Time-Limited $\mathcal{H}_2$-Optimal Model Order Reduction

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Abstract

In this paper, we investigate a time-limited $\mathcal{H}_2$-model order reduction method for linear dynamical systems. For this, we propose a time-limited $\mathcal{H}_2$-norm and show its connection with the time-limited Gramians. We then derive first-order conditions for optimality of reduced-order systems with respect to the time-limited $\mathcal{H}_2$-norm. Based on these optimality conditions, we propose an iterative correction scheme to construct reduced-order systems, which, upon convergence, nearly satisfy these conditions. Furthermore, a diagnostic measure is proposed for how close the obtained reduced-order system is to optimality. We test the efficiency of the proposed iterative scheme using various numerical examples and illustrate that the newly proposed iterative method can lead to a better reduced-order models compared to the unrestricted *iterative rational Krylov subspace algorithm* in a finite time interval of interest.

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1 Introduction

We consider a continuous linear time-invariant (LTI) system as follows:

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \\ y(t) = Cx(t), \quad t \geq 0, \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ are constant matrices. Generally, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ denote the state, control input and the quantity of interest (output vector) at time t, respectively, and in the most cases, the dimension of the state vector is much larger than the number of control inputs and outputs, i.e., $n \gg m, p$.

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We also assume that the matrix A is Hurwitz, meaning that $\Lambda(A) \subset \mathbb{C}_- = \{z \in \mathbb{C} : \Re(z) < 0\}$, where $\Lambda(\cdot)$ denotes the spectrum of a matrix and $\Re(\cdot)$ represents the real part of a complex number. Due to the large dimension of system (1), it is numerically very expensive to simulate the system for various control inputs and perform engineering studies such as optimal control and optimization. One approach to overcome such an issue is *model order reduction* (MOR), where we aim at constructing a reduced-order system as follows:

$$\hat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \quad \hat{x}(0) = 0, \\ \hat{y}(t) = \hat{C}\hat{x}(t), \quad t \geq 0, \end{cases} \tag{2}$$

where $\hat{A} \in \mathbb{R}^{r\times r}$, $\hat{B} \in \mathbb{R}^{r\times m}$, and $\hat{C} \in \mathbb{R}^{p\times r}$ with $r \ll n$ such that $y \approx \hat{y}$ in an appropriate norm for all admissible control inputs u. In the literature, there is a huge collection of available methods, allowing us to construct such reduced-order systems, e.g., see [1, 4, 15].

Most of the methods for linear systems such as balanced truncation, e.g., see [1, 11] and the iterative rational Krylov subspace algorithm [8] aim at constructing a reduced-order system, which is good for an infinite time horizon. In other words, the output of system (1) is very well approximated by the output of (2) on the time interval $[0, \infty)$. However, there are several practical applications, for example, a finite-time optimal control problem, where one is interested in approximating the output y on a finite time interval, e.g., $[0, \bar{T}]$, meaning that

$$y \approx \hat{y} \quad \text{on} \quad [0, \bar{T}]. \tag{3}$$

Due to the relation (3), we can expect a better reduced-order system in the time interval $[0, \bar{T}]$ as compared to unconstrained MOR approaches for a given order of the reduced system. A MOR technique for such a problem, called balanced truncation, was first proposed in [7] and further studies were carried out in [9, 14]. In this work, we consider a similar time-limited problem, aiming to extend the Wilson conditions [17] and first-order optimality conditions [8, 10, 17].

For this, we first propose the time-limited $\mathcal{H}_2$-norm for linear systems in Section 2 and provide different representations of the metric induced by this norm, which are based on time-limited Gramians. Then, we define the problem setting for time-limited MOR as an optimization problem. Subsequently, in Section 3, we extend the Wilson conditions to time-limited linear systems and derive first-order optimality conditions, which minimize the time-limited $\mathcal{H}_2$-norm of the error system. Based on these conditions, we propose an iterative scheme, which, upon convergence, aims at constructing a reduced-order system that nearly satisfies the optimality conditions. Later on, we derive expressions, revealing how close the obtained reduced systems via the proposed iterative scheme are to optimality. In Section 4, we illustrate the efficiency of the proposed iterative scheme by three benchmark numerical examples for linear systems. Finally, we conclude the paper with a short summary and an outlook for future work.

2 Time-Limited $\mathcal{H}_2$-Norm and Problem Setting

In this section, we first define the time-limited $\mathcal{H}_2$-norm for linear systems and show its relation to the output error. Furthermore, we provide different representations for the time-limited $\mathcal{H}_2$-norm using time-limited Gramians and then define the time-limited $\mathcal{H}_2$-model reduction problem for linear systems. Before we proceed further, we note important relations between the Kronecker product, the vectorization and the trace of a matrix. These are:

$$\operatorname{vec}(XYZ) = (Z^* \otimes X)\operatorname{vec}(Y), \tag{4a}$$
$$\operatorname{tr}(XYZ) = (\operatorname{vec}(X^*))^* (I \otimes Y)\operatorname{vec}(Z), \tag{4b}$$

where X, Y and Z are matrices of suitable dimensions; $\operatorname{vec}(\cdot)$ and $\operatorname{tr}(\cdot)$ denote, respectively, the vectorization and the trace of a matrix; $\otimes$ represents the Kronecker product of two matrices, and * denotes the transpose of a matrix or a vector.

We investigate a MOR problem for the large-scale system (1) and seek to construct a reduced-order system (2). Since our goal is to construct a good approximation of the system (1) on a finite time interval $[0, \bar{T}]$, where $\bar{T} > 0$ is the terminal time, we first investigate a worst case error between the output of the system (2) and the output of (1) on $[0, \bar{T}]$. In order to find a bound for the error between the output y of the original model and the output $\hat{y}$ of the reduced system, the arguments from the case of having an infinite time horizon are used, see, e.g., [1, 8]. Similar estimates can also be found, e.g., in [5, 6, 13], where $\mathcal{H}_2$-error bounds for more general stochastic systems are derived. There, reduced-order systems are obtained by applying balanced truncation.

Next, we make use of the explicit representations for the outputs as follows:

$$y(t) = C\int_0^t \mathrm{e}^{A(t-s)}\, Bu(s)ds \quad \text{and} \quad \hat{y}(t) = \hat{C}\int_0^t \mathrm{e}^{\hat{A}(t-s)}\, \hat{B}u(s)ds,$$

thus yielding

$$\begin{aligned}
\|y(t) - \hat{y}(t)\|_2 &= \left\| C\int_0^t \mathrm{e}^{A(t-s)}\, Bu(s)ds - \hat{C}\int_0^t \mathrm{e}^{\hat{A}(t-s)}\, \hat{B}u(s)ds \right\|_2 \\
&\leq \int_0^t \left\| \left(C\,\mathrm{e}^{A(t-s)}\, B - \hat{C}\,\mathrm{e}^{\hat{A}(t-s)}\, \hat{B} \right) u(s) \right\|_2 ds \\
&\leq \int_0^t \left\| C\,\mathrm{e}^{A(t-s)}\, B - \hat{C}\,\mathrm{e}^{\hat{A}(t-s)}\, \hat{B} \right\|_F \|u(s)\|_2\, ds.
\end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|y(t)-\hat{y}(t)\|_2 &\leq \left(\int_0^t \left\| C\,\mathrm{e}^{A(t-s)}\,B - \hat{C}\,\mathrm{e}^{\hat{A}(t-s)}\,\hat{B}\right\|_F^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \|u(s)\|_2^2\,ds\right)^{\frac{1}{2}} \\ &\leq \left(\int_0^t \left\| C\,\mathrm{e}^{As}\,B - \hat{C}\,\mathrm{e}^{\hat{A}s}\,\hat{B}\right\|_F^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \|u(s)\|_2^2\,ds\right)^{\frac{1}{2}} \\ &\leq \left(\int_0^{\bar{T}} \left\| C\,\mathrm{e}^{As}\,B - \hat{C}\,\mathrm{e}^{\hat{A}s}\,\hat{B}\right\|_F^2 ds\right)^{\frac{1}{2}} \|u\|_{L^2_{\bar{T}}}\end{aligned}$$

for $t \in [0, \bar{T}]$. Hence,

$$\max_{t\in[0,\bar{T}]} \|y(t)-\hat{y}(t)\|_2 \leq \left\|\Sigma - \hat{\Sigma}\right\|_{\mathcal{H}_{2,\bar{T}}} \|u\|_{L^2_{\bar{T}}}\,, \tag{5}$$

where $\left\|\Sigma - \hat{\Sigma}\right\|_{\mathcal{H}_{2,\bar{T}}} := \left(\int_0^{\bar{T}} \left\| C\,\mathrm{e}^{As}\,B - \hat{C}\,\mathrm{e}^{\hat{A}s}\,\hat{B}\right\|_F^2 ds\right)^{\frac{1}{2}}$. We call $\|\cdot\|_{\mathcal{H}_{2,\bar{T}}}$ the time-limited $\mathcal{H}_2$-norm, since $\left\|\Sigma - \hat{\Sigma}\right\|_{\mathcal{H}_{2,\bar{T}}}$ provides the time-domain representation of the metric induced by the $\mathcal{H}_2$-norm if $\bar{T} \to \infty$. We can see from (5) that the time-limited $\mathcal{H}_2$-norm is an upper bound for the induced norm from $L^2_{\bar{T}}$ to $L^\infty_{\bar{T}}$ of the input-output operator. Whether the time-limited $\mathcal{H}_2$-norm coincides with this induced operator norm, e.g., for single-input-multiple-output or multiple-input-single-output systems, is unlike the regular $\mathcal{H}_2$-case still an open question.

The time-limited $\mathcal{H}_2$-error can also be expressed with the help of the time-limited reachability and observability Gramians. We refer, e.g., to [7] for a further discussion of these Gramians. In order to show the Gramian based representations of the time-limited $\mathcal{H}_2$-norm of the error system, we first provide the following lemma.

Lemma 2.1. *Let $A_1 \in \mathbb{R}^{d_1\times d_1}$, $A_2 \in \mathbb{R}^{d_2\times d_2}$ with $\Lambda(A_1)\cap\Lambda(-A_2) = \emptyset$, $K_1 \in \mathbb{R}^{d_1\times d_3}$ and $K_2 \in \mathbb{R}^{d_2\times d_3}$. Then,*

$$X = \int_0^{\bar{T}} \mathrm{e}^{A_1 s}\,K_1 K_2^*\,\mathrm{e}^{A_2^* s}\,ds,$$

where $X \in \mathbb{R}^{d_1\times d_2}$, uniquely solves the Sylvester equation

$$A_1 X + X A_2^* = -K_1 K_2^* + \mathrm{e}^{A_1\bar{T}}\,K_1 K_2^*\,\mathrm{e}^{A_2^*\bar{T}}\,. \tag{6}$$

Proof. This result is a consequence of the product rule. Setting $g_1(t) := \mathrm{e}^{A_1 t}\,K_1$ and $g_2(t) := K_2^*\,\mathrm{e}^{A_2^* t}$, it holds that

$$\begin{aligned}g_1(\bar{T})g_2(\bar{T}) - g_1(0)g_2(0) &= \int_0^{\bar{T}} dg_1(s)g_2(s) + \int_0^{\bar{T}} g_1(s)dg_2(s) \\ &= A_1 \int_0^{\bar{T}} g_1(s)g_2(s)ds + \int_0^{\bar{T}} g_1(s)g_2(s)ds\; A_2^* = A_1 X + X A_2^*,\end{aligned}$$

since $dg_2(s) = g_2(s)A_2^* ds$ and $dg_1(s) = A_1 g_1(s) ds$. Furthermore, using (4a), we can equivalently write (6) as follows:

$$\underbrace{(I_{d_2} \otimes A_1 + A_2 \otimes I_{d_1})}_{=:\mathcal{A}_\otimes} \operatorname{vec}(X) = \operatorname{vec}(R_{12}), \tag{7}$$

where R_{12} is the right-hand side of (6), and I_q denotes the identity matrix of size $q \times q$. Note that, the eigenvalues of $\mathcal{A}_\otimes$ are given by $\mu_1^{(i)} + \mu_2^{(j)}$, where $\mu_1^{(i)}$ is the ith eigenvalue of A_1 and $\mu_2^{(j)}$ the jth eigenvalue of A_2. Due the assumption on the spectra of A_1 and A_2, the matrix $\mathcal{A}_\otimes$ is invertible, hence a unique solution to (7). □

The next proposition shows that the time-limited error can be expressed with the help of time-limited Gramians. This result is used later on in order to derive first-order necessary conditions for a minimal error in the time-limited $\mathcal{H}_2$-norm.

Proposition 2.2. *Let Σ and $\hat{\Sigma}$ be the original and reduced-order systems as defined in* (1) *and* (2)*, respectively. Then, the time-limited $\mathcal{H}_2$-norm of the error system $\Sigma - \hat{\Sigma}$ is given by*

$$\left\|\Sigma - \hat{\Sigma}\right\|_{\mathcal{H}_{2,\bar{T}}}^2 = \operatorname{tr}(CP_{\bar{T}}C^*) + \operatorname{tr}(\hat{C}\hat{P}_{\bar{T}}\hat{C}^*) - 2\operatorname{tr}(CP_{2,\bar{T}}\hat{C}^*), \tag{8}$$

where $P_{\bar{T}}, P_{2,\bar{T}}$ and $\hat{P}_{\bar{T}}$, respectively, satisfy

$$AP_{\bar{T}} + P_{\bar{T}}A^* = -BB^* + \mathrm{e}^{A\bar{T}}\, BB^*\, \mathrm{e}^{A^*\bar{T}}, \tag{9a}$$
$$AP_{2,\bar{T}} + P_{2,\bar{T}}\hat{A}^* = -B\hat{B}^* + \mathrm{e}^{A\bar{T}}\, B\hat{B}^*\, \mathrm{e}^{\hat{A}^*\bar{T}}, \text{ and} \tag{9b}$$
$$\hat{A}\hat{P}_{\bar{T}} + \hat{P}_{\bar{T}}\hat{A}^* = -\hat{B}\hat{B}^* + \mathrm{e}^{\hat{A}\bar{T}}\, \hat{B}\hat{B}^*\, \mathrm{e}^{\hat{A}^*\bar{T}}. \tag{9c}$$

Proof. The definition of the Frobenius norm and the linearity of the integral yield

$$\begin{aligned}
\left\|\Sigma - \hat{\Sigma}\right\|_{\mathcal{H}_{2,\bar{T}}}^2 &= \int_0^{\bar{T}} \left\|C\,\mathrm{e}^{As}\,B - \hat{C}\,\mathrm{e}^{\hat{A}s}\,\hat{B}\right\|_F^2 ds \\
&= \int_0^{\bar{T}} \operatorname{tr}\left(C\,\mathrm{e}^{As}\,BB^*\,\mathrm{e}^{A^*s}\,C^*\right) ds + \int_0^{\bar{T}} \operatorname{tr}\left(\hat{C}\,\mathrm{e}^{\hat{A}s}\,\hat{B}\hat{B}^*\,\mathrm{e}^{\hat{A}^*s}\,\hat{C}^*\right) ds \\
&\quad - 2\int_0^{\bar{T}} \operatorname{tr}\left(C\,\mathrm{e}^{As}\,B\hat{B}^*\,\mathrm{e}^{\hat{A}^*s}\,\hat{C}^*\right) ds \\
&= \operatorname{tr}\left(CP_{\bar{T}}C^*\right) + \operatorname{tr}\left(\hat{C}\hat{P}_{\bar{T}}\hat{C}^*\right) - 2\ \operatorname{tr}\left(CP_{2,\bar{T}}\hat{C}^*\right)
\end{aligned}$$

with $P_{\bar{T}} := \int_0^{\bar{T}} \mathrm{e}^{As}\,BB^*\,\mathrm{e}^{A^*s}\,ds$, $P_{2,\bar{T}} := \int_0^{\bar{T}} \mathrm{e}^{As}\,B\hat{B}^*\,\mathrm{e}^{\hat{A}^*s}\,ds$, $\hat{P}_{\bar{T}} := \int_0^{\bar{T}} \mathrm{e}^{\hat{A}s}\,\hat{B}\hat{B}^*\,\mathrm{e}^{\hat{A}^*s}\,ds$. Due to Lemma 2.1, it can easily be shown that $P_{\bar{T}}, P_{2,\bar{T}}$ and $\hat{P}_{\bar{T}}$ are the solutions to (9a), (9b) and (9c), respectively. □

The result of Proposition 2.2 has the same structure as the error expression in [14], where the case of time-limited balanced truncation has been investigated. Moreover, if we take the limit $\bar{T} \to \infty$ in (8), we obtain a representation for the full $\mathcal{H}_2$-error that is derived e.g., in [1]. The next proposition shows that the time-limited $\mathcal{H}_2$-norm of the error system as in Proposition 2.2 can be rewritten using the time-limited observability Gramians.

Proposition 2.3. *Let Σ and $\hat{\Sigma}$ be the original and reduced-order systems as defined in* (1) *and* (2)*, respectively. Moreover, let $P_{\bar{T}}, P_{2,\bar{T}}$ and $\hat{P}_{\bar{T}}$ be the solutions to* (9a)*,* (9b) *and* (9c)*, respectively. Then, the following holds:*

$$\begin{aligned}
\operatorname{tr}(CP_{\bar{T}}C^*) &= \operatorname{tr}(B^*Q_{\bar{T}}B), \\
\operatorname{tr}(\hat{C}\hat{P}_{\bar{T}}\hat{C}^*) &= \operatorname{tr}(\hat{B}^*\hat{Q}_{\bar{T}}\hat{B}), \\
\operatorname{tr}(CP_{2,\bar{T}}\hat{C}^*) &= \operatorname{tr}(\hat{B}^*Q_{2,\bar{T}}B),
\end{aligned}$$

where the matrices $Q_{\bar{T}}, Q_{2,\bar{T}}$ and $\hat{Q}_{\bar{T}}$, respectively, satisfy

$$A^*Q_{\bar{T}} + Q_{\bar{T}}A = -C^*C + \mathrm{e}^{A^*\bar{T}}\, C^*C\, \mathrm{e}^{A\bar{T}}, \tag{10a}$$

$$\hat{A}^*Q_{2,\bar{T}} + Q_{2,\bar{T}}A = -\hat{C}^*C + \mathrm{e}^{\hat{A}^*\bar{T}}\, \hat{C}^*C\, \mathrm{e}^{A\bar{T}}, \quad \textit{and} \tag{10b}$$

$$\hat{A}^*\hat{Q}_{\bar{T}} + \hat{Q}_{\bar{T}}\hat{A} = -\hat{C}^*\hat{C} + \mathrm{e}^{\hat{A}^*\bar{T}}\, \hat{C}^*\hat{C}\, \mathrm{e}^{\hat{A}\bar{T}}. \tag{10c}$$

Proof. We insert the integral representations of $P_{\bar{T}}, P_{2,\bar{T}}$ and $\hat{P}_{\bar{T}}$ and use basic properties of the trace operator. Thus,

$$\begin{aligned}
\operatorname{tr}(CP_{\bar{T}}C^*) &= \int_0^{\bar{T}} \operatorname{tr}(C\,\mathrm{e}^{As}\,BB^*\,\mathrm{e}^{A^*s}\,C^*)ds = \int_0^{\bar{T}} \operatorname{tr}(B^*\,\mathrm{e}^{A^*s}\,C^*C\,\mathrm{e}^{As}\,B)ds, \\
\operatorname{tr}(\hat{C}\hat{P}_{\bar{T}}\hat{C}^*) &= \int_0^{\bar{T}} \operatorname{tr}(\hat{C}\,\mathrm{e}^{\hat{A}s}\,\hat{B}\hat{B}^*\,\mathrm{e}^{\hat{A}^*s}\,\hat{C}^*)ds = \int_0^{\bar{T}} \operatorname{tr}(\hat{B}^*\,\mathrm{e}^{\hat{A}^*s}\,\hat{C}^*\hat{C}\,\mathrm{e}^{\hat{A}s}\,\hat{B})ds, \\
\operatorname{tr}(CP_{2,\bar{T}}\hat{C}^*) &= \int_0^{\bar{T}} \operatorname{tr}(C\,\mathrm{e}^{As}\,B\hat{B}^*\,\mathrm{e}^{\hat{A}^*s}\,\hat{C}^*)ds = \int_0^{\bar{T}} \operatorname{tr}(\hat{B}^*\,\mathrm{e}^{\hat{A}^*s}\,\hat{C}^*C\,\mathrm{e}^{As}\,B)ds.
\end{aligned}$$

Let us define $Q_{\bar{T}} := \int_0^{\bar{T}} \mathrm{e}^{A^*s}\,C^*C\,\mathrm{e}^{As}\,ds$, $Q_{2,\bar{T}} := \int_0^{\bar{T}} \mathrm{e}^{\hat{A}^*s}\,\hat{C}^*C\,\mathrm{e}^{As}\,ds$ and $\hat{Q}_{\bar{T}} := \int_0^{\bar{T}} \mathrm{e}^{\hat{A}^*s}\,\hat{C}^*\hat{C}\,\mathrm{e}^{\hat{A}s}\,ds$. Furthermore, by using Lemma 2.1, we can show that the $Q_{\bar{T}}$, $Q_{2,\bar{T}}$ and $\hat{Q}_{\bar{T}}$ satisfy the equations given in (10). □

From inequality (5), it can be seen that it makes sense to minimize $\left\|\Sigma - \hat{\Sigma}\right\|^2_{\mathcal{H}_{2,\bar{T}}}$ with respect to the reduced-order matrices $\hat{A}$, $\hat{B}$ and $\hat{C}$ since a small $\mathcal{H}_{2,\bar{T}}$-error ensures a small output error. Due to the fact that $\left\|\Sigma - \hat{\Sigma}\right\|_{\mathcal{H}_{2,\bar{T}}}$ is increasing in $\bar{T}$, the time-limited error is less or equal to the error in the full $\mathcal{H}_2$-norm $\|\cdot\|_{\mathcal{H}_{2,\infty}}$. Thus, $\|\cdot\|_{\mathcal{H}_{2,\bar{T}}}$ provides a more accurate bound than $\|\cdot\|_{\mathcal{H}_{2,\infty}}$ for the output error in (5). By minimizing $\|\cdot\|_{\mathcal{H}_{2,\bar{T}}}$, we can expect to find a better reduce-order system in the time-interval $[0, \bar{T}]$ than the case of a locally optimal reduced-order system with respect to $\|\cdot\|_{\mathcal{H}_{2,\infty}}$.

3 First-Order Necessary Conditions for $\mathcal{H}_{2,T}$ Optimal Model Order Reduction

In this section, we begin by deriving first-order necessary conditions for time-limited $\mathcal{H}_2$-optimal reduced-order systems. In other words, our aim is to construct a reduced-order system $\hat{\Sigma}$ of order r as in (2), such that it minimizes $\|\Sigma - \hat{\Sigma}\|^2_{\mathcal{H}_{2,\bar{T}}} =: \mathcal{E}$, where Σ is the original system as in (1). An expression for $\mathcal{E}$ is given in (8). Since the term $\operatorname{tr}(CP_{\bar{T}}C^*)$ in (8) does not depend on the reduced-order matrices, we focus on minimizing the expression

$$\mathcal{E}_r := \operatorname{tr}(\hat{C}\hat{P}_{\bar{T}}\hat{C}^*) - 2\operatorname{tr}(CP_{2,\bar{T}}\hat{C}^*). \tag{11}$$

Before proceeding further, we assume that the matrix $\hat{A}$ in (2) is diagonalizable, i.e., there exists an invertible matrix S such that $\hat{A} = S^{-1}DS$, where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_r)$. Using the matrix S as a state-space transformation of (2), the term (11) can be equivalently rewritten as

$$\begin{aligned}\mathcal{E}_r &= \operatorname{tr}(\hat{C}S^{-1}S\hat{P}_{\bar{T}}S^*S^{-*}\hat{C}^*) - 2\operatorname{tr}(CP_{2,\bar{T}}S^*S^{-*}\hat{C}^*)\\ &= \operatorname{tr}(\tilde{C}\tilde{P}_{\bar{T}}\tilde{C}^*) - 2\operatorname{tr}(C\tilde{P}_{2,\bar{T}}\tilde{C}^*),\end{aligned} \tag{12}$$

where $\tilde{C} = \hat{C}S^{-1}$, $\tilde{P}_{\bar{T}} = S\hat{P}_{\bar{T}}S^*$, $\tilde{P}_{2,\bar{T}} = P_{2,\bar{T}}S^*$, and $(\cdot)^{-*}$ denotes the inverse of a matrices transposed, i.e, $\big((\cdot)^{-1}\big)^*$. Furthermore, it can be shown that the matrices $\tilde{P}_{\bar{T}}$ and $\tilde{P}_{2,\bar{T}}$ are the solutions to

$$A\tilde{P}_{2,\bar{T}} + \tilde{P}_{2,\bar{T}}D = -B\tilde{B}^* + \mathrm{e}^{A\bar{T}}\,B\tilde{B}^*\,\mathrm{e}^{D\bar{T}}, \tag{13a}$$

$$D\tilde{P}_{\bar{T}} + \tilde{P}_{\bar{T}}D = -\tilde{B}\tilde{B}^* + \mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^*\,\mathrm{e}^{D\bar{T}}, \tag{13b}$$

respectively, where $\tilde{B} = S\hat{B}$. More precisely, equation (13a) is obtained by multiplying (9b) with S^* from the right-hand side, and (13b) is derived by multiplying (9c) with S and S^* from the left-hand and right-hand sides, respectively, and by using the relation $\mathrm{e}^{\hat{A}\bar{T}} = S^{-1}\,\mathrm{e}^{D\bar{T}}\,S$.

In order to find first-order necessary conditions that minimize error expression (12), we compute the partial derivatives of the form $\partial_x \operatorname{tr}(\tilde{C}\tilde{P}_{\bar{T}}\tilde{C}^*)$ and $\partial_x \operatorname{tr}(C\tilde{P}_{2,\bar{T}}\tilde{C}^*)$ and then set

$$\partial_x \operatorname{tr}(\tilde{C}\tilde{P}_{\bar{T}}\tilde{C}^*) = 2\partial_x \operatorname{tr}(C\tilde{P}_{2,\bar{T}}\tilde{C}^*),$$

where $x \in \{\lambda_i, \tilde{c}_{ki}, \tilde{b}_{ij}\}$, $i \in \{1, \ldots, r\}$, $j \in \{1, \ldots, m\}$, $k \in \{1, \ldots, p\}$ and $\tilde{c}_{ki}$, $\tilde{b}_{ij}$ being ki-th and ij-th elements of the matrices $\tilde{C}$ and $\tilde{B}$, respectively.

Before proceeding further, we note that with e_i, we denote the i-th column of the identity matrix of suitable dimension which is clear from the context. Next, we aim at deriving optimality conditions with respect to $\tilde{c}_{ki}$. Towards this, we first note that

$$\begin{aligned}\partial_{\tilde{c}_{ki}} \operatorname{tr}(\tilde{C}\tilde{P}_{\bar{T}}\tilde{C}^*) &= \partial_{\tilde{c}_{ki}} \operatorname{tr}(\tilde{C}^*\tilde{C}\tilde{P}_{\bar{T}})\\ &= \operatorname{tr}((\partial_{\tilde{c}_{ki}}\tilde{C}^*)\tilde{C}\tilde{P}_{\bar{T}} + \tilde{C}^*(\partial_{\tilde{c}_{ki}}\tilde{C})\tilde{P}_{\bar{T}}) = \operatorname{tr}(e_ie_k^*\tilde{C}\tilde{P}_{\bar{T}} + \tilde{C}^*e_ke_i^*\tilde{P}_{\bar{T}})\\ &= 2e_k^*\tilde{C}\tilde{P}_{\bar{T}}e_i,\end{aligned}$$

where we have used the linearity of the trace, the product rule and the fact that $\tilde{P}_{\bar{T}}$ does not depend on $\tilde{C}$. Since

$$\partial_{\tilde{c}_{ki}} \operatorname{tr}(C\tilde{P}_{2,\bar{T}}\tilde{C}^*) = \operatorname{tr}(C\tilde{P}_{2,\bar{T}}e_ie_k^*) = e_k^*C\tilde{P}_{2,\bar{T}}e_i,$$

the optimality condition with respect to $\tilde{c}_{ki}$ is $e_k^*\tilde{C}\tilde{P}_{\bar{T}}e_i = e_k^*C\tilde{P}_{2,\bar{T}}e_i$ for all $i \in \{1,\dots,r\}$, $k \in \{1,\dots,p\}$. Hence, we obtain

$$\tilde{C}\tilde{P}_{\bar{T}} = C\tilde{P}_{2,\bar{T}}. \tag{14}$$

We now derive the partial derivatives of the error expression (12) with respect to $\tilde{b}_{ij}$. To simplify this procedure, we first rewrite (12) by making use of Proposition 2.3 as follows:

$$\begin{aligned}\mathcal{E}_r = \operatorname{tr}(\tilde{C}\tilde{P}_{\bar{T}}\tilde{C}^*) - 2\operatorname{tr}(C\tilde{P}_{2,\bar{T}}\tilde{C}^*) &= \operatorname{tr}(\hat{B}^*\hat{Q}_{\bar{T}}\hat{B}) - 2\operatorname{tr}(\hat{B}^*Q_{2,\bar{T}}B)\\ &= \operatorname{tr}(\tilde{B}^*\tilde{Q}_{\bar{T}}\tilde{B}) - 2\operatorname{tr}(\tilde{B}^*\tilde{Q}_{2,\bar{T}}B),\end{aligned}$$

where $\tilde{Q}_{\bar{T}} = S^{-*}\hat{Q}_{\bar{T}}S^{-1}$ and $\tilde{Q}_{2,\bar{T}} = S^{-*}\hat{Q}_{2,\bar{T}}$, and the matrices $\tilde{Q}_{\bar{T}}$ and $\tilde{Q}_{2,\bar{T}}$ satisfy

$$D\tilde{Q}_{2,\bar{T}} + \tilde{Q}_{2,\bar{T}}A = -\tilde{C}^*C + \mathrm{e}^{D\bar{T}}\,\tilde{C}^*C\,\mathrm{e}^{D\bar{T}}, \tag{15a}$$
$$D\tilde{Q}_{\bar{T}} + \tilde{Q}_{\bar{T}}D = -\tilde{C}^*\tilde{C} + \mathrm{e}^{D\bar{T}}\,\tilde{C}^*\tilde{C}\,\mathrm{e}^{D\bar{T}}, \tag{15b}$$

respectively. Again, (15a) is obtained by multiplying (10b) with S^{-*} from the left-hand side, and we find (15b) by multiplying (10c) with S^{-*} from the left-hand side and with S^{-1} from the right-hand side. Thus, we have

$$\begin{aligned}\partial_{\tilde{b}_{ij}} \operatorname{tr}(\tilde{B}\tilde{B}^*\tilde{Q}_{\bar{T}}) &= \operatorname{tr}((\partial_{\tilde{b}_{ij}}\tilde{B})\tilde{B}^*\tilde{Q}_{\bar{T}} + \tilde{B}(\partial_{\tilde{b}_{ij}}\tilde{B}^*)\tilde{Q}_{\bar{T}}) = \operatorname{tr}(e_ie_j^*\tilde{B}^*\tilde{Q}_{\bar{T}} + \tilde{B}e_je_i^*\tilde{Q}_{\bar{T}})\\ &= 2e_i^*\tilde{Q}_{\bar{T}}\tilde{B}e_j\end{aligned}$$

using that $\tilde{Q}_{\bar{T}}$ does not depend on $\tilde{B}$ or $\tilde{b}_{ij}$. Since

$$\partial_{\tilde{b}_{ij}} \operatorname{tr}(\tilde{B}^*\tilde{Q}_{2,\bar{T}}B) = \operatorname{tr}(e_je_i^*\tilde{Q}_{2,\bar{T}}B) = e_i^*\tilde{Q}_{2,\bar{T}}Be_j,$$

it is necessary that $e_i^*\tilde{Q}_{\bar{T}}\tilde{B}e_j = e_i^*\tilde{Q}_{2,\bar{T}}Be_j$ for $i \in \{1,\dots,r\}$, $j \in \{1,\dots,m\}$, which can be equivalently written as

$$\tilde{Q}_{\bar{T}}\tilde{B} = \tilde{Q}_{2,\bar{T}}B. \tag{16}$$

Next, we first introduce the following lemma in order to derive an optimality condition with respect to the eigenvalues λ_i of $\hat{A}$.

Lemma 3.1. *The partial derivatives* $X^{(i)} := \partial_{\lambda_i}\tilde{P}_{\bar{T}}$ *and* $X_2^{(i)} := \partial_{\lambda_i}\tilde{P}_{2,\bar{T}}$ *solve*

$$DX^{(i)} + X^{(i)}D = -e_ie_i^*\tilde{P}_{\bar{T}} - \tilde{P}_{\bar{T}}e_ie_i^* + \bar{T}e_ie_i^*\,\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^*\,\mathrm{e}^{D\bar{T}} + \bar{T}\,\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^*\,\mathrm{e}^{D\bar{T}}\,e_ie_i^*, \tag{17}$$
$$AX_2^{(i)} + X_2^{(i)}D = -\tilde{P}_{2,\bar{T}}e_ie_i^* + \bar{T}\,\mathrm{e}^{A\bar{T}}\,B\tilde{B}^*\,\mathrm{e}^{D\bar{T}}\,e_ie_i^*, \tag{18}$$

respectively.

Proof. The derivative of the left-hand side of (13a) is

$$AX_2^{(i)} + X_2^{(i)} D + \tilde{P}_{2,\bar{T}} e_i e_i^*,$$

which is obtained by applying the product rule. The derivative of the corresponding right-hand side is

$$\mathrm{e}^{A\bar{T}} B\tilde{B}^* \partial_{\lambda_i} \mathrm{e}^{D\bar{T}} = \mathrm{e}^{A\bar{T}} B\tilde{B}^* \mathrm{e}^{D\bar{T}} e_i e_i^* \bar{T},$$

because $\partial_{\lambda_i} \mathrm{e}^{D\bar{T}} = \partial_{\lambda_i} \operatorname{diag}(\mathrm{e}^{\lambda_1 \bar{T}}, \dots, \mathrm{e}^{\lambda_i \bar{T}}, \dots, \mathrm{e}^{\lambda_r \bar{T}}) = \operatorname{diag}(0, \dots, \bar{T}\,\mathrm{e}^{\lambda_i \bar{T}}, \dots, 0)$. This yields (18). Applying ∂_{λ_i} to the left-hand side of (13b) provides

$$e_i e_i^* \tilde{P}_{\bar{T}} + DX^{(i)} + X^{(i)} D + \tilde{P}_{\bar{T}} e_i e_i^*$$

again using the product rule. Doing the same with the corresponding right-hand side, we have

$$\begin{aligned} \partial_{\lambda_i}(\mathrm{e}^{D\bar{T}} \tilde{B}\tilde{B}^* \mathrm{e}^{D\bar{T}}) &= (\partial_{\lambda_i} \mathrm{e}^{D\bar{T}})\tilde{B}\tilde{B}^* \mathrm{e}^{D\bar{T}} + \mathrm{e}^{D\bar{T}} \tilde{B}\tilde{B}^* (\partial_{\lambda_i} \mathrm{e}^{D\bar{T}}) \\ &= \bar{T} e_i e_i^* \mathrm{e}^{D\bar{T}} \tilde{B}\tilde{B}^* \mathrm{e}^{D\bar{T}} + \mathrm{e}^{D\bar{T}} \tilde{B}\tilde{B}^* \mathrm{e}^{D\bar{T}} e_i e_i^* \bar{T}. \end{aligned}$$

This proves (17). □

Before we proceed further, let us introduce the infinite Gramian $\tilde{Q}_\infty$, which we define as the solution to

$$D\tilde{Q}_\infty + \tilde{Q}_\infty D = -\tilde{C}^* \tilde{C}. \tag{19}$$

It is unique if and only if D and $-D$ have no common eigenvalues. Using (19) , we obtain

$$\partial_{\lambda_i} \operatorname{tr}(\tilde{C}\tilde{P}_{\bar{T}}\tilde{C}^*) = \operatorname{tr}(\tilde{C}^*\tilde{C}X^{(i)}) = -\operatorname{tr}([D\tilde{Q}_\infty + \tilde{Q}_\infty D]X^{(i)}) = -\operatorname{tr}(\tilde{Q}_\infty [X^{(i)} D + DX^{(i)}]).$$

Using Lemma 3.1, we get

$$\begin{aligned} \partial_{\lambda_i} \operatorname{tr}(\tilde{C}\tilde{P}_{\bar{T}}\tilde{C}^*) &= \operatorname{tr}(\tilde{Q}_\infty [e_i e_i^* \tilde{P}_{\bar{T}} + \tilde{P}_{\bar{T}} e_i e_i^* - \bar{T} e_i e_i^* \mathrm{e}^{D\bar{T}} \tilde{B}\tilde{B}^* \mathrm{e}^{D\bar{T}} - \bar{T}\,\mathrm{e}^{D\bar{T}} \tilde{B}\tilde{B}^* \mathrm{e}^{D\bar{T}} e_i e_i^*]) \\ &= 2 e_i^* \tilde{Q}_\infty [\tilde{P}_{\bar{T}} - \bar{T}\,\mathrm{e}^{D\bar{T}} \tilde{B}\tilde{B}^* \mathrm{e}^{D\bar{T}}] e_i. \end{aligned}$$

Assuming that D and $-A$ have no common eigenvalues, we define the infinite cross Gramian, $\tilde{Q}_{2,\infty}$ which satisfies

$$D\tilde{Q}_{2,\infty} + \tilde{Q}_{2,\infty} A^* = -\tilde{C}^* C. \tag{20}$$

Hence, it holds that

$$\begin{aligned} \partial_{\lambda_i} \operatorname{tr}(C\tilde{P}_{2,\bar{T}}\tilde{C}^*) &= \operatorname{tr}(\tilde{C}^* C X_2^{(i)}) = -\operatorname{tr}([D\tilde{Q}_{2,\infty} + \tilde{Q}_{2,\infty} A] X_2^{(i)}) \\ &= -\operatorname{tr}(\tilde{Q}_{2,\infty}[X_2^{(i)} D + AX_2^{(i)}]) = \operatorname{tr}(\tilde{Q}_{2,\infty}[\tilde{P}_{2,\bar{T}} - \bar{T}\,\mathrm{e}^{A\bar{T}} B\tilde{B}^* \mathrm{e}^{D\bar{T}}] e_i e_i^*) \\ &= e_i^* \tilde{Q}_{2,\infty} [\tilde{P}_{2,\bar{T}} - \bar{T}\,\mathrm{e}^{A\bar{T}} B\tilde{B}^* \mathrm{e}^{D\bar{T}}] e_i \end{aligned}$$

applying Lemma 3.1 again. This leads to the third optimality condition, which is

$$e_i^* \tilde{Q}_{2,\infty}[\tilde{P}_{2,\bar{T}} - \bar{T}\,\mathrm{e}^{A\bar{T}}\,B\tilde{B}^*\,\mathrm{e}^{D\bar{T}}]e_i = e_i^* \tilde{Q}_\infty[\tilde{P}_{\bar{T}} - \bar{T}\,\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^*\,\mathrm{e}^{D\bar{T}}]e_i \tag{21}$$

for all $i \in \{1, \ldots, r\}$.

Below, the generalized optimality conditions are summarized that have been derived above. Additionally, we provide an equivalent Kronecker formulation in the next theorem that is useful for the error analysis in the optimality conditions.

Theorem 3.2. *Let the reduced-order system (2) be a locally optimal approximation to the original system (1) with respect to $\|\cdot\|_{\mathcal{H}_{2,\bar{T}}}$. Then, according to (14), (16) and (21), it holds that*

$$\tilde{C}\tilde{P}_{\bar{T}} = C\tilde{P}_{2,\bar{T}}, \tag{22}$$

$$\tilde{Q}_{\bar{T}}\tilde{B} = \tilde{Q}_{2,\bar{T}}B, \tag{23}$$

$$e_i^* \tilde{Q}_{2,\infty}[\tilde{P}_{2,\bar{T}} - \bar{T}\,\mathrm{e}^{A\bar{T}}\,B\tilde{B}^*\,\mathrm{e}^{D\bar{T}}]e_i = e_i^* \tilde{Q}_\infty[\tilde{P}_{\bar{T}} - \bar{T}\,\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^*\,\mathrm{e}^{D\bar{T}}]e_i \tag{24}$$

for all $i \in \{1, \ldots, r\}$, where e_i is the ith unit vector and the matrices $\tilde{P}_{2,\bar{T}}$, $\tilde{P}_{\bar{T}}$, $\tilde{Q}_{2,\bar{T}}$, $\tilde{Q}_{\bar{T}}$, $\tilde{Q}_\infty$ and $\tilde{Q}_{2,\infty}$ are the solutions to (13a), (13b), (15a), (15b), (19) and (20), respectively.

Equivalent to (22), (23) and (24), we have

$$\begin{aligned}
&(I \otimes \hat{C})\left[(I \otimes \hat{A}) + (D \otimes I)\right]^{-1}(\mathrm{e}^{D\bar{T}}\,\tilde{B} \otimes \mathrm{e}^{\hat{A}\bar{T}}\,\hat{B} - \tilde{B} \otimes \hat{B})\operatorname{vec}(I)\\
&= (I \otimes C)\left[(I \otimes A) + (D \otimes I)\right]^{-1}(\mathrm{e}^{D\bar{T}}\,\tilde{B} \otimes \mathrm{e}^{A\bar{T}}\,B - \tilde{B} \otimes B)\operatorname{vec}(I),
\end{aligned} \tag{25}$$

$$\begin{aligned}
&(\hat{B}^* \otimes I)\left[(I \otimes D) + (\hat{A}^* \otimes I)\right]^{-1}(\mathrm{e}^{\hat{A}^*\bar{T}}\,\hat{C}^* \otimes \mathrm{e}^{D\bar{T}}\,\tilde{C}^* - \hat{C}^* \otimes \tilde{C}^*)\operatorname{vec}(I)\\
&= (B^* \otimes I)\left[(I \otimes D) + (A^* \otimes I)\right]^{-1}(\mathrm{e}^{A^*\bar{T}}\,C^* \otimes \mathrm{e}^{D\bar{T}}\,\tilde{C}^* - C^* \otimes \tilde{C}^*)\operatorname{vec}(I)
\end{aligned} \tag{26}$$

and for all $i \in \{1, \ldots, r\}$

$$\begin{aligned}
&\operatorname{vec}^*(I)(\hat{C} \otimes \tilde{C})\left[(I \otimes D) + (\hat{A} \otimes I)\right]^{-1}(I \otimes e_i e_i^*)\\
&\quad\times\left(\left[(I \otimes D) + (\hat{A} \otimes I)\right]^{-1}(\mathrm{e}^{\hat{A}\bar{T}}\,\hat{B} \otimes \mathrm{e}^{D\bar{T}}\,\tilde{B} - \hat{B} \otimes \tilde{B}) - (\bar{T}\,\mathrm{e}^{\hat{A}\bar{T}}\,\hat{B} \otimes \mathrm{e}^{D\bar{T}}\,\tilde{B})\right)\operatorname{vec}(I)\\
&= \operatorname{vec}^*(I)(C \otimes \tilde{C})\left[(I \otimes D) + (A \otimes I)\right]^{-1}(I \otimes e_i e_i^*)\\
&\quad\times\left(\left[(I \otimes D) + (A \otimes I)\right]^{-1}(\mathrm{e}^{A\bar{T}}\,B \otimes \mathrm{e}^{D\bar{T}}\,\tilde{B} - B \otimes \tilde{B}) - (\bar{T}\,\mathrm{e}^{A\bar{T}}\,B \otimes \mathrm{e}^{D\bar{T}}\,\tilde{B})\right)\operatorname{vec}(I).
\end{aligned} \tag{27}$$

Proof. Applying the vec operator to (14) leads to the following equivalent formulation:

$$\operatorname{vec}(\tilde{C}\tilde{P}_{\bar{T}}) = \operatorname{vec}(C\tilde{P}_{2,\bar{T}}).$$

Now, using the vectorization of (13b) and the relation in (4a), we obtain

$$\begin{aligned}\operatorname{vec}(\tilde{C}\tilde{P}_{\bar{T}}) &= (I\otimes\tilde{C})\operatorname{vec}(\tilde{P}_{\bar{T}}) = (I\otimes\tilde{C})\,[(I\otimes D)+(D\otimes I)]^{-1}\operatorname{vec}(\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^*\,\mathrm{e}^{D\bar{T}} - \tilde{B}\tilde{B}^*)\\ &= (I\otimes\tilde{C})\,[(I\otimes D)+(D\otimes I)]^{-1}\,(\mathrm{e}^{D\bar{T}}\,\tilde{B}\otimes\mathrm{e}^{D\bar{T}}\,\tilde{B}-\tilde{B}\otimes\tilde{B})\operatorname{vec}(I).\end{aligned}$$

Since $(I\otimes\tilde{C}) = (I\otimes\hat{C})(I\otimes S)^{-1}$ and $(\mathrm{e}^{D\bar{T}}\,\tilde{B}\otimes\mathrm{e}^{D\bar{T}}\,\tilde{B}-\tilde{B}\otimes\tilde{B}) = (I\otimes S^{-1})^{-1}(\mathrm{e}^{D\bar{T}}\,\tilde{B}\otimes\mathrm{e}^{\hat{A}\bar{T}}\,\hat{B}-\tilde{B}\otimes\hat{B})$, we get

$$\operatorname{vec}(\tilde{C}\tilde{P}_{\bar{T}}) = (I\otimes\hat{C})\,\Big[(I\otimes\hat{A})+(D\otimes I)\Big]^{-1}(\mathrm{e}^{D\bar{T}}\,\tilde{B}\otimes\mathrm{e}^{\hat{A}\bar{T}}\,\hat{B}-\tilde{B}\otimes\hat{B})\operatorname{vec}(I).$$

With the help of (13a), the vectorization of $C\tilde{P}_{2,\bar{T}}$ is given by

$$\begin{aligned}\operatorname{vec}(C\tilde{P}_{2,\bar{T}}) &= (I\otimes C)\operatorname{vec}(\tilde{P}_{2,\bar{T}}) = (I\otimes C)\,[(I\otimes A)+(D\otimes I)]^{-1}\operatorname{vec}(\mathrm{e}^{A\bar{T}}\,B\tilde{B}^*\,\mathrm{e}^{D\bar{T}} - B\tilde{B}^*)\\ &= (I\otimes C)\,[(I\otimes A)+(D\otimes I)]^{-1}\,(\mathrm{e}^{D\bar{T}}\,\tilde{B}\otimes\mathrm{e}^{A\bar{T}}\,B-\tilde{B}\otimes B)\operatorname{vec}(I)\end{aligned}$$

applying (4a) again, thus (25) follows. Condition (16) is equivalent to

$$\operatorname{vec}(\tilde{Q}_{\bar{T}}\tilde{B}) = \operatorname{vec}(\tilde{Q}_{2,\bar{T}}B),$$

and with property (4a), it holds that

$$\begin{aligned}\operatorname{vec}(\tilde{Q}_{\bar{T}}\tilde{B}) &= (\tilde{B}^*\otimes I)\operatorname{vec}(\tilde{Q}_{\bar{T}})\\ &= (\tilde{B}^*\otimes I)\,[(I\otimes D)+(D\otimes I)]^{-1}\,(\mathrm{e}^{D\bar{T}}\,\tilde{C}^*\otimes\mathrm{e}^{D\bar{T}}\,\tilde{C}^*-\tilde{C}^*\otimes\tilde{C}^*)\operatorname{vec}(I)\end{aligned}$$

inserting the vectorized representation of (15b). Using the identities $(\tilde{B}^*\otimes I) = (\hat{B}^*\otimes I)(S^{-*}\otimes I)^{-1}$ and $(\mathrm{e}^{D\bar{T}}\,\tilde{C}^*\otimes\mathrm{e}^{D\bar{T}}\,\tilde{C}^*-\tilde{C}^*\otimes\tilde{C}^*) = (S^*\otimes I)^{-1}(\mathrm{e}^{\hat{A}^*\bar{T}}\,\hat{C}^*\otimes\mathrm{e}^{D\bar{T}}\,\tilde{C}^*-\hat{C}^*\otimes\tilde{C}^*)$ yields

$$\operatorname{vec}(\tilde{Q}_{\bar{T}}\tilde{B}) = (\hat{B}^*\otimes I)\,\Big[(I\otimes D)+(\hat{A}^*\otimes I)\Big]^{-1}(\mathrm{e}^{\hat{A}^*\bar{T}}\,\hat{C}^*\otimes\mathrm{e}^{D\bar{T}}\,\tilde{C}^*-\hat{C}^*\otimes\tilde{C}^*)\operatorname{vec}(I).$$

Vectorizing (15a) leads to

$$\operatorname{vec}(\tilde{Q}_{2,\bar{T}}\tilde{B}) = (B^*\otimes I)\,[(I\otimes D)+(A^*\otimes I)]^{-1}\,(\mathrm{e}^{A^*\bar{T}}\,C^*\otimes\mathrm{e}^{D\bar{T}}\,\tilde{C}^*-C^*\otimes\tilde{C}^*)\operatorname{vec}(I),$$

leading to (26). Condition (21) is equivalent to

$$\operatorname{tr}([\tilde{P}_{2,\bar{T}}-\bar{T}\,\mathrm{e}^{A\bar{T}}\,B\tilde{B}^*\,\mathrm{e}^{D\bar{T}}]e_ie_i^*\tilde{Q}_{2,\infty}) = \operatorname{tr}([\tilde{P}_{\bar{T}}-\bar{T}\,\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^*\,\mathrm{e}^{D\bar{T}}]e_ie_i^*\tilde{Q}_{\infty})$$

for every $i\in\{1,\dots,r\}$. Taking (4b) into account, we can express the trace using the vec operator as follows:

$$\operatorname{tr}([\tilde{P}_{\bar{T}}-\bar{T}\,\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^*\,\mathrm{e}^{D\bar{T}}]e_ie_i^*\tilde{Q}_{\infty}) = \operatorname{vec}^*(\tilde{P}_{\bar{T}}-\bar{T}\,\mathrm{e}^{D\bar{T}}\,\tilde{B}\tilde{B}^*\,\mathrm{e}^{D\bar{T}})(I\otimes e_ie_i^*)\operatorname{vec}(\tilde{Q}_{\infty}). \tag{28}$$

With the above arguments, we see that the vectorization of (19) yields

$$\operatorname{vec}(\tilde{Q}_\infty) = -(S^{-*} \otimes I) \left[(I \otimes D) + (\hat{A}^* \otimes I)\right]^{-1} (\hat{C}^* \otimes \tilde{C}^*) \operatorname{vec}(I). \tag{29}$$

Before we proceed further, we need the following two relations:

$$(S^{-1} \otimes I) \operatorname{vec}(\bar{T} \, \mathrm{e}^{D\bar{T}} \, \tilde{B}\tilde{B}^* \, \mathrm{e}^{D\bar{T}}) = (\bar{T} \, \mathrm{e}^{\hat{A}\bar{T}} \, \hat{B} \otimes \mathrm{e}^{D\bar{T}} \, \tilde{B}) \operatorname{vec}(I), \tag{30}$$

$$(S^{-1} \otimes I) \operatorname{vec}(\tilde{P}_{\bar{T}}) = \left[(I \otimes D) + (\hat{A} \otimes I)\right]^{-1} (\mathrm{e}^{\hat{A}\bar{T}} \, \hat{B} \otimes \mathrm{e}^{D\bar{T}} \, \tilde{B} - \hat{B} \otimes \tilde{B}) \operatorname{vec}(I). \tag{31}$$

We insert (29) into (28) and obtain

$$\begin{aligned}
&\operatorname{tr}([\tilde{P}_{\bar{T}} - \bar{T} \, \mathrm{e}^{D\bar{T}} \, \tilde{B}\tilde{B}^* \, \mathrm{e}^{D\bar{T}}] e_i e_i^* \tilde{Q}_\infty) \\
&= \operatorname{vec}^*(\tilde{P}_{\bar{T}} - \bar{T} \, \mathrm{e}^{D\bar{T}} \, \tilde{B}\tilde{B}^* \, \mathrm{e}^{D\bar{T}}) (S^{-*} \otimes I)(I \otimes e_i e_i^*) \left[-(I \otimes D) - (\hat{A}^* \otimes I)\right]^{-1} \\
&\quad \times (\hat{C}^* \otimes \tilde{C}^*) \operatorname{vec}(I).
\end{aligned}$$

We apply (30) and (31) to the above identity. This leads to the following:

$$\begin{aligned}
&\operatorname{tr}([\tilde{P}_{\bar{T}} - \bar{T} \, \mathrm{e}^{D\bar{T}} \, \tilde{B}\tilde{B}^* \, \mathrm{e}^{D\bar{T}}] e_i e_i^* \tilde{Q}_\infty) \\
&= \operatorname{vec}^*(I) \left[(\hat{B}^* \, \mathrm{e}^{\hat{A}^*\bar{T}} \otimes \tilde{B}^* \, \mathrm{e}^{D\bar{T}} - \hat{B}^* \otimes \tilde{B}^*) \left[(I \otimes D) + (\hat{A}^* \otimes I)\right]^{-1} - (\bar{T}\hat{B}^* \, \mathrm{e}^{\hat{A}^*\bar{T}} \otimes \tilde{B}^* \, \mathrm{e}^{D\bar{T}})\right] \\
&\quad \times (I \otimes e_i e_i^*) \left[-(I \otimes D) - (\hat{A}^* \otimes I)\right]^{-1} (\hat{C}^* \otimes \tilde{C}^*) \operatorname{vec}(I)
\end{aligned}$$

Using (4b) and evaluating the expression

$$\operatorname{tr}([\tilde{P}_{2,\bar{T}} - \bar{T} \, \mathrm{e}^{A\bar{T}} \, B\tilde{B}^* \, \mathrm{e}^{D\bar{T}}] e_i e_i^* \tilde{Q}_{2,\infty}) = \operatorname{vec}^*(\tilde{P}_{2,\bar{T}}^* - \bar{T} \, \mathrm{e}^{D\bar{T}} \, \tilde{B}B^* \, \mathrm{e}^{A^*\bar{T}})(I \otimes e_i e_i^*) \operatorname{vec}(\tilde{Q}_{2,\infty}),$$

further by inserting the vectorized form of the matrices yields (27). □

Remark. *The Wilson conditions (22), (23) and (24) are based on the finite time Gramians. Alternatively, we refer [16], where interpolation-based first-order necessary $\mathcal{H}_{2,T}$ optimality conditions are derived.*

Furthermore, we would like to mention that there are several other extensions of the Wilson conditions of first-order ODEs, e.g., to bilinear systems [2, 18], to quadratic-bilinear systems [3], to delay systems [12].

Having derived optimality conditions, in the following, we propose an iterative algorithm, see Algorithm 1, which we refer to as *time-limited IRKA-type algorithm.* Like IRKA for linear systems in [8], this algorithm is based on Sylvester equations. To be more precise, the projection matrices V and W that are used to determine the reduced system (2) in Algorithm 1 are computed from (13a) and (15a). In comparison to the classical IRKA, the time-limited scheme is characterized by an additional term in the right-hand side of the Sylvester equations.

Algorithm 1 Time-limited IRKA-type Algorithm

Input: The system matrices: A, B, C.

Output: The reduced matrices: $\hat{A}, \hat{B}, \hat{C}$.

Make an initial guess for the reduced matrices $\hat{A}, \hat{B}, \hat{C}$.

while not converged **do**

Perform the spectral decomposition of $\hat{A}$ and define:
$D = S\hat{A}S^{-1}, \ \tilde{B} = S\hat{B}, \ \tilde{C} = \hat{C}S^{-1}$.

Solve for V and W:
$-VD - AV = B\tilde{B}^* - e^{A\bar{T}}B\tilde{B}^*e^{D\bar{T}}$,
$-WD - A^*W = C^*\tilde{C} - e^{A^*\bar{T}}C^*\tilde{C}e^{D\bar{T}}$.

$V = \mathrm{orth}\,(V)$ and $W = \mathrm{orth}\,(W)$, where $\mathrm{orth}\,(\cdot)$ returns an orthonormal basis for the range of a matrix.

Determine the reduced matrices:
$\hat{A} = (W^*V)^{-1}W^*AV, \qquad \hat{B} = (W^*V)^{-1}W^*B, \qquad \hat{C} = CV$.

end while

As for IRKA, the connection between Algorithm 1 and the error measure that is aimed to be minimized is through the solutions of the underlying Sylvester equations which also enter in a Gramian based representation of the respective error measure. In the time-limited framework, the error measure is the $\mathcal{H}_{2,T}$-metric, for which the Gramian based representation is given in (8) or alternatively through Proposition 2.3.

With this choice of the Sylvester equations, the first-order $\mathcal{H}_{2,T}$ optimality conditions as presented in Theorem 3.2 are aimed to be satisfied which is true for classical IRKA ($\bar{T} \to \infty$). However, we would like to point out that the proposed algorithm in general does not construct reduced-order systems which satisfy the first-order necessary conditions for optimality. Thus, our next goal is to derive expressions, which allow us to estimate how close the obtained reduced-order systems, corresponding to Algorithm 1, are to optimality.

Theorem 3.3. *Let $\hat{A}$, $\hat{B}$ and $\hat{C}$ be the reduced-order matrices computed by Algorithm 1. Let E_c be the difference between the left-hand and the right-hand sides of (25). Then,*

$$E_c = (I \otimes \hat{C}) \left[(I \otimes \hat{A}) + (D \otimes I) \right]^{-1} (\mathrm{e}^{D\bar{T}}\, \tilde{B} \otimes (W^*V)^{-1}W^*(\mathrm{e}^{A\,\Pi\,\bar{T}} - \mathrm{e}^{A\bar{T}})B)\, \mathrm{vec}(I),$$

where the projection matrix Π is defined as $\Pi := V(W^*V)^{-1}W^*$. Moreover, the deviation E_b between both sides of (26) is given by

$$E_b = (\hat{B}^* \otimes I) \left[(I \otimes D) + (\hat{A}^* \otimes I) \right]^{-1} (V^*(\mathrm{e}^{A^*\,\Pi^*\,\bar{T}} - \mathrm{e}^{A^*\bar{T}})C^* \otimes \mathrm{e}^{D\bar{T}}\, \tilde{C}^*)\, \mathrm{vec}(I).$$

For all $i = 1, \ldots, r$ the error E_λ^i in (27) is $E_\lambda^i = E_{\lambda,1}^i + E_{\lambda,2}^i$, where

$$
\begin{aligned}
E_{\lambda,1}^i = &\operatorname{vec}^*(I)(\hat{C} \otimes \tilde{C}) \left[(I \otimes D) + (\hat{A} \otimes I)\right]^{-1} (I \otimes e_i e_i^*) \\
&\times \Bigg(\left[(I \otimes D) + (\hat{A} \otimes I)\right]^{-1} ((W^*V)^{-1} W^* (\mathrm{e}^{A\,\Pi\,\bar{T}} - \mathrm{e}^{A\bar{T}}) B \otimes \mathrm{e}^{D\bar{T}}\,\tilde{B}) \\
&\qquad - (\bar{T}(W^*V)^{-1} W^* (\mathrm{e}^{A\,\Pi\,\bar{T}} - \mathrm{e}^{A\bar{T}}) B \otimes \mathrm{e}^{D\bar{T}}\,\tilde{B}) \Bigg) \operatorname{vec}(I)
\end{aligned}
$$

and the second term is given by

$$
\begin{aligned}
E_{\lambda,2}^i = &\operatorname{vec}^*(I)(C\,\mathrm{e}^{A\bar{T}} \otimes \tilde{C}\,\mathrm{e}^{D\bar{T}}) \\
&\times \left[(V \otimes I) \left[(I \otimes D) + (\hat{A} \otimes I)\right]^{-1} ((W^*V)^{-1} W^* \otimes I) - [(I \otimes D) + (A \otimes I)]^{-1} \right] \\
&\times (I \otimes e_i e_i^*) \left[[(I \otimes D) + (A \otimes I)]^{-1} (\mathrm{e}^{A\bar{T}}\, B \otimes \mathrm{e}^{D\bar{T}}\,\tilde{B} - B \otimes \tilde{B}) - (\bar{T}\,\mathrm{e}^{A\bar{T}}\, B \otimes \mathrm{e}^{D\bar{T}}\,\tilde{B}) \right] \\
&\times \operatorname{vec}(I).
\end{aligned}
$$

Proof. The result is proved in the Appendix. □

Theorem 3.3 allows us to point out the cases in which Algorithm 1 works well. The method is expected to perform well whenever the error expressions E_b, E_c and E_λ^i are small. By Theorem 3.3, the error in the optimality condition (25) is bounded as follows:

$$
\|E_c\|_2 \le \sqrt{m} k_c \left\| \mathrm{e}^{D\bar{T}}\,\tilde{B} \right\|_2 \left\| (W^*V)^{-1} W^* (\mathrm{e}^{A\,\Pi\,\bar{T}} - \mathrm{e}^{A\bar{T}}) B \right\|_2,
$$

where $k_c = \left\| (I \otimes \hat{C}) \left[(I \otimes \hat{A}) + (D \otimes I)\right]^{-1} \right\|_2$. Thus, the norm of the error $\|E_c\|_2$ is small if $\left\| (W^*V)^{-1} W^* (\mathrm{e}^{A\,\Pi\,\bar{T}} - \mathrm{e}^{A\bar{T}}) B \right\|_2$ is small. This is, e.g., given if the columns of $(\mathrm{e}^{A\,\Pi\,\bar{T}} - \mathrm{e}^{A\bar{T}}) B$ are close to the kernel of W^*. At the same time

$$
\left\| \mathrm{e}^{D\bar{T}}\,\tilde{B} \right\|_2 \le \mathrm{e}^{\lambda_{\max}\bar{T}} \left\| \tilde{B} \right\|_2
$$

should not be too large which is given if the largest eigenvalue $\lambda_{\max}$ of $\hat{A}$ is small enough or ideally negative (asymptotic stability of the reduced system).

Similar conclusions can be made by looking at E_b. It is bounded by

$$
\|E_b\|_2 \le \sqrt{p} k_b \left\| \tilde{C}\,\mathrm{e}^{D\bar{T}} \right\|_2 \left\| C(\mathrm{e}^{\Pi\,A\bar{T}} - \mathrm{e}^{A\bar{T}}) V \right\|_2
$$

with $k_b = \left\| (\hat{B}^* \otimes I) \left[(I \otimes D) + (\hat{A}^* \otimes I)\right]^{-1} \right\|_2$. Hence, if $\left\| C(\mathrm{e}^{\Pi\,A\bar{T}} - \mathrm{e}^{A\bar{T}}) V \right\|_2$ is small, then condition (26) is approximately satisfied. This is true if $(\mathrm{e}^{A^*\,\Pi^*\,\bar{T}} - \mathrm{e}^{A^*\bar{T}}) C^*$ is close to the kernel of V^*.

Now, let us finally discuss when the deviation in (27) is close to zero. The term $\left|E^i_{\lambda,1}\right|$ can be bounded in a similar way as $\|E_c\|_2$ such that it is also small if again $\left\|(W^*V)^{-1}W^*(\mathrm{e}^{A\,\Pi\,\bar{T}} - \mathrm{e}^{A\bar{T}})B\right\|_2$ is neglectable, whereas for $\left|E^i_{\lambda,2}\right|$ it is required to have the product

$$\begin{aligned}&\left\|C\,\mathrm{e}^{A\bar{T}}\right\|_2 \left\|\tilde{C}\,\mathrm{e}^{D\bar{T}}\right\|_2 \\ &\quad\times \left\|(V\otimes I)\left[(I\otimes D)+(\hat{A}\otimes I)\right]^{-1}((W^*V)^{-1}W^*\otimes I)-[(I\otimes D)+(A\otimes I)]^{-1}\right\|_2\end{aligned}$$

small. This is ensured for sufficiently large $\bar{T}$ if A is an asymptotically stable matrix. Of course, there can be cases, in which $\mathrm{e}^{A\bar{T}}$ is still large notwithstanding asymptotic stability (e.g. $\bar{T}$ is relatively small). Consequently, a larger error $\left|E^i_{\lambda,2}\right|$ is possible.

Remark. *One of main bottleneck in order to apply Algorithm 1 is the product of the matrix exponential to vectors. In the literature, several methods have been proposed to tackle such a problem in a numerically efficient way. For a brief discussion on it, we refer the readers to [9] and references therein. However, we would like to point out here that in our numerical experiments, we compute products such as $\mathrm{e}^{A}b$ exactly (given a matrix A and a matrix/vector b), rather than inexactly as proposed, e.g., in [9] since the considered problems are of moderate sizes.*

4 Numerical Experiments

In this section, we investigate the efficiency of the time-limited IRKA inspired algorithm, see Algorithm 1, and compare it with conventional IRKA (unbounded time), see [8]. All the experiments are done in MATLAB® 8.0.0.783 (R2012b) on a machine Intel®Xeon®CPU X5650 @ 2.67GHz with 48 GB RAM. We run both iterative algorithms until the relative change in the eigenvalues of $\hat{A}$ becomes less a tolerance of 10^{-8}. We initialize conventional IRKA randomly, and we use the reduced-order system obtained by conventional IRKA as an initial guess for Algorithm 1. In Table 1, we list the examples used in order to compare the algorithms. For all examples, we compare the impulse responses of the systems, which is simulated using the `impulse` command from MATLAB. To quantify the quality of reduced-order systems, we determine either the absolute or the relative error, depending on weather the impulse response crosses zero or not. We define the absolute $\mathcal{E}^{(a)}(t)$ and relative errors $\mathcal{E}^{(r)}(t)$, respectively, as follows:

$$\mathcal{E}^{(a)}(t) := \|y^{(\delta)}(t) - y_r^{(\delta)}(t)\| \quad \text{and} \quad \mathcal{E}^{(r)}(t) := \frac{\|y^{(\delta)}(t) - y_r^{(\delta)}(t)\|}{\|y(t)\|}, \tag{32}$$

where $y^{(\delta)}$ and $y_r^{(\delta)}$ are the impulses responses of original and reduced-order systems, respectively. In addition to this, we numerically examine how far away the reduced-order systems due to IRKA and Algorithm 1 are from satisfying the optimality conditions

Example	n	m	p
Heat equation	200	1	1
Clamped beam model	348	1	1
Component 1r of the International Space Station	270	3	3

Table 1: A list of examples with their dimensions (n), the number of inputs (m) and outputs (p). These examples are taken from `http://slicot.org/20-site/126-benchmark-examples-for-model-reduction`.

Method	$\mathcal{E}_c$	$\mathcal{E}_b$	$\mathcal{E}_\lambda$
IRKA	2.7×10^{-3}	2.7×10^{-3}	9.10×10^{-3}
TL-IRKA	1.39×10^{-4}	1.39×10^{-4}	1.58×10^{-1}

Table 2: Heat example: relative errors in satisfying the optimality conditions.

(25)–(27). To measure this, we first define the following quantities:

$$\mathcal{E}_c = \|\mathcal{R}_l^{(c)} - \mathcal{R}_r^{(c)}\|/\|\mathcal{R}_l^{(c)}\|, \tag{33a}$$

$$\mathcal{E}_b = \|\mathcal{R}_l^{(b)} - \mathcal{R}_r^{(b)}\|/\|\mathcal{R}_l^{(b)}\|, \tag{33b}$$

$$\mathcal{E}_\lambda = \max_i (\mathcal{R}_{\lambda_i}), \qquad \mathcal{R}_{\lambda_i} = \left|\mathcal{R}_l^{(\lambda_i)} - \mathcal{R}_r^{(\lambda_i)}\right| / \left|\mathcal{R}_l^{(\lambda_i)}\right|, \tag{33c}$$

where $\mathcal{R}_l^{(c)}$ and $\mathcal{R}_r^{(c)}$ are the left and right-hand sides of (25); $\mathcal{R}_l^{(b)}$ and $\mathcal{R}_r^{(b)}$ are the left and right-hand sides of (26); $\mathcal{R}_l^{(\lambda_i)}$ and $\mathcal{R}_r^{(\lambda_i)}$ are the left and right-hand sides of (27); $\max(\cdot)$ denotes the maximum.

In the following, we discuss each of these examples in detail. Beginning with the heat example, we compute the reduced-order systems by employing conventional IRKA and Algorithm 1 of order $r = 5$. We consider the terminal time $\bar{T} = 1$. In Figure 1, we compare the impulse responses, which shows that Algorithm 1 yields a reduced-order system, replicating the systems dynamics better in the time interval $[0, \bar{T}]$. We observe that Algorithm 1 takes 23 iterations to converge. Furthermore, as it has been noted in Section 3, Algorithm 1 does not yield a reduced-order system, satisfying the optimality conditions. Thus, in Table 2 we measure the error of the reduced-order systems obtained via IRKA and Algorithm 1 in the optimality conditions as described in (33). The table shows that for the heat example, Algorithm 1 does a better job in satisfying the two optimality conditions, and in contrast the third condition is satisfied better by the reduced-order system due to conventional IRKA. However, when we compare the time limited $\mathcal{H}_2$-norm of the error by using the reduced-order systems obtained by IRKA and Algorithm 1, then we observe, see Table 3, that Algorithm 1 yields a better reduced-order system with respect to $\mathcal{H}_{2,\bar{T}}$.

As a second example, we have taken a beam model which is reduced to the order $r = 10$ using IRKA and Algorithm 1. For this, we set the terminal time to $\bar{T} = 2$. Here, we notice that Algorithm 1 takes 28 iterations to converge. Next, we compare the

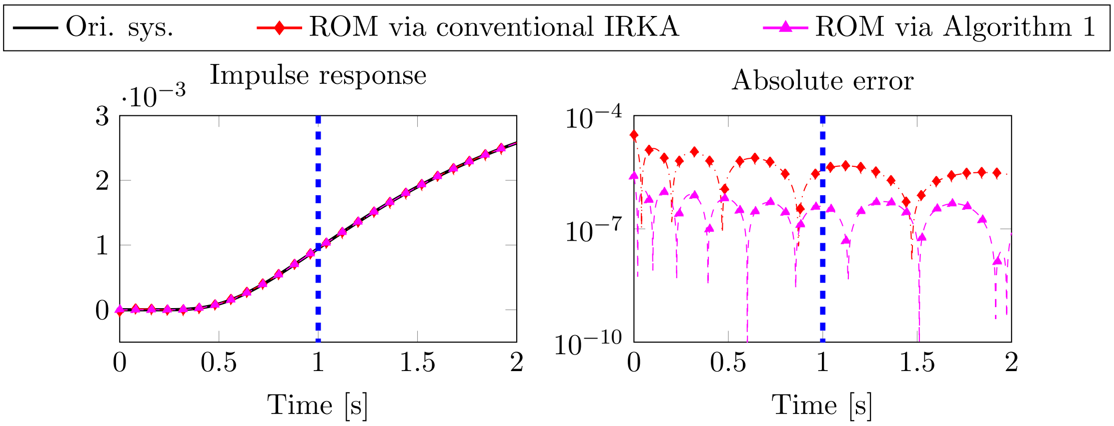

Figure 1: Heat example: a comparison of the impulse response of the original system and reduced-order system obtained via IRKA and Algorithm 1.

Method	Relative $\mathcal{H}_{2,\bar{T}}$ – error
IRKA	4.65×10^{-3}
TL-IRKA	8.77×10^{-5}

Table 3: Heat example: relative $\mathcal{H}_{2,\bar{T}}$-error comparison.

Method	$\mathcal{E}_c$	$\mathcal{E}_b$	$\mathcal{E}_\lambda$
IRKA	5.96×10^{-2}	5.96×10^{-2}	9.47×10^{-2}
TL-IRKA	3.94×10^{-4}	3.94×10^{-4}	1.26×10^{-1}

Table 4: Beam example: relative error in satisfying the optimality conditions.

Method	Relative $\mathcal{H}_{2,\bar{T}}$ – error
IRKA	6.98×10^{-3}
TL-IRKA	6.05×10^{-4}

Table 5: Beam example: relative $\mathcal{H}_{2,\bar{T}}$-error comparison.

impulse responses of the original and reduced-order systems in Figure 2. Clearly, we observe that Algorithm 1 produces a better reduced-order system as compared to IRKA at least within the time interval of interest. Furthermore, in Table 4, we measure the error of the obtained reduced-order systems in the optimality conditions, where we make a similar observation as in the heat example. We also compare the time limited $\mathcal{H}_2$-norm of the error by using the reduced-order systems obtained by IRKA and Algorithm 1 in Table 5, and observe that Algorithm 1 yields a better reduced-order system with respect to $\mathcal{H}_{2,\bar{T}}$.

Lastly, we present the results for the model of a space station. We first set the terminal time to $\bar{T} = 1$. For this example, we construct reduced systems of order $r = 20$

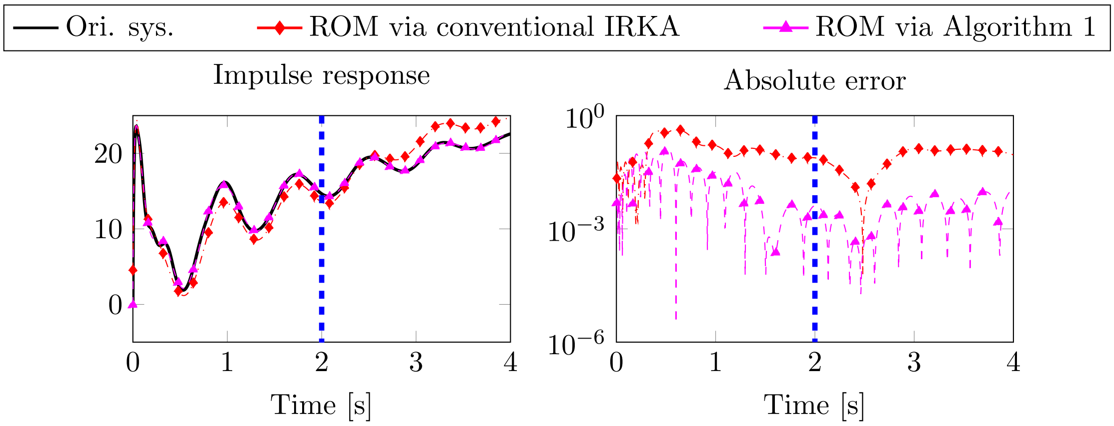

Figure 2: Beam example: a comparison of the impulse response of the original system and reduced-order system obtained via IRKA and Algorithm 1.

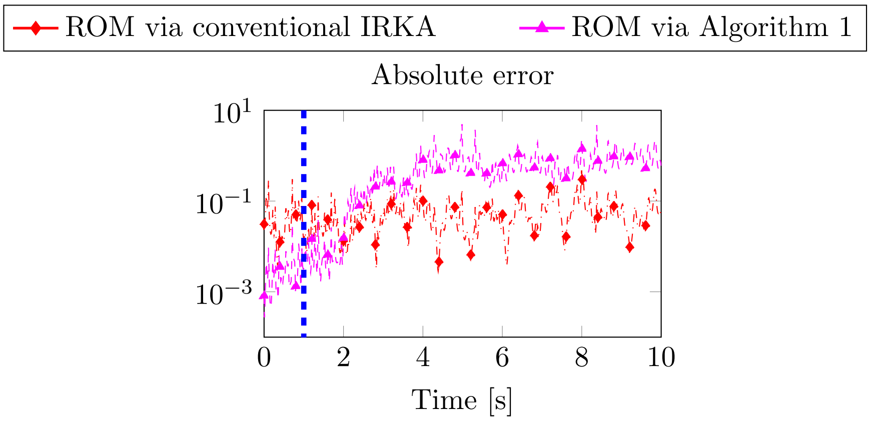

Figure 3: ISS example: a comparison of the impulse response of the original system and reduced-order system obtained via IRKA and Algorithm 1.

via IRKA and Algorithm 1 and compare the quality of them using the impulse response. Firstly, we note that Algorithm 1 takes 11 iterations to converge. Since the example has 3 inputs and 3 outputs, for brevity we refrain from plotting the impulse response, but we rather plot the norm absolute error which is shown in Figure 3. We observe that Algorithm 1 constructs a reduced-order system which replicates the dynamics better within the time interval of interest. For this example, we again compute how far away the reduced-order systems are from satisfying the optimality conditions exactly in Table 6. For this example as well, Algorithm 1 does a better job than IRKA in satisfying the first two conditions, but fails to perform better for the third conditions. However, importantly, Algorithm 1 yields a better reduced-order system. Like the previous two example, we observe that the time limited $\mathcal{H}_2$-norm of the error system for Algorithm 1 is better compared to IRKA.

Method	$\mathcal{E}_c$	$\mathcal{E}_b$	$\mathcal{E}_\lambda$
IRKA	2.61×10^{-1}	1.62×10^{-1}	1.08×10^{-1}
TL-IRKA	6.00×10^{-2}	5.43×10^{-3}	4.46×10^{-1}

Table 6: ISS example: relative error in satisfying the optimality conditions.

Method	Relative $\mathcal{H}_{2,\bar{T}}$ − error
IRKA	2.29×10^{-2}
TL-IRKA	6.87×10^{-5}

Table 7: ISS example: relative $\mathcal{H}_{2,\bar{T}}$-error comparison.

5 Conclusions

In this work, we have studied model order reduction of large-scale linear time-invariant systems. We have showed that the error between the original and reduced-order system on a finite time interval can be bounded using the time-limited $\mathcal{H}_2$-norm. Next, we have derived first-order optimality conditions, which are necessary for the time-limited $\mathcal{H}_2$-norm of the error system to be minimal. Based on these optimality conditions, we have proposed an iterative scheme which is inspired by the iterative rational Krylov algorithm [8]. Moreover, we have proposed a disgnostic measure, showing how close the resulted reduced-order systems are to optimality. We have concluded this paper by comparing conventional IRKA, an algorithm leading to a good reduced system on an infinite time horizon, with the proposed iterative scheme in several numerical experiments. The simulations have showed that time-limited IRKA can outperform IRKA on the finite time interval of interest.

As we have seen, the proposed iterative-type algorithm for the time-limited problem does not satisfy the optimality conditions exactly. Therefore, it would be worthwhile to come up with an improved algorithm, allowing us to construct a reduced-order system which satisfy the derived optimality conditions exactly

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A Proof of Theorem 3.3

The left side of (25) can be expressed as

$$(I \otimes \hat{C}) \left[(I \otimes \hat{A}) + (D \otimes I)\right]^{-1} (\mathrm{e}^{D\bar{T}}\, \tilde{B} \otimes (W^*V)^{-1}W^*\, \mathrm{e}^{A\bar{T}}\, B - \tilde{B} \otimes \hat{B})\, \mathrm{vec}(I) + E_c,$$

where we apply the identity $\mathrm{e}^{\hat{A}\bar{T}}\,\hat{B} = (W^*V)^{-1}W^*\,\mathrm{e}^{A\,\Pi\,\bar{T}}\,B$ which is obtained by using the series representation of the matrix exponential $\mathrm{e}^{\hat{A}\bar{T}}$, the definitions of $\hat{A}$, $\hat{B}$ in Algorithm 1 and $\left[(W^*V)^{-1}W^*AV\right]^k(W^*V)^{-1}W^* = (W^*V)^{-1}W^*\left[AV(W^*V)^{-1}W^*\right]^k$ for $k \in \mathbb{N}$. We set $\hat{K} := (I \otimes \hat{A}) + (D \otimes I)$ and $K := (I \otimes A) + (D \otimes I)$ and obtain

$$
\begin{aligned}
&(I \otimes \hat{C})\hat{K}^{-1}(\mathrm{e}^{D\bar{T}}\,\tilde{B} \otimes (W^*V)^{-1}W^*\,\mathrm{e}^{A\bar{T}}\,B - \tilde{B} \otimes \hat{B})\,\mathrm{vec}(I)\\
&= (I \otimes \hat{C})\hat{K}^{-1}(I \otimes (W^*V)^{-1}W^*)(\mathrm{e}^{D\bar{T}}\,\tilde{B} \otimes \mathrm{e}^{A\bar{T}}\,B - \tilde{B} \otimes B)\,\mathrm{vec}(I)\\
&= (I \otimes \hat{C})\hat{K}^{-1}(I \otimes (W^*V)^{-1}W^*)K\,\mathrm{vec}(V)\\
&= (I \otimes \hat{C})\hat{K}^{-1}(I \otimes (W^*V)^{-1}W^*)K\,\mathrm{vec}(V(W^*V)^{-1}W^*V)\\
&= (I \otimes \hat{C})\hat{K}^{-1}(I \otimes (W^*V)^{-1}W^*)K(I \otimes V(W^*V)^{-1}W^*)\,\mathrm{vec}(V)\\
&= (I \otimes \hat{C})\hat{K}^{-1}\hat{K}(I \otimes (W^*V)^{-1}W^*)\,\mathrm{vec}(V)\\
&= (I \otimes C)(I \otimes V)(I \otimes (W^*V)^{-1}W^*)\,\mathrm{vec}(V) = (I \otimes C)\,\mathrm{vec}(V)\\
&= (I \otimes C)K^{-1}(\mathrm{e}^{D\bar{T}}\,\tilde{B} \otimes \mathrm{e}^{A\bar{T}}\,B - \tilde{B} \otimes B)\,\mathrm{vec}(I),
\end{aligned}
$$

where the last term above is the right side of (25). The left-hand side of (26) is given by

$$
(\hat{B}^* \otimes I)\left[(I \otimes D) + (\hat{A}^* \otimes I)\right]^{-1}(V^*\,\mathrm{e}^{A^*\bar{T}}\,C^* \otimes \mathrm{e}^{D\bar{T}}\,\tilde{C}^* - \hat{C}^* \otimes \tilde{C}^*)\,\mathrm{vec}(I) + E_b,
$$

taking the identity $\mathrm{e}^{\hat{A}^*\bar{T}}\,\hat{C}^* = V^*\,\mathrm{e}^{A^*\,\Pi^*\,\bar{T}}\,C^*$ into account. So, by setting $\hat{K}_2 := (I \otimes D) + (\hat{A} \otimes I)$ and $K_2 := (I \otimes D) + (A \otimes I)$, we have

$$
\begin{aligned}
&(\hat{B}^* \otimes I)\hat{K}_2^{-*}(V^*\,\mathrm{e}^{A^*\bar{T}}\,C^* \otimes \mathrm{e}^{D\bar{T}}\,\tilde{C}^* - \hat{C}^* \otimes \tilde{C}^*)\,\mathrm{vec}(I)\\
&= (\hat{B}^* \otimes I)\hat{K}_2^{-*}(V^* \otimes I)(\mathrm{e}^{A^*\bar{T}}\,C^* \otimes \mathrm{e}^{D\bar{T}}\,\tilde{C}^* - C^* \otimes \tilde{C}^*)\,\mathrm{vec}(I)\\
&= (\hat{B}^* \otimes I)\hat{K}_2^{-*}(V^* \otimes I)K_2^*\,\mathrm{vec}(W^*)\\
&= (\hat{B}^* \otimes I)\hat{K}_2^{-*}(V^* \otimes I)K_2^*\,\mathrm{vec}(W^*V(W^*V)^{-1}W^*)\\
&= (\hat{B}^* \otimes I)\hat{K}_2^{-*}(V^* \otimes I)K_2^*(W(W^*V)^{-*}V^* \otimes I)\,\mathrm{vec}(W^*)\\
&= (\hat{B}^* \otimes I)\hat{K}_2^{-*}\hat{K}_2^*(V^* \otimes I)\,\mathrm{vec}(W^*)\\
&= (B^* \otimes I)(W(W^*V)^{-*} \otimes I)(V^* \otimes I)\,\mathrm{vec}(W^*) = (B^* \otimes I)\,\mathrm{vec}(W^*)\\
&= (B^* \otimes I)K_2^{-*}(\mathrm{e}^{A^*\bar{T}}\,C^* \otimes \mathrm{e}^{D\bar{T}}\,\tilde{C}^* - C^* \otimes \tilde{C}^*)\,\mathrm{vec}(I)
\end{aligned}
\tag{34}
$$

which is the right-hand side of (26). The left-hand side of (27) is given by

$$
\begin{aligned}
E^i_{\lambda,1} + \mathrm{vec}^*(I)(\hat{C} \otimes \tilde{C})\hat{K}_2^{-1}(I \otimes e_i e_i^*)\Big(&\hat{K}_2^{-1}((W^*V)^{-1}W^*\,\mathrm{e}^{A\bar{T}}\,B \otimes \mathrm{e}^{D\bar{T}}\,\tilde{B} - \hat{B} \otimes \tilde{B})\\
&-(\bar{T}(W^*V)^{-1}W^*\,\mathrm{e}^{A\bar{T}}\,\hat{B} \otimes \mathrm{e}^{D\bar{T}}\,\tilde{B})\Big)\,\mathrm{vec}(I).
\end{aligned}
$$

For the term right of $(I \otimes e_i e_i^*)$ it holds that

$$
\Big[\hat{K}_2^{-1}((W^*V)^{-1}W^*\,\mathrm{e}^{A\bar{T}}\,B \otimes \mathrm{e}^{D\bar{T}}\,\tilde{B} - \hat{B} \otimes \tilde{B})
$$

$$
\begin{aligned}
&\quad -(\bar{T}(W^*V)^{-1}W^* \, \mathrm{e}^{A\bar{T}} \, \hat{B} \otimes \mathrm{e}^{D\bar{T}} \, \tilde{B})\Big] \operatorname{vec}(I) \\
&= \hat{K}_2^{-1}((W^*V)^{-1}W^* \otimes I)(\mathrm{e}^{A\bar{T}} \, B \otimes \mathrm{e}^{D\bar{T}} \, \tilde{B} - B \otimes \tilde{B}) \operatorname{vec}(I) \\
&\quad - (\bar{T}(W^*V)^{-1}W^* \, \mathrm{e}^{A\bar{T}} \, B \otimes \mathrm{e}^{D\bar{T}} \, \tilde{B}) \operatorname{vec}(I) \\
&= \hat{K}_2^{-1}((W^*V)^{-1}W^* \otimes I)K_2 \operatorname{vec}(V^*) - (\bar{T}(W^*V)^{-1}W^* \, \mathrm{e}^{A\bar{T}} \, B \otimes \mathrm{e}^{D\bar{T}} \, \tilde{B}) \operatorname{vec}(I) \\
&= \hat{K}_2^{-1}((W^*V)^{-1}W^* \otimes I)K_2 \operatorname{vec}(V^*W(W^*V)^{-*}V^*) \\
&\quad - (\bar{T}(W^*V)^{-1}W^* \, \mathrm{e}^{A\bar{T}} \, B \otimes \mathrm{e}^{D\bar{T}} \, \tilde{B}) \operatorname{vec}(I) \\
&= \hat{K}_2^{-1}((W^*V)^{-1}W^* \otimes I)K_2(V(W^*V)^{-1}W^* \otimes I) \operatorname{vec}(V^*) \\
&\quad - (\bar{T}(W^*V)^{-1}W^* \, \mathrm{e}^{A\bar{T}} \, B \otimes \mathrm{e}^{D\bar{T}} \, \tilde{B}) \operatorname{vec}(I) \\
&= ((W^*V)^{-1}W^* \otimes I) \operatorname{vec}(V^*) - (\bar{T}(W^*V)^{-1}W^* \, \mathrm{e}^{A\bar{T}} \, B \otimes \mathrm{e}^{D\bar{T}} \, \tilde{B}) \operatorname{vec}(I) \\
&= (W^*V)^{-1}W^* \otimes I) \Big[K_2^{-1}(\mathrm{e}^{A\bar{T}} \, B \otimes \mathrm{e}^{D\bar{T}} \, \tilde{B} - B \otimes \tilde{B}) - (\bar{T} \, \mathrm{e}^{A\bar{T}} \, B \otimes \mathrm{e}^{D\bar{T}} \, \tilde{B})\Big] \operatorname{vec}(I).
\end{aligned}
$$

Since $((W^*V)^{-1}W^* \otimes I)$ and $(I \otimes e_i e_i^*)$ commute, it remains to analyze the following term

$$
\operatorname{vec}^*(I)(\hat{C} \otimes \tilde{C})\hat{K}_2^{-1}((W^*V)^{-1}W^* \otimes I) = \Big[(W(W^*V)^{-*} \otimes I)\hat{K}_2^{-*}(\hat{C}^* \otimes \tilde{C}^*) \operatorname{vec}(I)\Big]^*.
$$

We add a zero such that

$$
\begin{aligned}
&(W(W^*V)^{-*} \otimes I)\hat{K}_2^{-*}(\hat{C}^* \otimes \tilde{C}^*) \operatorname{vec}(I) \\
&= (W(W^*V)^{-*} \otimes I)\hat{K}_2^{-*}(V^* \otimes I))[(C^* \otimes \tilde{C}^*) - (\mathrm{e}^{A^*\bar{T}} \, C^* \otimes \mathrm{e}^{D\bar{T}} \, \tilde{C}^*)] \operatorname{vec}(I) \\
&\quad + (W(W^*V)^{-*} \otimes I)\hat{K}_2^{-*}(V^* \otimes I))(\mathrm{e}^{A^*\bar{T}} \, C^* \otimes \mathrm{e}^{D\bar{T}} \, \tilde{C}^*) \operatorname{vec}(I).
\end{aligned}
$$

Using the same steps as in (34), we find

$$
\begin{aligned}
&(W(W^*V)^{-*} \otimes I)\hat{K}_2^{-*}(V^* \otimes I))[(C^* \otimes \tilde{C}^*) - (\mathrm{e}^{A^*\bar{T}} \, C^* \otimes \mathrm{e}^{D\bar{T}} \, \tilde{C}^*)] \operatorname{vec}(I) \\
&= K_2^{-*}[(C^* \otimes \tilde{C}^*) - (\mathrm{e}^{A^*\bar{T}} \, C^* \otimes \mathrm{e}^{D\bar{T}} \, \tilde{C}^*)] \operatorname{vec}(I).
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&\operatorname{vec}^*(I)(\hat{C} \otimes \tilde{C})\hat{K}_2^{-1}((W^*V)^{-1}W^* \otimes I) = \operatorname{vec}^*(I)(C \otimes \tilde{C})K_2^{-1} \\
&+ \operatorname{vec}^*(I)(C \, \mathrm{e}^{A\bar{T}} \otimes \tilde{C} \, \mathrm{e}^{D\bar{T}}) \Big[(V \otimes I)\hat{K}_2^{-1}((W^*V)^{-1}W^* \otimes I) - K_2^{-1}\Big].
\end{aligned} \tag{35}
$$

The term in (35) provides $E_{\lambda,2}^i$ which concludes the proof. □

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