} + \mathcal{O}\left(1/N_f\right). \tag{3.5}$$

Therefore, every time $Z_K K$ appears in the argument of Σ_0 and V_0, it can be replaced by $K\left(1 - \frac{K}{\epsilon}\right)^{-1}$; the additional contributions are higher order in $1/N_f$. For Eq. (3.4), we arrive at

$$\begin{aligned}&\text{div}\left\{Z_S Z_K \Pi^{(1)}(p^2,\epsilon)K\right\} \\ &\quad= \frac{K}{\epsilon} + \sum_{n=1}^{\infty} K^{n+1}\text{div}\left\{2\Pi_{\text{F}}^{(1)}(p^2,\epsilon)\left(1 - \frac{K}{\epsilon}\right)^{-n}\left[\Sigma^{(n)}(p^2,\epsilon) - V^{(n)}(p^2,\epsilon)\right]\right\}.\end{aligned} \tag{3.6}$$

Similarly, the second term of Eq. (3.1) reads

$$\frac{1}{N_f}\text{div}\left\{\sum_{n=2}^{\infty} Z_S (Z_S^{-1}K)^n \Pi^{(n)}(p^2,\epsilon)\right\} = \frac{1}{N_f}\sum_{n=2}^{\infty} K^n \text{div}\left\{\left(1 - \frac{K}{\epsilon}\right)^{1-n}\Pi^{(n)}(p^2,\epsilon)\right\}. \tag{3.7}$$

Altogether, we can write Z_S as

$$Z_S \;=\; 1 - \frac{K}{\epsilon} - \frac{1}{N_f}\sum_{n=2}^{\infty} K^n \left\{\left(1 - \frac{K}{\epsilon}\right)^{1-n}\left(2\Pi_{\text{F}}^{(1)}\left[\Sigma^{(n-1)} - V^{(n-1)}\right] + \Pi^{(n)}\right)\right\}, \tag{3.8}$$

where the explicit functional dependence on (p^2,ϵ) has been omitted to lighten the notation. Using the binomial expansion,

$$\left(1 - \frac{K}{\epsilon}\right)^{1-n} = \sum_{i=0}^{\infty}\binom{n+i-2}{i}\frac{K^i}{\epsilon^i} \tag{3.9}$$

and performing a shift in the summation, $n \to n-i$, we find our final expression for Z_S:

$$Z_S = 1 - \frac{K}{\epsilon} - \frac{1}{N_f} \sum_{n=2}^{\infty} K^n \text{div} \left\{ \sum_{i=0}^{n-2} \begin{pmatrix} n-2 \\ i \end{pmatrix} \frac{1}{\epsilon^i} \left(2\Pi_{\text{F}}^{(1)} \left(\Sigma^{(n-i-1)} - V^{(n-i-1)} \right) + \Pi^{(n-i)} \right) \right\}. \tag{3.10}$$

We notice that Eq. (3.10) differs essentially from its counterpart in the QED [7] because of the contribution from the fermion self-energy and the vertex, which exactly cancel in QED because of the Ward identity.

The expression for Z_F can be derived from Eq. (2.8) in a similar manner:

$$\begin{aligned} Z_F &= 1 - \frac{1}{N_f} \sum_{n=1}^{\infty} \text{div} \left\{ (Z_K K)^n \, \Sigma^{(n)}(p^2, \epsilon) \right\} \\ &= 1 - \frac{1}{N_f} \sum_{n=1}^{\infty} K^n \text{div} \left\{ \left(1 - \frac{K}{\epsilon} \right)^{-n} \Sigma^{(n)}(p^2, \epsilon) \right\} \\ &= 1 - \frac{1}{N_f} \sum_{n=1}^{\infty} K^n \text{div} \left\{ \sum_{i=0}^{n-1} \begin{pmatrix} n-1 \\ i \end{pmatrix} \frac{1}{\epsilon^i} \Sigma^{(n-i)}(p^2, \epsilon) \right\}, \end{aligned} \tag{3.11}$$

where we have again performed the same shift $n \to n-i$ in the last line. The derivation of Z_V is completely analogous, and we can readily write the expression for Z_V:

$$Z_V = 1 - \frac{1}{N_f} \sum_{n=1}^{\infty} K^n \text{div} \left\{ \sum_{i=0}^{n-1} \begin{pmatrix} n-1 \\ i \end{pmatrix} \frac{1}{\epsilon^i} V^{(n-i)}(p^2, \epsilon) \right\}. \tag{3.12}$$

4 Resummation

In this section we provide closed formulas for Eqs (3.10), (3.11), and (3.12).

4.1 The vertex

By explicit computation, the n-loop contribution to V_0 is

$$\begin{aligned} V^{(n)}(p^2, \epsilon) = &\frac{(-1)^n}{4} \left(\frac{1}{(4\pi)^{d/2-2}} \right)^n \left(\frac{G(1,1)}{2} \right)^{n-1} (-p^2)^{n(d/2-2)} \\ &\times G\left(1, 1-(n-1)(d/2-2)\right), \end{aligned} \tag{4.1}$$

where $G(n_1, n_2)$ is defined in Eq. (A.2). We notice that, as in Ref. [7], Eq. (4.1) allows for the following expansion:

$$V^{(n)}(p^2, \epsilon) = (-1)^n \frac{1}{n\epsilon^n} \frac{v(p^2, \epsilon, n)}{2}, \tag{4.2}$$

where

$$v(p^2, \epsilon, n) = \sum_{j=0}^{\infty} v_j(p^2, \epsilon)(n\epsilon)^j, \tag{4.3}$$

and $v_j(p^2,\epsilon)$ are regular in the limit $\epsilon \to 0$ for all j. In particular, $v_0(\epsilon)$ is independent of p^2 and is explicitly given by

$$v_0(\epsilon) = \frac{2\Gamma(2-\epsilon)}{\Gamma\left(1-\frac{\epsilon}{2}\right)^2 \Gamma\left(2-\frac{\epsilon}{2}\right)\Gamma\left(\frac{\epsilon}{2}\right)\epsilon}. \tag{4.4}$$

Substituting Eqs (4.1) and (4.2) in Eq. (3.12), we find:

$$Z_V = 1 - \frac{1}{N_f}\sum_{n=1}^{\infty}(-K)^n \mathrm{div}\left\{\sum_{j=0}^{n-1}\frac{1}{\epsilon^{n-j}}\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i(n-i)^{j-1}\frac{v_j(p^2,\epsilon)}{2}\right\}. \tag{4.5}$$

Then, by using the result of Ref. [7],

$$\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i(n-i)^{j-1} = -\delta_{j,0}\frac{(-1)^n}{n},\ j=0,\dots,n-1, \tag{4.6}$$

Eq. (4.5) gets simplified to

$$Z_V = 1 + \frac{1}{2N_f}\sum_{n=1}^{\infty}\frac{K^n}{\epsilon^n}\frac{v_0(\epsilon)}{n}. \tag{4.7}$$

Expanding $v_0(\epsilon)$ as

$$v_0(\epsilon) = \sum_{j=0}^{\infty} v_0^{(j)}\epsilon^j \tag{4.8}$$

and keeping only the $1/\epsilon$ pole of Eq. (4.7), we find the closed formula for Z_V:

$$Z_V = 1 + \frac{1}{2\epsilon N_f}\sum_{n=1}^{\infty}\frac{K^n}{n}v_0^{(n-1)} = 1 + \frac{1}{2\epsilon N_f}\int_0^K v_0(t)\mathrm{d}t. \tag{4.9}$$

4.2 The fermion self-energy

The n-loop contribution to Σ_0 is found to be

$$\begin{aligned}\Sigma^{(n)}(p^2,\epsilon) = &-\frac{(-1)^n}{8}\left(\frac{1}{(4\pi)^{d/2-2}}\right)^n\left(\frac{G(1,1)}{2}\right)^{n-1}(-p^2)^{n(d/2-2)} \\ &\times\left[G(1,1-(n-1)(d/2-2)) - G(1,-(n-1)(d/2-2))\right].\end{aligned} \tag{4.10}$$

Similarly to Eq. (4.1), Eq. (4.10) can be expanded as

$$\Sigma^{(n)}(p^2,\epsilon) = -(-1)^n\frac{1}{n\epsilon^n}\frac{\sigma(p^2,\epsilon,n)}{4}, \tag{4.11}$$

where

$$\sigma(n,\epsilon,p^2) = \sum_{j=0}^{\infty}\sigma_j(p^2,\epsilon)(n\epsilon)^j, \tag{4.12}$$

and $\sigma_j(p^2,\epsilon)$ are regular for $\epsilon \to 0$. Again, $\sigma_0(\epsilon)$ is independent of p^2, and it is given by

$$\sigma_0(\epsilon) = -\frac{2^{5-\epsilon}\Gamma\left(\frac{3}{2}-\frac{\epsilon}{2}\right)}{\sqrt{\pi}(4-\epsilon)\Gamma\left(-\frac{\epsilon}{2}\right)\epsilon}\frac{\sin\left(\frac{\pi\epsilon}{2}\right)}{\pi\epsilon}. \tag{4.13}$$

Using the same procedure as in the previous section, we find that only $\sigma_0(\epsilon)$ contributes to Z_F. Keeping only the $1/\epsilon$ pole, the closed formula for Z_F is

$$Z_F = 1 - \frac{1}{4\epsilon N_f}\int_0^K \sigma_0(t)\mathrm{d}t. \tag{4.14}$$

4.3 The scalar self-energy

The evaluation of the bubble diagrams in Fig. 1a is quite cumbersome and is discussed in Appendix A.2. Here, we notice that the expression for $\Pi^{(n)}(p^2,\epsilon)$, $n \geq 2$, allows for the following expansion:

$$\Pi^{(n)} = -\frac{3}{2}\frac{(-1)^n}{n(n-1)\epsilon^n}\pi(p^2,\epsilon,n), \tag{4.15}$$

where

$$\pi(p^2,\epsilon,n) = \sum_{j=0}^{\infty}\pi_j(p^2,\epsilon)(n\epsilon)^j, \tag{4.16}$$

and $\pi_j(p^2,\epsilon)$ are regular for $\epsilon \to 0$. Similarly to the previous cases, $\pi_0(\epsilon)$ is independent of p^2.

In view of Eq. (3.10), we define

$$2\Pi_{\mathrm{F}}^{(1)}(p^2,\epsilon)\left(\Sigma^{(n-1)}(p^2,\epsilon) - V^{(n-1)}(p^2,\epsilon)\right) + \Pi^{(n)}(p^2,\epsilon) \equiv \frac{(-1)^n}{n(n-1)\epsilon^n}\xi(p^2,\epsilon,n), \tag{4.17}$$

where

$$\xi(p^2,\epsilon,n) \equiv n\epsilon\,\Pi_{\mathrm{F}}^{(1)}\left(\frac{\sigma(p^2,\epsilon,n-1)}{2} + v(p^2,\epsilon,n-1)\right) - \frac{3}{2}\pi(p^2,\epsilon,n), \tag{4.18}$$

and

$$\xi(p^2,\epsilon,n) = \sum_{j=0}^{\infty}\xi_j(p^2,\epsilon)(n\epsilon)^j, \tag{4.19}$$

with $\xi_j(\epsilon,p^2)$ regular for $\epsilon \to 0$ for all j. In particular, $\xi_0(\epsilon)$ is independent of p^2 and is explicitly given by

$$\xi_0(\epsilon) = -\frac{(1-\epsilon)\Gamma(4-\epsilon)}{\Gamma\left(2-\frac{\epsilon}{2}\right)\Gamma\left(3-\frac{\epsilon}{2}\right)\pi\epsilon}\sin\left(\frac{\pi\epsilon}{2}\right) \tag{4.20}$$

Then, using the above definitions, Eq. (3.10) can be written as

$$\begin{aligned} Z_S &= 1 - \frac{K}{\epsilon} - \frac{1}{N_f}\sum_{n=2}^{\infty}K^n \mathrm{div}\left\{\sum_{i=0}^{n-2}\binom{n-2}{i}\frac{1}{\epsilon^i}\frac{(-1)^{n-i}}{(n-i)(n-i-1)\epsilon^{n-i}}\xi(p^2,\epsilon,n-i)\right\} \\ &= 1 - \frac{K}{\epsilon} - \frac{1}{N_f}\sum_{n=2}^{\infty}(-K)^n \mathrm{div}\left\{\sum_{j=0}^{n-1}\frac{1}{\epsilon^{n-j}}\xi_j(p^2,\epsilon)\sum_{i=0}^{n-2}\binom{n-2}{i}(-1)^i\frac{(n-i)^{j-1}}{(n-i-1)}\right\}. \end{aligned} \tag{4.21}$$

Moreover, we find that

$$\sum_{i=0}^{n-2}\binom{n-2}{i}(-1)^i\frac{(n-i)^{j-1}}{(n-i-1)} = \begin{cases}\frac{(-1)^n}{n} & j=0 \\ \frac{(-1)^n}{n-1} & j=1,\ldots,n-1\end{cases}, \tag{4.22}$$

and therefore the expression for Z_S can be significantly simplified:

$$
\begin{aligned}
Z_S &= 1 - \frac{K}{\epsilon} - \frac{1}{N_f}\sum_{n=2}^{\infty} K^n \,\mathrm{div}\left\{\frac{1}{\epsilon^n}\left(\frac{\xi_0(\epsilon)}{n} + \frac{1}{n-1}\sum_{j=1}^{n-1}\xi_j(p^2,\epsilon)\epsilon^j\right)\right\} \\
&= 1 - \frac{K}{\epsilon} - \frac{1}{N_f}\sum_{n=2}^{\infty} K^n \,\mathrm{div}\left\{\frac{1}{\epsilon^n}\left(\frac{\xi_0(\epsilon)}{n} + \frac{1}{n-1}\sum_{j=1}^{\infty}\xi_j(p^2,\epsilon)\epsilon^j\right)\right\} \\
&= 1 - \frac{K}{\epsilon} - \frac{1}{N_f}\sum_{n=2}^{\infty} K^n \,\mathrm{div}\left\{\frac{1}{\epsilon^n}\left(\frac{\xi_0(\epsilon)}{n} + \frac{\xi(p^2,\epsilon,1)-\xi_0(\epsilon)}{n-1}\right)\right\},
\end{aligned}
\tag{4.23}
$$

where in the second line we extended the sum over j up to ∞ without affecting the result, since all the terms for $j > n-1$ are finite. The function $\xi(p^2,\epsilon,1)$, corresponding to

$$
\xi(p^2,\epsilon,1) \equiv \sum_{j=0}^{\infty}\xi_j(p^2,\epsilon)\epsilon^j,
\tag{4.24}
$$

can be evaluated by taking in Eq. (4.18) the limit $n \to 1$, although the latter is formally defined for $n \geq 2$. We find the following expression:

$$
\xi(p^2,\epsilon,1) = -\frac{\Gamma(4-\epsilon)}{\Gamma\left(2-\frac{\epsilon}{2}\right)\Gamma\left(3-\frac{\epsilon}{2}\right)\pi\epsilon}\sin\left(\frac{\pi\epsilon}{2}\right) \equiv \xi(\epsilon,1).
\tag{4.25}
$$

Few comments are in order: Eq. (4.25) ensures that Z_S is independent of the external momentum p^2, as it should. This result comes from an exact cancellation among the different contributions of the scalar self-energy, the fermion self-energy, and the vertex in Eq. (4.18). In particular, we find that

$$
\begin{aligned}
\pi(p^2,\epsilon,1) &= \frac{2}{3}\left(\frac{\sigma(p^2,\epsilon,0)}{2} + v(p^2,\epsilon,0)\right)\left[1 + 1\cdot\epsilon\,\Pi_{\mathrm{F}}^{(1)}(p^2,\epsilon)\right] \\
&= \frac{2}{3}\left(\frac{\sigma_0(\epsilon)}{2} + v_0(\epsilon)\right)\left[1 + \epsilon\,\Pi_{\mathrm{F}}^{(1)}(p^2,\epsilon)\right],
\end{aligned}
\tag{4.26}
$$

and therefore

$$
\xi(\epsilon,1) = -\frac{\sigma_0(\epsilon)}{2} - v_0(\epsilon),
\tag{4.27}
$$

which is equivalent to Eq. (4.25). Interestingly, Eq. (4.26) only holds for $n = 1$. All in all, the p^2 independence of Eq. (4.25) provides a non-trivial check for our computation. Moreover, we see that

$$
\xi_0(\epsilon) = (1-\epsilon)\xi(\epsilon,1).
\tag{4.28}
$$

We are now ready to resum the series in Eq. (4.23). By expanding $\xi_0(\epsilon)$ as

$$
\xi_0(\epsilon) = \sum_{j=0}^{\infty}\xi_0^{(j)}\epsilon^j,
\tag{4.29}
$$

the $\frac{1}{n}$ term in Eq. (4.23) is given by

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{K^n}{\epsilon^n}\frac{\xi_0(\epsilon)}{n} &= \frac{1}{\epsilon}\sum_{n=2}^{\infty}\frac{K^n}{\epsilon^n}\frac{\xi_0^{(n-1)}}{n} + \text{higher poles} \\ &= \frac{1}{\epsilon}\left(\sum_{n=0}^{\infty}K^{n+1}\frac{\xi_0^{(n)}}{n+1} - K\xi_0^{(0)}\right) + \text{higher poles} \\ &= \frac{1}{\epsilon}\int_0^K \left[\xi_0(t)-\xi_0(0)\right]\mathrm{d}t + \text{higher poles.}\end{aligned} \tag{4.30}$$

As for the $\frac{1}{n-1}$ term, using $\xi_0(\epsilon) = (1-\epsilon)\xi(\epsilon,1)$ and expanding $\xi(\epsilon,1)$ as

$$\xi(\epsilon,1) = \sum_{j=0}^{\infty}\tilde{\xi}^{(n)}\epsilon^j, \tag{4.31}$$

we find

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{K^n}{\epsilon^n}\frac{\epsilon\xi(\epsilon,1)}{n-1} &= \frac{K}{\epsilon}\sum_{n=0}^{\infty}\frac{K^{n+1}}{n+1}\tilde{\xi}^{(n)} + \text{higher poles} \\ &= \frac{K}{\epsilon}\int_0^K \xi(t,1)\mathrm{d}t + \text{higher poles.}\end{aligned} \tag{4.32}$$

Finally, the closed formula for Z_S reads

$$Z_S = 1 - \frac{K}{\epsilon} - \frac{1}{\epsilon N_f}\int_0^K \left[\xi_0(t) - \xi_0(0) + \xi(t,1)K\right]\mathrm{d}t. \tag{4.33}$$

5 The β-function

Using the results of the previous section together with Eq. (2.5), we can finally proceed to evaluating the β-function. First, we find that

$$G_1(K) = K + \frac{1}{N_f}\int_0^K \left(\xi_0(t) - \xi_0(0) + \xi(t,1)K + \frac{\sigma_0(t)}{2} + v_0(t)\right)\mathrm{d}t. \tag{5.1}$$

Now, it is straightforward to compute the β-function:

$$\beta(K) = K^2 + \frac{K^2}{N_f}\left\{-\xi_0(0) + \xi(K,1) + \frac{\sigma_0(K)}{2} + v_0(K) + \int_0^K \xi(t,1)\mathrm{d}t\right\}. \tag{5.2}$$

Recalling Eq. (4.27) and using $\xi_0(0) = -\frac{3}{2}$, Eq. (5.2) can be further simplified to

$$\frac{\beta(K)}{K^2} = 1 + \frac{1}{N_f}\left\{\frac{3}{2} + \int_0^K \xi(t,1)\mathrm{d}t\right\}. \tag{5.3}$$

Finally, by comparison with Eq. (2.3), we see that $F_0 = 1$ and

$$F_1(K) = \frac{3}{2} + \int_0^K \xi(t,1)\mathrm{d}t. \tag{5.4}$$

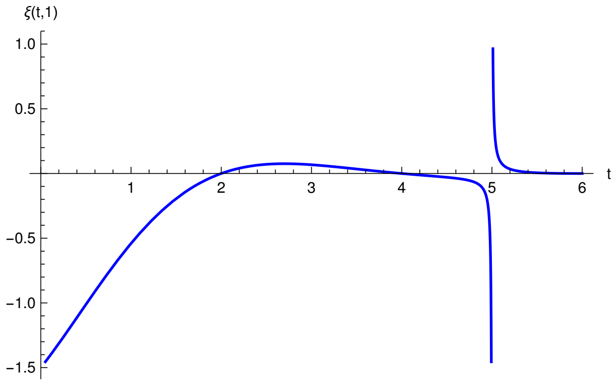


Figure 2: The function $\xi(t, 1)$.

We plot the integrand, $\xi(t, 1)$, in Fig. 2. We have checked that our β-function agrees at the leading order in N_f up to four-loop level by comparing with the result of Ref. [19], and with the result extracted from the critical exponents in Gross–Neveu–Yukawa model computed using a different technique [15].

Finally, let us comment on the pole structure: the integrand, $\xi(t, 1)$, has the first pole occuring at $t = 5$, which results in a logarithmic singularity for $F_1(K)$ around $K = 5$. Due to the sign of $\xi(t, 1)$, we see that $F_1(K)$ approaches large negative values for $K \to 5^-$. This suggests the existence of a UV fixed point at $K_{\rm UV} \lesssim 5$ such that $F_1(K_{\rm UV}) = -N_f$.

6 Conclusions

We have computed the leading $1/N_f$ contribution for the β-function in Yukawa theory with N_f fermion flavours coupling to a real scalar. We obtained a closed form expression for the β-function up to order $\mathcal{O}(1/N_f)$. This expression has a finite radius of convergence, and the first singularity occurs at $K = 5$.

The present result adds an interesting ingredient to models with a large number of fermions, and makes a contribution to better understand the UV behaviour of gauge-Yukawa theories.

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A Loop integrals

We here provide some explicit formulas. We follow closely the notations of Ref. [20].

A.1 The vertex and the fermion self-energy

As shown in Eqs (4.1) and (4.10), the 1PI vertex and the fermion self-energy involve only the function $G(n_1, n_2)$, independently of the number of bubbles. This corresponds to the one-loop integral

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{D_1^{n_1} D_2^{n_2}} = \mathrm{i}\frac{1}{(4\pi)^{d/2}}(-p^2)^{d/2-n_1-n_2}(-1)^{n_1+n_2}G(n_1,n_2), \tag{A.1}$$

where $D_1 = (k+p)^2$ and $D_2 = k^2$. Explicitly,

$$G(n_1,n_2) = \frac{\Gamma(-d/2+n_1+n_2)\Gamma(d/2-n_1)\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d-n_1-n_2)}. \tag{A.2}$$

A.2 The scalar self-energy

Unlike the 1PI vertex and the fermion self-energy, the n-loop contribution to the scalar self-energy, Π_0, indicated by $\Pi^{(n)}$, cannot be written in terms of $G(n_1, n_2)$ functions only. In fact, $\Pi^{(n)}$ is given by ($n \geq 2$):

$$\begin{aligned} p^2\Pi^{(n)}(p^2,\epsilon) = &-(4\pi^2)^2(-1)^n\left(\frac{1}{(4\pi)^{d/2-2}}\frac{G(1,1)}{2}\right)^{n-2}(-1)^\alpha\int\frac{d^dk_1}{(2\pi)^d}\int\frac{d^dk_2}{(2\pi)^d} \\ &\left\{\frac{6}{(p+k_1)^2k_2^2((k_1-k_2)^2)^{1-\alpha}} - \frac{2}{k_1^2(p+k_1)^2k_2^2((k_1-k_2)^2)^{-\alpha}}\right. \\ &-\frac{2p^2}{k_1^2(p+k_1)^2k_2^2((k_1-k_2)^2)^{1-\alpha}} + \frac{2p^2}{k_1^4(p+k_1)^2k_2^2((k_1-k_2)^2)^{-\alpha}} \\ &\left.-\frac{2p^2}{k_1^2(k_1+p)^2(k_2+p)^2k_2^2((k_1-k_2)^2)^{-\alpha}}\right\}, \end{aligned} \tag{A.3}$$

where $\alpha = (n-2)(d/2-2) = -(n-2)\epsilon/2$. Eq. (A.3) requires two-loop integrals which can be performed according to the formula in Ref. [20]:

$$\int\frac{d^dk_1}{(2\pi)^d}\int\frac{d^dk_2}{(2\pi)^d}\frac{1}{D_1^{n_1}D_2^{n_2}D_3^{n_3}D_4^{n_4}D_5^{n_5}} = (-1)^{1+\sum n_i}\frac{\pi^d(-p^2)^{d-\sum n_i}}{(2\pi)^{2d}}G(n_1,n_2,n_3,n_4,n_5), \tag{A.4}$$

where $D_1 = (k_1+p)^2$, $D_2 = (k_2+p)^2$, $D_3 = k_1^2$, $D_4 = k_2^2$, $D_5 = (k_1-k_2)^2$. The functions $G(n_1,n_2,n_3,n_4,n_5)$ are symmetric with respect to the following index exchanges: $(1 \leftrightarrow 2, 3 \leftrightarrow 4)$ and $(1 \leftrightarrow 3, 2 \leftrightarrow 4)$. Moreover, they reduce to a product of $G(n_1, n_2)$ if at least one of the entries is zero:

$$G(n_1,n_2,n_3,n_4,0) = G(n_1,n_3)G(n_2,n_4), \tag{A.5}$$

$$G(0,n_2,n_3,n_4,n_5) = G(n_3,n_5)G(n_2,n_3+n_4+n_5-d/2). \tag{A.6}$$

It turns out that the first four integrals in Eq. (A.3) can always be written in terms of $G(n_1, n_2)$ making use of Eqs (A.5) and (A.6).

However, the last integral in Eq. (A.3) involves $G(1,1,1,1,(n-2)\epsilon/2)$ and, for $n > 2$, its expression can be obtained in terms of hypergeometric functions ${}_3F_2$ by means of the Gegenbauer technique [21]. We have evaluated the function $G(1,1,1,1,(n-2)\epsilon/2)$ recursively according to Eqs (2.19) and (2.21) in Ref. [20].

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