THE ZILCH ELECTROMAGNETIC CONSERVATION LAW
IN VARIATIONAL CHARACTERISTIC FORM

SAJAD AGHAPOUR, LARS ANDERSSON, AND KJELL ROSQUIST

ABSTRACT. In this paper we consider the zilch conservation laws for Maxwell theory and demonstrate that in the duality-symmetric version of Maxwell theory, the zilch arises as a Noether current for a variational symmetry of the duality symmetric Lagrangian which we identify through an application of the reverse of the Noether theorem. A variational symmetry leaves Lagrangian invariant up to a total divergence, without restricting to solutions of the field equations. This fact was previously known only for the so-called chirality current, i.e. the 00-component of zilch.

1. INTRODUCTION

Conservation laws of Maxwell's electromagnetic field equations, while being an old subject[1] going back more than a hundred years, have in the last decades received renewed attention driven by new developments in both theory and experiment. It was noted by Lipkin [9] that the pseudoscalar $C = \mathbf{E} \cdot \nabla \times \mathbf{E} + \mathbf{B} \cdot \nabla \times \mathbf{B}$, now referred to as the *optical chirality* [19], satisfies a differential conservation law. He also showed, referring to an observer 4-velocity $u^a = (1, 0, 0, 0)$, that C is the component $Z_{000} = Z_{abc}u^a u^b u^c$ of a conserved Lorentz covariant tensor named the zilch tensor Z_{abc}.[2] This tensor represents a collection of conserved currents parametrized by its first two indices. The pseudoscalar nature of C makes it suitable for use in experimental investigations involving chiral electromagnetic fields such as circularly polarized light [19]. In [17], the authors found that the zilch tensor can encode information about the topology of electromagnetic field lines.

While the zilch tensor has been known for several decades as a conserved quantity, our understanding of its relation to a symmetry of the Maxwell action still contains some gaps. In particular, a symmetry generator which leaves the standard Maxwell action invariant and which has the zilch as its Noether current has not been given in the literature. In earlier work, authors have considered symmetries leading to conservation of the scalar part C (Deser and Teitelboim [6] and Philbin [14]) and of a vector part (Calkin [4]) of the zilch tensor. Przanowski et al. [16] related the first order action of [6] to the zilch by a third order symmetry generator. Both Morgan and Joseph [11] and Sudbery [18] use a vector Lagrangian which is just the nonzero trace of the zilch tensor. Using the field tensor as the independent variable, it gives field equations which are equivalent to the Maxwell equations. They showed that the Lagrangian constructed in that way is related to the zilch tensor by spacetime translation symmetries. The significance of this result is not clear to us.

As is well-known, Noether proved that a symmetry of an action leads to a conservation law. She also proved the less frequently used converse that a conservation law for some Euler-Lagrange equations emanates from a symmetry of the corresponding Lagrangian density. We refer to Olver's book for a comprehensive account of Noether's theorems [12, see especially Theorem 5.58]. This equivalence

---

[1]See [2] for historical references.

[2]Cartesian coordinates will be used throughout.

between symmetries and conservation laws requires the consideration of *generalized symmetries*, the generators of which depend not only on the dependent variables themselves but also on their derivatives up to some (finite) order (see (1.4) below).

The main focus in this work is a clarification of the variational nature of the zilch conservation law by writing it in *characteristic form* [12], symbolically as $\mathrm{div} Z = Q \cdot \Delta$, and in full as

$$\frac{\partial Z_{ab}{}^{c}}{\partial x^{c}} = Q_{abc}\Delta^{c} \tag{1.1}$$

where the *characteristic* Q_{abc} represents the *ab*-parametrized components of the symmetry generator and $\Delta^{a} = \mathsf{E}^{a}(L)$ are the Euler-Lagrange expressions for a Lagrangian $L(A)$ as a function of the vector potential. The variation of the Lagrangian under the symmetry generator $Q_{abc}\,\partial/\partial A_{c}$ must then be zero modulo a total divergence [12, Sect. 5.3]. The Maxwell equations are invariant under the 1-dimensional group of duality transformations (see e.g. [2])

$$\begin{aligned} F_{ab} &\to \quad F_{ab}\cos\alpha + {}^{\star}F_{ab}\sin\alpha \\ {}^{\star}F_{ab} &\to -F_{ab}\sin\alpha + {}^{\star}F_{ab}\cos\alpha \end{aligned} \tag{1.2}$$

It can be shown that the duality transformation (1.2) leads to the conservation of the physically significant helicity current, cf. [3,5,6]. See also [2] and references therein. As discussed below, the local symmetries form a Lie algebra. Therefore in order to relate the duality symmetry to other symmetries using the Lie algebra, it is necessary to use a duality-symmetric Lagrangian to represent also other symmetries. When studying the variational aspects of the zilch symmetry we will discuss two alternative duality-symmetric Lagrangians, one real and one complex described in Sect.3. These Lagrangians depend on two vector potentials, the standard one, A_{a}, and a dual "magnetic" one, C_{a}. Since the two formulations are essentially equivalent we treat the real formulation in detail and give a shortened version of the complex formulation for completeness.

Cameron and Barnett [5] discussed the variation of such a duality-symmetric Lagrangian with respect to a transformation which in our formalism corresponds to characteristics of the form $Q_{abc} = F_{c(a,b)}$. In doing so, they showed that the Lagrangian was left invariant, but only modulo the Euler-Lagrange equations (see [5, Sect.4]). Therefore their analysis did not in fact establish the variational character of the transformation according to the Noether theory as described above. Nevertheless, we will show that the transformation they discussed is a variational symmetry as defined in [12, Definition 5.51].

In addition to the works mentioned above, a number of authors have discussed and classified symmetries and conservation laws of the Maxwell equations without considering possible relations to a variational formulation, notably Fushchich and Nikitin [7], Pohjanpelto [15], Anco and Pohjanpelto [1].

An important aspect of the characteristic form relation (1.1) is that it is an identity. In this work we will be careful to make the important distinction in the Noether theory between equations which are only valid modulo solutions of the field equations (then said to be valid *on-shell*) and those that are identities (then said to be valid *off-shell*). For example, equation (1.1) implies the on-shell equation

$$\frac{\partial Z_{ab}{}^{c}}{\partial x^{c}} \doteq 0 \;, \tag{1.3}$$

where the notation "$\doteq$" signifies the on-shell restriction of the equality. When dealing with symmetries and conservation laws, a common situation is that a given Lagrangian has a known symmetry and one can then derive a corresponding conservation law more or less straightforwardly using Noether's theory. It may happen though, as is the case with the zilch tensor, that a conservation law is known but not

the responsible Noether symmetry. In this latter case one can look for a relation of the type (1.1) and in that way identify the corresponding variational symmetry. However, finding the symmetry using this reverse Noether procedure is not algorithmic and can be quite difficult. The underlying reason is that neither the symmetry nor the conservation law is uniquely defined. Rather they are only determined up to equivalence. The difference between any two equivalent characteristics is said be trivial if it is zero on-shell. For conservation laws, there are two kinds of trivialities. The difference between two conserved quantities is said be trivial of the first kind if it zero on-shell and trivial of the second kind if its divergence is identically zero regardless of any field equations. The problem is that in general, the characteristic form (1.1) of a conservation law is not preserved if the conserved quantity is replaced by an equivalent one.

The mathematical technique adopted in this note is the jet space formalism (Olver [12]) in which a symmetry is represented by a linear differential operator. These symmetries are generated by vector fields on a (finite-dimensional) jet space. They form a Lie algebra[3] and the symmetry structure can in principle be understood by exploring that algebra without the need for a Hamiltonian formulation. For the Maxwell theory, the basic n-jet space has the form $M^{(n)} = M \times U_n$ where $M = (x^a, A_a)$ consists of the spacetime coordinates and the components of the 4-potential while each U_k ($k \leq n$) consists of the components of the k'th order derivatives of A_a. The elements of $M^{(n)}$ are to be considered as independent in calculations. By including derivatives of any finite order, this formalism is not restricted to point symmetries but can also handle generalized symmetries as needed for the zilch tensor for example. In general, a symmetry generator has the form

$$\xi^a \frac{\bar{\partial}}{\bar{\partial} x^a} + \phi_a \frac{\partial}{\partial A_a} \tag{1.4}$$

where ξ^a and ϕ_a are functions on M for point symmetries and on some jet space $M^{(n)}$ for generalized symmetries. The bars on the first derivative in (1.4) serves to indicate that $\bar{\partial}/\bar{\partial} x^a$ is restricted to act only on explicit occurrences of x^a.[4] For a point symmetry, v is a vector field on M which generates a Lie group of transformations. Any symmetry can be represented by a single field $Q_a = \phi_a - A_{a,b}\xi^b$ with the corresponding generator

$$v = Q_a \frac{\partial}{\partial A_a} \tag{1.5}$$

where the characteristic Q_a is a jet space function. It could depend on the coordinates x^a, A_a and derivatives of A_a up to a finite order. For the action of a symmetry on the Lagrangian or the Euler-Lagrange equations one needs to prolong the symmetry (1.5) (or (1.4)) to take into account its action also on the derivatives of A_a. For functions on a jet space $M^{(n)}$, that prolongation is given by

$$\mathsf{pr}^{(n)}v = v + \sum_{k=1}^{n} Q_{a,b_1...b_k} \frac{\partial}{\partial A_{a,b_1...b_k}} \; . \tag{1.6}$$

The condition for v to be a variational symmetry for a Lagrangian L is

$$\mathsf{pr}^{(n)}v(L) = U^a{}_{,a} \tag{1.7}$$

---

[3]See also [15] for a jet space tutorial adapted to the Maxwell equations.

[4]For reference we note that Olver in [12] uses an opposite convention and instead refers to the standard partial coordinate derivative as a "total derivative" D_a. Specifically, the relation to our notation is $(\partial/\partial x^a, \bar{\partial}/\bar{\partial} x^a) \leftrightarrow (D_a, \partial/\partial x^a)$. Our use of the comma notation will always stand for the standard partial derivative, as in $A_{a,b} = \partial A_a/\partial x^b$ for example.

for some jet space field U^a [12, Ch.5]. We emphasize that (1.7) must be an identity, i.e. be valid off-shell. The corresponding conserved current is then given by

$$J^a = -Q_b \frac{\partial L}{\partial A_{b,a}} + U^a \,. \tag{1.8}$$

In the case of the zilch tensor, it was first established as a collection of conserved currents by Lipkin [9] without any known responsible symmetry. In general, if a conserved current is a variational symmetry for a Lagrangian L, the conservation law takes the form

$$J^a{}_{,a} = Q_a \mathcal{E}^a \tag{1.9}$$

where the characteristic Q_a and $\mathcal{E}^a \doteq 0$ are jet space functions. The simplest case occurs when the relation has the characteristic form

$$J^a{}_{,a} = Q_a \mathsf{E}^a(L) \,. \tag{1.10}$$

This form of the conservation law therefore neatly expresses the two-way nature of Noether's theorem:

variational symmetry $\Leftrightarrow$ conservation law

For currents depending on higher derivatives of the dependent variables, such as the zilch tensor, the conservation law corresponding to (1.9) can also include derivatives of the field equations. In that case one can recover the responsible symmetry generator by performing partial integrations leading to a modified current $\tilde{J}^a$ obeying (1.10). The modified current will then differ from the original by a part $K^a = \tilde{J}^a - J^a$, which vanishes on-shell, $K^a \doteq 0$. The difference K^a itself is said to be a trivial conservation law of the first kind and the conservation laws J and $\tilde{J}$ are then said to be equivalent. This procedure will be carried out explicitly for the zilch tensor in Sect.3 where the characteristic form for the zilch conservation law of the type (1.10) will be established.

2. The zilch tensor

In this section we display for reference the basic algebraic and differential properties of the zilch tensor. Its original covariant form given by Lipkin was subsequently simplified by Morgan [10] and Kibble [8] to the form

$$Z_{abc} = {}^\star F_{ad} F^d{}_{b,c} - F_{ad} {}^\star F^d{}_{b,c} \,. \tag{2.1}$$

Although not apparent from this form, the zilch tensor is symmetric and traceless with respect to its first two indices

$$Z_{abc} = Z_{(ab)c} \,, \quad Z^a{}_{ac} = 0 \,. \tag{2.2}$$

To show this, it is convenient to express (2.1) in the (vector valued) matrix form as

$$Z_c = {}^\star F F_{,c} - F {}^\star F_{,c} \tag{2.3}$$

where F has components $F^a{}_b$. Following Kibble [8] we use the following identity (valid for any antisymmetric matrix) to verify (2.2)

$$F {}^\star F = \tfrac{1}{4} \operatorname{tr}(F {}^\star F) \,. \tag{2.4}$$

Differentiating this expression gives

$$F' {}^\star F + F {}^\star F' = \tfrac{1}{2} \operatorname{tr}({}^\star F F') \,. \tag{2.5}$$

We can then write the zilch tensor in the form

$$Z_c = {}^\star F F_{,c} + F_{,c} {}^\star F - \tfrac{1}{2} \operatorname{tr}({}^\star F F_{,c}) = \{{}^\star F, F_{,c}\} - \tfrac{1}{2} \operatorname{tr}({}^\star F F_{,c}) \tag{2.6}$$

using the anticommutator[5] notation in the last equality. In components this expression reads

$$Z_{abc} = 2\,{}^{\star}F^d{}_{(a}F_{b)d,c} - \tfrac{1}{2}g_{ab}\,{}^{\star}F^d{}_e\,F^e{}_{d,c} \tag{2.7}$$

in which the symmetry of its first two indices is manifest. The traceless property with respect to its first two indices also follows directly.

To show the validity of the conservation law $Z_{ab}{}^c{}_{,c} \doteq 0$, Kibble used a third form of the zilch tensor which follows from (2.1) and (2.2) and is given by[6] [8]

$$Z_{abc} = {}^{\star}F^d{}_{(a}F_{b)d,c} - F^d{}_{(a}\,{}^{\star}F_{b)d,c} \ . \tag{2.8}$$

In matrix notation this corresponds to

$$Z_c = \tfrac{1}{2}\{{}^{\star}F, \partial_c F\} - \tfrac{1}{2}\{F, \partial_c\,{}^{\star}F\} \ . \tag{2.9}$$

Taking the divergence gives

$$Z^c{}_{,c} = \tfrac{1}{2}\{{}^{\star}F, \Box F\} - \tfrac{1}{2}\{F, \Box\,{}^{\star}F\} \ . \tag{2.10}$$

where the box operator stands for the d'Alembertian $\Box = g^{ab}\partial_a\partial_b$. Expressing this divergence in components, the zilch conservation law then reads

$$Z_{ab}{}^c{}_{,c} = {}^{\star}F^d{}_{(a}\,\Box F_{b)d} - F^d{}_{(a}\,\Box\,{}^{\star}F_{b)d} \doteq 0 \ . \tag{2.11}$$

The conclusion that the divergence vanishes on solutions of the Maxwell equations follows from the fact that both $\Box F_{ab}$ and $\Box\,{}^{\star}F_{ab}$ vanish on-shell.

3. Duality-symmetric Maxwell Lagrangians

In this section we discuss two versions of duality-symmetric Lagrangians for the Maxwell theory, one real and one complex. In the duality-symmetric theory, the degrees of freedom are doubled. It becomes equivalent to standard Maxwell theory only after imposing a duality-constraint, see below. Our aim is to find the symmetry associated to the conservation of the zilch tensor and showing that it is a variational symmetry, i.e. the action which gives rise to the Maxwell equation is invariant under that symmetry. It turns out that the standard Lagrangian fails to be invariant under the zilch symmetry but the duality-symmetric Lagrangian is invariant. We will further demonstrate that the zilch tensor is equivalent to the Noether current associated to this variational symmetry applied to the duality-symmetric Lagrangian.

The standard Maxwell Lagrangian is given by

$$L = \tfrac{1}{4}\,\mathrm{tr}(F^2) = \tfrac{1}{4}F^a{}_b F^b{}_a = -\kappa^{abcd}A_{a,b}A_{c,d} \tag{3.1}$$

where $F_{ab} = -2A_{[a,b]}$ is the field tensor expressed in terms of the 4-potential A_a and $\kappa_{abcd} = g_{c[a}g_{b]d}$ is the antisymmetry projector.[7] The Euler-Lagrange equations then become

$$\mathsf{E}^a(L) = -F^{ab}{}_{,b} = 0 \tag{3.2}$$

where E^a is the (first order) Euler operator

$$\mathsf{E}^a = \frac{\partial}{\partial A_a} - \partial_b\frac{\partial}{\partial A_{a,b}} \tag{3.3}$$

The zilch tensor is invariant under the duality transformation (1.2) but the standard Maxwell Lagrangian (3.1) is not. To treat the zilch conservation law, we will consider two versions of duality-symmetric Lagrangians, one using a two-potential

---

[5]For any two matrices H and K the anticommutator is defined by $\{H, K\} = HK + KH$.

[6]Kibble [8] expressed this form in a different somewhat non-standard notation.

[7]This tensor projects out the antisymmetric part of any index pair, e.g. $M_{[ab]} = \kappa_{ab}{}^{cd}M_{cd}$. It has the same index symmetries as the Riemann tensor, $\kappa_{abcd} = \kappa_{cdab} = \kappa_{[ab]cd}$ and $\kappa_{a[bcd]} = 0$.

real formalism and one using complex variables, both of these introduced in [3]. The complex field tensor is defined by

$$\mathcal{F}_{ab} = \tfrac{1}{2}(F_{ab} + i{}^\star F_{ab}) \ . \tag{3.4}$$

The duality transformation now takes the simple form

$$\mathcal{F}_{ab} \to e^{-i\alpha}\,\mathcal{F}_{ab} \ . \tag{3.5}$$

To go from the standard Lagrangian to one which is duality-symmetric we can start from the relations

$$F^2 = (\mathcal{F} + \bar{\mathcal{F}})^2 = X_1 + X_2 \tag{3.6}$$

where

$$X_1 = \mathcal{F}^2 + \bar{\mathcal{F}}^2 = \tfrac{1}{2}[F^2 - ({}^\star F)^2] \tag{3.7}$$

$$X_2 = 2\mathcal{F}\bar{\mathcal{F}} = \tfrac{1}{2}[F^2 + ({}^\star F)^2] \tag{3.8}$$

using the fact that $\mathcal{F}$ and $\bar{\mathcal{F}}$ commute which in turn follows from the following identity, valid for any antisymmetric matrix

$$F\,{}^\star F = \tfrac{1}{4}\operatorname{tr}(F\,{}^\star F)\,\mathrm{I} \tag{3.9}$$

where I is the unit matrix. Note that X_2 is duality-invariant and is essentially the (symmetric) stress-energy tensor which is given by

$$T = -2\mathcal{F}\bar{\mathcal{F}} = -\tfrac{1}{2}(F^2 + {}^\star F^2) \ . \tag{3.10}$$

It follows that its trace vanishes identically. Therefore, in the standard Maxwell theory we have that $X_1 = 4L$. However, we can have a consistent duality-symmetric formulation if we treat F_{ab} and $G_{ab} := {}^\star F_{ab}$ as independent quantities. They must then also have independent potentials leading to the introduction of a magnetic potential C_a such that $G_{ab} = -2C_{[a,b]}$. Then, formally, $\operatorname{tr} X_2$ does not vanish as long as the identification $G_{ab} = {}^\star F_{ab}$ is not imposed. Referring to (3.6) we will therefore used scaled versions of $\operatorname{tr} T$ as duality-symmetric Lagrangians in the following. In addition to the duality transformation (1.2), a corresponding transformation at the potential level is also needed for expressions involving the potentials. It has the analogous form

$$\begin{aligned} A_a &\to \ \ A_a\cos\alpha + C_a\sin\alpha \\ C_a &\to -A_a\sin\alpha + C_a\cos\alpha \end{aligned} \tag{3.11}$$

We emphasize that due to the additional degrees of freedom represented by C_a and its corresponding field strength G_{ab}, the duality-symmetric formulation of Maxwell theory considered below is an extension of standard Maxwell theory, which becomes equivalent to the standard theory only after imposing the duality constraint

$$G_{ab} = *F_{ab} \ . \tag{3.12}$$

4. Real duality-symmetric formulation of Maxwell theory

Based on the above discussion (cf. also [3,5]) we define the real duality-symmetric Lagrangian

$$\mathcal{L}_{\mathrm{R}} = -\tfrac{1}{8}(F_{ab}F^{ab} + G_{ab}G^{ab}) = -\tfrac{1}{2}\kappa^{abcd}(A_{a,b}A_{c,d} + C_{a,b}C_{c,d}) \ . \tag{4.1}$$

Using the notations $M^a = \mathsf{E}_{\mathrm{A}}^a(\mathcal{L}_{\mathrm{R}})$ and $N^a = \mathsf{E}_{\mathrm{C}}^a(\mathcal{L}_{\mathrm{R}})$, the Euler-Lagrange expressions are

$$\begin{aligned} M_a &= \tfrac{1}{2}F^b{}_{a,b} = \tfrac{1}{4}\Box A_a - \tfrac{1}{4}(\operatorname{div} A)_{,a} \ , \\ N_a &= \tfrac{1}{2}G^b{}_{a,b} = \tfrac{1}{4}\Box C_a - \tfrac{1}{4}(\operatorname{div} C)_{,a} \ . \end{aligned} \tag{4.2}$$

We now introduce the following expression,

$$\mathcal{Z}_c = \tfrac{1}{2}\{G, \partial_c F\} - \tfrac{1}{2}\{F, \partial_c G\} \ . \tag{4.3}$$

which upon imposing the duality constraint (3.12) becomes the zilch tensor (2.9). In component form, (4.3) is

$$\mathcal{Z}_{abc} = G^d{}_{(a} F_{b)d,c} - F^d{}_{(a} G_{b)d,c} \ . \tag{4.4}$$

Its divergence is then given by

$$\mathcal{Z}^c{}_{,c} = \tfrac{1}{2}\{G, \Box F\} - \tfrac{1}{2}\{F, \Box G\} \tag{4.5}$$

and in components by

$$\mathcal{Z}_{ab}{}^c{}_{,c} = G^d{}_{(a} \Box F_{b)d} - F^d{}_{(a} \Box G_{b)d} \, . \tag{4.6}$$

4.1. Characteristic form of the Zilch conservation law. Using the relations

$$\Box F_{ab} = -4 M_{[a,b]} \ , \qquad \Box G_{ab} = -4 N_{[a,b]} \ , \tag{4.7}$$

we can write the divergence of the zilch tensor in the form

$$\mathcal{Z}_{ab}{}^c{}_{,c} = -4 G_{(a}{}^d \kappa_{b)d}{}^{ef} M_{e,f} + 4 F_{(a}{}^d \kappa_{b)d}{}^{ef} N_{e,f} \ . \tag{4.8}$$

We note that this equation is not in the desired characteristic form since it is not proportional to the Euler-Lagrange expressions but rather to the derivatives of those expressions. However, we can deal with this issue by performing partial integrations leading to

$$\mathcal{Z}_{ab}{}^c{}_{,c} = -(4 G_{(a}{}^e \kappa_{b)e}{}^{cd} M_c)_{,d} - 2\, G_{c(a,b)} M^c + (4 F_{(a}{}^e \kappa_{b)e}{}^{cd} N_c)_{,d} + 2\, F_{c(a,b)} N^c \, . \tag{4.9}$$

This relation can be expressed in the characteristic form

$$\tilde{\mathcal{Z}}_{ab}{}^c{}_{,c} = -2\, G_{c(a,b)} M^c + 2\, F_{c(a,b)} N^c \tag{4.10}$$

for the modified zilch tensor

$$\tilde{\mathcal{Z}}_{ab}{}^c = \mathcal{Z}_{ab}{}^c - 4 G_{(a}{}^e \kappa_{b)e}{}^{cd} M_d + 4 F_{(a}{}^e \kappa_{b)e}{}^{cd} N_d \ . \tag{4.11}$$

which is equal to the zilch tensor $\mathcal{Z}_{ab}{}^c$ on-shell in the duality-symmetric theory. The difference $\mathcal{Z}_{ab}{}^c - \tilde{\mathcal{Z}}_{ab}{}^c$ is itself trivially conserved being zero on-shell (referred to as triviality of the first kind in [12]).

We can now identify the coefficients of M^c and N^c in (4.10) with the components of the characteristic as

$$Q_{abc} = -2\, G_{c(a,b)} \ , \qquad P_{abc} = 2\, F_{c(a,b)} \ , \tag{4.12}$$

leading to the Noether symmetry candidate

$$\begin{aligned} v_{ab} &= Q_{abc}\, \frac{\partial}{\partial A_c} + P_{abc}\, \frac{\partial}{\partial C_c} \\ &= -2\, G_{c(a,b)}\, \frac{\partial}{\partial A_c} + 2\, F_{c(a,b)}\, \frac{\partial}{\partial C_c} \ , \end{aligned} \tag{4.13}$$

which is invariant with respect to the duality transformation (3.11). The infinitesimal transformations of potential fields under (4.13) are

$$A_c \to A_c - \zeta^{ab}\, G_{ca,b} \tag{4.14a}$$

$$C_c \to C_c + \zeta^{ab}\, F_{ca,b} \tag{4.14b}$$

from which, using Bianchi identities, we have

$$F_{cd} \to F_{cd} + \zeta^{ab}\, G_{cd,ab} \tag{4.15a}$$

$$G_{cd} \to G_{cd} - \zeta^{ab}\, F_{cd,ab} \tag{4.15b}$$

for an infinitesimal symmetric matrix ζ^{ab} of parameters. The transformation generated by (4.14), (4.15) is the same as in [5, Eq. (6.20), (6.21)]. See [5, Sect. 6.3] for discussion and references. These symmetries arise from the modified zilch tensor $\tilde{\mathcal{Z}}_{ab}{}^c$ in the duality-symmetric theory, defined in (4.10). As we shall demonstrate in the next section, the action of the symmetry (4.14), (4.15) on the duality-symmetric Lagrangian is a total divergence. As emphasized above for Eq. (1.7) this must hold off-shell in order to qualify as a variational symmetry in accordance with the Noether theorem.

4.2. **Zilch-related variational symmetry in the real formulation.** We show that the duality-symmetric Lagrangian (4.1) is invariant under the symmetry generated by the generalized vector field (4.13) and the conserved current derived from that symmetry is indeed the zilch current. The prolongation of (4.13) acting on the duality symmetric Lagrangian gives

$$\mathsf{pr}v_{ab}(\mathcal{L}) = Q_{abc,d}\frac{\partial \mathcal{L}}{\partial A_{c,d}} + P_{abc,d}\frac{\partial \mathcal{L}}{\partial C_{c,d}} = \mathcal{U}_{ab}{}^c{}_{,c} \tag{4.16}$$

where

$$\mathcal{U}_{ab}{}^c = -\tfrac{1}{2}\,\delta_{(a}{}^c\,G^{de}{}_{,b)}F_{de} + \tfrac{1}{2}\delta_{(a}{}^c\,F^{de}{}_{,b)}G_{de} \tag{4.17}$$

and we have used the identities

$$G^{cd}\,F_{bd,c,a} = \tfrac{1}{2}\,G^{cd}\,F_{cd,a,b} \quad \text{and} \quad F^{cd}\,G_{bd,c,a} = \tfrac{1}{2}\,F^{cd}\,G_{cd,a,b}\,, \tag{4.18}$$

which are consequences of the Bianchi identities $F_{[ab,c]} = G_{[ab,c]} = 0$. The fact that (4.16) is a total divergence, implies that the conservation of the zilch tensor comes from a variational symmetry of the duality-symmetric Lagrangian. We note that equation (4.16) is valid off-shell in the duality-symmetric theory, as required in the Noether theory. We note that Cameron and Barnett in [5] considered only symmetries valid on-shell.

The Noether current generated by v_{ab} is

$$\begin{aligned}
\mathcal{Z}'_{ab}{}^c &= -Q_{abd}\frac{\partial \mathcal{L}}{\partial A_{d,c}} - P_{abd}\frac{\partial \mathcal{L}}{\partial C_{d,c}} + \mathcal{U}_{ab}{}^c \\
&= -F^{cd}G_{d(a,b)} + G^{cd}F_{d(a,b)} + \mathcal{U}_{ab}{}^c\,,
\end{aligned} \tag{4.19}$$

where Q_{abc}, P_{abc} are given in (4.12), and $U_{ab}{}^c$ is the expression under divergence in (4.16). The current (4.19) is equivalent to the modified zilch tensor (4.11), which can be seen from the fact that their difference

$$\mathcal{Z}'_{ab}{}^c - \tilde{\mathcal{Z}}_{ab}{}^c = 2\left(F_{(a}{}^{[c}G_{b)}{}^{d]} - \delta_{(a}{}^{[c}G_{b)e}F^{d]e} + \delta_{(a}{}^{[c}F_{b)e}G^{d]e}\right)_{,d} \tag{4.20}$$

is obviously a trivial current, i.e. one whose divergence (with respect to index c) vanishes identically (triviality of the second kind).

Another fact about the current (4.19) is that for solutions of Maxwell equations, for which $G_{ab} = {}^\star F_{ab}$, the last term $\mathcal{U}_{ab}{}^c$ vanishes, as can be seen by a short calculation. The remaining terms in (4.19) result in

$$-F^{cd\star}F_{d(a,b)} + {}^\star F^{cd}F_{d(a,b)}\,, \tag{4.21}$$

which we shall denote by $Z'_{ab}{}^c$. It is an equivalent form of the zilch tensor introduced by Anco and Pojanpelto[8] in [1]. From (4.20) and (4.11), one can find the difference

---

[8]Our definition of $Z'_{ab}{}^c$ is related to Anco and Pojanpelto's by an overall minus sign and reordering the indices to agree with our convention for the conservation law $Z'_{ab}{}^c{}_{,c} \doteq 0$.

between the above current and the form of zilch tensor introduced by Kibble in (2.1) as

$$Z'_{ab}{}^c - Z_{ab}{}^c = -4^\star F_{(a}{}^e \kappa_{b)e}{}^{cd} M_d + 2 \left(F_{(a}{}^{[c\star} F_{b)}{}^{d]} - \delta_{(a}{}^{[c\star} F_{b)e} F^{d]e} + \delta_{(a}{}^{[c} F_{b)e}{}^\star F^{d]e} \right)_{,d} .$$ (4.22)

These two forms of the zilch tensor are equivalent in the sense that they contain the same information (cf. [13, p.143]). This is expressed by the relations

$$Z'_{abc} = Z_{c(ab)}$$ (4.23)

and its converse

$$Z_{abc} = -2Z'_{abc} + 3Z'_{(abc)} .$$ (4.24)

5. Complex duality-symmetric formulation of Maxwell theory

Using a complex vector potential $\mathcal{A}_a = A_a + iC_a$, Maxwell's equations can be derived from the duality-symmetric Lagrangian (*cf.* [3])

$$\mathcal{L} = -\tfrac{1}{2}\mathcal{F}_{ab}\bar{\mathcal{F}}^{ab} = -\tfrac{1}{2}\kappa^{abcd}\mathcal{A}_{a,b}\bar{\mathcal{A}}_{c,d}$$ (5.1)

where $\mathcal{F}_{ab} = -\mathcal{A}_{[a,b]}$ and the bar denotes complex conjugation. In [3], the real and imaginary parts of $\mathcal{A}_a$ were taken as the basic field variables. Here, we will instead use $\mathcal{A}_a$ and $\bar{\mathcal{A}}_a$ as the basic fields. Since they represent linearly independent combinations of A_a and C_a, they can be used as alternative field variables. They carry the same number of degrees of freedom (4 each) and are treated as formally independent. Defining $\mathcal{M}^a = \mathsf{E}^a_{\mathcal{A}}(\mathcal{L})$, where $\mathsf{E}_{\mathcal{X}}$ is the Euler operator for a variable $\mathcal{X}$, the Euler-Lagrange expressions are given by

$$\mathcal{M}^a = -\frac{\partial}{\partial x^b}\frac{\partial\mathcal{L}}{\partial\mathcal{A}_{a,b}} = \tfrac{1}{2}\bar{\mathcal{F}}^{ba}{}_{,b}$$ (5.2)

and $\mathsf{E}^a_{\bar{\mathcal{A}}}(\mathcal{L}) = \bar{\mathcal{M}}^a$. Expressed in terms of the potentials they become

$$\mathcal{M}_a = \tfrac{1}{4}\Box\bar{\mathcal{A}}_a - \tfrac{1}{4}(\operatorname{div}\bar{\mathcal{A}})_{,a} .$$ (5.3)

5.1. **Complex formulation of zilch.**
The complex duality-symmetric formulation of zilch conservation parallels that of the real formulation in Sect. 4. Rewriting the zilch expression (4.3) in terms of $\mathcal{F}^a{}_b$ gives

$$\mathcal{Z}_c = i\{\bar{\mathcal{F}}, \partial_c\mathcal{F}\} + \text{c.c.}$$ (5.4)

or in component form

$$\mathcal{Z}_{abc} = 2i\bar{\mathcal{F}}^d{}_{(a}\mathcal{F}_{b)d,c} + \text{c.c.}$$ (5.5)

where "c.c." stands for the complex conjugate of the preceding expression after the equality sign. Its divergence is then given by

$$\mathcal{Z}^c{}_{,c} = i\{\bar{\mathcal{F}}, \Box\mathcal{F}\} + \text{c.c.}$$ (5.6)

or in components

$$\mathcal{Z}_{ab}{}^c{}_{,c} = 2i\bar{\mathcal{F}}^d{}_{(a}\Box\mathcal{F}_{b)d} + \text{c.c.}$$ (5.7)

5.2. **Characteristic form of the Zilch conservation law in the complex formulation.**
Using the relation

$$\Box\mathcal{F}_{ab} = -4\bar{\mathcal{M}}_{[a,b]},$$ (5.8)

and its complex conjugate, we express (5.7) in the form (cf. (4.9))

$$\mathcal{Z}_{ab}{}^c{}_{,c} = -4i\mathcal{F}_a{}^d \mathcal{M}_{[d,b]} - 4i\mathcal{M}_{[a,d]} \mathcal{F}^d{}_b + \text{c.c.} .$$ (5.9)

Performing partial integration now leads to a conservation law in the characteristic form

$$\tilde{\mathcal{Z}}_{ab}{}^c{}_{,c} = \mathcal{Q}_{ab}{}^c\mathcal{M}_c + \text{c.c.}$$ (5.10)

where

$$\tilde{\mathcal{Z}}_{abc} = \mathcal{Z}_{abc} - 8i\mathcal{F}_{e(a}\kappa_{b)}{}^{ecd}\mathcal{M}_d \tag{5.11}$$

and

$$\mathcal{Q}_{abc} = -4i\mathcal{F}_{c(a,b)} \tag{5.12}$$

is the characteristic. The corresponding Noether symmetry candidate is then given by

$$v_{ab} = \mathcal{Q}_{abc}\frac{\partial}{\partial\mathcal{A}_c} + \bar{\mathcal{Q}}_{abc}\frac{\partial}{\partial\bar{\mathcal{A}}_c} \ , \tag{5.13}$$

which is manifestly duality invariant.

5.3. **Zilch-related variational symmetry in the complex formulation.** We show that the duality-symmetric Lagrangian (5.1) is invariant under the symmetry generated by the generalized vector field (5.13) and the conserved current derived from that symmetry is indeed the zilch current. The prolongation of (5.13) acting on duality-symmetric Lagrangian gives

$$\mathsf{pr}v_{ab}(\mathcal{L}) = \mathcal{Q}_{abc,d}\frac{\partial\mathcal{L}}{\partial\mathcal{A}_{c,d}} + \bar{\mathcal{Q}}_{abc,d}\frac{\partial\mathcal{L}}{\partial\bar{\mathcal{A}}_{c,d}} = \mathcal{U}_{ab}{}^c{}_{,c} \ , \tag{5.14}$$

where

$$\mathcal{U}_{ab}{}^c = -i\delta_{(a}{}^c\mathcal{F}^{de}{}_{,b)}\bar{\mathcal{F}}_{de} + \text{c.c.} \tag{5.15}$$

and we have used the identity $\bar{\mathcal{F}}^{cd}\mathcal{F}_{bd,c,a} = \frac{1}{2}\bar{\mathcal{F}}^{cd}\mathcal{F}_{cd,a,b}$, which is the consequence of the Bianchi identity $\mathcal{F}_{[ab,c]} = 0$. To verify that this symmetry results in the zilch tensor as its conserved current, we note that the conserved current is given by

$$\mathcal{Z}_{ab}{}^c = -\mathcal{Q}_{abd}\frac{\partial\mathcal{L}}{\partial\mathcal{A}_{d,c}} - \bar{\mathcal{Q}}_{abd}\frac{\partial\mathcal{L}}{\partial\bar{\mathcal{A}}_{d,c}} + \mathcal{U}_{ab}{}^c. \tag{5.16}$$

To convert this to a current for solutions of the Maxwell equations, we need to impose the duality constraint (3.12) which breaks the independence of $\mathcal{F}_{ab}$ and $\bar{\mathcal{F}}_{ab}$ that has been assumed in this section. One then finds that on-shell $\mathcal{U}_{ab}{}^c$ vanishes and the remaining terms in (5.16) result in the equivalent zilch tensor

$$Z'_{ab}{}^c = 2i\bar{\mathcal{F}}_{d(a,b)}\mathcal{F}^{dc} - 2i\mathcal{F}_{d(a,b)}\bar{\mathcal{F}}^{dc} \ . \tag{5.17}$$

6. Concluding remarks

We have discussed how the conservation of the zilch tensor is related to a variational symmetry in accordance with the Noether theory. The fact that the zilch tensor is conserved has been known since its discovery by Lipkin [9]. In this paper, we have derived the variational symmetry generator for a version of the zilch tensor in the duality-symmetric theory. The existence of this symmetry generator is guaranteed by Noether's theorem. The proof of its variational nature in accordance with (1.9) is given by (4.16). We note that the generator itself was given by Cameron and Barnett [5], however without proof of its variational nature.

To achieve the direct correspondence between the symmetry generator and the conservation law in the characteristic form (1.10), it was necessary to augment the zilch tensor by a trivial addition which vanishes on-shell, see (4.11). This feature is a reflection of the fact that symmetries and corresponding conservation laws are in general only determined as equivalence classes of symmetries and conservation laws. Members of an equivalence class of symmetries or conservation laws can differ by trivial symmetries or trivial conservation laws, respectively [12]. Finally, we note that the considerations here do not rule out the possibility that the standard Lagrangian could be invariant under a symmetry that generates the zilch tensor as a Noether current.

Appendix A. Alternative forms of the zilch tensor and their equivalence

For ease of reference, we collect here the different forms of the zilch tensor used in the paper. In section 2 we discussed the three versions of the zilch tensor introduced by Kibble [8] in standard Maxwell theory,

$$Z_{abc} = {}^\star F_{ad} F^d{}_{b,c} - F_{ad} {}^\star F^d{}_{b,c} \tag{A.1}$$

$$= 2 {}^\star F^d{}_{(a} F_{b)d,c} - \tfrac{1}{2} g_{ab} {}^\star F^d{}_e F^e{}_{d,c} \tag{A.2}$$

$$= {}^\star F^d{}_{(a} F_{b)d,c} - F^d{}_{(a} {}^\star F_{b)d,c} . \tag{A.3}$$

Applying the symmetry transformation associated to the conservation of the zilch tensor according to Noether's theorem, we encountered in (4.21) another version of the zilch tensor,

$$Z'_{ab}{}^c = {}^\star F^{cd} F_{d(a,b)} - F^{cd} {}^\star F_{d(a,b)} , \tag{A.4}$$

cf. [1]. As we stated in section 4.2, these are equivalent currents:

$$Z'_{ab}{}^c - Z_{ab}{}^c = -4 {}^\star F_{(a}{}^e \kappa_{b)e}{}^{cd} M_d + 2 \left(F_{(a}{}^{[c\star} F_{b)}{}^{d]} - \delta_{(a}{}^{[c\star} F_{b)e} F^{d]e} + \delta_{(a}{}^{[c} F_{b)e} {}^\star F^{d]e} \right)_{,d} . \tag{A.5}$$

The original definition of zilch tensor was made by Lipkin [9] in a form that can be written more compactly (using the dual field) as

$$ {}^\star F^{cd} F_{d(a,b)} - F^{cd} {}^\star F_{d(a,b)} + \left({}^\star F_{(a}{}^c F_{b)}{}^d - F_{(a}{}^c {}^\star F_{b)}{}^d \right)_{,d} . \tag{A.6}$$

in which the third term is a trivial current and the first two terms coincide with Z' in (A.4).

Appendix B. $3+1$ decomposition of the zilch tensor

The components of Kibble's form (2.1) of the zilch tensor are

$$Z_{000} = u^a u^b u^c Z_{abc} = \mathbf{E} \cdot \dot{\mathbf{B}} - \mathbf{B} \cdot \dot{\mathbf{E}} \doteq -\mathbf{E} \cdot (\nabla \times \mathbf{E}) - \mathbf{B} \cdot (\nabla \times \mathbf{B}) \tag{B.1}$$

$$Z_{00i} = u^a u^b e_i^c Z_{abc} = (E^j B_{j,i} - B^j E_{j,i}) \doteq Z_{i00} + [(\mathbf{E} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{E}]_i \tag{B.2}$$

$$Z_{i00} = e_i^a u^b u^c Z_{abc} = (\mathbf{E} \times \dot{\mathbf{E}} + \mathbf{B} \times \dot{\mathbf{B}})_i \doteq [\mathbf{E} \times (\nabla \times \mathbf{B}) - \mathbf{B} \times (\nabla \times \mathbf{E})]_i \tag{B.3}$$

$$\begin{aligned} Z_{i0j} = Z_{0ij} &= u^a e_i^b e_j^c Z_{abc} = \epsilon_{ikl}(E^k E^l{}_{,j} + B^k B^l{}_{,j}) \\ &= -\delta_{ij}(\mathbf{E} \cdot (\nabla \times \mathbf{E}) + \mathbf{B} \cdot (\nabla \times \mathbf{B})) + (\nabla \times E)_i E_j + (\nabla \times B)_i B_j \\ &\quad - \epsilon_i{}^{kl}(E_{j,k} E_l + B_{j,k} B_l) \\ &\doteq Z_{ij0} - E_i(\nabla \times E)_j - B_i(\nabla \times B)_j - \epsilon_i{}^{kl}(E_{j,k} E_l + B_{j,k} B_l) \end{aligned} \tag{B.4}$$

$$\begin{aligned} Z_{ij0} = e_i^a e_j^b u^c Z_{abc} &= \delta_{ij}(\mathbf{E} \cdot \dot{\mathbf{B}} - \mathbf{B} \cdot \dot{\mathbf{E}}) + 2\dot{E}_{(i} B_{j)} - 2\dot{B}_{(i} E_{j)} \\ &\doteq \delta_{ij} Z_{000} + 2(\nabla \times B)_{(i} B_{j)} + 2(\nabla \times E)_{(i} E_{j)} \end{aligned} \tag{B.5}$$

$$\begin{aligned} Z_{ijk} = e_i^a e_j^b e_k^c Z_{abc} &= \delta_{ij}(E^l B_{l,k} - B^l E_{l,k}) - 2E_{(i} B_{j),k} + 2B_{(i} E_{j),k} \\ &= \delta_{ij} Z_{00k} - 2E_{(i} B_{j),k} + 2B_{(i} E_{j),k} \end{aligned} \tag{B.6}$$

References

[1] S. C. Anco and J. Pohjanpelto. Classification of local conservation laws of maxwell's equations. *Acta Applicandae Mathematica*, 69(3):285–327, Dec 2001.

[2] L. Andersson, S. Aghapour, and R. Bhattacharyya. Helicity and spin conservation in Maxwell theory and Linearized Gravity. 2018.

[3] K. Y. Bliokh, A. Y. Bekshaev, and F. Nori. Dual electromagnetism: helicity, spin, momentum and angular momentum. *New Journal of Physics*, 15(3):033026, mar 2013.

[4] M. G. Calkin. An invariance property of the free electromagnetic field. *American Journal of Physics*, 33(11):958–960, 1965.

[5] R. P. Cameron and S. M. Barnett. Electric-magnetic symmetry and noether's theorem. *New Journal of Physics*, 14(12):123019, dec 2012.

[6] S. Deser and C. Teitelboim. Duality transformations of abelian and non-abelian gauge fields. *Phys. Rev. D*, 13:1592–1597, Mar 1976.

[7] V. Fushchich and A. Nikitin. New and old symmetries of the maxwell and dirac equations. *Soviet Journal of Particles and Nuclei*, 14(1):1–22, 1983.

[8] T. W. B. Kibble. Conservation laws for free fields. *Journal of Mathematical Physics*, 6(7):1022–1026, 1965.

[9] D. M. Lipkin. Existence of a new conservation law in electromagnetic theory. *Journal of Mathematical Physics*, 5(5):696–700, 1964.

[10] T. A. Morgan. Two classes of new conservation laws for the electromagnetic field and for other massless fields. *Journal of Mathematical Physics*, 5(11):1659–1660, 1964.

[11] T. A. Morgan and D. W. Joseph. Tensor lagrangians and generalized conservation laws for free fields. *Il Nuovo Cimento (1955-1965)*, 39(2):494–503, Sep 1965.

[12] P. J. Olver. *Applications of Lie Groups to Differential Equations*. Springer-Verlag, 2nd ed. edition, 1993.

[13] R. Penrose and W. Rindler. *Spinors and space-time*, volume 1: *Two-spinor calculus and relativistic fields*. Cambridge University Press, 1986.

[14] T. G. Philbin. Lipkin's conservation law, noether's theorem, and the relation to optical helicity. *Phys. Rev. A*, 87:043843, Apr 2013.

[15] J. Pohjanpelto. *Symmetries, Conservation Laws and Maxwell's Equations*, pages 560–589. World Scientific, 1995.

[16] M. Przanowski, B. Rajca, and J. Tosiek. On some conservation laws in the maxwell electrodynamics in vacuum. *Acta Physica Polonica. Series B*, 25(7):1065–1077, 1994.

[17] G. Smith and P. Strange. Lipkin's conservation law in vacuum electromagnetic fields. *Journal of Physics A: Mathematical and Theoretical*, 51(43):435204, sep 2018.

[18] A. Sudbery. A vector lagrangian for the electromagnetic field. *Journal of Physics A: Mathematical and General*, 19(2):L33–L36, feb 1986.

[19] Y. Tang and A. E. Cohen. Optical chirality and its interaction with matter. *Phys. Rev. Lett.*, 104:163901, Apr 2010.

(S. Aghapour) Department of Physics, Sharif University of Technology, Azadi Ave. Tehran, Iran and School of Physics, Institute for Research in Fundamental Sciences (IPM), P.O.Box 19395-5531, Tehran, Iran
   E-mail address: s_aghapour@physics.sharif.edu

(L. Andersson) Albert Einstein Institute, Am Mühlenberg 1, D-14476 Potsdam, Germany
   E-mail address: laan@aei.mpg.de

(K. Rosquist) Department of Physics, Stockholm University, SE-106 91 Stockholm, Sweden
   E-mail address, K. Rosquist: kr@fysik.su.se