

An $SO(3)$ -monopole cobordism formula relating Donaldson and Seiberg–Witten invariants

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Paul Feehan dedicates this monograph to the memory of his parents, Martin and Odilla Feehan and to his wife, Julia Stanat, and daughter, Odilla Yseult Tyler Feehan.
Thomas Leness dedicates this monograph to his wife, Kirsten Wood.

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ABSTRACT. We prove an analogue of the Kotschick–Morgan Conjecture in the context of $\mathrm{SO}(3)$ monopoles, obtaining a formula relating the Donaldson and Seiberg–Witten invariants of smooth four-manifolds using the $\mathrm{SO}(3)$ -monopole cobordism. The main technical difficulty in the $\mathrm{SO}(3)$ -monopole program relating the Seiberg–Witten and Donaldson invariants has been to compute intersection pairings on links of strata of reducible $\mathrm{SO}(3)$ monopoles, namely the moduli spaces of Seiberg–Witten monopoles lying in lower-level strata of the Uhlenbeck compactification of the moduli space of $\mathrm{SO}(3)$ monopoles [27]. In this monograph, we prove — modulo a gluing theorem which is an extension of our earlier work in [24] — that these intersection pairings can be expressed in terms of topological data and Seiberg–Witten invariants of the four-manifold. Our proofs that the $\mathrm{SO}(3)$ -monopole cobordism yields both the Superconformal Simple Type Conjecture of Moore, Mariño, and Peradze [65, 64] and Witten’s Conjecture [103] in full generality for all closed, oriented, smooth four-manifolds with $b_1 = 0$ and odd $b^+ \geq 3$ appear in [25, 23].

Contents

Preface	ix
Acknowledgments	x
Chapter 1. Introduction	1
1.1. Summary of main results	1
1.2. Outline of the argument	4
1.2.1. Problem of overlaps	5
1.2.2. Overlap space and overlap maps	6
1.2.3. Associativity of splicing maps	7
1.2.4. Instanton moduli space with spliced ends	8
1.2.5. Space of global splicing data	9
1.2.6. Definition of link of a subspace of a moduli space of ideal Seiberg–Witten monopoles	9
1.2.7. Computation of intersection numbers with the link of the moduli space of ideal Seiberg–Witten monopoles	9
1.3. Kotschick–Morgan Conjecture	10
1.4. Outline of the monograph	11
Chapter 2. Preliminaries	13
2.1. The moduli space of $SO(3)$ monopoles	13
2.1.1. Clifford modules	13
2.1.2. $SO(3)$ monopoles	14
2.2. Stratum of anti-self-dual or zero-section solutions	16
2.3. Strata of Seiberg–Witten or reducible solutions	16
2.3.1. Seiberg–Witten monopoles	17
2.3.2. Seiberg–Witten invariants	17
2.3.3. Reducible $SO(3)$ monopoles	18
2.3.4. Circle actions	19
2.3.5. The virtual normal bundle of the Seiberg–Witten moduli space	19
2.4. Cohomology classes on the moduli space of $SO(3)$ monopoles	21
2.5. Donaldson invariants	22
2.6. Links and the cobordism	23
Chapter 3. Diagonals of symmetric products of manifolds	25
3.1. Definitions	25
3.1.1. Subgroups of the symmetric group	25
3.1.2. Definition of the diagonals	26
3.1.3. Strata of the symmetric product	27

3.2.	Incidence relations among diagonals and strata	27
3.3.	Normal bundles of diagonals and strata	28
3.4.	Enumeration of the strata	30
Chapter 4.	A partial Thom–Mather structure on symmetric products	31
4.1.	Introduction	31
4.2.	Diagonals in products of \mathbb{R}^4	32
4.3.	Families of metrics	35
4.4.	Overlap maps	37
4.4.1.	The downwards overlap map	38
4.4.2.	The upwards overlap map	38
4.4.3.	Commuting overlap maps	39
4.4.4.	The projection maps	40
4.5.	Construction of the families of locally flattened metrics	40
4.6.	Normal bundles of strata of $\text{Sym}^\ell(X)$	42
4.7.	The tubular distance function	44
4.8.	Decomposition of the strata	47
Chapter 5.	The instanton moduli space with spliced ends	51
5.1.	Introduction	51
5.2.	Connections over the four-dimensional sphere	53
5.3.	Strata containing the product connection	54
5.3.1.	Tubular neighborhoods	55
5.4.	The splicing map with the product connection over \mathbb{R}^4	57
5.5.	Composition of splicing maps	60
5.5.1.	Definition of the overlap data	61
5.5.2.	Equality of splicing maps	63
5.5.3.	Symmetric group actions and quotients	66
5.6.	The spliced end of the instanton moduli space	67
5.7.	Tubular neighborhoods of the instanton moduli space with spliced ends	74
5.8.	Isotopy of the spliced end of the instanton moduli space	76
5.9.	Properties of the instanton moduli space with spliced ends	78
Chapter 6.	The space of global splicing data	79
6.1.	Introduction	79
6.2.	Splicing data	80
6.2.1.	Background pairs	80
6.2.2.	Riemannian metrics	80
6.2.3.	Frame bundles	80
6.2.4.	Group actions on the frame bundles	81
6.2.5.	Space of splicing data	81
6.3.	The flattening map on pairs	82
6.4.	The crude splicing map	86
6.4.1.	The standard splicing map	87
6.4.2.	Construction of the crude splicing map	88
6.4.3.	Properties of the crude splicing map	89

6.5. Overlap spaces and maps	90
6.5.1. The overlap space	90
6.5.2. The upwards overlap map	92
6.5.3. Downwards overlap map	93
6.5.4. Equality of splicing maps	94
6.6. Construction of the space of global splicing data	96
6.7. Thom–Mather structures on the space of global splicing data	99
6.8. Global splicing map	105
6.9. Projections onto symmetric products	106
Chapter 7. Obstruction bundle	109
7.1. Introduction	109
7.2. Infinite-rank obstruction pseudo-bundle	111
7.3. Background obstruction bundle	112
7.4. Equivariant Dirac index bundle	114
7.5. The action of $\text{Spin}^u(4)$	114
7.6. Pseudo-bundle over the instanton moduli space with spliced ends	115
7.6.1. Pseudo-bundles and overlap data	116
7.7. Instanton obstruction pseudo-bundle	120
7.7.1. The frame bundles	120
7.7.2. Splicing map	121
7.7.3. Overlap space and overlap maps	122
7.8. Local gluing hypothesis for $\text{SO}(3)$ monopoles	124
7.9. Notes on the justification of the local gluing hypothesis	125
7.9.1. Construction of a virtual neighborhood for the moduli space of $\text{SO}(3)$ monopoles near a top-level singular stratum of $\text{SO}(3)$ monopoles	126
7.9.2. Virtual neighborhoods for the moduli space of anti-self-dual connections	127
7.9.3. Extrinsic virtual neighborhoods for the moduli space of anti-self-dual connections and gluing	130
7.9.4. Construction of a virtual neighborhood for the moduli space of $\text{SO}(3)$ monopoles near a lower-level singular stratum of $\text{SO}(3)$ monopoles	132
Chapter 8. Link of an ideal Seiberg–Witten moduli space	135
8.1. Definition of the link of an ideal Seiberg–Witten moduli space	135
8.1.1. The virtual link of an ideal Seiberg–Witten moduli space	135
8.1.2. The link of an ideal Seiberg–Witten moduli space	139
8.1.3. A subspace of the virtual link of an ideal Seiberg–Witten moduli space	140
8.1.4. Orientations of the link of an ideal Seiberg–Witten moduli space	140
8.1.5. An equality of intersection numbers provided by the $\text{SO}(3)$ -monopole cobordism	141
8.2. Fiber bundle structure of the instanton component of the link of an ideal Seiberg– Witten moduli space	141
8.3. Boundaries of components of links of ideal Seiberg–Witten moduli spaces	143
Chapter 9. Cohomology and duality	149
9.1. Introduction	149
9.2. Definitions	150

9.2.1.	Subspaces and maps	150
9.2.2.	The incidence locus	151
9.2.3.	Cohomology classes	152
9.3.	Fundamental class of the virtual link of the ideal moduli space of Seiberg–Witten monopoles	155
9.4.	Computation of the μ -classes	157
9.4.1.	Geometric representatives and cocycles	157
9.4.2.	Cocycles as pullbacks	158
9.4.3.	Computations of cocycles	159
9.5.	Relative Euler class of the obstruction pseudo-bundle	164
9.5.1.	Euler class of the Seiberg–Witten component of the obstruction pseudo-bundle	164
9.5.2.	Local Euler class of the instanton component of the obstruction bundle	164
9.5.3.	Global Euler class of the instanton component of the obstruction bundle	170
9.5.4.	Relative Euler classes	171
9.6.	Duality and the link of an ideal Seiberg–Witten moduli space	173
9.6.1.	The initial duality	174
9.6.2.	Extension of the cocycles	175
9.7.	Reduction to the subspace $\mathbf{BL}_{t,s}^{\text{vir}}$	178
Chapter 10.	Computation of the intersection numbers	185
10.1.	Introduction	185
10.2.	Quotient space of $\mathbf{BL}_{t,s}^{\text{vir}}$	187
10.2.1.	Quotient maps	188
10.2.2.	Construction of the local quotient	191
10.2.3.	Global quotient of $\mathbf{BL}_{t,s}^{\text{vir}}$	195
10.3.	Homology and cohomology classes of the quotient	197
10.4.	Fiber bundles and pushforwards	198
10.5.	Computations of intersection numbers on $\bar{\mathbf{L}}_{t,s}$	202
10.6.	Proofs of the main theorems	204
Chapter 11.	Kotschick–Morgan Conjecture	209
11.1.	Cobordisms and reducible connections	209
11.2.	Cohomology classes on the cobordism	210
11.3.	Neighborhoods of gauge-equivalence classes of ideal reducible connections	211
11.3.1.	Kuranishi model for a neighborhood of a reducible connection	211
11.3.2.	Crude splicing maps	212
11.3.3.	Overlap spaces and maps	213
11.3.4.	Definition of the neighborhood of a gauge-equivalence class of an ideal reducible connection	214
11.3.5.	Thom–Mather structures on $\widehat{\mathcal{U}}_{\kappa}^w(L)/S^1$	215
11.3.6.	Global projection map for $\widehat{\mathcal{U}}_{\kappa}^w(L)/S^1$	215
11.3.7.	Global splicing map on $\widehat{\mathcal{U}}_{\kappa}^w(L)/S^1$	215
11.3.8.	Obstruction bundle on $\widehat{\mathcal{U}}_{\kappa}^w(L)/S^1$	215
11.3.9.	Gluing hypothesis	216
11.4.	Cohomology classes on the space of global splicing data	216

11.5. Definition of the link of a gauge-equivalence class of an ideal reducible connection	216
11.6. Computations of the difference term	217
Glossary of Notation	219
Index	223
Bibliography	225

Preface

The $\mathrm{SO}(3)$ -monopole cobordism formula is an equality between the Donaldson invariant of a closed, smooth four-manifold, X , and a universal expression involving the Seiberg–Witten invariants of X , the intersection form of X , and other homotopy invariants of X . In this monograph, we prove that this formula follows from certain properties of the gluing map constructed in [24].

Our proof that the $\mathrm{SO}(3)$ -monopole cobordism formula implies Witten’s Conjecture [103] relating the Donaldson and Seiberg–Witten invariants for four-manifolds of simple type which satisfy a ‘geography inequality’ or which are ‘abundant’ (as defined in [20]) appeared in [33]. Our proofs that the $\mathrm{SO}(3)$ -monopole cobordism yields both the Superconformal Simple Type Conjecture of Moore, Mariño, and Peradze [65, 64] and Witten’s Conjecture [103] in full generality for all closed, oriented, smooth four-manifolds with $b_1(X) = 0$, odd $b^+(X) \geq 3$, and Seiberg–Witten simple type have appeared in [25] and [23], respectively.

The monopole cobordism formula has an important feature in common with that conjectured by Kotschick and Morgan [52] concerning the wall-crossing property for Donaldson invariants: both formulae give an equality relating the invariants that involves unknown coefficients depending only on the homotopy type of the manifold. Göttsche [41] and Göttsche and Zagier [45] assumed the validity of the Kotschick–Morgan Conjecture to derive a wall-crossing formula, with all coefficients explicitly determined, and the resulting structure of Donaldson invariants for four-manifolds with $b^+ = 1$. We applied a broadly similar strategy in our articles [25, 23], but refer interested readers to those articles for further details.

The main technical difficulty arising in the $\mathrm{SO}(3)$ -monopole program to prove Witten’s Conjecture has been to compute intersection pairings on the links of strata of reducible $\mathrm{SO}(3)$ monopoles, namely the moduli spaces of Seiberg–Witten monopoles lying in lower-level strata of the Uhlenbeck compactification of the moduli space of $\mathrm{SO}(3)$ monopoles [27]. Our local gluing map [24] parameterizes a neighborhood of one of these strata. In this monograph, we solve the ‘overlap problem’ described in [32] and show how these local gluing maps fit together to describe a neighborhood of the union of these strata. This description allows us to prove that the desired intersection numbers can be expressed as a universal expression in the relevant Seiberg–Witten invariant, the intersection form of the manifold, and other homotopy invariants of the manifold appearing in the $\mathrm{SO}(3)$ -monopole cobordism formula.

Because the $\mathrm{SO}(3)$ -monopole cobordism formula has certain key features in common with that conjectured by Kotschick and Morgan [52], not only in the form of the result but also in the nature of the intersection pairings to be computed, we can use the techniques developed herein to prove that the Kotschick–Morgan Conjecture also follows by similar arguments and the form of our gluing theorem for anti-self-dual connections [21].

Since the first version of this monograph was circulated, applications of our results have appeared in the proof of Property P for knots by Kronheimer and Mrowka [54] and work of Sivek

on Donaldson invariants for symplectic four-manifolds [91]. An alternative approach to Witten’s Conjecture, inspired by results in physics, has been pursued by Göttsche, Nakajima, and Yoshioka [43, 42, 44], based in part on work of Mochizuki [69].

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CHAPTER 1

Introduction

1.1. Summary of main results

In [103], Witten defined the Seiberg–Witten invariants of smooth four-manifolds with $b_1(X) = 0$ and odd $b^+(X) > 0$ (based in part on earlier joint work with Seiberg [89]) and stated a conjecture, based on arguments from quantum field theory, relating the Seiberg–Witten and Donaldson invariants. Moore and Witten extended this conjecture to a formula for four-manifolds with $b_1(X) > 0$ and $b^+(X) = 0$ in [70]. All computed examples of Donaldson and Seiberg–Witten invariants satisfy these conjectures. However, these conjectures were based on mathematically non-rigorous arguments from quantum field theory and so the need for a mathematical explanation remained.

In [83], Pidstrigach and Tyurin outlined an $\mathrm{SO}(3)$ -monopole program with the goal of giving a mathematically rigorous proof of Witten’s Conjecture by using a moduli space of $\mathrm{SO}(3)$ monopoles. After the foundations of the $\mathrm{SO}(3)$ -monopole program were developed in [27, 28, 29, 18, 99], it soon became evident that the $\mathrm{SO}(3)$ -monopole program exhibited a fundamental difficulty — that of computing intersection numbers for links of certain singularities in the lower strata of the Uhlenbeck compactification of the moduli space of $\mathrm{SO}(3)$ monopoles. This difficulty echoes that arising in attempts [21, 41] to prove the conjecture of Kotschick and Morgan [52] concerning wall-crossing formulas for Donaldson invariants on four-manifolds with $b^+(X) = 1$, although there are many additional challenges in the case of the $\mathrm{SO}(3)$ -monopole program.

In this monograph, we complete the topological computation of these intersection numbers in the sense that we reduce the computation to a *local gluing theorem* which extends that of [24], stated here as Hypothesis 7.8.1.

THEOREM 1 ($\mathrm{SO}(3)$ -monopole cobordism formula). *Assume Hypothesis 7.8.1 holds. Let X be a closed, connected, oriented smooth four-manifold with $b_1(X) = 0$, odd $b^+(X) > 1$, Euler characteristic χ , and signature σ . Let $\Lambda, w \in H^2(X; \mathbb{Z})$ obey $w - \Lambda \equiv w_2(X) \pmod{2}$. Let δ, m be non-negative integers for which $m \leq \lceil \delta/2 \rceil$, where $\lceil \cdot \rceil$ denotes the greatest integer function, and $\delta \equiv -w^2 - \frac{3}{4}(\chi + \sigma) \pmod{4}$, with Λ and δ obeying $\delta < i(\Lambda)$, where $i(\Lambda) = \Lambda^2 - \frac{1}{4}(3\chi + 7\sigma)$. Then, for any $h \in H_2(X; \mathbb{R})$ and generator $x \in H_0(X; \mathbb{Z})$, one has the following expression for the Donaldson invariant,*

$$\begin{aligned}
 D_X^w(h^{\delta-2m}x^m) &= \sum_{\mathfrak{s} \in \mathrm{Spin}^c(X)} (-1)^{\frac{1}{4}(w-\Lambda+c_1(\mathfrak{s}))^2} SW_X(\mathfrak{s}) \\
 (1.1.1) \quad &\times \sum_{i=0}^{\min(\ell, \lceil \delta/2 \rceil - m)} (p_{\delta, \ell, m, i}(c_1(\mathfrak{s}), \Lambda) Q_X^i)(h),
 \end{aligned}$$

where Q_X is the intersection form on $H_2(X; \mathbb{R})$, and $\ell = \frac{1}{4}(\delta + (c_1(\mathfrak{s}) - \Lambda)^2 + \frac{3}{4}(\chi + \sigma))$, and $p_{\delta, \ell, m, i}(\cdot, \cdot)$ is a homogeneous polynomial of degree $\delta - 2m - 2i$ with coefficients which are universal functions of χ , σ , $c_1(\mathfrak{s})^2$, Λ^2 , $c_1(\mathfrak{s}) \cdot \Lambda$, δ , m , and ℓ .

Many significant results in low-dimensional topology can be proved using Theorem 1 and related results in this monograph. Foremost among these are our proof in [23] of the Superconformal Simple Type Conjecture of Moore, Mariño, and Peradze [65, 64], our proof of Witten’s Conjecture [103] for a large class of four-manifolds in [54, Corollary 7] and [33] and, finally, our proof in [25] of Witten’s Conjecture in full generality for all closed, oriented, smooth four-manifolds with $b_1(X) = 0$, odd $b^+(X) > 1$, and Seiberg–Witten simple type.

In addition, Kronheimer and Mrowka apply our Theorem 1 to give a proof of Property P for knots in [54]. Recall that Property P is the statement that $+1$ surgery on a non-trivial knot K in S^3 yields a manifold which is not a homotopy sphere. In [54, Theorem 6], Kronheimer and Mrowka employ Theorem 1 to prove that Witten’s Conjecture holds for a large family of manifolds¹. They then argue that a counterexample to Property P would allow them to construct a four-manifold with non-trivial Seiberg–Witten invariants but trivial Donaldson invariants. As such a four-manifold would contradict Theorem 1, there can be no counterexample to Property P.

Another result following from the methods in this monograph is the ‘Multiplicity Conjecture’, [20, Conjecture 3.1], stated here as Theorem 10.1.2. Recall from [20] that a four-manifold is ‘abundant’ if the orthogonal complement of the Seiberg–Witten basic classes, with respect to the intersection form, contains a hyperbolic summand. In collaboration with Kronheimer and Mrowka, we showed in [20] that, for abundant four-manifolds, this conjecture verified the ideas of Mariño, Moore, and Peradze from [64, 65] in which they proposed a lower bound on the multiplicity of the vanishing of the Seiberg–Witten series at zero in terms of $c(X) = -\frac{1}{4}(7\chi + 11\sigma)$ and from this bound derived a lower bound on the number of basic classes of a four-manifold also in terms of $c(X)$.

Sivek has applied Theorem 1 to show that symplectic four-manifolds with $b_1 = 0$ and odd $b^+ > 1$ have non-vanishing Donaldson invariants, and that the canonical class is always a Kronheimer–Mrowka basic class [91].

Finally, the Kotschick–Morgan Conjecture [52] for the wall-crossing formulas for Donaldson invariants of manifolds with $b^+ = 1$ also follows from the results in this monograph as described in Section 1.3.

Witten’s Conjecture for the relation between the Donaldson and Seiberg–Witten invariants is stated below. The definitions of all terms in its statement appear in Sections 2.5 and 2.3.2.

CONJECTURE 2 (Witten’s Conjecture). [103] Let X be a closed, connected, oriented, smooth four-manifold with $b_1(X) = 0$ and odd $b^+(X) > 1$. Assume that X has Seiberg–Witten simple type. Then X has Kronheimer–Mrowka simple type, the Kronheimer–Mrowka basic classes coincide with the Seiberg–Witten basic classes, and the Donaldson series of X is given by

$$(1.1.2) \quad \mathbf{D}_X^w(h) = 2^{2-c(X)} e^{Q_X(h)/2} \sum_{\mathfrak{s}} (-1)^{\frac{1}{2}(w^2 + w \cdot c_1(\mathfrak{s}))} \mathbf{SW}_X(\mathfrak{s}) e^{\langle c_1(\mathfrak{s}), h \rangle},$$

¹There is a slight difference between Theorem 1 as presented here and the version used in [54]. Specifically, the bound on the degree of the Donaldson invariant used in [54, Theorem 6] uses $i(\Lambda) = -\frac{1}{4}(\chi + \sigma)$ instead of the correct value of $i(\Lambda) = -\frac{1}{4}(3\chi + 7\sigma)$. We believe that this difference is due to a typographical error in an earlier draft of this monograph. The version given here is correct, but the error in no way effects any of the results in [54].

where $c(X) = -\frac{1}{4}(7\chi + 11\sigma)$.

Detailed introductions of the $\mathrm{SO}(3)$ -monopole program have appeared in [26, 31, 32]. We now give a brief summary of these ideas here. Recall that a spin^c structure on X is determined by a complex rank-four vector bundle, $W \rightarrow X$, and a Clifford multiplication map, $\rho : T^*X \rightarrow \mathrm{Hom}(W)$. A spin^u structure \mathfrak{t} on X , as defined in [28] or Section 2.1.1 here, is given by (ρ_V, V) , where $V = W \otimes E$, and $\rho_V = \rho \otimes \mathrm{id}_E$, and (ρ, W) is a spin^c structure, and $E \rightarrow X$ is a complex rank-two vector bundle,

An $\mathrm{SO}(3)$ monopole for a spin^u structure, $\mathfrak{t} = (\rho_V, W \otimes E)$, is a pair, (A, Φ) , where A is a unitary connection on E and Φ is a section of $V^+ := W^+ \otimes E$, satisfying equations which can be thought of as a higher-rank version of the Seiberg–Witten equations, and $W = W^+ \oplus W^-$ is the usual splitting of W into the complex rank-two vector bundles of positive and negative spinors. The moduli space, $\mathcal{M}_{\mathfrak{t}}$, is the space of $\mathrm{SO}(3)$ monopoles modulo gauge equivalence. For generic perturbations, $\mathcal{M}_{\mathfrak{t}}$ is a smooth manifold away from two types of singular subspaces which are also identified as fixed points of an S^1 action on $\mathcal{M}_{\mathfrak{t}}$. The first type of singular subspace is identified with the moduli space of anti-self-dual connections [28, Equation (3.5)]. The second type of singular subspace, that of reducible $\mathrm{SO}(3)$ monopoles, is identified in [28, Lemma 3.13] with the Seiberg–Witten moduli space, $M_{\mathfrak{s}}$, where the spin^u structure \mathfrak{t} admits a splitting $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s} \otimes L$. The $\mathrm{SO}(3)$ -monopole program aims to use $\mathcal{M}_{\mathfrak{t}}$ as a cobordism between the links of these singularities. Because $\mathcal{M}_{\mathfrak{t}}$ is not compact in general, this cobordism does not provide any useful homological information.

The moduli space, $\mathcal{M}_{\mathfrak{t}}$, admits an Uhlenbeck compactification, $\bar{\mathcal{M}}_{\mathfrak{t}}$, which is contained in the space of ideal monopoles,

$$\bigcup_{\ell=0}^N \mathcal{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^{\ell}(X),$$

where $\mathcal{M}_{\mathfrak{t}(\ell)}$ is a moduli space of $\mathrm{SO}(3)$ monopoles for a ‘lower charge’ spin^u structure, $\mathfrak{t}(\ell)$, and $\mathrm{Sym}^{\ell}(X)$ is the ℓ -th symmetric product of X . The space, $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$, provides a compact and oriented cobordism between a link of the moduli space of anti-self-dual connections and links, $\bar{\mathcal{L}}_{\mathfrak{t},\mathfrak{s}}$, of subspaces of reducible $\mathrm{SO}(3)$ monopoles. However, additional reducible $\mathrm{SO}(3)$ monopoles, in the form of subspaces of ideal Seiberg–Witten monopoles,

$$(1.1.3) \quad M_{\mathfrak{s}} \times \mathrm{Sym}^{\ell}(X) \subset \mathcal{M}_{\mathfrak{t}(\ell)} \times \mathrm{Sym}^{\ell}(X),$$

appear in the lower levels of the Uhlenbeck compactification. The intersection number of certain geometric representatives of cohomology classes with the link of the anti-self-dual connections yields a multiple of the Donaldson invariant. The main technical result of this monograph, Theorem 10.1.1, is a formula for the intersection number of these geometric representatives with the link, $\bar{\mathcal{L}}_{\mathfrak{t},\mathfrak{s}}$, of the subspace (1.1.3) of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$. This formula expresses the intersection number in terms of universal functions which, while not explicit, do not depend on the manifold, X . The cobordism provided by $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ then yields the equality (1.1.1) between the aforementioned multiple of the Donaldson invariant and the sum, over spin^c structures \mathfrak{s} , of these intersection numbers.

We note that a generalization of Theorem 10.1.1 can also be proved by the methods of this monograph, where the assumptions that $b_1(X) = 0$ or $z = h^{\delta-2m}x^m$ are relaxed, but the resulting expression becomes considerably more complicated.

1.2. Outline of the argument

As described above, the problem we address in this monograph is the computation of the intersection numbers,

$$(1.2.1) \quad \# (\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}_{t,s}),$$

where $\bar{\mathcal{V}}(z)$ and $\bar{\mathcal{W}}$ are the geometric representatives mentioned above and $\bar{\mathbf{L}}_{t,s}$ is the link in $\bar{\mathcal{M}}_t/S^1$ of the Seiberg–Witten singularities, $M_s \times \text{Sym}^\ell(X)$, appearing in (1.1.3). A summary of our approach to this computation has appeared in [32]. We give a short sketch of our method here.

For the simplest case, when $M_s \subset \mathcal{M}_t$, computations of the intersection number (1.2.1) appeared in [29] (where $\ell = 0$) while computations of (1.2.1) for singularities of the form $M_s \times X$ appeared in [30] (where $\ell = 1$). While it should be possible to adapt the techniques of [59] to directly compute (1.2.1) for singularities of the form $M_s \times \text{Sym}^2(X)$, the difficulty increases rapidly as ℓ becomes larger and direct calculations appear intractable when $\ell \geq 3$.

Our gluing result [24, Theorem 1.1] can be used to define *local gluing maps* which parameterize neighborhoods of the strata,

$$(M_s \times \Sigma) \cap \bar{\mathcal{M}}_t \subset \left(M_s \times \text{Sym}^\ell(X) \right) \cap \bar{\mathcal{M}}_t,$$

in $\bar{\mathcal{M}}_t$, where Σ is a stratum of the symmetric product, $\text{Sym}^\ell(X)$. A local gluing map is then a composition of a *local splicing map* and a certain perturbation of the image of the splicing map, called the *solution map*. The local splicing map is defined, roughly, by patching together solutions of the SO(3)-monopole equations over X and over the four-sphere, S^4 , with its standard round metric of radius one using cut-off functions. Let $[A_0, \Phi_0] \in M_s$ be a gauge-equivalence class of SO(3) monopoles over X and let $\mathbf{x} \in \Sigma \subset \text{Sym}^\ell(X)$ be represented by a set of distinct points with multiplicity, $\{x_1, \dots, x_r\}$, for $1 \leq r \leq \ell$. We refer to (A_0, Φ_0) as the *background pair* and x_i as the *splicing points*. Let $(A_i, 0)$ for $i = 1, \dots, r$ be SO(3) monopoles over S^4 ; such solutions are given by an anti-self-dual connection, A_i , on an SU(2)-bundle over S^4 , with second Chern (or ‘instanton’) number equal to the multiplicity of x_i , and the zero section. Then, using local trivializations of the bundle supporting the pair (A_0, Φ_0) near the points x_i and cut-off functions, one can patch together the SO(3)-monopoles, (A_0, Φ_0) and $(A_i, 0)$, to form a pair, (A', Φ') , which is equal to (A_0, Φ_0) away from the points x_i and equal to $(A_i, 0)$ near x_i . We denote this pair schematically by

$$(1.2.2) \quad (A', \Phi') = (A_0, \Phi_0) \#_{x_1} (A_1, 0) \#_{x_2} (A_2, 0) \# \cdots \#_{x_r} (A_r, 0).$$

This construction of the pair (A', Φ') is used to define the splicing map. Roughly speaking, the domain of this splicing map is then given by a fiber bundle,

$$(1.2.3) \quad \text{Gl}(\Sigma) = \text{Fr}(\Sigma) \times_{G(\Sigma)} M(\Sigma) \rightarrow M_s \times \Sigma,$$

with a well-understood principal bundle, $\text{Fr}(\Sigma)$, and structure group, $G(\Sigma)$. The fiber, $M(\Sigma)$, is a product of moduli spaces of anti-self-dual connections over S^4 . In the preceding description of the spliced pair, (A', Φ') , the points $[A_0, \Phi_0]$ and \mathbf{x} lie in the base of the bundle (1.2.3) and the points $[A_i, 0]$ lie in the fiber, $M(\Sigma)$. The bundle $\text{Fr}(\Sigma)$ arises from the trivializations of the bundles used in the construction of (A', Φ') .

The pair (A', Φ') is only approximately a solution to the SO(3)-monopole equations. There is a section, χ_Σ , of an *obstruction pseudo-bundle*², $\Upsilon_\Sigma \rightarrow \text{Gl}(\Sigma)$, whose fibers are finite-dimensional

²See Definition 7.1.1 for an explanation of the term ‘pseudo-bundle’.

vector spaces, such that the *solution map* defined by

$$(A', \Phi') \mapsto (A', \Phi') + (a, \phi),$$

where $(a, \phi) := \wp(A', \Phi')$ is the solution to a system of partial differential equations given by

$$\mathfrak{S}((A', \Phi') + \wp(A', \Phi')) \in \Upsilon|_{(A', \Phi')},$$

where $(A, \Phi) \mapsto \mathfrak{S}(A, \Phi)$ is the map defined by the expression on the left-hand side of SO(3)-monopole equations (2.1.10). The *local obstruction section* is defined by

$$\chi_{\Sigma}(A', \Phi') = \mathfrak{S}((A', \Phi') + \wp(A', \Phi')),$$

and hence the solution map identifies the zero locus of the obstruction section, $\chi_{\Sigma}^{-1}(0)$, with a subset of \mathcal{M}_t .

There are two main sources of difficulty in using the parametrization of a neighborhood of $M_{\mathfrak{s}} \times \Sigma$ provided by the gluing map in computing the intersection pairings (1.2.1) of singularities of the form (1.1.3) when $\ell > 1$. The first difficulty is the presence of ‘higher-charge’ moduli spaces of anti-self-dual connections on S^4 in the fiber $M(\Sigma)$. It would be an interesting question to see if the work of [76, 77, 61] could be adapted to give the necessary information for the $G(\Sigma)$ -equivariant cohomology ring of $M(\Sigma)$, but one which is beyond the scope of this monograph. Such computations are carried out for the case $\ell = 2$ in [59]. We do not address that problem in this monograph. Instead, we use a pushforward-pullback argument (see Section 10.5) to isolate the topology of these fibers as universal polynomials of the type appearing in (1.1.1). The second difficulty, which we do address in this monograph, is the ‘overlap problem’ arising from the presence of more than one stratum $\Sigma \subset \text{Sym}^{\ell}(X)$ and the resulting need for more than one gluing map to parameterize a neighborhood of $M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X)$.

1.2.1. Problem of overlaps. When $\ell \geq 2$, the singularity, $M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X)$, has more than one stratum. Hence, to compute the intersection pairing (1.2.1) when $\ell \geq 2$, we must understand the overlap of the images of the gluing maps with domains given by the fiber bundles $\text{Gl}(\Sigma)$ and $\text{Gl}(\Sigma')$ of (1.2.3) for different strata Σ and Σ' of $\text{Sym}^{\ell}(X)$. Merely adding up the intersection numbers in the open sets parameterized by each gluing map could yield the wrong answer because these open sets overlap and we might be counting each intersection point more than once. Examples of this type of problem have appeared in [79, 58, 59, 102].

Our approach is to describe the overlaps of the images of the gluing maps (compositions of splicing and solution maps) by defining so-called *crude splicing maps*. Because we can write the splicing map explicitly while the solution map is defined by the Implicit Function Theorem, it is easier to compare two splicing maps than to compare two gluing maps. Because the SO(3)-monopole gluing maps defined in [24] only identify the zero locus of the obstruction section in $\text{Gl}(\Sigma)$ with an open subspace of \mathcal{M}_t , the intersection of the images of $\text{Gl}(\Sigma)$ and $\text{Gl}(\Sigma')$ under the gluing map could be quite complicated and need not be given by an open subspace of $\text{Gl}(\Sigma)$ or $\text{Gl}(\Sigma')$ and need not share any of the fiber bundle properties which we wish to use in our computation. Instead, using a deformation of the splicing map and of the fiber of the bundle $\text{Gl}(\Sigma)$ to construct the crude splicing map, we are able to ensure that the overlap of the images of two splicing maps is a subbundle of each image.

Given two strata Σ and Σ' of $\text{Sym}^\ell(X)$, with $\Sigma \subset \text{cl } \Sigma'$ (where $\text{cl } \Sigma$ denotes the closure in $\text{Sym}^\ell(X)$ of a stratum $\Sigma \subset \text{Sym}^\ell(X)$), and crude splicing maps

$$\gamma''_{\Sigma'} : \text{Gl}(\Sigma') \rightarrow \mathcal{E}_t, \quad \gamma''_{\Sigma} : \text{Gl}(\Sigma) \rightarrow \mathcal{E}_t,$$

we control the overlap of their images by defining a space of *overlap data*, $\text{Gl}(\Sigma, \Sigma')$, and maps

$$\rho''_{\Sigma, \Sigma'} : \text{Gl}(\Sigma, \Sigma') \rightarrow \text{Gl}(\Sigma'), \quad \rho^d_{\Sigma, \Sigma'} : \text{Gl}(\Sigma, \Sigma') \rightarrow \text{Gl}(\Sigma),$$

such that the diagram

$$(1.2.4) \quad \begin{array}{ccc} \text{Gl}(\Sigma, \Sigma') & \xrightarrow{\rho''_{\Sigma, \Sigma'}} & \text{Gl}(\Sigma') \\ \rho^d_{\Sigma, \Sigma'} \downarrow & & \gamma''_{\Sigma'} \downarrow \\ \text{Gl}(\Sigma) & \xrightarrow{\gamma''_{\Sigma}} & \mathcal{E}_t \end{array}$$

commutes and such that

$$\text{Im}(\gamma''_{\Sigma'}) \cap \text{Im}(\gamma''_{\Sigma}) = \text{Im}(\gamma''_{\Sigma'} \circ \rho''_{\Sigma, \Sigma'}) = \text{Im}(\gamma''_{\Sigma} \circ \rho^d_{\Sigma, \Sigma'}).$$

Moreover, the upwards and downwards overlap maps, $\rho''_{\Sigma, \Sigma'}$ and $\rho^d_{\Sigma, \Sigma'}$, are both fiber bundle maps. The diagram (1.2.4) is then used to define the space of *global splicing data* as a pushout of the spaces of local splicing data, $\text{Gl}(\Sigma)$. Once this is accomplished, the rest of the computation of the intersection number (1.2.1) is largely a formal accounting of choices of cohomology classes with compact support and a use of the familiar pullback-pushforward technique.

1.2.2. Overlap space and overlap maps. We now sketch the construction of the overlap space, $\text{Gl}(\Sigma, \Sigma')$, and overlap maps, $\rho''_{\Sigma, \Sigma'}$ and $\rho^d_{\Sigma, \Sigma'}$. First, we note that the overlap of the images will be empty unless $\Sigma \subset \text{cl } \Sigma'$ or $\Sigma' \subset \text{cl } \Sigma$. We will assume that the first case holds, thus $\Sigma \subset \text{cl } \Sigma'$, and will refer to Σ as the *lower* stratum.

There is an open neighborhood, $\nu(\Sigma, \Sigma') \subset \Sigma'$, of Σ which can be thought of as a tubular neighborhood of Σ in Σ' . We then define the space of *overlap data*, $\text{Gl}(\Sigma, \Sigma')$, to be the restriction of the bundle $\text{Gl}(\Sigma')$ in (1.2.3) to

$$M_{\mathfrak{s}} \times \nu(\Sigma, \Sigma') \subset M_{\mathfrak{s}} \times \Sigma'.$$

The map $\rho''_{\Sigma, \Sigma'}$ is the inclusion of the bundles. The definition of $\rho^d_{\Sigma, \Sigma'}$ will require a redefinition of the fibers $M(\Sigma)$.

The *overlap map*, $\rho^d_{\Sigma, \Sigma'}$, will map the fiber of the composition,

$$(1.2.5) \quad \text{Gl}(\Sigma, \Sigma') \rightarrow M_{\mathfrak{s}} \times \nu(\Sigma, \Sigma') \rightarrow M_{\mathfrak{s}} \times \Sigma,$$

to the fiber of the projection $\text{Gl}(\Sigma) \rightarrow M_{\mathfrak{s}} \times \Sigma$ from (1.2.3). The fiber of the map $\nu(\Sigma, \Sigma') \rightarrow \Sigma$ is, up to a local trivialization, a collection of points in \mathbb{R}^4 . As an example, the fiber of the normal bundle of the diagonal $\Delta \subset X^2$ is, after specifying a trivialization of the tangent bundle of X at the point given by the point in Δ , a pair of points in \mathbb{R}^4 with a mass-centering condition. The fiber of the composition (1.2.5) is then given by $M(\Sigma')$ and a collection of points in \mathbb{R}^4 . Hence, the map $\rho^d_{\Sigma, \Sigma'}$ must take the points in \mathbb{R}^4 and the connections in $M(\Sigma')$ and map them to elements of $M(\Sigma)$.

The strata of $\text{Sym}^\ell(X)$ are in bijective correspondence with partitions of ℓ . A stratum Σ is *lower* than a stratum Σ' if and only if the partition determining Σ' is a refinement of the partition

determining Σ . By *refinement*, we mean that if Σ is given by a partition $\ell = \kappa_1 + \cdots + \kappa_s$, then Σ' is given by a partition $\sum_{i,j} \kappa_{i,j}$, where $\kappa_i = \sum_j \kappa_{i,j}$. If Σ and Σ' are given by the partitions above, then a point in the fiber of $\nu(\Sigma, \Sigma') \rightarrow \Sigma$ over $\mathbf{x} \in \Sigma$ consists of clusters of points $y_{i,j} \in X$ near each point $x_i \in X$ defining \mathbf{x} . The points $y_{i,j}$ correspond to the weights $\kappa_{i,j}$ making up the partition $\sum_j \kappa_{i,j}$ of κ_i . With a trivialization of the tangent bundle at each x_i , the points $y_{i,j}$ can be considered as points in \mathbb{R}^4 . Consequently, the data in the fiber of (1.2.5), points in \mathbb{R}^4 , and connections in $M(\Sigma')$, comprise the domain of a gluing map parameterizing a neighborhood (in the Uhlenbeck compactification of a moduli space of framed, mass-centered, anti-self-dual connections on S^4) of the points,

$$(1.2.6) \quad [\Theta] \times \Sigma(\kappa_{i,j}),$$

where Θ is the product connection and

$$\Sigma(\kappa_{i,j}) \subset \text{Sym}^{\kappa_i}(\mathbb{R}^4) - \{c\}, \quad \text{where } c = [0, \dots, 0],$$

is the stratum given by the partition $\kappa_i = \sum_j \kappa_{i,j}$. We refer to the strata (1.2.6) as *product connection strata*. Hence, it would seem natural to choose $\rho_{\Sigma, \Sigma'}^d$ to be the gluing map and identify the data in the fiber of (1.2.5) with an element of $M(\Sigma)$. However, the above definition will *not* make the diagram (1.2.4) commute.

1.2.3. Associativity of splicing maps. The commutativity of the diagram (1.2.4) can be understood as an associativity property of the splicing construction as follows. Analogous properties of gluing maps for gradient flow trajectories are discussed in [9, Theorem 2.1], [85], and [101].

Let Σ and Σ' be two strata of $\text{Sym}^\kappa(\mathbb{R}^4)$. Strata of such a symmetric product are specified by partitions of κ . If Σ is the lower stratum, that is, $\Sigma \subset \text{cl}\Sigma'$, then the partition defining Σ' is a refinement of the partition $\kappa = \kappa_1 + \cdots + \kappa_s$ defining Σ in the sense that $\kappa_i = \kappa_{i,1} + \cdots + \kappa_{i,r_i}$, where Σ' is given by the partition $\kappa = \sum_{i,j} \kappa_{i,j}$.

Let (A_0, Φ_0) be the background pair and $A_{i,j}$ be the connections over S^4 given by the point in the fiber $M(\Sigma')$. Let $\mathbf{x} \in \Sigma$ be given by the set of points $\{x_1, \dots, x_r\}$. Let $\mathbf{z} \in \nu(\Sigma, \Sigma') \subset \Sigma'$ lying over \mathbf{x} be given by points $z_{i,j} \in X$ corresponding to points $y_{i,j} \in \mathbb{R}^4$. Let Θ be the product connection on S^4 and identify the points $y_{i,j}$ with points in S^4 by stereographic projection. We define connections with charge κ_i on S^4 by applying the splicing construction of (1.2.2) just to the connection,

$$(1.2.7) \quad A'_i = \Theta \#_{y_{i,1}} A_{i,1} \#_{y_{i,2}} A_{i,2} \# \cdots \#_{y_{i,r_i}} A_{i,r_i}.$$

If we consider the splicing construction as defining a binary operation and the pair $(\Theta, 0)$ as a unit for this binary operation, then the following can be seen as an associativity equality.

$$(1.2.8) \quad \begin{aligned} & (A_0, \Phi_0) \#_{x_1} (A'_1, 0) \#_{x_2} (A'_2, 0) \# \cdots \#_{x_r} (A'_r, 0) \\ &= (A_0, \Phi_0) \#_{z_{1,1}} (A_{1,1}, 0) \# \cdots \#_{z_{1,r_1}} (A_{1,r_1}, 0) \# \cdots \#_{z_{s,1}} A_{s,1} \# \cdots \#_{z_{s,r_s}} (A_{s,r_s}, 0). \end{aligned}$$

We wish to define the map $\rho_{\Sigma, \Sigma'}^d$ in the diagram (1.2.4) by the construction of the connections A'_i . We can see as follows that the diagram (1.2.4) will commute if the equality (1.2.8) holds. The composition $\gamma''_{\Sigma'} \circ \rho_{\Sigma, \Sigma'}^d$ corresponds to the right-hand-side of the equality (1.2.8) and is equal to the map obtained by splicing the pairs at the points in X defined by an element of $\nu(\Sigma, \Sigma')$. The composition $\gamma''_{\Sigma} \circ \rho_{\Sigma, \Sigma'}^d$ corresponds to the left-hand-side of the equality (1.2.8) and is equal to the composition of two splicing maps.

There are two problems with the preceding strategy. First, the connections A'_i are not anti-self-dual and thus do not define an element of $M(\Sigma)$. Second, the equality (1.2.8) does not hold in general. In the sequel, we describe how we correct these problems.

1.2.4. Instanton moduli space with spliced ends. To overcome the first difficulty mentioned above, we shall change the fiber $M(\Sigma)$ as follows. Recall that $M_\kappa^s(S^4)$ denotes the moduli space of charge- κ , mass-centered, anti-self-dual, framed $SU(2)$ -connections over S^4 and that $\bar{M}_\kappa^s(S^4)$ denotes its Uhlenbeck compactification. We shall redefine the fiber,

$$M(\Sigma) = \prod_{i=1}^r \bar{M}_{\kappa_i}^s(S^4),$$

by replacing $\bar{M}_{\kappa_i}^s(S^4)$ in the preceding product with an *instanton moduli space with spliced ends*, $\bar{M}_{\text{SPL},\kappa_i}^s$, whose construction we now indicate.

Recall that for each stratum, $\Sigma(\kappa_{i,j}) \subset \text{Sym}^{\kappa_i}(\mathbb{R}^4) - \{c\}$, there is a gluing map,

$$(1.2.9) \quad \prod_j \bar{M}_{\kappa_{i,j}}^s(S^4) \times \Sigma(\kappa_{i,j}) \rightarrow \bar{M}_{\kappa_i}^s(S^4),$$

parameterizing a neighborhood of the stratum (1.2.6) in $\bar{M}_{\kappa_i}^s(S^4)$. We define the instanton moduli space with spliced ends, $\bar{M}_{\text{SPL},\kappa_i}^s$, using induction on κ_i by replacing the image of the above gluing map by the image of the corresponding splicing map with domain,

$$\bar{M}_{\text{SPL},\kappa_{i,j}}^s \times \Sigma(\kappa_{i,j}).$$

We now describe how the images of these splicing maps fit together with each other and with the rest of $\bar{M}_{\kappa_i}^s(S^4)$ to form a smoothly stratified space. By *smoothly stratified space*, we mean a stratified space in the sense of [98, Definition 3.4] where each stratum is a smooth manifold. Such a space is called a Whitney pre-stratified space satisfying the condition of the frontier in [66, p. 480].

For these splicing maps, the background connection is the product connection, Θ , and the metric is flat. Under these conditions, the analogue of the associative equality (1.2.8) for splicing connections holds and the analogue of the diagram (1.2.4) commutes. This implies that the union, over strata in $\text{Sym}^{\kappa_i}(\mathbb{R}^4) - \{c\}$, of the images of these splicing maps forms a smoothly stratified space $W(\kappa_i)$ whose image, $W'(\kappa_i)$, under the gluing deformation is a neighborhood of $\{\Theta\} \times \text{Sym}^{\kappa_i}(\mathbb{R}^4) - \{c\}$ in $\bar{M}_{\kappa_i}^s(S^4)$. The gluing deformation defines a smoothly stratified isotopy R_t , $0 \leq t \leq 1$, in the sense of [66, p. 482], between $W(\kappa_i)$ and $W'(\kappa_i)$. For neighborhoods³ $W_1(\kappa_i) \sqsubset W_2(\kappa_i) \sqsubset W(\kappa_i)$ of the product connection strata (1.2.6) and $f : W(\kappa_i) \rightarrow [0, 1]$ satisfying $W_1(\kappa_i) \subset f^{-1}(0)$ and $W(\kappa_i) \setminus W_2(\kappa_i) \subset f^{-1}(1)$, let $W''(\kappa_i)$ be the image of $W(\kappa_i)$ under the map $R_{f(\cdot)}(\cdot)$. Defining $\bar{M}_{\text{SPL},\kappa_i}^s$ by

$$(\bar{M}_{\kappa_i}^s(S^4) \setminus W'(\kappa_i)) \cup W''(\kappa_i),$$

then yields the desired smoothly stratified space with an Uhlenbeck neighborhood of the product connection strata given by the image of a splicing map. By redefining $M(\Sigma)$ with the aid of the above construction of the instanton moduli space with spliced ends, we can define $\rho_{\Sigma,\Sigma'}^d$ using a splicing map.

³If U, U' are subsets of a topological space, S , we write $U' \sqsubset U$ if there is a continuous function $f : S \rightarrow [0, 1]$ such that $f(U') = 1$ and $f(S \setminus U) = 0$.

1.2.5. Space of global splicing data. With the redefinition of $M(\Sigma)$ described in Section 1.2.4, the only remaining problem in making the diagram (1.2.4) commute is the possible failure of the equality (1.2.8) when the Riemannian metric on X and the background connection, A_0 , are not flat.

These problems can be overcome by performing the splicing operation with respect to perturbations of the metric and background connection which are flat near the splicing points. Because the metric and connection A_0 need not be globally flat, this deformation will vary with the splicing point. This process defines the crude splicing maps, γ''_Σ , appearing in (1.2.4).

The space of global splicing data, $\bar{\mathcal{M}}_{t,s}^{\text{vir}}$, is the pushout of the spaces $\text{Gl}(\Sigma)$ by the diagram (1.2.4). That is, $\bar{\mathcal{M}}_{t,s}^{\text{vir}}$ is the union of the spaces $\text{Gl}(\Sigma)$ subject to the relation that for $\mathbf{A} \in \text{Gl}(\Sigma, \Sigma')$, we have $\rho_{\Sigma, \Sigma'}^d(\mathbf{A}) = \rho_{\Sigma, \Sigma'}^u(\mathbf{A})$.

1.2.6. Definition of link of a subspace of a moduli space of ideal Seiberg–Witten monopoles. The commutativity of the diagram (1.2.4) allows us to define a smoothly stratified space, $\bar{\mathcal{M}}_{t,s}^{\text{vir}}$, as the union of the images of the crude splicing maps discussed above. This space is then the union of the cone bundle neighborhoods defined by the domains (1.2.3) of the crude splicing maps. These local cone bundle structures allow us to define a subspace, $\mathbf{L}_{t,s}^{\text{vir}} \subset \bar{\mathcal{M}}_{t,s}^{\text{vir}}/S^1$, as the union of smoothly stratified, codimension-one subspaces, $\mathbf{L}(\Sigma) \subset \text{Gl}(\Sigma)/S^1$, each of which is a subbundle of the restriction of $\text{Gl}(\Sigma)/S^1$ to a compact subspace, $K_\Sigma \Subset \Sigma$, namely

$$(1.2.10) \quad \mathbf{L}(\Sigma) = \text{Fr}(\Sigma)|_{M_s \times K_\Sigma} \times_{G(\Sigma) \times S^1} \partial M(\Sigma),$$

where $\partial M(\Sigma) \subset M(\Sigma)$ is a smoothly stratified, codimension-one subspace. The link, $\bar{\mathbf{L}}_{t,s}$, will be defined, roughly, as the intersection of $\mathbf{L}_{t,s}^{\text{vir}}$ with the zero-locus of a section of a vector bundle, referred to as the *obstruction bundle*, over $\bar{\mathcal{M}}_{t,s}^{\text{vir}}$. The intersection, $\mathbf{L}(\Sigma) \cap \mathbf{L}(\Sigma')$, will be a smoothly stratified, codimension-one subspace of $\mathbf{L}_{t,s}^{\text{vir}}$ and can be described as

$$(1.2.11) \quad \begin{aligned} \mathbf{L}(\Sigma) \cap \mathbf{L}(\Sigma') &= \text{Fr}(\Sigma')|_{M_s \times \partial_\Sigma K_{\Sigma'}} \times_{G(\Sigma') \times S^1} \partial M(\Sigma') \\ &= \text{Fr}(\Sigma)|_{K_\Sigma} \times_{G(\Sigma) \times S^1} \partial_{\Sigma'} \partial M(\Sigma), \end{aligned}$$

where $\partial_\Sigma K_{\Sigma'} \subset K_{\Sigma'}$ is a codimension-one boundary admitting a fibration, $\partial_\Sigma K_{\Sigma'} \rightarrow K_\Sigma$, and $\partial_{\Sigma'} \partial M(\Sigma) \subset \partial M(\Sigma)$ is a smoothly stratified, codimension-one subspace.

1.2.7. Computation of intersection numbers with the link of the moduli space of ideal Seiberg–Witten monopoles. Duality arguments allow us to convert the intersection number (1.2.1) into a pairing of cohomology classes, which we write schematically as

$$\langle \bar{\mu} \smile e, [\mathbf{L}_{t,s}^{\text{vir}}] \rangle,$$

where $\bar{\mu}$ is a cohomology class to which the intersection of the geometric representatives are dual and e denotes the Euler class of the obstruction bundle. We observe that the cohomology classes $\bar{\mu}$ and e are generated by cohomology classes pulled back from $M_s \times K_\Sigma$ and by the first Chern class of the S^1 action. We wish to write this pairing as a sum,

$$\sum_i \langle \bar{\mu} \smile e, [\mathbf{L}(\Sigma_i)] \rangle,$$

over the subspaces $\mathbf{L}(\Sigma_i) \subset \mathbf{L}_{t,s}^{\text{vir}}$ and apply a pushforward-pullback argument to the diagram,

$$(1.2.12) \quad \begin{array}{ccc} \mathbf{L}(\Sigma) = \text{Fr}(\Sigma) \times_{G(\Sigma) \times S^1} \partial M(\Sigma) & \longrightarrow & \text{EG}(\Sigma) \times_{G(\Sigma) \times S^1} \partial M(\Sigma) \\ \downarrow & & \downarrow \\ M_s \times K_\Sigma & \longrightarrow & \text{BG}(\Sigma) \end{array}$$

To do so, we must select a representative of the cohomology class $\bar{\mu} \smile e$ with compact support away from the boundaries of $\mathbf{L}(\Sigma)$ that are given by $\mathbf{L}(\Sigma) \cap \mathbf{L}(\Sigma')$. We specify such a choice in a manner similar to that introduced in [79]. We define quotients $q_\Sigma : \mathbf{L}(\Sigma) \rightarrow \widehat{\mathbf{L}}(\Sigma)$ with the following properties:

- (1) The map $q_\Sigma : \mathbf{L}(\Sigma) \rightarrow \widehat{\mathbf{L}}(\Sigma)$ is injective on the interior of $\mathbf{L}(\Sigma)$.
- (2) The restrictions of q_Σ and $q_{\Sigma'}$ to the intersection $\mathbf{L}(\Sigma) \cap \mathbf{L}(\Sigma')$ are equal.
- (3) There is a cohomology class $\widehat{\mu} \smile \widehat{e}$ on $\widehat{\mathbf{L}}(\Sigma)$ such that the restriction of $\bar{\mu} \smile e$ to $\mathbf{L}(\Sigma)$ equals $q_\Sigma^*(\widehat{\mu} \smile \widehat{e})$.
- (4) Each quotient admits an (orbifold) fiber bundle structure,

$$\widehat{\mathbf{L}}(\Sigma) \rightarrow M_s \times \text{cl}(\Sigma),$$

with the same structure group $G(\Sigma)$.

We construct the quotient $\widehat{\mathbf{L}}(\Sigma)$ by replacing the restricted bundle $\text{Fr}(\Sigma)|_{M_s \times K_\Sigma}$ with the extension of that bundle over $M_s \times \text{cl}(\Sigma)$. The equality (1.2.11) allows us to do this and satisfy the second requirement above simultaneously by replacing the fibers $\partial M(\Sigma)$ with a quotient of this fiber. Then, we can write

$$\langle \bar{\mu} \smile e, [\mathbf{L}_{t,s}^{\text{vir}}] \rangle = \sum_i \langle \widehat{\mu} \smile \widehat{e}, [\widehat{\mathbf{L}}(\Sigma_i)] \rangle$$

and apply the pushforward-pullback argument to the diagram analogous to (1.2.12) for $\widehat{\mathbf{L}}(\Sigma)$. Because the topology of the fiber bundle $\text{Fr}(\Sigma)$ depends only on quantities described following (1.1.1), such a pushforward-pullback argument will yield the desired result.

1.3. Kotschick–Morgan Conjecture

We now describe how the Kotschick–Morgan Conjecture [41, 52] also follows from the methods in this monograph.

If $b^+(X) = 1$, the Donaldson invariant depends on the metric. If $\omega^+(g)$ indicates the unique (once an orientation for $H^+(X)$ is specified) unit-length, self-dual, harmonic two-form, the Donaldson invariant will change when this harmonic form ‘crosses a wall’ for reasons we now describe.

For manifolds with $b^+(X) > 1$, one proves that the Donaldson invariant is independent of the metric (see [15, Theorem 9.2.12]) by considering the cobordism defined by

$$\bar{M}_\kappa^w(g_I) = \{[A, \mathbf{x}, t] : [A, \mathbf{x}] \in \bar{M}_\kappa^w(g_t) \text{ and } t \in [-1, 1]\},$$

where $\bar{M}_\kappa^w(g_t)$ is the Uhlenbeck compactification of the moduli space of connections which are anti-self-dual with respect to the Riemannian metric g_t and g_t is a smooth path of Riemannian metrics on X connecting two different Riemannian metrics, g_{-1} and g_1 . A Donaldson invariant defined by the metric g_t is given by a pairing

$$\langle \bar{\mu}, [\bar{M}_\kappa^w(g_t)] \rangle.$$

The cohomology classes, namely the μ -classes [15, 53], used in defining the Donaldson invariants extend over the cobordism $\bar{M}_\kappa^w(g_I)$. Because the boundaries of this cobordism are $\bar{M}_\kappa^w(g_{\pm 1})$, we can write

$$(1.3.1) \quad \langle \bar{\mu}, [\bar{M}_\kappa^w(g_1)] \rangle - \langle \bar{\mu}, [\bar{M}_\kappa^w(g_{-1})] \rangle = \langle \delta^* \bar{\mu}, [\bar{M}_\kappa^w(g_I)] \rangle = 0,$$

proving that the Donaldson invariant for the metric g_1 is equal to that for the metric g_{-1} .

If $b^+(X) = 1$, the above argument fails because the μ -classes do not extend over the cobordism. The cobordism may contain singularities of the form

$$(1.3.2) \quad \{[A_L]\} \times \text{Sym}^\ell(X) \subset \bar{M}_\kappa^w(g_t),$$

where A_L is a connection which is reducible with respect to a reduction of an $\text{SO}(3)$ -bundle, $\mathfrak{g} = \mathbb{R} \oplus L$, and L is a complex line bundle with $c_1(L) \smile \omega^+(g_t) = 0$. For each of these families of reducible singularities appearing in the cobordism, the change in the Donaldson invariant, as expressed by the left-hand-side of (1.3.1), will be given by

$$(1.3.3) \quad \langle \bar{\mu}, [\partial \bar{U}_\kappa^w(L)] \rangle,$$

where $\partial \bar{U}_\kappa^w(L)$ is the link of the set of singularities (1.3.2) in the cobordism $\bar{M}_\kappa^w(g_I)$.

In [52, Conjectures 6.2.1 and 6.2.2], Kotschick and Morgan proposed that the pairing (1.3.3) was given by a polynomial in $c_1(L)$ and the intersection form Q_X and that the coefficients of this polynomial depended only on the homotopy type of X . Assuming this conjecture, Göttsche was able to compute an explicit formula for the pairing (1.3.3) in [41].

This conjecture follows almost immediately from the methods of this monograph and an analogue of Hypothesis 7.8.1 for the anti-self-dual equations. Our gluing theorem for anti-self-dual connections [21] (based in turn on [15, 12, 72, 94, 95, 96, 97]), parameterize neighborhoods of the strata $[A_L] \times \Sigma$, by bundles that are almost identical to the bundles, $\text{Gl}(\Sigma)$, described in (1.2.3). The argument described in the preceding sections, where we employ the overlap spaces and instanton moduli space with spliced ends to create a space of global splicing data, can be adapted — essentially without change — to define a link, $\partial \bar{U}_\kappa^w(L)$, to which we can apply the quotient arguments in Section 1.2.7, thus proving the Kotschick–Morgan Conjecture.

1.4. Outline of the monograph

We give a brief outline of our monograph. A review of notation and definitions from our previous articles in this sequence appears in Chapter 2. Chapter 3 contains a description of a stratification of $\text{Sym}^\ell(X)$, and definitions of normal bundles of the strata of $\text{Sym}^\ell(X)$. In Chapter 4, we show that the projection maps of the normal bundles introduced in Chapter 3 satisfy Thom–Mather type equalities on the overlap of their images. In Chapter 5, we define the spliced-ends moduli space of anti-self-dual connections over S^4 and this is used in Chapter 6 to define the space of global splicing data. An analogous construction of an obstruction bundle over the space of global splicing data is carried out in Chapter 7. In Chapter 8, we define the link of a Seiberg–Witten moduli space contained in a lower level of the Uhlenbeck compactification of the moduli space of $\text{SO}(3)$ monopoles and describe the fiber bundle structure of the subspaces of the link described in (1.2.10). We use cohomological computations in Chapter 9 to compute the pullbacks of the relevant cohomology classes to the space of global splicing data. In Chapter 10, we perform the remaining computations required to prove the main results of the monograph. Finally, in Chapter 11, we show how the arguments of this monograph yield a proof of the Kotschick–Morgan Conjecture.

CHAPTER 2

Preliminaries

In this chapter, we introduce the notation and review the definitions from our preceding work on $\mathrm{SO}(3)$ monopoles, specifically [28, 29]. We begin in Section 2.1 by recalling the definition of the moduli space of $\mathrm{SO}(3)$ monopoles and its basic properties [27, 18]. In Section 2.2, we describe the stratum of zero-section monopoles, that is, anti-self-dual connections. In Section 2.3, we discuss the strata of reducible, or Seiberg–Witten monopoles, together with their ‘virtual’ neighborhoods and normal bundles. In Section 2.4, we define the cohomology classes which will be paired with the links of the anti-self-dual and Seiberg–Witten moduli spaces. In Section 2.5, we review the definition of the Donaldson series. Lastly, in Section 2.6, we describe the basic relation between the pairings with links of the anti-self-dual and Seiberg–Witten moduli spaces provided by the $\mathrm{SO}(3)$ -monopole cobordism.

2.1. The moduli space of $\mathrm{SO}(3)$ monopoles

Throughout this monograph, (X, g) will denote a closed, connected, oriented, smooth, Riemannian four-manifold.

2.1.1. Clifford modules. Let V be a Hermitian vector bundle over (X, g) and let $\rho : T^*X \rightarrow \mathrm{End}_{\mathbb{C}}(V)$ be a real-linear map satisfying

$$(2.1.1) \quad \rho(\alpha)^2 = -g(\alpha, \alpha)\mathrm{id}_V \quad \text{and} \quad \rho(\alpha)^\dagger = -\rho(\alpha), \quad \alpha \in C^\infty(T^*X).$$

The map ρ uniquely extends to a linear isomorphism, $\rho : \Lambda^\bullet(T^*X) \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \mathrm{End}_{\mathbb{C}}(V)$, and gives V the structure of a Hermitian Clifford module for the complex Clifford algebra $\mathcal{C}\ell(T^*X)$. There is a splitting $V = V^+ \oplus V^-$, where V^\mp are the ± 1 eigenspaces of $\rho(\mathrm{vol})$. A unitary connection A on V is *spin* if

$$(2.1.2) \quad [\nabla_A, \rho(\alpha)] = \rho(\nabla\alpha) \quad \text{on } C^\infty(V),$$

for any $\alpha \in C^\infty(T^*X)$, where ∇ is the Levi-Civita connection.

A Hermitian Clifford module $\mathfrak{s} = (\rho, W)$ is a *spin^c structure* when W has complex rank four; it defines a class

$$(2.1.3) \quad c_1(\mathfrak{s}) = c_1(W^+),$$

which is an integral lift of the second Stiefel-Whitney class. If $L \rightarrow X$ is a complex line bundle, we write $\mathfrak{s} \otimes L$ for the *spin^c structure* $(\rho \otimes \mathrm{id}_L, W \otimes L)$.

We call a Hermitian Clifford module $\mathfrak{t} = (\rho, V)$ a *spin^u structure* when V has complex rank eight. Recall that $\mathfrak{g}_{\mathfrak{t}} \subset \mathfrak{su}(V)$ is the $\mathrm{SO}(3)$ subbundle given by the span of the sections of the bundle $\mathfrak{su}(V)$ which commute with the action of $\mathcal{C}\ell(T^*X)$ on V . We obtain a splitting,

$$(2.1.4) \quad \mathfrak{su}(V^+) \cong \rho(\Lambda^+) \oplus i\rho(\Lambda^+) \otimes_{\mathbb{R}} \mathfrak{g}_{\mathfrak{t}} \oplus \mathfrak{g}_{\mathfrak{t}},$$

and similarly for $\mathfrak{su}(V^-)$. The fibers V_x^+ define complex lines whose tensor-product square is $\det(V_x^+)$ and thus a complex line bundle over X ,

$$(2.1.5) \quad \det^{\frac{1}{2}}(V^+).$$

A spin^u structure \mathfrak{t} thus defines characteristic classes,

$$(2.1.6) \quad c_1(\mathfrak{t}) = \frac{1}{2}c_1(V^+), \quad p_1(\mathfrak{t}) = p_1(\mathfrak{g}_{\mathfrak{t}}), \quad \text{and} \quad w_2(\mathfrak{t}) = w_2(\mathfrak{g}_{\mathfrak{t}}).$$

Given a spin^c bundle W , one has an isomorphism $V \cong W \otimes_{\mathbb{C}} E$ of Hermitian Clifford modules, where E is a complex rank-two Hermitian vector bundle [28, Lemma 2.3]; then

$$(2.1.7) \quad \mathfrak{g}_{\mathfrak{t}} = \mathfrak{su}(E) \quad \text{and} \quad \det^{\frac{1}{2}}(V^+) = \det(W^+) \otimes_{\mathbb{C}} \det(E).$$

If for $\mathfrak{t} = (\rho, V)$, there is a spin^c structure $\mathfrak{s} = (\rho_W, W)$ and the bundle V admits a splitting $V \cong W \oplus W \otimes L$ and $\rho = \rho_W \oplus \rho_W \otimes \text{id}_L$, then we write $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{s} \otimes L$.

2.1.2. SO(3) monopoles. We fix a smooth unitary connection A_{Λ} on the line bundle $\det^{\frac{1}{2}}(V^+)$, let $k \geq 2$ be an integer, and let $\mathcal{A}_{\mathfrak{t}}$ be the affine space of L^2_k spin connections¹ on V which induce the connection $2A_{\Lambda}$ on $\det(V^+)$. If A is a spin connection on V , then it defines an SO(3) connection, \hat{A} , on the subbundle $\mathfrak{g}_{\mathfrak{t}} \subset \mathfrak{su}(V)$ [28, Lemma 2.5]; conversely, every SO(3) connection on $\mathfrak{g}_{\mathfrak{t}}$ lifts to a unique spin connection on V inducing the connection $2A_{\Lambda}$ on $\det(V^+)$ [28, Lemma 2.11].

Let $\mathcal{G}_{\mathfrak{t}}$ denote the group of L^2_{k+1} unitary automorphisms of V which commute with $\mathcal{C}\ell(T^*X)$ and which have Clifford-determinant one (see [28, Definition 2.6]). Define

$$(2.1.8) \quad \tilde{\mathcal{E}}_{\mathfrak{t}} = \mathcal{A}_{\mathfrak{t}} \times L^2_k(X, V^+) \quad \text{and} \quad \mathcal{E}_{\mathfrak{t}} = \tilde{\mathcal{E}}_{\mathfrak{t}}/\mathcal{G}_{\mathfrak{t}}.$$

The action of $\mathcal{G}_{\mathfrak{t}}$ on V induces an adjoint action on $\text{End}_{\mathbb{C}}(V)$, acting as the identity on $\rho(\Lambda_{\mathbb{C}}^{\bullet}) \subset \text{End}_{\mathbb{C}}(V)$ and inducing an adjoint action on $\mathfrak{g}_{\mathfrak{t}} \subset \text{End}_{\mathbb{C}}(V)$ (see [28, Lemma 2.7]). The space $\tilde{\mathcal{E}}_{\mathfrak{t}}$ carries a circle action induced by scalar multiplication on V :

$$(2.1.9) \quad S^1 \times V \rightarrow V, \quad (e^{i\theta}, \Phi) \mapsto e^{i\theta} \Phi.$$

Because this action commutes with that of $\mathcal{G}_{\mathfrak{t}}$, the action (2.1.9) also defines an action on $\mathcal{E}_{\mathfrak{t}}$. Note that $-1 \in S^1$ acts trivially on $\mathcal{E}_{\mathfrak{t}}$. Let $\mathcal{E}_{\mathfrak{t}}^0 \subset \mathcal{E}_{\mathfrak{t}}$ be the subspace represented by pairs whose spinor components are not identically zero, let $\mathcal{E}_{\mathfrak{t}}^* \subset \mathcal{E}_{\mathfrak{t}}$ be the subspace represented by pairs where the induced SO(3) connections on $\mathfrak{g}_{\mathfrak{t}}$ are irreducible, and let $\mathcal{E}_{\mathfrak{t}}^{*,0} = \mathcal{E}_{\mathfrak{t}}^* \cap \mathcal{E}_{\mathfrak{t}}^0$.

We call a pair (A, Φ) in $\tilde{\mathcal{E}}_{\mathfrak{t}}$ an SO(3) *monopole* if

$$(2.1.10) \quad \mathfrak{S}(A, \Phi) := \begin{pmatrix} \text{ad}^{-1}(F_A^+) - \tau \rho^{-1}(\Phi \otimes \Phi^*)_{00} \\ D_A \Phi + \rho(\vartheta) \end{pmatrix} = 0.$$

Here, $D_A = \rho \circ \nabla_A : C^\infty(X, V^+) \rightarrow C^\infty(X, V^-)$ is the Dirac operator; the isomorphism $\text{ad} : \mathfrak{g}_{\mathfrak{t}} \rightarrow \mathfrak{so}(\mathfrak{g}_{\mathfrak{t}})$ identifies the self-dual component of the curvature F_A^+ , a section of $\Lambda^+ \otimes \mathfrak{so}(\mathfrak{g}_{\mathfrak{t}})$, with $\text{ad}^{-1}(F_A^+)$, a section of $\Lambda^+ \otimes \mathfrak{g}_{\mathfrak{t}}$; the section τ of $\text{GL}(\Lambda^+)$ is a perturbation close to the identity; the perturbation ϑ is a complex one-form close to zero; $\Phi^* \in \text{Hom}(V^+, \mathbb{C})$ is the pointwise Hermitian dual $\langle \cdot, \Phi \rangle$ of Φ ; and $(\Phi \otimes \Phi^*)_{00}$ is the component of the section $\Phi \otimes \Phi^*$ of $i\mathfrak{u}(V^+)$ lying in $\rho(\Lambda^+) \otimes \mathfrak{g}_{\mathfrak{t}}$ with respect to the splitting $\mathfrak{u}(V^+) = i\mathbb{R} \oplus \mathfrak{su}(V^+)$ and decomposition (2.1.4) of $\mathfrak{su}(V^+)$.

¹We adopt the notation of Freed and Uhlenbeck [34] for Sobolev spaces.

Equation (2.1.10) is invariant under the action of \mathcal{G}_t . We let $\mathcal{M}_t \subset \mathcal{C}_t$ be the subspace represented by pairs satisfying equation (2.1.10) and write

$$(2.1.11) \quad \mathcal{M}_t^* = \mathcal{M}_t \cap \mathcal{C}_t^*, \quad \mathcal{M}_t^0 = \mathcal{M}_t \cap \mathcal{C}_t^0, \quad \text{and} \quad \mathcal{M}_t^{*,0} = \mathcal{M}_t \cap \mathcal{C}_t^{*,0}.$$

Since equation (2.1.10) is invariant under the circle action induced by scalar multiplication on V , the subspaces (2.1.11) of \mathcal{C}_t are also invariant under this action.

THEOREM 2.1.1. (See Feehan [18, Theorem 1.1] and Teleman [99].) *Let X be a closed, oriented, smooth four-manifold and let V be a complex rank-eight, Hermitian vector bundle over X . Then for parameters $(\rho, g, \tau, \vartheta)$, which are generic in the sense of [18], and $\mathfrak{t} = (\rho, V)$, the space $\mathcal{M}_t^{*,0}$ is a smooth manifold of the expected dimension,*

$$(2.1.12) \quad \dim \mathcal{M}_t^{*,0} = d(\mathfrak{t}) = d_a(\mathfrak{t}) + 2n_a(\mathfrak{t}), \quad \text{where } d_a(\mathfrak{t}) = -2p_1(\mathfrak{t}) - \frac{3}{2}(\chi(X) + \sigma(X)),$$

$$n_a(\mathfrak{t}) = \frac{1}{4}(p_1(\mathfrak{t}) + c_1(\mathfrak{t})^2 - \sigma(X)),$$

and $\chi(X)$ is the Euler characteristic and $\sigma(X)$ is the signature of X .

For the remainder of the article, we assume that the *perturbation parameters* in (2.1.10) are chosen as indicated in Theorem 2.1.1. For each non-negative integer ℓ , let $\mathfrak{t}(\ell) = (\rho, V_\ell)$ be the spin^u structure characterized by

$$(2.1.13) \quad c_1(V_\ell) = c_1(V), \quad p_1(\mathfrak{g}_{t(\ell)}) = p_1(\mathfrak{g}_t) + 4\ell, \quad \text{and} \quad w_2(\mathfrak{g}_{t(\ell)}) = w_2(\mathfrak{g}_t).$$

We let $\bar{\mathcal{M}}_t$ denote the closure of \mathcal{M}_t in the space of *ideal SO(3) monopoles*,

$$(2.1.14) \quad I\mathcal{M}_t = \bigsqcup_{\ell=0}^{\infty} (\mathcal{M}_{t(\ell)} \times \text{Sym}^\ell(X)),$$

with respect to an *Uhlenbeck topology* [27, Definition 4.19] defined by the following notion of convergence.

DEFINITION 2.1.2. A sequence $\{[A_\alpha, \Phi_\alpha]\}_{\alpha \in \mathbb{N}} \subset \mathcal{C}_t$ converges to an ideal pair $[A_0, \Phi_0, \mathbf{x}] \in \mathcal{C}_{t(\ell)} \times \text{Sym}^\ell(X)$ if

- There is a sequence of $L^2_{k+1, \text{loc}}$ spin^u bundle isomorphisms, $u_\alpha : V_t|_{X-\mathbf{x}} \rightarrow V_{t(\ell)}|_{X-\mathbf{x}}$, such that the sequence of pairs $u_\alpha(A_\alpha, \Phi_\alpha)$ converges as $\alpha \rightarrow \infty$ to (A_0, Φ_0) in $L^2_{k, \text{loc}}$ over $X - \mathbf{x}$ and
- The sequence of measures, $|F_{\hat{A}_\alpha}|^2 d\text{vol}$ converges as $\alpha \rightarrow \infty$ in the weak-star topology to the measure $|F_{\hat{A}}|^2 d\text{vol} + 8\pi^2 \sum_{x \in \mathbf{x}} \delta_x$, where δ_x is the Dirac delta measure centered at $x \in X$.

We refer to neighborhoods in the topology defined by Uhlenbeck convergence as *Uhlenbeck neighborhoods*.

We call the intersection of $\bar{\mathcal{M}}_t$ with $\mathcal{M}_{t(\ell)} \times \text{Sym}^\ell(X)$ its ℓ -th *level*. We define the space $\bar{\mathcal{C}}_t$ to be the set

$$(2.1.15) \quad \bar{\mathcal{C}}_t := \bigsqcup_{\ell=0}^{\infty} (\mathcal{C}_{t(\ell)} \times \text{Sym}^\ell(X)),$$

with the topology given by Definition 2.1.2.

THEOREM 2.1.3. (See Feehan and Leness [27, Theorem 1.1].) *Let X be a closed Riemannian four-manifold with spin^u structure \mathfrak{t} . Then there is a positive integer N , depending at most on the scalar curvature of X , the curvature of the chosen unitary connection on $\det(V^+)$, and $p_1(\mathfrak{t})$, such that the Uhlenbeck closure $\bar{\mathcal{M}}_{\mathfrak{t}}$ of $\mathcal{M}_{\mathfrak{t}}$ in $\sqcup_{\ell=0}^N (\mathcal{M}_{\mathfrak{t}(\ell)} \times \text{Sym}^{\ell}(X))$ is a second-countable, compact, Hausdorff space. The space $\bar{\mathcal{M}}_{\mathfrak{t}}$ carries a continuous circle action, which restricts to the circle action defined on $\mathcal{M}_{\mathfrak{t}_\ell}$ for each level.*

2.2. Stratum of anti-self-dual or zero-section solutions

From equation (2.1.10), we see that the stratum of $\mathcal{M}_{\mathfrak{t}}$ represented by pairs with zero spinor is identified with

$$(2.2.1) \quad \{A \in \mathcal{A}_{\mathfrak{t}} : F_{\hat{A}}^+ = 0\} / \mathcal{G}_{\mathfrak{t}} \cong M_{\kappa}^w(X, g),$$

the moduli space of g -anti-self-dual connections on the $\text{SO}(3)$ bundle, $\mathfrak{g}_{\mathfrak{t}}$, where $\kappa = -\frac{1}{4}p_1(\mathfrak{t})$ and $w \equiv w_2(\mathfrak{t}) \pmod{2}$. For a generic Riemannian metric g , the space $M_{\kappa}^w(X, g)$ is a smooth manifold of the expected dimension, $-2p_1(\mathfrak{t}) - \frac{3}{2}(\chi + \sigma) = d_a(\mathfrak{t})$.

As explained in [28, Section 3.4.1], it is desirable to choose $w \pmod{2}$ so as to exclude points in $\bar{\mathcal{M}}_{\mathfrak{t}}$ with associated flat $\text{SO}(3)$ connections, so we have a *disjoint* union,

$$(2.2.2) \quad \bar{\mathcal{M}}_{\mathfrak{t}} \cong \bar{\mathcal{M}}_{\mathfrak{t}}^{*,0} \sqcup \bar{M}_{\kappa}^w \sqcup \bar{\mathcal{M}}_{\mathfrak{t}}^{\text{red}},$$

where $\bar{\mathcal{M}}_{\mathfrak{t}}^* \subset \bar{\mathcal{M}}_{\mathfrak{t}}$ is the subspace represented by triples whose associated $\text{SO}(3)$ connections are irreducible, $\bar{\mathcal{M}}_{\mathfrak{t}}^0 \subset \bar{\mathcal{M}}_{\mathfrak{t}}$ is the subspace represented by triples whose spinors are not identically zero, $\bar{\mathcal{M}}_{\mathfrak{t}}^{*,0} = \bar{\mathcal{M}}_{\mathfrak{t}}^* \cap \bar{\mathcal{M}}_{\mathfrak{t}}^0$, while $\bar{\mathcal{M}}_{\mathfrak{t}}^{\text{red}} \subset \bar{\mathcal{M}}_{\mathfrak{t}}$ is the subspace $\bar{\mathcal{M}}_{\mathfrak{t}} - \bar{\mathcal{M}}_{\mathfrak{t}}^*$ represented by triples whose associated $\text{SO}(3)$ connections are reducible. We recall the

DEFINITION 2.2.1. (See Feehan and Leness [29, Definition 3.20].) A class $v \in H^2(X; \mathbb{Z}/2)$ is *good* if no integral lift of v is torsion.

If $w \pmod{2}$ is good, then the union (2.2.2) is disjoint, as desired. In practice, rather than constraining $w \pmod{2}$ itself, we use the blow-up trick of [73], replacing X with the blow-up, $X \# \overline{\mathbb{C}\mathbb{P}^2}$, and replacing w by $w + \text{PD}[e]$ (where $e \in H_2(X; \mathbb{Z})$ is the exceptional class and $\text{PD}[e]$ denotes its Poincaré dual), noting that $w + \text{PD}[e] \pmod{2}$ is always good, and define gauge-theoretic invariants of X in terms of moduli spaces on $X \# \overline{\mathbb{C}\mathbb{P}^2}$. When $w \pmod{2}$ is good, we define [28, Definition 3.7] the link of \bar{M}_{κ}^w in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ by

$$(2.2.3) \quad \bar{\mathbf{L}}_{\mathfrak{t},\kappa}^w = \{[A, \Phi, \mathbf{x}] \in \bar{\mathcal{M}}_{\mathfrak{t}}/S^1 : \|\Phi\|_{L^2}^2 = \varepsilon\},$$

where ε is a small positive constant; for generic ε , the link, $\bar{\mathbf{L}}_{\mathfrak{t},\kappa}^w$, is a smoothly-stratified, codimension-one subspace of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$.

2.3. Strata of Seiberg–Witten or reducible solutions

We call a pair $(A, \Phi) \in \tilde{\mathcal{C}}_{\mathfrak{t}}$ *reducible* if the connection A on V respects a splitting,

$$(2.3.1) \quad V = W \oplus W \otimes L = W \otimes (\underline{\mathbb{C}} \oplus L),$$

for some spin^c structure $\mathfrak{s} = (\rho, W)$ and complex line bundle L , in which case $c_1(L) = c_1(\mathfrak{t}) - c_1(\mathfrak{s})$. A spin connection A on V is reducible with respect to the splitting (2.3.1) if and only if \hat{A} is reducible with respect to the splitting $\mathfrak{g}_{\mathfrak{t}} \cong \underline{\mathbb{R}} \oplus L$, [28, Lemma 2.9]. If A is reducible, we can write

$A = B \oplus B \otimes A_L$, where B is a spin connection on W and A_L is a unitary connection on L ; then $\hat{A} = d_{\mathbb{R}} \oplus A_L$ and $A_L = A_{\Lambda} \otimes (B^{\det})^*$, where B^{\det} is the connection on $\det(W^+)$ induced by B on W and $d_{\mathbb{R}}$ is the product connection on the line bundle $\underline{\mathbb{R}} := X \times \mathbb{R}$.

2.3.1. Seiberg–Witten monopoles. Given a spin^c structure $\mathfrak{s} = (\rho, W)$ on X , let $\mathcal{A}_{\mathfrak{s}}$ denote the affine space of L_k^2 spin connections on W . Let $\mathcal{G}_{\mathfrak{s}}$ denote the group of L_{k+1}^2 unitary automorphisms of W , commuting with $\mathcal{C}\ell(T^*X)$, which we identify with $L_{k+1}^2(X, S^1)$. We then define

$$(2.3.2) \quad \tilde{\mathcal{C}}_{\mathfrak{s}} = \mathcal{A}_{\mathfrak{s}} \times L_k^2(W^+) \quad \text{and} \quad \mathcal{C}_{\mathfrak{s}} = \tilde{\mathcal{C}}_{\mathfrak{s}} / \mathcal{G}_{\mathfrak{s}},$$

where $\mathcal{G}_{\mathfrak{s}}$ acts on $\tilde{\mathcal{C}}_{\mathfrak{s}}$ by

$$(2.3.3) \quad (s, (B, \Psi)) \mapsto s(B, \Psi) = (B - (s^{-1}ds)\text{id}_W, s\Psi).$$

We call a pair $(B, \Psi) \in \tilde{\mathcal{C}}_{\mathfrak{s}}$ a Seiberg–Witten monopole if

$$(2.3.4) \quad \begin{aligned} \text{Tr}(F_B^+) - \tau\rho^{-1}(\Psi \otimes \Psi^*)_0 - F^+(A_{\Lambda}) &= 0, \\ D_B\Psi + \rho(\vartheta)\Psi &= 0, \end{aligned}$$

where $\text{Tr} : \mathfrak{u}(W^+) \rightarrow i\mathbb{R}$ is defined by the trace on 2×2 complex matrices, $(\Psi \otimes \Psi^*)_0$ is the component of the section $\Psi \otimes \Psi^*$ of $i\mathfrak{u}(W^+)$ contained in $i\mathfrak{su}(W^+)$, $D_B : C^\infty(W^+) \rightarrow C^\infty(W^-)$ is the Dirac operator, and A_{Λ} is a unitary connection on a line bundle with first Chern class $\Lambda \in H^2(X; \mathbb{Z})$. The perturbations are chosen so that solutions to equation (2.3.4) are identified with reducible solutions to (2.1.10) when $c_1(t) = \Lambda$. Let $\tilde{M}_{\mathfrak{s}} \subset \tilde{\mathcal{C}}_{\mathfrak{s}}$ be the subspace cut out by equation (2.3.4) and denote the moduli space of Seiberg–Witten monopoles by $M_{\mathfrak{s}} = \tilde{M}_{\mathfrak{s}} / \mathcal{G}_{\mathfrak{s}}$.

2.3.2. Seiberg–Witten invariants. We let $\mathcal{C}_{\mathfrak{s}}^0 \subset \mathcal{C}_{\mathfrak{s}}$ be the open subspace represented by pairs whose spinor components are not identically zero and define a complex line bundle over $\mathcal{C}_{\mathfrak{s}}^0 \times X$ by

$$(2.3.5) \quad \mathbb{L}_{\mathfrak{s}} = \mathcal{C}_{\mathfrak{s}}^0 \times_{\mathcal{G}_{\mathfrak{s}}} \mathbb{C},$$

where $\mathbb{C} = X \times \mathbb{C}$ and $s \in \mathcal{G}_{\mathfrak{s}}$ acts on $(B, \Psi) \in \tilde{\mathcal{C}}_{\mathfrak{s}}$ and $(x, \zeta) \in \mathbb{C}$ by

$$(2.3.6) \quad ((B, \Psi), (x, \zeta)) \mapsto (s(B, \Psi), (x, s(x)^{-1}\zeta)).$$

Define

$$(2.3.7) \quad \mathbb{A}_2(X) = \text{Sym}(H_0(X; \mathbb{R})) \otimes \Lambda^\bullet(H_1(X; \mathbb{R}))$$

to be the graded algebra, with $z = \beta_1\beta_2 \cdots \beta_r$ having total degree $\deg(z) = \sum_p(2 - i_p)$, when $\beta_p \in H_{i_p}(X; \mathbb{R})$. Then the map,

$$(2.3.8) \quad \mu_{\mathfrak{s}} : H_\bullet(X; \mathbb{R}) \rightarrow H^{2-\bullet}(\mathcal{C}_{\mathfrak{s}}^0; \mathbb{R}), \quad \beta \mapsto c_1(\mathbb{L}_{\mathfrak{s}}) / \beta,$$

extends in the usual way to a homomorphism of graded real algebras,

$$\mu_{\mathfrak{s}} : \mathbb{A}_2(X) \rightarrow H^\bullet(\mathcal{C}_{\mathfrak{s}}^0; \mathbb{R}).$$

If $x \in H_0(X; \mathbb{Z})$ denotes the positive generator, we set

$$(2.3.9) \quad \mu_{\mathfrak{s}} = c_1(\mathbb{L}_{\mathfrak{s}}) / x \in H^2(\mathcal{C}_{\mathfrak{s}}^0; \mathbb{Z}).$$

Equivalently, $\mu_{\mathfrak{s}}$ is the first Chern class of the S^1 base-point fibration over $\mathcal{C}_{\mathfrak{s}}^0$. If $b_1(X) = 0$, then $c_1(\mathbb{L}_{\mathfrak{s}}) = \mu_{\mathfrak{s}} \times 1$ by [28, Lemma 2.14].

For $b^+(X) > 0$ and generic Riemannian metrics on X , the space $M_{\mathfrak{s}}$ contains zero-section pairs if and only if $c_1(\mathfrak{s}) - \Lambda$ is a torsion class by [71, Proposition 6.3.1]. If $M_{\mathfrak{s}}$ contains no zero-section pairs then, for generic perturbations, it is a compact, orientable, smooth manifold of dimension

$$(2.3.10) \quad d_s(\mathfrak{s}) = \dim M_{\mathfrak{s}} = \frac{1}{4}(c_1(\mathfrak{s})^2 - 2\chi - 3\sigma).$$

Let $\tilde{X} = X \# \overline{\mathbb{C}\mathbb{P}^2}$ denote the blow-up of X with exceptional class $e \in H_2(\tilde{X}; \mathbb{Z})$ and denote its Poincaré dual by $\text{PD}[e] \in H^2(\tilde{X}; \mathbb{Z})$. Let $\mathfrak{s}^{\pm} = (\tilde{\rho}, \tilde{W})$ denote the spin^c structure on \tilde{X} with $c_1(\mathfrak{s}^{\pm}) = c_1(\mathfrak{s}) \pm \text{PD}[e]$ obtained by splicing the spin^c structure $\mathfrak{s} = (\rho, W)$ on X with the spin^c structure on $\overline{\mathbb{C}\mathbb{P}^2}$ with first Chern class $\pm \text{PD}[e]$. (See [29, Section 4.5] for an explanation of the relation between spin^c structures on X and \tilde{X} .) Now

$$c_1(\mathfrak{s}) \pm \text{PD}[e] - \Lambda \in H^2(\tilde{X}; \mathbb{Z})$$

is not a torsion class and so — for $b^+(X) > 0$, generic Riemannian metrics on X and related metrics on the connected sum \tilde{X} — the moduli spaces $M_{\mathfrak{s}^{\pm}}(\tilde{X})$ contain no zero-section pairs. Thus, for our choice of generic perturbations, the moduli spaces $M_{\mathfrak{s}^{\pm}}(\tilde{X})$ are compact, oriented, smooth manifolds, both of dimension $\dim M_{\mathfrak{s}}(X)$.

For $b_1(X) = 0$ and odd $b^+(X) > 1$, we define the *Seiberg–Witten invariant* by [29, Section 4.1]

$$(2.3.11) \quad SW_X(\mathfrak{s}) = \langle \mu_{\mathfrak{s}^+}^d, [M_{\mathfrak{s}^+}(\tilde{X})] \rangle = \langle \mu_{\mathfrak{s}^-}^d, [M_{\mathfrak{s}^-}(\tilde{X})] \rangle,$$

where $2d = d_s(\mathfrak{s}) = d_s(\mathfrak{s}^{\pm})$. When $b_2^+(X) = 1$ the pairing on the right-hand side of definition (2.3.11) depends on the chamber in the positive cone of $H^2(\tilde{X}; \mathbb{R})$ determined by the period point of the Riemannian metric on \tilde{X} . The definition of the Seiberg–Witten invariant for this case is also given in [29, Section 4.1]: we assume that the class $w_2(X) - \Lambda \pmod{2}$ is good to avoid technical difficulties involved in chamber specification. Since $w \equiv w_2(X) - \Lambda \pmod{2}$, this coincides with the constraint we use to define the Donaldson invariants in Section 2.5 when $b^+(X) = 1$. We refer to [29, Lemma 4.1 and Remark 4.2] for a comparison of the chamber structures required for the definition of Donaldson and Seiberg–Witten invariants when $b^+(X) = 1$.

We say that $c_1(\mathfrak{s})$ is a *Seiberg–Witten basic class* if the map $\mu_{\mathfrak{s}}$ is non-trivial. The manifold X is said to have *Seiberg–Witten simple type* if all basic classes satisfy $d_s(\mathfrak{s}) = 0$.

2.3.3. Reducible $\text{SO}(3)$ monopoles. If $\mathfrak{t} = (\rho, V)$ and $\mathfrak{s} = (\rho, W)$ with $V = W \oplus W \otimes L$, then there is a smooth embedding

$$(2.3.12) \quad \tilde{\mathcal{C}}_{\mathfrak{s}} \hookrightarrow \tilde{\mathcal{C}}_{\mathfrak{t}}, \quad (B, \Psi) \mapsto (B \oplus B \otimes A_{\Lambda} \otimes B^{\text{det},*}, \Psi \oplus 0),$$

which is gauge-equivariant with respect to the homomorphism

$$(2.3.13) \quad \varrho : \mathcal{G}_{\mathfrak{s}} \hookrightarrow \mathcal{G}_{\mathfrak{t}}, \quad s \mapsto \text{id}_W \otimes \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}.$$

According to [28, Lemma 3.13], the map (2.3.12) defines a topological embedding $M_{\mathfrak{s}}^0 \hookrightarrow \mathcal{M}_{\mathfrak{t}}$, where $M_{\mathfrak{s}}^0 = M_{\mathfrak{s}} \cap \mathcal{C}_{\mathfrak{s}}^0$ and an embedding of $M_{\mathfrak{s}}$ if $w_2(\mathfrak{t}) \neq 0$ or $b_1(X) = 0$; its image in $\mathcal{M}_{\mathfrak{t}}$ is represented by pairs which are reducible with respect to the splitting $V = W \oplus W \otimes L$. Henceforth, we shall not distinguish between $M_{\mathfrak{s}}$ and its image in $\mathcal{M}_{\mathfrak{t}}$ under this embedding.

If $\mathfrak{g}_{\mathfrak{t}(\ell)} \cong i\mathbb{R} \oplus L$ where $c_1(L) = c_1(\mathfrak{t}) - c_1(\mathfrak{s})$, then $p_1(\mathfrak{g}_{\mathfrak{t}(\ell)}) = (c_1(\mathfrak{t}) - c_1(\mathfrak{s}))^2$. Hence, for

$$(2.3.14) \quad \ell(\mathfrak{t}, \mathfrak{s}) = \frac{1}{4} \left((c_1(\mathfrak{t}) - c_1(\mathfrak{s}))^2 - p_1(\mathfrak{t}) \right),$$

the embedding (2.3.12) gives an inclusion of $M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X)$ into $I\mathcal{M}_{\mathfrak{t}}$, where $\ell = \ell(\mathfrak{t}, \mathfrak{s})$.

2.3.4. Circle actions. When $V = W \oplus W \otimes L$ and $\mathfrak{t} = (\rho, V)$, the space $\tilde{\mathcal{C}}_{\mathfrak{t}}$ inherits a circle action defined by

$$(2.3.15) \quad S^1 \times V \rightarrow V, \quad (e^{i\theta}, \Psi \oplus \Psi') \mapsto \Psi \oplus e^{i\theta} \Psi',$$

where $\Psi \in C^{\infty}(W)$ and $\Psi' \in C^{\infty}(W \otimes L)$. With respect to the splitting $V = W \oplus W \otimes L$, the actions (2.3.15) and (2.1.9) are related by

$$(2.3.16) \quad \begin{pmatrix} 1 & 0 \\ 0 & e^{i2\theta} \end{pmatrix} = e^{i\theta} u, \quad \text{where } u = \varrho_{\mathfrak{s}}(e^{-i\theta}) := \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \in \mathcal{G}_{\mathfrak{t}},$$

and so, when we pass to the induced circle actions on the quotient $\mathcal{C}_{\mathfrak{t}} = \tilde{\mathcal{C}}_{\mathfrak{t}}/\mathcal{G}_{\mathfrak{t}}$, the actions (2.3.15) and (2.1.9) on $\mathcal{C}_{\mathfrak{t}}$ differ only in their multiplicity. Recall [28, Lemma 3.11] that the image in $\tilde{\mathcal{C}}_{\mathfrak{t}}$ of the map (2.3.12) contains all pairs which are fixed by the circle action (2.3.15).

When $V = W \oplus W \otimes L$, the bundle $\mathfrak{g}_{\mathfrak{t}}$ defined prior to (2.1.4) admits a splitting

$$(2.3.17) \quad \mathfrak{g}_{\mathfrak{t}} \cong i\mathbb{R} \oplus L,$$

where $\mathbb{R} = X \times \mathbb{R}$. The action (2.3.15) induces an S^1 action on $\mathfrak{g}_{\mathfrak{t}}$ given, for $z \in L$ and $\zeta \in i\mathbb{R}$, by

$$(2.3.18) \quad (e^{i\theta}, (\zeta, z)) \mapsto (\zeta, e^{-2i\theta} z),$$

as described in [30, Equation 3.56].

2.3.5. The virtual normal bundle of the Seiberg–Witten moduli space. Suppose $\mathfrak{t} = (\rho, V)$ and $\mathfrak{s} = (\rho, W)$, with $V = W \oplus W \otimes L$, so we have a topological embedding $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}$; we assume $M_{\mathfrak{s}}$ contains no zero-section monopoles. Recall from [28, Section 3.5] that there exist finite-rank, complex vector bundles,

$$(2.3.19) \quad \pi_{\Xi} : \Xi_{\mathfrak{t}, \mathfrak{s}} \rightarrow M_{\mathfrak{s}} \quad \text{and} \quad \pi_N : N_{\mathfrak{t}, \mathfrak{s}} \rightarrow M_{\mathfrak{s}},$$

with $\Xi_{\mathfrak{t}, \mathfrak{s}} \cong M_{\mathfrak{s}} \times \mathbb{C}^{r_{\Xi}}$, called the *obstruction bundles* and *virtual normal bundles* of $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}$, respectively. For a small enough positive radius ε , there are a topological embedding [28, Theorem 3.21] of an open tubular neighborhood,

$$(2.3.20) \quad \gamma_{\mathfrak{s}} : N_{\mathfrak{t}, \mathfrak{s}}(\varepsilon) \hookrightarrow \mathcal{C}_{\mathfrak{t}},$$

and a smooth section $\chi_{\mathfrak{s}}$ of the pulled-back complex vector bundle,

$$(2.3.21) \quad \pi_N^* \Xi_{\mathfrak{t}, \mathfrak{s}} \rightarrow N_{\mathfrak{t}, \mathfrak{s}}(\varepsilon),$$

such that the restriction of $\gamma_{\mathfrak{s}}$ yields a homeomorphism

$$(2.3.22) \quad \gamma_{\mathfrak{s}} : \chi_{\mathfrak{s}}^{-1}(0) \cap N_{\mathfrak{t}, \mathfrak{s}}(\varepsilon) \cong \mathcal{M}_{\mathfrak{t}} \cap \gamma_{\mathfrak{s}}(N_{\mathfrak{t}, \mathfrak{s}}(\varepsilon)),$$

restricting to a diffeomorphism on the complement of $M_{\mathfrak{s}}$ and identifying $M_{\mathfrak{s}}$ with its image in $\mathcal{M}_{\mathfrak{t}}$ under the embedding (2.3.12). We often refer to the image $\gamma_{\mathfrak{s}}(N_{\mathfrak{t}, \mathfrak{s}}(\varepsilon))$ as a *virtual moduli space*.

Our terminology is loosely motivated by that of [46] and [86], where the goal (translated to our setting) would be to construct a *virtual fundamental class* for $\mathcal{M}_{\mathfrak{t}}$, given by the cap product of the fundamental class of a virtual space containing $\mathcal{M}_{\mathfrak{t}}$ with the Euler class of a vector bundle over this virtual space, where $\mathcal{M}_{\mathfrak{t}}$ is the zero locus of a (possibly non-transversally vanishing) section. Here, $\gamma_{\mathfrak{s}}(N_{\mathfrak{t}, \mathfrak{s}}(\varepsilon))$ plays the role of the virtual space and (the pullback of) $\Xi_{\mathfrak{t}, \mathfrak{s}}$ the vector bundle with zero section yielding (an open neighborhood in) $\mathcal{M}_{\mathfrak{t}}$. Then, $N_{\mathfrak{t}, \mathfrak{s}}$ is the normal bundle of

$M_{\mathfrak{s}} \hookrightarrow \gamma_{\mathfrak{s}}(N_{\mathfrak{t},\mathfrak{s}}(\varepsilon))$, while $[N_{\mathfrak{t},\mathfrak{s}}] - [\Xi_{\mathfrak{t},\mathfrak{s}}]$ would more properly be called the ‘virtual normal bundle’ of $M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{\mathfrak{t}}$, in the language of K -theory.

By [28, Equations (2.47) and (3.35)], the negative of the index of the $\mathrm{SO}(3)$ -monopole elliptic deformation complex at a reducible solution can be written as

$$(2.3.23) \quad \dim \mathcal{M}_{\mathfrak{t}} = 2n_s(\mathfrak{t}, \mathfrak{s}) + d_s(\mathfrak{s}),$$

where $d_s(\mathfrak{s})$ is the expected dimension of the Seiberg–Witten moduli space $M_{\mathfrak{s}}$ (see equation (2.3.10)), while $n_s(\mathfrak{t}, \mathfrak{s}) = n'_s(\mathfrak{t}, \mathfrak{s}) + n''_s(\mathfrak{t}, \mathfrak{s})$ is minus the complex index of the normal deformation operator [28, Equations (3.71) and (3.72)], with

$$(2.3.24) \quad \begin{aligned} n'_s(\mathfrak{t}, \mathfrak{s}) &= -(c_1(\mathfrak{t}) - c_1(\mathfrak{s}))^2 - \frac{1}{2}(\chi + \sigma), \\ n''_s(\mathfrak{t}, \mathfrak{s}) &= \frac{1}{8}(c_1(\mathfrak{s}) - 2c_1(\mathfrak{t}))^2 - \sigma. \end{aligned}$$

If r_{Ξ} is the complex rank of $\Xi_{\mathfrak{t},\mathfrak{s}} \rightarrow M_{\mathfrak{s}}$, and $r_N(\mathfrak{t}, \mathfrak{s})$ is the complex rank of $N_{\mathfrak{t},\mathfrak{s}} \rightarrow M_{\mathfrak{s}}$, then

$$(2.3.25) \quad r_N(\mathfrak{t}, \mathfrak{s}) = n_s(\mathfrak{t}, \mathfrak{s}) + r_{\Xi},$$

as we can see from the dimension relation (2.3.23) and the topological model (2.3.22).

The map (2.3.20) is S^1 -equivariant when S^1 acts trivially on $M_{\mathfrak{s}}$, by scalar multiplication on the fibers of $N_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$, and by the action (2.3.15) on $\mathcal{C}_{\mathfrak{t}}$. The bundle (2.3.21) and section $\chi_{\mathfrak{s}}$ are S^1 -equivariant if S^1 acts on $N_{\mathfrak{t},\mathfrak{s}}$ and the fibers of $\gamma_{\mathfrak{s}}^* \Xi_{\mathfrak{t},\mathfrak{s}}$ by scalar multiplication.

Let $\tilde{N}_{\mathfrak{t},\mathfrak{s}} \rightarrow \tilde{M}_{\mathfrak{s}}$ be the pullback of $N_{\mathfrak{t},\mathfrak{s}}$ by the projection $\tilde{M}_{\mathfrak{s}} \rightarrow M_{\mathfrak{s}} = \tilde{M}_{\mathfrak{s}}/\mathcal{G}_{\mathfrak{s}}$, so $\tilde{N}_{\mathfrak{t},\mathfrak{s}}$ is a $\mathcal{G}_{\mathfrak{s}}$ -equivariant bundle, where $\mathcal{G}_{\mathfrak{s}}$ acts on the base $\tilde{M}_{\mathfrak{s}}$ by the usual gauge group action (2.3.3) and the induced action on the total space,

$$(2.3.26) \quad \tilde{N}_{\mathfrak{t},\mathfrak{s}} \subset \tilde{M}_{\mathfrak{s}} \times L_k^2(\Lambda^1 \otimes_{\mathbb{R}} L) \oplus L^2(W^+ \otimes L) \subset \tilde{M}_{\mathfrak{s}} \times L_k^2(\Lambda^1 \otimes_{\mathbb{R}} \mathfrak{g}_{\mathfrak{t}}) \oplus L^2(V^+),$$

via the embedding (2.3.13) of $\mathcal{G}_{\mathfrak{s}}$ into $\mathcal{G}_{\mathfrak{t}}$ and the splittings $\mathfrak{g}_{\mathfrak{t}} \cong \mathbb{R} \oplus L$ [28, Lemma 3.10] and $V = W \oplus W \otimes L$. Thus, $s \in \mathcal{G}_{\mathfrak{s}}$ acts by scalar multiplication by s^{-2} on sections of $\Lambda^1 \otimes_{\mathbb{R}} L$ and by s^{-1} on sections of $W^+ \otimes L$ [28, Section 3.5.4].

For a small enough positive ε , there is a smooth embedding [28, Section 3.5.4] of the open tubular neighborhood $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$,

$$(2.3.27) \quad \tilde{\gamma}_{\mathfrak{s}} : \tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon) \rightarrow \tilde{\mathcal{C}}_{\mathfrak{t}},$$

which is gauge-equivariant with respect to the preceding action of $\mathcal{G}_{\mathfrak{s}}$, and covers the topological embedding (2.3.20). The map (2.3.27) is S^1 -equivariant, where S^1 acts trivially on $\tilde{M}_{\mathfrak{s}}$, by scalar multiplication on the fibers of $\tilde{N}_{\mathfrak{t},\mathfrak{s}}(\varepsilon)$, and by the action (2.3.15) on $\mathcal{C}_{\mathfrak{t}}$. We note that the map (2.3.27) is also S^1 -equivariant with respect to the action (2.1.9) on $\mathcal{C}_{\mathfrak{t}}$, if S^1 acts on $\tilde{N}_{\mathfrak{t},\mathfrak{s}}$ by

$$(2.3.28) \quad (e^{i\theta}, (B, \Psi, \beta, \psi)) \mapsto \varrho(e^{i\theta})(B, \Psi, e^{2i\theta}\beta, e^{2i\theta}\psi) = (B, e^{i\theta}\Psi, \beta, e^{i\theta}\psi),$$

where $(B, \Psi) \in \tilde{M}_{\mathfrak{s}}$ and $(\beta, \psi) \in L_k^2(\Lambda^1 \otimes_{\mathbb{R}} L) \oplus L^2(W^+ \otimes L)$, so $(B, \Psi, \beta, \psi) \in \tilde{N}_{\mathfrak{t},\mathfrak{s}}$, and $\varrho : \mathcal{G}_{\mathfrak{s}} \rightarrow \mathcal{G}_{\mathfrak{t}}$ is the homomorphism (2.3.13). This equivariance follows from the relation (2.3.16) between the actions (2.3.15) and (2.1.9).

We note that the Chern character of the bundle $N_{\mathfrak{t},\mathfrak{s}}$ is computed in [28, Theorem 3.29] while the Segre classes of this bundle are computed, under some additional assumptions, in [29, Lemma 4.11].

2.4. Cohomology classes on the moduli space of SO(3) monopoles

The identity (1.1.1) arises as an equality between pairings of suitable cohomology classes with a link in $\bar{\mathcal{M}}_t^{*,0}/S^1$ of the anti-self-dual moduli subspace and with the links of the Seiberg–Witten moduli subspaces in $\bar{\mathcal{M}}_t/S^1$. We now review the definitions of these cohomology classes and their dual geometric representatives given in [29, Section 3].

The first kind of cohomology class is defined on \mathcal{M}_t^*/S^1 , via the associated SO(3) bundle,

$$(2.4.1) \quad \mathbb{F}_t = \tilde{\mathcal{C}}_t^*/S^1 \times_{\mathcal{G}_t} \mathfrak{g}_t \rightarrow \mathcal{C}_t^*/S^1 \times X.$$

The group \mathcal{G}_t acts diagonally in (2.4.1), with \mathcal{G}_t acting on the left on \mathfrak{g}_t . We define [29, Section 3.1]

$$(2.4.2) \quad \mu_p : H_\bullet(X; \mathbb{R}) \rightarrow H^{4-\bullet}(\mathcal{C}_t^*/S^1; \mathbb{R}), \quad \beta \mapsto -\frac{1}{4}p_1(\mathbb{F}_t)/\beta.$$

On restriction to $M_\kappa^w \hookrightarrow \mathcal{M}_t$, the cohomology classes $\mu_p(\beta)$ coincide with those used in the definition of Donaldson invariants [29, Lemma 3.1]. Define

$$(2.4.3) \quad \mathbb{A}(X) = \text{Sym}(H_{\text{even}}(X; \mathbb{R})) \otimes \Lambda^\bullet(H_{\text{odd}}(X; \mathbb{R}))$$

to be the graded algebra, with $z = \beta_1\beta_2 \cdots \beta_r$ having total degree $\deg(z) = \sum_p(4 - i_p)$, when $\beta_p \in H_{i_p}(X; \mathbb{R})$. Then μ_p extends in the usual way to a homomorphism of graded real algebras,

$$(2.4.4) \quad \mu_p : \mathbb{A}(X) \rightarrow H^\bullet(\mathcal{C}_t^*/S^1; \mathbb{R}),$$

which preserves degrees. Next, we define a complex line bundle over $\mathcal{C}_t^{*,0}/S^1$,

$$(2.4.5) \quad \mathbb{L}_t = \mathcal{C}_t^{*,0} \times_{(S^1, \times_{-2})} \mathbb{C},$$

where the S^1 action is given, for $[A, \Phi] \in \mathcal{C}_t^{*,0}$ and $\zeta \in \mathbb{C}$, by

$$(2.4.6) \quad ([A, \Phi], \zeta) \mapsto ([A, e^{i\theta}\Phi], e^{2i\theta}\zeta).$$

Then we define the second kind of cohomology class on $\mathcal{M}_t^{*,0}/S^1$ by

$$(2.4.7) \quad \mu_c = c_1(\mathbb{L}_t) \in H^2(\mathcal{C}_t^{*,0}/S^1; \mathbb{R}).$$

For monomials $z \in \mathbb{A}(X)$, we constructed [29, Section 3.2] geometric representatives $\mathcal{V}(z)$ dual to $\mu_p(z)$ (following the discussion in [53]) and \mathcal{W} dual to μ_c , defined on \mathcal{M}_t^*/S^1 and $\mathcal{M}_t^{*,0}/S^1$, respectively; their closures in $\bar{\mathcal{M}}_t/S^1$ are denoted by $\bar{\mathcal{V}}(z)$ and $\bar{\mathcal{W}}$ [29, Definition 3.14]. When

$$(2.4.8) \quad \deg(z) + 2\eta = \dim(\mathcal{M}_t^{*,0}/S^1) - 1,$$

and $\deg(z) \geq \dim M_\kappa^w$ it follows from [29, Section 3.3] that the intersection

$$(2.4.9) \quad \bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathcal{M}}_t^{*,0}/S^1,$$

is an oriented one-manifold (not necessarily connected) whose closure in $\bar{\mathcal{M}}_t/S^1$ can only intersect $(\bar{\mathcal{M}}_t - \mathcal{M}_t)/S^1$ at points in $\mathcal{M}_t^{\text{red}} \cong \cup(M_\mathfrak{s} \times \text{Sym}^\ell(X))$, where the union is over $\ell \geq 0$ and $\mathfrak{s} \in \text{Spin}^c(X)$ [29, Corollary 3.18].

2.5. Donaldson invariants

We first recall the definition [53, Section 2] of the Donaldson series when $b_1(X) = 0$ and $b^+(X) > 1$ is odd, so that, in this case, $\chi(X) + \sigma(X) \equiv 0 \pmod{4}$. See also Section 3.4.2 in [29], especially for a definition of the Donaldson invariants when $b^+(X) = 1$. For any choice of $w \in H^2(X; \mathbb{Z})$, the Donaldson invariant is a linear function,

$$D_X^w : \mathbb{A}(X) \rightarrow \mathbb{R}.$$

Let $\tilde{X} = X \# \overline{\mathbb{C}\mathbb{P}^2}$ be the blow-up of X and let $e \in H_2(\tilde{X}; \mathbb{Z})$ be the exceptional class, with Poincaré dual $\text{PD}[e] \in H^2(\tilde{X}; \mathbb{Z})$. If $z \in \mathbb{A}(X)$ is a monomial, we define $D_X^w(z) = 0$ unless

$$(2.5.1) \quad \deg(z) \equiv -2w^2 - \frac{3}{2}(\chi + \sigma) \pmod{8}.$$

If $\deg(z)$ obeys equation (2.5.1), we let $\kappa \in \frac{1}{4}\mathbb{Z}$ be defined by

$$\deg(z) = 8\kappa - \frac{3}{2}(\chi + \sigma).$$

There exists an $\text{SO}(3)$ bundle over \tilde{X} with first Pontrjagin number $-4\kappa - 1$ and second Stiefel–Whitney class $w + \text{PD}[e] \pmod{2}$. One then defines the Donaldson invariant on monomials by

$$(2.5.2) \quad D_X^w(z) = \# \left(\bar{\mathcal{V}}(ze) \cap \bar{M}_{\kappa+1/4}^{w+\text{PD}[e]}(\tilde{X}) \right),$$

and extends D_X^w to a real linear function on $\mathbb{A}(X)$. Note that $w + \text{PD}[e] \pmod{2}$ is good in the sense of Definition 2.2.1. If $w' \equiv w \pmod{2}$, then [14]

$$(2.5.3) \quad D_X^{w'} = (-1)^{\frac{1}{4}(w'-w)^2} D_X^w.$$

The Donaldson series is a formal power series,

$$(2.5.4) \quad \mathbf{D}_X^w(h) = D_X^w\left(\left(1 + \frac{1}{2}x\right)e^h\right), \quad h \in H_2(X; \mathbb{R}).$$

By equation (2.5.1), the series \mathbf{D}_X^w is even if

$$-w^2 - \frac{3}{4}(\chi + \sigma) \equiv 0 \pmod{2},$$

and odd otherwise. A four-manifold has Kronheimer–Mrowka simple type if for some w and all $z \in \mathbb{A}(X)$,

$$D_X^w(x^2 z) = 4D_X^w(z).$$

According to [53, Theorem 1.7], when X has Kronheimer–Mrowka simple type the series $\mathbf{D}_X^w(h)$ is an analytic function of h and there are finitely many characteristic cohomology classes, K_1, \dots, K_m in $H^2(X; \mathbb{Z})$ (the Kronheimer–Mrowka basic classes), and non-zero rational numbers, a_1, \dots, a_m (independent of w), so that

$$\mathbf{D}_X^w(h) = e^{\frac{1}{2}h \cdot h} \sum_{i=1}^r (-1)^{\frac{1}{2}(w^2 + K_i \cdot w)} a_i e^{\langle K_i, h \rangle}.$$

Witten’s Conjecture [103] then relates the Donaldson and Seiberg–Witten series for four-manifolds of simple type.

When $b^+(X) = 1$, the pairing on the right-hand side of definition (2.5.2) depends on the chamber in the positive cone of $H^2(\tilde{X}; \mathbb{R})$ determined by the period point of the Riemannian metric on \tilde{X} ,

just as in the case of Seiberg–Witten invariants described in Section 2.3.2. We refer to Section 3.4.2 in [29] for a detailed discussion of this case and, as in [29], we assume that the class $w \pmod{2}$ is good in order to avoid technical difficulties involved in chamber specification.

2.6. Links and the cobordism

Since the ends of the components of the one-manifold (2.4.9) either lie near M_κ^w or $M_\mathfrak{s} \times \text{Sym}^\ell(X)$, for some \mathfrak{s} and $\ell(\mathfrak{t}, \mathfrak{s}) \geq 0$ for which $\mathfrak{t}(\ell) = \mathfrak{s} \oplus \mathfrak{s} \otimes L$, we have when $w_2(\mathfrak{t})$ is good in the sense of Definition 2.2.1 that (see Theorem 8.1.9)

$$(2.6.1) \quad \#(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{n_a-1} \cap \bar{\mathbf{L}}_{\mathfrak{t}, \kappa}^w) = - \sum_{\mathfrak{s} \in \text{Spin}^c(X)} \#(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{n_a-1} \cap \bar{\mathbf{L}}_{\mathfrak{t}, \mathfrak{s}}),$$

where $\bar{\mathbf{L}}_{\mathfrak{t}, \kappa}^w$ is the link of \bar{M}_κ^w in $\bar{\mathcal{M}}_\mathfrak{t}/S^1$ (see [28, Definition 3.7]) and where $\bar{\mathbf{L}}_{\mathfrak{t}, \mathfrak{s}}$ is empty if $\ell(\mathfrak{t}, \mathfrak{s}) < 0$ and is the boundary of an open neighborhood of the Seiberg–Witten stratum $M_\mathfrak{s} \times \text{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_\mathfrak{t}/S^1$ if $\ell = \ell(\mathfrak{t}, \mathfrak{s}) \geq 0$. By construction, the intersection of $\bar{\mathbf{L}}_{\mathfrak{t}, \mathfrak{s}}$ with the top stratum of $\bar{\mathcal{M}}_\mathfrak{t}/S^1$ is a smoothly-stratified space and the intersection of $\bar{\mathbf{L}}_{\mathfrak{t}, \mathfrak{s}}$ with the geometric representatives is in the top stratum of $\bar{\mathbf{L}}_{\mathfrak{t}, \mathfrak{s}}$. The precise definition of $\bar{\mathbf{L}}_{\mathfrak{t}, \mathfrak{s}}$ is given in [28, Definition 3.22] for $\ell = 0$ and in Definition 8.1.3 for $\ell \geq 1$.

When $\deg(z) = \dim M_\kappa^w$ and $n_a(\mathfrak{t}) > 0$, the intersection of the one-manifold (2.4.9) with the link $\bar{\mathbf{L}}_{\mathfrak{t}, \kappa}^w$ is given by [29, Lemma 3.30]

$$(2.6.2) \quad 2^{1-n_a} \#(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{n_a-1} \cap \bar{\mathbf{L}}_{\mathfrak{t}, \kappa}^w) = \#(\bar{\mathcal{V}}(z) \cap \bar{M}_\kappa^w).$$

Applying this identity to the blow-up, $X \# \overline{\mathbb{C}\mathbb{P}^2}$, when $n_a(\mathfrak{t}) > 0$, we recover the Donaldson invariant $D_X^w(z)$ on the right-hand side of (2.6.2) via definition (2.5.2).

REMARK 2.6.1. If in (2.6.2) we have $\deg(z) > \dim M_\kappa^w$ and we replace $\bar{\mathcal{W}}^{n_a-1}$ with $\bar{\mathcal{W}}^\eta$ where η satisfies (2.4.8), then the intersection number vanishes. If $n_a(\mathfrak{t}) \leq 0$ and $\deg(z)$ still satisfies (2.4.8) with $\eta = 0$, then the intersection number in (2.6.2) is a constant times the spin polynomial invariant [84].

CHAPTER 3

Diagonals of symmetric products of manifolds

To describe the link of the family of singularities, $M_5 \times \text{Sym}^\ell(X)$, in $\bar{\mathcal{M}}_t$, we will need to describe the strata of the symmetric product $\text{Sym}^\ell(X)$, the normal bundles of the strata, and their incidence relations.

3.1. Definitions

In this section, we introduce some vocabulary for describing the action of the symmetric group on X^ℓ , the fixed point sets of this action, and the resulting strata of the symmetric product $\text{Sym}^\ell(X)$. For basic definitions of the vocabulary of group actions, we refer to [100, pp. 2–8].

3.1.1. Subgroups of the symmetric group. For any set P , let \mathfrak{S}_P be the group of bijections, $\sigma : P \rightarrow P$. For $P = N_\ell = \{1, \dots, \ell\}$, we will write \mathfrak{S}_ℓ for the symmetric group on ℓ elements. We will write *partitions* of N_ℓ as $\mathcal{P} = \{P_1, \dots, P_{r(\mathcal{P})}\}$ where $P_i \subset N_\ell$. We refer to the number of sets in \mathcal{P} as the *length* of \mathcal{P} . Each such partition of N_ℓ gives a partition of ℓ : $\ell = |P_1| + \dots + |P_{r(\mathcal{P})}|$. The symmetric group \mathfrak{S}_ℓ acts on the set of partitions of N_ℓ by

$$(3.1.1) \quad (\sigma, \mathcal{P}) \rightarrow \sigma(\mathcal{P}) := \{\sigma(P_1), \dots, \sigma(P_{r(\mathcal{P})})\}.$$

The orbits of this action are distinguished by the partitions of ℓ . Let $\Gamma(\mathcal{P}) < \mathfrak{S}_\ell$ be the stabilizer of a partition \mathcal{P} , considered as an ordered set, with respect to the action (3.1.1).

LEMMA 3.1.1. *Let \mathcal{P} be a partition of N_ℓ . Then the subgroup $\Gamma(\mathcal{P})$ of \mathfrak{S}_ℓ is given by the image of the inclusion,*

$$(3.1.2) \quad \prod_{P \in \mathcal{P}} \mathfrak{S}_P \rightarrow \mathfrak{S}_\ell.$$

PROOF. Let $\Gamma'(\mathcal{P})$ be the image of the homomorphism (3.1.2). There is then an inclusion $\Gamma'(\mathcal{P}) \rightarrow \Gamma(\mathcal{P})$. This map is surjective because any $\sigma \in \mathfrak{S}_\ell$ which preserves the subsets $P_1, \dots, P_{r(\mathcal{P})}$ is a permutation of each of these subsets and thus in $\Gamma'(\mathcal{P})$. \square

LEMMA 3.1.2. *Let $\mathfrak{S}(\mathcal{P}) \leq \mathfrak{S}_\ell$ be the normalizer of $\Gamma(\mathcal{P})$. Then for all $\sigma \in \mathfrak{S}(\mathcal{P})$ and all $P \in \mathcal{P}$, one has that $\sigma(P) \in \mathcal{P}$.*

PROOF. Assume that there are $\sigma \in \mathfrak{S}(\mathcal{P})$ and $P \in \mathcal{P}$ with $\sigma(P) \notin \mathcal{P}$. If $|P| = 1$, then there is a $P' \in \mathcal{P}$ with $\sigma(P) \subsetneq P'$. The strictness of the preceding inclusion implies that $|P'| > 1$. Thus, there are $a, b \in P'$ with $a \in \sigma(P)$ and $b \notin \sigma(P)$. By Lemma 3.1.1, the transposition $(a \ b)$ is in $\Gamma(\mathcal{P})$ and by the definition of $\mathfrak{S}(\mathcal{P})$ as the normalizer of $\Gamma(\mathcal{P})$, we have $\sigma^{-1}(a \ b)\sigma = (c \ d) \in \Gamma(\mathcal{P})$. However, $c = \sigma^{-1}(a) \in P$ while $d = \sigma^{-1}(b) \notin P$, so $(c \ d) \notin \Gamma(\mathcal{P})$. This contradiction proves the lemma if $|P| = 1$.

If $|P| > 1$, then there are $a, b \in \sigma(P)$ with $a \neq b$ and $P' \in \mathcal{P}$ with $a \in P'$ and $b \notin P'$. By Lemma 3.1.1, the transposition $(a \ b)$ is not in $\Gamma(\mathcal{P})$. However, because $\sigma^{-1}(a), \sigma^{-1}(b) \in P$, the transposition $\sigma^{-1}(a \ b)\sigma = (\sigma^{-1}(a) \ \sigma^{-1}(b))$ is in $\Gamma(\mathcal{P})$, contradicting the definition of $\mathfrak{S}(\mathcal{P})$ as the normalizer of $\Gamma(\mathcal{P})$. \square

Lemma 3.1.2 shows that $\mathfrak{S}(\mathcal{P})$ acts via (3.1.1) as a permutation of \mathcal{P} . If we define

$$(3.1.3) \quad W(\mathcal{P}) := \mathfrak{S}(\mathcal{P})/\Gamma(\mathcal{P}),$$

then $W(\mathcal{P})$ will act freely on \mathcal{P} .

3.1.2. Definition of the diagonals. We define the strata of $\text{Sym}^\ell(X)$ by the quotients of diagonals in X^ℓ by the permutation action of \mathfrak{S}_ℓ on X^ℓ . For a partition \mathcal{P} of N_ℓ , define

$$(3.1.4) \quad \begin{aligned} \Delta^\circ(X^\ell, \mathcal{P}) &:= \{(x_1, \dots, x_\ell) \in X^\ell : x_i = x_j \text{ if and only if } \exists P \in \mathcal{P} \text{ with } i, j \in P\}, \\ \Delta(X^\ell, \mathcal{P}) &:= \{(x_1, \dots, x_\ell) \in X^\ell : x_i = x_j \text{ if } \exists P \in \mathcal{P} \text{ with } i, j \in P\}. \end{aligned}$$

The preceding subspaces are related by

$$(3.1.5) \quad \text{cl}_{X^\ell} \Delta^\circ(X^\ell, \mathcal{P}) = \Delta(X^\ell, \mathcal{P}).$$

The *big* (respectively, *small*) *diagonal* in X^ℓ comprises the set of points $(x_1, \dots, x_\ell) \in X^\ell$ with $x_i = x_j$ for at least one (respectively, all) pair i, j . We can identify $\Delta^\circ(X^\ell, \mathcal{P})$ with the complement of the big diagonal in X^r (where r is the length of \mathcal{P}) as follows. Write \mathcal{P} as an ordered collection $P_1, \dots, P_{r(\mathcal{P})}$. Then define an embedding,

$$(3.1.6) \quad \iota_{\mathcal{P}} : X^r \setminus \bigcup_{i < j} \{x_i = x_j\} \rightarrow \Delta^\circ(X^\ell, \mathcal{P}) \subset X^\ell,$$

by $\iota(x_1, \dots, x_r) = (y_1, \dots, y_\ell)$, where $y_k = x_i$ for $k \in P_i$. The map $\iota_{\mathcal{P}}$ is a diffeomorphism.

LEMMA 3.1.3. *Let \mathcal{P} be a partition of N_ℓ . Let \mathfrak{S}_ℓ act on X^ℓ by the permutation action,*

$$(3.1.7) \quad (\sigma, (x_1, \dots, x_n)) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

Then the following hold:

- (1) *The diagonal $\Delta(X^\ell, \mathcal{P})$ is the fixed-point set of the group $\Gamma(\mathcal{P})$;*
- (2) *For $\mathbf{x} \in X^\ell$, one has $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$ if and only if $\text{Stab}_{\mathbf{x}} = \Gamma(\mathcal{P})$;*
- (3) *If $\sigma \in \mathfrak{S}_\ell$ then $\sigma(\Delta^\circ(X^\ell, \mathcal{P})) \subset \Delta^\circ(X^\ell, \mathcal{P})$ if and only if $\sigma \in \mathfrak{S}(\mathcal{P})$.*

PROOF. That $\Delta(X^\ell, \mathcal{P})$ is contained in the fixed-point set of $\Gamma(\mathcal{P})$ follows immediately from the definitions. The fixed-point set of \mathfrak{S}_i acting on X^i is the small diagonal. Thus, by the characterization of $\Gamma(\mathcal{P})$ in Lemma 3.1.1, if $\mathbf{x} \in X^\ell$ has the property that $\sigma(\mathbf{x}) = \mathbf{x}$ for all $\sigma \in \Gamma(\mathcal{P})$, then $x_i = x_j$ for all $i, j \in P$ for every $P \in \mathcal{P}$. Hence, the fixed-point set of $\Gamma(\mathcal{P})$ is contained in $\Delta(X^\ell, \mathcal{P})$, proving Item (1).

Item (1) implies that $\Gamma(\mathcal{P}) \subset \text{Stab}_{\mathbf{x}}$ for $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$. If $\sigma \in \text{Stab}_{\mathbf{x}}$ and $\sigma \notin \Gamma(\mathcal{P})$, then there are $P \in \mathcal{P}$ and $i \in P$ such that $\sigma(i) \notin P$. But $\sigma(\mathbf{x}) = \mathbf{x}$ then implies that $x_{\sigma(i)} = x_i$, which contradicts the fact that $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$. Therefore, $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$ implies that $\text{Stab}_{\mathbf{x}} = \Gamma(\mathcal{P})$. If $\text{Stab}_{\mathbf{x}} = \Gamma(\mathcal{P})$, then Item (1) implies that $\mathbf{x} \in \Delta(X^\ell, \mathcal{P})$. If $\mathbf{x} \notin \Delta^\circ(X^\ell, \mathcal{P})$ then there are $i \in P_a$ and $j \in P_b$, $P_a \neq P_b \in \mathcal{P}$, with $x_i = x_j$. Consequently, for $\sigma = (i \ j) \in \mathfrak{S}_\ell$, one has $\sigma(\mathbf{x}) = \mathbf{x}$ but $\sigma \notin \Gamma(\mathcal{P})$, a contradiction, and this proves Item (2).

Item (3) follows immediately from the definition of $\mathfrak{S}(\mathcal{P})$ as the normalizer of $\Gamma(\mathcal{P})$, Item (2), and the relation $\text{Stab}_{\sigma(\mathbf{x})} = \sigma \text{Stab}_{\mathbf{x}} \sigma^{-1}$ (see [100, p. 3]). \square

3.1.3. Strata of the symmetric product. Let $\tilde{\pi}_\ell : X^\ell \rightarrow \text{Sym}^\ell(X) = X^\ell/\mathfrak{S}_\ell$ be the projection. Define the *stratum of the symmetric product* $\text{Sym}^\ell(X)$ corresponding to a partition \mathcal{P} by

$$(3.1.8) \quad \Sigma(X^\ell, \mathcal{P}) := \tilde{\pi}_\ell \left(\Delta^\circ(X^\ell, \mathcal{P}) \right).$$

Lemma 3.1.3 implies that

$$(3.1.9) \quad \Sigma(X^\ell, \mathcal{P}) = \Delta^\circ(X^\ell, \mathcal{P})/\mathfrak{S}(\mathcal{P}) = \Delta^\circ(X^\ell, \mathcal{P})/W(\mathcal{P}).$$

The result below follows immediately from the definitions.

LEMMA 3.1.4. *Let \mathcal{P} be a partition of N_ℓ and let $[\mathcal{P}]$ denote the orbit of \mathcal{P} under \mathfrak{S}_ℓ . Then,*

$$\tilde{\pi}_\ell^{-1} \left(\Sigma(X^\ell, \mathcal{P}) \right) = \bigcup_{\mathcal{P}' \in [\mathcal{P}]} \Delta^\circ(X, \mathcal{P}').$$

3.2. Incidence relations among diagonals and strata

We now describe incidence relations among the diagonals of X^ℓ and the resulting relations among the strata Σ of $\text{Sym}^\ell(X)$.

DEFINITION 3.2.1. Let \mathcal{P} and \mathcal{P}' be partitions of N_ℓ . We say \mathcal{P}' is a *refinement* of \mathcal{P} , or $\mathcal{P} < \mathcal{P}'$, if for every $P' \in \mathcal{P}'$ there is a $P \in \mathcal{P}$ with $P' \subseteq P$.

LEMMA 3.2.2. *If \mathcal{P} and \mathcal{P}' are partitions of N_ℓ then $\Delta^\circ(X^\ell, \mathcal{P}) \subset \Delta(X, \mathcal{P}')$ if and only if $\mathcal{P} < \mathcal{P}'$.*

PROOF. If \mathcal{P}' is a refinement of \mathcal{P} and $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$, then for every $P' \in \mathcal{P}'$ and $i, j \in P'$, there is a $P \in \mathcal{P}$ with $i, j \in P' \subset P$. Since $i, j \in P \in \mathcal{P}$, we have $x_i = x_j$ so $\mathbf{x} \in \Delta(X, \mathcal{P}')$, proving one implication.

Conversely, if $\Delta^\circ(X^\ell, \mathcal{P}) \subset \Delta(X, \mathcal{P}')$ and $P' \in \mathcal{P}'$ then, because every $\mathbf{x} = (x_1, \dots, x_\ell) \in \Delta^\circ(X^\ell, \mathcal{P})$ is also in $\Delta(X, \mathcal{P}')$, we see that for every $i, j \in P'$, we have $x_i = x_j$. This implies (by the only if in the definition of $\Delta^\circ(X^\ell, \mathcal{P})$) that there must be a $P \in \mathcal{P}$ with $i, j \in P$. Thus, $P' \subset P$ and $\mathcal{P} < \mathcal{P}'$. \square

Now, we compare the incidence relations with the strata of $\text{Sym}^\ell(X)$. Recall that for a partition \mathcal{P} of N_ℓ , we use $[\mathcal{P}]$ to denote the orbit of \mathcal{P} under the action (3.1.1) of \mathfrak{S}_ℓ on the set of partitions of N_ℓ . For partitions \mathcal{P} and \mathcal{P}' of N_ℓ , define

$$(3.2.1) \quad [\mathcal{P} < \mathcal{P}'] := \{\mathcal{P}'' \in [\mathcal{P}'] : \mathcal{P} < \mathcal{P}''\}.$$

We now give an example to show that while $\mathfrak{S}(\mathcal{P})$ acts on $[\mathcal{P} < \mathcal{P}']$ by the action (3.1.1), this action need not be transitive.

EXAMPLE 3.2.3. Consider the partitions $\mathcal{P} = \{P_1, P_2\}$, where $P_1 = \{1, 2\}$ and $P_2 = \{3, 4, 5\}$, and $\mathcal{P}' = \{Q_1, Q_2, Q_3, Q_4\}$, where $Q_1 = \{1\}$, $Q_2 = \{2\}$, $Q_3 = \{3\}$, and $Q_4 = \{4, 5\}$. Observe that $\mathcal{P} < \mathcal{P}'$ because $Q_1, Q_2 \subset P_1$ and $Q_3, Q_4 \subset P_2$. Define $\sigma := (1\ 4)(2\ 5) \in \mathfrak{S}_5$. Then $\sigma(\mathcal{P}') = \mathcal{P}'' = \{R_1, R_2, R_3, R_4\}$, where $R_1 = \{1, 2\}$, $R_2 = \{3\}$, $R_3 = \{4\}$ and $R_4 = \{5\}$. Because

$R_1 = P_1$ and $R_2, R_3, R_4 \subset P_2$, we have $\mathcal{P} < \mathcal{P}'$. Thus, $\mathcal{P}', \sigma(\mathcal{P}') \in [\mathcal{P} < \mathcal{P}']$ but $\sigma \notin \mathfrak{S}(\mathcal{P})$ because σ does not satisfy the conclusion of Lemma 3.1.2.

The next lemma follows from the definition of $\Sigma(X^\ell, \mathcal{P})$ (see (3.1.8)) as well as Lemma 3.2.2 and the third assertion of Lemma 3.1.4.

LEMMA 3.2.4. *If \mathcal{P} and \mathcal{P}' are partitions of N_ℓ , then*

$$\Sigma(X^\ell, \mathcal{P}) \subset \text{cl}_{\text{Sym}^\ell(X)} \Sigma(X^\ell, \mathcal{P}')$$

if and only if there is a $\mathcal{P}'' \in [\mathcal{P}']$ with $\mathcal{P} < \mathcal{P}''$.

3.3. Normal bundles of diagonals and strata

We now introduce the normal bundles of the diagonals $\Delta^\circ(X^\ell, \mathcal{P})$ of X^ℓ .

LEMMA 3.3.1. *Let \mathcal{P} be any partition of N_ℓ . The tangent bundle of the submanifold $\Delta^\circ(X^\ell, \mathcal{P})$ is given by*

$$(3.3.1) \quad T\Delta^\circ(X^\ell, \mathcal{P}) = \left\{ (v_1, \dots, v_\ell) \in TX^\ell : v_i = v_j \iff \exists P \in \mathcal{P} \text{ with } i, j \in P \right\}.$$

If g is any metric on X , then the orthogonal complement of $T\Delta^\circ(X^\ell, \mathcal{P})$ with respect to the metric on X^ℓ given by the ℓ -fold product of g is the following normal bundle of $\Delta^\circ(X^\ell, \mathcal{P})$,

$$(3.3.2) \quad \tilde{\nu}(X^\ell, \mathcal{P}) := \left\{ (v_1, \dots, v_\ell) \in TX^\ell|_{\Delta^\circ(X^\ell, \mathcal{P})} : \sum_{i \in P} v_i = 0 \text{ for all } P \in \mathcal{P} \right\}.$$

PROOF. Differentiate any path in $\Delta^\circ(X^\ell, \mathcal{P})$ to see the form of $T\Delta^\circ(X^\ell, \mathcal{P})$ appearing in equation (3.3.1). For $\vec{v} = (v_1, \dots, v_\ell) \in T\Delta^\circ(X^\ell, \mathcal{P})$, define $v_P = v_i$ for any i with $i \in P$. If $\vec{v} = (v_1, \dots, v_\ell) \in T\Delta^\circ(X^\ell, \mathcal{P})$ and $\vec{w} = (w_1, \dots, w_\ell) \in \tilde{\nu}(X^\ell, \mathcal{P})$, then the equality,

$$\vec{v} \cdot \vec{w} = \sum_i v_i \cdot w_i = \sum_{P \in \mathcal{P}} v_P \cdot \left(\sum_{i \in P} w_i \right) = 0,$$

which holds for any product metric on X^ℓ , implies that the bundle in equation (3.3.2) is contained in the orthogonal complement of $T\Delta^\circ(X^\ell, \mathcal{P})$. Since the above inclusion must hold for any value of v_P , we see that any element of the orthogonal complement of $T\Delta^\circ(X^\ell, \mathcal{P})$ must satisfy the equations defining the bundle in (3.3.2). \square

Lemma 3.3.1 leads to the following description of a neighborhood of $\Delta^\circ(X^\ell, \mathcal{P})$.

LEMMA 3.3.2. *There is an open neighborhood, $\tilde{\mathcal{O}}(X^\ell, \mathcal{P}) \subset \tilde{\nu}(X^\ell, \mathcal{P})$, of the zero-section and an $\mathfrak{S}(\mathcal{P})$ -equivariant exponential map identifying $\tilde{\mathcal{O}}(X^\ell, \mathcal{P})$ with an open neighborhood, $\tilde{\mathcal{U}}(X^\ell, \mathcal{P})$, of $\Delta^\circ(X^\ell, \mathcal{P})$ in X^ℓ .*

REMARK 3.3.3. If \mathcal{P} is the partition of N_ℓ given by $\ell = 1 + 1 + \dots + 1$, so $\Delta^\circ(X^\ell, \mathcal{P})$ is the complement of the big diagonal in X^ℓ , then the normal bundle (3.3.2) is trivial in the sense that it is equal to $\Delta^\circ(X^\ell, \mathcal{P})$. The diagonal $\Delta^\circ(X^\ell, \mathcal{P})$ is an open, dense subspace of X^ℓ and equal to the open neighborhood $\tilde{\mathcal{U}}(X^\ell, \mathcal{P})$ of Lemma 3.3.2.

There is an obvious $\mathfrak{S}(\mathcal{P})$ action on the normal bundle $\tilde{\nu}(X^\ell, \mathcal{P})$. We will write $\nu(X^\ell, \mathcal{P})$ for the quotient $\tilde{\nu}(X^\ell, \mathcal{P})/\mathfrak{S}(\mathcal{P})$. Because the exponential map is $\mathfrak{S}(\mathcal{P})$ -equivariant, equation (3.1.9) yields

LEMMA 3.3.4. *Continue the notation of Lemma 3.3.2. If $\mathcal{O}(X^\ell, \mathcal{P}) = \tilde{\mathcal{O}}(X^\ell, \mathcal{P})/\mathfrak{S}(\mathcal{P})$, then a neighborhood,*

$$\mathcal{U}(X^\ell, \mathcal{P}) := \tilde{\mathcal{U}}(X^\ell, \mathcal{P})/\mathfrak{S}(\mathcal{P}),$$

of $\Sigma(X^\ell, \mathcal{P})$ in $\text{Sym}^\ell(X)$ is homeomorphic to $\mathcal{O}(X^\ell, \mathcal{P})$.

We now identify the intersection of $\Delta^\circ(X, \mathcal{P}')$ with $\tilde{\mathcal{U}}(X^\ell, \mathcal{P})$.

LEMMA 3.3.5. *If $\mathcal{P} < \mathcal{P}'$ are partitions of N_ℓ and*

$$(3.3.3) \quad \tilde{\nu}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}') := \left\{ (v_1, \dots, v_\ell) \in \tilde{\nu}(X^\ell, \mathcal{P}) : v_i = v_j \iff \exists P' \in \mathcal{P}' \text{ with } i, j \in P' \right\},$$

then the intersection of $\Delta^\circ(X^\ell, \mathcal{P}')$ with $\tilde{\mathcal{U}}(X^\ell, \mathcal{P})$ is homeomorphic to

$$\tilde{\mathcal{O}}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}', g_{\mathcal{P}}) := \tilde{\nu}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}') \cap \tilde{\mathcal{O}}(X^\ell, g_{\mathcal{P}}).$$

PROOF. This follows immediately from the equivariance of the exponential map. \square

Thus, a neighborhood of the lower diagonal, $\Delta^\circ(X^\ell, \mathcal{P})$, in $\Delta^\circ(X^\ell, \mathcal{P}')$ can be described by the bundle $\tilde{\nu}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}')$. However, the end of the stratum $\Sigma(X^\ell, \mathcal{P}')$ near the lower stratum, $\Sigma(X^\ell, \mathcal{P})$, can be more complicated than a quotient of $\tilde{\nu}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}')$ by a subgroup of $\mathfrak{S}(\mathcal{P})$. Indeed, $\tilde{\nu}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}')$ need not be an invariant subspace of $\tilde{\nu}(X^\ell, \mathcal{P})$ under the action of $\mathfrak{S}(\mathcal{P})$. Recall from Lemma 3.1.4 that $\tilde{\pi}_\ell^{-1}(\Sigma(X^\ell, \mathcal{P}'))$ comprises not just $\Delta^\circ(X^\ell, \mathcal{P}')$ but rather all the diagonals given by a partition in $[\mathcal{P}']$, the orbit of \mathcal{P}' under the action (3.1.1). Therefore, while we can cover $\Sigma(X^\ell, \mathcal{P})$ by $\Delta^\circ(X^\ell, \mathcal{P})$, to account for all the ends of $\Delta^\circ(X^\ell, \mathcal{P}')$ which get mapped by $\tilde{\pi}_\ell$ to a neighborhood of the lower stratum, $\Sigma(X^\ell, \mathcal{P})$, we must consider all diagonals $\Delta^\circ(X^\ell, \mathcal{P}'')$ where $\mathcal{P}'' \in [\mathcal{P}']$ and $\mathcal{P} < \mathcal{P}''$. That is, we must consider all partitions $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$, where $[\mathcal{P} < \mathcal{P}']$ is defined in (3.2.1). The next lemma follows from Lemmas 3.3.2 and 3.3.5.

LEMMA 3.3.6. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_ℓ . Let $[\mathcal{P} < \mathcal{P}']$ be the set of partitions defined in (3.2.1). Then a neighborhood of $\Sigma(X^\ell, \mathcal{P})$ in $\Sigma(X^\ell, \mathcal{P}')$ is homeomorphic to a neighborhood of the zero-section in*

$$(3.3.4) \quad \nu(X^\ell, \mathcal{P} \rightarrow [\mathcal{P}']) := \tilde{\nu}(X^\ell, \mathcal{P} \rightarrow [\mathcal{P}'])/\mathfrak{S}(\mathcal{P})$$

where

$$\tilde{\nu}(X^\ell, \mathcal{P} \rightarrow [\mathcal{P}']) := \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \tilde{\nu}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}'').$$

REMARK 3.3.7. As noted in Example 3.2.3, $\mathfrak{S}(\mathcal{P})$ need not act transitively on $[\mathcal{P} < \mathcal{P}']$. The orbits of $\mathfrak{S}(\mathcal{P})$ in $[\mathcal{P} < \mathcal{P}']$ will enumerate the components of the end of $\Sigma(X^\ell, \mathcal{P}')$ in a neighborhood of $\Sigma(X^\ell, \mathcal{P})$.

3.4. Enumeration of the strata

We will often need to give arguments by induction on strata of $\text{Sym}^\ell(X)$. To that end, we now give a method of enumerating these strata. Choose a representative, \mathcal{P} , from each orbit of partitions, $[\mathcal{P}]$, under the action (3.1.1). We refer to the partition $\mathcal{P}_0 = \{N_\ell\}$ as the *crudest* partition. Enumerate the remaining representatives in such a way that

$$\Sigma(X^\ell, \mathcal{P}_i) \subset \text{cl} \left(\Sigma(X^\ell, \mathcal{P}_j) \right) \quad \text{only if } i < j.$$

Write these partitions as $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_r$ so that $\Sigma(X^\ell, \mathcal{P}_0)$ is the lowest stratum and $\Sigma(X^\ell, \mathcal{P}_r)$ is the highest.

CHAPTER 4

A partial Thom–Mather structure on symmetric products

In this chapter, we define neighborhoods of the strata of $\text{Sym}^\ell(X)$ defined in Section 3.1.2, projections of these neighborhoods onto the strata, and vector-valued tubular distance functions on these neighborhoods.

4.1. Introduction

A topological space Z is *smoothly stratified* if it admits a decomposition into a locally finite collection of locally closed subspaces, $Z = \cup_i \Sigma_i$, each of which is a smooth manifold and which satisfy the *condition of the frontier*: $\Sigma \cap \text{cl}(\Sigma_j) \neq \emptyset$ if and only if $\Sigma \subset \text{cl}(\Sigma_j)$. Such spaces are called Whitney pre-stratified in [66, p. 480] (see also, [39, p. 36]). A *Thom–Mather* stratification on a smoothly-stratified space Z with strata Σ_i consists of *tubular neighborhood structures*, by which we mean neighborhoods, U_i of Σ_i , with *tubular neighborhood projections*, $\pi_i : U_i \rightarrow \Sigma_i$, and *tubular distance functions*, $t_i : U_i \rightarrow [0, 1)$, satisfying the following properties:

- (1) For all i , $t_i^{-1}(0) = \Sigma_i$.
- (2) For $\Sigma_i \subset \text{cl}_Z(\Sigma_j)$ the *compatibility conditions*

$$(4.1.1) \quad \begin{aligned} \pi_i \circ \pi_j &= \pi_i, \\ t_i \circ \pi_j &= t_i, \end{aligned}$$

wherever these compositions are defined.

- (3) The map $\pi_i \times t_i : U_i \rightarrow \Sigma_i \times [0, 1)$ is a submersion.

These conditions are called the *control conditions* of a Thom–Mather stratification (see [39, p. 42]). Because $\text{Sym}^\ell(X)$ is the quotient of the smooth manifold X^ℓ by the action of a finite group, it admits the structure of a *Whitney stratification* (for example, see [82, Theorem 4.3.7]) and hence a Thom–Mather stratification (see [39, p. 42]).

In this chapter, we give an explicit construction of a *partial tubular neighborhood structure* on the stratification of $\text{Sym}^\ell(X)$ defined in Section 3.1.2. By partial tubular neighborhood structure on a smoothly-stratified space, Z , with strata, Σ_i , we mean neighborhoods, U_i of Σ_i , with projection maps, $\pi_i : U_i \rightarrow \Sigma_i$, and *vector-valued tubular distance functions*, $\vec{t}_i : U_i \rightarrow [0, 1)^{n_i}$, with $\Sigma_i = \vec{t}_i^{-1}(\vec{0})$. We show these maps satisfy the first compatibility condition in (4.1.1) and how to control the failure of the second compatibility condition.

We shall need the explicit construction of these projection and distance functions to compare them with the functions of a partial tubular neighborhood structure on neighborhoods of $M_5 \times \text{Sym}^\ell(X)$. In addition, this presentation serves as an introduction to a technique for constructing partial tubular neighborhood structures which we shall employ again in later chapters in less familiar settings.

To construct the partial tubular neighborhood structure on $\text{Sym}^\ell(X)$, we begin by constructing one on X^ℓ for the stratification given by the diagonals of X^ℓ . In Section 4.2 we define subspaces of $(\mathbb{R}^4)^\ell$ which will appear as the fibers of the normal bundles defined in Section 3.3. We also define stratifications of these subspaces of $(\mathbb{R}^4)^\ell$ and tubular neighborhoods of these stratifications. To use the normal bundles of Section 3.3 to construct partial tubular neighborhood structures on $\text{Sym}^\ell(X)$, we define an exponential map with a varying metric in Section 4.3. We describe the overlap of the exponential maps from these normal bundles in Section 4.4. In Section 4.5 we construct families of metrics so that the projection maps defined by the associated exponential maps satisfy the first equation in (4.1.1). Then, we show how this partial tubular neighborhood structure on X^ℓ descends to one on $\text{Sym}^\ell(X)$ in Section 4.6. In Section 4.7, we define the vector-valued tubular distance functions and describe their failure to satisfy the second equation in (4.1.1). This description and the analogous description for the vector-valued tubular distance functions for $M_\mathfrak{s} \times \text{Sym}^\ell(X)$ will allow us to define the link of $M_\mathfrak{s} \times \text{Sym}^\ell(X)$ in (8.1.5) as a union of fiber bundles as described in Proposition 8.2.1 which will enable the pushforward-pullback arguments of Sections 10.4, 10.5 and 10.6. Finally, in Section 4.8, we use the partial tubular neighborhood structure to produce a partition of $\text{Sym}^\ell(X)$ into compact subsets of these tubular neighborhoods. These compact subsets will be used in our construction of the link of $M_\mathfrak{s} \times \text{Sym}^\ell(X)$ and to give a criterion for patching together maps to $\text{Sym}^\ell(X)$ which are homotopic but not equal.

4.2. Diagonals in products of \mathbb{R}^4

The fiber of the normal bundle of the diagonal $\Delta^\circ(X^\ell, \mathcal{P})$ is given by the product, over $P \in \mathcal{P}$, of the subspace of mass-centered points in $\bigoplus_{i \in P} \mathbb{R}^4$ (see (3.3.2)). Hence, for any subset P of N_ℓ , define

$$(4.2.1) \quad z_P : \bigoplus_{i \in P} \mathbb{R}^4 \ni (v_i)_{i \in P} \mapsto \sum_i v_i \in \mathbb{R}^4 \quad \text{and} \quad Z_P := z_P^{-1}(0) \subset \bigoplus_{i \in P} \mathbb{R}^4.$$

We will write z_ℓ for z_{N_ℓ} . We define the *cone point*,

$$(4.2.2) \quad c_P \in Z_P,$$

to be the zero vector in $\bigoplus_{i \in P} \mathbb{R}^4$. Note that if $|P| = 1$, then $Z_P = \{c_P\}$. The fiber of $\tilde{\nu}(X^\ell, \mathcal{P})$ is then given by (compare (3.3.2))

$$(4.2.3) \quad Z(\mathcal{P}) = \prod_{P \in \mathcal{P}} Z_P.$$

For partitions $\mathcal{P} < \mathcal{P}'$ of N_ℓ , define a partition of each $P \in \mathcal{P}$ by

$$(4.2.4) \quad \mathcal{P}'_P := \{P' \in \mathcal{P}' : P' \subseteq P\}.$$

If $P \in \mathcal{P} \cap \mathcal{P}'$, then $\mathcal{P}'_P = \{P\}$. We define

$$(4.2.5) \quad \Delta^\circ(Z_P, \mathcal{P}'_P) := \{(v_i)_{i \in P} \in Z_P : v_i = v_j \text{ if and only if } \exists P' \in \mathcal{P}'_P \text{ with } i, j \in P'\},$$

and observe that the fiber of the bundle $\tilde{\nu}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}')$ in (3.3.3) is given by

$$(4.2.6) \quad Z(\mathcal{P}, \mathcal{P}') = \prod_{P \in \mathcal{P}} \Delta^\circ(Z_P, \mathcal{P}'_P).$$

Note that if \mathcal{P}'_P is given by one set, P , as is the case when $P \in \mathcal{P} \cap \mathcal{P}'$, then $\Delta^\circ(Z_P, \mathcal{P}'_P)$ is a single point, given by $|P|$ copies of the zero vector in \mathbb{R}^4 .

We define a *cone parameter function* on Z_P by the map

$$(4.2.7) \quad t_{P,0} : Z_P \ni (v_i)_{i \in P} \mapsto \sum_{i \in P} |v_i|^2 \in [0, \infty).$$

Then $t_{P,0}^{-1}(0) = c_P$ and there is a deformation retraction of Z_P to c_P given by

$$(4.2.8) \quad r_P : Z_P \times [0, 1] \ni ((v_i)_{i \in P}, s) \mapsto (sv_i)_{i \in P} \in Z_P.$$

Observe that $r_P(\cdot, 1)$ is the identity map, while $r_P(\cdot, 0) = c_P$. Moreover,

$$(4.2.9) \quad t_{P,0} \circ r_P(\cdot, s) = s^2 t_{P,0}(\cdot).$$

Hence, there is a homeomorphism between Z_P and the cone on $t_{P,0}^{-1}(1)$.

The following lemma describes the normal bundle of $\Delta^\circ(Z_P, \mathcal{P}'_P) \subset z_P^{-1}(0)$.

LEMMA 4.2.1. *Let \mathcal{P}'_P be a partition of P . Then the restriction of the map*

$$(4.2.10) \quad e(Z_P, \mathcal{P}'_P) : \Delta^\circ(Z_P, \mathcal{P}'_P) \times \prod_{P' \in \mathcal{P}'_P} Z_{P'} \rightarrow Z_P$$

defined by

$$(4.2.11) \quad \left((v_i)_{i \in P}, (z_j)_{\substack{j \in P' \\ P' \in \mathcal{P}'_P}} \right) \mapsto (v_i + z_i)_{i \in P}$$

to a sufficiently small neighborhood $\mathcal{O}(Z_P, \mathcal{P}'_P)$ of the subspace

$$\Delta^\circ(Z_P, \mathcal{P}'_P) \times \prod_{P' \in \mathcal{P}'_P} \{c_{P'}\}$$

is a homeomorphism onto its image, $\mathcal{U}(Z_P, \mathcal{P}'_P)$.

PROOF. We observe that the map (4.2.11) can be inverted by the *center of mass map*,

$$(4.2.12) \quad \begin{aligned} (w_i)_{i \in P} &\rightarrow \left((v_i)_{i \in P}, (z_j)_{\substack{j \in P' \\ P' \in \mathcal{P}'_P}} \right), \\ v_i &= \frac{1}{|P'|} \sum_{j \in P'} w_j, \quad \text{for } i \in P', \\ z_j &= w_j - v_j. \end{aligned}$$

(Note that the above map is an inverse because $\sum_{j \in P'} z_j = 0$ by the definition of $Z_{P'}$.) However, the map (4.2.12) only takes values in

$$\Delta^\circ(Z_P, \mathcal{P}'_P) \times \prod_{P' \in \mathcal{P}'_P} Z_{P'}$$

if we restrict to a sufficiently small neighborhood $\mathcal{O}(Z_P, \mathcal{P}'_P)$. \square

REMARK 4.2.2. As in Remark 3.3.3, if \mathcal{P}'_P is the finest partition of P , made of subsets all of size one, then for all $P' \in \mathcal{P}'_P$, the point $Z_{P'}$ is the cone point and $e(Z_P, \mathcal{P}'_P)$ is the identity map. In this case, $\Delta^\circ(Z_P, \mathcal{P}'_P) = \mathcal{O}(Z_P, \mathcal{P}'_P)$ is an open and dense subset of Z_P .

REMARK 4.2.3. Observe that the map $e(Z_P, \mathcal{P}'_P)$ is equivariant with respect to the \mathbb{R}^+ actions given by scalar multiplication on all factors in the domain,

$$\left((v_i)_{i \in P}, (z_j)_{\substack{j \in P' \\ P' \in \mathcal{P}'_P}} \right) \mapsto \left((sv_i)_{i \in P}, (sz_j)_{\substack{j \in P' \\ P' \in \mathcal{P}'_P}} \right),$$

and the map $r_P(\cdot, s)$ on Z_P .

Because Z_P is a completely normal topological space (see [75, Section 32, Example 4]), we can find open neighborhoods $U(Z_P, \mathcal{P}'_P)$ of $\Delta^\circ(Z_P, \mathcal{P}'_P)$ for every partition \mathcal{P}'_P of P such that $U(Z_P, \mathcal{P}'_P) \cap U(Z_P, \mathcal{P}''_P)$ is empty unless $\mathcal{P}'_P < \mathcal{P}''_P$ or $\mathcal{P}''_P < \mathcal{P}'_P$. By Remark 4.2.3, we can assume that the neighborhoods $U(Z_P, \mathcal{P}'_P)$ are closed under the maps $r_P(\cdot, s)$ for $s \in [0, 1]$. The normal bundle structure given by the map (4.2.11) then defines a tubular neighborhood projection,

$$(4.2.13) \quad \pi(Z_P, \mathcal{P}'_P) : U(Z_P, \mathcal{P}'_P) \rightarrow \Delta^\circ(Z_P, \mathcal{P}'_P),$$

given by the v_i component of the map (4.2.12):

$$(4.2.14) \quad (w_i)_{i \in P} \mapsto (v_i)_{i \in P}, \quad \text{where } v_i := \frac{1}{|P'|} \sum_{j \in P'} w_j \text{ for } i \in P' \in \mathcal{P}'_P.$$

In the case of the top stratum described in Remark 4.2.2, the map $\pi(Z_P, \mathcal{P}'_P)$ is just the identity map. By construction, the maps $\pi(Z_P, \mathcal{P}'_P)$ commute with the map r_P of (4.2.8) in the sense that

$$(4.2.15) \quad r_P(\pi(Z_P, \mathcal{P}'_P)(\cdot), s) = \pi(Z_P, \mathcal{P}'_P)(r_P(\cdot, s)),$$

for all $s \in [0, 1]$.

The following lemma shows that the maps $\pi(Z_P, \mathcal{P}'_P)$ satisfy the first equality in (4.1.1).

LEMMA 4.2.4. *If $\mathcal{P}'_P < \mathcal{P}''_P$ are partitions of P , then on the intersection $U(Z_P, \mathcal{P}'_P) \cap U(Z_P, \mathcal{P}''_P)$, the following equality holds:*

$$(4.2.16) \quad \pi(Z_P, \mathcal{P}'_P) \circ \pi(Z_P, \mathcal{P}''_P) = \pi(Z_P, \mathcal{P}'_P).$$

PROOF. The lemma follows immediately from the definition (4.2.14). \square

We use the above set of tubular neighborhoods to define a deformation of the scale function, $t_{P,0}$, which is constant on the fibers of the maps $\pi(Z_P, \mathcal{P}'_P)$.

LEMMA 4.2.5. *There are*

- (1) *A smooth function, $t(Z_P) : t_{P,0}^{-1}([0, 1]) \subset Z_P \rightarrow [0, \infty)$,*
- (2) *Open neighborhoods, $U'(Z_P, \mathcal{P}'_P) \sqsubset U(Z_P, \mathcal{P}'_P) \cap t_{P,0}^{-1}([0, 1])$, of the diagonals, $\Delta^\circ(Z_P, \mathcal{P}'_P) \cap t_{P,0}^{-1}([0, 1])$,*

such that the following hold:

- (1) *If $c_P \in Z_P$ is the cone point in (4.2.2), then $t(Z_P)^{-1}(0) = c_P$.*
- (2) *If r_P is the map in (4.2.8) then, for all $s \in [0, 1]$,*

$$(4.2.17) \quad t(Z_P)(r_P(\cdot, s)) = s^2 t(Z_P)(\cdot).$$

- (3) *For every partition \mathcal{P}'_P of P of length greater than one,*

$$(4.2.18) \quad t(Z_P) \circ \pi(Z_P, \mathcal{P}'_P) = t(Z_P) \quad \text{on } U'(Z_P, \mathcal{P}'_P).$$

- (4) *The function, $t(Z_P)$, is invariant with respect to the permutation action of \mathfrak{S}_P and the diagonal rotation action of $\text{SO}(4)$ on Z_P .*

PROOF. Enumerate the conjugacy classes of partitions of P , $[\mathcal{P}_0], \dots, [\mathcal{P}_n]$ in the manner described in Section 3.4. That is, for $\mathcal{P} \in [\mathcal{P}_i]$ and $\mathcal{P}' \in [\mathcal{P}_j]$, we have $\Delta(Z_P, \mathcal{P}) \subset \text{cl } \Delta(Z_P, \mathcal{P}')$ only if $i < j$. We will construct a sequence of functions, t_0, \dots, t_n , such that t_i satisfies (4.2.18) with respect to $\pi(Z_P, \mathcal{P})$ for any partition $\mathcal{P} \in [\mathcal{P}_j]$ with $j < i$.

Begin with $t_0 = t_{P,0}$. This function is invariant with respect to the actions of $\text{SO}(4)$ and \mathfrak{S}_P and satisfies (4.2.17).

Assume we have defined functions, $t_i : Z_P \rightarrow [0, \infty)$, satisfying Items (1), (2), and (4) of the lemma and satisfying (4.2.18) for all partitions $\mathcal{P} \in [\mathcal{P}_j]$ with $j < i$.

There are neighborhoods

$$\mathcal{U}_1 \sqsubset \mathcal{U}_2 \sqsubset \mathcal{U}(Z_P, \mathcal{P}_i)$$

of $\Delta^\circ(Z_P, \mathcal{P}_i)$ which are closed under the $\text{SO}(4)$ action and under the action of \mathbb{R}^+ given by $r_P(\cdot, s)$. Let \mathcal{V}_i be the orbit of \mathcal{U}_i under the action of \mathfrak{S}_P . By shrinking \mathcal{U}_i if necessary, we can assume that either $\mathcal{U}_i = \sigma(\mathcal{U}_i)$ or $\mathcal{U}_i \cap \sigma(\mathcal{U}_i) = \emptyset$ for all $\sigma \in \mathfrak{S}_P$. Let $\beta_i : Z_P \rightarrow [0, 1]$ be a function, invariant under the actions of $\text{SO}(4)$, \mathfrak{S}_P , and \mathbb{R}^+ , supported on \mathcal{V}_2 , and satisfying $\mathcal{V}_1 \subset \beta_i^{-1}(1)$. Denote

$$\pi_i := \coprod_{\mathcal{P} \in [\mathcal{P}_i]} \pi(Z_P, \mathcal{P}) : \mathcal{V}_i \rightarrow \bigsqcup_{\mathcal{P} \in [\mathcal{P}_i]} \Delta^\circ(Z_P, \mathcal{P}).$$

Observe that the map, π_i , is equivariant with respect to the actions of $\text{SO}(4)$, \mathfrak{S}_P , and \mathbb{R}^+ . On \mathcal{V}_2 , we define

$$t_{i+1}(\cdot) = \beta_i(\cdot)(t_i \circ \pi_i)(\cdot) + (1 - \beta_i(\cdot))t_i(\cdot).$$

Extend t_{i+1} to Z_P by setting $t_{i+1} := t_i$ on $Z_P - \mathcal{V}_2$.

By the invariance of β_i and t_i with respect to the actions of $\text{SO}(4)$ and \mathfrak{S}_P , and the equivariance of π_i with respect to these actions, the function t_{i+1} is invariant with respect to these actions. The same argument shows that t_{i+1} satisfies (4.2.17), that is, $t_{i+1}(r_P(\cdot, s)) = s^2 t_{i+1}(\cdot)$.

For $j < i$ and $\mathcal{P} \in [\mathcal{P}_j]$, the function t_i is constant on the fibers of $\pi(Z_P, \mathcal{P})$ by our induction hypothesis. The relation (4.2.16) implies that on $\mathcal{V}_2 \cap \mathcal{U}'(X_P, \mathcal{P})$, the fibers of π_i are contained in those of $\pi(Z_P, \mathcal{P})$ and thus $t_i \circ \pi_i$ equals t_i on $\mathcal{V}_2 \cap \mathcal{U}'(X_P, \mathcal{P})$. This implies that on $\mathcal{V}_2 \cap \mathcal{U}'(X_P, \mathcal{P})$ any convex linear combination of $t_i \circ \pi_i$ and t_i equals t_i . Hence for any $j < i$ and any $\mathcal{P} \in [\mathcal{P}_j]$, the restriction of t_{i+1} to $\mathcal{U}(X_P, \mathcal{P})$ equals the restriction of t_i . By definition, this equality also holds on the complement of \mathcal{V}_2 . Thus, t_{i+1} satisfies (4.2.18) for any $\mathcal{P} \in [\mathcal{P}_j]$ and $j < i$. We further observe that on $\mathcal{V}_1 \subset \beta_i^{-1}(1)$, we have $t_{i+1} = t_i \circ \pi_i$ while the equality $\pi_i \circ \pi_i = \pi_i$ implies that

$$(t_{i+1} \circ \pi_i)(\cdot) = (\beta_i \circ \pi_i)(\cdot)(t_i \circ \pi_i)(\cdot) + (1 - (\beta_i \circ \pi_i)(\cdot))(t_i \circ \pi_i)(\cdot) = (t_i \circ \pi_i)(\cdot).$$

Hence, if we define $\mathcal{U}'(Z_P, \mathcal{P}) = \mathcal{V}_1 \cap \mathcal{U}(Z_P, \mathcal{P})$ for $\mathcal{P} \in [\mathcal{P}_i]$, then (4.2.18) holds, completing the induction. \square

4.3. Families of metrics

A Riemannian metric, g , on X determines a product Riemannian metric on X^ℓ and thus exponential maps of the normal bundles of the diagonal. We will use a non-standard diffeomorphism of a neighborhood of the zero section in the normal bundle $\tilde{\nu}(X^\ell, \mathcal{P})$ with a neighborhood of $\Delta^\circ(X^\ell, \mathcal{P})$ in X^ℓ . This diffeomorphism will be defined by varying the Riemannian metric on X

and hence the resulting product metric on X^ℓ . We begin by noting that it is possible to define such an exponential map with a varying metric.

LEMMA 4.3.1. *If $g_{\mathcal{P}}$ is a smooth family of metrics on X parameterized by $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$, then there is an open neighborhood $\tilde{\mathcal{O}}(X^\ell, g_{\mathcal{P}}) \subset \tilde{\nu}(X^\ell, \mathcal{P})$ of the zero section and a smooth embedding,*

$$(4.3.1) \quad e(X^\ell, g_{\mathcal{P}}) : \tilde{\mathcal{O}}(X^\ell, g_{\mathcal{P}}) \subset \tilde{\nu}(X^\ell, \mathcal{P}) \rightarrow X^\ell,$$

parameterizing an open neighborhood, $\tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}})$, of $\Delta^\circ(X^\ell, \mathcal{P})$ in X^ℓ such that the restriction of the map $e(X^\ell, g_{\mathcal{P}})$ to a fiber $\tilde{\mathcal{O}}(X^\ell, g_{\mathcal{P}})|_{\mathbf{x}}$ is given by the exponential map defined by the ℓ -fold product of the metric $g_{\mathcal{P}, \mathbf{x}}$.

PROOF. We claim that the derivative of the map $e(X^\ell, g_{\mathcal{P}})$ at a point p on the zero-section of $\tilde{\nu}(X^\ell, \mathcal{P})$ is the identity. We identify this zero section with $\Delta^\circ(X^\ell, \mathcal{P})$ throughout this discussion. Over a point p in this zero section, the tangent bundle of $\tilde{\nu}(X^\ell, \mathcal{P})$ admits a decomposition into the direct sum of $\tilde{\nu}(X^\ell, \mathcal{P})|_p$ and $T_p\Delta^\circ(X^\ell, \mathcal{P})$. By definition of the exponential map, $De(X^\ell, g_{\mathcal{P}})_p(v) = v$ for any $v \in \tilde{\nu}(X^\ell, \mathcal{P})|_p$. To evaluate $De(X^\ell, g_{\mathcal{P}})_p(w)$ for $w \in T_p\Delta^\circ(X^\ell, \mathcal{P})$, let $\gamma : [0, 1] \rightarrow \Delta^\circ(X^\ell, \mathcal{P})$ be a smooth path with $\gamma'(0) = w$. Then for all $t \in [0, 1]$, we have $e(X^\ell, g_{\mathcal{P}})(\gamma(t)) = \gamma(t)$. Differentiating the preceding equality yields $De(X^\ell, g_{\mathcal{P}})_p(w) = w$, so $De(X^\ell, g_{\mathcal{P}})_p$ is the identity. Hence, $e(X^\ell, g_{\mathcal{P}})$ defines an embedding of a neighborhood of the zero section. \square

We can assume that the neighborhood $\tilde{\mathcal{O}}(X^\ell, g_{\mathcal{P}})$ is small enough to have the following property.

LEMMA 4.3.2. *Continue the hypotheses of Lemma 4.3.1. The open neighborhood $\tilde{\mathcal{O}}(X^\ell, g_{\mathcal{P}})$ can be chosen so that the images of $\pi_i \circ e(X^\ell, g_{\mathcal{P}})$ and $\pi_j \circ e(X^\ell, g_{\mathcal{P}})$ are disjoint for $i \neq j$, where $\pi_i : X^\ell \rightarrow X$ is projection onto the i -th factor.*

PROOF. For $(x_P)_{P \in \mathcal{P}} \in \Delta^\circ(X^\ell, \mathcal{P})$, the points x_P are distinct. Hence, we can choose balls around the origin in $T_{x_P}X$ whose image under the exponential map are disjoint, thus satisfying the conclusion of the lemma. \square

The image of the exponential map in (4.3.1) is a tubular neighborhood of $\Delta^\circ(X^\ell, \mathcal{P})$. Such a tubular neighborhood, $\tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}}) \subset X^\ell$, defines a projection,

$$(4.3.2) \quad \pi(X^\ell, g_{\mathcal{P}}) : \tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}}) \rightarrow \Delta^\circ(X^\ell, \mathcal{P}),$$

by the composition of $e(X^\ell, g_{\mathcal{P}})^{-1}$ and the projection $\tilde{\nu}(X^\ell, \mathcal{P}) \rightarrow \Delta^\circ(X, \mathcal{P})$.

REMARK 4.3.3. If \mathcal{P} is the finest partition as described in Remark 3.3.3, then the exponential map $e(X^\ell, g_{\mathcal{P}})$ and the projection map $\pi(X^\ell, g_{\mathcal{P}})$ both equal the identity map.

For any smooth manifold, M , of dimension n , let $\text{Fr}_{\text{GL}}(TM)$ denote the $\text{GL}(n)$ -bundle of frames of tangent vectors. Define

$$\text{Fr}_{\text{GL}}(TX^\ell, \mathcal{P}) := \{(F_1, \dots, F_\ell) \in \text{Fr}(TX^\ell)|_{\Delta^\circ(X^\ell, \mathcal{P})} : F_i = F_j \text{ for all } i, j \in P \in \mathcal{P}\}.$$

If $g_{\mathcal{P}}$ is a family of metrics parameterized by $\Delta^\circ(X^\ell, \mathcal{P})$, then we define $\text{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}})$ to be the subbundle of $\text{Fr}_{\text{GL}}(TX^\ell, \mathcal{P})$ of frames such that if $(F_1, \dots, F_\ell) \in \text{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}})|_{\mathbf{x}}$, then F_i is orthonormal with respect to $g_{\mathcal{P}, \mathbf{x}}$. The structure group of $\text{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}})$ is

$$(4.3.3) \quad \tilde{G}(T, \mathcal{P}) := \text{SO}(4)^{r(\mathcal{P})} = \prod_{P \in \mathcal{P}} \text{SO}(4), \quad \text{where } \mathcal{P} = \{P_1, \dots, P_{r(\mathcal{P})}\}.$$

Recalling the definition of $Z(\mathcal{P})$ from (4.2.3), we have the identity,

$$(4.3.4) \quad \tilde{\nu}(X^\ell, \mathcal{P}) = \text{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}}) \times_{\tilde{G}(T, \mathcal{P})} Z(\mathcal{P}).$$

Lemma 3.3.5 implies the following.

LEMMA 4.3.4. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_ℓ . Let $\tilde{\nu}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}')$ be the bundle defined in (3.3.3). Then the restriction of the exponential map $e(X^\ell, g_{\mathcal{P}})$ to*

$$(4.3.5) \quad \tilde{\mathcal{O}}(X^\ell, g_{\mathcal{P}}) \cap \tilde{\nu}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}')$$

takes values in $\Delta(X, \mathcal{P}')$.

We have the following relation between the bundles $\text{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}})$ for different partitions, \mathcal{P} .

LEMMA 4.3.5. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_ℓ . Let $\tilde{\mathcal{U}}(X, \mathcal{P} \rightarrow \mathcal{P}', g_{\mathcal{P}}) \subset \Delta^\circ(X^\ell, \mathcal{P}')$ be the image of the restriction of the exponential map $e(X^\ell, g_{\mathcal{P}})$ to the domain in (4.3.5). Let $g_{\mathcal{P}}$ and $g_{\mathcal{P}'}$ be smooth families of metrics parameterized by $\Delta^\circ(X^\ell, \mathcal{P})$ and $\Delta^\circ(X, \mathcal{P}')$ respectively. Assume that $g_{\mathcal{P}', \mathbf{x}'} = g_{\mathcal{P}, \mathbf{x}}$ if $\pi(X^\ell, g_{\mathcal{P}})(\mathbf{x}') = \mathbf{x}$. Then,*

$$(4.3.6) \quad \text{Fr}(TX^\ell, \mathcal{P}', g_{\mathcal{P}'})|_{\tilde{\mathcal{U}}(X, \mathcal{P} \rightarrow \mathcal{P}', g_{\mathcal{P}})} \\ \cong \text{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}}) \times_{\tilde{G}(T, \mathcal{P})} \prod_{P \in \mathcal{P}} \left(\Delta^\circ(Z_P, \mathcal{P}'_P) \times \tilde{G}(T, \mathcal{P}'_P) \right).$$

PROOF. By Lemma 4.3.4, a neighborhood of $\Delta^\circ(X^\ell, \mathcal{P})$ in $\Delta^\circ(X^\ell, \mathcal{P}')$ is diffeomorphic to a neighborhood of the zero section in

$$(4.3.7) \quad \tilde{\nu}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}') \cong \text{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}}) \times_{\tilde{G}(T, \mathcal{P})} \prod_{P \in \mathcal{P}} \Delta^\circ(Z_P, \mathcal{P}'_P) \subset \nu(X^\ell, \mathcal{P}, g_{\mathcal{P}})$$

by the exponential map $e(X^\ell, g_{\mathcal{P}})$. The isomorphism (4.3.6) is then defined by the obvious parallel translation of the frames in $\text{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}})$ to the point in $\tilde{\mathcal{U}}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}')$ given by the data in (4.3.7). \square

4.4. Overlap maps

Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_ℓ . To define tubular neighborhoods and projection maps which satisfy the conditions (4.1.1), we define a space to control the overlap of the tubular neighborhoods, $\mathcal{U}(X, \mathcal{P}, g_{\mathcal{P}})$ and $\mathcal{U}(X, \mathcal{P}', g_{\mathcal{P}'})$, as follows. We begin by defining an overlap fiber bundle,

$$(4.4.1) \quad \tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}') := \text{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}}) \times_{\tilde{G}(T, \mathcal{P})} \prod_{P \in \mathcal{P}} \left(\Delta^\circ(Z_P, \mathcal{P}'_P) \times \prod_{P' \in \mathcal{P}'_P} Z_{P'} \right).$$

The cone points, $c_{P'} \in Z_{P'}$, define a subspace of $\tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}')$ which is identified with $\tilde{\nu}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}')$ by (4.3.7). Neighborhoods in $\Delta^\circ(Z_P, \mathcal{P}'_P)$ of the cone points $c_P \in \Delta(Z_P, \mathcal{P}'_P)$ define ‘neighborhoods of the zero section’ in $\tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}')$.

We will define an open neighborhood, $\tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}})$, of the zero section in $\tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}')$ and maps, $\rho_{\mathcal{P}, \mathcal{P}'}^{X, u}$ and $\rho_{\mathcal{P}, \mathcal{P}'}^{X, d}$, so that the diagram,

$$(4.4.2) \quad \begin{array}{ccc} \tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}}) & \xrightarrow{\rho_{\mathcal{P}, \mathcal{P}'}^{X, u}} & \tilde{\mathcal{O}}(X^\ell, g_{\mathcal{P}'}) \\ \rho_{\mathcal{P}, \mathcal{P}'}^{X, d} \downarrow & & e(X^\ell, g_{\mathcal{P}'}) \downarrow \\ \tilde{\mathcal{O}}(X^\ell, g_{\mathcal{P}}) & \xrightarrow{e(X^\ell, g_{\mathcal{P}})} & X^\ell \end{array}$$

commutes and controls the overlaps of the images of the exponential maps, $e(X^\ell, g_{\mathcal{P}})$ and $e(X^\ell, g_{\mathcal{P}'})$.

4.4.1. The downwards overlap map. The downwards overlap map,

$$(4.4.3) \quad \rho_{\mathcal{P}, \mathcal{P}'}^{X, d} : \tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}}) \rightarrow \tilde{\mathcal{O}}(X^\ell, g_{\mathcal{P}})$$

is given by the fiberwise inclusions defined by the map in equation (4.2.10),

$$e(Z_P, \mathcal{P}'_P) : \Delta^\circ(Z_P, \mathcal{P}'_P) \times \prod_{P' \in \mathcal{P}'_P} Z_{P'} \rightarrow Z_P.$$

Note that if $P \in \mathcal{P} \cap \mathcal{P}'$, then $\Delta^\circ(Z_P, \mathcal{P}'_P)$ is a point and $e(Z_P, \mathcal{P}'_P)$ is the identity map on Z_P . Because $e(Z_P, \mathcal{P}'_P)$ is SO(4)-equivariant, the product,

$$\prod_{P \in \mathcal{P}} e(Z_P, \mathcal{P}'_P) : \prod_{P \in \mathcal{P}} \left(\Delta^\circ(Z_P, \mathcal{P}'_P) \times \prod_{P' \in \mathcal{P}'_P} Z_{P'} \right) \rightarrow \prod_{P \in \mathcal{P}} Z_P$$

is $G(T, \mathcal{P})$ -equivariant and thus extends to the domain and range given in (4.4.3). The open subspace, $\tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}})$, must satisfy

$$(4.4.4) \quad \tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}}) \subseteq (\rho_{\mathcal{P}, \mathcal{P}'}^{X, d})^{-1} \left(\tilde{\mathcal{O}}(X^\ell, g_{\mathcal{P}}) \right)$$

for the composition in the diagram (4.4.2) to be defined.

4.4.2. The upwards overlap map. We define the upwards overlap map,

$$(4.4.5) \quad \rho_{\mathcal{P}, \mathcal{P}'}^{X, u} : \tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}') \rightarrow \tilde{\mathcal{O}}(X^\ell, g_{\mathcal{P}'})$$

as follows. Let

$$(4.4.6) \quad \pi_1 : \tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}') \rightarrow \tilde{\nu}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}')$$

be the projection which is obvious from the definitions (4.3.7) and (4.4.1) (delete the factors of $Z_{P'}$). We assume that the open set $\tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}})$ satisfies

$$(4.4.7) \quad \tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}}) \subseteq \pi_1^{-1} \left(\tilde{\mathcal{O}}(X^\ell, g_{\mathcal{P}}) \right).$$

Then the composition $e(X^\ell, g_{\mathcal{P}}) \circ \pi_1$ is defined on $\tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}})$ and takes values in $\Delta^\circ(X^\ell, \mathcal{P}')$.

We now define the map $\rho_{\mathcal{P}, \mathcal{P}'}^{X, u}$. We make the assumption on the families of metrics $g_{\mathcal{P}}$ and $g_{\mathcal{P}'}$ appearing in Lemma 4.3.5. Given a point in $\tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}')$ lying over $\mathbf{y} = (y_P) \in \Delta^\circ(X^\ell, \mathcal{P})$, define a point $\mathbf{y}' = (y_{P'})_{P' \in \mathcal{P}'} \in \Delta^\circ(X^\ell, \mathcal{P}')$ by the composition,

$$e(X^\ell, g_{\mathcal{P}}) \circ \pi_1 : \tilde{\nu}(X^\ell, \mathcal{P}, \mathcal{P}') \rightarrow \Delta^\circ(X^\ell, \mathcal{P}').$$

Then, for $P' \in \mathcal{P}'_P$ (where $P \in \mathcal{P}$), define $F_{P'}$ to be the parallel translation of the frame $F_P \in \text{Fr}(TX)|_{y_P}$ from y_P to $y'_{P'}$ using the Levi-Civita connection defined by the metric $g_{\mathcal{P},\mathbf{y}}$. Leave the data in $Z_{P'}$ alone. The result is an element of

$$\text{Fr}(TX^\ell, \mathcal{P}', g_{\mathcal{P}'})|_{\mathbf{y}'} \times_{G(T, \mathcal{P}')} \prod_{P' \in \mathcal{P}'} Z_{P'} = \tilde{v}(X^\ell, \mathcal{P}')|_{\mathbf{y}'},$$

thus defining the map $\rho_{\mathcal{P}, \mathcal{P}'}^{X, u}$. Observe that if $P \in \mathcal{P} \cap \mathcal{P}'$, the parallel translation for the P -th component will be trivial because $\Delta^\circ(Z_P, \mathcal{P}'_P)$ is $|P|$ -copies of the zero vector.

4.4.3. Commuting overlap maps. We now define conditions on the metrics $g_{\mathcal{P}}$ and $g_{\mathcal{P}'}$ which ensure that the diagram (4.4.2) commutes. The diagram will commute if we chose the families of metrics $g_{\mathcal{P}}$ and $g_{\mathcal{P}'}$ in such a way as to eliminate holonomy. To do this, we introduce the notion of a *locally flattened metric*.

DEFINITION 4.4.1. If $A \subset X^\ell$, then the *support* of A in X is the union of the images of A under the projection maps, $X^\ell \rightarrow X$. Let $g_{\mathcal{P}}$ be a smooth family of metrics on X parameterized by $\Delta^\circ(X^\ell, \mathcal{P})$. Let $\tilde{\mathcal{U}}' \subseteq \tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}})$ be a neighborhood of $\Delta^\circ(X^\ell, \mathcal{P})$. The family of metrics $g_{\mathcal{P}}$ is *locally flat with respect to $\tilde{\mathcal{U}}'$* at $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$ if the metric $g_{\mathcal{P}, \mathbf{x}}$ is flat on the support of $\tilde{\mathcal{U}}' \cap \pi(X^\ell, g_{\mathcal{P}})^{-1}(\mathbf{x})$ in X . The family of metrics is *locally flat with respect to $\tilde{\mathcal{U}}'$* if this holds for all $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$.

The following lemma shows that the diagram (4.4.2) commutes when the metrics are locally flat in the sense of Definition 4.4.1.

LEMMA 4.4.2. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_ℓ . Assume that the smooth families of metrics $g_{\mathcal{P}}$ and $g_{\mathcal{P}'}$ satisfy*

- (1) *The metrics $g_{\mathcal{P}}$ are flat with respect to a neighborhood $\tilde{\mathcal{U}}' \subseteq \tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}})$,*
- (2) *For $\mathbf{y}' \in \tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}}) \cap \Delta^\circ(X^\ell, \mathcal{P}')$ with $\pi(X^\ell, g_{\mathcal{P}})(\mathbf{y}') = \mathbf{y}$, the metrics satisfy $g_{\mathcal{P}, \mathbf{y}} = g_{\mathcal{P}', \mathbf{y}'}$.*

Then there is a neighborhood $\tilde{\mathcal{O}}'(X, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}})$ of the zero section in the bundle $\tilde{v}(X^\ell, \mathcal{P}, \mathcal{P}')$ defined in (4.4.1) such that the diagram (4.4.2) commutes when restricted to $\tilde{\mathcal{O}}'(X, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}})$.

PROOF. We observe that for the compositions

$$e(X^\ell, g_{\mathcal{P}}) \circ \rho_{\mathcal{P}, \mathcal{P}'}^{X, d} \quad \text{and} \quad e(X^\ell, g_{\mathcal{P}'}) \circ \rho_{\mathcal{P}, \mathcal{P}'}^{X, u}$$

to be defined, the neighborhood $\tilde{\mathcal{O}}'(X, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}})$ must satisfy the inclusion relations (4.4.4) and (4.4.7). However, we replace the condition (4.4.4) with the stronger requirement that

$$(4.4.8) \quad \tilde{\mathcal{O}}'(X, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}}) \subseteq (e(X^\ell, g_{\mathcal{P}}) \circ \rho_{\mathcal{P}, \mathcal{P}'}^{X, d})^{-1}(\tilde{\mathcal{U}}').$$

On any open subset satisfying the above condition, the composition

$$e(X^\ell, g_{\mathcal{P}}) \circ \rho_{\mathcal{P}, \mathcal{P}'}^{X, d}$$

is defined by the exponential map for the metric $g_{\mathcal{P}, \mathbf{y}}$ of the vectors obtained by appropriately adding vectors in the fiber of $\tilde{v}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}})$. The composition,

$$e(X^\ell, g_{\mathcal{P}'}) \circ \rho_{\mathcal{P}, \mathcal{P}'}^{X, u},$$

is defined by first parallel translating the vectors defined by the elements of $Z_{\mathcal{P}'}$ from \mathbf{y} to the nearby points in the support of \mathbf{y}' , and then exponentiating. By (4.4.8) and the first assumption on the metrics $g_{\mathcal{P}}$ and $g_{\mathcal{P}'}$, the metric $g_{\mathcal{P},\mathbf{y}}$ is flat where this parallel translation takes place and hence the two compositions are equal. \square

4.4.4. The projection maps. We now show that the commutativity of the diagram (4.4.2) will imply that the first Thom–Mather property holds.

LEMMA 4.4.3. *Continue the hypotheses and notation of Lemma 4.4.2. Then, upon restriction to $\tilde{\mathcal{U}}' \cap \pi(X^\ell, g_{\mathcal{P}'})^{-1}(\Delta^\circ(X^\ell, \mathcal{P}'))$, the following equality holds:*

$$(4.4.9) \quad \pi(X^\ell, g_{\mathcal{P}}) \circ \pi(X^\ell, g_{\mathcal{P}'}) = \pi(X^\ell, g_{\mathcal{P}}).$$

PROOF. The subspace $\tilde{\mathcal{U}}' \cap \pi(X^\ell, g_{\mathcal{P}'})^{-1}(\Delta^\circ(X^\ell, \mathcal{P}'))$ is in the image of the composition $e(X^\ell, g_{\mathcal{P}}) \circ \rho_{\mathcal{P},\mathcal{P}'}^{X,d}$. The equality follows from the observation that the pullback of $\pi(X^\ell, g_{\mathcal{P}'})$ by $\rho_{\mathcal{P},\mathcal{P}'}^{X,d}$ is given by the projection π_1 of (4.4.6). \square

4.5. Construction of the families of locally flattened metrics

We now construct the families of metrics satisfying the conditions of Lemma 4.4.2 and thus yielding projection maps $\pi(X^\ell, g_{\mathcal{P}})$ which satisfy (4.4.9). The first step is to construct a locally flat family of metrics.

LEMMA 4.5.1. *If $g_{\mathcal{P}}$ is a family of metrics smoothly parameterized by $\Delta^\circ(X^\ell, \mathcal{P})$, then there is a family of smooth metrics $\bar{g}_{\mathcal{P}}$ smoothly parameterized by $\Delta^\circ(X^\ell, \mathcal{P})$ such that the following hold:*

- (1) $\bar{g}_{\mathcal{P}}$ is locally flat with respect to a tubular neighborhood $\tilde{\mathcal{U}}' \subseteq \tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}})$.
- (2) For every $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$, the metric $\bar{g}_{\mathcal{P},\mathbf{x}}$ is equal to $g_{\mathcal{P},\mathbf{x}}$ on the complement of the support of $\tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}}) \cap \pi(X^\ell, g_{\mathcal{P}})^{-1}(\mathbf{x})$.
- (3) The families $g_{\mathcal{P}}$ and $\bar{g}_{\mathcal{P}}$ are C^1 close within the support of $\tilde{\mathcal{U}}'$.
- (4) The exponential maps $e(X^\ell, g_{\mathcal{P}})$ and $e(X^\ell, \bar{g}_{\mathcal{P}})$ are equal.
- (5) If $g_{\mathcal{P}}$ is already flat with respect to $\tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}})$ at $\mathbf{x} \in \Delta^\circ(X^\ell, \mathcal{P})$, then $\bar{g}_{\mathcal{P},\mathbf{x}} = g_{\mathcal{P},\mathbf{x}}$.

PROOF. Let $\tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}})$ be the tubular neighborhood of $\Delta^\circ(X^\ell, \mathcal{P})$ defined by the family of metrics, $g_{\mathcal{P}}$, so

$$e(X^\ell, g_{\mathcal{P}}) : \tilde{\mathcal{O}}(X^\ell, g_{\mathcal{P}}) \subset \tilde{\nu}(X^\ell, \mathcal{P}) \cong \tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}}).$$

For $P \in \mathcal{P}$, let $s_P : \Delta^\circ(X^\ell, \mathcal{P}) \rightarrow [0, \infty)$ be a smooth function such that

$$(4.5.1) \quad \{(v_1, \dots, v_\ell) \in \tilde{\nu}(X^\ell, \mathcal{P})|_{\mathbf{x}} : |v_i| \leq 4s_P(\mathbf{x}) \text{ for all } i \in P \text{ and } P \in \mathcal{P}\} \subset \mathcal{O}(\mathcal{P}).$$

For $\mathbf{x} \in \Delta^\circ(\mathcal{P})$ and $(F_P)_{P \in \mathcal{P}} \in \text{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}})|_{\mathbf{x}}$, the frame F_P and the exponential map of the metric $g_{\mathcal{P},\mathbf{x}}$ identify $B(0, 4s_P(\mathbf{x})) \subset \mathbb{R}^4$ with $B_{g_{\mathcal{P},\mathbf{x}}}(x_P, 4s_P(\mathbf{x})) \subset X$. Let δ_{x_P} denote the flat metric on $B_{g_{\mathcal{P},\mathbf{x}}}(x_P, 4s_P(\mathbf{x}))$ given by the pushforward of the Euclidean metric on $B(0, 4s_P(\mathbf{x}))$ by the exponential map. For $\lambda \in (0, 1]$, let $\chi_\lambda : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function such that $\chi_\lambda(t) = 0$ for $t \leq \frac{1}{2}\lambda$ and $\chi_\lambda(t) = 1$ for $t \geq \lambda$. We define $\bar{g}_{\mathcal{P},\mathbf{x}}$ by

$$(4.5.2) \quad \bar{g}_{\mathcal{P},\mathbf{x}} := \begin{cases} g_{\mathcal{P},\mathbf{x}} & \text{on } X \setminus \cup_i B_{g_{\mathcal{P},\mathbf{x}}}(x_P, 4s_P(\mathbf{x})), \\ \chi_{4s_P(\mathbf{x})}(|x|)g_{\mathcal{P},\mathbf{x}} + (1 - \chi_{4s_P(\mathbf{x})}(|x|))\delta_{x_P} & \text{on } \Omega_{g_{\mathcal{P},\mathbf{x}}}(x_P, 2s_P(\mathbf{x}), 4s_P(\mathbf{x})), \\ \delta_{x_P} & \text{on } B_{g_{\mathcal{P},\mathbf{x}}}(x_P, 2s_P(\mathbf{x})), \end{cases}$$

where $x(\cdot)$ denotes the local coordinate chart defined by the exponential map for $g_{\mathcal{P},\mathbf{x}}$.

The resulting family of metrics, $\bar{g}_{\mathcal{P}}$, is locally flat with respect to the tubular neighborhood $\tilde{\mathcal{U}}'$ defined by replacing $4s_P(\mathbf{x})$ with $2s_P(\mathbf{x})$ in (4.5.1). Item (2) follows immediately from the construction (4.5.2) while Item (3) follows from the expression for a metric in geodesic normal coordinates [5, Definition 1.24, Proposition 1.25, and Corollary 1.32].

For x in the support of \mathbf{x} , the radial geodesics from x are the same for $g_{\mathcal{P},\mathbf{x}}$ and for $\bar{g}_{\mathcal{P},\mathbf{x}}$. These radial geodesics from x define the exponential maps $e(X^\ell, g_{\mathcal{P}})$ and $e(X^\ell, \bar{g}_{\mathcal{P}})$ and hence these maps are equal.

Because the metric $g_{\mathcal{P},\mathbf{x}}$ has not been changed at x_P , the frame bundles do not change. If the metric $g_{\mathcal{P},\mathbf{x}}$ is flat on $B(x_P, 4s_P(\mathbf{x}))$, then it is given by the Euclidean metric in geodesic normal coordinates by [93, Theorem 7.11]. Thus, $\delta_{x_P} = g_{\mathcal{P},\mathbf{x}}$ and the interpolation in (4.5.2) does not change the metric. \square

We now construct the desired families of metrics.

LEMMA 4.5.2. *For each partition \mathcal{P} of N_ℓ , there is a smooth family of metrics $g_{\mathcal{P}}$ on X defining tubular neighborhoods $\mathcal{U}(X^\ell, g_{\mathcal{P}})$ by their exponential maps, such that the following hold:*

- (1) *The family $g_{\mathcal{P}}$ is locally flat with respect to a tubular neighborhood $\tilde{\mathcal{U}}'(X^\ell, g_{\mathcal{P}}) \subset \tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}})$.*
- (2) *If $\mathcal{P} < \mathcal{P}'$, then for all $\mathbf{x}' \in \Delta^\circ(X^\ell, \mathcal{P}') \cap \tilde{\mathcal{U}}'(X^\ell, g_{\mathcal{P}})$, we have $g_{\mathcal{P}',\mathbf{x}'} = g_{\mathcal{P},\mathbf{x}}$, where $\mathbf{x} = \pi(X^\ell, g_{\mathcal{P}})(\mathbf{x}')$.*
- (3) *For $\sigma \in \mathfrak{S}_\ell$, we have $g_{\mathcal{P},\mathbf{x}} = g_{\sigma(\mathcal{P}),\sigma(\mathbf{x})}$.*

PROOF. The proof is by induction on the partitions, \mathcal{P} . We will construct the family of metrics, $g_{\mathcal{P}}$, for one partition in the orbit, $[\mathcal{P}]$, and then use Item (3) in the lemma to define $g_{\mathcal{P}}$ for all other $\mathcal{P}' \in [\mathcal{P}]$.

Choose a Riemannian metric, g , on X . To every partition \mathcal{P} of N_ℓ , define an initial family of metrics $g_{\mathcal{P}}$ by setting $g_{\mathcal{P},\mathbf{x}} := g$.

Let \mathcal{P}_0 be the crudest partition. Redefine $g_{\mathcal{P}_0}$ by applying the flattening procedure of Lemma 4.5.1. The redefined family of metrics $g_{\mathcal{P}_0}$ is locally flat with respect to the tubular neighborhood $\tilde{\mathcal{U}}'(X^\ell, \mathcal{P}_0, g_{\mathcal{P}_0})$ given in Lemma 4.5.1.

Assume that families of metrics, $g_{\mathcal{P}}$, and tubular neighborhoods, $\tilde{\mathcal{U}}'(X^\ell, g_{\mathcal{P}})$, satisfying the conclusions of the lemma have been defined for all $\mathcal{P} < \mathcal{P}'$. There are two families of metrics on X parameterized by the intersection,

$$(4.5.3) \quad \Delta^\circ(X^\ell, \mathcal{P}') \cap \mathcal{U}(X^\ell, g_{\mathcal{P}}),$$

namely, the initial family, $g_{\mathcal{P}'}$, and the family, $\pi(X^\ell, g_{\mathcal{P}})^* g_{\mathcal{P}}$, which is defined by

$$\pi(X^\ell, g_{\mathcal{P}})^* g_{\mathcal{P},\mathbf{x}'} := g_{\mathcal{P},\mathbf{x}}, \quad \text{where } \pi(X^\ell, g_{\mathcal{P}})(\mathbf{x}') = \mathbf{x}.$$

For $\mathcal{P}_1 < \mathcal{P}'$ and $\mathcal{P}_2 < \mathcal{P}'$ and by shrinking the tubular neighborhoods when necessary, observe that if the intersection,

$$(4.5.4) \quad \Delta^\circ(X^\ell, \mathcal{P}') \cap \tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}_1}) \cap \tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}_2}),$$

is non-empty, then there is a relation $\mathcal{P}_1 < \mathcal{P}_2 < \mathcal{P}'$. Consequently, the relations,

$$\pi(X^\ell, g_{\mathcal{P}_1}) \circ \pi(X^\ell, g_{\mathcal{P}_2}) = \pi(X^\ell, g_{\mathcal{P}_1}),$$

(which follow by induction and Lemma 4.4.3) and the identity,

$$g_{\mathcal{P}_2} = \pi(X^\ell, g_{\mathcal{P}_1})^* g_{\mathcal{P}_1},$$

imply that the families of metrics,

$$(4.5.5) \quad \pi(X^\ell, g_{\mathcal{P}_1})^* g_{\mathcal{P}_1} \quad \text{and} \quad \pi(X^\ell, g_{\mathcal{P}_2})^* g_{\mathcal{P}_2},$$

are equal on (4.5.4). Thus we can form a new family of metrics by interpolating between $g_{\mathcal{P}}$ on

$$\Delta^\circ(X, \mathcal{P}') \setminus \bigcup_{\mathcal{P} < \mathcal{P}'} \tilde{\mathcal{U}}'(X^\ell, g_{\mathcal{P}}),$$

and the pullback metrics (4.5.5). By further shrinking the tubular neighborhoods, $\tilde{\mathcal{U}}'(X^\ell, g_{\mathcal{P}})$, for $\mathcal{P} < \mathcal{P}'$ and taking a sufficiently small neighborhood, $\tilde{\mathcal{U}}'(X^\ell, g_{\mathcal{P}'})$, we can assume this interpolated family is locally flat with respect to $\tilde{\mathcal{U}}'(X^\ell, g_{\mathcal{P}'})$ on

$$(4.5.6) \quad \Delta^\circ(X^\ell, \mathcal{P}') \cap \left(\bigcup_{\mathcal{P} < \mathcal{P}'} \tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}}) \right).$$

Now apply the flattening procedure of Lemma 4.5.1 to this interpolated family. Because this family is already flat with respect to $\tilde{\mathcal{U}}'(X^\ell, g_{\mathcal{P}'})$ for \mathbf{x} in the open space (4.5.6), the flattening procedure does not change the interpolated family there. Hence, the new flattened family satisfies Items (1) and (2), completing the induction. \square

4.6. Normal bundles of strata of $\text{Sym}^\ell(X)$

We now use the \mathfrak{S}_ℓ -equivariance of the construction of the previous section to define normal bundles of the strata $\Sigma(X^\ell, \mathcal{P})$ in $\text{Sym}^\ell(X)$. Recall from Lemma 3.3.6 that to describe the end of $\Sigma(X^\ell, \mathcal{P}')$ near $\Sigma(X^\ell, \mathcal{P})$, we have to consider the ends of the diagonals $\Delta^\circ(X^\ell, \mathcal{P}'')$ near $\Delta^\circ(X^\ell, \mathcal{P})$ for all $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$.

The analogue of the bundle (4.4.1) appropriate for this discussion is

$$(4.6.1) \quad \begin{aligned} & \tilde{\nu}(X^\ell, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}}) \\ & := \text{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}}) \times_{\tilde{G}(T, \mathcal{P})} \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}} \left(\Delta^\circ(Z_P, \mathcal{P}''_P) \times \prod_{Q \in \mathcal{P}''_P} Z_Q \right), \end{aligned}$$

where we use \bigsqcup to denote disjoint union. The symmetric group $\mathfrak{S}(\mathcal{P})$ acts on the spaces,

$$\text{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}}), \quad \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}} \Delta^\circ(Z_P, \mathcal{P}''_P), \quad \text{and} \quad \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}} \prod_{Q \in \mathcal{P}''_P} Z_Q,$$

by permuting the factors. The diagonal action of $\mathfrak{S}(\mathcal{P})$ on the preceding three spaces defines an action on $\tilde{\nu}(X^\ell, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}})$. We define

$$(4.6.2) \quad \begin{aligned} & \nu(X^\ell, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}}) \\ & := \text{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}}) \times_{G(T, \mathcal{P})} \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}} \left(\Delta^\circ(Z_P, \mathcal{P}''_P) \times \prod_{Q \in \mathcal{P}''_P} Z_Q \right) \end{aligned}$$

where, if $\tilde{G}(T, \mathcal{P})$ as defined as in (4.3.3),

$$G(T, \mathcal{P}) := \tilde{G}(T, \mathcal{P}) \rtimes \mathfrak{S}(\mathcal{P}),$$

where \rtimes denotes the semi-direct product, [56, p. 59] (compare the structure group appearing on [35, p. 252]). Thus,

$$\nu(X^\ell, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}}) = \tilde{\nu}(X^\ell, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}}) / \mathfrak{S}(\mathcal{P}).$$

We define

$$(4.6.3) \quad \begin{aligned} \tilde{\mathcal{O}}(X^\ell, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}}) &\subseteq \tilde{\nu}(X^\ell, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}}), \\ \mathcal{O}(X^\ell, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}}) &= \tilde{\mathcal{O}}(X^\ell, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}}) / \mathfrak{S} \subseteq \nu(X^\ell, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}}) \end{aligned}$$

by analogy with $\tilde{\mathcal{O}}(X^\ell, \mathcal{P}, \mathcal{P}', g_{\mathcal{P}})$ in (4.4.4).

We define a downwards overlap map,

$$(4.6.4) \quad \rho_{\mathcal{P}, [\mathcal{P}']}^d : \nu(X^\ell, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}}) \rightarrow \mathcal{O}(X^\ell, g_{\mathcal{P}}) = \tilde{\mathcal{O}}(X^\ell, g_{\mathcal{P}}) / \mathfrak{S}(\mathcal{P}),$$

by extending the map of fibers given by

$$\prod_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}} e(Z_P, \mathcal{P}''_P),$$

over the domain. We have used the convention that if $f_\alpha : X_\alpha \rightarrow Y_\alpha$ is a family of continuous maps, then $\prod_\alpha f_\alpha : \sqcup_\alpha X_\alpha \rightarrow \sqcup_\alpha Y_\alpha$ is the map defined on the disjoint unions by this family. Note that the quotient in (4.6.4) could be taken to be by $W(\mathcal{P})$ instead of $\mathfrak{S}(\mathcal{P})$, as $\Gamma(\mathcal{P})$ acts trivially. This phenomenon will occur frequently, but as it will not affect the discussion we will not mention it again.

To define an upwards overlap map, we define

$$(4.6.5) \quad \mathcal{O}(X^\ell, [\mathcal{P} < \mathcal{P}'], g_{\mathcal{P}''}) := \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \tilde{\mathcal{O}}(X^\ell, \mathcal{P}'', g_{\mathcal{P}''}) / \mathfrak{S}(\mathcal{P}).$$

The upwards transition map is then given by

$$(4.6.6) \quad \rho_{\mathcal{P}, [\mathcal{P}']}^u : \mathcal{O}(X^\ell, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}}) \rightarrow \mathcal{O}(X^\ell, [\mathcal{P} < \mathcal{P}'], g_{\mathcal{P}''}), \quad \rho_{\mathcal{P}, [\mathcal{P}']}^u = \prod_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \rho_{\mathcal{P}, [\mathcal{P}']}^u.$$

The equivariance of the exponential map $e(X^\ell, g_{\mathcal{P}})$ with respect to the symmetric group action implies that $e(X^\ell, g_{\mathcal{P}})$ defines an embedding,

$$(4.6.7) \quad e(X^\ell, g_{\mathcal{P}}) : \mathcal{O}(X^\ell, g_{\mathcal{P}}) \rightarrow \text{Sym}^\ell(X).$$

This equivariance and Item (3) of Lemma 4.5.2 then imply that the exponential maps, $e(X^\ell, g_{\mathcal{P}''})$, define an embedding,

$$e(X^\ell, g_{[\mathcal{P} < \mathcal{P}']}) = \prod_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} e(X^\ell, g_{\mathcal{P}''}) : \mathcal{O}(X^\ell, [\mathcal{P} < \mathcal{P}'], g_{\mathcal{P}''}) \rightarrow \text{Sym}^\ell(X).$$

We then have the

PROPOSITION 4.6.1. *For each partition \mathcal{P} of N_ℓ , let $g_{\mathcal{P}}$ be the family of metrics constructed in Lemma 4.5.2. Then the image of the map (4.6.7) defines a homeomorphism onto a neighborhood $\mathcal{U}(X^\ell, g_{\mathcal{P}})$ of $\Sigma(X^\ell, \mathcal{P})$. For partitions $\mathcal{P} < \mathcal{P}'$ of N_ℓ , there is a commutative diagram,*

$$(4.6.8) \quad \begin{array}{ccc} \mathcal{O}(X^\ell, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}}) & \xrightarrow{\rho_{\mathcal{P}, [\mathcal{P}']}^{X, u}} & \mathcal{O}(X^\ell, [\mathcal{P} < \mathcal{P}'], g_{\mathcal{P}''}) \\ \rho_{\mathcal{P}, [\mathcal{P}']}^{X, d} \downarrow & & e(X^\ell, g_{[\mathcal{P} < \mathcal{P}']}) \downarrow \\ \mathcal{O}(X^\ell, g_{\mathcal{P}}) & \xrightarrow{e(X^\ell, g_{\mathcal{P}'})} & \text{Sym}^\ell(X) \end{array}$$

such that any point in $\mathcal{U}(X^\ell, g_{\mathcal{P}}) \cap \mathcal{U}(X^\ell, g_{\mathcal{P}'})$ is in the image of the compositions $e(X^\ell, g_{\mathcal{P}}) \circ \rho_{\mathcal{P}, [\mathcal{P}']}^d$ and $e(X^\ell, g_{[\mathcal{P} < \mathcal{P}']}) \circ \rho_{\mathcal{P}, [\mathcal{P}']}^u$.

PROOF. The proposition follows immediately from the \mathfrak{S}_ℓ -equivariance of the constructions of Sections 4.4 and 4.5. \square

We note that the projection maps, $\pi(X^\ell, g_{\mathcal{P}}) : \tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}}) \rightarrow \Delta^\circ(X^\ell, \mathcal{P})$, are $\mathfrak{S}(\mathcal{P})$ -equivariant and thus define projection maps,

$$\pi(X^\ell, g_{\mathcal{P}}) : \mathcal{U}(X^\ell, g_{\mathcal{P}}) \rightarrow \Sigma(X^\ell, \mathcal{P}),$$

on the quotient. These maps on the quotient still satisfy (4.4.9).

4.7. The tubular distance function

Rather than a single tubular distance function, $t_{\mathcal{P}} : \mathcal{U}(X^\ell, g_{\mathcal{P}}) \rightarrow [0, \infty)$, we find it convenient to define several functions, one for each $P \in \mathcal{P}$, whose common zero locus is $\Sigma(X^\ell, \mathcal{P})$. The functions, $t(Z_P)$, defined in Lemma 4.2.5 are $\text{SO}(4)$ -equivariant and thus define functions,

$$t(Z_P) : \text{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}}) \times_{\tilde{G}(T, \mathcal{P})} \prod_{P \in \mathcal{P}} Z_P \rightarrow [0, \infty).$$

For $I^{\mathcal{P}} = \prod_{P \in \mathcal{P}} [0, 1]$, we define

$$(4.7.1) \quad \vec{t}(X^\ell, g_{\mathcal{P}}) := \left(t(Z_P) \circ e(X^\ell, g_{\mathcal{P}})^{-1} \right)_{P \in \mathcal{P}} : \tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}}) \rightarrow I^{\mathcal{P}}.$$

If \mathcal{P} is the finest partition described in Remark 3.3.3, then the function, $\vec{t}(X^\ell, g_{\mathcal{P}})$, is the zero map.

To define hypersurfaces and disk bundles in $\mathcal{U}(X^\ell, g_{\mathcal{P}})$, we will use the following ‘square’ sets,

$$(4.7.2) \quad \begin{aligned} D(\mathcal{P}, \varepsilon) &= \{(t_P)_{P \in \mathcal{P}} : 0 \leq t_P < \varepsilon \text{ for all } P \in \mathcal{P}\}, \\ \bar{D}(\mathcal{P}, \varepsilon) &= \{(t_P)_{P \in \mathcal{P}} : 0 \leq t_P \leq \varepsilon \text{ for all } P \in \mathcal{P}\}, \\ \partial \bar{D}(\mathcal{P}, \varepsilon) &= \{(t_P)_{P \in \mathcal{P}} \in \bar{D}(\mathcal{P}, \varepsilon) : t_P = \varepsilon \text{ for some } P \in \mathcal{P}\}, \end{aligned}$$

to define tubular neighborhoods of the diagonals. The symmetric group, $\mathfrak{S}(\mathcal{P})$, acts on $I^{\mathcal{P}}$ and thus on the sets (4.7.2) by permuting the factors. Observe that $\vec{t}(X^\ell, g_{\mathcal{P}})$ is equivariant with respect to the action of $\mathfrak{S}(\mathcal{P})$ and thus defines a map,

$$\vec{t}(X^\ell, g_{\mathcal{P}}) : \mathcal{U}(X^\ell, g_{\mathcal{P}}) \rightarrow I^{\mathcal{P}} / \mathfrak{S}(\mathcal{P}).$$

For $s \in [0, 1]$, we will write $s\vec{t}(X^\ell, g_{\mathcal{P}})$ for the function $(st_P)_{P \in \mathcal{P}}$ and $\vec{t}^P(X^\ell, g_{\mathcal{P}})$ for the P -th component of $\vec{t}(X^\ell, g_{\mathcal{P}})$. The identification of $\mathcal{U}(X^\ell, g_{\mathcal{P}})$ with an open subspace of the bundle $\nu(X^\ell, \mathcal{P})$ then gives the following result.

LEMMA 4.7.1. *For every partition \mathcal{P} of N_ℓ , there is a smooth map,*

$$r(X^\ell, g_{\mathcal{P}}) : \tilde{\mathcal{U}}''(X^\ell, g_{\mathcal{P}}) \times [0, 1] \rightarrow \tilde{\mathcal{U}}''(X^\ell, g_{\mathcal{P}}),$$

such that $r(X^\ell, g_{\mathcal{P}})(\cdot, s)$ is a homeomorphism for all $s \in (0, 1]$ and $r(X^\ell, g_{\mathcal{P}})$ satisfies the following properties:

- (1) $r(X^\ell, g_{\mathcal{P}})(\cdot, 1)$ is the identity map.
- (2) $r(X^\ell, g_{\mathcal{P}})(\cdot, 0) = \pi(X^\ell, g_{\mathcal{P}})(\cdot)$.
- (3) For all $s \in [0, 1]$,

$$(4.7.3) \quad \vec{t}(X^\ell, g_{\mathcal{P}}) \circ r(X^\ell, g_{\mathcal{P}})(\cdot, s) = s^2 \vec{t}(X^\ell, g_{\mathcal{P}})(\cdot).$$

If $\mathcal{P} < \mathcal{P}'$, then on $\tilde{\mathcal{U}}''(X^\ell, g_{\mathcal{P}}) \cap \tilde{\mathcal{U}}''(X^\ell, g_{\mathcal{P}'})$ we have

$$(4.7.4) \quad \pi(X^\ell, g_{\mathcal{P}}) \circ r(X^\ell, g_{\mathcal{P}'}) (\cdot, s) = \pi(X^\ell, g_{\mathcal{P}})(\cdot),$$

for all $s \in [0, 1]$.

PROOF. If \mathcal{P} is the finest partition, as described in Remark 3.3.3, set $r(X^\ell, g_{\mathcal{P}})(\cdot, s)$ equal to the identity map for all s . Otherwise, we define the map $r(X^\ell, g_{\mathcal{P}})$ by pushing forward a deformation retraction, $r_\nu(X^\ell, \mathcal{P})$, on $\nu(X^\ell, \mathcal{P})$ to $\mathcal{U}(X^\ell, \mathcal{P})$ by the exponential map, $e(X^\ell, g_{\mathcal{P}})$. Let $r_P : Z_P \times [0, 1] \rightarrow Z_P$ be the deformation retraction defined in (4.2.8). If $\Delta_{\mathcal{P}} : [0, 1] \rightarrow \prod_{P \in \mathcal{P}} [0, 1]$ is the diagonal inclusion and $\text{id}_{Z, P} : Z_P \rightarrow Z_P$ is the identity map, then the composition

$$\begin{array}{c} (\prod_{P \in \mathcal{P}} Z_P) \times [0, 1] \\ \downarrow \prod_{P \in \mathcal{P}} \text{id}_{Z, P} \times \Delta_{\mathcal{P}} \\ \prod_{P \in \mathcal{P}} (Z_P \times [0, 1]) \\ \downarrow \prod_{P \in \mathcal{P}} r_P \\ \prod_{P \in \mathcal{P}} Z_P \end{array}$$

defines a $G(T, \mathcal{P})$ -equivariant deformation retraction and hence a deformation retraction,

$$r_\nu(X^\ell, \mathcal{P}) : \nu(X^\ell, \mathcal{P}) \times [0, 1] \rightarrow \nu(X^\ell, \mathcal{P}).$$

If $r(X^\ell, g_{\mathcal{P}})$ is defined by pushing $r_\nu(X^\ell, \mathcal{P})$ forward by $e(X^\ell, g_{\mathcal{P}})$, then $r(X^\ell, g_{\mathcal{P}})$ immediately satisfies Items (1) and (2) of the lemma. The property (4.7.3) follows from the definitions of $r(X^\ell, g_{\mathcal{P}})$ and $\vec{t}(X^\ell, g_{\mathcal{P}})$ and (4.2.17). Finally, (4.7.4) follows immediately from the definition of $r(X^\ell, g_{\mathcal{P}})$ as a map of fibers of $\nu(X^\ell, \mathcal{P})$. \square

The following result describes the failure of the second Thom–Mather conditions in (4.1.1) for these maps.

LEMMA 4.7.2. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_ℓ and let $r(X^\ell, g_{\mathcal{P}'})$ be the map defined in Lemma 4.7.1. If $\mathbf{x} \in \mathcal{U}(X^\ell, g_{\mathcal{P}}) \cap \mathcal{U}(X^\ell, g_{\mathcal{P}'})$ and $s \in [0, 1]$, then the following hold:*

- (1) If $P \notin \mathcal{P} \cap \mathcal{P}'$, then $\vec{t}^P(X^\ell, g_{\mathcal{P}})(\mathbf{x}) = \vec{t}^P(X^\ell, g_{\mathcal{P}})(r(X^\ell, g_{\mathcal{P}'}) (\mathbf{x}, s))$.
- (2) If $P \in \mathcal{P} \cap \mathcal{P}'$, then $s^2 \vec{t}^P(X^\ell, g_{\mathcal{P}})(\mathbf{x}) = \vec{t}^P(X^\ell, g_{\mathcal{P}})(r(X^\ell, g_{\mathcal{P}'}) (\mathbf{x}, s))$.

(3) If $P \in \mathcal{P} \cap \mathcal{P}'$, then $\vec{t}^P(X^\ell, g_{\mathcal{P}})(\mathbf{x}) = \vec{t}^P(X^\ell, g_{\mathcal{P}'}) (\mathbf{x})$.

PROOF. We prove Item (1) by showing that if $P \notin \mathcal{P} \cap \mathcal{P}'$, then the functions $t(Z_P)$ are constant on the fibers of $\pi(X^\ell, g_{\mathcal{P}'})$ which contain the paths $s \mapsto r(X^\ell, g_{\mathcal{P}'})(\cdot, s)$. The definition of the upwards overlap map $\rho_{\mathcal{P}, [\mathcal{P}']}^u$ implies that pulling the map $\pi(X^\ell, g_{\mathcal{P}'})$ back by $\rho_{\mathcal{P}, [\mathcal{P}']}^u$ gives the projection map,

$$\bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}''} \Delta^\circ(Z_P, \mathcal{P}''_P) \times \prod_{Q \in \mathcal{P}''_P} Z_Q \rightarrow \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}''} \Delta^\circ(Z_P, \mathcal{P}''_P).$$

From the definition of the downwards overlap map, $\rho_{\mathcal{P}, [\mathcal{P}']}^d$, in (4.6.4), we see that the image of the restriction of $\rho_{\mathcal{P}, [\mathcal{P}']}^d$ to the above fiber is given by

$$\bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}''} \text{Im}(e(Z_P, \mathcal{P}''_P)) = \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}''} \mathcal{U}(Z_P, \mathcal{P}''_P) \subset \prod_{P \in \mathcal{P}} Z_P,$$

where the set $\mathcal{U}(Z_P, \mathcal{P}'_P)$ is defined prior to (4.2.13). We focus on the restriction of $\pi(X^\ell, g_{\mathcal{P}'}) \circ \rho_{\mathcal{P}, [\mathcal{P}']}^u$ to one of the components in the domain above and see that the result holds for each such restriction. Pushing this restriction of $\pi(X^\ell, g_{\mathcal{P}'}) \circ \rho_{\mathcal{P}, [\mathcal{P}']}^u$ forward by $\rho_{\mathcal{P}, [\mathcal{P}']}^d$ to $\prod_{P \in \mathcal{P}} \mathcal{U}(Z_P, \mathcal{P}'_P)$ gives the map,

$$\prod_{P \in \mathcal{P}} \pi(Z_P, \mathcal{P}'_P) : \prod_{P \in \mathcal{P}} \mathcal{U}(Z_P, \mathcal{P}'_P) \rightarrow \prod_{P \in \mathcal{P}} Z_P,$$

where $\pi(Z_P, \mathcal{P}'_P)$ is defined in (4.2.14). The property (4.2.18) then implies that $t(Z_P)$ is constant on the fibers of $\pi(Z_P, \mathcal{P}'_P)$ for all $P \notin \mathcal{P} \cap \mathcal{P}'$.

If $P \in \mathcal{P} \cap \mathcal{P}'$, the equality (4.2.17), namely, $t(Z_P)(r_P(\cdot, s)) = s^2 t(Z_P)(\cdot)$, then yields Item (2).

Item (3) follows by observing that if $P \in \mathcal{P} \cap \mathcal{P}'$, then both $\rho_{\mathcal{P}, [\mathcal{P}']}^u$ and $\rho_{\mathcal{P}, [\mathcal{P}']}^d$ are defined by the identity map on Z_P and the P -th components of $\vec{t}(X^\ell, g_{\mathcal{P}}) \circ \rho_{\mathcal{P}, [\mathcal{P}']}^d$ and $\vec{t}(X^\ell, g_{\mathcal{P}'}) \circ \rho_{\mathcal{P}, [\mathcal{P}']}^u$ are both defined by $t(Z_P)$ and hence are equal. \square

Although the preceding lemma showed that the second Thom–Mather identity in (4.1.1) does not hold, the following lemma shows that for $\mathcal{P} < \mathcal{P}'$, the restriction of the projection $\pi(X^\ell, g_{\mathcal{P}'})$ to a sufficiently small tubular neighborhood of $\Delta^\circ(X^\ell, \mathcal{P}')$ preserves the tubular neighborhood defined by $\vec{t}(X^\ell, g_{\mathcal{P}})$.

LEMMA 4.7.3. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_ℓ . Let $\hat{D}(\mathcal{P}, \varepsilon)$ denote either $D(\mathcal{P}, \varepsilon)$ or $\bar{D}(\mathcal{P}, \varepsilon)$. If $\mathbf{x} \in \mathcal{U}(X^\ell, g_{\mathcal{P}}) \cap \mathcal{U}(X^\ell, g_{\mathcal{P}'})$, then the following hold:*

- (1) *If $\vec{t}(X^\ell, g_{\mathcal{P}})(\mathbf{x}) \in \hat{D}(\mathcal{P}, \varepsilon)$, then $(\vec{t}(X^\ell, g_{\mathcal{P}}) \circ \pi(X^\ell, g_{\mathcal{P}'}))(\mathbf{x}) \in \hat{D}(\mathcal{P}, \varepsilon)$.*
- (2) *If $\varepsilon' < \varepsilon$, and $\vec{t}(X^\ell, g_{\mathcal{P}'}) (\mathbf{x}) \in \bar{D}(\mathcal{P}', \varepsilon')$, and $(\vec{t}(X^\ell, g_{\mathcal{P}}) \circ \pi(X^\ell, g_{\mathcal{P}'}))(\mathbf{x}) \in \hat{D}(\mathcal{P}, \varepsilon)$, then $\vec{t}(X^\ell, g_{\mathcal{P}})(\mathbf{x}) \in \hat{D}(\mathcal{P}, \varepsilon)$.*

PROOF. Item (1) follows immediately from Items (1) and (2) of Lemma 4.7.2. To prove Item (2), we note that if $(\vec{t}(X^\ell, g_{\mathcal{P}}) \circ \pi(X^\ell, g_{\mathcal{P}'}))(\mathbf{x}) \in \hat{D}(\mathcal{P}, \varepsilon)$ and $\vec{t}(X^\ell, g_{\mathcal{P}})(\mathbf{x}) \notin \hat{D}(\mathcal{P}, \varepsilon)$, then Item (1) of Lemma 4.7.2 implies that there must be a $P \in \mathcal{P} \cap \mathcal{P}'$ such that the P -th component of $\vec{t}(X^\ell, g_{\mathcal{P}})(\mathbf{x})$ is greater than or equal to ε . However, Item (3) of Lemma 4.7.2 and the assumption $\vec{t}(X^\ell, g_{\mathcal{P}'}) (\mathbf{x}) \in \bar{D}(\mathcal{P}', \varepsilon')$ imply that the P -th component of $\vec{t}(X^\ell, g_{\mathcal{P}})(\mathbf{x})$ is less than or equal to ε' , which is strictly less than ε . This contradiction proves Item (2). \square

4.8. Decomposition of the strata

We now construct decompositions of $\text{Sym}^\ell(X)$ and $\Sigma(X^\ell, \mathcal{P})$ to be used in the construction of the link of $M_5 \times \text{Sym}^\ell(X)$ appearing in equation (2.6.1).

LEMMA 4.8.1. *Enumerate the strata of $\text{Sym}^\ell(X)$ as described in Section 3.4 using partitions $\mathcal{P}_0, \dots, \mathcal{P}_n$. To each partition \mathcal{P}_i , there is a small, positive parameter ε_i satisfying $\varepsilon_i > \varepsilon_j$ for $i < j$ such that if we define*

$$T_i := \mathcal{U}(X^\ell, \mathcal{P}_i) \cap \bar{t}(X^\ell, g_{\mathcal{P}_i})^{-1}(D(\mathcal{P}_i, \varepsilon_i)) \setminus \bigcup_{j < i} \bar{t}(X^\ell, g_{\mathcal{P}_j})^{-1}(D(\mathcal{P}_j, \varepsilon_j)), \quad \text{for } i \neq n,$$

$$T_n := \Sigma(X^\ell, \mathcal{P}_n) \setminus \bigcup_{j < n} \bar{t}(X^\ell, g_{\mathcal{P}_j})^{-1}(D(\mathcal{P}_j, \varepsilon_j)),$$

then for $i \neq n$, we have $T_i \in \mathcal{U}(X^\ell, g_{\mathcal{P}_i})$, and $\text{Sym}^\ell(X) = \cup_i T_i$, and this union is disjoint.

PROOF. The lemma follows immediately from the definitions. \square

The decomposition of $\text{Sym}^\ell(X)$ in the previous lemma leads to decompositions of the strata $\Sigma(X^\ell, \mathcal{P}_i)$.

LEMMA 4.8.2. *Continue the hypotheses of Lemma 4.8.1 and let ε_i be the parameter constructed in Lemma 4.8.1. For one of the partitions, \mathcal{P}_k , in the given enumeration, define*

$$T_{k,j} := \text{cl}\left(\Sigma(X^\ell, \mathcal{P}_k)\right) \cap T_j, \quad \text{and} \quad K_k := \text{cl}\left(\Sigma(X^\ell, \mathcal{P}_k)\right) \setminus \bigcup_{i < k} T_{k,i}.$$

Then the following hold.

- (1) *There is an equality,*

$$\text{cl}\left(\Sigma(X^\ell, \mathcal{P}_k)\right) = K_k \cup \left(\bigcup_{j < k} T_{k,j}\right),$$

where the union on the right-hand side is disjoint.

- (2) *K_k is compact.*
 (3) *The restriction of $\pi(X^\ell, \mathcal{P}_j)$ to $T_{k,j}$ takes values in K_j .*

PROOF. The first assertion follows immediately from Lemma 4.8.1. We note that K_k is compact by observing that it is a closed subset of the compact set $\text{cl}\Sigma(X^\ell, \mathcal{P}_k)$.

We now prove Item (3). If $\mathbf{x} \in T_{k,j}$, then Item (1) in Lemma 4.7.3 implies that $\mathbf{x}' = \pi(X^\ell, \mathcal{P}_k)(\mathbf{x}) \in \bar{t}(X^\ell, \mathcal{P}_j)^{-1}(D(\mathcal{P}_k, \varepsilon_j))$. If $\pi(X^\ell, \mathcal{P}_k)(\mathbf{x}) \notin K_j$, then there is an index $i < j$ such that $\bar{t}(X^\ell, \mathcal{P}_i)(\mathbf{x}') \in D(\mathcal{P}_i, \varepsilon_i)$. However, $\mathbf{x} \in T_{k,j}$ implies that $\bar{t}(X^\ell, \mathcal{P}_i)(\mathbf{x}') \notin D(\mathcal{P}_i, \varepsilon_i)$ contradicting Item (2) in Lemma 4.7.3. \square

The spaces $T_i \subset \text{Sym}^\ell(X)$ admit deformation retractions to $K_i \subset \Sigma(X^\ell, \mathcal{P}_i)$ as described in the following lemma.

LEMMA 4.8.3. *Continue the hypotheses and notation of Lemma 4.8.1. If the parameters $\varepsilon_0, \dots, \varepsilon_n$ defining T_i in Lemma 4.8.1, are sufficiently small then, for each partition $\mathcal{P}_i \neq \mathcal{P}_n$, there is a map,*

$$r_i : \text{Sym}^\ell(X) \rightarrow \text{Sym}^\ell(X),$$

satisfying the following properties, for $\pi_i = \pi(X^\ell, g_{\mathcal{P}_i})$:

- (1) r_i is homotopic to the identity.
- (2) The restriction of r_i to T_i is equal to $\pi(X^\ell, g_{\mathcal{P}_i})$.
- (3) For each partition \mathcal{P}_j , we have $r_i(\Sigma(X^\ell, \mathcal{P}_j)) \subset \text{cl } \Sigma(X^\ell, \mathcal{P}_j)$.
- (4) For $j < i$, we have $\pi(X^\ell, g_{\mathcal{P}_j}) \circ r_i = \pi(X^\ell, g_{\mathcal{P}_j})$.
- (5) For $j < i$, the restriction of r_i to the complement of $\vec{t}(X^\ell, g_{\mathcal{P}_i})^{-1}(D(\mathcal{P}_i, \varepsilon_j))$ is equal to the identity map.
- (6) For $j < i$ and A_j equal to $\vec{t}(X^\ell, g_{\mathcal{P}_j})^{-1}(\bar{D}(\mathcal{P}_j, \varepsilon_j))$ or $\vec{t}(X^\ell, g_{\mathcal{P}_j})^{-1}(D(\mathcal{P}_j, \varepsilon_j))$, we have $r_i(\mathbf{x}) \in A_j$ if and only if $x \in A_j$.
- (7) For $j < i$, we have $r_i(T_j) \subset T_j$.
- (8) $r_i(K_k \cup \bigcup_{j=k-1}^{i+1} T_{k,j}) = K_k \cup \bigcup_{j=k-1}^i T_{k,j}$.
- (9) The restriction of r_i to the complement of $\bigcup_{j \leq i} \vec{t}(X^\ell, g_{\mathcal{P}_j})^{-1}(\bar{D}(\mathcal{P}_j, \varepsilon_j))$ is injective.

PROOF. If the parameters ε_i are sufficiently small, then we can find neighborhoods,

$$\mathcal{O}_1(X^\ell, g_{\mathcal{P}_i}) \sqsubset \mathcal{O}_2(X^\ell, g_{\mathcal{P}_i}) \sqsubset \mathcal{O}(X^\ell, g_{\mathcal{P}_i}),$$

of $\Sigma(X^\ell, \mathcal{P}_i)$ such that

$$T_i = e(X^\ell, g_{\mathcal{P}_i})(\mathcal{O}_1(X^\ell, g_{\mathcal{P}_i})) - \bigcup_{j < i} \vec{t}(X^\ell, g_{\mathcal{P}_j})^{-1} D(\mathcal{P}_j, \varepsilon_j),$$

and for all $j < i$,

$$(4.8.1) \quad e(X^\ell, g_{\mathcal{P}_i})(\mathcal{O}_2(X^\ell, g_{\mathcal{P}_i})) \sqsubset \vec{t}(X^\ell, g_{\mathcal{P}_i})^{-1} D(\mathcal{P}_i, \varepsilon_j).$$

There is a continuous function, $p : \mathcal{O}(X^\ell, g_{\mathcal{P}_i}) \rightarrow [0, 1]$, with $p^{-1}(0) = \mathcal{O}_1(X^\ell, g_{\mathcal{P}_i})$ and $p^{-1}(1) = \mathcal{O}(X^\ell, g_{\mathcal{P}_i}) - \mathcal{O}_2(X^\ell, g_{\mathcal{P}_i})$. Setting $p_e := p \circ e(X^\ell, g_{\mathcal{P}_i})^{-1}$, define r_i on $\mathcal{U}(X^\ell, g_{\mathcal{P}_i})$ by

$$r_i(\cdot) := r(X^\ell, g_{\mathcal{P}_i})(\cdot, p_e(\cdot)),$$

and extend it as the identity map on $\text{Sym}^\ell(X) - \mathcal{U}(X^\ell, g_{\mathcal{P}_i})$. Items (1) and (2) follow immediately from the definition. Item (3) follows from noting that the maps r_P and thus $r(X^\ell, g_{\mathcal{P}})$ preserve the diagonals. Item (4) follows from (4.7.4). Item (5) follows from the construction of r_i and the inclusion (4.8.1).

We now prove Item (6). If $\mathbf{x} \in A_j$, then Items (1) and (2) of Lemma 4.7.2 imply that $r(X^\ell, g_{\mathcal{P}_i})(\mathbf{x}, s) \in A_j$ for all $s \in [0, 1]$ and thus $r_i(\mathbf{x}) \in A_j$. If $r_i(\mathbf{x}) \in A_j$ and $\mathbf{x} \notin A_j$, then $\vec{t}(X^\ell, g_{\mathcal{P}_j})(\mathbf{x}) \neq \vec{t}(X^\ell, g_{\mathcal{P}_j})(r_i(\mathbf{x}))$. Item (1) of Lemma 4.7.2 implies that

$$\vec{t}^P(X^\ell, g_{\mathcal{P}_j})(\mathbf{x}) = \vec{t}^P(X^\ell, g_{\mathcal{P}_j})(r_i(\mathbf{x})) \leq \varepsilon_j,$$

for all $P \in \mathcal{P}_i \cap \mathcal{P}_j$. Thus $\mathbf{x} \notin A_j$ implies that there is a $P \in \mathcal{P}_i \cap \mathcal{P}_j$ with $\vec{t}^P(X^\ell, g_{\mathcal{P}_j})(\mathbf{x}) \geq \varepsilon_j$. Item (3) in Lemma 4.7.2 then implies that $\vec{t}^P(X^\ell, g_{\mathcal{P}_i})(\mathbf{x}) \geq \varepsilon_j$, so $\mathbf{x} \notin \vec{t}(X^\ell, g_{\mathcal{P}_i})^{-1}(D(\mathcal{P}_i, \varepsilon_j))$ and thus, by Item (5), $r_i(\mathbf{x}) = \mathbf{x}$, a contradiction.

Item (7) follows from the definition of T_i and Item (6). Item (8) follows from the definition of $T_{k,j}$ and Item (6). Item (9) follows from the definition of r_i and Item (6). \square

In the following lemma, we show how to use the maps r_i to construct a global map to $\text{Sym}^\ell(X)$ from a collection of maps to $\text{Sym}^\ell(X)$ which are approximately equal.

LEMMA 4.8.4. *Continue the hypotheses of Lemma 4.8.3. Let $\{U_i\}_{i=0}^n$ be an open cover of a topological space Y with $U_i \cap U_j = \emptyset$ unless $i \leq j$. For $i = 0, \dots, n$, let $f_i : U_i \rightarrow \text{Sym}^\ell(X)$ be a continuous map satisfying the following properties:*

- (1) *If $j < k$, then $f_k(U_k \cap U_j) \subset \cup_{i \leq j} T_i$,*
- (2) *If $i < j$, then $\pi_i \circ f_j|_{U_i \cap U_j} = f_i|_{U_i \cap U_j}$.*

Then there is a map $F : Y \rightarrow \text{Sym}^\ell(X)$ such that $F|_{U_i}$ is homotopic to f_i .

PROOF. Define $m_i := r_0 \circ r_1 \circ \dots \circ r_i$ and let m_{-1} be the identity map. Each map m_i is homotopic to the identity by Item (1) in Lemma 4.8.3. If $g_i = m_{i-1} \circ f_i$, so $g_0 = f_0$, then g_i is homotopic to f_i . We claim that the maps g_i and g_j are equal on $U_i \cap U_j$ and thus define the global map F .

Arguing by upwards induction, assume that $g_i|_{U_i \cap U_j} = g_j|_{U_i \cap U_j}$ for all $i, j < k$. We now prove that if $j < k$, then $g_k|_{U_j \cap U_k} = g_j|_{U_j \cap U_k}$. If $x \in U_j \cap U_k$, then by property (1) there is an index $i \leq j$ such that $f_k(x) \in T_i$.

Item (7) in Lemma 4.8.3 and the fact that $f_k(x) \in T_i$ imply that

$$(4.8.2) \quad (r_{i+1} \circ \dots \circ r_{k-1} \circ f_k)(x) \in T_i.$$

Hence,

$$\begin{aligned} g_k(x) &= (m_{i-1} \circ r_i \circ r_{i+1} \circ \dots \circ r_{k-1} \circ f_k)(x) \\ &= (m_{i-1} \circ \pi_i \circ r_{i+1} \circ \dots \circ r_{k-1} \circ f_k)(x) \quad (\text{by Item (2) in Lemma 4.8.3 and (4.8.2)}) \\ &= (m_{i-1} \circ \pi_i \circ f_k)(x) \quad (\text{by Item (4) in Lemma 4.8.3}) \\ &= (m_{i-1} \circ f_i)(x) \quad (\text{by property (2)}) \\ &= g_i(x). \end{aligned}$$

By induction, we have $g_i(x) = g_j(x)$ which, together with the previous equality $g_k(x) = g_i(x)$, gives the equality $g_j(x) = g_k(x)$ required to complete the proof. \square

CHAPTER 5

The instanton moduli space with spliced ends

The gluing maps defined in [24] provide a tubular neighborhood structure for each stratum $M_{\mathfrak{s}} \times \Sigma$ of $M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X)$ just as the exponential map $e(X^{\ell}, g_{\mathcal{P}})$ did for the stratum $\Sigma(X^{\ell}, \mathcal{P})$. In this analogy, $\bar{M}_{\kappa}^{s, \natural}(S^4, \delta)$ plays the same role as the space Z_P , making up the fiber of the tubular neighborhood. Here, $\bar{M}_{\kappa}^{s, \natural}(S^4, \delta)$ denotes the Uhlenbeck compactification of the moduli space of framed anti-self-dual connections on S^4 which are mass-centered and have scale less than δ , where the mass and scale are defined in (5.2.2) and (5.2.3) respectively. However, it will require much more work to establish the commutativity of the diagram analogous to (4.4.2) for the tubular neighborhood structure of $M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X)$. As we described in our Introduction, we will work with splicing maps instead of gluing maps to establish such an equality because the explicit definition of the splicing map allows us to verify equalities analogous to (4.4.2). In order to use splicing maps instead of gluing maps, we must introduce a deformation of $\bar{M}_{\kappa}^{s, \natural}(S^4, \delta)$, which we call the *instanton moduli space with spliced ends*. The new space can still be used to define the domain of the gluing map of Hypothesis 7.8.1 because the connections in this new space satisfy the same crucial estimates (for example, Item (4) in Theorem 5.1.1) as the connections in $\bar{M}_{\kappa}^{s, \natural}(S^4, \delta)$.

We will be working with topological spaces which are *smoothly stratified* as in the definition given in Section 4.1. A continuous map between such spaces is smoothly stratified if it maps each smooth stratum into a smooth stratum and if the restriction of the map to each smooth stratum of the domain is a smooth map.

5.1. Introduction

The instanton moduli space with spliced ends will be identical to $\bar{M}_{\kappa}^{s, \natural}(S^4, \delta)$ except on an Uhlenbeck neighborhood of the *punctured, centered symmetric product*,

$$(5.1.1) \quad [\Theta] \times \left(\text{Sym}_{\delta}^{\kappa, \natural}(\mathbb{R}^4) \setminus c_{\kappa} \right),$$

where Θ is the product connection on $S^4 \times \text{SU}(2)$, and $\text{Sym}_{\delta}^{\kappa, \natural}(\mathbb{R}^4)$ is the subspace of mass-centered points in the κ -th symmetric product of \mathbb{R}^4 with a scale constraint (see (5.3.1)), and $c_{\kappa} = [0, \dots, 0] \in \text{Sym}^{\kappa}(\mathbb{R}^4)$ is given by κ copies of the origin.

A neighborhood of the strata (5.1.1) in $\bar{M}_{\kappa}^{s, \natural}(S^4, \delta)$ is parameterized by the union of the images of the gluing maps $\gamma_{\Theta, \Sigma}$, as Σ varies over the strata in (5.1.1). As described in Section 5.3, the space $\text{Sym}_{\delta}^{\kappa, \natural}(\mathbb{R}^4)$ is the quotient of a subspace $Z_{\kappa}(\delta) \subset (\mathbb{R}^4)^{\kappa}$ by the symmetric group action. If Σ is a stratum of $\text{Sym}_{\delta}^{\kappa, \natural}(\mathbb{R}^4)$ and \mathcal{P} is a partition of N_{κ} corresponding to Σ , so $\Sigma = \Delta^{\circ}((\mathbb{R}^4)^{\kappa}, \mathcal{P}) \cap Z_{\kappa}(\delta) / \mathfrak{S}(\mathcal{P})$ as in (5.3.2), then the gluing map is a smoothly stratified embedding [22],

$$(5.1.2) \quad \gamma_{\Theta, \mathcal{P}} : \left(\Delta^{\circ}(\mathbb{R}^{4\kappa}, \mathcal{P}) \cap Z_{\kappa}(\delta) \right) \times_{\mathfrak{S}(\mathcal{P})} \prod_{P \in \mathcal{P}} \bar{M}_{|P|}^{s, \natural}(S^4, \delta_P) \rightarrow \bar{M}_{\kappa}^{s, \natural}(S^4, \delta).$$

The instanton moduli space with spliced ends, $\bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta)$, will be defined by replacing the image of $\gamma_{\Theta, \mathcal{P}}$ with the image of the splicing map $\gamma'_{\Theta, \mathcal{P}}$ on the interior of this tubular neighborhood. The connections defining this deformation of the instanton moduli space will be almost anti-self-dual as measured by the norm $\|\cdot\|_{L^{\sharp,2}(X)}$, which on a four-manifold X was defined in [96, Equation (6.1a)] and [19, Equation (4.3)] by

$$(5.1.3) \quad \|a\|_{L^{\sharp,2}(X)} := \|a\|_{L^2(X)} + \sup_{x \in X} \|\text{dist}^{-2}(x, \cdot)|a|\|_{L^1(X)}.$$

We construct $\bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta)$ and summarize its properties in the following

THEOREM 5.1.1. *Let $\bar{\mathcal{B}}_{\kappa}^s(S^4, \delta)$ be the Uhlenbeck extension of the framed quotient space defined in (5.2.4). There exists a smoothly-stratified subspace, $\bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta) \subset \bar{\mathcal{B}}_{\kappa}^s(S^4, 2\delta)$, closed under the $\text{SO}(3) \times \text{SO}(4)$ action defined in (5.2.5) and (5.2.6), satisfying the following properties:*

- (1) *There is a smoothly-stratified, $\text{SO}(3) \times \text{SO}(4)$ -equivariant homeomorphism,*

$$(5.1.4) \quad \bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta) \cong \bar{M}_{\kappa}^{s,\natural}(S^4, \delta),$$

which is the identity on the levels,

$$[\Theta] \times \text{Sym}_{\delta}^{\kappa,\natural}(\mathbb{R}^4),$$

where $\text{Sym}_{\delta}^{\kappa,\natural}(\mathbb{R}^4) \subset \text{Sym}^{\kappa}(\mathbb{R}^4)$ is defined in (5.3.1).

- (2) *There is an Uhlenbeck neighborhood in the sense of Definition 2.1.2, namely W_{κ} , in $\bar{\mathcal{B}}_{\kappa}^s(S^4, 2\delta)$ of the punctured, centered symmetric product (5.1.1) such that*

$$\bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta) \setminus W_{\kappa} = \bar{M}_{\kappa}^{s,\natural}(S^4, \delta) \setminus W_{\kappa}.$$

- (3) *For each stratum $[\Theta] \times \Sigma$ in (5.1.1), there is an Uhlenbeck neighborhood, $W(\Sigma) \subset \bar{\mathcal{B}}_{\kappa}^s(S^4, \delta)$, of the stratum*

$$(5.1.5) \quad [\Theta] \times \Sigma$$

and an open neighborhood, $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, \delta) \subset \nu(\Theta, \mathcal{P}, \delta)$, of the stratum (5.6.1) in the domain of the splicing map $\gamma'_{\Theta, \mathcal{P}}$ defined in (5.4.2), such that

$$(5.1.6) \quad \bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta) \cap W(\Sigma) = \gamma'_{\Theta, \mathcal{P}}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, \delta)).$$

- (4) *For each integer $\kappa \geq 0$, there is a constant, $C = C(\kappa)$, such that for all $[A, \mathbf{x}] \in \bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta)$, one has*

$$\|F_A^+\|_{L^{\sharp,2}(S^4)} \leq C\delta,$$

where the norm $\|\cdot\|_{L^{\sharp,2}}$ is defined in (5.1.3).

We construct $\bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta)$ by induction on $\kappa \in \mathbb{N}$. We first construct what we call the ‘spliced end’ of $\bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta)$. The *spliced end* will be an Uhlenbeck neighborhood of the punctured, centered symmetric product (5.1.1) in $\bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta)$. We shall define the spliced end of $\bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta)$ to be the union of the images of splicing maps (5.4.2), $\gamma'_{\Theta, \mathcal{P}}$, where \mathcal{P} is a partition of N_{κ} corresponding to a stratum Σ in $\text{Sym}_{\delta}^{\kappa,\natural}(\mathbb{R}^4) - \{c_{\kappa}\}$ and where the domain of the splicing map is not defined by the moduli spaces, $\bar{M}_{\kappa_i}^{s,\natural}(S^4, \delta_i)$, with smaller instanton number but rather by the instanton moduli spaces with spliced ends, $\bar{M}_{\text{SPL},\kappa_i}^{s,\natural}(S^4, \delta_i)$, with smaller instanton number. A technical result on the equality of two compositions of splicing maps, Proposition 5.5.3, shows that the images of the

different splicing maps intersect in open sets, proving that the union of these images, W_κ , is a smoothly stratified space.

After reviewing the properties of connections on S^4 in Section 5.2, we adapt the notation of Chapter 3 to describe the product-connection strata, (5.1.1), in Section 5.3. In Section 5.4, we define the relevant splicing maps. In Section 5.5, we prove the crucial technical result, Proposition 5.5.3, that provides control over the overlap of two splicing maps. In Section 5.6, we finish the first stage of the proof of Theorem 5.1.1 by constructing W_κ in Proposition 5.6.1.

The second stage of the proof of Theorem 5.1.1 appears in Section 5.8, where we construct an isotopy of the complement of a neighborhood of the punctured, centered symmetric product in W_κ to the actual moduli space, $\bar{M}_\kappa^{s,\natural}(S^4, \delta)$. This isotopy is defined by the composition of the isotopy defined by the gluing map and the isotopy defined by the centering map. That such an isotopy exists follows immediately from the estimates on $\|F_A^+\|_{L^{\sharp,2}(S^4)}$ for a connection, A , in the spliced end; most of the work in Section 5.8 lies in constructing the parameter on the spliced end for the isotopy. Finally, in Section 5.9 we construct a cone parameter on the instanton moduli space with spliced end and prove that this space has the structure of a *Whitney stratified space* in the sense of [39, Section 1.2].

5.2. Connections over the four-dimensional sphere

We begin by reviewing some standard definitions concerning connections over S^4 ; similar definitions appear in [21, 24].

For an integer $\kappa \geq 1$, let $\mathcal{B}_\kappa(S^4)$ denote the quotient, $\mathcal{A}_\kappa/\mathcal{G}_\kappa$, of the affine space, \mathcal{A}_κ , of $L_2^2 \text{SU}(2)$ connections on a Hermitian smooth vector bundle $E_\kappa \rightarrow S^4$ with $c_2(E_\kappa) = \kappa$, modulo the action of the group, $\mathcal{G}_\kappa = \text{Aut}(E_\kappa)$, of $L_3^2 \text{SU}(2)$ gauge transformations of E_κ . Let $\mathcal{B}_\kappa^s(S^4) \rightarrow \mathcal{B}_\kappa(S^4)$ be the principal $\text{SO}(3)$ -bundle defined by

$$\mathcal{B}_\kappa^s(S^4) := (\mathcal{A}_\kappa(S^4) \times P_\kappa|_s) / \mathcal{G}_\kappa,$$

where $P_\kappa = \text{Fr}(E_\kappa)$, the principal $\text{SU}(2)$ -bundle of $\text{SU}(2)$ frames for E_κ and $s \in S^4$ is the South Pole, identified with the point at infinity in $S^4 = \mathbb{R}^4 \cup \{\infty\}$. Let

$$(5.2.1) \quad \bar{\mathcal{B}}_\kappa(S^4) = \bigsqcup_{\ell=0}^{\kappa} \left(\mathcal{B}_{\kappa-\ell}(S^4) \times \text{Sym}^\ell(\mathbb{R}^4) \right),$$

denote the space of ideal connections on E_κ given the topology induced by Uhlenbeck convergence as defined in Definition 2.1.2. We use $\text{Sym}^\ell(\mathbb{R}^4)$ in (5.2.1) rather than $\text{Sym}^\ell(S^4)$ to simplify the definitions (5.2.2) and (5.2.3) below. We will be working with ideal connections with finite scale so using $\text{Sym}^\ell(\mathbb{R}^4)$ does not omit relevant ideal connections. We write elements of $\bar{\mathcal{B}}_\kappa(S^4)$ as $[A, \mathbf{x}]$, where $[A] \in \mathcal{B}_{\kappa-\ell}(S^4)$ and $\mathbf{x} \in \text{Sym}^\ell(\mathbb{R}^4)$.

If $n \in S^4$ denotes the North Pole, identified with the origin in $S^4 = \mathbb{R}^4 \cup \{\infty\}$, let $y(\cdot) = \varphi_n^{-1} : S^4 \setminus \{s\} \rightarrow \mathbb{R}^4$ be the coordinate chart given by a stereographic projection from the South Pole with $\varphi_n(0) = n \in S^4$. For a point $[A, \mathbf{x}] \in \bar{\mathcal{B}}_\kappa(S^4)$ with $\mathbf{x} = [x_1, \dots, x_\ell]$, we define its *center of mass* by

$$(5.2.2) \quad z[A, \mathbf{x}] := \left(\int_{\mathbb{R}^4} |\varphi_n^* F_A|^2 d^4 y \right)^{-1} \int_{\mathbb{R}^4} y |\varphi_n^* F_A|^2 d^4 y + \sum_{i=1}^{\ell} x_i \in \mathbb{R}^4,$$

and the *scale* by

$$(5.2.3) \quad \lambda[A, \mathbf{x}]^2 := \left(\int_{\mathbb{R}^4} |\varphi_n^* F_A|^2 d^4 y \right)^{-1} \int_{\mathbb{R}^4} |y - z[A, \mathbf{x}]|^2 |\varphi_n^* F_A|^2 d^4 y + \sum_{i=1}^{\ell} |x_i|^2 \in [0, \infty).$$

For a constant $\varepsilon > 0$, we define

$$\bar{\mathcal{B}}_{\kappa}(S^4, \varepsilon) := \lambda^{-1}([0, \varepsilon]).$$

We continue to denote by $z[\cdot]$ and $\lambda[\cdot]$ the pull-back of those functions from $\mathcal{B}_{\kappa}(S^4)$ to $\mathcal{B}_{\kappa}^s(S^4)$. If $[A, \mathbf{x}] \in \bar{\mathcal{B}}_{\kappa}(S^4, \varepsilon)$ and $\varepsilon \ll 1$, then the support of \mathbf{x} is disjoint from the south pole. Hence, the quotient space, $\mathcal{B}_{\kappa}^s(\varepsilon)$, is continuously embedded in the quotient space of pairs of *ideal* connections and frames,

$$(5.2.4) \quad \begin{aligned} \bar{\mathcal{B}}_{\kappa}^s(S^4, \varepsilon) &:= \lambda^{-1}[0, \varepsilon] \cap \left(\bigsqcup_{\ell=0}^{\kappa} \left(\mathcal{B}_{\kappa-\ell}^s(S^4) \times \text{Sym}^{\ell}(\mathbb{R}^4) \right) \right), \\ \bar{\mathcal{B}}_{\kappa}^{s, \natural}(S^4, \varepsilon) &= z^{-1}(0) \cap \bar{\mathcal{B}}_{\kappa}^s(S^4, \varepsilon), \end{aligned}$$

and $\mathcal{B}_{\kappa}^{s, \natural}(S^4, \varepsilon)$ is continuously embedded in the quotient space of pairs of ideal *mass-centered* connections and frames. If Θ is the product connection, we call $[\Theta, F^s, c_{\kappa}] \in \mathcal{B}_0^s(S^4) \times \text{Sym}^{\kappa}(\mathbb{R}^4)$ the *cone point* of $\bar{\mathcal{B}}_{\kappa}^{s, \natural}(S^4, \varepsilon)$.

The group $\text{SU}(2)$ acts on $\bar{\mathcal{B}}_{\kappa}^{s, \natural}(S^4, \varepsilon)$ by the action of $\text{SU}(2)$ on the frame in $\text{Fr}(E_{\kappa})|_s$. (Because connections that are mass-centered at the North Pole and have uniformly positive scale cannot bubble over the South Pole — see [17] for an explanation based on the Chebychev Inequality — the bundle fibers $E_{\kappa-\ell}$ for $0 \leq \ell \leq \kappa$ can all be identified with $E_{\kappa}|_s$.) The stabilizer of any point in $\bar{\mathcal{B}}_{\kappa}^{s, \natural}(S^4, \varepsilon)$ contains $\{\pm \text{id}\} \subset \text{SU}(2)$, so this action of $\text{SU}(2)$ on $\bar{\mathcal{B}}_{\kappa}^{s, \natural}(S^4, \varepsilon)$ descends to an action of $\text{SO}(3) = \text{SU}(2)/\{\pm \text{id}\}$ on $\bar{\mathcal{B}}_{\kappa}^{s, \natural}(S^4, \varepsilon)$,

$$(5.2.5) \quad \text{SO}(3) \times \bar{\mathcal{B}}_{\kappa}^{s, \natural}(S^4, \varepsilon) \rightarrow \bar{\mathcal{B}}_{\kappa}^{s, \natural}(S^4, \varepsilon)$$

which we refer to as the action on the frame.

The group $\text{SO}(4)$ acts on S^4 by pull-back of its action on \mathbb{R}^4 via stereographic projection, $S^4 \setminus \{s\} \rightarrow \mathbb{R}^4$. The action of $\text{SO}(4)$ on S^4 induces an action on $\bar{\mathcal{B}}_{\kappa}^{s, \natural}(S^4, \varepsilon)$,

$$(5.2.6) \quad \text{SO}(4) \times \bar{\mathcal{B}}_{\kappa}^{s, \natural}(S^4, \varepsilon) \rightarrow \bar{\mathcal{B}}_{\kappa}^{s, \natural}(S^4, \varepsilon),$$

by pull-back of the connection and frame for $E_{\kappa}|_s$ and the rotation action on the points in $\text{Sym}^{\kappa}(\mathbb{R}^4)$ (see [96, Section 4, p. 343] or [17, Section 3.2]).

5.3. Strata containing the product connection

We now adapt the framework of Chapter 3 to describe the product-connection strata in (5.1.1). Define $\text{Sym}^{\kappa, \natural}(\mathbb{R}^4)$ to be the quotient of the zero locus,

$$Z_{\kappa} := z_{\kappa}^{-1}(0),$$

of the center-of-mass map, $z_{\kappa} : \bigoplus_{i \in N_{\kappa}} \mathbb{R}^4 \rightarrow \mathbb{R}^4$ (defined following (4.2.1)), by the symmetric group, \mathfrak{S}_{κ} ,

$$\text{Sym}^{\kappa, \natural}(\mathbb{R}^4) := Z_{\kappa} / \mathfrak{S}_{\kappa} = z_{\kappa}^{-1}(0) / \mathfrak{S}_{\kappa}.$$

We define a scale for elements of $\text{Sym}^\kappa(\mathbb{R}^4)$ by

$$(5.3.1) \quad \lambda[v_1, \dots, v_\kappa]^2 := \sum_{i=1}^{\kappa} |v_i|^2,$$

and set

$$\begin{aligned} Z_\kappa(\delta) &:= z_\kappa^{-1}(0) \cap \lambda^{-1}([0, \delta]), \\ \text{Sym}_\delta^{\kappa, \natural}(\mathbb{R}^4) &:= Z_\kappa(\delta) / \mathfrak{S}_\kappa. \end{aligned}$$

We will define strata of $\text{Sym}_\delta^{\kappa, \natural}(\mathbb{R}^4)$ to be quotients of the intersections of the diagonals in $(\mathbb{R}^4)^\kappa$ with $Z_\kappa(\delta)$. For any partition \mathcal{P} of N_κ , we define

$$(5.3.2) \quad \begin{aligned} \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) &:= \Delta^\circ(Z_\kappa, \mathcal{P}) \cap Z_\kappa(\delta), \\ \Sigma(Z_\kappa(\delta), \mathcal{P}) &:= \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) / \mathfrak{S}(\mathcal{P}) \subset \text{Sym}_\delta^{\kappa, \natural}(\mathbb{R}^4), \end{aligned}$$

where $\Delta^\circ(Z_\kappa, \mathcal{P})$ is defined in (4.2.5). Let $\Sigma(Z_\kappa(\delta), \mathcal{P})$ be the image of $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$ under the projection $Z_\kappa(\delta) \rightarrow \text{Sym}_\delta^{\kappa, \natural}(\mathbb{R}^4)$. We then have the

LEMMA 5.3.1. *If $\kappa \geq 1$ is an integer and \mathcal{P} and \mathcal{P}' are partitions of N_κ and $\tilde{\pi}_\kappa : Z_\kappa \subset \oplus_{i \in N_\kappa} \mathbb{R}^4 \rightarrow \text{Sym}^{\kappa, \natural}(\mathbb{R}^4)$ is the projection, then the following hold:*

- (1) $\Sigma(Z_\kappa(\delta), \mathcal{P}) = \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) / \mathfrak{S}(\mathcal{P}) = \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) / W(\mathcal{P})$.
- (2) $\tilde{\pi}_\kappa^{-1}(\Sigma(Z_\kappa(\delta), \mathcal{P})) = \sqcup_{\mathcal{P}' \in [\mathcal{P}]} \Delta^\circ(Z_\kappa(\delta), \mathcal{P}')$.
- (3) $\Sigma(Z_\kappa(\delta), \mathcal{P}) \subset \text{cl}(\Sigma(Z_\kappa(\delta), \mathcal{P}'))$ if and only if there is $\mathcal{P}'' \in [\mathcal{P}]$ such that $\mathcal{P}'' < \mathcal{P}$.

If $(v_1, \dots, v_\kappa) \in \Delta^\circ(Z_\kappa, \mathcal{P})$ and $P \in \mathcal{P}$, then $v_i = v_j$ for $i, j \in P$. Thus, we will write $v_P = v_i$ for any $i \in P$. With this notation, we will write $(y_1, \dots, y_\kappa) = (y_P)_{P \in \mathcal{P}}$ for an element of $\Delta^\circ(Z_\kappa, \mathcal{P})$.

5.3.1. Tubular neighborhoods. Because $Z_\kappa(\delta)$ is an open subspace of Z_κ , the tubular neighborhood of the diagonals in (5.3.2) can be described by the restriction of the tubular neighborhoods described in Lemma 4.2.1, as we formally state in the following lemma.

LEMMA 5.3.2. *Let \mathcal{P} be a partition of N_κ . For $P \subseteq N_\kappa$, let $Z_P \subset \oplus_{i \in P} \mathbb{R}^4$ be the subspace of mass-centered sets of points defined in (4.2.1). The normal bundle of the diagonal $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$ in Z_κ is*

$$(5.3.3) \quad \tilde{\nu}(Z_\kappa(\delta), \mathcal{P}) = \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times \prod_{P \in \mathcal{P}} Z_P.$$

There is an open neighborhood, $\tilde{\mathcal{O}}(Z_\kappa(\delta), \mathcal{P})$, of the zero section in $\tilde{\nu}(Z_\kappa(\delta), \mathcal{P})$ such that the restriction of the exponential map $e(Z_\kappa, \mathcal{P})$ defined in (4.2.10) to $\tilde{\mathcal{O}}(Z_\kappa(\delta), \mathcal{P})$ is injective and $\mathfrak{S}(\mathcal{P})$ -equivariant.

Let $\tilde{\mathcal{U}}(Z_\kappa(\delta), \mathcal{P})$ be the image of $\tilde{\mathcal{O}}(Z_\kappa(\delta), \mathcal{P})$ under the exponential map (4.2.10). By the $\mathfrak{S}(\mathcal{P})$ -equivariance of the exponential map $e(Z_\kappa(\delta), \mathcal{P})$, there is a homeomorphism,

$$(5.3.4) \quad \mathcal{O}(Z_\kappa(\delta), \mathcal{P}) \cong \mathcal{U}(Z_\kappa(\delta), \mathcal{P}),$$

where $\mathcal{O}(Z_\kappa(\delta), \mathcal{P}) = \tilde{\mathcal{O}}(Z_\kappa(\delta), \mathcal{P}) / \mathfrak{S}(\mathcal{P})$ and $\mathcal{U}(Z_\kappa(\delta), \mathcal{P}) = \tilde{\mathcal{U}}(Z_\kappa(\delta), \mathcal{P}) / \mathfrak{S}(\mathcal{P})$.

To describe the pre-image of $\Delta^\circ(Z_\kappa(\delta), \mathcal{P}')$ under $e(Z_\kappa, \mathcal{P})$, where $\mathcal{P} < \mathcal{P}'$, we introduce the following notation.

LEMMA 5.3.3. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_κ . For $P \in \mathcal{P}$, let \mathcal{P}'_P be the partition of P defined in (4.2.4). Then there is an inclusion,*

$$(5.3.5) \quad \tilde{\nu}(Z_\kappa(\delta), \mathcal{P} \rightarrow \mathcal{P}') = \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times \prod_{P \in \mathcal{P}} \Delta^\circ(Z_P, \mathcal{P}'_P) \hookrightarrow \tilde{\nu}(Z_\kappa(\delta), \mathcal{P}),$$

with the following significance. If we define

$$(5.3.6) \quad \tilde{\mathcal{O}}(Z_\kappa(\delta), \mathcal{P} \rightarrow \mathcal{P}') := \tilde{\mathcal{O}}(Z_\kappa(\delta), \mathcal{P}) \cap \tilde{\nu}(Z_\kappa(\delta), \mathcal{P} \rightarrow \mathcal{P}'),$$

then

$$\tilde{\mathcal{O}}(Z_\kappa(\delta), \mathcal{P} \rightarrow \mathcal{P}') = e(Z_\kappa, \mathcal{P})^{-1}(\Delta^\circ(Z_\kappa(\delta), \mathcal{P}')).$$

PROOF. We observe that

$$\Delta^\circ(Z_P, \mathcal{P}'_P) = \{(v_i)_{i \in P} \in Z_P : v_i = v_j \iff \exists P' \in \mathcal{P}'_P \text{ with } i, j \in P'\}.$$

Because \mathcal{P} is a partition of N_κ , there is an isomorphism,

$$\bigoplus_{P \in \mathcal{P}} \bigoplus_{i \in P} \mathbb{R}^4 \cong \bigoplus_{i \in N_\kappa} \mathbb{R}^4,$$

which induces an inclusion,

$$(5.3.7) \quad \prod_{P \in \mathcal{P}} \Delta^\circ(Z_P, \mathcal{P}'_P) \ni ((v_i)_{i \in P, P \in \mathcal{P}}) \mapsto (v_i)_{i \in N_\kappa} \in Z_\kappa.$$

By the $\mathfrak{S}(\mathcal{P})$ -equivariance of the exponential map, $e(Z_\kappa, \mathcal{P})$, an element of the pre-image of $\Delta^\circ(Z_\kappa(\delta), \mathcal{P}')$ under $e(Z_\kappa, \mathcal{P})$ can be written as

$$((x_1, \dots, x_\kappa), (v_1, \dots, v_\kappa)) \in \Delta^\circ(Z_\kappa, \mathcal{P}) \times \bigoplus_{i \in N_\kappa} \mathbb{R}^4$$

where $(v_1, \dots, v_\kappa) \in \bigoplus_{i \in N_\kappa} \mathbb{R}^4$ satisfies

- (1) $\sum_{i \in P} v_i = 0$ for all $P \in \mathcal{P}$.
- (2) For all i, j , we have $v_i = v_j$ if and only if there exists $P' \in \mathcal{P}'$ with $i, j \in P'$.

Then one observes that the set of elements $(v_1, \dots, v_\kappa) \in \bigoplus_{i \in N_\kappa} \mathbb{R}^4$ satisfying these two conditions is precisely the image of the inclusion (5.3.7). \square

By Items (2) and (3) in Lemma 5.3.1, to describe the end of $\Sigma(Z_\kappa(\delta), \mathcal{P}')$ near $\Sigma(Z_\kappa(\delta), \mathcal{P})$, we must describe not only the end of $\Delta^\circ(Z_\kappa(\delta), \mathcal{P}')$ near $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$, as is done in Lemma 5.3.3, but also the ends of $\Delta^\circ(Z_\kappa(\delta), \mathcal{P}'')$ near $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$ for all $\mathcal{P}'' \in [\mathcal{P}']$ with $\mathcal{P} < \mathcal{P}''$.

Thus, define

$$(5.3.8) \quad \tilde{\nu}(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}']) := \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \left(\prod_{P \in \mathcal{P}} \Delta^\circ(Z_P, \mathcal{P}''_P) \right).$$

There is an action of $\mathfrak{S}(\mathcal{P})$ on $\tilde{\nu}(Z_\kappa, [\mathcal{P} < \mathcal{P}'])$ defined by the standard action on $\Delta^\circ(Z_\kappa, \mathcal{P})$ and the action of $\mathfrak{S}(\mathcal{P})$ on $[\mathcal{P} < \mathcal{P}']$. Although $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$ is not a subset of $\tilde{\nu}(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])$, we use the phrase *a neighborhood of $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$ in $\nu(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])$* to refer to the obvious parallel with subspaces of $\nu(Z_\kappa(\delta), \mathcal{P})$.

LEMMA 5.3.4. *Let \mathcal{P} and \mathcal{P}' be partitions of N_κ with $\mathcal{P} < \mathcal{P}'$. Then a neighborhood of $\Sigma(Z_\kappa(\delta), \mathcal{P})$ in $\Sigma(Z_\kappa(\delta), \mathcal{P}')$ is homeomorphic to a neighborhood, $\mathcal{O}(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])$, of the zero section in*

$$\tilde{\nu}(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])/\mathfrak{S}(\mathcal{P}).$$

PROOF. The bundle $\tilde{\nu}(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])$ is the union of the bundles $\tilde{\nu}(Z_\kappa(\delta), \mathcal{P} \rightarrow \mathcal{P}'')$ defined in (5.3.5) as \mathcal{P}'' varies in $[\mathcal{P} < \mathcal{P}']$ along the subspace $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$ of each of these bundles. Therefore, a neighborhood of $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$ in

$$(5.3.9) \quad \bigcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \Delta^\circ(Z_\kappa(\delta), \mathcal{P}'')$$

is homeomorphic to a neighborhood of the zero-section in $\nu(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])$. By the characterization of the pre-image of $\Sigma(Z_\kappa(\delta), \mathcal{P})$ under the projection $Z_\kappa(\delta) \rightarrow \text{Sym}_\delta^{\mathfrak{h}, \kappa}(\mathbb{R}^4)$ in Item (2) of Lemma 5.3.1, a neighborhood of $\Sigma(Z_\kappa(\delta), \mathcal{P})$ in $\Sigma(Z_\kappa(\delta), \mathcal{P}')$ is homeomorphic to the quotient of (5.3.9) by $\mathfrak{S}(\mathcal{P})$, thus proving the lemma. \square

5.4. The splicing map with the product connection over \mathbb{R}^4

In this section, we define the basic operation of splicing onto the product connection as used in [24]. Our main result here is in Lemma 5.5.2, where we show that the composition of two splicing maps is equal to a single splicing map on suitably small open sets.

Let \mathcal{P} be a partition of N_κ . We define the splicing map on an open subspace of

$$\Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times_{\mathfrak{S}(\mathcal{P})} \prod_{P \in \mathcal{P}} \tilde{\mathcal{B}}_{|P|}^{s, \mathfrak{h}}(S^4, \delta_P),$$

where the constants δ_P are invariant under the action of $\mathfrak{S}(\mathcal{P})$ and where $\mathfrak{S}(\mathcal{P})$ acts on the generalized connections by permuting them, sending an element of $\tilde{\mathcal{B}}_{|P|}^{s, \mathfrak{h}}(S^4, \delta_P)$ to an element of $\tilde{\mathcal{B}}_{|\sigma(P)|}^{s, \mathfrak{h}}(S^4, \delta_{|\sigma(P)|})$. Note that elements of $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$ can be written as $(y_P)_{P \in \mathcal{P}}$, where $y_P \in \mathbb{R}^4$, while elements of $\prod_{P \in \mathcal{P}} \tilde{\mathcal{B}}_{|P|}^{s, \mathfrak{h}}(S^4, \delta_P)$ can be written as $([A_P, F_P^s, \mathbf{x}_P])_{P \in \mathcal{P}}$, where $[A_P, F_P^s, \mathbf{x}_P] \in \tilde{\mathcal{B}}_{|P|}^{s, \mathfrak{h}}(S^4, \delta_P)$. The open subspace on which the splicing map will be defined is

$$(5.4.1) \quad \tilde{\mathcal{O}}(\Theta, \mathcal{P}, \delta) := \left\{ (y_P, [A_P, F_P^s, \mathbf{x}_P])_{P \in \mathcal{P}} \in \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times \prod_{P \in \mathcal{P}} \tilde{\mathcal{B}}_{|P|}^s(S^4, \delta_P) : \right. \\ \left. 8\sqrt{\lambda[A_P, \mathbf{x}_P]} + 8\sqrt{\lambda[A_{P'}, \mathbf{x}_{P'}]} < \text{dist}(y_P, y_{P'}), \forall P \neq P' \right\}.$$

We define the splicing map,

$$(5.4.2) \quad \gamma'_{\Theta, \mathcal{P}} : \mathcal{O}(\Theta, \mathcal{P}, \delta) := \tilde{\mathcal{O}}(\Theta, \mathcal{P}, \delta)/\mathfrak{S}(\mathcal{P}) \rightarrow \tilde{\mathcal{B}}_\kappa^s(S^4, 2\delta),$$

as follows. For $P \in \mathcal{P}$, let $E_P \rightarrow S^4$ be the Hermitian vector bundle supporting the connection A_P and let $\text{Fr}(E_P)$ be the bundle of unitary frames of E_P . The framed connection $[A_P, F_P^s]$ defines a section $\phi(A_P, F_P^s)$ of $\text{Fr}(E_P)$ over the complement of the North Pole by parallel translation of F_P^s with respect to the connection A_P , along great circles from the South Pole. Let $\Theta(A_P, F_P^s)$ be the product connection defined by this section. Note that if A_P is flat on the domain of $\phi(A_P, F_P^s)$, then $\Theta(A_P, F_P^s) = A_P$.

If Θ denotes the product connection on $S^4 \times \mathbb{C}^2$, then the product connection $\Theta(A_P, F_P^s)$ is identified with the restriction of Θ to the domain of $\phi(A_P, F_P^s)$ as follows. The section $\phi(A_P, F_P^s)$ gives a trivialization of the bundle E and thus identifies it with the restriction of the product bundle $S^4 \times \mathbb{C}^2$ to the domain of the section. Under this identification the connection Θ is identified with $\Theta(A_P, F_P^s)$.

We also note that given any two connections, A_1 and A_2 , a convex linear combination of them, $tA_1 + (1-t)A_2$, also defines a connection. For example, if A_1 is a connection which equals the product connection in a particular trivialization and $A_2 = A_1 + a_2$, we would write the connection one form for $tA_1 + (1-t)A_2$ in this trivialization as $(1-t)a_2$.

Given $(y, \lambda) \in \mathbb{R}^4 \times (0, \infty)$, let $c_{y, \lambda} : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be defined by

$$c_{y, \lambda}(z) := (z - y)/\lambda,$$

and so $c_{y, \lambda}$ maps the ball $B(y, \lambda)$ onto $B(0, 1)$.

Let $\beta : \mathbb{R} \rightarrow [0, 1]$ be a smooth function satisfying $\beta(x) = 0$ for $x \leq 1/2$ and $\beta(x) = 1$ for $x \geq 1$. Define $\chi : \mathbb{R}^4 \rightarrow [0, 1]$ by $\chi(x) = \beta(|x|)$ and $\chi_{y, \lambda} : \mathbb{R}^4 \rightarrow [0, 1]$ by $\chi_{y, \lambda}(x) = (c_{y, \lambda}^* \chi)(x) = \beta(|x - y|/\lambda)$. The function $1 - \chi_{y, \lambda}$ is thus supported on the ball $B(y, \lambda)$ and equal to one on the ball $B(y, \frac{1}{2}\lambda)$.

For $\mathbf{y} = (y_P)_{P \in \mathcal{P}} \in \Delta^\circ(Z_\kappa(\delta), \mathcal{P})$, we define the splicing map (5.4.2) by setting

$$(5.4.3) \quad \gamma'_{\Theta, \mathcal{P}}(\mathbf{y}, ([A_P, F_P^s, \mathbf{x}_P]_{P \in \mathcal{P}})) := [A', F^s, \mathbf{x}'],$$

where, for $\lambda_P = \lambda([A_P, F_P^s, \mathbf{x}_P])$, we define the framed connection $[A', F^s]$ by

$$(5.4.4) \quad A' = \begin{cases} \Theta & \text{on } \mathbb{R}^4 \setminus \cup_{P \in \mathcal{P}} B(y_P, 4\sqrt{\lambda_P}), \\ (1 - \chi_{y_P, 4\sqrt{\lambda_P}})c_{y_P, 1}^* A_P + \chi_{y_P, 4\sqrt{\lambda_P}} c_{y_P, 1}^* \Theta(A_P, F_P^s) & \text{on } \Omega(y_P; 2\sqrt{\lambda_P}, 4\sqrt{\lambda_P}), \\ c_{y_P, 1}^* A_P & \text{on } B(y_P, 2\sqrt{\lambda_P}), \end{cases}$$

with the frame F^s in $[A', F^s]$ being given by the canonical frame for the trivialization associated with the product connection, Θ . If the points \mathbf{x}_P in (5.4.3) are given by points $x_{P,i} \in \mathbb{R}^4$ with multiplicities $\kappa_{P,i}$, then the point \mathbf{x}' in the expression (5.4.3) is defined by the points $c_{y_P, 1}^{-1}(x_{P,i})$ with multiplicities $\kappa_{P,i}$.

Note that we have not computed the behavior of the function $\lambda \circ \gamma'_{\Theta, \mathcal{P}}$ precisely, so we cannot assert that the image of $\gamma'_{\Theta, \mathcal{P}}$ is contained in $\mathcal{B}_\kappa^s(2\delta)$. However, that containment will follow from the continuity of λ and of $\gamma'_{\Theta, \mathcal{P}}$ with respect to Uhlenbeck limits if we shrink the domain $\mathcal{O}(\Theta, \mathcal{P}, \delta)$ by requiring the scales λ_P to be sufficiently small.

We have the following result regarding the behavior of this splicing map at the cone point of $\bar{\mathcal{B}}_{|P|}^s(\delta_P)$.

LEMMA 5.4.1. *For any partition \mathcal{P} of N_κ , the map $\gamma'_{\Theta, \mathcal{P}}$ is smoothly stratified. If $c_{|P|} \in \text{Sym}_\delta^{|P|, \natural}(\mathbb{R}^4)$ denotes the cone point, then for any $\mathbf{y} \in \Delta^\circ(Z_\kappa(\delta), \mathcal{P})$,*

$$(5.4.5) \quad \gamma'_{\Theta, \mathcal{P}}(\mathbf{y}, ([\Theta, F^s, c_{|P|}]_{P \in \mathcal{P}})) = [\Theta, F^s, \mathbf{y}].$$

PROOF. The fact that the map $\gamma'_{\Theta, \mathcal{P}}$ is smooth when restricted to the subspaces defined by the strata of $\bar{\mathcal{B}}_{|P|}^{s, \natural}(\delta_P)$ is clear from the definition. If a sequence

$$\{[A_{P, \alpha}, F_{P, \alpha}^s, \mathbf{x}_{P, \alpha}]\}_{\alpha=1}^\infty \subset \bar{\mathcal{B}}_{|P|}^{s, \natural}(\delta_P)$$

converges in the Uhlenbeck topology to $[A_{P,0}, F_{P,0}^s, \mathbf{x}_{P,0}] \in \bar{\mathcal{B}}_{|P|}^{s, \natural}(\delta_P)$, then by the definition of Uhlenbeck convergence,

$$\lim_{\alpha \rightarrow \infty} \lambda[A_{P,\alpha}, F_{P,\alpha}^s, \mathbf{x}_{P,\alpha}] = \lambda[A_{P,0}, F_{P,0}^s, \mathbf{x}_{P,0}] := \lambda_{P,0}.$$

By (5.2.3), none of the points in $\mathbf{x}_{P,0}$ can lie outside of the ball $B(0, \lambda_{P,0}) \subset B(0, 2\sqrt{\lambda_{P,0}})$ and so the sequence of connections $\{A_{P,\alpha}\}$ cannot “bubble” outside of the ball $B(0, 2\sqrt{\lambda_{P,0}})$. These observations and the construction of $\gamma'_{\Theta, \mathcal{P}}$ in (5.4.4) imply that $\gamma'_{\Theta, \mathcal{P}}$ is continuous with respect to these Uhlenbeck limits.

The equality (5.4.5) follows immediately from the construction (5.4.4). \square

It is important that our constructions in Section 5.6 be equivariant with respect to the $\mathrm{SO}(3) \times \mathrm{SO}(4)$ actions on the space of connections defined in (5.2.5) and (5.2.6), so that the instanton moduli space with spliced ends can be used in the space of gluing data. The action of $\mathrm{SO}(3) \times \mathrm{SO}(4)$ on the domain (5.4.1) is defined as follows. For $R \in \mathrm{SO}(4)$, let \tilde{R}_P be any automorphism of $\mathrm{Fr}(E_P)$ covering $R : S^4 \rightarrow S^4$. For $A \in \mathrm{SO}(3)$ and $R \in \mathrm{SO}(4)$, define

$$(5.4.6) \quad (((y_P)_{P \in \mathcal{P}}, [A_P, F_P^s, \mathbf{x}_P]_{P \in \mathcal{P}}), A, R) \mapsto \left((Ry_P)_{P \in \mathcal{P}}, [(\tilde{R}_P^{-1})^* A_P, \tilde{R}_P F_P^s A^{-1}, R\mathbf{x}_P]_{P \in \mathcal{P}} \right).$$

We have the following lemma.

LEMMA 5.4.2. *Let \mathcal{P} be a partition of N_κ . The splicing map $\gamma'_{\Theta, \mathcal{P}}$ defined in (5.4.3) is equivariant with respect to the $\mathrm{SO}(3) \times \mathrm{SO}(4)$ actions defined in (5.4.6) on its domain and in (5.2.5) and (5.2.6) on its image.*

PROOF. The equivariance of $\gamma'_{\Theta, \mathcal{P}}$ with respect to the $\mathrm{SO}(3)$ actions follows immediately from the construction (5.4.4).

The construction of the splicing map in (5.4.4) defines, for each point

$$\mathbf{A} = ((y_P)_{P \in \mathcal{P}}, [A_P, F_P^s, \mathbf{x}_P]_{P \in \mathcal{P}}) \in \tilde{\mathcal{O}}(\Theta, \mathcal{P}, \delta),$$

an embedding of principal bundles,

$$\iota_{\mathbf{A}} : \bigsqcup_{P \in \mathcal{P}} \mathrm{Fr}(E_P)|_{B(0, 4\sqrt{\lambda_P})} \rightarrow \mathrm{Fr}(E_\kappa),$$

covering the embedding $B(0, 4\sqrt{\lambda_P}) \rightarrow S^4$ given by $x_P \mapsto y_P + x_P$ on the P -th component. If $\tilde{R}_\kappa : \mathrm{Fr}(E_\kappa) \rightarrow \mathrm{Fr}(E_\kappa)$ is a bundle map covering $R : S^4 \rightarrow S^4$, then $\tilde{R} \circ \iota_{\mathbf{A}}$ equals, up to a gauge transformation, the embedding $\iota_{R\mathbf{A}}$, where

$$R\mathbf{A} = \left((Ry_P)_{P \in \mathcal{P}}, [(\tilde{R}_P^{-1})^* A_P, \tilde{R}_P F_P^s, R\mathbf{x}_P]_{P \in \mathcal{P}} \right).$$

The desired equivariance, $(\tilde{R}_\kappa^{-1})^* \gamma'_{\Theta, \mathcal{P}}(\mathbf{A}) = \gamma'_{\Theta, \mathcal{P}}(R\mathbf{A})$, then follows by considering the images of the horizontal distributions defining the connections under these embeddings. \square

We will also require the following observation about centering maps.

LEMMA 5.4.3. *Let $c_{y,\lambda} : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be defined by $c_{y,\lambda}(z) = (z - y)/\lambda$. For any $\lambda_1, \lambda_2 > 0$ and $x_1, x_2 \in \mathbb{R}^4$, we have:*

$$\begin{aligned} c_{x_2, \lambda_2} \circ c_{x_1, \lambda_1} &= c_{x_1 + \lambda_1 x_2, \lambda_1 \lambda_2}, \\ c_{x_2, \lambda_2}^* \chi_{x_1, \lambda_1} &= \chi_{x_2 + \lambda_2 x_1, \lambda_1 \lambda_2}. \end{aligned}$$

If one splices a mass-centered connection, of charge κ and scale λ at a point $y \in \mathbb{R}^4$, with the product connection it is not immediately clear what the scale of the resulting connection will be. However, the limit of the scale of the spliced connection (as λ tends to zero) will be $\sqrt{\kappa|y|^2}$. The following exploitation of this fact will be used in Lemma 5.5.2 to prove that when composing two splicing maps— for a suitable open set of splicing data — the annulus on which the second splicing map interpolates between connections (see the second line of (5.4.4)) does not intersect the annuli of the first splicing map.

LEMMA 5.4.4. *Let Θ be the product connection and $c_{|P|} \in \text{Sym}_\delta^{|P|, \natural}(\mathbb{R}^4)$ the cone point. If*

$$(5.4.7) \quad T(\Theta, \mathcal{P}, \delta) := \left\{ (y_P, [\Theta, F_P^s, c_{|P|}])_{P \in \mathcal{P}} \in \mathcal{O}(\Theta, \mathcal{P}, \delta) \right\},$$

then, for $\Sigma(Z_\kappa(\delta), \mathcal{P})$ as defined in (5.3.2), we have

$$(5.4.8) \quad \gamma'_{\Theta, \mathcal{P}}(T(\Theta, \mathcal{P}, \delta)) = [\Theta] \times \Sigma(Z_\kappa(\delta), \mathcal{P}),$$

and there is an open neighborhood, $D(\Theta, \mathcal{P}, \delta)$, of $T(\Theta, \mathcal{P}, \delta)$ in $\mathcal{O}(\Theta, \mathcal{P}, \delta)$, such that for every

$$\mathbf{A} := (y_P, [A_P, F_P^s, \mathbf{x}_P])_{P \in \mathcal{P}} \in D(\Theta, \mathcal{P}, \delta)$$

and for every $P \in \mathcal{P}$, we have

$$(5.4.9) \quad B(y_P, 4\sqrt{\lambda_P}) \Subset B(0, 2\sqrt{\lambda}),$$

and

$$(5.4.10) \quad B(y_P, \frac{1}{4}\lambda_P^{1/3}) \Subset B(0, \frac{1}{8}\lambda^{1/3}),$$

where $\lambda_P = \lambda([A_P, F_P^s, \mathbf{x}_P])$ and $\lambda = \lambda(\gamma'_{\Theta, \mathcal{P}}(\mathbf{A}))$.

PROOF. The equality (5.4.8) follows immediately from Lemma 5.4.1.

The functions λ_P vanish on $T(\Theta, \mathcal{P}, \delta)$ while by Lemma 5.4.1, the composition $\lambda \circ \gamma'_{\Theta, \mathcal{P}}$ is equal to the square root of $\sum_{P \in \mathcal{P}} |P||x_P|^2$ and thus is non-zero on $T(\Theta, \mathcal{P}, \delta)$. The continuity of the functions involved then yields the existence of the desired open subspace. \square

5.5. Composition of splicing maps

To construct the instanton moduli space with spliced ends, we need to show that the union of the images of the splicing maps $\gamma'_{\Theta, \mathcal{P}}$ form a smoothly-stratified space. Let $\mathcal{P}, \mathcal{P}'$ be partitions of N_κ with $\mathcal{P} < \mathcal{P}'$. For each $P \in \mathcal{P}$, let \mathcal{P}'_P be the partition of P defined in (4.2.4). We will need to understand the overlaps of the images of the maps $\gamma'_{\Theta, \mathcal{P}}$ and $\gamma'_{\Theta, \mathcal{P}'}$. To that end, we define a space $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ with maps to $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \delta)$ and $\tilde{\mathcal{O}}(\Theta, \mathcal{P}', \delta)$ so that the following diagram commutes:

$$(5.5.1) \quad \begin{array}{ccc} \tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta) & \xrightarrow{\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}} & \tilde{\mathcal{O}}(\Theta, \mathcal{P}, \delta) \\ \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u} \downarrow & & \gamma'_{\Theta, \mathcal{P}} \downarrow \\ \tilde{\mathcal{O}}(\Theta, \mathcal{P}', \delta) & \xrightarrow{\gamma'_{\Theta, \mathcal{P}'}} & \tilde{\mathcal{B}}_\kappa^s(S^4, 2\delta) \end{array}$$

The superscripts d and u appearing in the maps $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}$ and $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}$ suggest “down” and “up”, respectively, as \mathcal{P} and \mathcal{P}' correspond to the lower and upper strata.

5.5.1. Definition of the overlap data. We now define the objects appearing in the diagram (5.5.1). The open set $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ will be defined to be a subspace of

$$(5.5.2) \quad \tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta) := \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times \prod_{P \in \mathcal{P}} \left(\Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}'_P) \times \prod_{P' \in \mathcal{P}'_P} \bar{\mathcal{B}}^s_{|P'|}(S^4, \delta_{P'}) \right),$$

where the constants δ_P and $\delta_{P'}$ will not need to be defined explicitly. The open set $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ will be defined by the conditions (5.5.6), (5.5.7), (5.5.9), and (5.5.10).

An element $\sigma \in \mathfrak{S}(\mathcal{P})$ with $\sigma(\mathcal{P}') = \mathcal{P}''$ defines a bijection,

$$(5.5.3) \quad \sigma : \tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta) \rightarrow \tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}'', \delta),$$

by the diagonal action on each of the factors. That is, σ defines bijections,

$$\begin{aligned} \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) &\rightarrow \Delta^\circ(Z_\kappa(\delta), \mathcal{P}), \\ \Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}'_P) &\rightarrow \Delta^\circ(Z_{|\sigma(P)|}(\delta_{\sigma(P)}), \mathcal{P}''_{\sigma(P)}), \\ \bar{\mathcal{B}}^s_{|P'|}(S^4, \delta_{P'}) &\rightarrow \bar{\mathcal{B}}^s_{|\sigma(P')|}(S^4, \delta_{\sigma(P')}), \end{aligned}$$

which define the desired bijection (5.5.3).

The following maps are given by the obvious projections onto the factors of $\tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$:

$$(5.5.4) \quad \begin{aligned} \pi_{\mathcal{P}, \mathbf{x}} : \tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta) &\rightarrow \Delta^\circ(Z_\kappa(\delta), \mathcal{P}), \\ \pi_{\mathcal{P}'_P, \mathbf{x}} : \tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta) &\rightarrow \Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}'_P), \\ \pi_{\mathbf{x}} : \tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta) &\rightarrow \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times \prod_{P \in \mathcal{P}} \Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}'_P), \\ \pi_P : \tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta) &\rightarrow \Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}'_P) \times \prod_{P' \in \mathcal{P}'_P} \bar{\mathcal{B}}^s_{|P'|}, \\ \pi_{P'} : \tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta) &\rightarrow \bar{\mathcal{B}}^s_{|P'|}(\delta_P). \end{aligned}$$

We use the inclusion given in (5.3.5),

$$\tilde{\nu}(Z_\kappa(\delta), \mathcal{P} \rightarrow \mathcal{P}') = \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times \prod_{P \in \mathcal{P}} \Delta^\circ(Z_P(\delta_P), \mathcal{P}'_P) \rightarrow \tilde{\nu}(Z_\kappa(\delta), \mathcal{P}),$$

to view $\tilde{\nu}(Z_\kappa(\delta), \mathcal{P} \rightarrow \mathcal{P}')$ as a subspace of $\tilde{\nu}(Z_\kappa(\delta), \mathcal{P})$. By Lemma 5.3.3, the image of the restriction of the exponential map $e(Z_\kappa, \mathcal{P})$ in (4.2.10) to elements of $\tilde{\nu}(Z_\kappa(\delta), \mathcal{P} \rightarrow \mathcal{P}')$ is contained in $\Delta^\circ(Z_\kappa(\delta), \mathcal{P}')$. We would like to define the map $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}$ appearing in the diagram (5.5.1) by

$$(5.5.5) \quad \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u} := (e(Z_\kappa, \mathcal{P}) \circ \pi_{\mathbf{x}}) \times \prod_{P' \in \mathcal{P}'} \pi_{P'}.$$

That is, the map $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}$ leaves the connection data in $\nu(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ unchanged and maps the points in $\tilde{\nu}(Z_\kappa(\delta), \mathcal{P} \rightarrow \mathcal{P}')$ to the diagonal $\Delta^\circ(Z_\kappa(\delta), \mathcal{P}')$.

For the composition $e(Z_\kappa, \mathcal{P}) \circ \pi_{\mathbf{x}}$ in (5.5.5) to be defined, we must restrict the domain of $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}$ to an open subspace $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ of $\tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ satisfying

$$(5.5.6) \quad \tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta) \subset \pi_{\mathbf{x}}^{-1} \left(\tilde{\mathcal{O}}(Z_\kappa(\delta), \mathcal{P} \rightarrow \mathcal{P}') \right),$$

where $\tilde{\mathcal{O}}(Z_\kappa(\delta), \mathcal{P} \rightarrow \mathcal{P}')$ is defined in (5.3.6).

To define the composition $\gamma'_{\Theta, \mathcal{P}'} \circ \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}$ appearing in the diagram (5.5.1), the open subspace $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ must also satisfy

$$(5.5.7) \quad \tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta) \subset (\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u})^{-1} \left(\tilde{\mathcal{O}}(\Theta, \mathcal{P}', \delta) \right),$$

where $\tilde{\mathcal{O}}(\Theta, \mathcal{P}', \delta)$, defined in (5.4.1), is the domain of $\gamma'_{\Theta, \mathcal{P}'}$.

Before we proceed to define $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}$, we note that if $P \in \mathcal{P} \cap \mathcal{P}'$, then the partition \mathcal{P}'_P is just the set P and $\Delta^\circ(Z_{|P|}, \mathcal{P}'_P)$ is a single point (the zero vector). In this case, we define the splicing map,

$$\gamma'_{\Theta, \mathcal{P}'_P} : \Delta^\circ(Z_{|P|}(\delta), \mathcal{P}'_P) \times \bar{\mathcal{B}}_{|P|}^s(S^4, \delta_P) \rightarrow \bar{\mathcal{B}}_{|P|}^s(S^4, 2\delta),$$

to be the projection onto $\bar{\mathcal{B}}_{|P|}^s(S^4, \delta_P) \subset \bar{\mathcal{B}}_{|P|}^s(S^4, 2\delta)$. The map $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}$ of (5.5.1) will then be defined by

$$(5.5.8) \quad \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d} := \pi_{\mathcal{P}, \mathbf{x}} \times \left(\prod_{P \in \mathcal{P}} \gamma'_{\Theta, \mathcal{P}'_P} \circ \pi_P \right).$$

That is, the map $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}$ leaves the point in $\Delta^\circ(Z_\kappa, \mathcal{P})$ unchanged and for each $P \in \mathcal{P}$ applies the splicing map $\gamma'_{\Theta, \mathcal{P}'_P}$ to the space

$$\Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}'_P) \times \prod_{P' \in \mathcal{P}'_P} \bar{\mathcal{B}}_{|P'|}^s(S^4, \delta_{P'}).$$

For the map $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}$ to be defined on $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$, we need the image of π_P to be contained in the domain, $\mathcal{O}(\Theta, \mathcal{P}'_P, \delta_P)$, of the splicing map $\gamma'_{\Theta, \mathcal{P}'_P}$. However, we will make a stronger requirement (to be used in Lemma 5.5.2),

$$(5.5.9) \quad \tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta) \subset \pi_P^{-1} (D(\Theta, \mathcal{P}'_P, \delta_P)) \quad \text{for all } P \in \mathcal{P} \setminus \mathcal{P}',$$

where $D(\Theta, \mathcal{P}'_P, \delta_P)$ is the open subspace of $\mathcal{O}(\Theta, \mathcal{P}'_P, \delta_P)$ defined in Lemma 5.4.4. Since

$$D(\Theta, \mathcal{P}'_P, \delta_P) \subset \mathcal{O}(\Theta, \mathcal{P}'_P, \delta_P),$$

the condition (5.5.9) implies that $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}$ is defined on $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$.

Finally, for the composition $\gamma'_{\Theta, \mathcal{P}} \circ \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}$ to be well-defined, $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ must satisfy

$$(5.5.10) \quad \tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta) \subset \left(\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d} \right)^{-1} \left(\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \delta) \right).$$

In the proof of Lemma 5.5.1 below, we will show that the image of the restriction of $\gamma'_{\Theta, \mathcal{P}'} \circ \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}$ to $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ contains

$$(5.5.11) \quad \{[\Theta, \mathbf{x}] : \mathbf{x} \in e(Z_\kappa, \mathcal{P})(\mathcal{O}(Z_\kappa(\delta), \mathcal{P}, \mathcal{P}'))\}.$$

Hence, the image of $\gamma'_{\Theta, \mathcal{P}'} \circ \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}$ will contain the intersection of the stratum $[\Theta] \times \Sigma(Z_\kappa(\delta), \mathcal{P}')$ with the image of $\gamma'_{\Theta, \mathcal{P}}$.

LEMMA 5.5.1. *Let $\mathcal{P}, \mathcal{P}'$ be partitions of N_κ satisfying $\mathcal{P} < \mathcal{P}'$. Define a subspace of $\tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ by*

$$(5.5.12) \quad T(\Theta, \mathcal{P}, \mathcal{P}', \delta) := \pi_{\mathbf{x}}^{-1} \left(\tilde{\mathcal{O}}(Z_\kappa(\delta), \mathcal{P} \rightarrow \mathcal{P}') \right) \cap \left(\bigcap_{P' \in \mathcal{P}'} \pi_{P'}^{-1}([\Theta, c_{|P'|}] \right).$$

Then the following hold:

- (1) $(\gamma'_{\Theta, \mathcal{P}'} \circ \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u})(T(\Theta, \mathcal{P}, \mathcal{P}', \delta)) = \left\{ [\Theta, \mathbf{x}] : \mathbf{x} \in e(Z_\kappa, \mathcal{P}) \left(\tilde{\mathcal{O}}(Z_\kappa(\delta), \mathcal{P} \rightarrow \mathcal{P}') \right) \right\},$
- (2) $(\gamma'_{\Theta, \mathcal{P}} \circ \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d})(T(\Theta, \mathcal{P}, \mathcal{P}', \delta)) = \left([\Theta] \times e(Z_\kappa, \mathcal{P}) \left(\tilde{\mathcal{O}}(Z_\kappa(\delta), \mathcal{P} \rightarrow \mathcal{P}') \right) \right).$

Moreover, there is an open neighborhood, $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$, of $T(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ in $\tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ satisfying the conditions (5.5.6), (5.5.7), (5.5.9), and (5.5.10).

PROOF. Item (1) follows from the definitions of the spaces and maps and the value of the splicing map given in (5.4.5). Equation (5.4.5) implies that

$$\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d} \Big|_{T(\Theta, \mathcal{P}, \mathcal{P}', \delta)} = \pi_{\mathbf{x}} \Big|_{T(\Theta, \mathcal{P}, \mathcal{P}', \delta)}.$$

The preceding identity and the equality,

$$(5.5.13) \quad \pi_{\mathbf{x}}(T(\Theta, \mathcal{P}, \mathcal{P}', \delta)) = \tilde{\mathcal{O}}(Z_\kappa(\delta), \mathcal{P} \rightarrow \mathcal{P}'),$$

which follows immediately from the definition of $T(\Theta, \mathcal{P}, \mathcal{P}', \delta)$, yields Item (2).

Equation (5.5.13) implies that there is an open neighborhood of $T(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ in the space $\tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ satisfying (5.5.6). For every $P' \in \mathcal{P}'$, we have

$$\pi_{P'}(T(\Theta, \mathcal{P}, \mathcal{P}', \delta)) = [\Theta, c_{|P'|}],$$

so $\pi_P(T(\Theta, \mathcal{P}, \mathcal{P}', \delta)) \subset T(\Theta, \mathcal{P}'_P, \delta_P) \subset D(\Theta, \mathcal{P}, \delta)$ for all $P \in \mathcal{P}$ and there is an open neighborhood of $T(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ in $\tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ satisfying (5.5.9).

Item (1) implies that $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}(T(\Theta, \mathcal{P}, \mathcal{P}', \delta)) \subset \tilde{\mathcal{O}}(\Theta, \mathcal{P}', \delta)$, so there is an open neighborhood of $T(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ in $\tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ satisfying (5.5.7).

Item (2) implies that $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}(T(\Theta, \mathcal{P}, \mathcal{P}', \delta))$ is contained in the domain of $\gamma'_{\Theta, \mathcal{P}'}$, namely, $\mathcal{O}(\Theta, \mathcal{P}, \delta)$, so there is an open neighborhood of $T(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ in $\tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ satisfying (5.5.10).

The intersection of these open neighborhoods of $T(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ in $\tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$, yields the desired open neighborhood $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$. \square

5.5.2. Equality of splicing maps. We now prove that the diagram (5.5.1) commutes. The key point in this proof is the restriction (5.5.9) on the domain $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ which, as shown in Lemma 5.4.4, ensures that the connections given by $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}$ are *already* equal to the product connection on the annuli on which the splicing map $\gamma'_{\Theta, \mathcal{P}}$ interpolates between them and the product connection. Hence, in the iterated splicing construction defining the composition $\gamma'_{\Theta, \mathcal{P}} \circ \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}$, the interpolation in the definition of $\gamma'_{\Theta, \mathcal{P}}$ does not change the connection. This iterated splicing procedure is thus equivalent to a single splicing, that given by the composition $\gamma'_{\Theta, \mathcal{P}'} \circ \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}$.

LEMMA 5.5.2. *Let \mathcal{P} and \mathcal{P}' be partitions of N_κ with $\mathcal{P} < \mathcal{P}'$. If $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ is the open subset defined in Lemma 5.5.1 of the space $\tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ defined in (5.5.2), then the following diagram commutes:*

$$(5.5.14) \quad \begin{array}{ccc} \tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta) & \xrightarrow{\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}} & \tilde{\mathcal{O}}(\Theta, \mathcal{P}', \delta) \\ \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d} \downarrow & & \gamma'_{\Theta, \mathcal{P}'} \downarrow \\ \tilde{\mathcal{O}}(\Theta, \mathcal{P}, \delta) & \xrightarrow{\gamma'_{\Theta, \mathcal{P}}} & \bar{\mathcal{B}}_\kappa^s(S^4, 2\delta) \end{array}$$

PROOF. To clarify the notation, we write Q for an element of the partition \mathcal{P}' . Denote

$$\mathbf{A} := \left((y_P)_{P \in \mathcal{P}}, \left((x_Q)_{Q \in \mathcal{P}'_P}, [A_Q, F_Q^s]_{Q \in \mathcal{P}'_P} \right)_{P \in \mathcal{P}} \right) \in \tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta),$$

where $(y_P)_{P \in \mathcal{P}} \in \Delta^\circ(Z_\kappa, \mathcal{P})$ and, by (5.5.9), $((x_Q)_{Q \in \mathcal{P}'_P}, [A_Q, F_Q^s]_{Q \in \mathcal{P}'_P}) \in D(\Theta, \mathcal{P}'_P, \delta_P)$. (For simplicity of exposition, we assume that $[A_Q, F_Q^s] \in \bar{\mathcal{B}}_{|Q|}^s(S^4, \delta_Q)$ has no ideal points; the proof is no more difficult without that assumption, but the notation becomes more opaque.)

By (5.5.5),

$$\begin{aligned} \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}(\mathbf{A}) &= \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u} \left((y_P)_{P \in \mathcal{P}}, \left((x_Q)_{Q \in \mathcal{P}'_P}, [A_Q, F_Q^s]_{Q \in \mathcal{P}'_P} \right)_{P \in \mathcal{P}} \right) \\ &= \left((x'_Q)_{Q \in \mathcal{P}'}, \left([A_Q, F_Q^s]_{Q \in \mathcal{P}'_P} \right) \right), \end{aligned}$$

where

$$(5.5.15) \quad x'_Q = y_P + x_Q \quad \text{if } Q \in \mathcal{P}'_P.$$

(Note that $x'_Q = y_P$ if $Q \in \mathcal{P} \cap \mathcal{P}'$ as x_Q is then the zero vector.) If we denote $\lambda_Q := \lambda([A_Q])$, then

$$(5.5.16) \quad \gamma'_{\Theta, \mathcal{P}'} \circ \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}(\mathbf{A}) = \begin{cases} \Theta & \text{on } \mathbb{R}^4 \setminus \cup_{P \in \mathcal{P}, Q \in \mathcal{P}'_P} B(x'_Q, 4\sqrt{\lambda_Q}), \\ (1 - \chi_{x'_Q, 4\sqrt{\lambda_Q}}) c_{x'_Q, 1}^* A_Q \\ + \chi_{x'_Q, 4\sqrt{\lambda_Q}} c_{x'_Q, 1}^* \Theta(A_Q, F_Q^s) & \text{on } \Omega(x'_Q; 2\sqrt{\lambda_Q}, 4\sqrt{\lambda_Q}), \\ c_{x'_Q, 1}^* A_Q & \text{on } B(x'_Q, 2\sqrt{\lambda_Q}). \end{cases}$$

We now compare (5.5.16) with $\gamma'_{\Theta, \mathcal{P}} \circ \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}(\mathbf{A})$.

For $P \in \mathcal{P} \setminus \mathcal{P}'$, we denote

$$[A_P, F_P^s] := \gamma'_{\Theta, \mathcal{P}'}(\pi_P(\mathbf{A})) = \gamma'_{\Theta, \mathcal{P}'} \left((x_Q)_{Q \in \mathcal{P}'_P}, [A_Q, F_Q^s]_{Q \in \mathcal{P}'_P} \right),$$

and $[A_P, F_P^s] := [A_Q, F_Q^s]$ for $P = Q \in \mathcal{P} \cap \mathcal{P}'$. Then by (5.5.8),

$$\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}(\mathbf{A}) = ((y_P)_{P \in \mathcal{P}}, ([A_P, F_P^s])_{P \in \mathcal{P}}) \in \mathcal{O}(\Theta, \mathcal{P}, \delta).$$

If we denote $\lambda'_P := \lambda([A_P])$, then

$$(5.5.17) \quad \gamma'_{\Theta, \mathcal{P}} \circ \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}(\mathbf{A}) = \begin{cases} \Theta & \text{on } \mathbb{R}^4 \setminus \cup_{P \in \mathcal{P}} B(y_P, 4\sqrt{\lambda'_P}), \\ (1 - \chi_{y_P, 4\sqrt{\lambda'_P}}) c_{y_P, 1}^* A_P & \\ + \chi_{y_P, 4\sqrt{\lambda'_P}} c_{y_P, 1}^* \Theta(A_P, F_P^s) & \text{on } \Omega(y_P; 2\sqrt{\lambda'_P}, 4\sqrt{\lambda'_P}), \\ c_{y_P, 1}^* A_P & \text{on } B(y_P, 4\sqrt{\lambda'_P}). \end{cases}$$

Over the balls $B(x_P, 4\sqrt{\lambda'_P})$, for $P \in \mathcal{P} \cap \mathcal{P}'$, and over $\mathbb{R}^4 \setminus \cup_{P \in \mathcal{P}} B(x_P, 4\sqrt{\lambda'_P})$, the connections (5.5.16) and (5.5.17) are identical. We thus focus our attention on the balls $B(x_P, 4\sqrt{\lambda'_P})$ for $P \in \mathcal{P} \setminus \mathcal{P}'$. For such a P , denoting $\lambda_Q := \lambda([A_Q])$ as done prior to (5.5.16), we have

$$(5.5.18) \quad A_P = \gamma'_{\Theta, \mathcal{P}'_P}(\pi_P(\mathbf{A})) = \begin{cases} \Theta & \text{on } \mathbb{R}^4 \setminus \cup_{Q \in \mathcal{P}'_P} B(x_Q, 4\sqrt{\lambda_Q}), \\ (1 - \chi_{x_Q, 4\sqrt{\lambda_Q}}) c_{x_Q, 1}^* A_Q & \\ + \chi_{x_Q, 4\sqrt{\lambda_Q}} c_{x_Q, 1}^* \Theta(A_Q, F_Q^s) & \text{on } \Omega(x_Q; 2\sqrt{\lambda_Q}, 4\sqrt{\lambda_Q}), \\ c_{x_Q, 1}^* A_Q & \text{on } B(x_Q, 2\sqrt{\lambda_Q}). \end{cases}$$

Because the points $(x_Q)_{Q \in \mathcal{P}'_P} \in \Delta^\circ(Z_P(\delta), \mathcal{P}'_P)$ and the connections $[A_Q, F_Q^s] \in \bar{\mathcal{B}}_{|Q|}^s(S^4)$ form a point in $D(\Theta, \mathcal{P}'_P, \delta_P)$ by the condition (5.5.9) in the construction of $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$, Lemma 5.4.4 and equation (5.4.9) imply that for all $Q \in \mathcal{P}'_P$ we have

$$B(x_Q, 4\sqrt{\lambda_Q}) \subset B(0, 2\sqrt{\lambda'_P}),$$

where λ'_P is defined prior to (5.5.17). Therefore (5.5.18) implies that A_P is already equal to the product connection Θ over

$$\mathbb{R}^4 \setminus B(0, 2\sqrt{\lambda'_P}),$$

so $c_{y_P, 1}^* A_P$ is equal to Θ over

$$\mathbb{R}^4 \setminus B(y_P, 2\sqrt{\lambda'_P}).$$

Hence, the convex combination in the second line of (5.5.17) is equal to Θ . Denoting $x'_Q = y_P + x_Q$ as in (5.5.15) and $\lambda_Q := \lambda([A_Q])$ as before (5.5.18), we examine the final line of (5.5.17) by applying Lemma 5.4.3 to rewrite (5.5.18) as

$$(5.5.19) \quad c_{y_P, 1}^* A_P = \begin{cases} \Theta & \text{on } \mathbb{R}^4 \setminus \cup_{Q \in \mathcal{P}'_P} B(x'_Q, 4\sqrt{\lambda_Q}), \\ (1 - \chi_{x'_Q, 4\sqrt{\lambda_Q}}) c_{x'_Q, 1}^* A_Q & \\ + \chi_{x'_Q, 4\sqrt{\lambda_Q}} c_{x'_Q, 1}^* \Theta(A_Q, F_Q^s) & \text{on } \Omega(x'_Q; 2\sqrt{\lambda_Q}, 4\sqrt{\lambda_Q}), \\ c_{x'_Q, 1}^* A_Q & \text{on } B(x'_Q, 2\sqrt{\lambda_Q}). \end{cases}$$

Consequently, we see that $\gamma'_{\Theta, \mathcal{P}} \circ \rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}(\mathbf{A})$ is given by

$$(5.5.20) \quad \begin{cases} \Theta & \text{on } \mathbb{R}^4 \setminus \cup_{P \in \mathcal{P}, Q \in \mathcal{P}'} B(x'_Q, 4\sqrt{\lambda_Q}), \\ (1 - \chi_{x'_Q, 4\sqrt{\lambda_Q}}) c_{x'_Q, 1}^* A_Q & \\ + \chi_{x'_Q, 4\sqrt{\lambda_Q}} c_{x'_Q, 1}^* \Theta(A_Q, F_Q^s) & \text{on } \Omega(x'_Q; 2\sqrt{\lambda_Q}, 4\sqrt{\lambda_Q}), \\ c_{x'_Q, 1}^* A_Q & \text{on } B(x'_Q, 2\sqrt{\lambda_Q}). \end{cases}$$

The conclusion of Lemma 5.5.2 now follows by comparing (5.5.20) with (5.5.16). \square

5.5.3. Symmetric group actions and quotients. We now make some observations on the action of the symmetric group necessary to define a quotient of the diagram (5.5.1) by the symmetric group. For $\mathcal{P} < \mathcal{P}'$, we recall from Lemma 5.3.4 that to describe a neighborhood of $\Sigma(Z_\kappa(\delta), \mathcal{P})$ in $\Sigma(Z_\kappa, \mathcal{P})$, we must describe all the diagonals $\Delta^\circ(Z_\kappa(\delta), \mathcal{P}'')$ where $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$. We define

$$(5.5.21) \quad \begin{aligned} \tilde{\nu}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) &:= \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}'', \delta), \\ \nu(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) &:= \tilde{\nu}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) / \mathfrak{S}(\mathcal{P}), \end{aligned}$$

where the group $\mathfrak{S}(\mathcal{P})$ acts on $\tilde{\nu}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ by the action defined in (5.5.3). If the open subspaces $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}'', \delta)$ defined in Lemma 5.5.1 are sufficiently small, then their union $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) \subset \tilde{\nu}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ will be closed under the action of $\mathfrak{S}(\mathcal{P})$ and we define

$$(5.5.22) \quad \mathcal{O}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) := \tilde{\mathcal{O}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) / \mathfrak{S}(\mathcal{P}).$$

By analogy with the definition of $T(\Theta, \mathcal{P}, \mathcal{P}'', \delta)$ in (5.5.12), we define

$$(5.5.23) \quad \begin{aligned} \tilde{T}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) &:= \{((y_P)_{P \in \mathcal{P}}, (x_Q)_{Q \in \mathcal{P}''}, ([\Theta, c_{|Q|}])_{Q \in \mathcal{P}''}) \in \tilde{\nu}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) : \\ &((y_P)_{P \in \mathcal{P}}, (x_Q)_{Q \in \mathcal{P}''}) \in \mathcal{O}(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])\}, \\ T(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) &:= \tilde{T}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) / \mathfrak{S}(\mathcal{P}). \end{aligned}$$

where $\mathcal{O}(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])$ is defined in Lemma 5.3.4.

Recall that we denote the disjoint union or coproduct of sets A_α by $\sqcup_\alpha A_\alpha$ and for maps $f_\alpha : A_\alpha \rightarrow B_\alpha$, the coproduct map $\coprod f_\alpha : \sqcup_\alpha A_\alpha \rightarrow \sqcup_\alpha B_\alpha$ is defined by f_α on the subset $A_\alpha \subset \sqcup_\alpha A_\alpha$. The coproduct of the maps $\rho_{\mathcal{P}, \mathcal{P}''}^{\Theta, u}$,

$$\coprod_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \rho_{\mathcal{P}, \mathcal{P}''}^{\Theta, u} : \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}'', \delta) \rightarrow \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \nu(\Theta, \mathcal{P}'', \delta)$$

is equivariant with respect to the $\mathfrak{S}(\mathcal{P})$ actions and thus defines a map on $\mathfrak{S}(\mathcal{P})$ quotients,

$$(5.5.24) \quad \begin{aligned} \rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, u} : \mathcal{O}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) &\rightarrow \nu(\Theta, [\mathcal{P} < \mathcal{P}'], \delta) \\ \nu(\Theta, [\mathcal{P} < \mathcal{P}'], \delta) &:= \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \nu(\Theta, \mathcal{P}'', \delta) \right) / \mathfrak{S}(\mathcal{P}). \end{aligned}$$

Similarly, the coproduct of the maps $\rho_{\mathcal{P},[\mathcal{P}']}^{\Theta,d}$ is equivariant with respect to the action of $\mathfrak{S}(\mathcal{P})$ and thus defines a map of the $\mathfrak{S}(\mathcal{P})$ quotients,

$$(5.5.25) \quad \rho_{\mathcal{P},[\mathcal{P}']}^{\Theta,d} : \mathcal{O}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) \rightarrow \mathcal{O}(\Theta, \mathcal{P}, \delta).$$

The coproduct of the splicing maps $\gamma'_{\Theta, \mathcal{P}''}$,

$$\coprod_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \gamma'_{\Theta, \mathcal{P}''} : \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \mathcal{O}(\mathbb{R}^4, \mathcal{P}'', \delta) \rightarrow \bar{\mathcal{B}}_{\kappa}^s(S^4, 2\delta)$$

is invariant under the action of $\mathfrak{S}(\mathcal{P})$ and thus defines a map on the $\mathfrak{S}(\mathcal{P})$ quotient

$$(5.5.26) \quad \gamma'_{\Theta, \mathcal{P}, [\mathcal{P}']} : \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \mathcal{O}(\mathbb{R}^4, \mathcal{P}'', \delta) \right) / \mathfrak{S}(\mathcal{P}) \rightarrow \bar{\mathcal{B}}_{\kappa}^s(S^4, 2\delta),$$

which has the same image as $\gamma'_{\Theta, \mathcal{P}'}$. Lemma 5.5.2 and the preceding discussion of symmetric group actions then yields the

PROPOSITION 5.5.3. *Let \mathcal{P} and \mathcal{P}' be partitions of N_{κ} with $\mathcal{P} < \mathcal{P}'$. If $\mathcal{O}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ is the open subspace defined in (5.5.22), then one obtains the commutative diagram,*

$$(5.5.27) \quad \begin{array}{ccc} \mathcal{O}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) & \xrightarrow{\rho_{\mathcal{P},[\mathcal{P}']}^{\Theta,u}} & \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \mathcal{O}(\mathbb{R}^4, \mathcal{P}'') \right) / \mathfrak{S}(\mathcal{P}) \\ \rho_{\mathcal{P},[\mathcal{P}']}^{\Theta,d} \downarrow & & \gamma'_{\Theta, \mathcal{P}, [\mathcal{P}']} \downarrow \\ \mathcal{O}(\Theta, \mathcal{P}, \delta) & \xrightarrow{\gamma'_{\Theta, \mathcal{P}}} & \bar{\mathcal{B}}_{\kappa}^s(S^4, 2\delta) \end{array}$$

where $\rho_{\mathcal{P},[\mathcal{P}']}^{\Theta,d}$ is defined in (5.5.25), $\gamma'_{\Theta, \mathcal{P}, [\mathcal{P}]}$ in (5.5.26), and $\rho_{\mathcal{P},[\mathcal{P}']}^{\Theta,u}$ in (5.5.24).

5.6. The spliced end of the instanton moduli space

We now construct the deformation of an Uhlenbeck neighborhood of the strata (5.1.1) in $\bar{M}_{\kappa}^{s,\natural}(S^4, \delta)$ by taking the union of the images of the splicing maps $\gamma'_{\Theta, \mathcal{P}}$ restricted to appropriate subspaces of $\mathcal{O}(\Theta, \mathcal{P}, \delta)$.

This construction is inductive. We therefore assume that the instanton moduli space with spliced ends, $\bar{M}_{\text{SPL},|P|}^{s,\natural}(\delta)$, with the properties stated in Theorem 5.1.1 has already been constructed for $|P| < \kappa$. As described in Section 5.1, the space $\bar{M}_{\text{SPL},\kappa}^{s,\natural}(\delta)$ differs from $\bar{M}_{\kappa}^{s,\natural}(\delta)$ only on Uhlenbeck neighborhoods of the strata containing the product connection described in (5.1.1). Such strata are determined by partitions $\mathcal{P} = \{P_1, \dots, P_r\}$ of N_{κ} with $r > 1$ so $|P_i| < \kappa$ for all i . Hence, the Uhlenbeck neighborhoods of these strata are parameterized by the gluing maps (5.1.2) whose domain includes moduli spaces $\bar{M}_{|P|}^{s,\natural}(\delta)$ where $|P| < \kappa$. The inductive hypothesis will thus allow us to replace these moduli spaces with $\bar{M}_{\text{SPL},|P|}^{s,\natural}(\delta)$. There are no partitions of $\kappa = 1$ with length greater than one, so defining $\bar{M}_{\text{SPL},1}^{s,\natural}(\delta) = \bar{M}_1^{s,\natural}(\delta)$ will complete the initial step of the induction. Therefore, the inductive hypothesis suffices to complete this construction.

The splicing maps will be restricted to the following spaces (where the precise choice of the constants $\delta_P > 0$ will not be relevant to the rest of the discussion),

$$(5.6.1) \quad \begin{aligned} \tilde{\mathcal{O}}^{\text{ASD}}(\Theta, \mathcal{P}, \delta) &:= \tilde{\mathcal{O}}(\Theta, \mathcal{P}, \delta) \cap \left(\Delta^\circ(Z_\kappa(2\delta), \mathcal{P}) \times \prod_{P \in \mathcal{P}} M_{\text{SPL}, |P|}^{s, \natural}(\delta_P) \right), \\ \mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, \delta) &:= \tilde{\mathcal{O}}^{\text{ASD}}(\Theta, \mathcal{P}, \delta) / \mathfrak{S}(\mathcal{P}). \end{aligned}$$

Observe that $\tilde{T}(\Theta, \mathcal{P}, \delta) \subset \tilde{\mathcal{O}}^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$, where $\tilde{T}(\Theta, \mathcal{P}, \delta)$ is defined in (5.4.7) and that the dimension of $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$ is equal to that of $\bar{M}_\kappa^{s, \natural}(S^4, \delta)$.

The spliced end of the instanton moduli space is constructed in the following proposition.

PROPOSITION 5.6.1. *Assume that the instanton moduli space with spliced ends, $\bar{M}_{\text{SPL}, \kappa'}^{s, \natural}(\delta)$, with the properties enumerated in Theorem 5.1.1 has already been constructed for all $\kappa' < \kappa$. Then for every partition \mathcal{P} of N_κ with length greater than one, there is an $\text{SO}(3) \times \text{SO}(4)$ -invariant open neighborhood, $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$, of $T(\Theta, \mathcal{P}, \delta)$ in $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$ such that*

$$(5.6.2) \quad W_\kappa := \bigcup_{\mathcal{P}} \gamma'_{\Theta, \mathcal{P}}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, \delta))$$

is a smoothly-stratified subspace of $\bar{\mathcal{B}}_\kappa^s(S^4, \delta)$, with every point $[A, F^s, \mathbf{x}] \in W_\kappa$ satisfying the estimate $\|F_A^+\|_{L^{\sharp, 2}(S^4)} \leq \varepsilon$, where $\|\cdot\|_{L^{\sharp, 2}}$ is the norm defined in (5.1.3) and ε is the constant appearing in [21, Proposition 7.6].

We will prove Proposition 5.6.1 by using Proposition 5.5.3 to describe the intersections of the images of the splicing maps. The transition maps $\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, u}$ and $\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, d}$ will be restricted to the following subspace of $\mathcal{O}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$,

$$(5.6.3a) \quad \begin{aligned} \tilde{\mathcal{O}}^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) &:= \tilde{\mathcal{O}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) \cap (\Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \\ &\times \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}''} \left(\Delta^\circ(Z_P(\delta_P), \mathcal{P}''_P) \times \prod_{Q \in \mathcal{P}''_P} \bar{M}_{\text{SPL}, |Q|}^{s, \natural}(\delta_Q) \right)), \end{aligned}$$

$$(5.6.3b) \quad \mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) := \tilde{\mathcal{O}}^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) / \mathfrak{S}(\mathcal{P}).$$

(Compare the definition (5.5.21).) Observe that $T(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) \subset \mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$, where $T(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ is well-defined in (5.5.23) because $[\Theta, c_{|Q|}] \in \bar{M}_{\text{SPL}, |Q|}^{s, \natural}(\delta_Q)$ for all $|Q| < \kappa$.

We must first verify that the transition maps $\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, u}$ and $\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, d}$ map $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ to $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}', \delta)$ and $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$ respectively.

LEMMA 5.6.2. *Assume that $\bar{M}_{\text{SPL}, \kappa'}^{s, \natural}(\delta)$ has been constructed so that it satisfies the properties enumerated in Theorem 5.1.1 for all $\kappa' < \kappa$. Let $T(\Theta, \mathcal{P}, [\mathcal{P}'], \delta) \subset \mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ be as defined in (5.5.23). Then there is an open neighborhood $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ of $T(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ in $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ such that the restriction of $\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, d}$ to $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ is an open embedding of $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ into $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$.*

PROOF. Induction and the property (5.1.6) in Theorem 5.1.1 imply that for each $P \in \mathcal{P} \setminus \mathcal{P}'$, there is an open neighborhood $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}'_P, \delta_P)$ of $T(\Theta, \mathcal{P}'_P, \delta)$ in $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}'_P, \delta)$ such that

$$\gamma'_{\Theta, \mathcal{P}'_P}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}'_P, \delta_P)) \subset \bar{M}_{\text{SPL}, |P|}^{s, \natural}(\delta_P).$$

Therefore, in the definition (5.6.3) of $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$, if we replace the factor

$$\Delta^\circ(Z_P(\delta_P), \mathcal{P}''_P) \times \prod_{Q \in \mathcal{P}''_P} \bar{M}_{\text{SPL}, |Q|}^{s, \natural}(\delta_Q)$$

with the open subspace,

$$\tilde{\mathcal{O}}_1^{\text{ASD}}(\Theta, \mathcal{P}'_P, \delta_P) \subset \Delta^\circ(Z_P(\delta_P), \mathcal{P}''_P) \times \prod_{Q \in \mathcal{P}''_P} \bar{M}_{\text{SPL}, |Q|}^{s, \natural}(\delta_Q),$$

then the resulting open subspace $\tilde{\mathcal{O}}_1^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ of $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ will satisfy

$$(5.6.4) \quad \rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, d}(\tilde{\mathcal{O}}_1^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)) \subset \mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, \delta).$$

(Note that we are not concerned with the subsets $P \in \mathcal{P} \cap \mathcal{P}'$ because $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}$ is the identity on these factors and so they do not affect the inclusion (5.6.4).)

The fact that the restriction of $\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, d}$ to $\tilde{\mathcal{O}}_1^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ gives an open embedding into $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$ follows from the assumption that $\gamma'_{\Theta, \mathcal{P}'_P}$ gives an open embedding of $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}'_P, \delta_P)$ into $\bar{M}_{\text{SPL}, |P|}^{s, \natural}(\delta_P)$. \square

We now prove an analogue of Lemma 5.6.2 for the map $\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, u}$. First, we must define the appropriate range of $\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, u}$, to take into account the appearance of the conjugate partitions $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$. Let

$$(5.6.5) \quad \mathcal{O}^{\text{ASD}}(\Theta, [\mathcal{P} < \mathcal{P}'], \delta) := \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}'', \delta) \right) / \mathfrak{S}(\mathcal{P}) \subset \nu(\Theta, [\mathcal{P} < \mathcal{P}'], \delta),$$

where $\nu(\Theta, [\mathcal{P} < \mathcal{P}'], \delta)$ is defined in (5.5.24). We then have the

LEMMA 5.6.3. *Continue the assumptions and hypotheses of Lemma 5.6.2. Then the restriction of the map $\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, u}$ to $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ is an open embedding into $\mathcal{O}^{\text{ASD}}(\Theta, [\mathcal{P} < \mathcal{P}'], \delta)$.*

PROOF. The restriction of the map $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}$ defines a map

$$\begin{aligned} & \tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta) \cap \left(\prod_{P \in \mathcal{P}} \left(\Delta^\circ(Z_P(\delta_P), \mathcal{P}'_P) \times \prod_{Q \in \mathcal{P}'_P} \bar{M}_{\text{SPL}, |Q|}^{s, \natural}(\delta_Q) \right) \right) \\ & \quad \downarrow \\ & \Delta^\circ(Z_P(\delta_P), \mathcal{P}') \times \prod_{P \in \mathcal{P}} \prod_{Q \in \mathcal{P}'_P} \bar{M}_{\text{SPL}, |Q|}^{s, \natural}(\delta_Q) \end{aligned}$$

By the definition of $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}$ in (5.5.5), this restriction is given by the product of the exponential map $e(Z_\kappa, \mathcal{P})$ with the projection maps π_Q onto $\bar{M}_{\text{SPL}, |Q|}^{s, \natural}(\delta_Q)$ defined in (5.5.4) and is thus an open

embedding. The conclusion of the lemma then follows immediately from the definition of $\rho_{\mathcal{P},[\mathcal{P}']}^{\Theta,u}$ in terms of the maps $\rho_{\mathcal{P},\mathcal{P}'}^{\Theta,u}$. \square

The following lemma yields the existence of suitably small, $\mathrm{SO}(3) \times \mathrm{SO}(4)$ -invariant neighborhoods of $T(\Theta, \mathcal{P}, \delta)$. For any subspace V of $\bar{\mathcal{B}}_{\kappa}^s(S^4)$, let $\mathrm{cl}(V)$ denote the closure of V in the Uhlenbeck topology.

LEMMA 5.6.4. *If V is any set in $\bar{\mathcal{B}}_{\kappa}^s(S^4)$ satisfying*

$$\mathrm{cl}(V) \cap \gamma'_{\Theta, \mathcal{P}}(T(\Theta, \mathcal{P}, \delta)) = \emptyset,$$

then there are $\mathrm{SO}(3) \times \mathrm{SO}(4) \times \mathfrak{S}(\mathcal{P})$ -invariant neighborhoods,

$$\mathcal{O}_{f,3}^{\mathrm{ASD}}(\Theta, \mathcal{P}, \delta) \subset \mathcal{O}_{f,2}^{\mathrm{ASD}}(\Theta, \mathcal{P}, \delta) \subset \mathcal{O}_{f,1}^{\mathrm{ASD}}(\Theta, \mathcal{P}, \delta),$$

of $T(\Theta, \mathcal{P}, \delta)$ in $\mathcal{O}^{\mathrm{ASD}}(\Theta, \mathcal{P}, \delta)$ such that the intersection,

$$\mathrm{cl}(V) \cap \gamma'_{\Theta, \mathcal{P}}(\mathcal{O}_{f,1}^{\mathrm{ASD}}(\Theta, \mathcal{P}, \delta)),$$

is empty and there is an inclusion,

$$\mathrm{cl}(V) \cap \mathrm{cl}(\gamma'_{\Theta, \mathcal{P}}(\mathcal{O}_{f,1}^{\mathrm{ASD}}(\Theta, \mathcal{P}, \delta))) \subset \mathrm{cl}(V) \cap \mathrm{cl}(\gamma'_{\Theta, \mathcal{P}}(T(\Theta, \mathcal{P}, \delta))).$$

PROOF. For $\mathbf{A} \in \mathcal{O}^{\mathrm{ASD}}(\Theta, \mathcal{P}, \delta)$ given by $\mathbf{A} = ((x_P)_{P \in \mathcal{P}}, ([A_P, F_P^s, \mathbf{y}_P])_{P \in \mathcal{P}})$, denote $\lambda_P(\mathbf{A}) := \lambda(A_P, \mathbf{y}_P)$. For $\mathbf{x} := (x_P)_{P \in \mathcal{P}} \in \Delta^\circ(Z_\kappa, \mathcal{P})$, define

$$V(\mathbf{x}, \mathcal{P}) := \{(\mathbf{x}, ([A_P, F_P^s, \mathbf{y}_P])_{P \in \mathcal{P}}) \in (\gamma'_{\Theta, \mathcal{P}})^{-1}(\mathrm{cl}(V))\}$$

and define $\lambda_V : \Delta^\circ(Z_\kappa, \mathcal{P}) \rightarrow (0, 1]$ by

$$\lambda_V(\mathbf{x})^2 := \min \left\{ \min_{\mathbf{A} \in V(\mathbf{x}, \mathcal{P})} \sum_{P \in \mathcal{P}} \lambda_P(\mathbf{A})^2, 1 \right\}.$$

By the hypothesis that $\mathrm{cl}(V)$ is disjoint from $\gamma'_{\Theta, \mathcal{P}}(T(\Theta, \mathcal{P}, \delta))$ and the observation that $T(\Theta, \mathcal{P}, \delta)$ is the zero locus of $\sum_{P \in \mathcal{P}} \lambda_P(\cdot)$, we see that λ_V is always positive on $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$. Let $f : \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \rightarrow (0, \varepsilon)$ be a continuous function with

$$(5.6.6) \quad f((x_P)_{P \in \mathcal{P}}) < \frac{1}{2} \lambda_V((x_P)_{P \in \mathcal{P}})^2,$$

for all $(x_P)_{P \in \mathcal{P}} \in \Delta^\circ(Z_\kappa(\delta), \mathcal{P})$. Then for $j = 1, 2, 3$, we define the neighborhood $\mathcal{O}_{f,j}^{\mathrm{ASD}}(\Theta, \mathcal{P}, \delta)$ by

$$\mathcal{O}_{f,j}^{\mathrm{ASD}}(\Theta, \mathcal{P}, \delta) := \left\{ \mathbf{A} = (\mathbf{x}, ([A_P, F_P^s, \mathbf{y}_P])_{P \in \mathcal{P}}) \in \mathcal{O}^{\mathrm{ASD}}(\Theta, \mathcal{P}, \delta) : \sum_{P \in \mathcal{P}} \lambda_P(\mathbf{A})^2 < f(\mathbf{x})/j \right\}.$$

These sets have the desired invariance and their images do not intersect $\mathrm{cl}(V)$. The existence of a smoothly-stratified function g_j supported in $\mathcal{O}_{f,j}^{\mathrm{ASD}}(\Theta, \mathcal{P}, \delta)$ with $g_j(\mathcal{O}_{f,j+1}^{\mathrm{ASD}}(\Theta, \mathcal{P}, \delta)) = 1$ (needed to prove $\mathcal{O}_{f,j+1}^{\mathrm{ASD}}(\Theta, \mathcal{P}, \delta) \subset \mathcal{O}_{f,j}^{\mathrm{ASD}}(\Theta, \mathcal{P}, \delta)$) follows from the definition of these sets.

To prove the final inclusion, let $\{\mathbf{A}(\alpha)\}_{\alpha=1}^\infty \subset \mathcal{O}_{f,1}^{\mathrm{ASD}}(\Theta, \mathcal{P}, \delta)$ be a sequence with

$$(5.6.7) \quad \lim_{\alpha \rightarrow \infty} \gamma'_{\Theta, \mathcal{P}}(\mathbf{A}(\alpha)) \in \mathrm{cl}(V).$$

If

$$\mathbf{A}(\alpha) = [\mathbf{x}(\alpha), ([A_P(\alpha), F_P^s(\alpha), \mathbf{y}_P(\alpha)])_{P \in \mathcal{P}}],$$

where $\mathbf{x}(\alpha) \in \Delta^\circ(Z_\kappa(\delta), \mathcal{P})$, then (5.6.7) implies that

$$(5.6.8) \quad \lim_{\alpha \rightarrow \infty} \sum_{P \in \mathcal{P}} \lambda_P(\mathbf{A}(\alpha))^2 = \lim_{\alpha \rightarrow \infty} \lambda_V(\mathbf{x}(\alpha))^2.$$

Because $\mathbf{A}(\alpha) \in \mathcal{O}_{f,1}^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$, we have

$$(5.6.9) \quad \sum_{P \in \mathcal{P}} \lambda_P(\mathbf{A}(\alpha))^2 < \frac{1}{2} \lambda_V(\mathbf{x}(\alpha))^2.$$

Combining (5.6.8) and (5.6.9) yields

$$\lim_{\alpha \rightarrow \infty} \sum_{P \in \mathcal{P}} \lambda_P^2(\mathbf{A}(\alpha)) = 0,$$

and so for all $P \in \mathcal{P}$,

$$\lim_{\alpha \rightarrow \infty} [A_P(\alpha), F_P^s(\alpha), \mathbf{y}_P(\alpha)] = [\Theta, F_P^s(\infty), c_P],$$

and thus,

$$\lim_{\alpha \rightarrow \infty} \gamma'_{\Theta, \mathcal{P}}(\mathbf{A}(\alpha)) = \lim_{\alpha \rightarrow \infty} \gamma'_{\Theta, \mathcal{P}}(\mathbf{x}(\alpha), ([\Theta, F_P^s(\infty), c_P])_{P \in \mathcal{P}}) \in \text{cl}(\gamma'_{\Theta, \mathcal{P}}(T(\Theta, \mathcal{P}, \delta))),$$

as asserted. \square

We now construct the subspaces $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$ referred to in Proposition 5.6.1. We will construct these subspaces to be \mathfrak{S}_κ -invariant in the sense that if $\mathcal{P}'' \in [\mathcal{P}]$, then the open subspaces $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$ and $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}'', \delta)$ will be identified by the natural action of \mathfrak{S}_κ on $\bigsqcup_{\mathcal{P}_i \in [\mathcal{P}_1]} \nu(\Theta, \mathcal{P}_i)$. Thus, defining $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}', \delta)$ for one $\mathcal{P}' \in [\mathcal{P}']$ suffices to define the space

$$(5.6.10) \quad \mathcal{O}_1^{\text{ASD}}(\Theta, [\mathcal{P} < \mathcal{P}']) = \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}'', \delta) \right) / \mathfrak{S}(\mathcal{P}) \subset \mathcal{O}^{\text{ASD}}(\Theta, [\mathcal{P} < \mathcal{P}'], \delta),$$

where $\mathcal{O}^{\text{ASD}}(\Theta, [\mathcal{P} < \mathcal{P}'], \delta)$ is defined in (5.6.5).

LEMMA 5.6.5. *Assume that $\bar{M}_{\text{SPL}, \kappa'}^{\text{s}, \natural}(\delta)$ has been constructed so that it satisfies the properties enumerated in Theorem 5.1.1 for all $\kappa' < \kappa$. Then for every partition \mathcal{P} of N_κ of length greater than one, there are open neighborhoods*

$$\mathcal{O}_3^{\text{ASD}}(\Theta, \mathcal{P}, \delta) \sqsubset \mathcal{O}_2^{\text{ASD}}(\Theta, \mathcal{P}, \delta) \sqsubset \mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$$

of $T(\Theta, \mathcal{P}, \delta)$ in $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$ such that the following hold for $j = 1, 2, 3$.

- (1) The neighborhoods $\mathcal{O}_j^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$ are closed under the $\text{SO}(3) \times \text{SO}(4)$ action given in (5.4.6).
- (2) If the partitions \mathcal{P}_1 and \mathcal{P}_2 are conjugate under the action of \mathfrak{S}_κ , then the neighborhoods $\mathcal{O}_j^{\text{ASD}}(\Theta, \mathcal{P}_1, \delta)$ and $\mathcal{O}_j^{\text{ASD}}(\Theta, \mathcal{P}_2, \delta)$ are identified by the natural action of \mathfrak{S}_κ on $\bigsqcup_{\mathcal{P}_i \in [\mathcal{P}_1]} \nu(\Theta, \mathcal{P}_i)$.
- (3) If there are no partitions $\mathcal{P}_1 \in [\mathcal{P}]$ and $\mathcal{P}_2 \in [\mathcal{P}']$ with $\mathcal{P}_1 < \mathcal{P}_2$ or $\mathcal{P}_2 < \mathcal{P}_1$, then

$$\gamma'_{\Theta, \mathcal{P}}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, \delta)) \cap \gamma'_{\Theta, \mathcal{P}'}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}', \delta)) = \emptyset.$$

- (4) If $\mathcal{P} < \mathcal{P}'$ then there exists an open neighborhood $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ of the subspace $T(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ of $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ such that

$$\begin{aligned}
(5.6.11) \quad & \gamma'_{\Theta, \mathcal{P}}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, \delta)) \cap \gamma'_{\Theta, \mathcal{P}'}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}', \delta)) \\
& \subseteq \gamma'_{\Theta, \mathcal{P}'}\left(\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, u}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta))\right) \\
& = \gamma'_{\Theta, \mathcal{P}}\left(\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, d}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta))\right).
\end{aligned}$$

PROOF. Enumerate the strata $\Sigma(Z_\kappa(\delta), \mathcal{P})$ in the manner described in Section 3.4 with conjugacy classes $[\mathcal{P}_0], \dots, [\mathcal{P}_n]$ so that

$$(5.6.12) \quad \Sigma(Z_\kappa(\delta), \mathcal{P}_i) \subset \text{cl}(\Sigma(Z_\kappa(\delta), \mathcal{P}_j)) \quad \text{only if } i < j.$$

Because the length of the partition \mathcal{P}_0 is one, we begin our induction with \mathcal{P}_1 and using induction on k , construct neighborhoods $\mathcal{O}_j^{\text{ASD}}(\Theta, \mathcal{P}_k, \delta)$ satisfying items (1) and (2). In addition, these inductively constructed neighborhoods will, for all $i < k$, satisfy Items (3) and (4) with $[\mathcal{P}_i] = [\mathcal{P}]$ and $[\mathcal{P}_k] = [\mathcal{P}']$. We note that the symmetry between \mathcal{P} and \mathcal{P}' in Items (3) and (4) implies that this will suffice to complete the induction and prove the lemma.

In each step of the induction, to ensure this construction satisfies Item (2), it suffices to construct $\mathcal{O}_j^{\text{ASD}}(\Theta, \mathcal{P}_k, \delta)$ for a single partition $\mathcal{P}' \in [\mathcal{P}_k]$ and for any other partition, $\mathcal{P}'' \in [\mathcal{P}_k]$, define $\mathcal{O}_j^{\text{ASD}}(\Theta, \mathcal{P}'', \delta)$ to be the image of $\mathcal{O}_j^{\text{ASD}}(\Theta, \mathcal{P}', \delta)$ under the action of \mathfrak{S}_κ .

For $k = 1$, take $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_1, \delta) = \mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}_1, \delta)$. The neighborhoods $\mathcal{O}_j^{\text{ASD}}(\Theta, \mathcal{P}_1, \delta)$ for $j = 2, 3$ can be constructed by replacing the coefficient 8 in the definition (5.4.1) with slightly larger coefficients. Item (2) holds from the definition of $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}_1, \delta)$ while Items (3) and (4) hold trivially.

By induction, assume that the open sets $\mathcal{O}_i^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta)$ satisfying Items (1)-(4) have been defined for all $i < k$; we will complete the induction by constructing $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_k, \delta)$. Note that the conclusion still holds if we shrink the sets $\mathcal{O}_i^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta)$ to smaller neighborhoods of $T(\Theta, \mathcal{P}_i, \delta)$. We now construct $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_k, \delta)$.

For each $i < k$, if there are no $\mathcal{P}' \in [\mathcal{P}_k]$ and $\mathcal{P} \in [\mathcal{P}_i]$ with $\mathcal{P} < \mathcal{P}'$, then by (5.4.8), (5.6.12), and Lemma 3.2.4,

$$(5.6.13) \quad \text{cl}(\gamma'_{\Theta, \mathcal{P}_i}(T(\Theta, \mathcal{P}_i, \delta))) \cap \gamma'_{\Theta, \mathcal{P}_k}(T(\Theta, \mathcal{P}_k, \delta)) = \emptyset,$$

$$(5.6.14) \quad \gamma'_{\Theta, \mathcal{P}_i}(T(\Theta, \mathcal{P}_i, \delta)) \cap \text{cl}(\gamma'_{\Theta, \mathcal{P}_k}(T(\Theta, \mathcal{P}_k, \delta))) = \emptyset.$$

Lemma 5.6.4 and (5.6.13) imply that there are $\text{SO}(3) \times \text{SO}(4)$ -invariant neighborhoods,

$$\mathcal{O}_3^{\text{ASD}}(\Theta, \mathcal{P}_k, \delta) \sqsubset \mathcal{O}_2^{\text{ASD}}(\Theta, \mathcal{P}_k, \delta) \sqsubset \mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_k, \delta),$$

of $T(\Theta, \mathcal{P}_k, \delta)$ such that

$$\text{cl}(\gamma'_{\Theta, \mathcal{P}_i}(T(\Theta, \mathcal{P}_i, \delta))) \cap \gamma'_{\Theta, \mathcal{P}_k}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_k, \delta)) = \emptyset.$$

Lemma 5.6.4 also ensures that

$$\begin{aligned}
(5.6.15) \quad & \text{cl}(\gamma'_{\Theta, \mathcal{P}_i}(T(\Theta, \mathcal{P}_i, \delta))) \cap \text{cl}(\gamma'_{\Theta, \mathcal{P}_k}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_k, \delta))) \\
& \subseteq \text{cl}(\gamma'_{\Theta, \mathcal{P}_i}(T(\Theta, \mathcal{P}_i, \delta))) \cap \text{cl}(\gamma'_{\Theta, \mathcal{P}_k}(T(\Theta, \mathcal{P}_k, \delta))).
\end{aligned}$$

Equations (5.6.15) and (5.6.14) yield

$$\begin{aligned} & \gamma'_{\Theta, \mathcal{P}_i}(T(\Theta, \mathcal{P}_i, \delta)) \cap \text{cl}(\gamma'_{\Theta, \mathcal{P}_k}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_k, \delta))) \\ & \subset \gamma'_{\Theta, \mathcal{P}_i}(T(\Theta, \mathcal{P}_i, \delta)) \cap \text{cl}(\gamma'_{\Theta, \mathcal{P}_k}(T(\Theta, \mathcal{P}_k, \delta))) = \emptyset. \end{aligned}$$

Thus, by Lemma 5.6.4 there are (smaller) $\text{SO}(3) \times \text{SO}(4)$ -invariant neighborhoods $\mathcal{O}_{j,2}^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta) \subset \mathcal{O}_j^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta)$ of $T(\Theta, \mathcal{P}_i, \delta)$ with $j = 1, 2, 3$ satisfying

$$\mathcal{O}_{3,2}^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta) \sqsubset \mathcal{O}_{2,2}^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta) \sqsubset \mathcal{O}_{1,2}^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta)$$

and

$$\gamma'_{\Theta, \mathcal{P}_i}(\mathcal{O}_{1,2}^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta)) \cap \text{cl}(\gamma'_{\Theta, \mathcal{P}_k}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_k, \delta))) = \emptyset,$$

as desired. Replace the open neighborhoods $\mathcal{O}_j^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta)$ with $\mathcal{O}_{j,2}^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta)$. For all other $\mathcal{P}'' \in [\mathcal{P}_i]$, replace $\mathcal{O}_j^{\text{ASD}}(\Theta, \mathcal{P}'', \delta)$ with the image of $\mathcal{O}_{j,2}^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta)$ under the action of \mathfrak{S}_κ . Then $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_k, \delta)$ satisfies Items (3) and (4) for \mathcal{P}_i with $i < k$ and no $\mathcal{P}' \in [\mathcal{P}_k]$ and $\mathcal{P} \in [\mathcal{P}_i]$ with $\mathcal{P} < \mathcal{P}'$.

If $[\mathcal{P}_i] < \mathcal{P}_k$, by conjugating \mathcal{P}_i if necessary and using the invariance of the neighborhoods $\mathcal{O}_j^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta)$ under the action of \mathfrak{S}_κ , we can assume that $\mathcal{P}_i < \mathcal{P}_k$. By the equality (5.4.8), the closure of

$$[\Theta] \times (\Sigma(Z_\kappa(\delta), \mathcal{P}_k) \setminus \text{Im}(e(Z_\kappa, \mathcal{P}_i))),$$

(where $e(Z_\kappa, \mathcal{P}_i)$ is the exponential map defined in equation (4.2.10)) does not intersect the image $\gamma'_{\Theta, \mathcal{P}_i}(T(\Theta, \mathcal{P}_i, \delta))$. Therefore, by applying Lemma 5.6.4 and shrinking $\mathcal{O}_j^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta)$, for $j = 1, 2, 3$, as above, we can assume that

$$\text{cl}(\gamma'_{\Theta, \mathcal{P}_i}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta))) \cap ([\Theta] \times (\Sigma(Z_\kappa, \mathcal{P}'') \setminus \text{Im}(e(Z_\kappa, \mathcal{P})))) = \emptyset$$

for all $\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_k]$ and hence

$$(5.6.16) \quad \text{cl}(\gamma'_{\Theta, \mathcal{P}}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta))) \cap \left(\left(\bigcup_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_k]} \gamma'_{\Theta, \mathcal{P}''}(T(\Theta, \mathcal{P}'', \delta)) \right) / \mathfrak{S}(\mathcal{P}) \right) \\ \sqsubset [\Theta] \times e(Z_\kappa(\delta), \mathcal{P}_i)(\mathcal{O}(Z_\kappa(\delta), [\mathcal{P}_i < \mathcal{P}_k])),$$

where $\mathcal{O}(Z_\kappa(\delta), [\mathcal{P}_i < \mathcal{P}_k])$ is defined in Lemma 5.3.4.

Let the spaces $T(\Theta, \mathcal{P}_i, [\mathcal{P}_k], \delta)$ and $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_i, [\mathcal{P}_k], \delta)$ be as defined in (5.5.23). The equality and inclusion,

$$\begin{aligned} & [\Theta] \times e(Z_\kappa, \mathcal{P}_i)(\mathcal{O}(Z_\kappa(\delta), [\mathcal{P}_i < \mathcal{P}_k])) \\ & = \gamma'_{\Theta, [\mathcal{P}_i < \mathcal{P}_k]}(\rho_{\mathcal{P}_i, [\mathcal{P}_k]}^{\Theta, u}(T(\Theta, \mathcal{P}_i, [\mathcal{P}_k], \delta))) \\ & \subset \gamma'_{\Theta, [\mathcal{P}_i < \mathcal{P}_k]}(\rho_{\mathcal{P}_i, [\mathcal{P}_k]}^{\Theta, u}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_i, [\mathcal{P}_k], \delta))), \end{aligned}$$

together with (5.6.16) then imply that

$$(5.6.17) \quad \text{cl}(\gamma'_{\Theta, \mathcal{P}_i}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta))) \setminus \gamma'_{\Theta, [\mathcal{P}_i < \mathcal{P}_k]}(\rho_{\mathcal{P}_i, [\mathcal{P}_k]}^{\Theta, u}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_i, [\mathcal{P}_k], \delta)))$$

is disjoint from $\gamma'_{\Theta, \mathcal{P}''}(T(\Theta, \mathcal{P}'', \delta))$ for all $\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_k]$. Lemma 5.6.4 gives $\text{SO}(3) \times \text{SO}(4) \times \mathfrak{S}(\mathcal{P})$ -invariant neighborhoods

$$\mathcal{O}_3^{\text{ASD}}(\Theta, [\mathcal{P}_i < \mathcal{P}_k], \delta) \sqsubset \mathcal{O}_2^{\text{ASD}}(\Theta, [\mathcal{P}_i < \mathcal{P}_k], \delta) \sqsubset \mathcal{O}_1^{\text{ASD}}(\Theta, [\mathcal{P}_i < \mathcal{P}_k], \delta),$$

of

$$\left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_k]} T(\Theta, \mathcal{P}'', \delta) \right) / \mathfrak{S}(\mathcal{P}_i) \quad \text{in} \quad \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_k]} \mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}'', \delta) \right) / \mathfrak{S}(\mathcal{P}_i)$$

whose image under $\gamma'_{\Theta, [\mathcal{P}_i < \mathcal{P}_k]}$ is disjoint from the set (5.6.17). This disjointness yields the inclusion,

$$\begin{aligned} & \gamma'_{\Theta, \mathcal{P}_i}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_i, \delta)) \cap \gamma'_{\Theta, \mathcal{P}''}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}'', \delta)) \\ & \subset \gamma'_{\Theta, [\mathcal{P}_i < \mathcal{P}_k]} \left(\rho_{\mathcal{P}_i, [\mathcal{P}_k]}^{\Theta, u}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}_i, [\mathcal{P}_k], \delta)) \right), \end{aligned}$$

for all $\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_k]$, as required to show that Items (3) and (4) hold. This completes the induction and hence the proof of the lemma. \square

PROOF OF PROPOSITION 5.6.1. For each partition \mathcal{P} of N_κ with length greater than one, let $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$ be the $\text{SO}(3) \times \text{SO}(4)$ invariant subspace of $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$ constructed in Lemma 5.6.5. For any two partitions \mathcal{P} and \mathcal{P}' , the images $\gamma'_{\Theta, \mathcal{P}}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, \delta))$ and $\gamma'_{\Theta, \mathcal{P}'}(\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}', \delta))$ are either disjoint or are given by the open subspaces described in Item (4) in Lemma 5.6.5. Hence, the overlaps are open subspaces and the union of the images, W_κ , has the desired properties.

The arguments in [21, Proposition 5.10] imply that if A' is defined by splicing connections A_P with scales δ_P for $P \in \mathcal{P}$ onto the product connection and if $\delta_P \leq \delta$, then

$$\|F_{A'}^+\|_{L^{\sharp, 2}(S^4)} \leq \delta + \sum_{P \in \mathcal{P}} \|F_{A_P}^+\|_{L^{\sharp, 2}(S^4)}.$$

The bound on $\|F_{A'}^+\|_{L^{\sharp, 2}(S^4)}$ for all $A' \in W_\kappa$ then follows by induction. \square

5.7. Tubular neighborhoods of the instanton moduli space with spliced ends

We now introduce the partial tubular neighborhood structure of the spliced end W_κ defined in (5.6.2). This structure will satisfy the first of the conditions in (4.1.1). In the following lemma, we introduce the projection map, analogous to the map π_i in (4.1.1).

LEMMA 5.7.1. *For each partition \mathcal{P} of N_κ with $|\mathcal{P}| > 1$, let $\mathcal{U}(\Theta, \mathcal{P}) \subset W_\kappa$ be the image of the open subspace $\mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$ defined in Lemma 5.6.5. Then there is an $\text{SO}(3) \times \text{SO}(4)$ -equivariant map,*

$$(5.7.1) \quad \pi(\Theta, \mathcal{P}) : \mathcal{U}(\Theta, \mathcal{P}) \rightarrow [\Theta] \times \Sigma(Z_\kappa(\delta), \mathcal{P}),$$

such that the following hold.

- (1) If $\mathcal{P}' \in [\mathcal{P}]$, then $\mathcal{U}(\Theta, \mathcal{P}) = \mathcal{U}(\Theta, \mathcal{P}')$, and $\Sigma(Z_\kappa(\delta), \mathcal{P}) = \Sigma(Z_\kappa(\delta), \mathcal{P}')$, and $\pi(\Theta, \mathcal{P}) = \pi(\Theta, \mathcal{P}')$.
- (2) On the overlap $\mathcal{U}(\Theta, \mathcal{P}) \cap \mathcal{U}(\Theta, \mathcal{P}')$, we have

$$(5.7.2) \quad \pi(\Theta, \mathcal{P}) \circ \pi(\Theta, \mathcal{P}') = \pi(\Theta, \mathcal{P}).$$

PROOF. The map $\pi(\Theta, \mathcal{P})$ is defined to be the composition of the projection,

$$\pi_{Z, \mathcal{P}} : \mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}, \delta) \rightarrow \Sigma(Z_\kappa(\delta), \mathcal{P}),$$

with the inverse of the splicing map, $\gamma'_{\Theta, \mathcal{P}}$. Item (1) follows from this definition.

The equality (5.7.2) follows from the assertion that the intersection of $\mathcal{U}(\Theta, \mathcal{P}) \cap \mathcal{U}(\Theta, \mathcal{P}')$ with a fiber of $\pi(\Theta, \mathcal{P}')$ is contained in a fiber of $\pi(\Theta, \mathcal{P})$. To prove this, first note that by (5.6.11),

$$\mathcal{U}(\Theta, \mathcal{P}) \cap \mathcal{U}(\Theta, \mathcal{P}') = \text{Im} \left(\gamma'_{\Theta, \mathcal{P}} \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, d} \right) = \text{Im} \left(\gamma'_{\Theta, \mathcal{P}'} \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, u} \right).$$

Consider the following diagram,

$$\begin{array}{ccc} \tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta) & \xrightarrow{\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, u}} & \mathcal{O}_1^{\text{ASD}}(\Theta, \mathcal{P}', \delta) \\ \pi_{\mathbf{x}} \downarrow & & \pi_{Z, \mathcal{P}'} \downarrow \\ \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times \prod_{P \in \mathcal{P}} \Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}'_P) & \xrightarrow{e(Z_\kappa(\delta), \mathcal{P})} & \Delta(Z_\kappa(\delta), \mathcal{P}') \\ p \downarrow & & \\ \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) & & \end{array}$$

where the map $\pi_{\mathbf{x}}$ is defined in (5.5.4) and the map p is the obvious projection. Note that $p \circ \pi_{\mathbf{x}} = \pi_{\mathcal{P}, \mathbf{x}}$, where $\pi_{\mathcal{P}, \mathbf{x}}$ is defined in (5.5.4). The diagram commutes by the definition of $\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, u}$ in (5.5.5). By the commutativity of this diagram and the injectivity of $e(Z_\kappa(\delta), \mathcal{P})$, for $\mathbf{y} \in \Delta(Z_\kappa(\delta), \mathcal{P}')$, and $\mathbf{x} \in \Delta^\circ(Z_\kappa(\delta), \mathcal{P})$, and $\tilde{\mathbf{x}} \in p^{-1}(\mathbf{x})$ with $e(Z_\kappa(\delta), \mathcal{P})(\tilde{\mathbf{x}}) = \mathbf{y}$,

$$(5.7.3) \quad \text{Im} \left(\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, u} \right) \cap \pi_{Z, \mathcal{P}'}^{-1}(\mathbf{y}) = \rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, u} \left(\pi_{\mathbf{x}}^{-1}(\tilde{\mathbf{x}}) \right) \subset \rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, u} \left((p \circ \pi_{\mathbf{x}})^{-1}(\mathbf{x}) \right).$$

Next, we observe that by the definition of $\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, d}$ in (5.5.8), for $\mathbf{x} \in \Delta^\circ(Z_\kappa(\delta), \mathcal{P})$,

$$(5.7.4) \quad \text{Im} \left(\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, d} \right) \cap \pi_{Z, \mathcal{P}}^{-1}(\mathbf{x}) = \rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, d} \left((p \circ \pi_{\mathbf{x}})^{-1}(\mathbf{x}) \right).$$

Combining the inclusion (5.7.3) with the equalities (5.7.4) and (5.5.27) yields

$$\gamma'_{\Theta, \mathcal{P}'} \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, u} \left(\pi_{\mathbf{x}}^{-1}(\tilde{\mathbf{x}}) \right) \subset \gamma'_{\Theta, \mathcal{P}'} \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, u} \left((p \circ \pi_{\mathbf{x}})^{-1}(\mathbf{x}) \right) = \gamma'_{\Theta, \mathcal{P}} \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, d} \left((\pi_{Z, \mathcal{P}})^{-1}(\mathbf{x}) \right),$$

which gives the inclusion of fibers required to prove (5.7.2). \square

The obvious identification,

$$(5.7.5) \quad \{[\Theta]\} \times Z_\kappa / \mathfrak{S}_\kappa \cong Z_\kappa / \mathfrak{S}_\kappa,$$

allows us to view the projection maps $\pi(Z_\kappa, \mathcal{P})$ defined in Section 4.2 as maps on the strata of W_κ containing the product connection. We identify the restriction of the projection $\pi(\Theta, \mathcal{P})$ with such a map in the following lemma.

LEMMA 5.7.2. *Under the identification (5.7.5), the restriction of the projection map $\pi(\Theta, \mathcal{P})$ in Lemma 5.7.1 to the intersection,*

$$\left(\{[\Theta]\} \times Z_\kappa / \mathfrak{S}_\kappa \right) \cap \mathcal{U}(\Theta, \mathcal{P}),$$

is equal to the map

$$\pi(Z_\kappa, \mathcal{P}) : U(Z_\kappa, \mathcal{P}) / \mathfrak{S}(\mathcal{P}) \rightarrow \Delta^\circ(Z_\kappa, \mathcal{P}) = \Sigma(Z_\kappa, \mathcal{P}) / \mathfrak{S}(\mathcal{P})$$

defined in (4.2.13).

PROOF. The conclusion follows by noting that the restriction of the splicing map $\gamma'_{\Theta, \mathcal{P}}$ to points where the connections being spliced in are the product connection is exactly the map $e(Z_\kappa, \mathcal{P})$ in (4.2.10). \square

5.8. Isotopy of the spliced end of the instanton moduli space

The final step in the proof of Theorem 5.1.1 is to use the isotopy provided by the gluing and centering maps to attach the spliced end, W_κ , to the original moduli space $\bar{M}_\kappa^{s, \natural}(S^4, \delta)$.

We first construct the isotopy of W_κ in $\bar{\mathcal{B}}_\kappa^s(S^4, \delta)$.

LEMMA 5.8.1. *For δ sufficiently small, there is a continuous, smoothly-stratified map,*

$$(5.8.1) \quad R : (-\infty, 1] \times W_\kappa \rightarrow \bar{\mathcal{B}}_\kappa^s(S^4, 2\delta),$$

such that the following hold.

- (1) For all $t \in (-\infty, \frac{1}{2}]$ and $[A, F^s, \mathbf{x}] \in W_\kappa$, we have $R(t, [A, F^s, \mathbf{x}]) = [A, F^s, \mathbf{x}]$.
- (2) For all $t \in [\frac{3}{4}, 1]$ and $[A, F^s, \mathbf{x}] \in W_\kappa$, we have $R(t, [A, F^s, \mathbf{x}]) \in \bar{M}_\kappa^s(\delta)$.
- (3) If we define

$$\|[A, F^s, \mathbf{x}]\|_{L^{\sharp, 2}(S^4)} := \|F^+(A)\|_{L^{\sharp, 2}(S^4)}$$

then for all $t \in (-\infty, 1]$ and $[A, F^s, \mathbf{x}] \in W_\kappa$,

$$\|R(t, [A, F^s, \mathbf{x}])\|_{L^{\sharp, 2}(S^4)} \leq \|[A, F^s, \mathbf{x}]\|_{L^{\sharp, 2}(S^4)}.$$

- (4) For all $[A, F^s, \mathbf{x}] \in W_\kappa$, we have $R(1, [A, F^s, \mathbf{x}]) \in \bar{M}_\kappa^{s, \natural}(\delta)$.
- (5) For all $t \in (-\infty, 1]$, the map $R(t, \cdot) : W_\kappa \rightarrow \bar{\mathcal{B}}_\kappa^s(S^4, \delta)$ is $\mathrm{SO}(3) \times \mathrm{SO}(4)$ -equivariant.

PROOF. The bound on the $L^{\sharp, 2}(S^4)$ -norm of the F_A^+ for all $[A, \mathbf{x}] \in W_\kappa$ appearing in Proposition 5.6.1 and [21, Proposition 7.6] imply that for δ small enough to satisfy [21, Proposition 7.6], there is a continuous, smoothly-stratified, $\mathrm{SO}(3) \times \mathrm{SO}(4)$ -equivariant embedding,

$$(5.8.2) \quad G : [0, 1] \times W_\kappa \ni (t, [A, F^s, \mathbf{x}]) \mapsto [A + a_t(A), F^s, \mathbf{x}] \in \bar{\mathcal{B}}_\kappa^{s, \natural}(S^4, 2\delta),$$

where $a_t(A) = d_A^{+, *}(v_t(A))$ satisfies

$$(5.8.3) \quad F^+(A + a_t(A)) = (1 - t)F_A^+.$$

Observe that $a_0(A) = 0$ while $A \mapsto A + a_1(A)$ is the gluing map defined by [21, Proposition 7.6]. For $t \in [\frac{1}{2}, \frac{3}{4}]$, we define

$$(5.8.4) \quad R(t, [A, F^s, \mathbf{x}]) := [A + a_{4(t-1/2)}(A), F^s, \mathbf{x}].$$

This smoothly-stratified isotopy satisfied Items (1) and (2). Although $R(3/4, [A, F^s, \mathbf{x}])$ will be anti-self-dual, it need not be mass-centered. Thus, define

$$w(A, F^s, \mathbf{x}) := z[A + a_1(A), F^s, \mathbf{x}].$$

Because, by [16, Lemma 4.4] and the definition (5.2.2),

$$z[c_{y,1}^* A, F^s, c_{y,1}^{-1}(\mathbf{x})] = z[A, F^s, \mathbf{x}] + y,$$

if we define

$$R(t, [A, F^s, \mathbf{x}]) := [c_{(3-4t)w(A, F^s, \mathbf{x}), 1}^* A, F^s, c_{(3-4t)w(A, F^s, \mathbf{x}), 1}^{-1}(\mathbf{x})],$$

for $t \in [3/4, 1]$, then $R(3/4, [A, F^s, \mathbf{x}]) = [A, F^s, \mathbf{x}]$ and

$$z(R(1, [A, F^s, \mathbf{x}])) = 0.$$

Consequently, the isotopy R satisfies Item (4).

Item (3) follows for $t \in [1/2, 3/4]$ by observing that equation (5.8.3) implies that

$$\|R(t, [A, F^s, \mathbf{x}])\|_{L^{\sharp, 2}(S^4)} = \|F^+(A + a_t(A))\|_{L^{\sharp, 2}(S^4)} = (1-t)\|F^+(A)\|_{L^{\sharp, 2}(S^4)}, \quad \text{for } t \in [1/2, 3/4].$$

For $t \in [3/4, 1]$, Item (3) follows by the invariance of the $L^{\sharp, 2}(S^4)$ norm under pullback by translations.

We now prove that Item (5) holds. The $\text{SO}(3)$ equivariance of R is immediate. For $S \in \text{SO}(4)$, the $\text{SO}(4)$ equivariance of (5.8.3) yields

$$\begin{aligned} F^+(S^*A + S^*a_t(A)) &= S^*F^+(A + a_t(A)) \\ &= (1-t)S^*F^+(A) \\ &= (1-t)F^+(S^*A) \\ &= F^+(S^*A + a_t(S^*A)). \end{aligned}$$

The uniqueness of the solution $a_t(A)$ to (5.8.3) then implies that $S^*a_t(A) = a_t(S^*A)$. This gives the $\text{SO}(4)$ equivariance of R for $t \in [1/2, 3/4]$. To prove the equivariance for $t \in [3/4, 1]$, we first note that

$$z[S^*A, \tilde{S}^{-1}F^s, S^{-1}\mathbf{x}] = S^{-1}z[A, F^s, \mathbf{x}],$$

and so, by the $\text{SO}(4)$ equivariance of $A \mapsto a_1(A)$,

$$(5.8.5) \quad w(S^*A, \tilde{S}^{-1}F^s, S^{-1}\mathbf{x}) = S^{-1}w(A, F^s, \mathbf{x}).$$

The equality $S \circ c_{S^{-1}w, 1} = c_{w, 1} \circ S$ implies that $c_{S^{-1}w, 1}^*S^* = S^*c_{w, 1}^*$. Combining this with (5.8.5) in the definition of $R(t, \cdot)$ for $3/4 \leq t \leq 1$ completes the proof of the $\text{SO}(4)$ equivariance of R and hence the proof of Item (5). \square

PROOF OF THEOREM 5.1.1. Let

$$\mathcal{U}_3 \sqsubset \mathcal{U}_2 \sqsubset W_\kappa$$

be the union, over partitions \mathcal{P} of N_κ of length greater than one, of the neighborhoods,

$$(5.8.6) \quad \gamma'_{\Theta, \mathcal{P}}(\mathcal{O}_j^{\text{ASD}}(\Theta, \mathcal{P}, \delta)),$$

constructed in Lemma 5.6.5, where W_κ is defined in (5.6.2). These sets are closed under the actions of $\text{SO}(3)$ and $\text{SO}(4)$ and form a neighborhood of $[\Theta] \times (Z_P - \{c_P\})/\mathfrak{S}_P$ in W_κ . Then, from the proof of [82, Theorem 1.3.13], there is a smoothly-stratified map¹ $\beta : W_\kappa \rightarrow [0, 1]$ with $\mathcal{U}_3 \subset \beta^{-1}(0)$ and $W_\kappa \setminus \mathcal{U}_2 \subset \beta^{-1}(1)$.

Define $R(W_\kappa)$ to be the image of W_κ under the map $R(\beta(\cdot), \cdot)$:

$$R(W_\kappa) := \{R(\beta([A, F^s, \mathbf{x}]), [A, F^s, \mathbf{x}]) : [A, F^s, \mathbf{x}] \in W_\kappa\}.$$

Define the instanton moduli space with spliced ends by

$$(5.8.7) \quad \bar{M}_{\text{SPL}, \kappa}^{s, \natural}(S^4, \delta) := \left(\bar{M}_\kappa^{s, \natural}(S^4, \delta) \setminus R(1, W_\kappa) \right) \cup R(W_\kappa).$$

The $\text{SO}(3) \times \text{SO}(4)$ equivariance of R , provided by Item (3), implies that the space (5.8.7) is closed under the $\text{SO}(3) \times \text{SO}(4)$ action. The isotopy R defines the $\text{SO}(3) \times \text{SO}(4)$ -equivariant homeomorphism in Item (1) of the theorem. The neighborhood W_κ appearing in Item (2) is given by $R(W_\kappa)$. The Uhlenbeck neighborhoods $W(\Sigma)$ appearing in Item (3) are given by the sets (5.8.6)

¹A map that is continuous and whose restriction to each stratum is smooth.

with $j = 3$. By the construction of the solution $a_t(A)$ in (5.8.3), the $\|\cdot\|_{L^{\sharp,2}(S^4)}$ -norm of the curvature decreases along the isotopy $R(t, \cdot)$ as t increases. The bounds appearing in (5.6.1) on this norm then imply that the space (5.8.7) satisfies Item (4). \square

5.9. Properties of the instanton moduli space with spliced ends

We now prove two properties of the instanton moduli space with spliced ends, $\bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta)$, defined in (5.8.7) which will be used in the Definition 8.1.3 of the link $\bar{\mathbf{L}}_{t,s}$ of the lower-level reducible $\text{SO}(3)$ monopoles (1.1.3) and of a fundamental class for $\bar{\mathbf{L}}_{t,s}$. To construct the link, we will require the following perturbation of the scale function that is constant along the fibers of the projections $\pi(\Theta, \mathcal{P})$.

LEMMA 5.9.1. *There are a continuous, $\text{SO}(3) \times \text{SO}(4)$ invariant map,*

$$(5.9.1) \quad \tilde{\lambda}_\kappa : \bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta) \rightarrow [0, 2\delta],$$

which is smooth on each stratum, and neighborhoods $\mathcal{U}'(\Theta, \mathcal{P}) \sqsubset \mathcal{U}(\Theta, \mathcal{P})$ of $[\Theta] \times \Sigma(Z_\kappa(\delta), \mathcal{P})$ with the following properties.

- (1) *The function $\tilde{\lambda}_\kappa$ is equal to the scale function λ defined in (5.2.3) on the complement of W_κ .*
- (2) *Under the identification (5.7.5), the restriction of $\tilde{\lambda}_\kappa$ to the strata containing the product connection in (5.7.5) is equal to the function $t(Z_\kappa)$ defined in Lemma 4.2.5.*
- (3) *For all partitions \mathcal{P} of N_κ with $|\mathcal{P}| > 1$, we have $\tilde{\lambda}_\kappa = \tilde{\lambda}_\kappa \circ \pi(\Theta, \mathcal{P})$ on $\mathcal{U}'(\Theta, \mathcal{P})$.*

PROOF. The function $\tilde{\lambda}_\kappa$ is constructed in a manner identical to that of $t(Z_P)$ in Lemma 4.2.5 with the scale function λ defined in (5.2.3) playing the roles of $t_{P,0}$ and the neighborhoods $\mathcal{U}(\Theta, \mathcal{P})$ and maps $\pi(\Theta, \mathcal{P})$ playing the role of $\mathcal{U}(Z_P, \mathcal{P})$ and $\pi(Z_P, \mathcal{P})$. \square

We will use the following lemma when constructing a fundamental class for the link of the strata (1.1.3) of gauge-equivalence classes of reducible ideal $\text{SO}(3)$ monopoles.

LEMMA 5.9.2. *The stratified-space $\bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta)$ is a Whitney-stratified space in the sense of [39, Section 1.2]. If $\Sigma \subset \bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta)$ is the complement of the top stratum, then there is a neighborhood U of Σ in $\bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta)$ of which Σ is a retraction.*

PROOF. The smoothly-stratified homeomorphism (5.1.4) between $\bar{M}_{\text{SPL},\kappa}^{s,\natural}(S^4, \delta)$ and $\bar{M}_\kappa^{s,\natural}(S^4, \delta)$ preserves strata. Hence, the lemma will follow by proving that the pair $(\bar{M}_\kappa^{s,\natural}(S^4, \delta), \Sigma')$ has the required properties, where Σ' is the complement of the top stratum of $\bar{M}_\kappa^{s,\natural}(S^4, \delta)$. The space $\bar{M}_\kappa^{s,\natural}(S^4, \delta)$ is identified with a subspace of a completion of the ADHM data in [63, pp. 470–471] or [15, Corollary 3.4.10]. In [60, Theorem 1.2], it is shown that this completion of the ADHM data is a semi-algebraic space (that is, defined by a finite set of polynomial inequalities on a Euclidean space) and hence Whitney-stratified (see [39, Section 1.2]). The same is true for the unframed space, $\bar{M}_\kappa^\natural(S^4, \delta)$. In [40, Proposition 5], Goresky proves that Whitney-stratified spaces can be given the structure of a simplicial complex in such a way that the complement of the top stratum is a subcomplex. The existence of the neighborhood U then follows from [92, Corollary 3.3.11]. \square

CHAPTER 6

The space of global splicing data

6.1. Introduction

Let \mathfrak{t} be a spin^u structure and \mathfrak{s} a spin^c structure such that the intersection

$$(6.1.1) \quad \left(M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X) \right) \cap \bar{\mathcal{M}}_{\mathfrak{t}}/S^1,$$

is non-empty, where $M_{\mathfrak{s}}$ is the subspace of reducible $\text{SO}(3)$ monopoles appearing in (1.1.3). In this chapter we construct the space of global splicing data discussed in Section 1.2.5 and which will be used to construct the link of the subspace (6.1.1). This space will be the union of spaces of splicing data, $\bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$, associated with diagonals $\Sigma(X^{\ell}, \mathcal{P})$ defined in (3.1.8) of $\text{Sym}^{\ell}(X)$ as \mathcal{P} varies over partitions of $N_{\ell} = \{1, \dots, \ell\}$. (Of course, we will do this equivariantly with respect to the symmetric group.) To each partition \mathcal{P} , we will define a *crude splicing map* which will be identical to the splicing map defined in [24, Equation (3.27)] except in the following respects:

- (1) The connections over S^4 being spliced are elements of the instanton moduli space with spliced ends defined in Theorem 5.1.1 rather than the moduli space of anti-self-dual connections on S^4 ; and
- (2) Each background pair, (A_0, Φ_0) , is ‘flattened’ in the sense that the connection A_0 is replaced with one which is flat while the section Φ_0 is multiplied by a cut-off function that is zero on balls of fixed radius around the splicing point rather than on balls whose radius tends to zero as the connection on S^4 becomes more concentrated.

The space of global splicing data is the union, over partitions of N_{ℓ} , of the images of the crude splicing maps. To ensure that this union is a smoothly-stratified space, we require the images of the crude splicing maps for different partitions to overlap in open subsets, as in the construction of the spliced end of the instanton moduli space, W_{κ} , in Section 5.6. In Proposition 6.5.3, we prove an analogue of Proposition 5.5.3, the result describing the overlaps of images of different splicing maps when splicing connections on S^4 with the product connection on S^4 . However, the proof of Proposition 5.5.3 relied on the flatness of the Riemannian metric on the manifold and of the product connection. We will use the method from Section 4.5 to produce the needed flatness of the metric on manifold. The condition (2) above on the crude splicing map gives us the flatness of the connection needed to prove Proposition 6.5.3.

After defining the domain of the splicing map in Section 6.2, we show how to flatten pairs as described above in Section 6.3 and thus define the crude splicing maps in Section 6.4. Given the analysis of the overlaps of the images of the crude splicing maps developed in Section 6.5, we construct the space of global splicing data in Section 6.6. In Section 6.7, we construct the partial Thom–Mather structure (as described in Section 4.1) which will be used in the construction of the link $\bar{\mathbf{L}}_{\mathfrak{t}, \mathfrak{s}}$ of the subspace (6.1.1). In Section 6.8, we describe a global splicing map whose image will

contain the subspace (6.1.1). Finally, in Section 6.9, we construct a map from the space of global splicing data to $\text{Sym}^\ell(X)$.

6.2. Splicing data

We begin by recalling the domains of the splicing maps from [24]. Assume that $M_{\mathfrak{s}} \times \text{Sym}^\ell(X)$ appears in the ℓ -th level of the space of ideal $\text{SO}(3)$ monopoles on \mathfrak{t} , namely $I\mathcal{M}_{\mathfrak{t}}$. Let $\mathfrak{t}(\ell)$ be the spin^u structure defined prior to equation (2.1.14). By generalizing the splicing map defined in (5.4.3), we will obtain a splicing map that will attach connections on S^4 to the ‘background pair’ $(A_0, \Phi_0) \in \tilde{\mathcal{C}}_{\mathfrak{t}(\ell)}$ to create a pair $(A, \Phi) \in \tilde{\mathcal{C}}_{\mathfrak{t}}$.

6.2.1. Background pairs. Let $\tilde{M}_{\mathfrak{s}} \subset \tilde{\mathcal{C}}_{\mathfrak{t}(\ell)}$ denote the image of the embedding (2.3.12) of the space of solutions to the perturbed Seiberg–Witten equations from [28, Lemma 3.12]. The quotient, $M_{\mathfrak{s}} = \tilde{M}_{\mathfrak{s}}/\mathcal{G}_{\mathfrak{s}}$, of $\tilde{M}_{\mathfrak{s}}$ by the group $\mathcal{G}_{\mathfrak{s}}$ of gauge transformations embeds in $\mathcal{C}_{\mathfrak{t}(\ell)}$. Let $N_{\mathfrak{t}(\ell),\mathfrak{s}} \rightarrow M_{\mathfrak{s}}$ be the virtual normal bundle discussed in Section 2.3.5 with an embedding of a δ -disk subbundle $N_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \subset N_{\mathfrak{t}(\ell),\mathfrak{s}} \rightarrow \mathcal{C}_{\mathfrak{t}(\ell)}$, as defined in (2.3.20). Let $\tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}} \rightarrow \tilde{M}_{\mathfrak{s}}$ be the pullback of $N_{\mathfrak{t}(\ell),\mathfrak{s}}$ by the projection $\tilde{M}_{\mathfrak{s}} \rightarrow M_{\mathfrak{s}}$. As in (2.3.27), there is an embedding $\tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \rightarrow \tilde{\mathcal{C}}_{\mathfrak{t}(\ell)}$ covering the embedding $N_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \rightarrow \mathcal{C}_{\mathfrak{t}(\ell)}$.

6.2.2. Riemannian metrics. Recall that in Lemma 4.5.2, we constructed for each partition \mathcal{P} of N_ℓ a smooth family of metrics $g_{\mathcal{P}}$ on X , parameterized by $\Delta^\circ(X^\ell, \mathcal{P}) \subset X^\ell$ and a tubular neighborhood $\tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}}) \subset X^\ell$ of $\Delta^\circ(X^\ell, \mathcal{P})$. The map $\pi(X^\ell, \mathcal{P}) : \tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}}) \rightarrow \Delta^\circ(X^\ell, \mathcal{P})$ denotes the projection map of this tubular neighborhood. We list here the relevant properties of this family of metrics:

- (1) For each $\mathbf{y} \in \Delta^\circ(X^\ell, \mathcal{P})$, the metric $g_{\mathcal{P},\mathbf{y}}$ is flat on the support of

$$\tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}}) \cap \pi(X^\ell, \mathcal{P})^{-1}(\mathbf{y}),$$

where support is defined in Definition 4.4.1.

- (2) For each pair of partitions $\mathcal{P} < \mathcal{P}'$ of N_ℓ and point $\mathbf{y}' \in \tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}}) \cap \Delta^\circ(X^\ell, \mathcal{P}')$ with $\pi(X^\ell, \mathcal{P})(\mathbf{y}') = \mathbf{y}$, the equality $g_{\mathcal{P},\mathbf{y}} = g_{\mathcal{P}',\mathbf{y}'}$ holds.
- (3) The metrics are \mathfrak{S}_ℓ -invariant in the sense that $g_{\mathcal{P},\mathbf{y}} = g_{\sigma(\mathcal{P}),\sigma(\mathbf{y})}$ for all $\sigma \in \mathfrak{S}_\ell$.

6.2.3. Frame bundles. For a partition \mathcal{P} of N_ℓ , let

$$\text{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}}) \rightarrow \Delta^\circ(X^\ell, \mathcal{P}),$$

be the fiber bundle defined just before (4.3.4). Similarly, we define

$$(6.2.1) \quad \text{Fr}(\mathfrak{g}_{\mathfrak{t}(\ell)}, \mathcal{P}) := \left\{ (F_1^{\mathfrak{g}}, \dots, F_\ell^{\mathfrak{g}}) \in \left(\prod_{i=1}^{\ell} \text{Fr}(\mathfrak{g}_{\mathfrak{t}(\ell)}) \right) \Big|_{\Delta^\circ(X^\ell, \mathcal{P})} : \right. \\ \left. F_i^{\mathfrak{g}} = F_j^{\mathfrak{g}} \iff \exists P \in \mathcal{P} \text{ with } i, j \in P \right\}.$$

We define the *gluing data bundle* by

$$(6.2.2) \quad \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) := \text{Fr}(TX, \mathcal{P}, g_{\mathcal{P}}) \times_{\Delta^\circ(X^\ell, \mathcal{P})} \text{Fr}(\mathfrak{g}_{\mathfrak{t}(\ell)}, \mathcal{P}).$$

Recall that we denote elements of $\Delta^\circ(X^\ell, \mathcal{P})$ by $(y_P)_{P \in \mathcal{P}}$, where $y_P \in X$. Similarly, we denote elements of the total space of the bundle $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ lying over $(y_P)_{P \in \mathcal{P}}$ by $(F_P^T, F_P^{\mathfrak{g}})_{P \in \mathcal{P}}$,

where $F_P^T \in \text{Fr}(TX)|_{y_P}$ and $F_P^{\mathfrak{g}} \in \text{Fr}(\mathfrak{g}_{t(\ell)})|_{y_P}$. We also write $(F_P^{T,\mathfrak{g}})_{P \in \mathcal{P}}$ for such an element of $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, g_{\mathcal{P}})$.

The structure groups of the bundle $\text{Fr}(X, \mathcal{P})$ over $\Delta^\circ(X^\ell, \mathcal{P})$ and over $\Sigma(X^\ell, \mathcal{P})$ are $\tilde{G}(\mathcal{P})$ and $G(\mathcal{P})/\Gamma(\mathcal{P})$, respectively, where

$$(6.2.3a) \quad \tilde{G}(\mathcal{P}) := \{((R_1, M_1), \dots, (R_\ell, M_\ell)) \in (\text{SO}(4) \times \text{SO}(3))^\ell : \\ R_i = R_j \text{ and } M_i = M_j, \forall i, j \in P \in \mathcal{P}\},$$

$$(6.2.3b) \quad G(\mathcal{P}) := \tilde{G}(\mathcal{P}) \rtimes \mathfrak{S}(\mathcal{P}),$$

where \rtimes denotes the semi-direct product [56, p. 59] (compare the structure group appearing on [35, p. 252]). Observe that for $\mathcal{P} < \mathcal{P}'$, there is an inclusion,

$$(6.2.4) \quad \tilde{G}(\mathcal{P}) \rightarrow \tilde{G}(\mathcal{P}'),$$

given by the inclusion of the diagonals. We can write elements of $\tilde{G}(\mathcal{P})$ as $(R_P, M_P)_{P \in \mathcal{P}}$. With this notation, the action of $G(\mathcal{P})$ on

$$(6.2.5) \quad \bar{M}(\mathcal{P}) := \prod_{P \in \mathcal{P}} \bar{M}_{\text{SPL}, |P|}^{s, \natural}(\delta)$$

is given by the factor (R_P, M_P) acting by the standard $\text{SO}(3) \times \text{SO}(4)$ action on $\bar{M}_{\text{SPL}, |P|}^{s, \natural}(\delta)$ and by the natural permutation action of $\mathfrak{S}(\mathcal{P})$.

6.2.4. Group actions on the frame bundles. In addition to the action of the structure groups $G(\mathcal{P})$ and $\tilde{G}(\mathcal{P})$ on the frame bundle $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$, there are also actions of the gauge group $\mathcal{G}_{\mathfrak{s}}$ and of S^1 on $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$. Both of these actions are diagonal actions defined on the factors of $\text{Fr}(\mathfrak{g}_{t(\ell)})$ in $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ by the S^1 action on $\mathfrak{g}_{t(\ell)}$ defined by the reduction of $\mathfrak{g}_{t(\ell)}$ to an S^1 bundle as in (2.3.17).

6.2.5. Space of splicing data. The splicing map will be defined on a subspace

$$\tilde{N}_{t(\ell), s}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \subset \tilde{N}_{t(\ell), s}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}),$$

where

$$(6.2.6) \quad \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) := \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \times_{G(\mathcal{P})} \bar{M}(\mathcal{P}).$$

The subspace $\mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ of $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ is an open neighborhood of

$$(6.2.7) \quad \Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) := \left\{ \left((F_P^{T,\mathfrak{g}})_{P \in \mathcal{P}}, ([\Theta, F_P^s, c_{|P|}])_{P \in \mathcal{P}} \right) \in \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \right\},$$

where Θ is the product connection over S^4 and $c_{|P|}$ is the cone point of $\text{Sym}^{|P|, \natural}(\mathbb{R}^4)$. The notation in (6.2.7) is motivated by the observation that because $\text{SO}(3) \times \text{SO}(4)$ acts trivially on $[\Theta, F_P^s, c_{|P|}]$, there is an identification,

$$(6.2.8) \quad \Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \cong \Sigma(X^\ell, \mathcal{P}),$$

where $\Sigma(X^\ell, \mathcal{P}) \subset \text{Sym}^\ell(X)$ is defined in (3.1.9). We define a subspace of $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ containing $\Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ by

$$(6.2.9) \quad T(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) := \left\{ \left((F_P^{T,\mathfrak{g}})_{P \in \mathcal{P}}, ([\Theta, F_P^s, \mathbf{v}_P])_{P \in \mathcal{P}} \right) \in \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \right\}.$$

One requirement in the definition of $\mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ will be that for any point,

$$((F_P^T, F_P^g)_{P \in \mathcal{P}}, ([A_P, F_P^s, \mathbf{v}_P])_{P \in \mathcal{P}}) \in \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}),$$

where F_P^T and F_P^g lie over $y_P \in X$ with $P' \neq P$, we have

$$(6.2.10) \quad 8\lambda([A_P, F_P^s, \mathbf{v}_P]) < \min_{P' \in \mathcal{P}, P' \neq P} d(y_P, y_{P'}),$$

where λ is the scale parameter (5.2.3). Finally, we define

$$(6.2.11) \quad \pi_\Sigma : \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \Sigma(X^\ell, \mathcal{P})$$

to be the obvious projection. The S^1 action on $\tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}$ defined in (2.3.28) defines an S^1 action,

$$(6.2.12) \quad S^1 \times \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}),$$

and we will see that all the constructions of this chapter are equivariant with respect to this S^1 action. The following analogue of [30, Lemma 3.6] gives two descriptions of the action (6.2.12).

LEMMA 6.2.1. *The following two circle actions on $\tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ are equivalent:*

- (1) *The action (2.3.28) on $\tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta)$ and the trivial action on $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$.*
- (2) *The diagonal action with weight two on the fibers of $\tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \rightarrow \tilde{M}_s$ and with the action on $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ described in Section 6.2.4 with weight two.*

PROOF. The proof of this lemma is identical to that of [30, Lemma 3.6] using the equality in (2.3.28) of S^1 actions. \square

We will need to refer to the S^1 action,

$$(6.2.13) \quad S^1 \times \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}),$$

given by the diagonal action with weight one on the fibers of $\tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \rightarrow \tilde{M}_s$ and with the action on $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ defined by the S^1 action described in Section 6.2.4 with weight one. The action described in Lemma 6.2.1 is the action (6.2.13) with weight two.

Each $\sigma \in \mathfrak{S}_\ell$ defines a smoothly-stratified diffeomorphism $\sigma : \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \sigma(\mathcal{P}))$, where $\sigma(\mathcal{P})$ is the partition defined in (3.1.1) by the following relabelling map,

$$(6.2.14) \quad \mathbf{A} := ((F_P^g, F_P^T), [A_P, F_P^s, \mathbf{v}_P])_{P \in \mathcal{P}} \\ \mapsto \sigma(\mathbf{A}) := \left((F_{\sigma(P)}^g, F_{\sigma(P)}^T), [A_{\sigma(P)}, F_{\sigma(P)}^s, \mathbf{v}_{\sigma(P)}] \right)_{P \in \mathcal{P}}.$$

Because the stabilizer $\mathfrak{S}(\mathcal{P})$ of \mathcal{P} under the action (3.1.1) is in the structure group $G(\mathcal{P})$ in (6.2.3b), if $\sigma \in \mathfrak{S}(\mathcal{P})$, then the map defined by σ by (6.2.14) is trivial.

6.3. The flattening map on pairs

The commutativity of the diagram (5.5.27) for the instanton moduli space with spliced ends depended on the flatness of the metric on \mathbb{R}^4 and on the flatness of the product connection Θ around the splicing points. The locally flat metrics $g_{\mathcal{P}}$ described in Section 6.2.2 have this property. We now introduce a method for achieving the same kind of local flatness for the background connection A_0 . This method also yields a way of cutting off the background section consistently from one stratum of the subspace (6.1.1) to another.

DEFINITION 6.3.1. A collection of smooth functions $s_P : \Delta^\circ(X^\ell, \mathcal{P}) \rightarrow (0, 1)$, indexed by $P \in \mathcal{P}$, is a *separating family* if the following hold:

- (1) The functions are $\mathfrak{S}(\mathcal{P})$ -invariant in the sense that $s_P = s_{\sigma(P)}$ and $s_P = s_P \circ \sigma$, for all $\sigma \in \mathfrak{S}(\mathcal{P})$.
- (2) For all $\mathbf{y} = (y_P)_{P \in \mathcal{P}} \in \Delta^\circ(X^\ell, \mathcal{P})$ and for all $P \neq P' \in \mathcal{P}$, the balls $B(y_P, 4s_P(\mathbf{y})^{1/2})$ and $B(y_{P'}, 4s_{P'}(\mathbf{y})^{1/2})$ are disjoint.

DEFINITION 6.3.2. Let $\{s_P\}$ be a separating family of smooth functions on $\Delta^\circ(X^\ell, \mathcal{P})$. A connection \hat{A} on a vector bundle $\mathfrak{g} \rightarrow X$ is *flat with respect to $\{s_P\}$* at $\mathbf{y} = (y_P)_{P \in \mathcal{P}} \in \Delta^\circ(X^\ell, \mathcal{P})$ if the connection \hat{A} is flat on

$$(6.3.1) \quad \bigcup_{P \in \mathcal{P}} B(y_P, 2s_P(\mathbf{y})^{1/2}).$$

A pair $(A, \Phi) \in \tilde{\mathcal{C}}_{\mathfrak{t}(\ell)}$ is *flat with respect to $\{s_P\}$* if \hat{A} is flat with respect to $\{s_P\}$ and Φ vanishes on

$$(6.3.2) \quad \bigcup_{P \in \mathcal{P}} B(y_P, 4s_P(\mathbf{y})^{1/3}).$$

Lemma 6.3.3 below provides the analogue of the flattening construction for metrics given in Lemma 4.5.1. Recall from Section 2.1.2 that for a spin^u structure \mathfrak{t} and spin^u connection A on the spin^u bundle $V_{\mathfrak{t}} \rightarrow X$ then \hat{A} denotes the unique connection induced by A on the $\mathfrak{so}(3)$ bundle $\mathfrak{g}_{\mathfrak{t}}$ appearing in (2.1.4).

LEMMA 6.3.3. *Let \mathcal{P} be a partition of N_ℓ . Let $\{s_P\}$ be a separating family of smooth functions on $\Delta^\circ(X^\ell, \mathcal{P})$. Then there is an S^1 -equivariant map,*

$$(6.3.3) \quad \Theta'_{\mathcal{P}} : \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times \Delta^\circ(X^\ell, \mathcal{P}) \rightarrow \tilde{\mathcal{C}}_{\mathfrak{t}(\ell)},$$

which is $\mathcal{G}_{\mathfrak{s}}$ -equivariant and descends to an S^1 -equivariant map,

$$N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times \Delta^\circ(X^\ell, \mathcal{P}) \rightarrow \mathcal{C}_{\mathfrak{t}(\ell)}.$$

In addition, for all $\mathbf{y} = (y_P)_{P \in \mathcal{P}} \in \Delta^\circ(X^\ell, \mathcal{P})$, if $(A', \Phi') = \Theta'_{\mathcal{P}}((A_0, \Phi_0), \mathbf{y})$, then

- (1) *The connection \hat{A}' is flat on*

$$\bigcup_{P \in \mathcal{P}} B(y_P, 2s_P(\mathbf{y})^{1/2})$$

and equal to \hat{A}_0 on the complement of

$$(6.3.4) \quad \bigcup_{P \in \mathcal{P}} B(y_P, 4s_P(\mathbf{y})^{1/2}).$$

- (2) *The section Φ' is identically zero on*

$$\bigcup_{P \in \mathcal{P}} B(y_P, 4s_P(\mathbf{y})^{1/3})$$

and is equal to Φ on the complement of

$$(6.3.5) \quad \bigcup_{P \in \mathcal{P}} B(y_P, 8s_P(\mathbf{y})^{1/3}).$$

If the connection \hat{A}_0 is already flat on the space (6.3.4), then $A' = A_0$. If the section Φ is already identically zero on the space (6.3.5), then $\Phi' = \Phi_0$.

PROOF. The map $\Theta_{\mathcal{P}}$ is initially defined as a map,

$$\Theta'_{\mathcal{P}} : \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \tilde{\mathcal{C}}_{\mathfrak{t}(\ell)},$$

as described in [24, Section 3.2]. Specifically, the frames and radial parallel translation define a product connection on the space (6.3.4). If \hat{A}_0 is flat on the space (6.3.4), this product connection is equal to \hat{A}_0 . We now use a cut-off function to interpolate between the product connection and \hat{A}_0 on the annuli around the points y_P , defining a new connection \hat{A}' which is flat on the inner balls and equal to \hat{A}_0 outside the space (6.3.4). Observe that the resulting connection \hat{A}' is independent of the choice of the frames used as it is equal to the connection obtained by splicing in the product connection on the space (6.3.4) which has stabilizer $\text{SO}(3)$ and is invariant under the rotation action on the tangent frame (see [15, Proposition 7.2.9]). Hence, the map $\Theta'_{\mathcal{P}}$ descends to the domain stated in the lemma.

The section Φ' is defined by multiplying Φ_0 by a cut-off function equal to one on the space (6.3.5) and vanishing on the balls of radius $4s_P(\mathbf{y})^{1/3}$. \square

To define the crude splicing maps in such a way that an analogue of Proposition 5.5.3 describes the overlaps of their images, we must redefine the flattening maps in such a way that for $\mathbf{y}' \in \Delta^\circ(X^\ell, \mathcal{P}')$ near $\mathbf{y} \in \Delta^\circ(X^\ell, \mathcal{P})$, the following equality holds:

$$\Theta_{\mathcal{P}'}(A_0, \Phi_0, \mathbf{y}') = \Theta_{\mathcal{P}}(A_0, \Phi_0, \mathbf{y}).$$

(Compare the preceding equality with the second property of the metrics $g_{\mathcal{P}}$ appearing in Section 6.2.2.) To this end, we introduce a refinement of the flattening map constructed in Lemma 6.3.3.

LEMMA 6.3.4. *Continue the assumptions of Lemma 6.3.3. Then, for every partition \mathcal{P} of N_ℓ there are separating functions $\{\tilde{s}_P\}$ and an S^1 -equivariant map,*

$$(6.3.6) \quad \Theta_{\mathcal{P}} : \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times \Delta^\circ(X^\ell, \mathcal{P}) \rightarrow \tilde{\mathcal{C}}_{\mathfrak{t}(\ell)},$$

which is $\mathfrak{S}(\mathcal{P})$ -invariant, $\mathcal{G}_\mathfrak{s}$ -equivariant, descends to the quotient,

$$N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times \Delta^\circ(X^\ell, \mathcal{P}) \rightarrow \mathcal{C}_{\mathfrak{t}(\ell)},$$

and has the properties that if $(A', \Phi') = \Theta_{\mathcal{P}}(A_0, \Phi_0, \mathbf{y})$, where $\mathbf{y} = (y_P)_{P \in \mathcal{P}} \in \Delta^\circ(X^\ell, \mathcal{P})$, then the following hold:

- (1) *If the symmetric group \mathfrak{S}_ℓ acts on the partitions of N_ℓ by the action (3.1.1) and on X^ℓ by the obvious permutation action, then for $\sigma \in \mathfrak{S}_\ell$,*

$$\Theta_{\sigma(\mathcal{P})}(A_0, \Phi_0, \sigma(\mathbf{y})) = \Theta(A_0, \Phi_0, \mathbf{y})$$

and the separating functions $\{\tilde{s}_{\sigma(P)}\}_{\sigma(P) \in \sigma(\mathcal{P})}$ associated to $\sigma(\mathcal{P})$ satisfy $\tilde{s}_{\sigma(P)} = \tilde{s}_P \circ \sigma$.

- (2) *The connection \hat{A}' is flat on*

$$\bigcup_{P \in \mathcal{P}} B(y_P, 2\tilde{s}_P(\mathbf{y})^{1/2}).$$

- (3) *The section Φ' vanishes on*

$$\bigcup_{P \in \mathcal{P}} B(y_P, 4\tilde{s}_P(\mathbf{y})^{1/3}).$$

Furthermore, if $\mathcal{P} < \mathcal{P}'$ and the tubular neighborhood $\tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}})$ of $\Delta^\circ(X^\ell, \mathcal{P})$ is suitably small, $\mathbf{y}' \in \tilde{\mathcal{U}}(X^\ell, g_{\mathcal{P}}) \cap \Delta^\circ(X^\ell, \mathcal{P}')$, and $\pi(X^\ell, g_{\mathcal{P}})(\mathbf{y}') = \mathbf{y} \in \Delta^\circ(X^\ell, \mathcal{P})$, then

$$(6.3.7) \quad \Theta_{\mathcal{P}}(A_0, \Phi_0, \mathbf{y}) = \Theta_{\mathcal{P}'}(A_0, \Phi_0, \mathbf{y}').$$

REMARK 6.3.5. The separating functions $\{\tilde{s}_P\}$ appearing in the statement of Lemma 6.3.4 differ from those appearing in Lemma 6.3.3 in that the functions $\{\tilde{s}_P\}$ satisfy the additional condition (6.3.9) relating the separating functions of one partition to another and which is used to derive (6.3.7).

PROOF. The proof is similar to the construction of the consistent flat families of metrics in Lemma 4.5.2. One constructs $\Theta_{\mathcal{P}}$ by induction on \mathcal{P} . As described in Section 3.4, the induction uses one partition from each orbit of the action of \mathfrak{S}_ℓ on the set of partitions of N_ℓ and extends the results to all partitions in the orbit so that Item (1) holds. Thus, Item (1) follows trivially from the method of induction.

For the crudest partition, $\mathcal{P}_0 := \{N_\ell\}$, of N_ℓ , define $\Theta_{\mathcal{P}_0} := \Theta'_{\mathcal{P}_0}$ (as constructed in Lemma 6.3.3).

Assume that $\Theta_{\mathcal{P}}$ satisfying the conclusions of the lemma has been constructed for all $\mathcal{P} < \mathcal{P}'$. To construct $\Theta_{\mathcal{P}'}$, we first define

$$\Theta_{\mathcal{P}', \mathcal{P}} : \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times \left(\mathcal{U}(X^\ell, g_{\mathcal{P}}) \cap \Delta^\circ(X^\ell, \mathcal{P}') \right) \rightarrow \tilde{\mathcal{C}}_{\mathfrak{t}(\ell)},$$

by

$$\Theta_{\mathcal{P}', \mathcal{P}}(A_0, \Phi_0, \mathbf{y}') := \Theta_{\mathcal{P}}(A_0, \Phi_0, \pi(X^\ell, g_{\mathcal{P}})(\mathbf{y}')).$$

Observe that if $\mathcal{P}_1 < \mathcal{P}$ and $\mathcal{P}_2 < \mathcal{P}$ and $\mathcal{U}(X^\ell, g_{\mathcal{P}_1}) \cap \mathcal{U}(X^\ell, g_{\mathcal{P}_2})$ is non-empty, then we can assume $\mathcal{P}_1 < \mathcal{P}_2 < \mathcal{P}$. The Thom–Mather condition (4.4.9) on the projections $\pi(X^\ell, g_{\mathcal{P}})$,

$$\pi(X^\ell, g_{\mathcal{P}_1}) \circ \pi(X^\ell, g_{\mathcal{P}_2}) = \pi(X^\ell, g_{\mathcal{P}_1}),$$

and the inductive assumption imply that for $\mathbf{y}' \in \mathcal{U}(X^\ell, g_{\mathcal{P}_1}) \cap \mathcal{U}(X^\ell, g_{\mathcal{P}_2}) \cap \Delta^\circ(X^\ell, \mathcal{P}')$ we have

$$\begin{aligned} \Theta_{\mathcal{P}', \mathcal{P}_2}(A_0, \Phi_0, \mathbf{y}') &= \Theta_{\mathcal{P}_2}(A_0, \Phi_0, \pi(X^\ell, g_{\mathcal{P}_2})(\mathbf{y}')) \\ &= \Theta_{\mathcal{P}_1}(A_0, \Phi_0, \pi(X^\ell, g_{\mathcal{P}_1}) \circ \pi(X^\ell, g_{\mathcal{P}_2})(\mathbf{y}')) \quad (\text{by inductive hypothesis and (6.3.7)}) \\ &= \Theta_{\mathcal{P}_1}(A_0, \Phi_0, \pi(X^\ell, g_{\mathcal{P}_1})(\mathbf{y}')) \quad (\text{by Thom–Mather property (4.4.9)}) \\ &= \Theta_{\mathcal{P}', \mathcal{P}_1}(A_0, \Phi_0, \mathbf{y}') \quad (\text{by the definition of } \Theta_{\mathcal{P}', \mathcal{P}_1}). \end{aligned}$$

Thus, if we define the ends of the diagonal $\Delta^\circ(X^\ell, \mathcal{P}')$ by

$$(6.3.8) \quad \mathcal{U}(\mathcal{P}') := \bigcup_{\mathcal{P} < \mathcal{P}'} \mathcal{U}(X^\ell, g_{\mathcal{P}}) \cap \Delta^\circ(X^\ell, \mathcal{P}'),$$

then the map

$$\begin{aligned} \Theta_{\mathcal{P}', \mathcal{U}} : \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times \mathcal{U}(\mathcal{P}') &\rightarrow \tilde{\mathcal{C}}_{\mathfrak{t}(\ell)}, \\ \Theta_{\mathcal{P}', \mathcal{U}}(A_0, \Phi_0, \mathbf{y}') &:= \Theta_{\mathcal{P}', \mathcal{P}}(A_0, \Phi_0, \mathbf{y}'), \quad \text{for } \mathbf{y}' \in \mathcal{U}(X^\ell, g_{\mathcal{P}}), \end{aligned}$$

is defined consistently for $\mathbf{y}' \in \mathcal{U}(X^\ell, g_{\mathcal{P}_1}) \cap \mathcal{U}(X^\ell, g_{\mathcal{P}_2}) \cap \Delta^\circ(X^\ell, \mathcal{P}')$. We next define

$$\Theta_{\mathcal{P}', 1} : \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times \Delta^\circ(X^\ell, \mathcal{P}') \rightarrow \tilde{\mathcal{C}}_{\mathfrak{t}(\ell)}$$

by

$$\Theta_{\mathcal{P}',1}(A_0, \Phi_0, \mathbf{y}') := (1 - \chi(\mathbf{y}'))(A_0, \Phi_0) + \chi(\mathbf{y}')\Theta_{\mathcal{P}',\mathcal{U}}(A_0, \Phi_0, \mathbf{y}'),$$

where $\chi : \Delta^\circ(X^\ell, \mathcal{P}) \rightarrow [0, 1]$ is an $\mathfrak{S}(\mathcal{P})$ -invariant function supported in (6.3.8) and equal to one on a neighborhood \mathcal{U}_2 of the end of $\Delta^\circ(X^\ell, \mathcal{P})$ (hence \mathcal{U}_2 is a proper subspace of $\mathcal{U}(\mathcal{P}')$). By shrinking the tubular neighborhoods $\mathcal{U}(X^\ell, g_{\mathcal{P}})$, we can find a family of separating functions $\{\tilde{s}_{P'}\}_{P' \in \mathcal{P}'}$, such that for $\mathbf{y}' = (y'_{P'})_{P' \in \mathcal{P}'} \in \mathcal{U}(X^\ell, g_{\mathcal{P}}) \cap \Delta^\circ(X^\ell, \mathcal{P}')$ with $\pi(X^\ell, g_{\mathcal{P}})(\mathbf{y}') = \mathbf{y} = (y_P)_{P \in \mathcal{P}}$, we have

$$(6.3.9) \quad \begin{aligned} B(y'_{P'}, 8\tilde{s}_{P'}(\mathbf{y}')^{1/3}) &\in B(y_P, 4\tilde{s}_P(\mathbf{y})^{1/3}), \\ B(y'_{P'}, 4\tilde{s}_{P'}(\mathbf{y}')^{1/2}) &\in B(y_P, 2\tilde{s}_P(\mathbf{y})^{1/2}), \quad \forall P' \in \mathcal{P}'_P. \end{aligned}$$

From the inclusions (6.3.9), the inductive hypothesis, and the final statement of Lemma 6.3.3, we have the equality,

$$\Theta'_{\mathcal{P}'}(\Theta_{\mathcal{P},1}(A_0, \Phi_0, \mathbf{y}'), \mathbf{y}') = \Theta_{\mathcal{P},1}(A_0, \Phi_0, \mathbf{y}'), \quad \forall \mathbf{y}' \in \mathcal{U}_2.$$

Thus, for $\mathbf{y}' \in \mathcal{U}_2$, the pair $\Theta_{\mathcal{P},1}(A_0, \Phi_0, \mathbf{y}')$ is already flat on the relevant balls and so the flattening construction of Lemma 6.3.3 does not change $\Theta_{\mathcal{P},1}(A_0, \Phi_0, \mathbf{y}')$. Therefore, if we define

$$\Theta_{\mathcal{P}'}(A_0, \Phi_0, \mathbf{y}') := \Theta'_{\mathcal{P}'}(\Theta_{\mathcal{P},1}(A_0, \Phi_0, \mathbf{y}'), \mathbf{y}'),$$

this flattening map will satisfy the condition (6.3.7), completing the construction of the map $\Theta_{\mathcal{P}'}$ and hence completing the induction. \square

6.4. The crude splicing map

The crude splicing map,

$$(6.4.1) \quad \gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}} : \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathfrak{g}_s} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \subset \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathfrak{g}_s} \bar{\mathfrak{G}}\mathfrak{l}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \bar{\mathcal{C}}_{\mathfrak{t}},$$

where $\bar{\mathcal{C}}_{\mathfrak{t}}$ is defined in (2.1.15) and $\bar{\mathfrak{G}}\mathfrak{l}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ is defined in (6.2.6), has a domain defined by a *separation condition*,

$$(6.4.2) \quad \begin{aligned} &\mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \\ &= \{(F_P^{T, \mathfrak{g}}, [A_P, F_P^s, \mathbf{v}_P])_{P \in \mathcal{P}} \in \bar{\mathfrak{G}}\mathfrak{l}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) : 8\lambda([A_P, F_P^s, \mathbf{v}_P])^{1/3} \leq s_P(F_P^{T, \mathfrak{g}})\}, \end{aligned}$$

where $\{s_P\}$ is the pullback of the family of separating functions $\{\tilde{s}_P\}$ constructed in Lemma 6.3.4 by the projection map, $\mathfrak{G}\mathfrak{l}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \Sigma(X^\ell, \mathcal{P})$. We will denote elements of the domain of the crude splicing map by $[(A_0, \Phi_0), \mathbf{A}]$ and use the following notation.

NOTATION 6.4.1. A point $[(A_0, \Phi_0), \mathbf{A}]$ in the domain of the map (6.4.2) is given by

- (1) A pair $(A_0, \Phi_0) \in \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta)$,
- (2) A point $(y_P)_{P \in \mathcal{P}} \in \Delta^\circ(X^\ell, \mathcal{P})$,
- (3) Frames $F_P^{\mathfrak{g}} \in \text{Fr}(\mathfrak{g}_{\mathfrak{t}(\ell)})|_{y_P}$,
- (4) Frames $F_P^T \in \text{Fr}(TX)|_{y_P}$,
- (5) Gauge-equivalence classes of framed, mass-centered connections, $[A_P, F_P^s] \in \bar{M}_{\text{SPL}, |P|}^{s, \mathfrak{h}}$, over S^4 .

The crude splicing map will be S^1 -equivariant with respect to the S^1 action on the domain given in Lemma 6.2.1 and the S^1 action on the range given by the action (2.1.9). Before defining the crude splicing map, we review the definition of the splicing map defined in [24, Section 3.2].

6.4.1. The standard splicing map. For clarity, we shall refer to the splicing map defined in [24, Section 3.2] as the *standard* splicing map. The *crude* splicing map will be similar to the standard splicing map, with some differences described below. We now sketch the construction in [24, Section 3.2] of a pair $(A', \Phi') = \gamma'_{t,s,\mathcal{P}}([(A_0, \Phi_0), \mathbf{A}])$, where $[(A_0, \Phi_0), \mathbf{A}]$ is a point in the domain of the crude splicing map, (6.4.2), given by the data in Notation 6.4.1.

Recall from Section 2.1.2 that for a spin^u connection A on a spin^u bundle V_t , we write \hat{A} for the connection induced by A on the $\mathfrak{so}(3)$ bundle \mathfrak{g}_t appearing in (2.1.4). If we restrict to the spin^u connections inducing a fixed connection on $\det(V_t)$, then the map $A \mapsto \hat{A}$ is a bijection.

For $[A_P, F_P^s] \in \bar{M}_{\text{SPL},|P|}^{s,\natural}$ as in Item (5) of Notation 6.4.1, the connection \hat{A}_P will be an orthogonal connection on the real rank-three, Riemannian vector bundle, $\mathfrak{g}_P \rightarrow S^4$. Similarly, for $(A_0, \Phi_0) \in \tilde{N}_{t(\ell),s}(\delta)$ as in Item (1) of Notation 6.4.1, \hat{A}_0 will be a connection on $\mathfrak{g}_{t(\ell)} \rightarrow X$. Define

$$(6.4.3) \quad \lambda_P := \lambda([A_P, F_P^s])$$

to be the scale of the connection A_P as defined in (5.2.3). For each $P \in \mathcal{P}$, let $\tau(A_P, F_P^s)$ be the trivialization of \mathfrak{g}_P over $S^4 \setminus \{n\}$ (where n is the North Pole) defined by parallel translation of the frame F_P^s along great circles. This trivialization defines a product (and hence flat) connection, $\Theta(A_P, F_P^s)$. Recall from Section 5.2, that the map $\varphi_n : \mathbb{R}^n \rightarrow S^4 \setminus \{s\}$ was defined by stereographic projection. Let $B(r) \subset \mathbb{R}^4$ be the open ball of radius r centered at the origin and let $\Omega(r_1, r_2) := B(r_2) \setminus \bar{B}(r_1)$. More generally, for $x \in X$, let $\Omega(x, r_1, r_2) := B(x, r_2) \setminus \bar{B}(x, r_1)$, where $B(x, r) \subset X$ denotes the open ball with center x and radius r defined by the Riemannian metric, g . For a smooth function $\beta : \mathbb{R} \rightarrow [0, 1]$ obeying $\beta(t) = 1$ for $t \leq 1$ and $\beta(t) = 0$ for $t \geq 2$, constant $\varepsilon > 0$, and point $x \in \mathbb{R}^4$, define $\chi_{x,\varepsilon} : \mathbb{R}^4 \rightarrow [0, 1]$ by $\chi_{x,\varepsilon} := \beta(|\cdot - x|/\varepsilon)$ and define $\chi_{n,\varepsilon} : S^4 \rightarrow [0, 1]$ by $\chi_{n,\varepsilon} := \chi_{0,\varepsilon} \circ \varphi_n^{-1}$. Thus, $\chi_{n,\frac{1}{4}\lambda^{1/2}}$ is a smooth cut-off function on S^4 that is equal to one on $\varphi_n(B(\frac{1}{4}\lambda^{1/2}))$ and is supported in $\varphi_n(B(\frac{1}{2}\lambda^{1/2}))$.

Define a connection on $\mathfrak{g}_P \rightarrow S^4$ by

$$(6.4.4) \quad \hat{A}'_P := \begin{cases} \Theta(A_P, F_P^s) & \text{on } S^4 \setminus \varphi_n(B(\frac{1}{2}\lambda_P^{1/2})), \\ (1 - \chi_{n,\frac{1}{4}\lambda_P^{1/2}})\Theta(A_P, F_P^s) + \chi_{n,\frac{1}{4}\lambda_P^{1/2}}\hat{A}_P & \text{on } \varphi_n(\Omega(\frac{1}{4}\lambda_P^{1/2}, \frac{1}{2}\lambda_P^{1/2})), \\ \hat{A}_P & \text{on } \varphi_n(B(\frac{1}{4}\lambda_P^{1/2})). \end{cases}$$

For $x \in X$, define $\chi_{x,\varepsilon} : X \rightarrow [0, 1]$ by $\chi_{x,\varepsilon} := \beta(\text{dist}_g(\cdot, x)/\varepsilon)$. Thus, $1 - \chi_{x,2\lambda^{1/2}}$ is a smooth cut-off function on X that is equal to one on $X \setminus B(x, 4\lambda^{1/2})$ and is equal to zero on $B(x, 2\lambda^{1/2})$. Parallel translation with respect to the connection \hat{A}_0 of the frame F_P^g (from Item (3) of Notation 6.4.1) along radial geodesics from the point y_P defines a trivialization $\tau(A_0, F_P^g)$ of the bundle $\mathfrak{g}_{t(\ell)}$ over a ball in X centered at the point y_P . Let $\Theta(A_0, F_P^g)$ be the product connection defined by this trivialization. We then define an orthogonal connection on $\mathfrak{g}_{t(\ell)}$ by

$$(6.4.5) \quad \hat{A}'_0 := \begin{cases} \hat{A}_0 & \text{on } X \setminus \cup_P B(x_P, 4\lambda_P^{1/2}), \\ (1 - \chi_{y_P,2\lambda_P^{1/2}})\hat{A}_0 + \chi_{y_P,2\lambda_P^{1/2}}\Theta(A_P, F_P^g) & \text{on } \Omega(y_P, 2\lambda_P^{1/2}, 4\lambda_P^{1/2}), \\ \Theta(A_0, F_P^g) & \text{on } B(y_P, 2\lambda_P^{1/2}). \end{cases}$$

The frame F_P^T from Item (4) of Notation 6.4.1, the exponential map around x_P , and stereographic projection identify the annuli $\Omega(x_P, \frac{1}{2}\lambda_P^{1/2}, 2\lambda_P^{1/2}) \subset X$ with the annuli $\varphi_n(\Omega(\frac{1}{2}\lambda_P^{1/2}, 2\lambda_P^{1/2})) \subset S^4$.

Identifying the trivializations $\tau(A_0, F_P^g)$ and $\tau(A_P, F_P^s)$ over these annuli allows us to glue the bundle $\mathfrak{g}_{\mathfrak{t}(\ell)}$ and bundles \mathfrak{g}_P (for $P \in \mathcal{P}$) to create a bundle isomorphic to $\mathfrak{g}_{\mathfrak{t}}$,

$$(6.4.6) \quad \mathfrak{g}_{\mathfrak{t}} \cong \mathfrak{g}_{\mathfrak{t}(\ell)}|_{X \setminus \cup_P B(y_P, \frac{1}{2}\lambda_P^{1/2})} \cup \bigcup_{P \in \mathcal{P}} \mathfrak{g}_P|_{\varphi_n(B(2\lambda_P^{1/2}))}.$$

Identifying these trivializations gives an identification of the product connections $\Theta(A_0, F_P^g)$ and $\Theta(A_P, F_P^s)$. Let $\Theta(A_0, F_P^g, F_P^T, A_P, F_P^s)$ be the product connection over $\Omega(x_P, \frac{1}{2}\lambda_P^{1/2}, 2\lambda_P^{1/2}) \subset X$ given by this identification. We define an orthogonal connection on $\mathfrak{g}_{\mathfrak{t}}$ by

$$(6.4.7) \quad \hat{A}' := \begin{cases} \hat{A}'_0 & \text{on } X \setminus \cup_P B(x_P, 4\lambda_P^{1/2}), \\ \Theta(A_0, F_P^g, F_P^T, A_P, F_P^s) & \text{on } \Omega(y_P, \frac{1}{2}\lambda_P^{1/2}, 2\lambda_P^{1/2}), \\ \hat{A}'_P & \text{on } B(y_P, 2\lambda_P^{1/2}). \end{cases}$$

If $x \in X$, then $\chi_{x, 4\lambda^{1/3}} : X \rightarrow [0, 1]$ is a smooth cut-off function such that $1 - \chi_{x, 4\lambda^{1/3}}$ is equal to one on $X \setminus B(x, 8\lambda^{1/3})$ and is equal to zero on $B(x, 4\lambda^{1/3})$. We define a section of $V_{\mathfrak{t}}$ by

$$(6.4.8) \quad \Phi' := \begin{cases} \Phi & \text{on } X \setminus \cup_P B(y_P, 8\lambda_P^{1/3}), \\ (1 - \chi_{y_P, 4\lambda_P^{1/3}})\Phi & \text{on } \Omega(y_P, 4\lambda_P^{1/3}, 8\lambda_P^{1/3}), \\ 0 & \text{on } \cup_P B(y_P, 4\lambda_P^{1/3}). \end{cases}$$

The *standard splicing map* has the same domain and range as the crude splicing map in (6.4.1). For the data \mathbf{A} given in Notation 6.4.1, the standard splicing map is defined by

$$\gamma'_{\mathfrak{t}, \mathcal{P}}([(A_0, \Phi_0), \mathbf{A}]) := [A', \Phi'],$$

where A' is the spin^u connection with \hat{A}' defined by (6.4.7) and Φ' is defined by (6.4.8).

6.4.2. Construction of the crude splicing map. The definition of the crude splicing map, $\gamma''_{\mathfrak{t}, \mathcal{P}}$, will differ from the preceding description of the standard splicing map, $\gamma'_{\mathfrak{t}, \mathcal{P}}$, in the following ways:

- (1) The metric on X used to identify geodesic balls in X with balls centered at the North Pole of S^4 is fixed for the standard splicing map. The metric on X so used in the definition of the crude splicing map varies in the smooth family of metrics $g_{\mathcal{P}}$ parameterized by the splicing point $\mathbf{y} \in \Delta^\circ(X^\ell, \mathcal{P})$ constructed in Lemma 4.5.2. Note that for $\mathbf{y} = (y_P)$, the balls of radius r centered at y_P defined by the Riemannian metrics g and $g_{\mathcal{P}, \mathbf{y}}$ are equal by Item (4) of Lemma 4.5.1.
- (2) For the definition of the crude splicing map, the trivializations of the background bundle, $\mathfrak{g}_{\mathfrak{t}(\ell)}$, are defined by parallel translation with respect to the flattened connection, \hat{A}''_0 where $(A''_0, \Phi'') = \Theta_{\mathcal{P}}(A_0, \Phi_0, \mathbf{x})$, instead of with respect to \hat{A}'_0 .
- (3) In the definition of the standard splicing map in [24, §3.2], the background connection A_0 is flattened over the balls

$$B(y_P, 2\lambda_P^{1/2}),$$

while the background section is multiplied by a cut-off function that is identically zero on

$$B(y_P, 4\lambda_P^{1/3}),$$

where λ_P is the scale of the connection on S^4 being spliced in at $x_P \in X$. The crude splicing map is defined instead by replacing the background pair (A_0, Φ_0) with the background pair $\Theta_{\mathcal{P}}(A_0, \Phi_0, \mathbf{y})$ constructed in Lemma 6.3.4. Hence, the background pair is flattened over balls whose radius does not depend on the scale of the connections over S^4 .

Let $[(A_0, \Phi_0), \mathbf{A}]$ be a point in the domain of $\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}$ in (6.4.1). We will continue to use the notation from Notation 6.4.1 for the data given by \mathbf{A} . For the flattening map $\Theta_{\mathcal{P}}$ in (6.3.6), let

$$(6.4.9) \quad (A''_0, \Phi'') := \Theta_{\mathcal{P}}(A_0, \Phi_0, (y_P)_{P \in \mathcal{P}}),$$

where (A_0, Φ_0) and $(y_P)_{P \in \mathcal{P}}$ are as in Items (1) and (2) of Notation 6.4.1, respectively. Let \hat{A}_P and λ_P be as defined in (6.4.4) and (6.4.3). Using the same notation as in (6.4.7), we define

$$(6.4.10) \quad \hat{A}'' := \begin{cases} \hat{A}''_0 & \text{on } X \setminus \cup_P B(x_P, 4\lambda_P^{1/2}), \\ \Theta(A''_0, F_P^{\mathfrak{g}}, F_P^T, A_P, F_P^s) & \text{on } \Omega(y_P, \frac{1}{2}\lambda_P^{1/2}, 2\lambda_P^{1/2}), \\ \hat{A}'_P & \text{on } B(y_P, 2\lambda_P^{1/2}). \end{cases}$$

Note, however, that the identification of $\Omega(y_P, \frac{1}{2}\lambda_P^{1/2}, 2\lambda_P^{1/2})$ and $\varphi_n(\Omega(\frac{1}{2}\lambda_P^{1/2}, 2\lambda_P^{1/2}))$ is obtained using φ_n and the exponential map around y_P given by the tangent frame F_P^T and the flattened metric $g_{\mathcal{P}, \mathbf{y}}$ constructed in Lemma 4.5.2. We then define the *crude splicing map* by

$$(6.4.11) \quad \gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}([(A_0, \Phi_0), \mathbf{A}]) := [A'', \Phi''],$$

where A'' is the spin^u connection inducing the connection \hat{A}'' appearing in (6.4.10).

6.4.3. Properties of the crude splicing map. The following equivariance of the crude splicing maps with respect to the symmetric group map (6.2.14) holds because the point $[A'', \Phi'']$ in (6.4.11) is independent of the labelling of the elements of the subsets $P \subset N_\ell$ making up the partition \mathcal{P} .

LEMMA 6.4.2. *If $\sigma \in \mathfrak{S}_\ell$ and $\mathbf{A} \in \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$, then*

$$\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}([(A_0, \Phi_0), \mathbf{A}]) = \gamma''_{\mathfrak{t}, \mathfrak{s}, \sigma(\mathcal{P})}([(A_0, \Phi_0), \sigma(\mathbf{A})]),$$

where $\sigma(\mathbf{A})$ is the image of \mathbf{A} under the map (6.2.14).

The following lemma gives a more explicit description of the crude splicing map when restricted to a subspace of its domain.

LEMMA 6.4.3. *Let $T(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \subset \bar{\mathbf{G}}\mathfrak{l}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ be the subspace defined in (6.2.9). Then there is a smoothly-stratified diffeomorphism*

$$k_{\mathcal{P}} : T(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \cong \tilde{\nu}(X^\ell, \mathcal{P})/\mathfrak{S}(\mathcal{P}),$$

where $\tilde{\nu}(X^\ell, \mathcal{P})$ is the normal bundle defined in (4.3.4). The restriction of the crude splicing map $\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}$ to

$$\tilde{N}_{(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_\mathfrak{s}} T(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$$

is given by

$$\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}([(A_0, \Phi_0), \mathbf{A}]) = \left(\Theta_{\mathcal{P}}((A_0, \Phi_0), (y_P)_{P \in \mathcal{P}}), e(X^\ell, g_{\mathcal{P}})(k_{\mathcal{P}}(\mathbf{A})) \right),$$

where $(A_0, \Phi_0) \in \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta)$, and $\mathbf{A} \in T(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ lies over $(y_P)_{P \in \mathcal{P}} \in \Delta^\circ(X^\ell, \mathcal{P})$, and $\Theta_{\mathcal{P}}$ is the flattening map (6.3.6), and $e(X^\ell, g_{\mathcal{P}})$ is the exponential map defined in (4.3.1), and $e(X^\ell, g_{\mathcal{P}})(k_{\mathcal{P}}(\mathbf{A}))$ represents an element of $\text{Sym}^\ell(X)$.

PROOF. By the definition (6.2.9), points in $T(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ can be written as

$$\mathbf{A} = ((F_P^T, F_P^g)_{P \in \mathcal{P}}, ([\Theta, F_P^s, \mathbf{v}_P]_{P \in \mathcal{P}})),$$

where F_P^T and F_P^g are the frames appearing in Notation 6.4.1 and Θ is the product connection on $S^4 \times \mathfrak{so}(3)$ with $[\Theta, F_P^s, \mathbf{v}_P] \in \bar{M}_{\text{SPL}, |P|}^{\mathfrak{s}, \mathfrak{h}}(\delta)$. The mass-centering condition on $[\Theta, F_P^s, \mathbf{v}_P]$ implies that $\mathbf{v}_P \in Z_P(\delta)$, where $Z_P(\delta)$ is defined following (5.3.1). Because the $\text{SO}(3)$ action on the frame is trivial on such elements of $\bar{M}_{\text{SPL}, |P|}^{\mathfrak{s}, \mathfrak{h}}(\delta)$, we obtain a smoothly-stratified diffeomorphism by setting

$$k_{\mathcal{P}}((F_P^T, F_P^g)_{P \in \mathcal{P}}, ([\Theta, F_P^s, \mathbf{v}_P]_{P \in \mathcal{P}})) := ((F_P^T, v_P)_{P \in \mathcal{P}}),$$

and comparing the definition of $\tilde{\nu}(X^\ell, \mathcal{P})$ in (4.3.4). The assertion regarding the crude splicing map follows immediately from its definition in (6.4.11), (6.4.9) and (6.4.10). \square

REMARK 6.4.4. In contrast to the conclusion of Lemma 6.4.3, the standard splicing map would satisfy, for $\mathbf{A} \in \Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$, the subspace defined in (6.2.7),

$$\gamma'_{X, \mathcal{P}}([(A_0, \Phi_0), \mathbf{A}]) = \left([A_0, \Phi_0], e(X^\ell, g_{\mathcal{P}})(k_{\mathcal{P}}(\mathbf{A})) \right),$$

that is, the background pair would not be flattened.

6.5. Overlap spaces and maps

As described in Section 1.2.1, we describe the intersection of the images of the crude splicing maps $\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}$ and $\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}'}$, for partitions $\mathcal{P} < \mathcal{P}'$ of N_ℓ , by introducing spaces of overlap data, $\text{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$, and defining maps from the space of overlap data to the domains of these crude splicing maps in such a way that the intersection of the images of $\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}$ and $\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}'}$ is described by a diagram similar to (1.2.4). The construction is analogous to that of the spliced end W_κ defined in (5.6.2), specifically the proof of the commutativity of the diagram (5.5.27).

6.5.1. The overlap space. Recall from (6.2.2) and (6.2.6) that the space of gluing data is a fiber bundle $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}') \rightarrow \Sigma(X^\ell, \mathcal{P}')$. For partitions $\mathcal{P} < \mathcal{P}'$ of N_ℓ , one can think of the space of overlap data associated to these partitions as the restriction of the fiber bundle $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}') \rightarrow \Sigma(X^\ell, \mathcal{P}')$ to the intersection of the tubular neighborhood $\mathcal{U}(X^\ell, \mathcal{P})$ of $\Sigma(X^\ell, \mathcal{P})$ with the stratum $\Sigma(X^\ell, \mathcal{P}')$. This construction will be similar to that of the space $\mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$ defined in (5.6.3b). Because our constructions must respect the action of the symmetric group, the space of overlap data must describe the intersection of the image of $\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}$ with $\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}''}$ for all $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$, where $[\mathcal{P} < \mathcal{P}']$ is defined in (3.2.1) to be the partitions conjugate under the symmetric group to \mathcal{P}' that are refinements of \mathcal{P} . (See the discussion prior to Lemma 3.3.6 for more on this requirement.) Then, we define the space of overlap data by

$$(6.5.1) \quad \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) := \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \times_{G(\mathcal{P})} \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}} (\Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}''_P) \times \bar{M}(\mathcal{P}''_P)),$$

where $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ is the gluing-data bundle defined in (6.2.2), and $\bar{M}(\mathcal{P}_P'')$ is the product of instanton moduli spaces with spliced ends defined in (6.2.5), and \mathcal{P}_P'' is the partition of P given by the subsets of P in \mathcal{P}'' , and the diagonal $\Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}_P'')$ is defined in (5.3.2). Recall that we denote elements of $\Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}_P'')$ by $(v_Q)_{Q \in \mathcal{P}_P''}$, where $v_Q \in \mathbb{R}^4$ (as described after Lemma 5.3.1). We then denote elements of $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ by

$$(6.5.2) \quad \mathbf{A}' = \left((F_P^T, F_P^g)_{P \in \mathcal{P}}, \left((v_Q)_{Q \in \mathcal{P}_P''}, ([A_Q, F_Q^s, \mathbf{v}_Q])_{Q \in \mathcal{P}_P''} \right)_{P \in \mathcal{P}} \right) \in \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']),$$

where $(F_P^T, F_P^g)_{P \in \mathcal{P}} \in \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$, and $(v_Q)_{Q \in \mathcal{P}_P''} \in \Delta^\circ(Z_{|P|}, \mathcal{P}_P'')$, and $[A_Q, F_Q^s, \mathbf{v}_Q] \in \bar{M}_{\text{SPL}, |Q|}^{s, \natural}(\delta)$.

The projection $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \Sigma(X^\ell, \mathcal{P})$ induces a projection,

$$(6.5.3) \quad \pi_\Sigma : \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \rightarrow \Sigma(X^\ell, \mathcal{P}).$$

Suppose Θ is the product connection on $S^4 \times \mathfrak{so}(3)$ and $c_{|Q|}$ is the cone point defined following (5.1.1). Because $[\Theta, F_Q^s, c_{|Q|}] \in \bar{M}_{\text{SPL}, |Q|}^{s, \natural}$ is a fixed point of the action of $\text{SO}(3) \times \text{SO}(4)$ on $\bar{M}_{\text{SPL}, |Q|}^{s, \natural}$ the subspace

$$(6.5.4) \quad \begin{aligned} & T(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \\ & := \left\{ \left((F_P^T, F_P^g)_{P \in \mathcal{P}}, \left((v_Q)_{Q \in \mathcal{P}_P''}, ([\Theta, F_Q^s, c_{|Q|}])_{Q \in \mathcal{P}_P''} \right)_{P \in \mathcal{P}} \right) \in \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \right\}, \end{aligned}$$

is identified with the subspace $\nu(X^\ell, \mathcal{P} \rightarrow [\mathcal{P}'])$ of the normal bundle $\nu(X^\ell, \mathcal{P})$ of $\Delta^\circ(X^\ell, \mathcal{P})$,

$$(6.5.5) \quad k_{\Sigma, \mathcal{P}, [\mathcal{P}']} : T(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \cong \nu(X^\ell, \mathcal{P} \rightarrow [\mathcal{P}']) \subset \nu(X^\ell, \mathcal{P}),$$

where $\nu(X^\ell, \mathcal{P} \rightarrow [\mathcal{P}'])$ was defined in (3.3.4) and which by (4.3.7) can be presented as

$$(6.5.6) \quad \nu(X^\ell, \mathcal{P} \rightarrow [\mathcal{P}']) \cong \text{Fr}(TX^\ell, \mathcal{P}, g_{\mathcal{P}}) \times_{G(T, \mathcal{P})} \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}} \Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}_P'').$$

Comparing the preceding presentation with (6.5.1), we see that there is a projection,

$$(6.5.7) \quad \pi_{\Sigma, \mathcal{P}, [\mathcal{P}']} : \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \rightarrow T(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']),$$

given by (using the same notation as in (6.5.4))

$$\begin{aligned} & \pi_{\Sigma, \mathcal{P}, [\mathcal{P}']} \left((F_P^T, F_P^g)_{P \in \mathcal{P}}, \left((v_Q)_{Q \in \mathcal{P}_P''}, ([A_Q, F_Q^s, \mathbf{x}_Q])_{Q \in \mathcal{P}_P''} \right)_{P \in \mathcal{P}} \right) \\ & = \left((F_P^T, F_P^g)_{P \in \mathcal{P}}, \left((v_Q)_{Q \in \mathcal{P}_P''}, ([\Theta, F_Q^s, c_{|Q|}])_{Q \in \mathcal{P}_P''} \right)_{P \in \mathcal{P}} \right). \end{aligned}$$

The composition of the projection (6.5.7) and the identification (6.5.5) defines a map,

$$(6.5.8) \quad \pi_{\nu(\Sigma)} = k_{\Sigma, \mathcal{P}, [\mathcal{P}']} \circ \pi_{\Sigma, \mathcal{P}, [\mathcal{P}']} : \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \rightarrow \nu(X^\ell, \mathcal{P} \rightarrow [\mathcal{P}']),$$

which we call the projection onto the normal bundle.

The S^1 action on $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ discussed in Section 6.2.4 is defined by the S^1 action on $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ given in Section 6.2.4. This circle action then gives an action of S^1 on $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ through its action on the factor $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ in the definition (6.5.1) of $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$. Hence, there is an S^1 action,

$$(6.5.9) \quad S^1 \times \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathfrak{g}_s} \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \rightarrow \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathfrak{g}_s} \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']),$$

defined in the same manner as the circle action (6.2.13).

6.5.2. The upwards overlap map. We now construct the upwards overlap map described in the diagram (1.2.4). This construction will be analogous to that of the map $\rho_{\mathcal{P},[\mathcal{P}']}^{\Theta,u}$ defined in (5.5.24). As mentioned in the beginning of the previous section, the image of this map will be contained in the union of the gluing data bundles corresponding to all partitions $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$,

$$(6.5.10) \quad \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, [\mathcal{P} < \mathcal{P}']) := \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}'') / \mathfrak{S}(\mathcal{P}),$$

where $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}'')$ is defined in (6.2.6) (compare the space (4.6.5) used to define the upwards transition map for the subspaces of $\text{Sym}^\ell(X)$) and $\mathfrak{S}(\mathcal{P})$ acts by permuting elements of $[\mathcal{P} < \mathcal{P}']$. For the space of overlap data $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ defined in (6.5.1), set

$$(6.5.11) \quad \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) := \pi_{\nu(\Sigma)}^{-1} \left(\mathcal{O}(X^\ell, g_{\mathcal{P}}) \cap \nu(X^\ell, \mathcal{P} \rightarrow [\mathcal{P}']) \right) \subset \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']).$$

We will define a subspace,

$$\tilde{N}(\delta) \times_{\mathcal{G}_s} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \subset \tilde{N}(\delta) \times_{\mathcal{G}_s} \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']),$$

and an *upwards overlap map*,

$$(6.5.12) \quad \rho_{\mathcal{P},[\mathcal{P}']}^{\mathfrak{t},\mathfrak{s},u} : \tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \rightarrow \tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, [\mathcal{P} < \mathcal{P}']),$$

will be defined, using the notation for points in the domain from (6.5.2), to act as the identity on the background pair $(A_0, \Phi_0) \in \tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta)$, the identity on the S^4 connections, and parallel translation of the frames (F_P^T, F_P^g) from $y_P \in X$ to each $x_Q \in X$ for $Q \in \mathcal{P}''_P$, where $(x_Q)_{Q \in \mathcal{P}''} \in \Delta^\circ(X^\ell, \mathcal{P}'')$ is the image of \mathbf{A}' under the composition $e(X^\ell, g_{\mathcal{P}}) \circ \pi_{\nu(\Sigma)}$. This parallel translation is done with respect to the locally flattened metric and background connection A''_0 . If $P \in \mathcal{P} \cap \mathcal{P}'$, then because $\Delta^\circ(Z_P(\delta), \mathcal{P}'_P)$ is given by $|P|$ -copies of the zero vector (see the sentence following (4.2.6)), we would have $x_Q = y_P$ and the parallel translation for this case would be given by the identity map.

We now give a formal definition of the upwards overlap map. Assume that x is a point in a geodesic ball centered at $y \in X$. Then for any connection A on a bundle over X and frame F of this bundle lying over y , let $T_{x,y}^A(F)$ denote parallel translation, respect to the connection A , of the frame F from y to x along the radial geodesic. We let $T_{x,y}^g$ denote parallel translation with respect to the Levi-Civita connection of a metric g on X .

Using the notation (6.5.2) for $\mathbf{A}' \in \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$, write

$$(6.5.13) \quad \pi_\Sigma(\mathbf{A}') = \mathbf{y} = (y_P)_{P \in \mathcal{P}} \quad \text{and} \quad e(X^\ell, g_{\mathcal{P}}) \circ \pi_{\nu(\Sigma)}(\mathbf{A}') = \mathbf{y}' = (x_Q)_{Q \in \mathcal{P}''}.$$

For $(A_0, \Phi_0) \in \tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta)$, let $\Theta_{\mathcal{P}}(A_0, \Phi_0, \mathbf{y}) = (A'_0, \Phi'_0)$ be as appearing in (6.3.6). Then,

$$(6.5.14) \quad \begin{aligned} & \rho_{\mathcal{P},[\mathcal{P}']}^{\mathfrak{t},\mathfrak{s},u} \left([(A_0, \Phi_0), \mathbf{A}'] \right) \\ &= \rho_{\mathcal{P},[\mathcal{P}']}^{\mathfrak{t},\mathfrak{s},u} \left(\left[(A_0, \Phi_0), (F_P^T, F_P^g)_{P \in \mathcal{P}}, \left((v_Q)_{Q \in \mathcal{P}''_P}, ([A_Q, F_Q^s, \mathbf{x}_Q])_{Q \in \mathcal{P}''_P} \right)_{P \in \mathcal{P}} \right] \right) \\ &:= \left[(A_0, \Phi_0), \left((T_{x_Q, y_P}^{g_{\mathcal{P}, \mathbf{y}}} (F_P^T), T_{x_Q, y_P}^{A'_0} (F_P^g))_{Q \in \mathcal{P}''}, ([A_Q, F_Q^s, \mathbf{x}_Q])_{Q \in \mathcal{P}''_P} \right) \right], \end{aligned}$$

where the indices P and Q appearing in the parallel translations $T_{x_Q, y_P}^{g_{\mathcal{P}, \mathbf{y}}}$ and $T_{x_Q, y_P}^{\hat{A}'_0}$ satisfy $Q \subseteq P$. The metric $g_{\mathcal{P}, \mathbf{y}}$ defining the parallel translation $T_{x_Q, y_P}^{g_{\mathcal{P}, \mathbf{y}}}$ is the locally flattened metric on X defined in Lemma 4.5.2. The proof of the following lemma is then straightforward.

LEMMA 6.5.1. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_ℓ . Then the map $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathbf{s}, u}$ defined above is an open embedding which is equivariant with respect to the S^1 actions defined by (6.2.13) and (6.5.9).*

6.5.3. Downwards overlap map. The downwards overlap map is an S^1 -equivariant map,

$$\rho_{\mathcal{P}, \mathcal{P}'}^{\mathbf{t}, \mathbf{s}, d} : \tilde{N}_{\mathbf{t}(\ell), \mathbf{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}_d(\mathbf{t}, \mathbf{s}, \mathcal{P}, [\mathcal{P}']) \rightarrow \tilde{N}_{\mathbf{t}(\ell), \mathbf{s}}(\delta) \times_{\mathcal{G}_s} \bar{\mathrm{Gl}}(\mathbf{t}, \mathbf{s}, \mathcal{P}),$$

defined similarly to the map $\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, d}$ defined in (5.5.25) and where the domain,

$$\mathcal{O}_d(\mathbf{t}, \mathbf{s}, \mathcal{P}, [\mathcal{P}']) \subset \mathcal{O}(\mathbf{t}, \mathbf{s}, \mathcal{P}, [\mathcal{P}']) \subset \bar{\mathrm{Gl}}(\mathbf{t}, \mathbf{s}, \mathcal{P}, [\mathcal{P}']),$$

will be defined in (6.5.18). We wish to define, for $\mathbf{A}' \in \bar{\mathrm{Gl}}(\mathbf{t}, \mathbf{s}, \mathcal{P}, [\mathcal{P}'])$, using the notation of (6.5.2), and $(A_0, \Phi_0) \in \tilde{N}_{\mathbf{t}(\ell), \mathbf{s}}(\delta)$,

$$\begin{aligned} & \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathbf{s}, d}([(A_0, \Phi_0), \mathbf{A}']) \\ (6.5.15) \quad &= \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathbf{s}, d} \left(\left[(A_0, \Phi_0), (F_P^T, F_P^g)_{P \in \mathcal{P}}, \left((v_Q)_{Q \in \mathcal{P}''_P}, ([A_Q, F_Q^s, \mathbf{x}_Q])_{Q \in \mathcal{P}''_P} \right)_{P \in \mathcal{P}} \right] \right) \\ &:= \left[(A_0, \Phi_0), (F_P^T, F_P^g)_{P \in \mathcal{P}}, \left(\gamma'_{\Theta, \mathcal{P}''_P} \left((v_Q)_{Q \in \mathcal{P}''_P}, ([A_Q, F_Q^s, \mathbf{x}_Q])_{Q \in \mathcal{P}''_P} \right) \right)_{P \in \mathcal{P}} \right], \end{aligned}$$

where $\gamma'_{\Theta, \mathcal{P}''_P}$ is the splicing map defined in (5.4.3) and we use the convention that $\gamma'_{\Theta, \mathcal{P}''_P}$ is the identity map when $P \in \mathcal{P} \cap \mathcal{P}''$. To motivate the definition of the domain $\mathcal{O}_d(\mathbf{t}, \mathbf{s}, \mathcal{P}, [\mathcal{P}'])$ and to show that $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathbf{s}, d}$ is a bundle map, we give the following description. If we define a map $\rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathbf{s}, d}$ on a subspace of the fiber of the map π_Σ defined in (6.5.3) by

$$(6.5.16) \quad \rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathbf{s}, d} := \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}} \gamma'_{\Theta, \mathcal{P}''_P},$$

(the f subscript standing for ‘fiber’) with domain and range,

$$(6.5.17) \quad \begin{array}{c} \mathcal{O} \subset \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}} \left(\Delta^\circ(Z_P(\delta_P), \mathcal{P}''_P) \times \prod_{Q \in \mathcal{P}''_P} \bar{M}_{\mathrm{SPL}, |Q|}^{s, \natural}(\delta_Q) \right) \\ \rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathbf{s}, d} \downarrow \\ \prod_{P \in \mathcal{P}} \bar{M}_{\mathrm{SPL}, |P|}^{s, \natural}(\delta_P) \end{array}$$

then by the $\mathrm{SO}(3) \times \mathrm{SO}(4)$ equivariance of $\gamma'_{\Theta, \mathcal{P}''_P}$ given by Lemma 5.4.2 and the $\mathfrak{S}(\mathcal{P})$ equivariance of the construction, we see that the map $\rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathbf{s}, d}$ is $G(\mathcal{P})$ -equivariant. The map $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathbf{s}, d}$ from (6.5.15) is then defined by the property that its restriction to the fiber of π_Σ equals $\rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathbf{s}, d}$.

The preceding description of $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathbf{s}, d}$ as the extension from the fiber of $\rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathbf{s}, d}$ implies that the domain of $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathbf{s}, d}$ must be contained in the set of points $\mathbf{A}' \in \bar{\mathrm{Gl}}(\mathbf{t}, \mathbf{s}, \mathcal{P}, [\mathcal{P}'])$ which lie in the

subset \mathcal{O} appearing in (6.5.17) of the fiber of the map π_Σ defined in (6.5.3). More precisely, we must assume that domain of $\rho_{\mathcal{P},[\mathcal{P}']}^{\mathbf{t},\mathbf{s},d}$ satisfies

$$(6.5.18) \quad \begin{aligned} & \mathcal{O}_d(\mathbf{t}, \mathbf{s}, \mathcal{P}, [\mathcal{P}']) \\ & \subseteq \left\{ \left((F_P^T, F_P^g)_{P \in \mathcal{P}}, \left((v_Q)_{Q \in \mathcal{P}''}, ([A_Q, F_Q^s, \mathbf{x}_Q])_{Q \in \mathcal{P}''} \right)_{P \in \mathcal{P}} \right) \in \bar{\mathrm{Gl}}(\mathbf{t}, \mathbf{s}, \mathcal{P}, [\mathcal{P}']) : \right. \\ & \quad \left. \left((v_Q)_{Q \in \mathcal{P}''}, ([A_Q, F_Q^s, \mathbf{x}_Q])_{Q \in \mathcal{P}''} \right) \in \mathcal{O}_1^{\mathrm{ASD}}(\Theta, \mathcal{P}''_P, \delta_P) \text{ for all } P \in \mathcal{P} \right\}, \end{aligned}$$

where $\mathcal{O}_1^{\mathrm{ASD}}(\Theta, \mathcal{P}''_P, \delta_P)$ is the domain of $\gamma'_{\Theta, \mathcal{P}''_P}$ defined in Lemma 5.6.5.

LEMMA 6.5.2. *If $\mathcal{O}_d(\mathbf{t}, \mathbf{s}, \mathcal{P}, [\mathcal{P}']) \subseteq \bar{\mathrm{Gl}}(\mathbf{t}, \mathbf{s}, \mathcal{P}, [\mathcal{P}'])$ is any open neighborhood of the subspace $T(\mathbf{t}, \mathbf{s}, \mathcal{P}, [\mathcal{P}'])$ satisfying (6.5.18), then the restriction of $\rho_{\mathcal{P},[\mathcal{P}']}^{\mathbf{t},\mathbf{s},d}$ to $\tilde{N}_{\mathbf{t}(\ell),\mathbf{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}_d(\mathbf{t}, \mathbf{s}, \mathcal{P}, [\mathcal{P}'])$ is an S^1 -equivariant open embedding, whose image is an open neighborhood of $N_{\mathbf{t}(\ell),\mathbf{s}}(\delta) \times \Sigma(\mathbf{t}, \mathbf{s}, \mathcal{P})$ in $\tilde{N}_{\mathbf{t}(\ell),\mathbf{s}}(\delta) \times_{\mathcal{G}_s} \bar{\mathrm{Gl}}(\mathbf{t}, \mathbf{s}, \mathcal{P})$ and where the S^1 actions are given by (6.2.13) and (6.5.9),*

PROOF. The conclusion follows immediately from Item (3) in Theorem 5.1.1 that the splicing maps $\gamma'_{\Theta, \mathcal{P}''_P}$ define open embeddings onto neighborhoods of the strata (5.1.5) in $\bar{M}_{\mathrm{SPL},|P|}^{\mathbf{s},\mathbf{d}}(\delta_P)$. \square

6.5.4. Equality of splicing maps. We now show how the constructions of the overlap maps, (6.5.12) and (6.5.15), and crude splicing maps (6.4.11) lead to a commutative diagram similar to the diagram (5.5.27) used to construct the spliced end W_κ defined in (5.6.2).

For $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$, let $\mathcal{O}(\mathbf{t}, \mathbf{s}, \mathcal{P}'') \subset \bar{\mathrm{Gl}}(\mathbf{t}, \mathbf{s}, \mathcal{P}'')$ be the open subspace defined in (6.4.2) by the separation condition. Because this separation condition is invariant under the action of $\mathfrak{S}(\mathcal{P})$ by Item (1) in Lemma 6.3.4, the collection of open subsets $\{\mathcal{O}(\mathbf{t}, \mathbf{s}, \mathcal{P}'')\}_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']}$ is $\mathfrak{S}(\mathcal{P})$ -invariant. Define

$$(6.5.19) \quad \mathcal{O}(\mathbf{t}, \mathbf{s}, [\mathcal{P} < \mathcal{P}']) := \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \mathcal{O}(\mathbf{t}, \mathbf{s}, \mathcal{P}'') \right) / \mathfrak{S}(\mathcal{P}) \subset \bar{\mathrm{Gl}}(\mathbf{t}, \mathbf{s}, [\mathcal{P} < \mathcal{P}']).$$

The $\mathfrak{S}(\mathcal{P})$ -invariance of the map,

$$\bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \gamma''_{\mathbf{t},\mathbf{s},\mathcal{P}''} : \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \tilde{N}_{\mathbf{t}(\ell),\mathbf{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}(\mathbf{t}, \mathbf{s}, \mathcal{P}'') \rightarrow \bar{\mathcal{C}}_{\mathbf{t}},$$

given by Lemma 6.4.2 then implies that it defines a map on the $\mathfrak{S}(\mathcal{P})$ -quotient,

$$(6.5.20) \quad \gamma''_{\mathbf{t},\mathbf{s},[\mathcal{P} < \mathcal{P}']} : \tilde{N}_{\mathbf{t}(\ell),\mathbf{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}(\mathbf{t}, \mathbf{s}, [\mathcal{P} < \mathcal{P}']) \rightarrow \bar{\mathcal{C}}_{\mathbf{t}}.$$

Given open subspaces, $\mathcal{O}(\mathbf{t}, \mathbf{s}, \mathcal{P}) \subset \bar{\mathrm{Gl}}(\mathbf{t}, \mathbf{s}, \mathcal{P})$ as in (6.4.2), and an $\mathfrak{S}(\mathcal{P})$ -invariant collection of subspaces, $\mathcal{O}(\mathbf{t}, \mathbf{s}, \mathcal{P}'')$ for $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$ as above, we define an open subspace,

$$(6.5.21) \quad \mathcal{O}_1(\mathbf{t}, \mathbf{s}, \mathcal{P}, [\mathcal{P}']) \subset \bar{\mathrm{Gl}}(\mathbf{t}, \mathbf{s}, \mathcal{P}, [\mathcal{P}']),$$

to be the subset of points satisfying the conditions (6.5.11), (6.5.18), and,

$$\mathcal{O}_1(\mathbf{t}, \mathbf{s}, \mathcal{P}, [\mathcal{P}']) \subseteq \left(\rho_{\mathcal{P},[\mathcal{P}']}^{\mathbf{t},\mathbf{s},u} \right)^{-1} \left(\mathcal{O}(\mathbf{t}, [\mathcal{P} < \mathcal{P}']) \right) \cap \left(\rho_{\mathcal{P},[\mathcal{P}']}^{\mathbf{t},\mathbf{s},d} \right)^{-1} \left(\mathcal{O}(\mathbf{t}, \mathbf{s}, \mathcal{P}) \right).$$

We then have the

PROPOSITION 6.5.3. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_ℓ . Assume that the families of metrics $g_{\mathcal{P}''}$ on X satisfy the conditions in Section 6.2.2. Let $\mathcal{O}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \subset \bar{\text{Gl}}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$ and $\mathcal{O}(\mathbf{t}, \mathfrak{s}, \mathcal{P}'') \subset \bar{\text{Gl}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}'')$ for $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$ be the open subspaces defined in (6.4.2). Let $\mathcal{O}(\mathbf{t}, \mathfrak{s}, [\mathcal{P} < \mathcal{P}'])$ and $\mathcal{O}_1(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ be the spaces defined in (6.5.19) and (6.5.21), respectively. Then the following diagram commutes,*

$$(6.5.22) \quad \begin{array}{ccc} \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}_1(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) & \xrightarrow{\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, u}} & \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}(\mathbf{t}, \mathfrak{s}, [\mathcal{P} < \mathcal{P}']) \\ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d} \downarrow & & \gamma''_{\mathbf{t}, \mathfrak{s}, [\mathcal{P} < \mathcal{P}']} \downarrow \\ \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) & \xrightarrow{\gamma''_{\mathbf{t}, \mathfrak{s}, \mathcal{P}}} & \bar{\mathcal{C}}_{\mathbf{t}} \end{array}$$

and the maps are S^1 -equivariant with respect to the action (6.5.9) with weight two on the domain of the overlap maps, the action (6.2.13) with weight two on the domains of the crude splicing maps, and the action (2.1.9) on $\bar{\mathcal{C}}_{\mathbf{t}}$.

PROOF. The conditions in the definition of the subspace $\mathcal{O}_1(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ imply that the compositions in the diagram (6.5.22),

$$(6.5.23) \quad \gamma''_{\mathbf{t}, \mathfrak{s}, [\mathcal{P} < \mathcal{P}']} \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, u} \quad \text{and} \quad \gamma''_{\mathbf{t}, \mathfrak{s}, \mathcal{P}} \circ (\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}),$$

are defined on $\tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}_1(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$.

The S^1 -equivariance of the overlap maps appears in Lemmas 6.5.1 and 6.5.2. The action (2.1.9) is just scalar multiplication on the section. Hence, the crude splicing maps will be equivariant if S^1 acts on their domains by scalar multiplication on the section of V_0^+ (where $\mathbf{t}(\ell) = (\rho, V_0^\pm)$). By Lemma 6.2.1, the action in (6.2.13) with weight two is scalar multiplication on the section of V_0^+ , giving the desired equivariance.

To see that the diagram (6.5.22) commutes, begin by restricting the compositions in (6.5.23) to

$$\mathcal{G}_s(A_0, \Phi_0) \times_{\mathcal{G}_s} \mathcal{O}_1(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])|_{\pi_\Sigma^{-1}(\mathbf{y})} \subset \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']),$$

for $(A_0, \Phi_0) \in \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta)$ and $\mathbf{y} \in \Delta^\circ(X^\ell, \mathcal{P})$. For $\mathbf{A}' \in \mathcal{O}_1(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])|_{\pi_\Sigma^{-1}(\mathbf{y})}$, we write

$$(6.5.24) \quad \begin{aligned} (A''_u, \Phi''_u) &:= (\gamma''_{\mathbf{t}, \mathfrak{s}, [\mathcal{P} < \mathcal{P}']} \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, u}) ([(A_0, \Phi_0), \mathbf{A}']), \\ (A''_d, \Phi''_d) &:= (\gamma''_{\mathbf{t}, \mathfrak{s}, \mathcal{P}} \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}) ([(A_0, \Phi_0), \mathbf{A}']). \end{aligned}$$

Thus, if $[A''_u, \Phi''_u] = [A''_d, \Phi''_d]$, then the diagram (6.5.22) commutes. The definitions of the overlap maps in (6.5.14) and (6.5.15) and of the crude splicing maps in (6.4.10) and (6.4.11) imply that A''_d and A''_u are defined respectively by splicing in connections on S^4 at the points in X given by $(y_P)_{P \in \mathcal{P}} = \mathbf{y} \in \Sigma(X, \mathcal{P})$ and at the points in X given by $(x_Q)_{Q \in \mathcal{P}'} = \mathbf{y}' \in \Sigma(X, \mathcal{P}')$ as defined in (6.5.13). Because $\pi(X^\ell, g_{\mathcal{P}})(\mathbf{y}') = \mathbf{y}$ by (6.5.13), Item (2) in Lemma 4.4.2 and the consistency of the families of metrics $g_{\mathcal{P}}$ and $g_{\mathcal{P}'}$ imply that $g_{\mathcal{P}, \mathbf{y}} = g_{\mathcal{P}', \mathbf{y}'}$. Similarly, the consistency condition (6.3.7) for the flattening maps constructed in Lemma 6.3.4 implies that

$$(A'_0, \Phi'_0) = \Theta_{\mathcal{P}}(A_0, \Phi_0, \mathbf{y}) = \Theta_{\mathcal{P}'}(A_0, \Phi_0, \mathbf{y}').$$

The definition in (6.4.11) and (6.4.9) of the section given by the crude splicing maps then imply that $\Phi''_u = \Phi_0 = \Phi''_d$.

Property (6.5.11) of $\mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ implies that \mathbf{y}' lies in the suitably small tubular neighborhood $\mathcal{U}(X^\ell, g_{\mathcal{P}})$ defined in Lemma 6.3.4. Thus, from (6.3.9) the inclusions

$$B(x_Q, 4\tilde{s}_{P'}(\mathbf{y}')^{1/2}) \Subset B(y_P, 2\tilde{s}_P(\mathbf{y})^{1/2}), \quad \text{for } Q \subseteq P,$$

hold for the balls where the splicing takes place. The definition of the crude splicing map in (6.4.10) implies that A''_u and A''_d both equal A'_0 , and thus each other, on the complement of the balls $B(y_P, 2\tilde{s}_P(\mathbf{y})^{1/2})$. Item (2) of Lemma 6.3.4 implies that the connection \hat{A}'_0 is flat on each ball $B(y_P, 2\tilde{s}_P(\mathbf{y})^{1/2})$ while Lemma 4.5.2 implies that the metric $g_{\mathcal{P}, \mathbf{y}}$ is flat on this set. Because the metric and the connection \hat{A}'_0 are flat on the union of the balls $B(y_P, 2\tilde{s}_P(\mathbf{y})^{1/2})$, the equality between the restrictions of A''_u and A''_d to these balls follows from Lemma 5.5.2. \square

6.6. Construction of the space of global splicing data

The construction of the space of global splicing data follows from Proposition 6.5.3 and the arguments on shrinking the neighborhoods $\mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ given in the construction of the spliced end W_κ in Section 5.6.

THEOREM 6.6.1. *Let \mathfrak{t} be a spin^u structure on X and \mathfrak{s} a spin^c structure with $M_{\mathfrak{s}} \times \text{Sym}^\ell(X) \subset I\mathcal{M}$. For a partition \mathcal{P} of N_ℓ , let $\Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ be the subspace defined in (6.2.7) of the gluing data space $\tilde{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ defined in (6.2.6). For every partition \mathcal{P} of N_ℓ , there is a neighborhood $\mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ of $\Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ in $\tilde{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ such that the intersection of the images of the crude splicing maps $\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}$ defined in (6.4.11),*

$$(6.6.1) \quad \gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}(N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})) \cap \gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}'}(N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}')),$$

is empty if the sets $[\mathcal{P} < \mathcal{P}']$ and $[\mathcal{P}' < \mathcal{P}]$ defined in (3.2.1) are empty while if $[\mathcal{P} < \mathcal{P}']$ is non-empty, then the intersection is contained in

$$\begin{aligned} & \gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}} \left(\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, d} \left(\tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}_1(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \right) \right) \\ &= \gamma''_{\mathfrak{t}, \mathfrak{s}, [\mathcal{P} < \mathcal{P}']} \left(\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, u} \left(\tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}_1(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \right) \right), \end{aligned}$$

where $\mathcal{O}_1(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ is the subspace appearing in Proposition 6.5.3 and defined in (6.5.21). If $\sigma \in \mathfrak{S}_\ell$ and $\mathcal{P}' = \sigma(\mathcal{P})$ as defined by (3.1.1), then images of $\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}$ and $\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}'}$ are equal.

PROOF. The proof follows the same argument as that of Items (3) and (4) in Lemma 5.6.5 for the construction of W_κ . The final sentence follows from Lemma 6.4.2. \square

We will require the following special case of Theorem 6.6.1.

LEMMA 6.6.2. *Continue the notation of Theorem 6.6.1. Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_ℓ and let $T(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \subset \tilde{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ be as defined in (6.5.4). There is a subspace $T^\theta(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \subset T(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ defined by the property*

$$(6.6.2) \quad \begin{aligned} & \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} T^\theta(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \\ &= \left(\tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} T(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \right) \cap \left(\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, d} \right)^{-1} \left(\tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \right), \end{aligned}$$

where $\mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ is as defined in Theorem 6.6.1. Then

$$\left(\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}} \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, d} \right) \left(N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times T^\theta(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \right) \subset \gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}'}(N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times \Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P}')).$$

PROOF. To see that the property (6.6.2) defines a subspace of $T(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$, we show that for $(A_0, \Phi_0) \in \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta)$ and $\mathbf{A}' \in T(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$, the validity of the inclusion

$$\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}([(A_0, \Phi_0), \mathbf{A}']) \in \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$$

does not depend on $(A_0, \Phi_0) \in \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}$. Observe that the map $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}$ is defined in (6.5.15) by the identity on $\tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta)$ and by the map $\rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}$ (defined in (6.5.17)) on the fibers of π_Σ . Because $\rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}$ does not depend on $(A_0, \Phi_0) \in \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta)$, the validity of the above inclusion does not depend on $(A_0, \Phi_0) \in \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta)$, as required.

The definition of $T(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ and $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, u}$ in (6.5.14) imply that

$$\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, u} \left(N_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times T^\theta(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \right) \subset (N_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times \Sigma(\mathbf{t}, \mathfrak{s}, \mathcal{P}')).$$

The lemma then follows from the commutativity result in Proposition 6.5.3. \square

We now define the *space of global splicing data* to be

$$(6.6.3) \quad \mathcal{M}_{\mathbf{t}, \mathfrak{s}}^{\text{vir}} := \bigsqcup_{\mathcal{P}} \left(\tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \right) / \sim,$$

where $\mathcal{O}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$ is the space appearing in Theorem 6.6.1 and points in

$$\tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \quad \text{and} \quad \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}(\mathbf{t}, \mathfrak{s}, \mathcal{P}')$$

are identified by the relation \sim if their images under the crude splicing maps $\gamma''_{\mathbf{t}, \mathfrak{s}, \mathcal{P}}$ and $\gamma''_{\mathbf{t}, \mathfrak{s}, \mathcal{P}'}$ defined in (6.4.11) are equal. Therefore the space $\mathcal{M}_{\mathbf{t}, \mathfrak{s}}^{\text{vir}}$ is, up to the action of \mathfrak{S}_ℓ on partitions, the homotopy pushout of the diagram (6.5.22) in the sense of the following lemma.

LEMMA 6.6.3. *If $\mathcal{P} < \mathcal{P}'$ are partitions of N_ℓ and*

$$[A_0, \Phi_0, \mathbf{A}_\mathcal{P}] \in \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \quad \text{and} \quad [A'_0, \Phi'_0, \mathbf{A}_{\mathcal{P}'}] \in \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}(\mathbf{t}, \mathfrak{s}, \mathcal{P}'),$$

then

$$(6.6.4) \quad [A_0, \Phi_0, \mathbf{A}_\mathcal{P}] \sim [A'_0, \Phi'_0, \mathbf{A}_{\mathcal{P}'}]$$

if and only if $[A_0, \Phi_0] = [A'_0, \Phi'_0]$ and there is a point $[A_0, \Phi_0, \mathbf{A}'] \in \tilde{N} \times_{\mathcal{G}_s} \mathcal{O}_1(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ satisfying

$$(6.6.5) \quad \begin{aligned} [A_0, \Phi_0, \mathbf{A}_\mathcal{P}] &= \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}([A_0, \Phi_0, \mathbf{A}']), \\ [A'_0, \Phi'_0, \mathbf{A}_{\mathcal{P}'}] &= \rho_{\mathcal{P}', [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, u}([A_0, \Phi_0, \mathbf{A}']). \end{aligned}$$

If $\sigma \in \mathfrak{S}_\ell$ and $\mathcal{P}' = \sigma(\mathcal{P})$ as defined by (3.1.1), then (6.6.4) holds if and only if $[(A'_0, \Phi'_0), \mathbf{A}_{\mathcal{P}'}] = [(A_0, \Phi_0), \sigma(\mathbf{A}_\mathcal{P})]$, where $\sigma(\mathbf{A}_\mathcal{P})$ is defined by (6.2.14).

Recall from [67, p. 43] that a pair of topological spaces (X, A) is a *neighborhood deformation retract* (NDR) pair if there are continuous maps $u : X \rightarrow [0, 1]$ and $h : X \times [0, 1] \rightarrow X$ with $A = u^{-1}(0)$, $h(x, 0) = x$ for all $x \in X$, $h(a, t) = a$ for all $(a, t) \in A \times [0, 1]$, and $h(x, 1) \in A$ if $u(x) < 1$.

LEMMA 6.6.4. *The space $\mathcal{M}_{\mathbf{t}, \mathfrak{s}}^{\text{vir}}$ defined by (6.6.3) is a smoothly-stratified space with the property that if $\Sigma \subset \mathcal{M}_{\mathbf{t}, \mathfrak{s}}^{\text{vir}}$ is the complement of the top stratum, then $(\mathcal{M}_{\mathbf{t}, \mathfrak{s}}^{\text{vir}}, \Sigma)$ is an NDR pair.*

PROOF. By Theorem 6.6.1 and Lemma 6.6.3, $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ is the union of open sets

$$(6.6.6) \quad \tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}),$$

as \mathcal{P} varies over partitions, attached to each other by the smoothly-stratified maps,

$$(6.6.7) \quad \begin{array}{c} \rho_{\mathcal{P},[\mathcal{P}']}^{\mathfrak{t},\mathfrak{s},d} \left(\tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}_1(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \right) \\ \rho_{\mathcal{P},[\mathcal{P}']}^{\mathfrak{t},\mathfrak{s},u} \circ (\rho_{\mathcal{P},[\mathcal{P}']}^{\mathfrak{t},\mathfrak{s},d})^{-1} \Big\downarrow \\ \rho_{\mathcal{P},[\mathcal{P}']}^{\mathfrak{t},\mathfrak{s},u} \left(\tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}_1(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \right) \end{array}$$

or, in the case when $\mathcal{P}' = \sigma(\mathcal{P})$, by the smoothly-stratified diffeomorphism (6.2.14). Thus, $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ is a smoothly-stratified space.

Lemma 5.9.2 implies that for $\mathcal{U} = \tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$, the pair $(\mathcal{U}, \Sigma \cap \mathcal{U})$ is an NDR pair. A pair (U, Σ) is an NDR pair if and only if the inclusion $i : \Sigma \rightarrow U$ is a cofibration by [67, p. 43] or [90, Theorem 7.1.10]. Thus, $(\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}, \Sigma)$ is a local cofibration in the sense that $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ admits a cover by open sets \mathcal{U} with the property that the inclusion $\Sigma \cap \mathcal{U} \rightarrow \mathcal{U}$ is a cofibration. Because a local cofibration is a global one by [10, Satz 2], the inclusion $\Sigma \rightarrow \bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ is a cofibration and hence an NDR pair. \square

Because the transition maps (6.6.7) are S^1 -equivariant by Lemmas 6.5.1 and 6.5.2, the S^1 actions defined in (6.2.13) on the open sets (6.6.6) determine an S^1 action,

$$(6.6.8) \quad S^1 \times \bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}} \rightarrow \bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}.$$

The description of the open cover in the preceding lemma then yields

COROLLARY 6.6.5. *The space $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ admits an S^1 -equivariant fibration,*

$$(6.6.9) \quad \pi_N : \bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}} \rightarrow N_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta),$$

and an S^1 -equivariant embedding,

$$(6.6.10) \quad \gamma''_{\mathcal{M}} : \bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}} \rightarrow \bar{\mathcal{E}}_{\mathfrak{t}},$$

such that the restriction of $\gamma''_{\mathcal{M}}$ to $\tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ is equal to $\gamma''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$.

PROOF. Each of the open sets (6.6.6) of $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ admits an S^1 -equivariant fibration,

$$\pi_{N,\mathcal{P}} : \tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow N_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta).$$

Because the transition maps (6.6.7) are equal to the identity on $\tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta)$ by the definitions (6.5.14) and (6.5.15), these local fibrations define the global fibration (6.6.9).

Similarly, the embeddings $\gamma''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ defined on the open sets (6.6.6) are equal on the intersections of these open sets by the commutativity of the diagram (6.5.22). Hence, the maps $\gamma''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ define the global map (6.6.10). The global map $\gamma''_{\mathcal{M}}$ is an embedding by the definition of the equivalence relation following (6.6.3).

Finally, we see that the maps π_N and $\gamma''_{\mathcal{M}}$ are S^1 -equivariant by observing that their local definitions are S^1 -equivariant and that the S^1 action (6.6.8) is defined by the S^1 actions on the open sets (6.6.6). \square

6.7. Thom–Mather structures on the space of global splicing data

On each open subspace,

$$(6.7.1) \quad \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) := \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \subset \bar{M}_{\mathbf{t}, \mathfrak{s}}^{\text{vir}},$$

there is a projection map,

$$(6.7.2) \quad \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}) : \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \rightarrow N_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times \Sigma(\mathbf{t}, \mathfrak{s}, \mathcal{P}),$$

defined by $\text{id}_N \times \pi_{\Sigma}$, where π_{Σ} is the restriction of the projection defined in (6.2.11) to $\mathcal{O}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$ and id_N is the identity on $\tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta)$.

To prove that the compatibility conditions (4.1.1) hold for the projection maps $\pi(\mathbf{t}, \mathfrak{s}, \mathcal{P})$, we make the following observations on the overlap maps. For $\mathcal{P} < \mathcal{P}'$, define

$$(6.7.3) \quad U_f(\mathcal{P}, [\mathcal{P}']) := \text{Im}(\rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}) \subset \bar{M}(\mathcal{P}),$$

where the map on fibers $\rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}$ is defined in (6.5.16). By the definition of $\rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}$, we have

$$U_f(\mathcal{P}, [\mathcal{P}']) \subseteq \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}} \mathcal{U}(\Theta, \mathcal{P}''_P) / \mathfrak{S}(\mathcal{P}),$$

where $\mathcal{U}(\Theta, \mathcal{P}''_P)$ is the neighborhood of $\{[\Theta]\} \times \Sigma(Z_P(\delta_P), \mathcal{P}''_P)$ defined in Lemma 5.7.1 and $\mathfrak{S}(\mathcal{P})$ acts by permuting the components of $U_f(\mathcal{P}, [\mathcal{P}'])$ in the disjoint union given by $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$. We use the convention that if $P \in \mathcal{P} \cap \mathcal{P}''$, then $\mathcal{U}(\Theta, \mathcal{P}''_P) = \bar{M}_{\text{SPL}, |P|}^{s, \natural}(\delta_P)$.

The projection maps,

$$\begin{aligned} \pi(\Theta, \mathcal{P}'_P) : \mathcal{U}(\Theta, \mathcal{P}'_P) \subset \bar{M}_{\text{SPL}, |P|}^{s, \natural}(\delta_P) &\rightarrow \{[\Theta]\} \times \Sigma(Z_P(\delta_P), \mathcal{P}'_P) \\ &\cong \Delta^\circ(Z_P(\delta_P), \mathcal{P}'_P) / \mathfrak{S}(\mathcal{P}'_P), \end{aligned}$$

defined in (5.7.1) are $\text{SO}(3) \times \text{SO}(4)$ -equivariant. We use the convention that if $P \in \mathcal{P} \cap \mathcal{P}'$, then $\pi(\Theta, \mathcal{P}'_P)$ is projection to the cone point, $\pi(\Theta, \mathcal{P}'_P)([A, F^s, \mathbf{x}]) = [\Theta, F^s, c_P]$. Together with the identity maps on $\tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta)$ and on $\text{Fr}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$, the maps $\pi(\Theta, \mathcal{P}''_P)$ (for $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$) define a map,

$$\begin{array}{c} \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \text{Fr}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \times_{G(\mathcal{P})} \prod_{P \in \mathcal{P}} \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \mathcal{U}(\Theta, \mathcal{P}''_P) \\ \downarrow p_{\mathcal{P}, [\mathcal{P}']} \\ \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \text{Fr}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \times_{G(\mathcal{P})} \prod_{P \in \mathcal{P}} \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \{[\Theta]\} \times \Sigma(Z_P(\delta_P), \mathcal{P}'_P) \end{array}$$

Observe that in the definition of the preceding map, the subgroup $\mathfrak{S}(\mathcal{P}'_P) < \mathfrak{S}(\mathcal{P}) < G(\mathcal{P})$ is contained in the stabilizer $\Gamma(\mathfrak{S})$ of the partition \mathcal{P} defined following (3.1.1) and thus acts trivially

on $\text{Fr}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$, giving the isomorphisms,

$$\begin{aligned} & \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s \times S^1} \text{Fr}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \times_{G(\mathcal{P})} \prod_{P \in \mathcal{P}} \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \{[\Theta]\} \times \Delta^\circ(Z_P(\delta_P), \mathcal{P}'_P) \\ & \cong \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s \times S^1} \text{Fr}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \times_{G(\mathcal{P})} \prod_{P \in \mathcal{P}} \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \{[\Theta]\} \times \Delta^\circ(Z_P(\delta_P), \mathcal{P}'_P) / \mathfrak{S}(\mathcal{P}'_P) \\ & \cong \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s \times S^1} \text{Fr}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \times_{G(\mathcal{P})} \prod_{P \in \mathcal{P}} \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \{[\Theta]\} \times \Sigma(Z_P(\delta_P), \mathcal{P}'_P). \end{aligned}$$

Thus we can rewrite the range of the map $p_{\mathcal{P}, [\mathcal{P}']}$ to give

$$(6.7.4) \quad \begin{array}{c} \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s \times S^1} \text{Fr}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \times_{G(\mathcal{P})} \prod_{P \in \mathcal{P}} \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \mathcal{U}(\Theta, \mathcal{P}''_P) \\ \downarrow p_{\mathcal{P}, [\mathcal{P}']} \\ \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s \times S^1} \text{Fr}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \times_{G(\mathcal{P})} \prod_{P \in \mathcal{P}} \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \{[\Theta]\} \times \Delta^\circ(Z_P(\delta_P), \mathcal{P}'_P) \end{array}$$

The inclusions,

$$\begin{aligned} & \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \mathcal{U}(\Theta, \mathcal{P}''_P) \subset \bar{M}_{\text{SPL}, |P|}^{s, \dagger}(\delta_P) \quad \text{and} \\ & \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \{[\Theta]\} \times \Sigma(Z_P(\delta_P), \mathcal{P}'_P) \subset \bar{M}_{\text{SPL}, |P|}^{s, \dagger}(\delta_P), \end{aligned}$$

imply that the map $p_{\mathcal{P}, [\mathcal{P}]}$ in (6.7.4) is defined on a subspace of $\tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$. Observe that the relation,

$$(6.7.5) \quad (\text{id}_N \times \pi_\Sigma) \circ p_{\mathcal{P}, [\mathcal{P}']} = \text{id}_N \times \pi_\Sigma,$$

holds where π_Σ is the projection given in (6.2.11) because $p_{\mathcal{P}, [\mathcal{P}]}$ respects the fibers of $\bar{\text{Gl}}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$.

For $\mathcal{P} < \mathcal{P}'$, the following lemma describes the projection map $\pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}')$ on the image of the overlap map $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}$.

LEMMA 6.7.1. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_ℓ . For the projection $\pi_{\Sigma, \mathcal{P}, [\mathcal{P}]}$ defined in (6.5.7) and resulting map,*

$$\text{id}_N \times \pi_{\Sigma, \mathcal{P}, [\mathcal{P}']} : \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \bar{\text{Gl}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \rightarrow N_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times T(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']),$$

the following relations hold on $\tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}_1(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$:

$$(6.7.6) \quad \begin{aligned} \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}') \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, u} &= \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, u} \circ (\text{id}_N \times \pi_{\Sigma, \mathcal{P}, [\mathcal{P}]}), \\ p_{\mathcal{P}, [\mathcal{P}']} \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d} &= \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d} \circ (\text{id}_N \times \pi_{\Sigma, \mathcal{P}, [\mathcal{P}]}). \end{aligned}$$

PROOF. The conclusion follows from the definition of the projection maps in (6.7.2), (6.7.4), and (6.5.7) and the construction of $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}$ in (6.5.15) and of $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, u}$ in (6.5.14). \square

We now prove that the projection maps $\pi(\mathbf{t}, \mathfrak{s}, \mathcal{P})$ satisfy the first compatibility condition in (4.1.1).

LEMMA 6.7.2. *If $\mathcal{P} < \mathcal{P}'$ are partitions of N_ℓ , then the following equality holds:*

$$(6.7.7) \quad \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \circ \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}') = \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \quad \text{on } \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \cap \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}').$$

PROOF. By Theorem 6.6.1, the intersection $\bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \cap \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}')$ is contained in the image of the overlap maps in the diagram (6.5.22). Therefore, using the notation of Lemma 6.6.3, every point $[A_0, \Phi_0, \mathbf{A}_\mathcal{P}] = [A_0, \Phi_0, \mathbf{A}_{\mathcal{P}'}]$ in the intersection is given by

$$[A_0, \Phi_0, \mathbf{A}_\mathcal{P}] = \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}([A_0, \Phi_0, \mathbf{A}']) = \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, u}([A_0, \Phi_0, \mathbf{A}']) = [A'_0, \Phi'_0, \mathbf{A}_{\mathcal{P}'}]$$

for some $[A_0, \Phi_0, \mathbf{A}'] \in \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_\mathfrak{s}} \mathcal{O}_1(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$. Therefore,

$$\begin{aligned} & \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}')([A'_0, \Phi'_0, \mathbf{A}_{\mathcal{P}'}]) \\ &= \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}') \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, u}([A_0, \Phi_0, \mathbf{A}']) \\ &= \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, u} \circ (\text{id}_N \times \pi_{\Sigma, \mathcal{P}, [\mathcal{P}']})([A_0, \Phi_0, \mathbf{A}']) \quad (\text{by (6.7.6)}) \\ &= \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d} \circ (\text{id}_N \times \pi_{\Sigma, \mathcal{P}, [\mathcal{P}']})([A_0, \Phi_0, \mathbf{A}']) \quad (\text{by the definition of } \sim \text{ following (6.6.3)}) \\ &= p_{\mathcal{P}, [\mathcal{P}']} \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}([A_0, \Phi_0, \mathbf{A}']) \quad (\text{by (6.7.6)}). \end{aligned}$$

The preceding equality implies that

$$\begin{aligned} & \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \circ \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}')([A'_0, \Phi'_0, \mathbf{A}_{\mathcal{P}'}]) \\ &= \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \circ p_{\mathcal{P}, [\mathcal{P}']} \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}([A_0, \Phi_0, \mathbf{A}']) \\ &= (\text{id}_N \times \pi_\Sigma) \circ p_{\mathcal{P}, [\mathcal{P}']} \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}([A_0, \Phi_0, \mathbf{A}']) \quad (\text{by the definition of } \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \text{ following (6.7.2)}) \\ &= (\text{id}_N \times \pi_\Sigma) \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}([A_0, \Phi_0, \mathbf{A}']) \quad (\text{by (6.7.5)}) \\ &= \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}([A_0, \Phi_0, \mathbf{A}']) \quad (\text{by the definition of } \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \text{ following (6.7.2)}) \\ &= \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P})([A_0, \Phi_0, \mathbf{A}_\mathcal{P}]), \end{aligned}$$

and which, together with the equality $[A_0, \Phi_0, \mathbf{A}_\mathcal{P}] = [A'_0, \Phi'_0, \mathbf{A}_{\mathcal{P}'}]$, proves (6.7.7). \square

We now construct a tubular distance function on $\bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$ in a manner similar to the definition of $\vec{t}(X^\ell, g_\mathcal{P})$ in (4.7.1). Because the functions $\tilde{\lambda}_P$ defined in Lemma 5.9.1 are $\text{SO}(4) \times \text{SO}(3)$ -invariant, the map

$$\prod_{P \in \mathcal{P}} \tilde{\lambda}_P : \bar{M}(\mathcal{P}) \rightarrow [0, 1]^\mathcal{P} := \prod_{P \in \mathcal{P}} [0, 1],$$

where $\bar{M}(\mathcal{P})$ is the fiber defined in (6.2.5) of $\bar{\text{Gl}}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$, extends from the fiber and defines a smoothly-stratified map

$$(6.7.8) \quad \vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) : \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \rightarrow [0, 1]^\mathcal{P} / \mathfrak{S}(\mathcal{P}).$$

That is, $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$ is the unique map whose restriction to each fiber $\bar{M}(\mathcal{P})$ is given by $\prod_{P \in \mathcal{P}} \tilde{\lambda}_P$. For $P \in \mathcal{P}$, we let $\vec{t}^P(\mathbf{t}, \mathfrak{s}, \mathcal{P})$ denote the P -th component of the map (6.7.8). We now describe the map $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}')$ on the intersection $\bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \cap \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}')$ where $\mathcal{P} < \mathcal{P}'$.

LEMMA 6.7.3. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_ℓ . Let \mathcal{O} be the domain of the map $\rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, d}$ defined on the fibers of $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ in (6.5.17). Then, for the structure group, $G(\mathcal{P})$, of the bundle $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ defined in (6.2.3b), there is a smoothly-stratified, $G(\mathcal{P})$ -invariant map,*

$$\vec{t}_f(\mathcal{P}, [\mathcal{P}']) : \rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, d}(\mathcal{O}) \subset \prod_{P \in \mathcal{P}} \bar{M}_{\text{SPL}, |P|}^{\mathfrak{s}, \mathfrak{d}}(\delta_P) \rightarrow \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} [0, 1]^{\mathcal{P}''} / \mathfrak{S}(\mathcal{P}),$$

whose extension to $\bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \cap \bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}')$ equals $\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}')$.

PROOF. We begin by defining a map,

$$\vec{t}_u : \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s \times S^1} \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \rightarrow \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} [0, 1]^{\mathcal{P}''} / \mathfrak{S}(\mathcal{P}),$$

as follows. For $(A_0, \Phi_0) \in \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta)$, and $(F_P^{T, \mathfrak{g}})_{P \in \mathcal{P}} \in \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$, and $v_P \in \Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}'_P)$, and $[A_Q, F_Z^{\mathfrak{s}}, \mathbf{x}_Q] \in \bar{M}_{\text{SPL}, |Q|}^{\mathfrak{s}, \mathfrak{d}}(\delta_Q)$, where $Q \in \mathcal{P}'_P$, we can write

$$[(A_0, \Phi_0), \mathbf{A}'] = \left[(A_0, \Phi_0), \left(F_P^{T, \mathfrak{g}}, v_P, ([A_Q, F_P^{\mathfrak{s}}, \mathbf{x}_Q])_{Q \in \mathcal{P}'_P} \right)_{P \in \mathcal{P}} \right] \in \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s \times S^1} \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']).$$

Define

$$\vec{t}_u([(A_0, \Phi_0), \mathbf{A}']) := (\tilde{\lambda}_Q([A_Q, F_P^{\mathfrak{s}}, \mathbf{x}_Q]))_{Q \in \mathcal{P}'}$$

The definitions of $\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}')$ and $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, u}$ (see (6.7.8) and (6.5.14)) imply that

$$(6.7.9) \quad \vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}') \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, u} = \vec{t}_u.$$

Hence, the composition $\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}') \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, u}$ is defined by a $G(\mathcal{P})$ -invariant map on the fibers of $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$. The downwards transition map defined in (6.5.15) is given by the $G(\mathcal{P})$ -equivariant map $\rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, d}$ on the fibers of $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ defined in (6.5.17). While the map $\rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, d}$ in (6.5.17) is not an embedding because of symmetric group actions, the map (6.7.9) is invariant under these symmetric group actions and thus defines a $G(\mathcal{P})$ -invariant map,

$$\vec{t}_{\mathcal{O}, f}(\mathcal{P}, [\mathcal{P}']) : \rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, d}(\mathcal{O}) \rightarrow \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} I^{\mathcal{P}''} / \mathfrak{S}(\mathcal{P}),$$

where \mathcal{O} is the domain of $\rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, d}$ appearing in the definition (6.5.17). From the definition of $\mathcal{O}_1(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ in (6.5.21), in particular the condition (6.5.18), the domain of $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, d}$ is contained in the union over the fibers of the map π_Σ defined in (6.5.3). Hence, the extension of the map $\vec{t}_{\mathcal{O}, f}(\mathcal{P}, [\mathcal{P}'])$ from the fibers of π_Σ defines a map on the domain of $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, d}$. Lemma 6.6.3 implies that $\bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \cap \bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}')$ equals the image of $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, d}$ and so this extension of $\vec{t}_{\mathcal{O}, f}(\mathcal{P}, [\mathcal{P}'])$ is the required map. \square

The following lemma identifies the restriction of $\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ to a subspace of $\bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ with $\vec{t}(X^\ell, g_\mathcal{P})$.

LEMMA 6.7.4. *Let $T(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ be the subspace defined in (6.5.4) and define*

$$\mathcal{O}_T(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) := \mathcal{O}_1(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \cap T(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']).$$

If $k_{\Sigma, \mathcal{P}, [\mathcal{P}']}$ is the smoothly-stratified diffeomorphism (6.5.5) and id_N denotes the identity map on $\tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta)$, then there is a smoothly-stratified embedding,

$$\text{id}_N \times (e(X^\ell, g_{\mathcal{P}}) \circ k_{\Sigma, \mathcal{P}, [\mathcal{P}']}) : \tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \mathcal{O}_T(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \rightarrow N_{\mathbf{t}(\ell), \mathfrak{s}}(\delta)/S^1 \times \Sigma(X^\ell, \mathcal{P}').$$

Let $\pi_2 : N_{\mathbf{t}(\ell), \mathfrak{s}}(\delta)/S^1 \times \Sigma(X^\ell, \mathcal{P}') \rightarrow \Sigma(X^\ell, \mathcal{P}')$ be the projection. Then the restriction of the map $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}$ to

$$\tilde{N}_{\mathbf{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \mathcal{O}_T(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']),$$

is equal to the composition,

$$\vec{t}(X^\ell, \mathcal{P}') \circ \pi_2 \circ \left(\text{id}_N \times e(X^\ell, g_{\mathcal{P}}) \circ k_{\Sigma, \mathcal{P}, [\mathcal{P}']} \right).$$

PROOF. The lemma follows immediately from the definition of $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}$ in terms of $\gamma'_{\Theta, \mathcal{P}'_P}$ in (6.5.15), the image of $\gamma'_{\Theta, \mathcal{P}'_P}$ on the domain $\mathcal{O}_T(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$, the definitions of $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$ and $\vec{t}(X^\ell, \mathcal{P}')$ in terms of $\tilde{\lambda}_P$ and $t(Z_P)$, in (6.7.8) and (4.7.1) respectively, and the equality in Item (2) of Lemma 5.9.1 between $\tilde{\lambda}_P$ and $t(Z_P)$. \square

The following lemma shows that the Thom–Mather relation $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \circ \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}') = \vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}')$ does not hold for the same reasons as the analogous relation described in the proof of Lemma 4.7.3.

LEMMA 6.7.5. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_ℓ . For each $\mathbf{A} \in \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \cap \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}')$, the following identities hold:*

- (1) *If $P \notin \mathcal{P} \cap \mathcal{P}'$, then $\vec{t}^P(\mathbf{t}, \mathfrak{s}, \mathcal{P})(\mathbf{A}) = \vec{t}^P(\mathbf{t}, \mathfrak{s}, \mathcal{P}')(\mathbf{A})$.*
- (2) *If $P \in \mathcal{P} \cap \mathcal{P}'$, then $\vec{t}^P(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \circ \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}')(\mathbf{A}) = 0$.*
- (3) *If $P \in \mathcal{P} \cap \mathcal{P}'$, then $\vec{t}^P(\mathbf{t}, \mathfrak{s}, \mathcal{P})(\mathbf{A}) = \vec{t}^P(\mathbf{t}, \mathfrak{s}, \mathcal{P}')(\mathbf{A})$.*

PROOF. By (6.7.6), we have the equality

$$\vec{t}^P(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \circ p_{\mathcal{P}, [\mathcal{P}']} \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d} = \vec{t}^P(\mathbf{t}, \mathfrak{s}, \mathcal{P}') \circ \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}') \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, u}$$

on $\bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \cap \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}')$. Thus, computing $\vec{t}^P(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \circ p_{\mathcal{P}, [\mathcal{P}']}$ will determine $\vec{t}^P(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \circ \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}')$. Item (3) of Lemma 5.9.1, the definition of $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$ in terms of $\tilde{\lambda}_P$ in (6.7.8), and the definition of $p_{\mathcal{P}, [\mathcal{P}]}$ in (6.7.4) in terms of the projection maps $\pi(\Theta, \mathcal{P}'_P)$ imply that $\vec{t}^P(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \circ p_{\mathcal{P}, [\mathcal{P}]}$ and $\vec{t}^P(\mathbf{t}, \mathfrak{s}, \mathcal{P}')$ will be equal for $P \notin \mathcal{P} \cap \mathcal{P}'$ while $\vec{t}^P(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \circ p_{\mathcal{P}, [\mathcal{P}]}$ will be zero for $P \in \mathcal{P} \cap \mathcal{P}'$. This yields Items (1) and (2).

Item (3) follows by observing that $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, u}$ and $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathbf{t}, \mathfrak{s}, d}$ are both given by the identity map on the factor $\bar{M}_{\text{SPL}, |P|}^{\mathfrak{s}, \dagger}(\delta)$ of the fibers (6.2.5) of $\mathcal{O}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$, and $\mathcal{O}(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$, and $\mathcal{O}(\mathbf{t}, \mathfrak{s}, [\mathcal{P} < \mathcal{P}'])$. \square

We obtain a weaker version of the second Thom–Mather relation from (4.1.1) in the following lemma and this will be sufficient to construct the link of $M_{\mathfrak{s}} \times \text{Sym}^\ell(X)$ in Section 8.1.

LEMMA 6.7.6. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_ℓ and let $\varepsilon > \varepsilon' > 0$. Assume that $\mathbf{A} \in \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \cap \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}')$. If $\hat{D}(\mathcal{P}, \varepsilon)$ denotes either of the spaces $D(\mathcal{P}, \varepsilon)$ or $\bar{D}(\mathcal{P}, \varepsilon)$ defined in (4.7.2), then the following hold:*

- (1) If $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P})(\mathbf{A}) \in \hat{D}(\mathcal{P}, \varepsilon)$, then $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \circ \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}')(\mathbf{A}) \in \hat{D}(\mathcal{P}, \varepsilon)$.
(2) If $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}')(\mathbf{A}) \in \bar{D}(\mathcal{P}', \varepsilon')$ and $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \circ \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}')(\mathbf{A}) \in \hat{D}(\mathcal{P}, \varepsilon)$, then $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P})(\mathbf{A}) \in \hat{D}(\mathcal{P}, \varepsilon)$.

PROOF. The conclusion follows from Lemma 6.7.6 by the same arguments used to prove Lemma 4.7.3 from Lemma 4.7.2. \square

In our construction of the link of $M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X)$ in $\bar{\mathcal{M}}_{\mathfrak{s}}^{\text{vir}}/S^1$, the subspace

$$(6.7.10) \quad \vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P})^{-1} \left(\hat{D}(\mathcal{P}, \varepsilon) \right) \subset \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$$

will play the role of a disk-bundle neighborhood of $N_{\mathfrak{s}}(\delta)/S^1 \times \Sigma(X^{\ell}, \mathcal{P})$ in $\bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$. Similarly, for partitions $\mathcal{P} < \mathcal{P}'$ of N_{ℓ} and the tubular distance function, $\vec{t}(X^{\ell}, g_{\mathcal{P}}) : \mathcal{U}(X^{\ell}, \mathcal{P}) \subset \text{Sym}^{\ell}(X) \rightarrow [0, 1]^{\mathcal{P}}$, defined in (4.7.1), the subspace

$$(6.7.11) \quad \Sigma(X^{\ell}, \mathcal{P}') \cap \left(\vec{t}(X^{\ell}, g_{\mathcal{P}})^{-1} \left(\hat{D}(\mathcal{P}, \varepsilon) \right) \right) \subset \Sigma(X^{\ell}, \mathcal{P}') \cap \mathcal{U}(X^{\ell}, \mathcal{P})$$

can be thought of as a disk-bundle neighborhood of $\Sigma(X^{\ell}, \mathcal{P})$ in $\Sigma(X^{\ell}, \mathcal{P}')$. The following corollary, to be used in Proposition 8.2.1 to establish the fiber-bundle structure of a subspace of the link, describes the intersection of two subspaces of the form (6.7.10) in terms of the subspace (6.7.11) and the projection map $\pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}') : \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}') \rightarrow N_{\mathfrak{s}}(\delta)/S^1 \times \Sigma(X^{\ell}, \mathcal{P}')$.

COROLLARY 6.7.7. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_{ℓ} . If*

$$D(X^{\ell}, \mathcal{P}, [\mathcal{P}'], \varepsilon) := N_{\mathfrak{s}}(\delta)/S^1 \times \left(\Sigma(X^{\ell}, \mathcal{P}') \cap \left(\vec{t}(X^{\ell}, g_{\mathcal{P}})^{-1} \left(\hat{D}(\mathcal{P}, \varepsilon) \right) \right) \right),$$

then for any $\varepsilon' \in (0, \varepsilon)$, we have

$$\begin{aligned} & \vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}')^{-1} \left(\bar{D}(\mathcal{P}', \varepsilon') \right) \cap \vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P})^{-1} \left(\bar{D}(\mathcal{P}, \varepsilon) \right) \\ &= \vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}')^{-1} \left(\bar{D}(\mathcal{P}', \varepsilon') \right) \cap \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}')^{-1} \left(D(X^{\ell}, \mathcal{P}, [\mathcal{P}'], \varepsilon) \right). \end{aligned}$$

PROOF. If

$$T(X^{\ell}, \mathcal{P}, [\mathcal{P}'], \varepsilon) := \left(N_{\mathfrak{s}}(\delta)/S^1 \times \Sigma(X^{\ell}, \mathcal{P}') \right) \cap \vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P})^{-1} \left(\bar{D}(\mathcal{P}, \varepsilon) \right),$$

then Lemma 6.7.6 implies that

$$\begin{aligned} & \left(\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}')^{-1} \left(\bar{D}(\mathcal{P}', \varepsilon') \right) \right) \cap \left(\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P})^{-1} \left(\bar{D}(\mathcal{P}, \varepsilon) \right) \right) \\ &= \left(\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}')^{-1} \left(\bar{D}(\mathcal{P}', \varepsilon') \right) \right) \cap \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}')^{-1} \left(T(X^{\ell}, \mathcal{P}, [\mathcal{P}'], \varepsilon) \right). \end{aligned}$$

The relation between $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$ and $\vec{t}(X^{\ell}, \mathcal{P})$ in Lemma 6.7.4 then implies that

$$T(X^{\ell}, \mathcal{P}, [\mathcal{P}'], \varepsilon) = D(X^{\ell}, \mathcal{P}, [\mathcal{P}'], \varepsilon),$$

completing the proof of the corollary. \square

6.8. Global splicing map

The global crude splicing map $\gamma''_{\mathcal{M}}$ in (6.6.10) is unsatisfactory because its image does not include $M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X)$ — see Lemma 6.4.3 and Remark 6.4.4. To remedy this, we define an isotopy of the embedding $\gamma''_{\mathcal{M}}$ to an embedding $\gamma'_{\mathcal{M}}$ with the desired image. The analytical underpinnings required to replace Hypothesis 7.8.1 by a theorem that yields its assertions will be provided in the forthcoming [22].

For each partition \mathcal{P} of N_{ℓ} , the splicing map defined in Section 6.4.1 and the crude splicing map defined in Section 6.4.2 give smoothly-stratified embeddings,

$$\gamma'_{\mathfrak{t},\mathfrak{s},\mathcal{P}} : \bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \bar{\mathcal{C}}_{\mathfrak{t}} \quad \text{and} \quad \gamma''_{\mathfrak{t},\mathfrak{s},\mathcal{P}} : \bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \bar{\mathcal{C}}_{\mathfrak{t}},$$

respectively, where $\bar{\mathcal{C}}_{\mathfrak{t}}$ is defined in (2.1.15). By its definition in (6.6.10), the restriction of the global crude splicing map to $\bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ equals $\gamma''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$. From the list of differences between the two splicing maps in the beginning of Section 6.4.2, one can see that the map $\gamma''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ equals the map defined by $\gamma'_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ using a locally flattened Riemannian metric $g_{\mathcal{P},\mathbf{x}}$ (where $\mathbf{x} \in \Sigma(X^{\ell}, \mathcal{P})$) and by replacing the pair $(A_0, \Phi_0) \in \bar{\mathcal{C}}_{\mathfrak{t}(\ell)}$ with the locally flattened pair $\Theta(A_0, \Phi_0, \mathbf{x})$. Thus, one can define an isotopy of smoothly-stratified embeddings between $\gamma'_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ and $\gamma''_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ by applying the construction of $\gamma'_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ with a convex linear combination of the Riemannian metrics g and $g_{\mathcal{P},\mathbf{x}}$ and of the pairs (A_0, Φ_0) and $\Theta(A_0, \Phi_0, \mathbf{x})$.

We write this isotopy as

$$\Gamma'_{\mathfrak{t},\mathfrak{s},\mathcal{P}} : \bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \times [0, 1] \rightarrow \bar{\mathcal{C}}_{\mathfrak{t}}, \quad \text{with} \quad \Gamma'_{\mathfrak{t},\mathfrak{s},\mathcal{P}}(\cdot, s) = \begin{cases} \gamma''_{\mathfrak{t},\mathfrak{s},\mathcal{P}} & \text{if } s = 0, \\ \gamma'_{\mathfrak{t},\mathfrak{s},\mathcal{P}} & \text{if } s = 1. \end{cases}$$

We now construct the function giving the parameter to give the isotopy. Let $f_{\mathcal{P}} : \Delta^{\circ}(X^{\ell}, \mathcal{P}) \rightarrow (0, 1]$ be a smooth, $\mathfrak{S}(\mathcal{P})$ -invariant function with the property that the rescaled map

$$\vec{t}_r(\mathfrak{t}, \mathfrak{s}, \mathcal{P})(\cdot) := \vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})(\cdot)(f_{\mathcal{P}} \circ \pi(\mathfrak{t}, \mathfrak{s}, \mathcal{P}))(\cdot),$$

(where the preceding multiplication means to multiply each factor of $\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ by the scalar $f_{\mathcal{P}} \circ \pi(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$) takes values in the subspace $\bar{D}(\mathcal{P}, \frac{1}{2}) \subset [0, 1]^{\mathcal{P}}$ defined in (4.7.2). Now, observe that if

$$\beta : [0, 1]^{\mathcal{P}} \rightarrow [0, 1],$$

is a smooth, $\mathfrak{S}(\mathcal{P})$ -invariant function, supported in $D(\mathcal{P}, \frac{1}{2})$ with $\beta(\bar{D}(\mathcal{P}, \frac{1}{4})) = 1$, then $\beta \circ \vec{t}_r(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ is identically equal to one on $\bar{\mathcal{U}}_1(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ and is identically equal to zero on $\bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \setminus \bar{\mathcal{U}}_2(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$, where the spaces $\bar{\mathcal{U}}_i(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$, for $i = 1, 2$, satisfy

$$N_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta)/S^1 \times \Sigma(X^{\ell}, \mathcal{P}) \sqsubset \bar{\mathcal{U}}_1(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \sqsubset \bar{\mathcal{U}}_2(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \sqsubset \bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}).$$

Hence, the function,

$$(6.8.1) \quad \beta_{\mathcal{P}} = \beta \circ \vec{t}_r(\mathfrak{t}, \mathfrak{s}, \mathcal{P}),$$

defines a global function on $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ satisfying

$$(6.8.2) \quad \bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}} \setminus \bar{\mathcal{U}}_2(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \subset \beta_{\mathcal{P}}^{-1}(0) \quad \text{and} \quad \bar{\mathcal{U}}_1(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \subset \beta_{\mathcal{P}}^{-1}(1).$$

We now describe how to patch the isotopies $\Gamma'_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ together to define a global isotopy of smoothly-stratified embeddings. Choose one representative \mathcal{P} in each conjugacy class $[\mathcal{P}]$ and enumerate these representatives in the manner described in Section 3.4. Assume $\mathcal{P}_0 = \{N_{\ell}\}$ is the crudest

partition and \mathcal{P}_m is the most refined partition. We define a finite sequence of smoothly-stratified embeddings,

$$\gamma'_{\mathcal{M},i} : \bar{\mathcal{M}}_{\mathbf{t},\mathfrak{s}}^{\text{vir}} \rightarrow \bar{\mathcal{C}}_{\mathbf{t}},$$

as follows. We begin with

$$(6.8.3) \quad \gamma'_{\mathcal{M},0}([(A_0, \Phi_0), \mathbf{A}]) := \begin{cases} \Gamma'_{\mathbf{t},\mathfrak{s},\mathcal{P}_0}([(A_0, \Phi_0), \mathbf{A}], \beta_{\mathcal{P}_0}([(A_0, \Phi_0), \mathbf{A}])) & \text{if } [(A_0, \Phi_0), \mathbf{A}] \in \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_0), \\ \gamma''_{\mathcal{M}}([(A_0, \Phi_0), \mathbf{A}]) & \text{if } [(A_0, \Phi_0), \mathbf{A}] \notin \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_0). \end{cases}$$

Observe that the restriction of $\gamma'_{\mathcal{M},0}$ to $\bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_0)$ is isotopic to $\gamma'_{\mathbf{t},\mathfrak{s},\mathcal{P}_0}$. We proceed by upwards induction on the enumerated partitions. Assume inductively that the restriction of $\gamma'_{\mathcal{M},i-1}$ to $\bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_i)$ is isotopic to $\gamma''_{\mathbf{t},\mathfrak{s},\mathcal{P}_i}$. Then, the construction of the isotopy $\Gamma'_{\mathbf{t},\mathfrak{s},\mathcal{P}_i}$ above can be modified to give an isotopy of smoothly-stratified embeddings of $\bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_i)$,

$$\hat{\Gamma}'_{\mathbf{t},\mathfrak{s},\mathcal{P}_i} : \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_i) \times [0, 1] \rightarrow \bar{\mathcal{C}}_{\mathbf{t}}, \quad \text{with} \quad \hat{\Gamma}'_{\mathbf{t},\mathfrak{s},\mathcal{P}_i}(\cdot, s) = \begin{cases} \gamma'_{\mathcal{M},i-1} & \text{if } s = 0, \\ \gamma'_{\mathbf{t},\mathfrak{s},\mathcal{P}_i} & \text{if } s = 1. \end{cases}$$

We then continue the induction by defining

$$(6.8.4) \quad \gamma'_{\mathcal{M},i}([(A_0, \Phi_0), \mathbf{A}]) := \begin{cases} \hat{\Gamma}'_{\mathbf{t},\mathfrak{s},\mathcal{P}_i}([(A_0, \Phi_0), \mathbf{A}], \beta_{\mathcal{P}_i}([(A_0, \Phi_0), \mathbf{A}])) & \text{if } [(A_0, \Phi_0), \mathbf{A}] \in \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_i), \\ \gamma'_{\mathcal{M},i-1}([(A_0, \Phi_0), \mathbf{A}]) & \text{if } [(A_0, \Phi_0), \mathbf{A}] \notin \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_i). \end{cases}$$

If \mathcal{P}_m is, as defined above, the most refined partition, we define the *global splicing map* by

$$(6.8.5) \quad \gamma'_{\mathcal{M}} := \gamma'_{\mathcal{M},m}.$$

The proof of the following proposition is then technical but straightforward.

PROPOSITION 6.8.1. *There is a smoothly-stratified, S^1 -equivariant embedding,*

$$\gamma'_{\mathcal{M}} : \bar{\mathcal{M}}_{\mathbf{t},\mathfrak{s}}^{\text{vir}} \rightarrow \bar{\mathcal{C}}_{\mathbf{t}},$$

such that the restriction of $\gamma'_{\mathcal{M}}$ to $N_{\mathbf{t}(\ell),\mathfrak{s}}(\delta) \times \text{Sym}^{\ell}(X)$ is equal to the product of the embedding $N_{\mathbf{t}(\ell),\mathfrak{s}}(\delta) \rightarrow \bar{\mathcal{C}}_{\mathbf{t}(\ell)}$ with the identity on $\text{Sym}^{\ell}(X)$.

6.9. Projections onto symmetric products

We now define a projection map,

$$(6.9.1) \quad \pi_X : \bar{\mathcal{M}}_{\mathbf{t},\mathfrak{s}}^{\text{vir}} \rightarrow \text{Sym}^{\ell}(X),$$

which will be used in describing cohomology classes on $\bar{\mathcal{M}}_{\mathbf{t},\mathfrak{s}}^{\text{vir}}$.

We will denote the projection π_{Σ} in (6.2.11) by

$$\pi_{\Sigma}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) : \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \Sigma(X^{\ell}, \mathcal{P}),$$

to avoid confusion between the projection maps of the different strata. These local projection maps do not agree on the intersections $\bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \cap \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}')$, but they can be related with the aid of the following lemma.

LEMMA 6.9.1. *If $\mathcal{P} < \mathcal{P}'$ are partitions of N_ℓ , then*

$$(6.9.2) \quad \pi(X^\ell, g_{\mathcal{P}}) \circ \pi_\Sigma(\mathbf{t}, \mathbf{s}, \mathcal{P}') = \pi_\Sigma(\mathbf{t}, \mathbf{s}, \mathcal{P}) \quad \text{on } \bar{\mathcal{U}}(\mathbf{t}, \mathbf{s}, \mathcal{P}) \cap \bar{\mathcal{U}}(\mathbf{t}, \mathbf{s}, \mathcal{P}').$$

PROOF. The conclusion follows immediately from Lemma 6.6.3 and the definitions of the projection maps. \square

We can now define the desired global projection to $\text{Sym}^\ell(X)$.

LEMMA 6.9.2. *There is an open neighborhood $\mathcal{U}_1 \subset \bar{\mathcal{M}}_{\mathbf{t}, \mathbf{s}}^{\text{vir}}$ of $N_{\mathbf{t}(\ell), \mathbf{s}}(\delta) \times \text{Sym}^\ell(X)$ and a projection map,*

$$\pi_X : \mathcal{U}_1 \rightarrow \text{Sym}^\ell(X),$$

such that the restriction of π_X to $\mathcal{U}_1 \cap \bar{\mathcal{U}}(\mathbf{t}, \mathbf{s}, \mathcal{P})$ is homotopic to $\pi_\Sigma(\mathbf{t}, \mathbf{s}, \mathcal{P})$.

PROOF. Enumerate the strata of $\text{Sym}^\ell(X)$ as described in Section 3.4 using partitions $\mathcal{P}_0, \dots, \mathcal{P}_n$. For $0 \leq i \leq n$, choose open neighborhoods $\mathcal{U}_1(\mathbf{t}, \mathbf{s}, \mathcal{P}_i) \subset \bar{\mathcal{U}}(\mathbf{t}, \mathbf{s}, \mathcal{P}_i)$ of $\Sigma(\mathbf{t}, \mathbf{s}, \mathcal{P}_i)$ such that, for $j < k$,

$$\pi(\mathbf{t}, \mathbf{s}, \mathcal{P}_k)(\bar{\mathcal{U}}_1(\mathbf{t}, \mathbf{s}, \mathcal{P}_j) \cap \bar{\mathcal{U}}_1(\mathbf{t}, \mathbf{s}, \mathcal{P}_k)) \subseteq \bigcup_{i < j} T_i,$$

where T_i is constructed in Lemma 4.8.1. The conclusion then follows from Lemma 4.8.4 and the equality (6.9.2). \square

CHAPTER 7

Obstruction bundle

7.1. Introduction

The gluing map described in Hypothesis 7.8.1 does not map all points in $\bar{\mathcal{M}}_{t,s}^{\text{vir}}$ to the moduli space \mathcal{M}_t . Instead, the intersection of \mathcal{M}_t with the image of the top stratum of $\bar{\mathcal{M}}_{t,s}^{\text{vir}}$ is diffeomorphic to the zero-locus of a section of a vector bundle over the top stratum of $\bar{\mathcal{M}}_{t,s}^{\text{vir}}$. To describe the intersection of the image of $\bar{\mathcal{M}}_{t,s}^{\text{vir}}$ under the gluing map with \mathcal{M}_t , we require the following notion of a pseudo-bundle.

DEFINITION 7.1.1. (See Schwartz [88].) Let Y be a stratified space. Assume that to every stratum Σ of Y , there is an open neighborhood, U_Σ , of Σ in Y and a strict deformation retraction, $r_\Sigma : U_\Sigma \rightarrow \Sigma$. A *pseudo-bundle* over Y is a space Υ with a surjective, continuous map $\pi_\Upsilon : \Upsilon \rightarrow Y$ satisfying the following conditions:

- (1) For every stratum Σ of Y , the subspace $\Upsilon_\Sigma = \pi_\Upsilon^{-1}(\Sigma)$ is a vector bundle with projection map given by the restriction of π_Υ .
- (2) For every stratum Σ of Y , there is an injective bundle map,

$$\iota_{\Upsilon,\Sigma} : r_\Sigma^* \Upsilon_\Sigma \rightarrow \Upsilon|_{U_\Sigma},$$

that is linear on each fiber and is the identity when restricted to Υ_Σ and which admits a left inverse,

$$\Pi_{\Upsilon,\Sigma} : \Upsilon|_{U_\Sigma} \rightarrow r_\Sigma^* \Upsilon_\Sigma,$$

that is a surjective vector bundle map.

A *section* of a pseudo-bundle $\pi_\Upsilon : \Upsilon \rightarrow Y$ is a continuous map $s : Y \rightarrow \Upsilon$ satisfying $\pi_\Upsilon \circ s = \text{id}_Y$.

In this chapter, we construct a pseudo-bundle (the *obstruction pseudo-bundle*) over $\bar{\mathcal{M}}_{t,s}^{\text{vir}}$ and then in Hypothesis 7.8.1 state the properties of a gluing map (to be proved in the forthcoming [22]) required to give a smoothly-stratified diffeomorphism between the zero-locus of a section of this obstruction pseudo-bundle (the *obstruction section*) and an open neighborhood of $M_s \times \text{Sym}^\ell(X)$ in \mathcal{M}_t .

For a point $[A', \Phi']$ in the image of the splicing map, let $D\mathfrak{S}|_{(A',\Phi')}$ denote the linearization of the $\text{SO}(3)$ -monopole equations at (A', Φ') with image in

$$(7.1.1) \quad L_{k-1}^2(\Lambda^+ \otimes \mathfrak{g}_t) \oplus L_{k-1}^2(V^-).$$

The fiber of the obstruction pseudo-bundle over $[A', \Phi']$ will be defined by assigning, \mathcal{G}_t -equivariantly, to each pair (A'', Φ'') in the \mathcal{G}_t -orbit a finite-rank subspace $\Upsilon_{(A'',\Phi'')}$ of the Hilbert space (7.1.1) which is close to the cokernel of $D\mathfrak{S}|_{(A'',\Phi'')}$ in the sense that the restriction of the L^2 -orthogonal projection onto $\text{Coker}(D\mathfrak{S}|_{(A'',\Phi'')})$ to $\Upsilon_{(A'',\Phi'')}$ is L^2 -close to the identity (see [24, Proposition 8.19]).

Let $[A', \Phi']$ be a point in the image of the splicing map, given by splicing $\mathrm{SO}(3)$ connections A_P over S^4 , for $P \in \mathcal{P}$, onto a background pair $(A_0, \Phi_0) \in \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta)$. In [24, Section 8], we constructed an embedding of vector spaces,

$$\varphi_{\mathcal{P}} : \mathrm{Coker}(D\mathfrak{S}|_{(A_0, \Phi_0)}) \oplus \left(\bigoplus_{P \in \mathcal{P}} \mathrm{Coker}(D\mathfrak{S}|_{A_P, 0}) \right) \rightarrow L_{k-1}^2(\Lambda^+ \otimes \mathfrak{gt}) \oplus L_{k-1}^2(V^-),$$

by multiplying elements of the domain by cut-off functions. We then showed that $\mathrm{Im}(\varphi_{\mathcal{P}})$ had the desired properties mentioned above and hence will define the fiber of the obstruction pseudo-bundle over $[A', \Phi']$. In this section, we show how to fit these fibers together as the point $[A', \Phi']$ and the partition \mathcal{P} vary to form a pseudo-bundle.

We refer to the component of $\mathrm{Im}(\varphi_{\mathcal{P}})$ given by $\varphi_{\mathcal{P}}(\mathrm{Coker}(D\mathfrak{S}|_{(A_0, \Phi_0)}))$ as the *background component*. If $M_{\mathfrak{s}}$ has positive dimension, then the possibility of spectral flow and the varying dimension of the cokernel of $D\mathfrak{S}|_{(A_0, \Phi_0)}$ prevent us from using this cokernel to form a vector bundle. Instead, we use the obstruction bundle $\Xi_{\mathfrak{t}(\ell), \mathfrak{s}} \rightarrow M_{\mathfrak{s}}$ described in Section 2.3.5. This obstruction bundle is a stabilization of the cokernel of $D\mathfrak{S}$ in the sense that it is a trivial bundle which surjects onto $\mathrm{Coker}(D\mathfrak{S}|_{(A_0, \Phi_0)})$ at each point $[A_0, \Phi_0] \in M_{\mathfrak{s}}$. For $[A', \Phi'] \in \bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \cap \bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}')$, the restrictions of the maps $\varphi_{\mathcal{P}}$ and $\varphi_{\mathcal{P}'}$ to $\Xi_{\mathfrak{t}(\ell), \mathfrak{s}}$ differ in the choice of cut-off functions just as the standard splicing maps $\gamma'_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}$ differed (see (6.4.8)). To fit these images together to form a global bundle, we use the flattening map for pairs, defined in Section 6.3, relying on the consistency property (6.3.7). The details of this construction appear in Section 7.3.

We refer to the component of $\mathrm{Im}(\varphi_{\mathcal{P}})$ given by $\varphi_{\mathcal{P}}(\bigoplus_{P \in \mathcal{P}} \mathrm{Coker}(D\mathfrak{S}|_{A_P, 0}))$ as the *instanton component*. The cokernel of $D\mathfrak{S}|_{A_P, 0}$ is isomorphic to the cokernel of the twisted Dirac operator $D_{A_P} : \Omega^0(V_{|P|}^+) \rightarrow \Omega^0(V_{|P|}^-)$, where $V_{|P|}$ is the spin^u structure on S^4 defined by the standard spin structure and the rank-two complex bundle $E_{|P|} \rightarrow S^4$. An argument using the Bochner–Weitzenböck formula for the Dirac operator and the fact that the standard round metric on S^4 has positive scalar curvature (see [27, Section 4.1]) ensures that spectral flow does not arise for this component of the obstruction and that the cokernel of the twisted Dirac operator forms a vector bundle (the index bundle) over $M_{|P|}^{s, \mathfrak{h}}(\delta_P)$, as described in Section 7.4. However, the index of the twisted Dirac operator drops on the lower-charge moduli spaces appearing in the lower strata of the Uhlenbeck compactification of the moduli space of anti-self-dual connections over S^4 . Note that this is a change in the index and not just a spectral flow problem which could be dealt with by the stabilization procedure used for the background obstruction. Hence the image space for the obstruction map will only be a pseudo-bundle in the sense of Definition 7.1.1. This change of index reflects the failure of $\bar{\mathcal{M}}_{\mathfrak{t}}$ to intersect the lower strata of $\bar{\mathcal{M}}_{\mathfrak{t}, \mathfrak{s}}^{\mathrm{vir}}$ in subspaces of the same codimension. One can express this failure in the language of intersection homology [37, 38, 6] by saying that the cycle $\bar{\mathcal{M}}_{\mathfrak{t}}$ has non-trivial perversity. However, in Section 9.5.4, we shall see that for computations in rational cohomology, the Euler class of the obstruction pseudo-bundle behaves in a manner similar to that of an actual vector bundle and there is no need to appeal to the framework of intersection-homology-valued characteristic classes.

The obstruction to forming a global obstruction pseudo-bundle from the images of $\varphi_{\mathcal{P}}$, restricted to the instanton component comes from comparing the images of $\varphi_{\mathcal{P}}$ and $\varphi_{\mathcal{P}'}$ for different partitions. We overcome this problem by adapting the method used in Chapters 5 and 6 to compare the images of the crude splicing maps $\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}$. In Section 7.6, we define the *spliced-ends obstruction*

pseudo-bundle,

$$(7.1.2) \quad \bar{\Upsilon}_{\text{SPL},\kappa}^i \rightarrow \bar{M}_{\text{SPL},\kappa}^{s,\natural}(\delta),$$

inductively by assuming that the bundle (7.1.2) exists for $\kappa' < \kappa$ and by replacing the index bundle of the twisted Dirac operator over the subspace $\mathcal{U}(\Theta, \mathcal{P}'_P)$ of the spliced end of $\bar{M}_{\text{SPL},\kappa}^{s,\natural}(\delta)$ with the image, under a splicing map similar to $\varphi_{\mathcal{P}'_P}$, of a direct sum of bundles,

$$\bigoplus_{P \in \mathcal{P}} \bar{\Upsilon}_{\text{SPL},|P|}^i.$$

Because this construction is equivariant with respect to group actions discussed in Section 7.5, the spliced-ends obstruction pseudo-bundle extends to a pseudo-bundle over $\mathcal{U}(t, \mathfrak{s}, \mathcal{P})$ with an embedding $\varphi_{t,\mathfrak{s},\mathcal{P}}$ replacing $\varphi_{\mathcal{P}}$. In Section 7.7, we give a proof, similar to that in Section 6.5, that the intersections of the images of splicing maps $\varphi_{t,\mathfrak{s},\mathcal{P}}$ and $\varphi_{t,\mathfrak{s},\mathcal{P}'}$ can be described by a push-out diagram, thus defining the instanton obstruction as a global pseudo-bundle. Finally, in Section 7.8, we state the properties of the local gluing map and obstruction section required to prove the main results of this monograph.

7.2. Infinite-rank obstruction pseudo-bundle

The monopole equations (2.1.10) define a section, \mathfrak{S} , of the infinite-rank obstruction pseudo-bundle,

$$(7.2.1) \quad \mathfrak{Y}_t := \bar{\mathcal{C}}_t \times_{\mathfrak{g}_t} L_{k-1}^2(\Lambda^+ \otimes \mathfrak{g}_t) \oplus L_{k-1}^2(V^-) \rightarrow \bar{\mathcal{C}}_t.$$

The S^1 action on $\bar{\mathcal{C}}_t$ lifts to an S^1 action on the space (7.2.1) given by the diagonal action of the S^1 action on $\bar{\mathcal{C}}_t$ defined in (2.3.15) and scalar multiplication on the element of $L_{k-1}^2(V^-)$.

We extend the map (7.2.1) to a map from a union of spaces to $\bar{\mathcal{C}}_t$ by setting

$$(7.2.2) \quad \bar{\mathfrak{Y}}_t := \bigsqcup_{\ell=0}^N \left(\mathfrak{Y}_{t(\ell)} \times \text{Sym}^\ell(X) \right) \rightarrow \bar{\mathcal{C}}_t = \bigsqcup_{\ell=0}^N \left(\bar{\mathcal{C}}_{t(\ell)} \times \text{Sym}^\ell(X) \right),$$

where N is the integer appearing in Theorem 2.1.3 and the space $\bar{\mathcal{C}}_t$ is defined in (2.1.15). The space (7.2.2) is endowed with a topology given by the following notion of convergence.

DEFINITION 7.2.1. A sequence $\{[A_\alpha, \Phi_\alpha, \mathbf{x}_\alpha, \Psi_\alpha]\}_{\alpha=1}^\infty \subset \bar{\mathfrak{Y}}_t$, where $\{[A_\alpha, \Phi_\alpha, \mathbf{x}_\alpha]\}_{\alpha=1}^\infty \subset \bar{\mathcal{C}}_t$ and $\{\Psi_\alpha\}_{\alpha=1}^\infty$ is a subset of the fiber of (7.2.1), converges to $[A_\infty, \Phi_\infty, \mathbf{x}_\infty, \Psi_\infty]$ as $\alpha \rightarrow \infty$ if

- (1) $\{[A_\alpha, \Phi_\alpha, \mathbf{x}_\alpha]\}_{\alpha=1}^\infty$ converges to $[A_\infty, \Phi_\infty, \mathbf{x}_\infty]$ in the Uhlenbeck topology on $\bar{\mathcal{C}}_t$.
- (2) If $u_\alpha : V_t|_{X \setminus \mathbf{x}} \rightarrow V_{t(\ell)}|_{X \setminus \mathbf{x}}$ are the spin^u bundle isomorphisms appearing in Definition 2.1.2 such that $\{u_\alpha(A_\alpha, \Phi_\alpha)\}_{\alpha=1}^\infty$ converges to (A_0, Φ_0) in $L_{k,\text{loc}}^2$ over $X \setminus \mathbf{x}$ as $\alpha \rightarrow \infty$, then $\{u_\alpha(\Psi_\alpha)\}_{\alpha=1}^\infty$ converges to Ψ_∞ in the $L_{k-1,\text{loc}}^2$ topology on $X \setminus \mathbf{x}_\infty$ as $\alpha \rightarrow \infty$.

The $\text{SO}(3)$ -monopole equations then extend to give a continuous section, $\mathfrak{S} : \bar{\mathcal{C}}_t \rightarrow \bar{\mathfrak{Y}}_t$.

We define an S^1 -invariant inner product on the fibers of $\bar{\mathfrak{Y}}_t$ by

$$(7.2.3) \quad \langle [A, \Phi, \mathbf{x}, \Psi_1], [A, \Phi, \mathbf{x}, \Psi_2] \rangle := (\Psi_1, \Psi_2)_{L^2(X)},$$

where $[A, \Phi, \mathbf{x}] \in \bar{\mathcal{C}}_t$ and $[A, \Phi, \mathbf{x}, \Psi_i] \in \bar{\mathfrak{Y}}_t$.

7.3. Background obstruction bundle

Recall from [24] or [30, Section 3.6.1] that the background component of the obstruction is constructed by splicing in sections of the obstruction bundle,

$$\pi_N^* \Xi_{\mathfrak{t}(\ell), \mathfrak{s}} \rightarrow N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta),$$

where $\pi_N : N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \rightarrow M_{\mathfrak{s}}$ is the projection and $\Xi_{\mathfrak{t}(\ell), \mathfrak{s}} \cong M_{\mathfrak{s}} \times \mathbb{C}^{r_{\mathfrak{s}}}$ is a product bundle. The embedding $N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \rightarrow \mathcal{C}_{\mathfrak{t}(\ell)}$ is covered by an embedding of vector bundles,

$$\begin{array}{ccc} \pi_N^* \Xi_{\mathfrak{t}(\ell), \mathfrak{s}} & \longrightarrow & \mathfrak{V}_{\mathfrak{t}(\ell)} \\ \downarrow & & \downarrow \\ N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) & \longrightarrow & \mathcal{C}_{\mathfrak{t}(\ell)} \end{array}$$

The preceding vector-bundle map is S^1 -equivariant with respect to the actions described in [30, Section 3.6.1] on the domain and the action on $\mathfrak{V}_{\mathfrak{t}(\ell)}$ described following (7.2.1). If $\pi_N^* \tilde{\Xi}_{\mathfrak{t}(\ell), \mathfrak{s}} \rightarrow \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta)$ is the pullback of $\pi_N^* \Xi_{\mathfrak{t}(\ell), \mathfrak{s}}$ by the projection $\tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \rightarrow \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta)/\mathcal{G}_{\mathfrak{s}} = N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta)$ (see (2.3.26)), so $\tilde{\Xi}_{\mathfrak{t}(\ell), \mathfrak{s}}/\mathcal{G}_{\mathfrak{s}} = \Xi_{\mathfrak{t}(\ell), \mathfrak{s}}$ then the preceding vector-bundle map is defined by a $\mathcal{G}_{\mathfrak{s}}$ to $\mathcal{G}_{\mathfrak{t}}$ equivariant map

$$\pi_N^* \tilde{\Xi}_{\mathfrak{t}(\ell), \mathfrak{s}} \rightarrow \tilde{\mathcal{C}}_{\mathfrak{t}(\ell)} \times L_{k-1}^2(\Lambda^+ \otimes \mathfrak{g}_{\mathfrak{t}(\ell)} \oplus V_{\mathfrak{t}(\ell)}^-),$$

with respect to the homomorphism $\varrho : \mathcal{G}_{\mathfrak{s}} \rightarrow \mathcal{G}_{\mathfrak{t}}$ defined in (2.3.13). Define

$$(7.3.1) \quad \Xi_{\mathfrak{s}}(\mathfrak{t}, \mathcal{P}) := \pi_N^* \tilde{\Xi}_{\mathfrak{t}(\ell), \mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \subset \pi_N^* \tilde{\Xi}_{\mathfrak{t}(\ell), \mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \bar{\mathcal{G}}\mathfrak{l}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}).$$

Because the S^1 action on $\pi_N^* \tilde{\Xi}_{\mathfrak{t}(\ell), \mathfrak{s}}$ described above commutes with the $\mathcal{G}_{\mathfrak{s}}$ action, this S^1 action defines an S^1 action on $\Xi_{\mathfrak{s}}(\mathfrak{t}, \mathcal{P})$ and covers the S^1 action described in Lemma 6.2.1. To embed $\Xi_{\mathfrak{s}}(\mathfrak{t}, \mathcal{P})$ in $\tilde{\mathfrak{V}}_{\mathfrak{t}}$, we now apply the crude splicing construction from Section 6.4. We define the *convex complement* of a function β to be $1 - \beta$. By multiplying the obstruction sections by the convex complements of the cut-off functions used to define the flattened section Φ' in the definition of the flattening map $\Theta_{\mathcal{P}}$, we define an S^1 -equivariant vector-bundle embedding,

$$(7.3.2) \quad \varphi_{\mathfrak{s}}''(\mathcal{P}) : \Xi_{\mathfrak{s}}(\mathfrak{t}, \mathcal{P}) \rightarrow \tilde{\mathfrak{V}}_{\mathfrak{t}},$$

covering the crude splicing map,

$$(7.3.3) \quad \begin{array}{ccc} \pi_N^* \tilde{\Xi}_{\mathfrak{t}(\ell), \mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) & \xrightarrow{\varphi_{\mathfrak{s}}''(\mathcal{P})} & \tilde{\mathfrak{V}}_{\mathfrak{t}} \\ \downarrow & & \downarrow \\ \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) & \xrightarrow{\gamma_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}''} & \bar{\mathcal{C}}_{\mathfrak{t}} \end{array}$$

The crude obstruction splicing maps $\varphi_{\mathfrak{s}}''(\mathcal{P}')$ are invariant under the action of the symmetric group and thus, for $\mathcal{P} < \mathcal{P}'$, define a crude obstruction splicing map,

$$\varphi_{\mathfrak{s}}''([\mathcal{P} < \mathcal{P}']) : \pi_N^* \tilde{\Xi}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, [\mathcal{P} < \mathcal{P}']) \rightarrow \tilde{\mathfrak{V}}_{\mathfrak{t}},$$

covering the crude splicing map $\gamma_{\mathfrak{t}, \mathfrak{s}, [\mathcal{P} < \mathcal{P}']}''$ defined in (6.5.20).

We can describe the overlaps of the images of the embeddings $\varphi_{\mathfrak{s}}''(\mathcal{P})$ exactly as was done in the construction of the space of global splicing data. The overlap maps, $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, u}$ and $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, d}$, appearing

in the diagram (6.5.22) are equal to the identity on the factor $\tilde{N}_{t(\ell),s}(\delta)$ in their domain. Hence, these overlap maps are covered by S^1 -equivariant bundle maps,

$$(7.3.4) \quad \begin{aligned} \rho_{\mathcal{P},[\mathcal{P}']}^{\Xi,t,s,u} &: \pi_N^* \tilde{\Xi}_{t(\ell),s} \times_{\mathcal{G}_s} \mathcal{O}(t, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \rightarrow \pi_N^* \tilde{\Xi}_{t(\ell),s} \times_{\mathcal{G}_s} \mathcal{O}(t, \mathfrak{s}, [\mathcal{P} < \mathcal{P}']), \\ \rho_{\mathcal{P},[\mathcal{P}']}^{\Xi,t,s,d} &: \pi_N^* \tilde{\Xi}_s \times_{\mathcal{G}_s} \mathcal{O}(t, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \rightarrow \pi_N^* \tilde{\Xi}_s \times_{\mathcal{G}_s} \mathcal{O}(t, \mathfrak{s}, \mathcal{P}), \end{aligned}$$

equal to the identity on $\pi_N^* \tilde{\Xi}_{t(\ell),s}$. Exactly as in the proof of Proposition 6.5.3 (specifically, the equality of the sections Φ_u'' and Φ_d'' appearing in (6.5.24)), we see that the diagram

$$(7.3.5) \quad \begin{array}{ccc} \pi_N^* \tilde{\Xi}_{t(\ell),s} \times_{\mathcal{G}_s} \mathcal{O}(t, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) & \xrightarrow{\rho_{\mathcal{P},[\mathcal{P}']}^{\Xi,t,s,u}} & \pi_N^* \tilde{\Xi}_{t(\ell),s} \times_{\mathcal{G}_s} \mathcal{O}(t, \mathfrak{s}, [\mathcal{P} < \mathcal{P}']) \\ \rho_{\mathcal{P},[\mathcal{P}']}^{\Xi,t,s,d} \downarrow & & \varphi_s''([\mathcal{P} < \mathcal{P}']) \downarrow \\ \pi_N^* \tilde{\Xi}_s \times_{\mathcal{G}_s} \mathcal{O}(t, \mathfrak{s}, \mathcal{P}) & \xrightarrow{\varphi_s''(\mathcal{P})} & \tilde{\mathfrak{Y}}_t \end{array}$$

commutes and covers the diagram (6.5.22).

Because the maps $\rho_{\mathcal{P},[\mathcal{P}']}^{\Xi,t,s,u}$ and $\rho_{\mathcal{P},[\mathcal{P}']}^{\Xi,t,s,d}$ in the diagram (7.3.5) are isomorphisms on the fibers of the bundles and cover open embeddings, we can define a vector bundle

$$(7.3.6) \quad \tilde{\Upsilon}_{t,s}^s \rightarrow \tilde{\mathcal{M}}_{t,s}^{\text{vir}}$$

as the union of the bundles $\Xi_s(t, \mathcal{P})$ as \mathcal{P} varies over the partitions \mathcal{P} of N_ℓ , patched together on the overlaps by the diagram (7.3.5). The embeddings $\varphi_s''(\mathcal{P})$ fit together to define an S^1 -equivariant embedding of vector bundles,

$$(7.3.7) \quad \varphi_s'' : \tilde{\Upsilon}_{t,s}^s \rightarrow \tilde{\mathfrak{Y}}_t,$$

covering the embedding given by the crude splicing maps, as shown in (7.3.3).

Like the crude splicing map, the embedding φ_s'' in (7.3.7) is not equal to the identity over $M_s \times \text{Sym}^\ell(X)$. We therefore define a new embedding of vector bundles in a manner similar to that used in defining the isotopies of Section 6.8. That is, let $\varphi_s'(\mathcal{P})$ be the standard slicing map,

$$\varphi_s'(\mathcal{P}) : \Xi_s(t, \mathfrak{s}, \mathcal{P}) \rightarrow \tilde{\mathfrak{Y}}_t,$$

defined by using the cut-off functions vanishing on balls $B(x_P, 4\lambda_P^{1/3})$. There is an isotopy through vector-bundle embeddings between $\varphi_s'(\mathcal{P})$ and φ_s'' given by taking convex linear combinations of the two sections parameterized by ‘time’. One then follows the inductive argument, using these isotopies through vector-bundle embeddings, defined like the isotopies in (6.8.3) and (6.8.4), to define a sequence of embeddings of vector bundles, $\varphi_{s,j}'' : \tilde{\Upsilon}_{t,s}^s \rightarrow \tilde{\mathfrak{Y}}_t$. In a manner similar to the definition of the global splicing map in (6.8.5), define

$$(7.3.8) \quad \varphi_s' : \tilde{\Upsilon}_{t,s}^s \rightarrow \tilde{\mathfrak{Y}}_t$$

to be the last embedding of vector bundles in the sequence $\varphi_{s,j}''$ of embeddings of vector bundles. The embedding of vector bundles φ_s' then covers the global splicing map $\gamma'_{\mathcal{M}}$ in (6.8.5).

7.4. Equivariant Dirac index bundle

To define the instanton component of the obstruction pseudo-bundle (7.1.2), we begin by defining the bundle $\text{Index}(\mathbf{D}_\kappa^*)$. Let $\mathfrak{s}_{S^4} = (\rho, S^+, S^-)$ be the standard spin structure on S^4 with respect to the round metric. Let $E_\kappa \rightarrow S^4$ be a complex Hermitian rank-two vector bundle with $c_2(E_\kappa) = \kappa$ and let $V_\kappa^\pm = S^\pm \otimes E_\kappa$ be the resulting spin^u structure on S^4 .

For $[A] \in M_\kappa^{s, \natural}(S^4, \delta)$, we define the Dirac operator

$$(7.4.1) \quad D_A : L_k^2(V_\kappa^+) \rightarrow L_{k-1}^2(V_\kappa^-)$$

to be the Dirac operator defined by tensor product of the spin connection induced by the Levi-Civita connection on \mathfrak{s}_{S^4} and of an $\text{SU}(2)$ connection A on E_κ . Let

$$\text{Index}(\mathbf{D}_\kappa^*) \rightarrow M_\kappa^{s, \natural}(S^4, \delta)$$

be the vector bundle defined by the cokernels of the Dirac operators (using $\text{Coker } D_A = \text{Ker } D_A^*$),

$$(7.4.2) \quad \text{Index}(\mathbf{D}_\kappa^*) := \{(A, F^s, \Psi) \in \tilde{M}_\kappa^{s, \natural}(S^4) \times \text{Fr}(E_\kappa)|_s \times L_{k-1}^2(S^- \otimes E_\kappa) : D_A^* \Psi = 0\} / \mathcal{G}_\kappa.$$

Because of the positive scalar curvature of the standard round metric on S^4 , the estimates in [27, Section 4.1] using a Bochner–Weitzenböck formula for the Dirac operator imply that $\text{Ker } D_A = 0$ for all $[A, F^s] \in M_\kappa^{s, \natural}(S^4, \delta)$. The fact that the index of the Dirac operator is constant with respect to $[A, F^s]$ then ensures that $\text{Index}(\mathbf{D}_\kappa^*)$ defines a vector bundle. The Chern class of this bundle is discussed in [3, 4]. A computation using the Atiyah–Singer Index Theorem shows that $\text{Index}(\mathbf{D}_\kappa^*)$ is a complex, rank- κ vector subbundle of the restriction of the Hilbert bundle,

$$(7.4.3) \quad \mathfrak{H}_\kappa := \mathcal{A}_\kappa^{\natural}(2\delta) \times \text{Fr}(E_\kappa)|_s \times_{\mathcal{G}_\kappa} L_{k-1}^2(V_\kappa^-) \rightarrow \mathcal{B}_\kappa^s(S^4, 2\delta),$$

to $M_{\text{SPL}, \kappa}^{s, \natural}(\delta)$. Using the same topology as that described for the infinite-dimensional obstruction bundle \mathfrak{H}_t in Definition 7.2.1, we define an extension of the Hilbert bundle (7.4.3) over the space (5.2.4) of ideal framed connections by

$$(7.4.4) \quad \bar{\mathfrak{H}}_\kappa := \bigsqcup_{\ell=0}^{\kappa} \mathfrak{H}_{\kappa-\ell} \times \text{Sym}^\ell(X).$$

If we define a subbundle of the restriction of $\bar{\mathfrak{H}}_\kappa$ to $\bar{M}_\kappa^{s, \natural}(S^4, \delta)$ by

$$(7.4.5) \quad \overline{\text{Index}}(\mathbf{D}_\kappa^*) := \{[A, F^s, \mathbf{x}, \Psi] \in \bar{\mathfrak{H}}_\kappa : [A, F^s, \mathbf{x}] \in \bar{M}_\kappa^{s, \natural}(S^4, \delta) \text{ and } D_A^* \Psi = 0\},$$

then the restriction of $\overline{\text{Index}}(\mathbf{D}_\kappa^*)$ to $M_\kappa^{s, \natural}(S^4, \delta)$ is exactly $\text{Index}(\mathbf{D}_\kappa^*)$. Because the index of the Dirac operator depends on κ , the rank of the fiber of the projection map $\overline{\text{Index}}(\mathbf{D}_\kappa^*) \rightarrow \bar{M}_\kappa^{s, \natural}(S^4, \delta)$ decreases on the lower strata of $\bar{M}_\kappa^{s, \natural}(S^4, \delta)$ and $\overline{\text{Index}}(\mathbf{D}_\kappa^*)$ is only a pseudo-bundle. The inclusion maps ι_Σ for this bundle, as required by Definition 7.1.1, are defined in [15, Proposition 7.1.32].

7.5. The action of $\text{Spin}^u(4)$

To splice elements of the instanton obstruction pseudo-bundle over S^4 onto X , we must use local trivializations of the spin^u bundles given by frames. Changing these frames is equivalent to changing the element of the obstruction pseudo-bundle being spliced in by an action which we now describe.

As in [30, Section 3.2], we denote

$$\text{Spin}^u(4) := (\text{U}(2) \times \text{Spin}^c(4)) / S^1.$$

The standard homomorphisms, $\text{Ad}_U : \text{U}(2) \rightarrow \text{SO}(3)$ and $\text{Ad}_S : \text{Spin}^c(4) \rightarrow \text{SO}(4)$ (see [57, Equation (D.2)]), define a homomorphism,

$$(7.5.1) \quad \text{Ad}^u : \text{Spin}^u(4) \ni [M, S] \mapsto (\text{Ad}_U(M), \text{Ad}_S(S)) \in \text{SO}(3) \times \text{SO}(4),$$

where $M \in \text{U}(2)$, $S \in \text{Spin}^c(4)$, and $[M, S] \in \text{Spin}^u(4)$. The composition of Ad^u with projection onto the factors $\text{SO}(3)$ and $\text{SO}(4)$ define a pair of surjective homomorphisms (see [30, Equation (3.13)]),

$$\text{Ad}_{\text{SO}(3)}^u : \text{Spin}^u(4) \rightarrow \text{SO}(3), \quad \text{and} \quad \text{Ad}_{\text{SO}(4)}^u : \text{Spin}^u(4) \rightarrow \text{SO}(4).$$

The action of $\text{SO}(4) \times \text{SO}(3)$ on $\bar{\mathcal{B}}_\kappa^s(S^4)$ is defined in [30, Section 3.3], with $\text{SO}(3)$ acting on the frame and $\text{SO}(4)$ acting by pulling the rotation action on \mathbb{R}^4 back to S^4 by stereographic projection. The $\text{SO}(4)$ action on S^4 induces an action on connections on bundles over S^4 by pullback. The action of $\text{Spin}^u(4)$ on $\bar{\mathfrak{Y}}_\kappa$ is defined in [30, Section 3.6.2] (see also [17, Section 3.2]). The map $\bar{\mathfrak{Y}}_\kappa \rightarrow \bar{\mathcal{B}}_\kappa^s(S^4)$ is equivariant with respect to the action of $\text{Spin}^u(4)$ on $\bar{\mathfrak{Y}}_\kappa$ and the action of $\text{Spin}^u(4)$ on $\bar{\mathcal{B}}_\kappa^s(S^4)$ defined by Ad^u and the action of $\text{SO}(4) \times \text{SO}(3)$ on $\bar{\mathcal{B}}_\kappa^s(S^4)$. The bundle $\text{Index}(\mathbf{D}_\kappa^*)$ is invariant under this action and thus is also a $\text{Spin}^u(4)$ -equivariant bundle.

7.6. Pseudo-bundle over the instanton moduli space with spliced ends

The obstruction pseudo-bundle over the instanton moduli space with spliced ends (7.1.2) will be defined to be identical to the index pseudo-bundle $\text{Index}(\bar{\mathbf{D}}_\kappa^*)$ defined in (7.4.5) on the complement of the spliced ends of $\bar{M}_{\text{SPL}, \kappa}^{s, \natural}(\delta)$. On the spliced end, W_κ , the obstruction pseudo-bundle will be constructed as the image of a splicing map which we now introduce.

Recall from (5.1.6) in Theorem 5.1.1 that the spliced end of $\bar{M}_{\text{SPL}, \kappa}^{s, \natural}(\delta)$ near the stratum containing the product connection,

$$[\Theta] \times \Sigma(Z_\kappa(\delta), \mathcal{P}),$$

where \mathcal{P} is a partition of N_κ with $|\mathcal{P}| > 1$, is given by the image of a splicing map,

$$\gamma'_{\Theta, \mathcal{P}} : \mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}) \subset \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times_{\mathfrak{S}(\mathcal{P})} \prod_{P \in \mathcal{P}} \bar{M}_{\text{SPL}, |P|}^{s, \natural}(\delta_P) \rightarrow \bar{M}_{\text{SPL}, \kappa}^{s, \natural}(\delta).$$

We will define the obstruction pseudo-bundle over the instanton moduli space with spliced ends inductively, just as we defined the instanton moduli space with spliced ends.

An element of the space $\bar{\mathfrak{Y}}_\kappa$ in (7.4.4) can be written as $[A, F^s, \mathbf{x}, \Psi]$ where $[A, F^s, \mathbf{x}] \in \bar{\mathcal{B}}_\kappa^s(S^4, 2\delta)$, and $\mathbf{x} \in \text{Sym}^\ell(\mathbb{R}^4)$, and $\Psi \in L_{k-1}^2(S^- \otimes E_{\kappa-\ell})$. For each partition \mathcal{P} of N_κ we define an embedding, covering the splicing map $\gamma'_{\Theta, \mathcal{P}}$, by

$$(7.6.1) \quad \varphi'_{\Theta, \mathcal{P}} : \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times_{\mathfrak{S}(\mathcal{P})} \prod_{P \in \mathcal{P}} \bar{\mathfrak{Y}}_{|P|} \rightarrow \bar{\mathfrak{Y}}_\kappa, \\ [x_P, [A_P, F_P^s, v_P, \Psi_P]]_{P \in \mathcal{P}} \mapsto \left(\gamma'_{\Theta, \mathcal{P}}([x_P, [A_P, F_P^s, v_P]]_{P \in \mathcal{P}}), \sum_{P \in \mathcal{P}} \chi_{x_P, \frac{1}{8}\lambda_P^{1/3}} \mathcal{C}_{x_P, 1}^* \Psi_P \right).$$

If $x_P \in \mathbb{R}^4$, then $\chi_{x_P, \frac{1}{8}\lambda_P^{1/3}} : \mathbb{R}^4 \rightarrow [0, 1]$ is a smooth cut-off function that is equal to one on $B(x_P, \frac{1}{8}\lambda_P^{1/3})$ and is supported in $B(x_P, \frac{1}{4}\lambda_P^{1/3})$ (see Section 6.4.1), where $\lambda_P = \lambda([A_P, F_P^s, \mathbf{x}_P])$.

The map $c_{x_P,1}$ is defined in Lemma 5.4.3. The definition (7.6.1) follows that in [24, Section 8.2 and Equation (8.32)].

Observe that the map (7.6.1) is equivariant with respect to the $\text{Spin}^u(4)$ action if $\text{Spin}^u(4)$ acts diagonally on the domain, by the projection $\text{Ad}_{\text{SO}(4)}^u$ to $\text{SO}(4)$ on the factor $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$. The properties of the obstruction pseudo-bundle over the instanton moduli space with spliced ends are given by the

THEOREM 7.6.1. *There exists a pseudo-bundle,*

$$(7.6.2) \quad \tilde{\Upsilon}_{\text{SPL},\kappa}^i \rightarrow \bar{M}_{\text{SPL},\kappa}^{s,\natural}(\delta),$$

which is a $\text{Spin}^u(4)$ -invariant subbundle of the space $\tilde{\mathfrak{Y}}_\kappa \rightarrow \bar{\mathcal{B}}_\kappa^{s,\natural}(S^4, 2\delta)$ and has the following properties:

- (1) *The restriction of $\tilde{\Upsilon}_{\text{SPL},\kappa}^i$ to the top stratum, $M_{\text{SPL},\kappa}^{s,\natural}(\delta)$, is a complex rank- κ vector bundle.*
- (2) *For any partition \mathcal{P} of κ with $|\mathcal{P}| > 1$, the restriction of $\tilde{\Upsilon}_{\text{SPL},\kappa}^i$ to the open neighborhood of $[\Theta] \times \Sigma(Z_\kappa(\delta), \mathcal{P})$ given in (5.1.6) is equal to the image of*

$$(7.6.3) \quad \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times_{\mathfrak{S}(\mathcal{P})} \prod_{P \in \mathcal{P}} \tilde{\Upsilon}_{\text{SPL},|P|}^i,$$

under the splicing map $\varphi'_{\Theta, \mathcal{P}}$.

- (3) *If $W_\kappa \subset \bar{M}_{\text{SPL},\kappa}^{s,\natural}(\delta)$ is as defined in (5.6.2), then the restriction of $\tilde{\Upsilon}_{\text{SPL},\kappa}^i$ to $\bar{M}_{\text{SPL},\kappa}^{s,\natural}(\delta) \setminus W_\kappa$ is given by the restriction of the pseudo-bundle $\overline{\text{Index}}(\mathbf{D}_\kappa^*)$ in (7.4.5) to $\bar{M}_{\text{SPL},\kappa}^{s,\natural}(\delta) \setminus W_\kappa$.*
- (4) *The restriction of $\tilde{\Upsilon}_{\text{SPL},\kappa}^i$ to the levels*

$$\bar{M}_{\text{SPL},\kappa}^{s,\natural}(\delta) \cap \left(M_{\text{SPL},\kappa-\ell}^{s,\natural}(\delta) \times \text{Sym}^\ell(\mathbb{R}^4) \right),$$

of $\bar{M}_{\text{SPL},\kappa}^{s,\natural}(\delta)$ is isomorphic to the pullback of $\tilde{\Upsilon}_{\text{SPL},\kappa-\ell}^i$ under the projection

$$M_{\text{SPL},\kappa-\ell}^{s,\natural}(\delta) \times \text{Sym}^\ell(\mathbb{R}^4) \rightarrow M_{\text{SPL},\kappa-\ell}^{s,\natural}(\delta).$$

- (5) *For any $[A', F^s, \mathbf{x}] \in \bar{M}_{\text{SPL},\kappa}^{s,\natural}(\delta)$, L^2 -orthogonal projection defines an isomorphism,*

$$\Upsilon_{\text{SPL},\kappa}^i|_{[A', F^s, \mathbf{x}]} \cong \text{Coker } D_{A_t},$$

for all $t \in [0, 1]$, where $A_t = A' + a_t(A')$ and $A = A' + a_1(A')$ is the point in the image of the gluing map appearing in (5.8.2) at A' .

REMARK 7.6.2. Observe that the rank of the fibers of $\tilde{\Upsilon}_{\text{SPL},\kappa}^i \rightarrow \bar{M}_{\text{SPL},\kappa}^{s,\natural}(\delta)$ varies with the stratum of $M_{\text{SPL},\kappa}^{s,\natural}(\delta)$ by Item (4). For this reason, $\tilde{\Upsilon}_{\text{SPL},\kappa}^i$ is a pseudo-bundle.

The proof of Theorem 7.6.1 occupies the rest of this section and is similar to the construction of the instanton moduli space with spliced ends in the proof of Theorem 5.1.1.

7.6.1. Pseudo-bundles and overlap data. If the restriction of the obstruction pseudo-bundle over the instanton moduli space with spliced ends to the image of the splicing map $\gamma'_{\Theta, \mathcal{P}}$ is to be given by the image of the splicing map $\varphi'_{\Theta, \mathcal{P}}$ restricted to the subspace (7.6.3) as asserted in Item (2) of Theorem 7.6.1, then we need to describe the overlaps of the images of the splicing maps $\varphi'_{\Theta, \mathcal{P}}$ and $\varphi'_{\Theta, \mathcal{P}'}$ where $\mathcal{P} < \mathcal{P}'$. This is undertaken by constructing a space of overlap data analogous to that appearing in Proposition 5.5.3.

We begin by initially considering the pseudo-bundle, $\tilde{\Upsilon}(\Theta, \mathcal{P}, \mathcal{P}')$, defined by restricting the pseudo-bundle,

$$(7.6.4) \quad \begin{array}{c} \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times \prod_{P \in \mathcal{P}} \left(\Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}'_P) \times \prod_{Q \in \mathcal{P}'_P} \tilde{\Upsilon}_{\text{SPL},|Q|}^i \right) \\ \downarrow \\ \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times \prod_{P \in \mathcal{P}} \left(\Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}'_P) \times \prod_{Q \in \mathcal{P}'_P} \bar{M}_{\text{SPL},|Q|}^{s,\natural}(\delta_Q) \right) \end{array}$$

to the intersection of its base with the space $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$ defined in Lemma 5.5.1. By induction on κ , the pseudo-bundle $\tilde{\Upsilon}_{\text{SPL},|Q|}^i$ is defined for all Q in a partition \mathcal{P} of N_κ of length greater than one. For all $P \in \mathcal{P}$, the splicing maps $\varphi_{\Theta, \mathcal{P}'_P}$ define maps,

$$\varphi_{\Theta, \mathcal{P}'_P} : \Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}'_P) \times \prod_{Q \in \mathcal{P}'_P} \tilde{\Upsilon}_{\text{SPL},|Q|}^i \rightarrow \tilde{\Upsilon}_{\text{SPL},|P|}^i,$$

covering the splicing map, $\gamma'_{\Theta, \mathcal{P}'_P}$. Hence, there is a map,

$$\rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, d} : \tilde{\Upsilon}(\Theta, \mathcal{P}, \mathcal{P}') \rightarrow \tilde{\Upsilon}(\Theta, \mathcal{P}) = \Delta^\circ(Z_\kappa(\delta), \mathcal{P}) \times \prod_{P \in \mathcal{P}} \tilde{\Upsilon}_{\text{SPL},|P|}^i,$$

defined by

$$(7.6.5) \quad \rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, d} := \text{id}_\Delta \times \prod_{P \in \mathcal{P}} \varphi'_{\Theta, \mathcal{P}'_P},$$

where id_Δ is the identity map on $\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$. By the definition of the overlap map $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}$ in (5.5.8), the map $\rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, d}$ fits into the following commutative diagram of maps of pseudo-bundles,

$$(7.6.6) \quad \begin{array}{ccc} \tilde{\Upsilon}(\Theta, \mathcal{P}, \mathcal{P}') & \xrightarrow{\rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, d}} & \tilde{\Upsilon}(\Theta, \mathcal{P}) \\ \downarrow & & \downarrow \\ \mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, \mathcal{P}', \delta) & \xrightarrow{\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}} & \mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}) \end{array}$$

The map $\rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, d}$ is not injective because of the action of the symmetric group, but this will not affect our construction.

We also define a map,

$$(7.6.7) \quad \rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, u} : \tilde{\Upsilon}(\Theta, \mathcal{P}, \mathcal{P}') \rightarrow \tilde{\Upsilon}(\Theta, \mathcal{P}'),$$

which fits into the following commutative diagram of maps of pseudo-bundles,

$$(7.6.8) \quad \begin{array}{ccc} \tilde{\Upsilon}(\Theta, \mathcal{P}, \mathcal{P}') & \xrightarrow{\rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, u}} & \tilde{\Upsilon}(\Theta, \mathcal{P}') \\ \downarrow & & \downarrow \\ \mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, \mathcal{P}', \delta) & \xrightarrow{\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}} & \mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}') \end{array}$$

The following lemma, analogous to Lemma 5.5.2, is the key to the construction of the bundle $\Upsilon_{\text{SPL}, \kappa}^i$.

LEMMA 7.6.3. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_κ with $|\mathcal{P}| > 1$. Then the following diagram of pseudo-bundles,*

$$(7.6.9) \quad \begin{array}{ccc} \tilde{\Upsilon}(\Theta, \mathcal{P}, \mathcal{P}') & \xrightarrow{\rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, d}} & \tilde{\Upsilon}(\Theta, \mathcal{P}) \\ \rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, u} \downarrow & & \varphi'_{\Theta, \mathcal{P}} \downarrow \\ \tilde{\Upsilon}(\Theta, \mathcal{P}') & \xrightarrow{\varphi'_{\Theta, \mathcal{P}'}} & \tilde{\mathfrak{X}}_\kappa \end{array}$$

covers the diagram (5.5.14), is $\text{Spin}^u(4)$ -equivariant, and commutes.

PROOF. The assertion that the diagram (7.6.9) covers the diagram (5.5.14) follows from the definition of the maps $\rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, d}$ and $\rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, u}$ (see the diagrams (7.6.6) and (7.6.8)) and the fact that the splicing maps $\varphi'_{\Theta, \mathcal{P}}$ cover the splicing maps $\gamma'_{\Theta, \mathcal{P}}$.

The $\text{Spin}^u(4)$ equivariance of the diagram (7.6.9) follows from that of the splicing maps $\varphi'_{\Theta, \mathcal{P}}$.

We now prove the commutativity of the diagram (7.6.9). Assume that a point in $\tilde{\Upsilon}(\Theta, \mathcal{P}, \mathcal{P}')$ is given by data consisting of points $(x_P)_{P \in \mathcal{P}} \in \Delta^\circ(Z_\kappa(\delta), \mathcal{P})$, points $(y_Q)_{Q \in \mathcal{P}'_P} \in \Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}'_P)$, and $[A_Q, F_Q^s, \mathbf{v}_Q, \Psi_Q] \in \tilde{\Upsilon}_{\text{SPL}, |Q|}$. We write

$$\begin{aligned} [A''_u, \Phi''_u, \mathbf{x}_u, \Psi_u] &= (\varphi'_{\Theta, \mathcal{P}'} \circ \rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, u}) \left(\left((x_P), ((y_Q), [A_Q, F_Q^s, \mathbf{v}_Q, \Psi_Q])_{Q \in \mathcal{P}'_P} \right)_{P \in \mathcal{P}} \right), \\ [A''_d, \Phi''_d, \mathbf{x}_d, \Psi_d] &= (\varphi'_{\Theta, \mathcal{P}} \circ \rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, d}) \left(\left((x_P), ((y_Q), [A_Q, F_Q^s, \mathbf{v}_Q, \Psi_Q])_{Q \in \mathcal{P}'_P} \right)_{P \in \mathcal{P}} \right). \end{aligned}$$

Because the diagram (7.6.9) covers the commuting diagram (5.5.14), to show that (7.6.9) commutes, we only need to prove that $\Psi_u = \Psi_d$.

From the definition of $\rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, u}$ in (7.6.7) and the definition of $\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}$ in (5.5.5), we have,

$$\rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, u} \left(\left((x_P), ((y_Q), [A_Q, F_Q^s, \mathbf{v}_Q, \Psi_Q])_{Q \in \mathcal{P}'_P} \right)_{P \in \mathcal{P}} \right) = (x_{P(Q)} + y_Q, [A_Q, F_Q^s, \mathbf{v}_Q, \Psi_Q])_{Q \in \mathcal{P}'},$$

where, for $Q \in \mathcal{P}'$, we write $P(Q) \in \mathcal{P}$ for the unique subset of N_ℓ with $Q \subseteq P$. If $\lambda_P = \lambda([A_P, F_P^s, \mathbf{x}'_P])$, the definition of $\varphi'_{\Theta, \mathcal{P}}$ then implies that

$$(7.6.10) \quad \Psi_u = \sum_{Q \in \mathcal{P}'} \chi_{x_{P(Q)} + y_Q, \frac{1}{8}\lambda_Q^{1/3} c_{x_{P(Q)} + y_Q, 1}^*} \Psi_Q.$$

From the definition of $\rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, d}$ in (7.6.5), if we write

$$\begin{aligned} \rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, d} \left(\left((x_P), ((y_Q), [A_Q, F_Q^s, \mathbf{v}_Q, \Psi_Q])_{Q \in \mathcal{P}'_P} \right)_{P \in \mathcal{P}} \right) \\ = ((x_P), [A_P, F_P^s, \mathbf{x}'_P, \Psi_P])_{P \in \mathcal{P}} \end{aligned}$$

then, for $\lambda_Q = \lambda([A_Q, F_Q^s, \mathbf{v}_Q])$,

$$\Psi_P = \sum_{Q \in \mathcal{P}'_P} \chi_{y_Q, \frac{1}{8}\lambda_Q^{1/3} c_{y_Q, 1}^*} \Psi_Q.$$

If $\lambda_P = \lambda([A_P, F_P^s, \mathbf{x}'_P])$, then the definition of $\varphi'_{\Theta, \mathcal{P}}$ implies that

$$\Psi_d = \sum_{P \in \mathcal{P}} \chi_{x_P, \frac{1}{8}\lambda_P^{1/3}} c_{x_P, 1}^* \Psi_P = \sum_{P \in \mathcal{P}} \sum_{Q \in \mathcal{P}'_P} \chi_{x_P, \frac{1}{8}\lambda_P^{1/3}} c_{x_P, 1}^* \left(\chi_{y_Q, \frac{1}{8}\lambda_Q^{1/3}} c_{y_Q, 1}^* \Psi_Q \right).$$

Applying Lemma 5.4.3 to the preceding equalities gives

$$(7.6.11) \quad \Psi_d = \sum_{Q \in \mathcal{P}'} \chi_{x_{P(Q)}, \frac{1}{8}\lambda_{P(Q)}^{1/3}} \chi_{x_{P(Q)} + y_Q, \frac{1}{8}\lambda_Q^{1/3}} c_{x_Q + y_Q, 1}^* \Psi_Q.$$

By the constraint (5.5.9), the points x_P and y_Q in the definition of the subspace $\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}')$ satisfy (see condition (5.4.10)),

$$B(x_{P(Q)} + y_Q, \frac{1}{4}\lambda_Q^{1/3}) \Subset B(x_{P(Q)}, \frac{1}{8}\lambda_{P(Q)}^{1/3}).$$

Because $\chi_{x, \frac{1}{8}\lambda^{1/3}}$ is supported on $B(x, \frac{1}{4}\lambda^{1/3})$ and equal to one on $B(x, \frac{1}{8}\lambda^{1/3})$, the preceding inclusion implies that $\chi_{x_{P(Q)}, \frac{1}{8}\lambda_{P(Q)}^{1/3}}$ is equal to one on the support of $\chi_{x_{P(Q)} + y_Q, \frac{1}{8}\lambda_Q^{1/3}}$, so (7.6.11) and (7.6.10) imply that

$$\Psi_d = \sum_{Q \in \mathcal{P}'} \chi_{x_{P(Q)} + y_Q, \frac{1}{8}\lambda_Q^{1/3}} c_{x_Q + y_Q, 1}^* \Psi_Q = \Psi_u,$$

as required to prove that the diagram (7.6.9) commutes. \square

PROOF OF THEOREM 7.6.1. The proof of this theorem is largely identical to the proof of Theorem 5.1.1 so we omit many of the details. We construct $\tilde{\Upsilon}_{\text{SPL}, \kappa}^i$ using induction on κ .

The induction begins with setting the restriction of $\tilde{\Upsilon}_{\text{SPL}, 1}^i$ to the top stratum of $\bar{M}_{\text{SPL}, 1}^{s, \natural}(\delta)$ equal to the bundle $\text{Index}(\mathbf{D}_1^*)$. Because there is, up to dilation and the $\text{SO}(3)$ action on the frame, only one gauge-equivalence class in $\bar{M}_{\text{SPL}, 1}^{s, \natural}(\delta)$, the bundle,

$$\overline{\text{Index}(\mathbf{D}_1^*)} \rightarrow \bar{M}_{\text{SPL}, 1}^{s, \natural}(\delta) = (0, \delta) \times \text{SO}(3),$$

is given by

$$(0, \delta) \times \text{SU}(2) \times_{\pm 1} \mathbb{C} \rightarrow (0, \delta) \times \text{SO}(3).$$

This bundle extends as a pseudo-bundle over the Uhlenbeck compactification, $c(\text{SO}(3))$, of $M_{\text{SPL}, 1}^{s, \natural}(\delta)$ as

$$c(\text{SU}(2) \times_{\pm 1} \mathbb{C}) \rightarrow c(\text{SO}(3)),$$

which satisfies the requirements of Theorem 7.6.1 for the case $\kappa = 1$.

Assuming that $\tilde{\Upsilon}_{\text{SPL}, \kappa'}^i$ has been constructed for all $\kappa' < \kappa$, we define a pseudo-bundle $\tilde{\Upsilon}^i(W_\kappa) \rightarrow W_\kappa$ over the spliced end by the image of the splicing maps $\varphi'_{\Theta, \mathcal{P}}$ on the domains in (7.6.3). This space is a pseudo-bundle because the overlap of the images of the maps $\varphi'_{\Theta, \mathcal{P}}$ is parameterized by Lemma 7.6.3.

The final property in Theorem 7.6.1 regarding the surjectivity of L^2 -orthogonal projection follows from the results of [24, Section 8] (see also [15, Proposition 7.1.32]). We can then define an isotopy R_D of $\tilde{\Upsilon}^i(W_\kappa)$ by, for $t \in [1/2, 3/4]$,

$$R_D(t, [A, F^s, \mathbf{x}, \Psi]) := (R(t, [A, F^s, \mathbf{x}]), (1 - 4(t - 1/2))\Psi + 4(t - 1/2)\Pi_{A+a(4(t-1/2), A)}\Psi)$$

where R is the isotopy defined in (5.8.4) and for $a(t, A) = a_t(A)$ defined by the gluing map appearing in (5.8.2),

$$G(t, [A, F^s, \mathbf{x}]) = [A + a(t, A), F^s, \mathbf{x}],$$

and where $\Pi_{A+a(t,A)}$ denotes L^2 -orthogonal onto $\text{Coker } D_{A+a(t,A)}$. For $t \in (-\infty, 1/2]$, the isotopy $R_D(t, \cdot)$ is equal to the identity and for $t \in [3/4, 1]$, the isotopy $R_D(\cdot, t)$ is defined by pulling back the section Ψ by the centering map.

The pseudo-bundle $\tilde{\Upsilon}_{\text{SPL}, \kappa}^i$ is then defined by the union of the images $\tilde{\Upsilon}^i(W_\kappa)$ under the map $R_D(\beta(\cdot, \cdot))$, where β is the function defined in the proof of Theorem 5.1.1, with $\text{Index}(\mathbf{D}_\kappa^*)$ restricted to the complement

$$\bar{M}_\kappa^{s, \natural}(S^4, \delta) \setminus R(1, W_\kappa),$$

as in the definition of $\bar{M}_{\text{SPL}, \kappa}^{s, \natural}(\delta)$ in the proof of Theorem 5.1.1. \square

7.7. Instanton obstruction pseudo-bundle

We now construct the instanton component of the obstruction pseudo-bundle and its embedding into the infinite-rank obstruction pseudo-bundle $\tilde{\mathfrak{Y}}_t$.

7.7.1. The frame bundles. For a spin^u structure $\mathfrak{t} = (\rho, V)$ over X , we defined a frame bundle in [30, Section 3.2] by

$$\text{Fr}_{\text{Cl}(T^*X)}(V) \rightarrow \text{Fr}(TX) \rightarrow X,$$

which is a $\text{U}(2)$ bundle over $\text{Fr}(TX)$ and a $\text{Spin}^u(4)$ bundle over X . The fiber over $F \in \text{Fr}(TX)|_x$ is given by the Clifford module isomorphisms from the Clifford module $\Delta \otimes \mathbb{C}^2$ to $V|_x$ with respect to the isomorphism $\mathbb{R}^4 \cong T^*X|_x$ defined by F , for each $x \in X$. The homomorphism $\text{Ad}^u : \text{Spin}^u(4) \rightarrow \text{SO}(3) \times \text{SO}(4)$ in (7.5.1) defines a bundle map,

$$\text{Fr}_{\text{Cl}(T^*X)}(V) \rightarrow \text{Fr}(\mathfrak{g}_t) \times_X \text{Fr}(TX).$$

By analogy with the definition of the gluing-data bundle $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ in (6.2.2), we define

$$(7.7.1) \quad \text{Fr}(V, \mathcal{P}, g_{\mathcal{P}}) \rightarrow \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}),$$

by

$$\text{Fr}(V, \mathcal{P}, g_{\mathcal{P}}) := \left\{ (F_1^V, \dots, F_\ell^V) \in \prod_{i=1}^{\ell} \text{Fr}_{\text{Cl}(T^*)}(V_{\mathfrak{t}(\ell)}) \right\}_{\Delta^\circ(X^\ell, \mathcal{P})} : \\ F_i^V = F_j^V \iff \exists P \in \mathcal{P} \text{ with } i, j \in P \}.$$

The structure group of the bundle $\text{Fr}(V, \mathcal{P}, g_{\mathcal{P}}) \rightarrow \Delta^\circ(X^\ell, \mathcal{P})$ is

$$(7.7.2) \quad \tilde{G}^V(\mathcal{P}) = \left\{ (G_1, \dots, G_\ell) \in \prod_{i=1}^{\ell} \text{Spin}^u(4) : G_i = G_j \iff \exists P \in \mathcal{P} \text{ with } i, j \in P \right\}.$$

The structure group of the bundle $\text{Fr}(V, \mathcal{P}, g_{\mathcal{P}}) \rightarrow \Sigma(X^\ell, \mathcal{P})$ is

$$(7.7.3) \quad G^V(\mathcal{P}) := \tilde{G}^V(\mathcal{P}) \rtimes \mathfrak{S}(\mathcal{P}) / \Gamma(\mathcal{P}).$$

We then define the instanton obstruction pseudo-bundle for the partition \mathcal{P} by

$$(7.7.4) \quad \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \times \text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \times \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}),$$

where

$$(7.7.5) \quad \text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) := \text{Fr}(V, \mathcal{P}, g_{\mathcal{P}}) \times_{G^V(\mathcal{P})} \prod_{P \in \mathcal{P}} \tilde{\Upsilon}_{\text{SPL}, |P|}^{s, \natural}.$$

We define an S^1 action on the bundle (7.7.4) covering the S^1 action on its base defined in Lemma 6.2.1 by the diagonal action on the factors $\tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta)$ and $\tilde{\Upsilon}_{\text{SPL}, |P|}^{s, \natural}$ in (7.7.4), where S^1 acts on $\tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta)$ by the action (2.3.28) and on $\tilde{\Upsilon}_{\text{SPL}, |P|}^{s, \natural}$ by scalar multiplication on the fibers of $\tilde{\Upsilon}_{\text{SPL}, |P|}^{s, \natural} \rightarrow \bar{M}_{\text{SPL}, |P|}^{s, \natural}$.

7.7.2. Splicing map. For each partition \mathcal{P} of N_{ℓ} , we define an S^1 -equivariant splicing map for the instanton obstruction,

$$(7.7.6) \quad \varphi_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}} : \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \bar{\mathfrak{X}}_{\mathfrak{t}},$$

which will cover the crude splicing map $\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}$. A choice of frames $(F_P^V)_{P \in \mathcal{P}} \in \text{Fr}(V, \mathcal{P})|_{\mathbf{y}}$, where $\mathbf{y} = (y_P)_{P \in \mathcal{P}}$, a spin^u connection A_0 on $V_{\mathfrak{t}(\ell)}$, an $\text{SO}(3)$ connection A_P on \mathfrak{g}_P , and a frame $F_P^s \in \text{Fr}(\mathfrak{g}_P)|_s$ determine embeddings,

$$(7.7.7) \quad \phi(A_0, F_P^V, A_P, F_P^s) : V|_{P|} |_{\varphi_n(B(8\lambda^{1/3}))} \cong V|_{B(y_P, 8\lambda^{1/3})},$$

as described in [30, Section 3.4.3]. If χ is a cut-off function supported on $\varphi_n(B(8\lambda^{1/3}))$ and Ψ is a section of $V|_{P|}$, then we write $\phi(A_0, F_P^V, A_P, F_P^s)(\chi\Psi)$ for the section of $V|_{B(y_P, 8\lambda^{1/3})}$ given by the isomorphism (7.7.7).

To ensure that the images of the different splicing maps fit together to form a vector pseudo-bundle, we introduce a flattening map on spin^u connections just as we did for Riemannian metrics on X and connections on $\mathfrak{g}_{\mathfrak{t}(\ell)}$ in Section 6.3. That is, given a spin^u connection A_0 on $V_{\mathfrak{t}(\ell)}$ we apply the construction of Lemma 6.3.4 and for each $\mathbf{y} \in \Sigma(X^{\ell}, \mathcal{P})$, we define a locally flattened spin^u connection $\Theta^u(A, \mathbf{y})$ on $V_{\mathfrak{t}(\ell)}$ satisfying the following properties:

- (1) For $\mathbf{y} = (y_P)_{P \in \mathcal{P}}$, the connection $\Theta^u(A_0, \mathbf{y})$ is flat on

$$\bigcup_{P \in \mathcal{P}} B(y_P, 4\tilde{s}_P(\mathbf{y})^{1/3}),$$

where \tilde{s}_P is the separating function defined in Lemma 6.3.4.

- (2) If $\mathcal{P} < \mathcal{P}'$ and $\mathbf{y} \in \Sigma(X^{\ell}, \mathcal{P})$ and $\mathbf{y}' \in \Sigma(X^{\ell}, \mathcal{P}') \cap \mathcal{U}(X^{\ell}, \mathcal{P})$ satisfy $\pi(X^{\ell}, \mathcal{P})(\mathbf{y}') = \mathbf{y}$, then

$$\Theta^u(A_0, \mathbf{y}) = \Theta^u(A_0, \mathbf{y}').$$

For $F^V = (F_P^V)_{P \in \mathcal{P}} \in \text{Fr}(V, \mathcal{P})|_{\mathbf{y}}$, define

$$\phi'(A_0, F_P^V, A_P, F_P^s) := \phi(\Theta^u(A_0, \mathbf{y}), F_P^V, A_P, F_P^s),$$

using the isomorphism (7.7.7) defined by the locally flattened connection $\Theta^u(A_0, \mathbf{y})$ on $V_{\mathfrak{t}(\ell)}$ instead of the connection A_0 .

For $[A_0, \Phi_0] \in N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \subset \mathcal{C}_{\mathfrak{t}(\ell)}$ and a point in $\text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ given by the data $(F_P^V)_{P \in \mathcal{P}} \in \text{Fr}(V, \mathcal{P}, g_{\mathcal{P}})$, and $[A_P, F_P^s, \mathbf{v}_P] \in \bar{M}_{\text{SPL}, |P|}^{s, \natural}(\delta_P)$, and $[A_P, F_P^s, \mathbf{v}_P, \Psi] \in \tilde{\Upsilon}_{\text{SPL}, |P|}$, and

$$\gamma''_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}((A_0, \Phi_0), F_P^V, [A_P, F_P^s, \mathbf{v}_P])_{P \in \mathcal{P}} = (A'', \Phi'', \mathbf{x}),$$

the instanton obstruction splicing map is then defined by

$$(7.7.8) \quad \begin{aligned} & \varphi_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}} \left([(A_0, \Phi_0), (F_P^V, [A_P, F_P^s, \mathbf{v}_P, \Psi_P])_{P \in \mathcal{P}}] \right) \\ & := \left[A'', \Phi'', \mathbf{x}, \sum_{P \in \mathcal{P}} \phi'(A_0, F_P^V, A_P, F_P^s)(\chi_{n, \frac{1}{8}\lambda_P^{1/3}} \Psi_P) \right], \end{aligned}$$

where $\chi_{n, \frac{1}{8}\lambda_P^{1/3}} : S^4 \rightarrow [0, 1]$ is a smooth cut-off function that is equal to one on $\varphi_n(B(\frac{1}{8}\lambda_P^{1/3}))$ and is supported in $\varphi_n(B(\frac{1}{4}\lambda_P^{1/3}))$ (see Section 6.4.1). This map is S^1 -equivariant with respect to the S^1 action on its domain described following (7.7.5) and the S^1 action on the image $\tilde{\mathfrak{Y}}_{\mathfrak{t}}$ defined following (7.2.2).

7.7.3. Overlap space and overlap maps. Following the method of previous sections, to parameterize the overlap of the images of the splicing maps $\varphi_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}$ and $\varphi_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}'}$, we introduce a space of overlap data and overlap maps. By analogy with the definition of the overlap space $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ in (6.5.1), we define

$$\text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \rightarrow \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$$

by

$$(7.7.9) \quad \begin{aligned} & \text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \\ & := \text{Fr}(V, \mathcal{P}, g_{\mathcal{P}}) \times_{G^V(\mathcal{P})} \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}} \left(\Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}''_P) \times \prod_{Q \in \mathcal{P}''_P} \tilde{\Upsilon}_{\text{SPL}, |Q|} \right). \end{aligned}$$

Define an S^1 action on $\text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ through the diagonal S^1 action on the factors of $\tilde{\Upsilon}_{\text{SPL}, |Q|}$ appearing in (7.7.9).

We define overlap maps,

$$\rho_{\mathcal{P}, \mathcal{P}'}^{V, d} : \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \rightarrow \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}),$$

covering the overlap maps $\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, d}$ defined in (6.5.15) by

$$(7.7.10) \quad \rho_{\mathcal{P}, \mathcal{P}'}^{V, d} := \text{id}_{\tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta)} \times \text{id}_{\text{Fr}(V)} \times \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}} \varphi_{\Theta, \mathcal{P}''_P},$$

where $\text{id}_{\text{Fr}(V)}$ is the identity map on $\text{Fr}(V, \mathcal{P}, g_{\mathcal{P}})$. Observe that $\rho_{\mathcal{P}, \mathcal{P}'}^{V, d}$ is S^1 -equivariant if S^1 acts on the domain by the diagonal action on $\tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta)$ and $\text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$, where S^1 acts on $\tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta)$ by the action (2.3.28) and on $\text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ by the action defined following (7.7.9) and S^1 acts on the image by the action defined following (7.7.5).

For

$$(7.7.11) \quad \begin{aligned} & \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \text{Ob}(\mathfrak{t}, \mathfrak{s}, [\mathcal{P} < \mathcal{P}']) \\ & := \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}}} \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}'') \right) / \mathfrak{S}(\mathcal{P}), \end{aligned}$$

we define an upwards overlap map,

$$(7.7.12) \quad \begin{array}{c} \tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) \\ \rho_{\mathcal{P},[\mathcal{P}']}^{V,u} \downarrow \\ \tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}'') \right) / \mathfrak{S}(\mathcal{P}) \end{array}$$

exactly as was done in (6.5.14). Recall that the map $\rho_{\mathcal{P},[\mathcal{P}']}^{\mathfrak{t},\mathfrak{s},u}$ was defined by a parallel translation of the frames in the domain $\mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}})$ and leaving the points in $\tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta)$ and the S^4 connections unchanged. Note that this parallel translation is carried out with respect to the locally flattened connection $\Theta^u(A_0, \mathbf{y})$. We define the map $\rho_{\mathcal{P},[\mathcal{P}']}^{V,u}$ in the same way, using the flattened spin^u connection A_0 to parallel translate the frames of $\text{Fr}_{\mathbb{C}\ell}(T^*X)(V_{\mathfrak{t}(\ell)})$ and leaving the elements of $\tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta)$ and of $\tilde{\Upsilon}_{\text{SPL},|Q|}$ unchanged. The map $\rho_{\mathcal{P},[\mathcal{P}']}^{V,u}$ is S^1 -equivariant when the S^1 actions on the range and domain are as described in the analogous assertion for $\rho_{\mathcal{P},[\mathcal{P}']}^{V,d}$.

By the invariance of the obstruction splicing maps under the action of the symmetric group, the splicing maps $\varphi_{\mathfrak{t},\mathfrak{s},\mathcal{P}''}$ for $\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']$ define a splicing map,

$$(7.7.13) \quad \varphi_{\mathfrak{t},\mathfrak{s},[\mathcal{P} < \mathcal{P}']} : \tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \text{Ob}(\mathfrak{t}, \mathfrak{s}, [\mathcal{P} < \mathcal{P}']) \rightarrow \tilde{\mathfrak{Y}}_{\mathfrak{t}},$$

which covers the crude splicing map $\gamma_{\mathfrak{t},\mathfrak{s},[\mathcal{P} < \mathcal{P}']}''$ in (6.5.20). We then have the following relation between the splicing maps $\varphi_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ and $\varphi_{\mathfrak{t},\mathfrak{s},[\mathcal{P} < \mathcal{P}']}$.

LEMMA 7.7.1. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_{ℓ} . Then the diagram*

$$(7.7.14) \quad \begin{array}{ccc} \tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}']) & \xrightarrow{\rho_{\mathcal{P},[\mathcal{P}']}^{V,u}} & \tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \text{Ob}(\mathfrak{t}, \mathfrak{s}, [\mathcal{P} < \mathcal{P}']) \\ \rho_{\mathcal{P},[\mathcal{P}']}^{V,d} \downarrow & & \varphi_{\mathfrak{t},\mathfrak{s},[\mathcal{P} < \mathcal{P}']} \downarrow \\ \tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) & \xrightarrow{\varphi_{\mathfrak{t},\mathfrak{s},\mathcal{P}}} & \tilde{\mathfrak{Y}}_{\mathfrak{t}} \end{array}$$

commutes, covers the diagram (6.5.22), and all the maps in the diagram are S^1 -equivariant.

PROOF. That the diagram (7.7.14) covers the diagram (6.5.22) follows immediately from the definitions of the overlap (see (7.7.10) and the paragraph following (7.7.12)) and obstruction splicing maps (see (7.7.8)). As in the proof of Proposition 6.5.3, we assume that the splicing maps $\varphi_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ and $\varphi_{\mathfrak{t},\mathfrak{s},\mathcal{P}'}$ are defined at points $\mathbf{y} \in \Sigma(X^{\ell}, \mathcal{P})$ and $\mathbf{y}' \in \Sigma(X^{\ell}, \mathcal{P}')$, respectively, where $\pi(X^{\ell}, \mathcal{P})(\mathbf{y}') = \mathbf{y}$. Thus, for $P' \in \mathcal{P}'$ and $P' \subseteq P \in \mathcal{P}$, the inclusion

$$B(y_{P'}, 8\tilde{s}(\mathbf{y}')^{1/3}) \subseteq B(y_P, 4\tilde{s}(\mathbf{y})^{1/2})$$

holds by (6.3.9). Because the Riemannian metric and spin^u connection used in the splicing argument are flat on the preceding balls, the conclusion of the lemma then follows from the argument establishing Lemma 7.6.3. \square

The commutativity of the diagram (7.7.14) and the definition of the space $\tilde{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ as the union of the spaces

$$\tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$$

implies that we can define a pseudo-bundle,

$$(7.7.15) \quad \tilde{\Upsilon}_{\mathfrak{t},\mathfrak{s}}^i \rightarrow \tilde{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}},$$

as the union of the pseudo-bundles

$$\tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}),$$

with the overlaps identified by the diagram (7.7.14). Because the maps in (7.7.14) are S^1 -equivariant and the S^1 actions in that diagram cover the S^1 actions on $\tilde{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$, the S^1 actions on $\tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times_{\mathcal{G}_s} \text{Ob}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ define a global S^1 action on $\tilde{\Upsilon}_{\mathfrak{t},\mathfrak{s}}^i$.

The splicing maps $\varphi_{\mathfrak{t},\mathfrak{s},\mathcal{P}}$ define a vector-bundle embedding of $\tilde{\Upsilon}_{\mathfrak{t},\mathfrak{s}}^i$ into $\tilde{\mathfrak{Y}}_{\mathfrak{t}}$ which covers the embedding of $\tilde{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ into $\tilde{\mathcal{C}}_{\mathfrak{t}}$ given by the crude splicing maps. Finally, we define an embedding,

$$(7.7.16) \quad \varphi'_i : \tilde{\Upsilon}_{\mathfrak{t},\mathfrak{s}}^i \rightarrow \tilde{\mathfrak{Y}}_{\mathfrak{t}},$$

covering the global splicing map $\gamma'_{\mathcal{M}}$ in (6.8.5) by replacing the triple $[A'', \Phi'', \mathbf{x}]$ in the definition (7.7.8) with the triple $[A', \Phi', \mathbf{x}]$ given by the global splicing map $\gamma'_{\mathcal{M}}$ of (6.8.5).

7.8. Local gluing hypothesis for $\text{SO}(3)$ monopoles

We can now state the relevant gluing hypothesis.

HYPOTHESIS 7.8.1 (Local gluing hypothesis). There is a continuous, S^1 -equivariant embedding,

$$(7.8.1) \quad \gamma_{\mathcal{M}} : \tilde{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}} \rightarrow \tilde{\mathcal{C}}_{\mathfrak{t}},$$

which is homotopic through S^1 -equivariant, continuous embeddings to the global splicing map $\gamma'_{\mathcal{M}}$, smooth on each stratum of $\tilde{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$, and equal to the identity on $N_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \times \text{Sym}^{\ell}(X)$. In addition, there are S^1 -equivariant sections χ_s and χ_i of the pseudo-bundles $\tilde{\Upsilon}_{\mathfrak{t},\mathfrak{s}}^s$ and $\tilde{\Upsilon}_{\mathfrak{t},\mathfrak{s}}^i$ with the following properties:

- (1) The restriction of the section $\bar{\chi} = \chi_s \oplus \chi_i$ of $\tilde{\Upsilon}_{\mathfrak{t},\mathfrak{s}}^s \oplus \tilde{\Upsilon}_{\mathfrak{t},\mathfrak{s}}^i$ to each stratum is smooth.
- (2) The restriction of the section $\bar{\chi}$ to each stratum of $\tilde{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ vanishes transversely.
- (3) If we pull back the fiber metric (7.2.3) to $\tilde{\Upsilon}_{\mathfrak{t},\mathfrak{s}}^s \oplus \tilde{\Upsilon}_{\mathfrak{t},\mathfrak{s}}^i$ by the splicing embeddings $\varphi'_s \oplus \varphi'_i$ defined in (7.3.8) and (7.7.16), then the L^2 norm of $\bar{\chi}$ is lower semi-continuous.
- (4) The restriction of $\gamma_{\mathcal{M}}$ to the zero-locus $\bar{\chi}^{-1}(0)$ is a homeomorphism between $\bar{\chi}^{-1}(0)$ and an open neighborhood of $M_s \times \text{Sym}^{\ell}(X)$ in $\tilde{\mathcal{M}}$.

REMARK 7.8.2. Recall from [62, p. 12] that a function f is lower semi-continuous if $f^{-1}((a, \infty))$ is open for all real a . A lower semi-continuous function $f(x)$ satisfies $\lim_{x \rightarrow x_0} f(x) \geq f(x_0)$ and we will use this property in Lemma 9.5.13 to obtain lower bounds on the fiber norm of the obstruction section on a relatively closed subspace of the top stratum of $\tilde{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$.

Although there are some important differences (which we explain further in Section 7.9), the gluing map and obstruction section in Hypothesis 7.8.1 are similar to those constructed in [24], as we now explain.

The gluing map in [24, Section 9] is the composition of the standard splicing map (defined in [24, Section 3.2] and here in Section 6.4.1) with domain given by the space $\tilde{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ defined in Section 6.7.1 and a solution map. Recall that the solution map,

$$(7.8.2) \quad (A', \Phi') \mapsto (A' + a(A', \Phi'), \Phi + \phi(A', \Phi')),$$

is defined on the image of the splicing map and has the property that $\mathfrak{S}(A' + a(A', \Phi'), \Phi + \phi(A', \Phi'))$ lies in a finite-dimensional obstruction space over (A', Φ') , and thus is a solution to the extended $\mathrm{SO}(3)$ -monopole equations, while (A, Φ) is an $\mathrm{SO}(3)$ monopole if and only $\mathfrak{S}(A, \Phi) = 0$, where \mathfrak{S} is as in (2.1.10). In [24, Section 8], this obstruction space over (A', Φ') is the span of the small-eigenvalue eigenvectors of the L^2 self-adjoint elliptic operator $(D\mathfrak{S}_{(A', \Phi')})(D\mathfrak{S}_{(A', \Phi')})^*$, where $D\mathfrak{S}_{(A', \Phi')}$ is the linearization of the map \mathfrak{S} at (A', Φ') . The map

$$(A', \Phi') \mapsto ((A', \Phi'), \mathfrak{S}(A' + a(A', \Phi'), \Phi + \phi(A', \Phi')))$$

then defines a section of a finite-rank vector bundle over the image of the slicing map whose zero locus is mapped to $\overline{\mathcal{M}}_t$ by the solution map.

To show that a solution map in the sense of (7.8.2) exists, we first prove in [24, Proposition 5.5] that for each pair (A', Φ') in the image of the splicing map, the error term $\mathfrak{S}(A', \Phi')$ is suitably small in the sense of norms defined in [19, 96]. Using techniques generalizing those introduced by Taubes in [94, 95, 96] and Donaldson and Kronheimer in [15, Section 7.2] (see also [72, 74]), we showed that for each point (A', Φ') in the image of the splicing map, there is a solution to the system of partial differential equations defining the solution map (see [24, Section 9.1]).

Our proof of existence of the solution map in [24] should extend to yield the desired gluing map, $\gamma_{\mathcal{M}}$, in Hypothesis 7.8.1 and identify the zero locus of the obstruction section, $\bar{\chi}$, with an open neighborhood in $\overline{\mathcal{M}}_t$. The smoothness of $\gamma_{\mathcal{M}}$ and the obstruction section χ asserted in Property (1) in Hypothesis 7.8.1 follow by the construction of the gluing map. Property (2), that the obstruction section vanishes transversely, follows from a formal argument and the result in [18] that the $\mathrm{SO}(3)$ -monopole map, \mathfrak{S} , vanishes transversely for appropriate choices of generic perturbations of the $\mathrm{SO}(3)$ -monopole equations.

We give proofs of the properties of the gluing map $\gamma_{\mathcal{M}}$ and obstruction section χ asserted by Hypothesis 7.8.1 in [22]. We expect that the continuity of the gluing map with respect to Uhlenbeck limits will follow along lines similar to the proof of the same property for the gluing map for anti-self-dual $\mathrm{SO}(3)$ connections given in [21]. This Uhlenbeck continuity will yield the Property (3) of the gluing map in Hypothesis 7.8.1. One must also show the map is injective and surjective where by the latter we mean Property (4) given in Hypothesis 7.8.1. In special cases, proofs of continuity with respect to Uhlenbeck limits, injectivity, and surjectivity for gluing maps for anti-self-dual $\mathrm{SO}(3)$ connections have been given in [15, Sections 7.2.5 and 7.2.6] and [94, 95, 96].

7.9. Notes on the justification of the local gluing hypothesis

The purpose of this section is to summarize the justification of Hypothesis 7.8.1 as a theorem, whose proof is provided by the authors in [22]. In our article [24], we discussed analytical issues specific to the problem of gluing $\mathrm{SO}(3)$ monopoles that are not present when gluing anti-self-dual $\mathrm{SO}(3)$ connections and we refer the reader to that discussion. Rather, our goal in this section is to describe modifications to our approach in [24] to gluing $\mathrm{SO}(3)$ monopoles. In [24], we had restricted our attention to the case of gluing anti-self-dual connections over S^4 and $\mathrm{SO}(3)$ monopoles over X that varied in an open neighborhood with the property that spectral flow for the Laplace operator, $d_{A, \Phi}^1 d_{A, \Phi}^{1, *}$, is small. This assumption is valid, for example, when considering small open neighborhoods of a *zero-dimensional* moduli space of Seiberg–Witten monopoles (equivalently, strata of reducible $\mathrm{SO}(3)$ monopoles), since those moduli spaces comprise finite sets

of isolated points (because those moduli spaces can be assumed to be compact and smooth). However, in this monograph and our previous articles such as [28] (where no bubbling is allowed) or [30] (where one bubble allowed), we must allow for moduli spaces of Seiberg–Witten moduli spaces that are *positive*-dimensional. In those cases, we cannot assume that spectral flow for $d_{A,\Phi}^1 d_{A,\Phi}^{1,*}$ is small and thus our method of gluing [24] does not apply without significant modification. Indeed, the rank of the obstruction vector bundle constructed in [24], by an analogue of the small-eigenvalue decomposition that Taubes developed in [95], necessarily jumps as eigenvalues of $d_{A,\Phi}^1 d_{A,\Phi}^{1,*}$ cross the small-eigenvalue cut-off parameter $\mu \in (0, 1]$ used to define the L^2 -orthogonal projections required to split the $\mathrm{SO}(3)$ -monopole equations, following the Kuranishi method [55].

In Section 7.9.1, we recall the main ideas from [28, Section 3] (see also Section 2.3.5 in this monograph) involved in our construction of a virtual neighborhood for the moduli space of $\mathrm{SO}(3)$ monopoles near a possibly singular¹ stratum of $\mathrm{SO}(3)$ monopoles² in the top level. This construction relies on a splitting of the $\mathrm{SO}(3)$ -monopole equations defined by a choice of abstract stabilizing bundle, Ξ , over an open neighborhood, \mathcal{U} , in the configuration space of pairs. When the stratum is in the top level, there is no bubbling and thus no need to glue in anti-self-dual connections over copies of S^4 . In Section 7.9.2, we describe two methods of constructing a virtual neighborhood for the moduli space of anti-self-dual connections near a possibly singular stratum of anti-self-dual connections³ in the top lower-level; we restrict our attention to the case of anti-self-dual connections so that we may focus on the essential ideas in a familiar and relatively simple case and relate our constructions to those of Donaldson and Kronheimer [15], an accessible reference. When the stratum is in a lower level, one has bubbling and thus one must glue in anti-self-dual connections over copies of S^4 in order to construct a virtual neighborhood; therefore in Section 7.9.3, we outline how to extend the construction of Section 7.9.2.2 to the case of gluing two families of anti-self-dual connections over a pair of four-manifolds, X_1 and X_2 , to produce new anti-self-dual connections over a connected sum, $X = X_1 \# X_2$. Finally, in Section 7.9.4, we very briefly outline the changes needed to extend our discussion of gluing anti-self-dual $\mathrm{SO}(3)$ connections to gluing $\mathrm{SO}(3)$ monopoles and thus construct a virtual neighborhood for the moduli space of $\mathrm{SO}(3)$ monopoles near a possibly singular stratum of $\mathrm{SO}(3)$ monopoles in a lower-level and complete the proof of Hypothesis 7.8.1 as a theorem.

7.9.1. Construction of a virtual neighborhood for the moduli space of $\mathrm{SO}(3)$ monopoles near a top-level singular stratum of $\mathrm{SO}(3)$ monopoles. We begin by recalling the basic outline, in the absence of bubbling, from our article [28, Section 3] for how to construct a virtual neighborhood for the moduli space of $\mathrm{SO}(3)$ monopoles near a top-level singular stratum of $\mathrm{SO}(3)$ monopoles. The ‘gluing map’ and ‘obstruction section’ are defined in terms of a Kuranishi model that describes how the Seiberg–Witten moduli space, M_5 , is contained in the top level, \mathcal{M}_t , of the Uhlenbeck compactification, $\bar{\mathcal{M}}_t$, of the moduli space of $\mathrm{SO}(3)$ monopoles.

We can assume that M_5 contains no zero-section pairs by [28, Corollary 3.3]. The Banach Lie group, \mathcal{G}_t , acts freely on the open subspace, $\mathcal{C}_t^0 \subset \mathcal{C}_t$, of non-zero-section pairs and the quotient, $\tilde{\mathcal{C}}_t^0 = \mathcal{C}_t^0 / \mathcal{G}_t$, is a smooth Banach manifold by [27, Proposition 2.8]. We have a smooth Hilbert

¹In the sense of Kodaira–Spencer deformation theory, rather than in the sense of bubbling or elliptic regularity theory.

²Reducible $\mathrm{SO}(3)$ monopoles or Seiberg–Witten monopoles in our application.

³Reducible anti-self-dual connections in the context of the proof of the Kotschick–Morgan conjecture [21, 52]

vector bundle, $\mathfrak{V} = \tilde{\mathcal{C}}_t^0 \times_{\mathfrak{g}_t} L^2(X; \Lambda^+ \otimes \mathfrak{g}_t \oplus W^- \otimes E)$ over \mathcal{C}_t^0 , as in [28, Equation (3.39)], where $t = (\rho, W \otimes E)$. We obtain a smooth splitting of Hilbert vector bundles, $\mathfrak{V} \upharpoonright \iota(\mathcal{C}_s^0) = \mathfrak{V}^t \oplus \mathfrak{V}^n$ for $\iota(\mathcal{C}_s) \subset \mathcal{C}_t$, following [28, Equation (3.40)], where $\mathfrak{s} = (\rho, W)$. We let $\mathcal{U} \subset \mathcal{C}_t^0$ denote an open subset that contains a compact subset of $\iota(\mathcal{C}_s^0)$. By [28, Theorem 3.19], there is a finite-rank, smooth, product vector subbundle, $\Xi \subset \mathfrak{V}$, over \mathcal{U} with the following properties:

- Ξ is S^1 equivariant with respect to the circle action specified in [28, Equation (3.2)] (implied by a splitting $E = \underline{\mathbb{C}} \oplus L$ and $W \otimes E = W \oplus W \otimes L$, with \mathbb{C}^* acting trivially on $\underline{\mathbb{C}} = X \times \mathbb{C}$ and by scalar multiplication on L) and given in this monograph by (2.3.15);
- $\Xi \upharpoonright \iota(\mathcal{C}_s^0)$ is a smooth complex vector bundle;
- The $\text{SO}(3)$ -monopole equations (2.1.10) (compare [28, Equation (2.32)]) define a smooth section, \mathfrak{S} , of $\mathfrak{V} \rightarrow \mathcal{C}_t^0$;
- If $\Pi_{\Xi} : \mathfrak{V} \rightarrow \Xi$ is the smooth map of vector bundles defined by L^2 -orthogonal projection and $\mathfrak{V} \upharpoonright \mathcal{U} = \Xi \oplus \Xi^{\perp}$ and $\Pi_{\Xi^{\perp}} := \text{id}_{\mathfrak{V}} - \Pi_{\Xi}$, then

$$\Pi_{\Xi^{\perp}} : \text{Ran}(D\mathfrak{S})_{A,\Phi} \rightarrow \Xi_{A,\Phi}^{\perp}$$

is a surjective map of fibers, for each $[A, \Phi] \in \mathcal{U}$, where

$$\mathfrak{V}_{A,\Phi} \subset L^2(X; \Lambda^+ \otimes \mathfrak{g}_t \oplus W^- \otimes E).$$

The $\text{SO}(3)$ -monopole section, \mathfrak{S} , of $\mathfrak{V} \rightarrow \mathcal{C}_t^0$ induces smooth sections,

$$\Pi_{\Xi} \circ \mathfrak{S} \quad \text{of} \quad \Xi \rightarrow \mathcal{U},$$

$$\Pi_{\Xi^{\perp}} \circ \mathfrak{S} \quad \text{of} \quad \Xi^{\perp} \rightarrow \mathcal{U},$$

with $\Pi_{\Xi^{\perp}} \circ \mathfrak{S}$ defining the virtual moduli space (denoted by $\mathcal{M}_t(\Xi, \mathfrak{s})$ in [28, Equation (3.41)]),

$$\mathcal{M}_{t,\mathfrak{s}}^{\text{vir}} = (\Pi_{\Xi^{\perp}} \circ \mathfrak{S})^{-1}(0) \subset \mathcal{C}_t^0.$$

Furthermore, our [28, Theorem 3.21] provides that:

- The Seiberg–Witten moduli space, $M_{\mathfrak{s}}$, is an S^1 -invariant, smooth submanifold of $\mathcal{M}_{t,\mathfrak{s}}^{\text{vir}}$ via the smooth embedding $\iota : \mathcal{C}_s^0 \rightarrow \mathcal{C}_t^0$;
- The restriction of \mathfrak{S} to $\mathcal{M}_{t,\mathfrak{s}}^{\text{vir}}$ takes values in Ξ and vanishes transversely on $\mathcal{M}_{t,\mathfrak{s}}^{\text{vir}} - \iota(M_{\mathfrak{s}})$;
- The smooth vector bundle, $N_{t,\mathfrak{s}} \rightarrow M_{\mathfrak{s}}$, given by [28, Equation (3.42)] (where it is denoted by $N_t(\Xi, \mathfrak{s})$) and in (2.3.19) in this monograph, and constructed from the deformation complex for \mathfrak{S} is the normal bundle for the submanifold, $\iota : M_{\mathfrak{s}} \hookrightarrow \mathcal{M}_{t,\mathfrak{s}}^{\text{vir}}$; the tubular map is equivariant with respect to the circle action on $N_{t,\mathfrak{s}}$ given by the trivial action on the base $M_{\mathfrak{s}}$ and complex multiplication on the fibers, and the circle action on $\mathcal{M}_{t,\mathfrak{s}}^{\text{vir}}$ induced from the S^1 action in [28, Equation (3.2)].

In the next subsection, we shall further explore certain aspects of the construction of virtual neighborhoods in the simpler setting of the moduli space of anti-self-dual connections.

7.9.2. Virtual neighborhoods for the moduli space of anti-self-dual connections.

There are essentially two ways of splitting the non-linear anti-self-dual or Yang–Mills equations in the presence of cokernel obstructions in their linearizations:

- (1) Taubes' intrinsic splitting for the Yang–Mills equations [96, Equation (8.4)] (compare [95, Equations (2.7) and (2.8)] for the anti-self-dual equation), and

- (2) Donaldson–Kronheimer’s extrinsic splitting for the anti-self-dual equation [15, Equations (7.26) or (7.27)].

These two approaches are inspired by the method employed by Kuranishi [55] in the deformation of complex structures and adapted by Atiyah, Hitchin, and Singer [2] to the anti-self-dual equation, but the technical differences between them are significant. In this subsection, we describe these methods in their original context for the anti-self-dual equation but they formally extend, with minor changes, to other non-linear equations arising in gauge theory, such as the Yang–Mills, SO(3)-monopole, and pseudo-holomorphic curve equations. Indeed, in Section 7.9.4, we outline such an extension for the SO(3)-monopole equations, as required by our application in this monograph.

An earlier variant of Method (1) was used by Taubes in [95] for his construction of solutions to the anti-self-dual equation and generalizations of his idea are described, for example, by the authors in [21, 24]. Method (1), proposed by Taubes [96] in the context of the Yang–Mills equations, was adapted to the anti-self-dual equation by Friedman and Morgan in [35, Section 3.4.6]. Method (2) is more abstract and, unlike Method (1), does not rely on an eigenvalue splitting for a Laplace operator depending on a connection, A , and thus is more suitable when spectral flow occurs as the connection A varies in a large neighborhood. In this subsection, we shall review these methods in their original setting of the anti-self-dual equation.

7.9.2.1. *Taubes’ intrinsic splitting.* Let G be a compact Lie group and P be a smooth principal G -bundle over a closed, oriented, smooth Riemannian four-manifold (X, g) and $\tilde{\mathcal{U}} \subset \mathcal{A}(P)$ be an open neighborhood in the pre-configuration space of all $W^{1,p}$ connections⁴ A on P (with $p \in (2, \infty)$). For Method (1), given a constant $\mu \in (0, 1]$, one aims to solve the following pair of equations,

$$(7.9.1a) \quad \Pi_\mu^\perp(A)F^+(A) = 0 \quad (\text{for } A \text{ in } \mathcal{U}),$$

$$(7.9.1b) \quad \Pi_\mu(A)F^+(A) = 0 \quad (\text{for } A \text{ in } \mathcal{U} \text{ and solving (7.9.1a)}),$$

where $\Pi_\mu(A) \in \mathcal{L}(W^{2,p}(X; \Lambda^+ \otimes \text{ad}P))$ is the finite-rank L^2 -orthogonal projection from $W^{2,p}(X; \Lambda^+ \otimes \text{ad}P)$ onto the subspace spanned by the eigenvectors of the unbounded operator,

$$d_A^+ d_A^{+,*} : L^2(X; \Lambda^+ \otimes \text{ad}P) \rightarrow L^2(X; \Lambda^+ \otimes \text{ad}P),$$

with eigenvalues less than $\mu/2$, while $\Pi_\mu^\perp(A) := 1 - \Pi_\mu(A)$; we choose μ so that no eigenvalue of $d_A^+ d_A^{+,*}$ lies in $[\mu/2, \mu]$. In applications of Method (1) and its variant in [95], such as in Feehan and Leness [21], the point $[A]$ varies in an open neighborhood $\mathcal{U} = \tilde{\mathcal{U}} / \text{Aut}(P)$ in the configuration space, $\mathcal{B}(P) := \mathcal{A}(P) / \text{Aut}(P)$, of all $W^{1,p}$ connections modulo the action of the group of $W^{2,p}$ gauge transformations, $\text{Aut}(P)$. We assume that each $[A] \in \mathcal{U}$ obeys $F^+(A) < \varepsilon$, for a small $\varepsilon \in (0, 1]$. We also assume that \mathcal{U} is small enough that the eigenvalues of $d_A^+ d_A^{+,*}$ do not cross μ — in other words, small enough that there is no spectral flow as $[A]$ varies in \mathcal{U} . The infinite-dimensional equation (7.9.1a) is often called the *extended anti-self-dual equation* and the finite-dimensional equation (7.9.1b) is often called the *balancing equation*.

7.9.2.2. *Donaldson–Kronheimer’s extrinsic splitting.* This method can be viewed as a more flexible version of Method (1) and is especially appropriate when the open $\text{Aut}(P)$ -invariant neighborhood, $\mathcal{U} \subset \mathcal{B}(P)$, cannot be assumed to be small and one must contend with spectral flow. If Ξ denotes both a finite-rank, product vector bundle over \mathcal{U} and (slightly abusing notation) its

⁴By analogy with the notation of Adams and Fournier [1], we define the $W^{k,p}$ Sobolev norm of $a \in C^\infty(X; \Lambda^l \otimes \text{ad}P)$ by requiring that $\nabla A_0^j a \in L^p(X; \Lambda^l \otimes \text{ad}P)$ for $j = 0, 1, \dots, k$, given a smooth reference connection on P , integers $k, l \geq 0$, and constant $p \in [1, \infty]$.

pullback to $\tilde{\mathcal{U}}$ (by the quotient map, $\pi : \mathcal{A}(P) \rightarrow \mathcal{B}(P)$), we let $\Pi_{\Xi}(A) : L^2(X; \Lambda^+ \otimes \text{ad}P) \rightarrow \Xi(A)$ and $\Pi_{\Xi}^{\perp}(A) := 1 - \Pi_{\Xi}(A) = \Pi_{\Xi^{\perp}}(A) : L^2(X; \Lambda^+ \otimes \text{ad}P) \rightarrow \Xi^{\perp}(A)$ denote the L^2 -orthogonal projection operators for $A \in \tilde{\mathcal{U}}$, where $L^2(X; \Lambda^+ \otimes \text{ad}P) = \Xi(A) \oplus \Xi^{\perp}(A)$. As we discuss in the following paragraph, one can construct the bundle Ξ so that the operators,

$$(7.9.2) \quad \Pi_{\Xi}^{\perp}(A) : \text{Ran } d_A^+ \rightarrow \Xi^{\perp}(A),$$

are surjective for all $[A] \in \mathcal{U}$.

For a small enough neighborhood \mathcal{U} , spectral flow is limited and the bundle Ξ can be constructed by decomposing the spectrum of $d_A^+ d_A^{+,*}$ into a set of small non-negative eigenvalues bounded above by $\mu/2$ and a set of eigenvalues bounded below by a uniform positive constant μ , just as in Method (1). One then defines $\Xi(A) := \Xi_{\mu}(A) \subset L^2(X; \Lambda^+ \otimes \text{ad}P)$, the subspace spanned by the eigenvectors of $d_A^+ d_A^{+,*}$ with eigenvalues less than $\mu/2$. For a large open neighborhood, \mathcal{U} , one can instead employ the method described by the authors in [28, Section 3] to construct Ξ in the context of the SO(3)-monopole equations (2.1.10).

Regardless of how the bundle Ξ is constructed, the smooth map,

$$\tilde{\mathcal{U}} \ni A \mapsto \Pi_{\Xi}^{\perp} F^+(A) := \Pi_{\Xi}^{\perp}(A) F^+(A) \in \Pi_{\Xi}^{\perp}(A) L^2(X; \Lambda^+ \otimes \text{ad}P),$$

defines a smooth section of the smooth vector bundle, $\Xi^{\perp} \rightarrow \mathcal{U}$. Moreover, if $F^+(A) = 0$, then

$$D(\Pi_{\Xi}^{\perp} F^+)(A)a = \Pi_{\Xi}^{\perp}(A) d_A^+ a, \quad \forall a \in W^{1,2}(X; \Lambda^1 \otimes \text{ad}P),$$

is the derivative of the smooth map, $\Pi_{\Xi}^{\perp} F^+$, at A in the direction a , noting that $d_A^+ a$ is the derivative of the smooth map, $A \mapsto F^+(A)$, at any connection A in the direction a . In particular, the operators

$$D(\Pi_{\Xi}^{\perp} F^+)(A) \in \mathcal{L} \left(W^{1,2}(X; \Lambda^1 \otimes \text{ad}P), \Pi_{\Xi}^{\perp}(A) \right)$$

are surjective⁵ for any $[A] \in \mathcal{U}$ with $F^+(A) = 0$. Hence, provided $\mathcal{U} \cap (F^+)^{-1}(0) \neq \emptyset$, we can arrange that the operators $D(\Pi_{\Xi}^{\perp} F^+)(A)$ are surjective for any $[A] \in \mathcal{U}$, by shrinking \mathcal{U} if necessary, and the set of solutions A to the extended anti-self-dual equation,

$$(7.9.3) \quad \Pi_{\Xi}^{\perp}(A) F^+(A) = 0 \quad (\text{for } A \in \tilde{\mathcal{U}}),$$

is a smooth submanifold of $\tilde{\mathcal{U}}$ by the Implicit Function Theorem. The quotient,

$$M(P, g; \Xi) := (\Pi_{\Xi}^{\perp} F^+)^{-1}(0) / \text{Aut}(P),$$

is a finite-dimensional smooth submanifold of \mathcal{U} away from points $[A]$ where the stabilizer of A in $\text{Aut}(P)$ is larger than the center of G .

When we solve the balancing equation,

$$(7.9.4) \quad \Pi_{\Xi}(A) F^+(A) = 0 \quad (\text{for } A \in \tilde{\mathcal{U}} \text{ solving (7.9.3)}),$$

to produce

$$\mathcal{U} \cap M(P, g) = M(P, g; \Xi) \cap (\Pi_{\Xi} F^+)^{-1}(0),$$

we recover an open neighborhood in the moduli space $M(P, g)$ of anti-self-dual connections on P .

⁵And thus admit right inverses, a construction favored by Donaldson and Kronheimer in [15, Section 7.2.2] as a way to convert the first-order anti-self-dual partial differential equation into a zeroth-order nonlinear integral equation that may be solved using the Quantitative Inverse Function Theorem.

Given an anti-self-dual ‘basepoint’ connection, A^b on P with $F^+(A^b) = 0$, and constant $\varepsilon \in (0, 1]$, we define an open ball,

$$B_{A^b}(\varepsilon) := \left\{ a \in W_{A^b}^{1,p}(X; \Lambda^1 \otimes \text{ad}P) : \|a\|_{W_{A^b}^{1,p}(X)} < \varepsilon \right\}.$$

The Implicit Function Theorem yields a $\text{Stab}(A^b)$ -equivariant embedding,

$$(7.9.5) \quad \gamma : B_{A^b}(\varepsilon) \cap \text{Ker} \left(d_{A^b}^* + \Pi_{\Xi}^\perp(A^b) d_{A^b}^+ \right) \ni a \mapsto \gamma(a) := A^b + a + \wp(a) \in \mathcal{A}(P),$$

where $\wp(a)$ is defined by solving (7.9.3), that is,

$$\Pi_{\Xi}^\perp(A^b + a + \wp(a)) F^+(A^b + a + \wp(a)) = 0.$$

Here, γ is the analogue of a gluing map in [35, 95] and

$$(7.9.6) \quad \chi = \Pi_{\Xi} F^+ : B_{A^b}(\varepsilon) \cap \text{Ker} \left(d_{A^b}^* + \Pi_{\Xi}^\perp(A^b) d_{A^b}^+ \right) \rightarrow \Pi_{\Xi}^\perp(A^b) L^2(X; \Lambda^+ \otimes \text{ad}P)$$

is the analogue of an $\text{Stab}(A^b)$ -equivariant obstruction map in [35, 95]. For small enough $\varepsilon \in (0, 1]$, one may replace $\Pi_{\Xi}^\perp(A^b) d_{A^b}^+$ by $d_{A^b}^+$ in the definitions of γ in (7.9.5) and χ in (7.9.6) and recover the standard Kuranishi model [15, Proposition 4.2.23] for an open neighborhood in $M(P, g)$ of a single point $[A^b]$. However, the advantage of the model (7.9.5), (7.9.6) is that it also adapts to the case where $[A^b]$ varies in a smooth stratum \mathcal{S} of $M(P, g)$ and where there would be spectral flow for the Laplace operator, $d_{A^b}^+ d_{A^b}^{+,*}$, so the bundle $\text{Ran} \Pi_{\mu}(A^b)$ would not have constant rank along \mathcal{S} for any choice of $\mu \in (0, 1]$, whereas — as we explain in [28, Section 3] — we may choose Ξ so that it has constant rank along \mathcal{S} and (7.9.2) holds. The ball $B_{A^b}(\varepsilon)$ would then be replaced by a relatively open tubular neighborhood, $N_{\mathcal{S}}(\varepsilon)$, of a closed smooth manifold, \mathcal{S} .

7.9.3. Extrinsic virtual neighborhoods for the moduli space of anti-self-dual connections and gluing. We shall omit discussions of the more technical analytic aspects of gluing anti-self-dual connections or $\text{SO}(3)$ monopoles described in [21, 24], complex structures, \mathbb{C}^* or S^1 or other group actions, or group equivariances, noting that our gluing constructions are natural and that all group actions and equivariances extend from the setting of [28] (no bubbling) to those considered here in the presence of bubbling.

We continue the notation and setup of Section 7.9.2.2 and first consider the problem of splicing a pair of families of connections, $A_i \in \tilde{\mathcal{U}}_i$, on smooth principal G -bundles P_i over closed, oriented, smooth Riemannian four-manifolds, (X_i, g_i) for $i = 1, 2$, that are connected by a small cylinder (or neck or tube) diffeomorphic to $S^3 \times (-1, 1)$ and whose neck size is defined by a small neck parameter, $\lambda \in (0, 1]$. Our development is inspired by that of Donaldson and Kronheimer [15, Section 7.2].

The open $\text{Aut}(P)$ -invariant neighborhoods, $\tilde{\mathcal{U}}_i \subset \mathcal{A}(P_i)$, project under the action of $\text{Aut}(P_i)$ onto open neighborhoods \mathcal{U}_i in the configuration spaces $\mathcal{B}(P_i) := \mathcal{A}(P_i) / \text{Aut}(P_i)$. We assume that each $[A_i] \in \mathcal{U}_i$ obeys $\|F^+(A_i)\|_{L^2(X_i)} < \varepsilon_i$, for suitably small $\varepsilon_i \in (0, 1]$. We let Ξ_i denote finite-rank, product vector bundles over \mathcal{U}_i and their pullbacks to $\tilde{\mathcal{U}}_i$ and let $\Pi_{\Xi_i}(A_i) : L^2(X_i; \Lambda^+ \otimes \text{ad}P_i) \rightarrow \Xi_i(A_i)$ and $\Pi_{\Xi_i}^\perp(A_i) := 1 - \Pi_{\Xi_i}(A_i) = \Pi_{\Xi_i^\perp}(A_i)$ denote L^2 -orthogonal projections for $i = 1, 2$, where $L^2(X_i; \Lambda^+ \otimes \text{ad}P_i) = \Xi_i(A_i) \oplus \Xi_i^\perp(A_i)$. As we explained in Section 7.9.2.2, one can construct the Ξ_i so that the operators,

$$(7.9.7) \quad \Pi_{\Xi_i}^\perp(A_i) : \text{Ran} d_{A_i}^+ \rightarrow \Xi_i^\perp(A_i),$$

are surjective for all $[A_i] \in \mathcal{U}_i$ and $i = 1, 2$. The smooth maps,

$$\tilde{\mathcal{U}}_i \ni A_i \mapsto \Pi_{\Xi_i}^\perp(A_i)F^+(A_i) \in \Pi_{\Xi_i}^\perp(A_i)L^2(X_i; \Lambda^+ \otimes \text{ad}P_i),$$

define smooth sections of the smooth vector bundles, $\Xi_i^\perp \rightarrow \mathcal{U}_i$. We can arrange that the operators $D(\Pi_{\Xi_i}^\perp F^+)(A_i)$ are surjective for any $[A_i] \in \mathcal{U}_i$, by shrinking \mathcal{U}_i if necessary, and the zero locus,

$$(\Pi_{\Xi_i}^\perp F^+)^{-1}(0) = \left\{ A_i \in \mathcal{U}_i : \Pi_{\Xi_i}^\perp F^+(A_i) = 0 \right\},$$

is a finite-dimensional, smooth submanifold of \mathcal{U}_i .

If $X := X_1 \# X_2$ denotes the connected sum of X_1 and X_2 along a small cylinder, then we may use splicing to construct approximate solutions, $A' := A_1 \# A_2$, of the extended anti-self-dual equation (7.9.3) on the smooth principal G -bundle $P = P_1 \# P_2$ over the closed, oriented, smooth Riemannian four-manifold (X, g) , where g coincides with the g_i outside the neck region, $\Xi := \Xi_1 \oplus \Xi_2$ is the induced finite-rank product vector bundle over X , the operators

$$\Pi_{\Xi}^\perp(A) : \text{Ran } d_A^+ \rightarrow \Xi^\perp(A)$$

are surjective (by construction) for all $[A] \in \mathcal{U}$, where $L^2(X; \Lambda^+ \otimes \text{ad}P) = \Xi(A) \oplus \Xi^\perp(A)$ and $\Pi_{\Xi}^\perp(A) := 1 - \Pi_{\Xi}(A) = \Pi_{\Xi^\perp}(A)$, and $\mathcal{U} \subset \mathcal{B}(P)$ is an open neighborhood containing $\mathcal{U}_1 \# \mathcal{U}_2$, the set of all gauge-equivalence classes of the spliced connections, $A_1 \# A_2$, and $\tilde{\mathcal{U}} \subset \mathcal{A}(P)$ is its $\text{Aut}(P)$ -invariant pullback. We now apply the (Quantitative) Implicit Function Theorem to construct solutions $a := \wp(A') \in W_{A'}^{1,p}(X; \Lambda^1 \otimes \text{ad}P)$ (for small enough parameters $\varepsilon_1, \varepsilon_2, \lambda$) to the extended anti-self-dual equation (7.9.3), namely

$$\Pi_{\Xi}^\perp F^+(A' + a) = 0 \quad (\text{for } A' \in \tilde{\mathcal{U}}).$$

Just as in the case of the component sections, $\Pi_{\Xi_i}^\perp F^+$, the zero locus,

$$M(P, g; \Xi) := \left\{ [A] \in \mathcal{U} : \Pi_{\Xi}^\perp F^+(A) = 0 \right\},$$

is a finite-dimensional, smooth submanifold of \mathcal{U} . By solving the balancing equation (7.9.4), namely

$$\Pi_{\Xi} F^+(A) = 0 \quad (\text{for } A \in \tilde{\mathcal{U}} \text{ solving (7.9.3)}),$$

to produce

$$\mathcal{U} \cap M(P, g) = M(P, g; \Xi) \cap (\Pi_{\Xi} F^+)^{-1}(0),$$

we recover an open neighborhood in the moduli space $M(P, g)$ of anti-self-dual connections on P .

Suppose now that we are given a pair of anti-self-dual basepoint connections, A_i^b on P_i with $F^+(A_i^b) = 0$, and constants $\varepsilon_i \in (0, 1]$ for $i = 1, 2$. For small enough $\varepsilon_1, \varepsilon_2, \lambda$, the Quantitative Implicit Function Theorem yields a $\text{Stab}(A_1^b) \times \text{Stab}(A_2^b)$ -equivariant embedding,

$$(7.9.8) \quad \gamma : B_{A_1^b}(\varepsilon_1) \cap \text{Ker} \left(d_{A_1^b}^* + \Pi_{\Xi_1}^\perp(A_1^b)d_{A_1^b}^+ \right) \times B_{A_2^b}(\varepsilon_2) \cap \text{Ker} \left(d_{A_2^b}^* + \Pi_{\Xi_2}^\perp(A_2^b)d_{A_2^b}^+ \right) \\ \times \text{Isom}_G(P_{x_1}, P_{x_2}) \ni (a_1, a_2, h) \mapsto A' + a + \wp(a) \in \mathcal{A}(P),$$

where $\wp(A')$ is defined by solving (7.9.3), that is,

$$\Pi_{\Xi}^\perp F^+(A' + \wp(A')) = 0,$$

and the approximately anti-self-dual connection, A' on P , is defined by splicing $A_i^b + a_i$ for $i = 1, 2$. Here, γ is the gluing map and

$$(7.9.9) \quad \begin{aligned} \chi : B_{A_1^b}(\varepsilon_1) \cap \text{Ker} \left(d_{A_1^b}^* + \Pi_{\Xi_1}^\perp(A_1^b)d_{A_1^b}^+ \right) \times B_{A_2^b}(\varepsilon_2) \cap \text{Ker} \left(d_{A_2^b}^* + \Pi_{\Xi_2}^\perp(A_2^b)d_{A_2^b}^+ \right) \\ \times \text{Isom}_G(P_{x_1}, P_{x_2}) \ni (a_1, a_2, h) \\ \mapsto \Pi_{\Xi} F^+(A' + \wp(A')) \in \Pi_{\Xi}^\perp(A' + \wp(A')) L^2(X; \Lambda^+ \otimes \text{ad}P) \end{aligned}$$

is the $\text{Stab}(A_1^b) \times \text{Stab}(A_2^b)$ -equivariant obstruction map. The existence and properties of γ in (7.9.8) and χ in (7.9.9) are established by Donaldson and Kronheimer in [15, Theorems 7.2.62 and 7.2.63]. The gluing map γ in (7.9.8) descends to the quotient by $\text{Stab}(A_1^b) \times \text{Stab}(A_2^b)$ to give a continuous embedding onto a relatively open subset of $\mathcal{B}(P)$ (respectively, smooth embedding away from singularities) and restricts to a continuous embedding from $\chi^{-1}(0)$ onto an open subset of $M(P, g)$ (respectively, smooth embedding away from singularities).

In the case where $[A_i^b]$ varies in smooth strata \mathcal{S}_i of $M(P_i, g)$ and where there would be spectral flow for the Laplace operators, $d_{A_i^b}^+, d_{A_i^b}^{+,*}$, so the bundles $\text{Ran} \Pi_{\mu_i}(A_i^b)$ would not have constant rank along \mathcal{S}_i for any choice of $\mu_i \in (0, 1]$, we may choose $\Xi \cong \Xi_1 \oplus \Xi_2$ so that it has constant rank along $\mathcal{S}_1 \times \mathcal{S}_2$ and (7.9.2) holds. The balls $B_{A_i^b}(\varepsilon)$ would then be replaced by a relatively open tubular neighborhoods, $N_{\mathcal{S}_i}(\varepsilon)$, of a closed smooth manifolds \mathcal{S}_i .

In applications to the proof of the Kotschick–Morgan Conjecture in this monograph, the above-mentioned strata, \mathcal{S} or $\mathcal{S}_1, \mathcal{S}_2$, in this context comprise isolated points corresponding to reducible anti-self-dual connections. However, in our application of these ideas to the proof of the Witten Conjecture, we must allow for positive-dimensional strata of reducible $\text{SO}(3)$ monopoles, that is, positive-dimensional moduli spaces of Seiberg–Witten monopoles.

When the neck parameter, λ , tends to zero, the pair (γ, χ) converges continuously to $(\gamma_1 \times \gamma_2, \chi_1 \times \chi_2)$. In typical applications (for example, see Donaldson and Kronheimer [15, Theorem 8.2.3 and Proposition 8.2.4] or our application to the proof of the Kotschick–Morgan Conjecture), we choose (X_2, g_2) to be (S^4, g_{round}) , the four-sphere equipped with its standard round metric, g_{round} , of radius one. The gluing theory summarized in this subsection yields an open neighborhood of a boundary point $([A_1, q_1], [A_2, q_2])$ in the *bubble-tree compactification* of $M(P_1, g_1)$, where $q_1 \in P_1|_{x_1}$ and $q_2 \in P_2|_s$ and the point $x_1 \in X_1$ is identified with the south pole $s \in S^4$. By ‘forgetting’ the additional data on S^4 (aside from $c_2(P_2)$ when $G = \text{SU}(2)$, which defines the multiplicity of the point x_1) and the frame $q_1 \in P_1|_{x_1}$ when the neck parameter λ becomes zero, the gluing theory yields an open neighborhood of a boundary point $([A_1], x_1)$ in the *Uhlenbeck compactification* of $M(P, g)$.

Lastly, the gluing theory for solutions to the anti-self-dual equation on the principal G -bundle $P = P_1 \# P_2$ over the connected sum $X = X_1 \# X_2$ summarized in this subsection extends to the case of *multiple connected sums* (see Donaldson and Kronheimer [15, Chapter 8] and Feehan and Leness [24]) and *tree connected sums* (see Feehan [17], Peng [80, 81], and Taubes [96]) required when building parameterizations for open neighborhoods of points in the boundary of the bubble-tree (and hence Uhlenbeck) compactification of $M(P_1, g_1)$.

7.9.4. Construction of a virtual neighborhood for the moduli space of $\text{SO}(3)$ monopoles near a lower-level singular stratum of $\text{SO}(3)$ monopoles. The gluing theory described in Section 7.9.3 for solutions to the anti-self-dual equation generalizes *formally* to the case of the

SO(3)-monopole equations (2.1.10). However, the analysis required to prove that the gluing map $\gamma_{\mathcal{M}}$ and obstruction map $\bar{\chi}$ have all the properties asserted by Hypothesis 7.8.1 is considerable and the details of that analysis do *not* extend in a straightforward manner from those previously encountered in the case of the anti-self-dual equation (for example, Donaldson and Kronheimer [15], Feehan [17], Feehan and Leness [21], Morgan and Mrowka [72], Mrowka [74], and Taubes [94, 95, 96]). We summarized the principal new analytical difficulties in our article [24] and address them fully in [22].

In this subsection, we outline a method for extending the construction of the stabilizing bundle Ξ for $\text{Ran } d_A^+$ to give a stabilizing bundle Ξ for $\text{Ran } d_{A,\Phi}^1$ and hence define a splitting of the SO(3)-monopole equations (2.1.10). If $(X, g) = (S^4, g_{\text{round}})$ and P is a principal SU(2)-bundle over S^4 and (ρ, W) is the standard spin^c structure over S^4 , we recall from [27, Remark 4.6] that if (A, Φ) is an SO(3) monopole on (P, W^+) , then $\Phi \equiv 0$ and $F_A^+ = 0$. The linearization, $d_{A,0}^1 = (D\mathfrak{G})(A, 0)$, of the SO(3)-monopole equations, $\mathfrak{G}(A, \Phi) = 0$, in (2.1.10) at an SO(3) monopole $(A, 0)$ over S^4 is given by [28, Equation (2.49)], namely,

$$(D\mathfrak{G})(A, 0) = (d_A^+, D_A),$$

where $D_A : C^\infty(S^4, W^+ \otimes E) \rightarrow C^\infty(S^4, W^- \otimes E)$ is the coupled Dirac operator (with L^2 -adjoint $D_A^* : C^\infty(S^4, W^- \otimes E) \rightarrow C^\infty(S^4, W^+ \otimes E)$) and, as usual, $d_A^+ : C^\infty(S^4; \Lambda^1 \otimes \text{ad}P) \rightarrow C^\infty(S^4; \Lambda^+ \otimes \text{ad}P)$ is the linearization of the anti-self-dual equation, $F_A^+ = 0$, at an anti-self-dual connection, A , and $E = P \times_{\text{std}} \mathbb{C}^2$. According to [24, Lemma 8.12], when $F_A^+ = 0$, the kernel of $D_A^* D_A$ (and thus D_A) is zero, the real dimension of the kernel of $D_A D_A^*$ (and thus D_A^*) is equal to twice the complex index of D_A^* , namely $\text{Index}_{\mathbb{C}} D_A^* = \dim_{\mathbb{C}} \text{Ker } D_A^* - \dim_{\mathbb{C}} \text{Ker } D_A = c_2(E)$ by [27, Proof of Proposition 2.28], and the least positive eigenvalue of the Dirac Laplacian, $D_A D_A^*$, is equal to 3. Hence, as A varies over $\tilde{M}(P, g_{\text{round}})$, the kernels, $\text{Ker } D_A^*$ (equivalently, the cokernels, $\text{Coker } D_A$), define a complex vector bundle, $\text{Ker } \mathbf{D}^* \cong \text{Coker } \mathbf{D}$, of complex rank $c_2(E)$ over $M(P, g_{\text{round}})$. As usual, $\text{Coker } d_A^+ = \{0\}$ when A is an anti-self-dual connection over S^4 by [2, Proof of Theorem 6.1] and thus $\text{Coker } d_{A,0}^1 \cong \text{Coker } D_A$.

We may extend the basic setting of Section 7.9.3 to the case of SO(3) monopoles and consider splicing a family of SO(3) monopoles, (A_1, Φ_1) , on (P_1, W_1^+) over $(X_1, g_1) = (X, g)$, and a family of zero-section SO(3) monopoles, $(A_2, 0)$, on (P_2, W_2^+) over $(X_2, g_2) = (S^4, g_{\text{round}})$ to form a family of approximate SO(3) monopoles, $(A', \Phi') = (A_1 \# A_2, \Phi_1 \# 0)$, on $(P, W^+) = (P_1 \# P_2, W_1^+ \# W_2^+)$ over the Riemannian connected sum, $(X \# S^4, g \# g_{\text{round}})$. We may choose the stabilizing bundle Ξ_1 for $\text{Ran } d_{A_1, \Phi_1}^1$ described in Section 7.9.1 and choose $\Xi_2 = \text{Coker } \mathbf{D}$ to form a stabilizing bundle, $\Xi = \Xi_1 \oplus \Xi_2$, for $\text{Ran } d_{A', \Phi'}^1$ over the connected sum, $X \# S^4$.

As in Section 7.9.3, the preceding construction extends to the case of multiple connected and tree connected sums. One difference in the case that all summands in a tree except $(X_1, g_1) = (X, g)$ comprise copies of (S^4, g_{round}) is that we may use the Taubes small-eigenvalue decomposition to construct obstruction bundles corresponding to connected sums of copies of (S^4, g_{round}) since there is no spectral flow for the Dirac Laplacians, $D_A D_A^*$, over (S^4, g_{round}) .

CHAPTER 8

Link of an ideal Seiberg–Witten moduli space

We now use the Thom–Mather structure defined in Section 6.7 to construct a *virtual link*, $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$, given by the boundary of a neighborhood of $M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}/S^1$. The *link*, $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$, appearing in (2.6.1), of $M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ is then the intersection of the virtual link $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ with the zero locus of the obstruction section $\bar{\chi}$ appearing in Hypothesis 7.8.1.

We construct the link in Section 8.1 using the tubular distance functions $\bar{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i)$ defined in (6.7.8). We use our understanding of the overlap maps to show that the virtual link $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ can be decomposed into closed subspaces, enumerated by the strata of $\text{Sym}^{\ell}(X)$. Each of these closed subspaces is a smoothly-stratified space. From this definition and a discussion of the orientation of the link, we prove Theorem 8.1.9 and thus the cobordism sum (2.6.1).

Our understanding of the overlaps of the spaces $\mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ allows us to prove in Section 8.2 that each of the aforementioned closed subspaces of the virtual link admits a fiber bundle structure. In Section 8.3, we describe the intersections of these subspaces of the link and the interaction of the intersections with the fiber bundle structures.

8.1. Definition of the link of an ideal Seiberg–Witten moduli space

We define the link $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ of an ideal Seiberg–Witten moduli space by first constructing a virtual link in Section 8.1.1. This virtual link is the union of closed subspaces, as described in (8.1.2), (8.1.4), and (8.1.5). We discuss the orientations of the link in Section 8.1.4 as needed to define the intersection numbers appearing in the terms of the cobordism sum in (2.6.1) in Theorem 8.1.9. Then in Section 8.2, we give a more detailed description of the closed subspaces of the link defined in (8.1.5), showing how they admit a fiber bundle structure which will be used in the pushforward-pullback computations of Chapter 10.

8.1.1. The virtual link of an ideal Seiberg–Witten moduli space. The virtual link is defined as the boundary of a neighborhood of $M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}/S^1$. The subspace $M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X)$ of $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}/S^1$ is the intersection of the zero-locus of the function defined by a norm on the fibers of the vector bundle $N_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \rightarrow M_{\mathfrak{s}}$,

$$(8.1.1) \quad t_N : \bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}/S^1 \rightarrow [0, \delta],$$

with the union of the zero-loci of the tubular distance function $\bar{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)$ defined on $\bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)$ in (6.7.8). We will define the virtual link as the union of two subspaces,

$$(8.1.2) \quad \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}} = \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},s} \cup \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},i}.$$

The first subspace, $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},s}$, is defined to be the codimension-one subspace,

$$(8.1.3) \quad \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},s} = t_N^{-1}(\delta) \subset \bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}/S^1,$$

in the sense that the intersection of $\bar{\mathbf{L}}_{\mathbf{t},\mathfrak{s}}^{\text{vir},s}$ with each smooth stratum of $\bar{\mathcal{M}}_{\mathbf{t},\mathfrak{s}}^{\text{vir}}/S^1$ is a codimension-one submanifold. To define the instanton component of the link, $\bar{\mathbf{L}}_{\mathbf{t},\mathfrak{s}}^{\text{vir},i}$, first enumerate the strata of $\text{Sym}^\ell(X)$ with partitions $\mathcal{P}_0, \dots, \mathcal{P}_r$ of N_ℓ as done in §3.4. Then, define

$$(8.1.4) \quad \bar{\mathbf{L}}_{\mathbf{t},\mathfrak{s}}^{\text{vir},i} := \bigcup_{j=0}^r \bar{\mathbf{L}}_{\mathbf{t},\mathfrak{s}}^{\text{vir},i}(\mathcal{P}_j)$$

with each subspace $\bar{\mathbf{L}}_{\mathbf{t},\mathfrak{s}}^{\text{vir},i}(\mathcal{P}_j)$ given as the pre-image of a codimension-one submanifold with corners (in the sense of [49]) in $[0, 1]^{|\mathcal{P}_j|}$ under $\bar{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)$. We now define the subspaces, $\bar{\mathbf{L}}_{\mathbf{t},\mathfrak{s}}^{\text{vir},i}(\mathcal{P}_j)$, of the instanton link.

Assign a small, generic constant ε_j to each such partition \mathcal{P}_j with $\varepsilon_j > \varepsilon_k$ for $j < k$ in a manner to be specified in Lemma 8.1.2. Then define a subspace of the instanton component of the link by

$$(8.1.5) \quad \bar{\mathbf{L}}_{\mathbf{t},\mathfrak{s}}^{\text{vir},i}(\mathcal{P}_j) := \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)/S^1 \cap \bar{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(\partial\bar{D}(\mathcal{P}, \varepsilon_j)) \setminus \bigcup_{k \neq j} \bar{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)^{-1}(D(\mathcal{P}, \varepsilon_k)).$$

Recall from (6.7.8) that the map $\bar{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)$ is defined by the functions $\tilde{\lambda}_P$ on the connections over S^4 making up the fiber of $\bar{\text{Gl}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \Sigma(X^\ell, \mathcal{P})$. Because of the separating condition (6.2.10) on the scales of these connections over S^4 in the definition of $\bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)$, the inclusion

$$\pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(\mathbf{x}) \cap \bar{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(\bar{D}(\mathcal{P}, \varepsilon_j)) \subseteq \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(\mathbf{x}) \cap \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)$$

need not hold as $\mathbf{x} \in \Sigma(X^\ell, \mathcal{P}_j)$ approaches lower strata in $\text{Sym}^\ell(X)$ and hence the subspace $\bar{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(\bar{D}(\mathcal{P}, \varepsilon_j))$ of $\bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)$ need not define a fiber bundle. The following lemma addresses this issue.

LEMMA 8.1.1. *Let \mathcal{P}_j be one of the partitions of N_ℓ chosen before (8.1.4). For any compact subset $K \in \Sigma(X^\ell, \mathcal{P}_j)$, there is a positive constant ε_j such that for all positive $\varepsilon \leq \varepsilon_j$, the set*

$$\bar{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(\bar{D}(\mathcal{P}_j, \varepsilon)) \cap \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(N_{\mathbf{t}(\ell),\mathfrak{s}}(\delta)/S^1 \times K)$$

is a proper subspace of

$$\bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)/S^1 \cap \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(N_{\mathbf{t}(\ell),\mathfrak{s}}(\delta)/S^1 \times K).$$

PROOF. The lemma follows immediately from the compactness of K . □

We now describe how to chose the decreasing sequence of constants appearing in (8.1.5). We say that a codimension-one submanifold N of a manifold M is *smoothly collared* if there is a smooth embedding, $N \times (-\varepsilon, \varepsilon) \rightarrow M$, whose restriction to $N \times \{0\}$ is the inclusion $N \rightarrow M$.

LEMMA 8.1.2. *There is a decreasing sequence of positive constants $\varepsilon_0 > \varepsilon_1 > \dots > \varepsilon_r$ such that the virtual link $\bar{\mathbf{L}}_{\mathbf{t},\mathfrak{s}}^{\text{vir}}$ defined in (8.1.2), (8.1.3), and (8.1.4) by these constants has the following properties:*

- (1) $\bar{\mathbf{L}}_{\mathbf{t},\mathfrak{s}}^{\text{vir}}$ is the boundary of a neighborhood of $M_{\mathfrak{s}} \times \text{Sym}^\ell(X)$ in $\bar{\mathcal{M}}_{\mathbf{t},\mathfrak{s}}^{\text{vir}}/S^1$.
- (2) The intersection of $\bar{\mathbf{L}}_{\mathbf{t},\mathfrak{s}}^{\text{vir}}$ with each stratum \mathcal{S} of $\bar{\mathcal{M}}_{\mathbf{t},\mathfrak{s}}^{\text{vir}}/S^1$ is a Whitney-stratified space whose top stratum is a codimension-one, collared submanifold of \mathcal{S} .
- (3) The intersection of $\bar{\mathbf{L}}_{\mathbf{t},\mathfrak{s}}^{\text{vir}}$ with the top stratum of $\bar{\mathcal{M}}_{\mathbf{t},\mathfrak{s}}^{\text{vir}}/S^1$ is a topological manifold.

PROOF. We work by induction on the partitions \mathcal{P}_j of N_ℓ used to enumerate the strata of $\text{Sym}^\ell(X)$ before (8.1.5). The lowest stratum, $\Sigma(X^\ell, \mathcal{P}_0)$, of $\text{Sym}^\ell(X)$ is necessarily compact and so there is a small positive constant ε_0 such that Lemma 8.1.1 applies with $K = \Sigma(X^\ell, \mathcal{P}_0)$.

Assume that we have defined $\varepsilon_0, \dots, \varepsilon_{k-1}$ such that for $0 \leq j \leq k-1$, ε_j is a constant sufficiently small that Lemma 8.1.1 applies to $K_j \Subset \Sigma(X^\ell, \mathcal{P}_j)$ where K_j is constructed as in Lemmas 4.8.1 and 4.8.2 using the constants ε_i with $i < j$. If we define $K_k \subset \Sigma(X^\ell, \mathcal{P}_k)$ as in Lemmas 4.8.1 and 4.8.2 applied to the constants $\varepsilon_0, \dots, \varepsilon_{k-1}$, then Lemma 4.8.2 implies that K_k is compact. We now select $\varepsilon_k < \varepsilon_{k-1}$ so Lemma 8.1.1 applies to K_k .

We note that for any $\varepsilon > 0$, the set $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P})^{-1}(\bar{D}(\mathcal{P}, \varepsilon))$ is a closed neighborhood of $N_{\mathbf{t}(\ell), \mathfrak{s}}(\delta)/S^1 \times \Sigma(X^\ell, \mathcal{P})$ in $\bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$. Hence,

$$(8.1.6) \quad \bigcup_j \vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(\bar{D}(\mathcal{P}_j, \varepsilon_j)) \\ = \bigcup_j \left(\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(\bar{D}(\mathcal{P}_j, \varepsilon_j)) \setminus \bigcup_{i \neq j} \vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_i)^{-1}(\bar{D}(\mathcal{P}_i, \varepsilon_i)) \right)$$

defines a closed neighborhood of $N_{\mathbf{t}(\ell), \mathfrak{s}}(\delta)/S^1 \times \text{Sym}^\ell(X)$ in $\bigcup_j \bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)/S^1$. The link (8.1.2) is the boundary of the preceding neighborhood, proving Item (1).

We prove that the intersection of the space (8.1.6) with each stratum of $\bar{\mathcal{M}}_{\mathbf{t}, \mathfrak{s}}^{\text{vir}}/S^1$ is a smooth manifold with corners using the following criterion. If Y is a smooth manifold and $f_j : Y \rightarrow \mathbb{R}$ is a smooth function for $1 \leq j \leq p$, then the subspace,

$$Z := \{y \in Y : f_j(y) \leq \varepsilon_j \text{ for } j = 1, \dots, p\},$$

for given constants $\varepsilon_1, \dots, \varepsilon_p$ will be a manifold with corners if for every collection of indices $1 \leq j_1, \dots, j_a \leq p$ and every $y \in Y$ satisfying $f_{j_k}(y) = \varepsilon_{j_k}$, the derivatives $(df_{j_1})_y, \dots, (df_{j_a})_y$ are linearly independent.

The level sets playing the role of $f_j^{-1}(\varepsilon_j)$ for the space (8.1.6) are the subspaces of $\bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P})/S^1$ defined by

$$(8.1.7) \quad H_P(\mathcal{P}, \varepsilon) := \vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P})^{-1} \left(\{(\lambda_Q)_{Q \in \mathcal{P}} : \lambda_P = \varepsilon\} \subset \mathbb{R}^{|\mathcal{P}|} \right).$$

From the definition (6.7.8) of $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$ in terms of the functions $\tilde{\lambda}_P$, the intersection of $H_P(\mathcal{P}, \varepsilon)$ with open sets of $\bar{\mathcal{U}}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$ of the form

$$(8.1.8) \quad N \times \mathcal{O}_P \times \prod_{P \in \mathcal{P}} \bar{M}_{\text{SPL}, |P|}^{s, \natural}(\delta),$$

where $N \subset N_{\mathbf{t}(\ell), \mathfrak{s}}(\delta)$ and $\mathcal{O}_P \subset \Delta^\circ(X^\ell, \mathcal{P})$ are open subsets, is given by the pre-image of $\tilde{\lambda}_P^{-1}(\varepsilon) \subset \bar{M}_{\text{SPL}, |P|}^{s, \natural}(\delta)$ under the obvious projection. Hence, for generic values of ε , the intersection of $H_P(\mathcal{P}, \varepsilon)$ with each stratum of $\bar{\mathcal{M}}_{\mathbf{t}, \mathfrak{s}}^{\text{vir}}/S^1$ will be a smooth, collared, codimension-one submanifold of that stratum. For any collection $P_1, \dots, P_s \in \mathcal{P}$, at any point in the intersection of

$$(8.1.9) \quad H_{P_1}(\mathcal{P}, \varepsilon) \cap \dots \cap H_{P_s}(\mathcal{P}, \varepsilon)$$

with an open set of the form (8.1.8), the pullbacks of the derivatives $d\tilde{\lambda}_{P_1}, \dots, d\tilde{\lambda}_{P_s}$ will be linearly independent at points in (8.1.9) by the product structure of (8.1.8).

To prove that the above criterion applies to more general intersections of the form

$$(8.1.10) \quad H_{P_{k_1}}(\mathcal{P}_{k_1}, \varepsilon_{k_1}) \cap \cdots \cap H_{P_{k_a}}(\mathcal{P}_{k_a}, \varepsilon_{k_a}),$$

we argue by induction on a . For $a = 1$, the result follows by the argument above given for intersections of the form (8.1.9). Now assume $a > 1$. After re-ordering the indices, we can assume that $\mathcal{P}_{k_1} \leq \mathcal{P}_{k_2} \leq \cdots \leq \mathcal{P}_{k_a}$. The case where $\mathcal{P}_{k_1} = \mathcal{P}_{k_2} = \cdots = \mathcal{P}_{k_a}$ has been done in (8.1.9). Then assume that $\mathcal{P}_{k_1} = \cdots = \mathcal{P}_{k_b} < \mathcal{P}_{k_{b+1}}$. By Theorem 6.6.1, points in $\mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_{k_b}) \cap \mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_{k_{b+1}})$ have neighborhoods which are smoothly-stratified diffeomorphic to an open subspace of (see (6.5.1))

$$(8.1.11) \quad N \times \mathcal{O} \times \prod_{P \in \mathcal{P}_{k_b}} \left(\Delta^\circ(Z_{|P|}(\delta_P), (\mathcal{P}_{k_{b+1}})_P) \times \prod_{P' \in (\mathcal{P}_{k_{b+1}})_P} \bar{M}_{\text{SPL}, |P'|}^{s, \natural}(\delta) \right),$$

where $N \subset N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta)$ and $\mathcal{O}_{\mathcal{P}} \subset \Delta^\circ(X^\ell, \mathcal{P}_{k_b})$ are open subsets. The space (8.1.11) is smoothly-stratified diffeomorphic to a product, $\mathcal{U}_\Sigma \times \mathcal{U}_M$, where

$$\begin{aligned} \mathcal{U}_\Sigma &:= N \times \mathcal{O} \times \prod_{P \in \mathcal{P}_{k_b}} \Delta^\circ(Z_{|P|}(\delta_P), (\mathcal{P}_{k_{b+1}})_P), \\ \mathcal{U}_M &:= \prod_{P \in \mathcal{P}_{k_b}} \prod_{P' \in (\mathcal{P}_{k_{b+1}})_P} \bar{M}_{\text{SPL}, |P'|}^{s, \natural}(\delta). \end{aligned}$$

We write p_Σ and p_M for the projections maps from the space (8.1.11) to \mathcal{U}_Σ and \mathcal{U}_M respectively.

From Lemma 6.7.3, for $u > b$ the restriction of $\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_{k_u})$ to an open subspace of the form (8.1.11) equals the pullback by p_M of a map on \mathcal{U}_M . By induction, the derivatives of the components of $\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_{k_u})$ defining $H_{P_{k_u}}(\mathcal{P}_{k_u}, \varepsilon_{k_u})$ are linearly independent at points in the intersection

$$(8.1.12) \quad H_{P_{k_{b+1}}}(\mathcal{P}_{k_{b+1}}, \varepsilon_{k_{b+1}}) \cap \cdots \cap H_{P_{k_a}}(\mathcal{P}_{k_a}, \varepsilon_{k_a}).$$

If $P_{k_v} \notin \mathcal{P}_{k_{b+1}}$ for all $1 \leq v \leq b$, then from Item (3) of Lemma 5.9.1 and the second equality of (6.7.6), the maps $\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_{k_v})$ are equal to the pullbacks of functions on \mathcal{U}_Σ by p_Σ . Again by induction, the derivatives of the components of $\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_{k_v})$ defining $H_{P_{k_v}}(\mathcal{P}_{k_v}, \varepsilon_{k_v})$ are linearly independent at points in the intersection

$$(8.1.13) \quad H_{P_{k_1}}(\mathcal{P}_{k_1}, \varepsilon_{k_1}) \cap \cdots \cap H_{P_{k_b}}(\mathcal{P}_{k_a}, \varepsilon_{k_b}).$$

Consequently, the derivatives of the functions defining the codimension-one submanifolds in (8.1.10) are linearly independent at points in the intersection (8.1.10) by the product structure $\mathcal{U}_\Sigma \times \mathcal{U}_M$ of (8.1.11).

If $P_{k_v} \in \mathcal{P}_{k_u}$ and $P_{k_v} = P_{k_u}$ for $1 \leq v \leq b < u$ then, because $\varepsilon_{k_v} \neq \varepsilon_{k_u}$, the intersection (8.1.10) is empty.

If $P_{k_v} \in \mathcal{P}_{k_u}$ and $P_{k_v} \neq P_{k_u}$ for $1 \leq v \leq b < u$ then, on the open set (8.1.11), the P_{k_v} -th components of $\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_{k_v})$ and $\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_{k_u})$ are equal. Thus, we can replace $H_{P_{k_v}}(\mathcal{P}_{k_v}, \varepsilon_{k_v})$ with $H_{P_{k_v}}(\mathcal{P}_{k_u}, \varepsilon_{k_v})$. Continue doing these replacements until the intersection is reduced to the previous case considered. This completes the induction and shows that the intersections (8.1.10) satisfy our criterion, proving that each stratum of the space (8.1.6) is a smooth manifold with corners.

The virtual link $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ is the boundary of the space (8.1.6). Hence, the intersection of $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ with the top stratum of $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}/S^1$ is the boundary of a smooth manifold with corners and hence a topological manifold, proving Item (3).

From the preceding argument, we see that for generic, decreasing values of $\varepsilon_1, \dots, \varepsilon_r$, the intersection of

$$(8.1.14) \quad \bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)/S^1 \cap \bar{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(\bar{D}(\mathcal{P}, \varepsilon_j)) \setminus \bigcup_{k \neq j} \bar{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)^{-1}(D(\mathcal{P}, \varepsilon_k))$$

with each stratum of $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}/S^1$ is a smooth manifold with corners. Then, considering $\partial\bar{D}(\mathcal{P}_j, \varepsilon_j)$ as a Whitney-stratified subspace of $\mathbb{R}^{|\mathcal{P}_j|}$ with the lower strata being given by the intersections $\cap_{\ell} H_{P_{\ell}}(\mathcal{P}_j, \varepsilon_j)$, the function $\bar{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)$ is transverse to $\partial\bar{D}(\mathcal{P}_j, \varepsilon_j)$ in the sense of [39, Definition 1.3.1] and so $\bar{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(\partial\bar{D}(\mathcal{P}_j, \varepsilon_j))$ is a Whitney-stratified subspace of the space (8.1.14), whose top stratum has a smooth collaring by [39, Section 1.5], proving Item (2). \square

8.1.2. The link of an ideal Seiberg–Witten moduli space. We can now give the

DEFINITION 8.1.3. The *link of an ideal Seiberg–Witten moduli space*, $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$, of $M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ is

$$\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}} = \bar{\chi}^{-1}(0) \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}},$$

where $\bar{\chi}$ is the obstruction section in Hypothesis 7.8.1 and $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ is the virtual link defined in (8.1.2).

From Item (2) of Hypothesis 7.8.1, the intersection of $\bar{\chi}^{-1}(0)$ with each stratum of $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}/S^1$ is a smooth submanifold of that stratum. For generic values of ε , the subspaces $H_P(\mathcal{P}, \varepsilon)$ from (8.1.7) defining the virtual link $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ will intersect $\bar{\chi}^{-1}(0)$ transversally. Hence, Lemma 8.1.2 yields:

LEMMA 8.1.4. *For generic values of the constants δ, ε_i used to define the virtual link in Lemma 8.1.2, the following hold:*

- (1) $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ is the boundary of a closed neighborhood of $M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X)$ in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$.
- (2) The intersection of $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ with each stratum \mathcal{S} of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ is a Whitney-stratified space whose top stratum is a codimension-one, collared submanifold of \mathcal{S} .
- (3) The intersection of $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ with the top stratum of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ is a topological manifold.

The proof of the following lemma then translates immediately from that of [30, Lemma 3.9].

LEMMA 8.1.5. *Assume $w \in H^2(X; \mathbb{Z})$ is such that $w \pmod{2}$ is good in the sense of Definition 2.2.1. Given a spin^u structure \mathfrak{t} on X and a spin^c structure \mathfrak{s} on X satisfying $\ell(\mathfrak{t}, \mathfrak{s}) \geq 0$ and $w_2(\mathfrak{t}) \equiv w \pmod{2}$, there are positive constants ε_0 and δ_0 such that the following hold for all generic choices of positive constants $\varepsilon_i \leq \varepsilon_0$ and $\delta \leq \delta_0$ defining $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$:*

- (1) $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ is disjoint from \bar{M}_{κ}^w and $\bar{\mathcal{M}}_{\mathfrak{t}}^{\text{red}}$ in the stratification (2.2.2) of $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$.
- (2) For all $z \in \mathbb{A}(X)$ and positive integers η satisfying (8.1.20), and for the geometric representatives $\bar{\mathcal{V}}(z)$ and $\bar{\mathcal{W}}$ discussed in Section 2.4, the intersection,

$$(8.1.15) \quad \bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\eta} \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}},$$

is a finite collection of points contained in the top stratum $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ of $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}} \subset \bar{\mathcal{M}}_{\mathfrak{t}}/S^1$.

8.1.3. A subspace of the virtual link of an ideal Seiberg–Witten moduli space. The deformation retraction $N_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta) \rightarrow M_{\mathfrak{s}}$ defines a deformation retraction of $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},i}$ to the subspace,

$$(8.1.16) \quad \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}} := \bigcup_{j=0}^r \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_j), \quad \text{where} \quad \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_j) := t_N^{-1}(0) \cap \mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},i}(\mathcal{P}_j),$$

where the function t_N is defined in (8.1.1). Essentially, $\bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_i)$ is defined by replacing $\tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta)$ with $\tilde{M}_{\mathfrak{s}}$ in the definition (8.1.5). In Section 9.7, we will show that the intersections with $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ in (2.6.1) can be replaced by intersections with $\bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$.

8.1.4. Orientations of the link of an ideal Seiberg–Witten moduli space. To define an intersection number using the intersection (8.1.15), it is necessary to discuss orientations of the link. An orientation for $\mathcal{M}_{\mathfrak{t}}$ determines one for $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ through the convention introduced in [29, Equations (2.16), (2.16) and (2.25)] by considering $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ as a boundary of $(\mathcal{M}_{\mathfrak{t}} \setminus \gamma_{\mathcal{M}}(\tilde{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}))/S^1$. Specifically, at a point $[A, \Phi] \in \mathbf{L}_{\mathfrak{t},\mathfrak{s}}$, if

- $\vec{r} \in T\mathcal{M}_{\mathfrak{t}}^{*,0}$ is an outward-pointing radial vector with respect to the open neighborhood $\mathcal{M}_{\mathfrak{t}} \cap \gamma(\tilde{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}})$ and spans a subspace complementary to the tangent space of $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$,
- $v_{S^1} \in T\mathcal{M}_{\mathfrak{t}}^{*,0}$ is tangent to the orbit of $[A, \Phi]$ under the (free) circle action (where $S^1 \subset \mathbb{C}$ has its usual orientation), and
- $\lambda_{\mathcal{M}} \in \det(T\mathcal{M}_{\mathfrak{t}}^{*,0})$ is an orientation for $T\mathcal{M}_{\mathfrak{t}}$ at $[A, \Phi]$,

then we define an orientation λ_L for the tangent space $T_{[A,\Phi]}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ when $[A, \Phi]$ is a point in the top stratum of $\bar{\mathcal{M}}_{\mathfrak{t}}$ by

$$(8.1.17) \quad \lambda_{\mathcal{M}} = -v_{S^1} \wedge \vec{r} \wedge \tilde{\lambda}_L,$$

where the lift $\tilde{\lambda}_L \in \Lambda^{\max-2}(T\mathcal{M}_{\mathfrak{t}}^{*,0})$ at the point $[A, \Phi] \in \mathcal{M}_{\mathfrak{t}}^{*,0}$ of

$$\lambda_L \in \det(T\mathbf{L}_{\mathfrak{t},\mathfrak{s}}) \subset \Lambda^{\max-1}(T(\mathcal{M}_{\mathfrak{t}}^{*,0}/S^1)),$$

obeys $\pi_*\tilde{\lambda}_L = \lambda_L$, where $\pi : \mathcal{M}_{\mathfrak{t}} \rightarrow \mathcal{M}_{\mathfrak{t}}/S^1$ denotes the quotient map.

DEFINITION 8.1.6. If O is an orientation for $\mathcal{M}_{\mathfrak{t}}$, we call the orientation for $\mathbf{L}_{\mathfrak{t},\mathfrak{s}}$ related to O by equation (8.1.17) the *boundary orientation defined by O* .

Given a homology orientation Ω , that is, an orientation for $H^+(X; \mathbb{R}) \oplus H^1(X; \mathbb{R})$ as defined prior to [53, Definition 2.1] and an integral lift $w \in H^2(X; \mathbb{Z})$ of $w_2(\mathfrak{t})$, an orientation $O^{\text{ASD}}(\Omega, w)$ for $\mathcal{M}_{\mathfrak{t}}$ is defined in [29, Definition 2.3].

The *standard orientation* for $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ is defined in [30, Definition 3.12]. The standard orientation arises from orientations of spaces used in the definition of $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ and thus is more natural for computations on this space. The standard and boundary orientations are related in the following

LEMMA 8.1.7. (See Feehan and Leness [30, Lemmas 3.13 and 3.14].) *Let \mathfrak{t} and $\mathfrak{t}(\ell)$ be spin^u structures on X satisfying*

$$p_1(\mathfrak{t}) = p_1(\mathfrak{t}(\ell)) - 4\ell, \quad c_1(\mathfrak{t}) = c_1(\mathfrak{t}(\ell)), \quad \text{and} \quad w_2(\mathfrak{t}) = w_2(\mathfrak{t}(\ell)).$$

Let Ω be a homology orientation and let $w \in H^2(X; \mathbb{Z})$ be an integral lift of $w_2(\mathfrak{t}) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$. If $\mathfrak{t}(\ell)$ admits a splitting $\mathfrak{t}(\ell) = \mathfrak{s} \oplus \mathfrak{s} \otimes L$, then the standard orientation for $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ and the boundary

orientation for $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ defined through the orientation $O^{\text{ASD}}(\Omega, w)$ for $\mathcal{M}_{\mathfrak{t}}$ differ by a factor of

$$(8.1.18) \quad (-1)^{\alpha_{\mathfrak{t}}(w,\mathfrak{s})}, \quad \text{where} \quad \alpha_{\mathfrak{t}}(w,\mathfrak{s}) := \frac{1}{4}(w - c_1(L))^2.$$

We note some alternative expressions for the change of orientation formula in (8.1.18), in particular one which matches that appearing in Conjecture 2. Recall from [36, Definition 1.2.8(c)] that $w \in H^2(X; \mathbb{Z})$ is *characteristic* if $w \cdot \alpha \equiv \alpha^2 \pmod{2}$ for all $\alpha \in H^2(X; \mathbb{Z})$.

LEMMA 8.1.8. *Continue the assumptions and notation of Lemma 8.1.7. Then,*

$$\alpha_{\mathfrak{t}}(w, \mathfrak{s}) \equiv \frac{1}{2} (w^2 + c_1(\mathfrak{s}) \cdot (w - c_1(\mathfrak{t}))) + \frac{1}{2} (\sigma - w^2) \pmod{2}.$$

If w is characteristic and $c_1(\mathfrak{t}) \cdot c_1(\mathfrak{s}) = 0$, then

$$\alpha_{\mathfrak{t}}(w, \mathfrak{s}) \equiv \frac{1}{2} (w^2 + c_1(\mathfrak{s}) \cdot w) \pmod{2}.$$

PROOF. The first equality is given by [29, Equation 4.62]. The second equality follows immediately from the first and the additional assumptions. \square

8.1.5. An equality of intersection numbers provided by the $\text{SO}(3)$ -monopole cobordism. For all $z \in \mathbb{A}(X)$ and positive integers η satisfying (8.1.20), and for the geometric representatives $\bar{\mathcal{V}}(z)$ and $\bar{\mathcal{W}}$ discussed in Section 2.4, Lemma 8.1.5 implies that we can define the intersection number,

$$(8.1.19) \quad \# (\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}),$$

to be the oriented count of points in the intersection (8.1.15), using the standard orientation of $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$. This yields the following equality of intersection numbers provided by the $\text{SO}(3)$ -monopole cobordism between links of the moduli space of anti-self-dual connections and links of the ideal moduli spaces of Seiberg–Witten moduli spaces.

THEOREM 8.1.9. *Let \mathfrak{t} be a spin^u structure on a closed, oriented, smooth four-manifold X . Let $z \in \mathbb{A}(X)$ and η be a non-negative integer satisfying*

$$(8.1.20) \quad \deg(z) + 2\eta = \dim \mathcal{M}_{\mathfrak{t}}^{*,0} - 2.$$

Assume that there is a class $w \in H^2(X; \mathbb{Z})$ satisfying $w_2(\mathfrak{t}) \equiv w \pmod{2}$ and which is good in the sense of Definition 2.2.1. Then the intersection numbers (8.1.19) obey

$$(8.1.21) \quad \# (\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\eta-1} \cap \bar{\mathbf{L}}_{\mathfrak{t},\kappa}^w) = - \sum_{\mathfrak{s} \in \text{Spin}^c(X)} (-1)^{\alpha_{\mathfrak{t}}(w,\mathfrak{s})} \# (\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\eta-1} \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}),$$

where $\bar{\mathbf{L}}_{\mathfrak{t},\kappa}^w$ is the link of the moduli space of anti-self-dual connections in $\bar{\mathcal{M}}_{\mathfrak{t}}/S^1$ specified in [28, Definition 3.7].

8.2. Fiber bundle structure of the instanton component of the link of an ideal Seiberg–Witten moduli space

We now describe the fiber bundle structure of the subspaces $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},i}(\mathcal{P}_j)$ (defined in (8.1.5)) of the link of an ideal Seiberg–Witten moduli space.

Recall from Lemma 6.7.3 that for non-negative integers $j \leq k \leq r$, the restriction of $\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)$ to $\bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j) \cap \bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)$ is given by the extension from the fibers of $\bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j) \rightarrow N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times \Sigma(X^\ell, \mathcal{P}_j)$ of the function

$$\vec{t}_f(\mathcal{P}_j, [\mathcal{P}_k]) : \bar{M}(\mathcal{P}_j) \rightarrow \bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_j < \mathcal{P}_k]} I^{\mathcal{P}''}.$$

(For $j = k$, recall that $[\mathcal{P}_j < \mathcal{P}_k] = \{\mathcal{P}_j\}$.) We note that the subset

$$\bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_j < \mathcal{P}_k]} D(\mathcal{P}'', \varepsilon_k) \subset \bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_j < \mathcal{P}_k]} I^{\mathcal{P}''}$$

is closed under the action of $\mathfrak{S}(\mathcal{P})$. For the constants $\varepsilon = (\varepsilon_0, \dots, \varepsilon_r)$ used in the definition of the link in Lemma 8.1.4, define

$$(8.2.1) \quad \bar{M}(\mathcal{P}_j, \varepsilon) := \bar{M}(\mathcal{P}_j) \cap \vec{t}_f(\mathcal{P}_j, [\mathcal{P}_j])^{-1}(\partial \bar{D}(\mathcal{P}_j, \varepsilon_j)) \\ \setminus \bigcup_{k > j} \vec{t}_f(\mathcal{P}_j, [\mathcal{P}_k])^{-1} \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_j < \mathcal{P}_k]} D(\mathcal{P}'', \varepsilon_k) \right).$$

We then have the following fiber bundle structure on the subspace $\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}, i}(\mathcal{P}_j)$ of the link.

PROPOSITION 8.2.1. *Let $\varepsilon = (\varepsilon_0, \dots, \varepsilon_r)$ be the generic constants defining the link in Lemma 8.1.4. Let $K_j \Subset \Sigma(X^\ell, \mathcal{P}_j)$ be the compact subset defined by those constants as in Lemma 4.8.2. For $0 \leq j \leq r$, the subspace $\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}, i}(\mathcal{P}_j)$ of $\bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)/S^1$ defined in (8.1.5) admits a fiber bundle structure,*

$$(8.2.2) \quad \bar{M}(\mathcal{P}_j, \varepsilon) \longrightarrow \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)|_{K_j} \times_{G(\mathcal{P}_j)} \bar{M}(\mathcal{P}_j, \varepsilon) \\ \pi(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j) \downarrow \\ N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta)/S^1 \times K_j$$

where $\bar{M}(\mathcal{P}_j, \varepsilon)$ is defined in (8.2.1).

PROOF. The proposition follows from our previously established results on the intersection,

$$(\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)^{-1}(D(\mathcal{P}_k, \varepsilon_k))) \cap (\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(\partial \bar{D}(\mathcal{P}_j, \varepsilon_j))),$$

as we now describe. For $j < k$, Lemma 6.7.3 equates $\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)$ and the extension of $\vec{t}_f(\mathcal{P}_j, [\mathcal{P}_k])$ and thus shows that the preceding intersection is given by the definition in (8.2.1). For $k < j$, Corollary 6.7.7 and the construction of K_j in Lemma 4.8.2 shows that removing the subset

$$\bigcup_{j < k} \vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)^{-1}(D(\mathcal{P}_k, \varepsilon_k))$$

from $\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)^{-1}(\partial \bar{D}(\mathcal{P}_j, \varepsilon_j))$ is equivalent to restricting the fiber bundle (8.2.2) to K_j . \square

The following lemma will be used to define quotients of the link $\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}$ in Chapter 10.

LEMMA 8.2.2. *The restriction of the $\text{SO}(3)$ action on $\bar{M}(\mathcal{P})$ defined by the diagonal action on the frames,*

$$(8.2.3) \quad (([A_P, F_P^{\mathfrak{s}}, \mathbf{x}_P])_{P \in \mathcal{P}}, M) \mapsto (([A_P, F_P^{\mathfrak{s}} M, \mathbf{x}_P])_{P \in \mathcal{P}}),$$

to $\bar{M}(\mathcal{P}, \varepsilon)$ (defined in (8.2.1)) is free, where $M \in \mathrm{SO}(3)$ and $[A_P, F_P^s, \mathbf{x}_P] \in \bar{M}_{\mathrm{SPL}, |P|}^{s, \natural}(\delta)$.

PROOF. The $\mathrm{SO}(3)$ action on $\bar{M}_{\mathrm{SPL}, |P|}^{s, \natural}(\delta)$ given by the action on the frame is free at all points except for points of the form $[\Theta, F^s, v_P]$, where Θ is the product connection. Thus a fixed point of the action (8.2.3) will be of the form, $A_{\mathcal{P}} = (([\Theta, F_P^s, \mathbf{x}_P])_{P \in \mathcal{P}})$. If $A_{\mathcal{P}} \in \bar{M}(\mathcal{P}, \varepsilon)$, then the requirement that $\vec{t}_f(\mathcal{P}, [\mathcal{P}])(A_{\mathcal{P}}) \in \partial \bar{D}(\mathcal{P}, \varepsilon)$ in (8.2.1) implies that not all the points \mathbf{x}_P can be cone points, c_P . Hence, for such a point $A_{\mathcal{P}} \in \bar{M}(\mathcal{P}, \varepsilon)$ and for any $F_{\mathcal{P}} \in \mathrm{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$, the point $[F_{\mathcal{P}}, A_{\mathcal{P}}]$ will lie in the subspace $T(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ defined in (6.5.4) for some partition \mathcal{P}' with $\mathcal{P} < \mathcal{P}'$. By a suitable choice of $F_{\mathcal{P}}$ (that is, one which lies over a point $\mathbf{x} \in \Sigma(X^\ell, \mathcal{P})$ sufficiently far from the lower strata in $\mathrm{Sym}^\ell(X)$), we can ensure that $[F_{\mathcal{P}}, A_{\mathcal{P}}] \in T^\theta(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$, where $T^\theta(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$ is defined in Lemma 6.6.2. By Lemma 6.6.2, Theorem 6.6.1, and the definition of the equivalence relation \sim in (6.6.3), there is a point $[F_{\mathcal{P}'}, A_{\mathcal{P}'}] \in \Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P}')$, where $\Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P}') \subset \bar{\mathrm{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}')$ is defined in (6.2.7), with

$$[(A_0, \Phi_0), F_{\mathcal{P}'}, A_{\mathcal{P}'}] \sim [(A_0, \Phi_0), F_{\mathcal{P}}, A_{\mathcal{P}}],$$

for any $[A_0, \Phi_0] \in N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta)$. Because $[F_{\mathcal{P}'}, A_{\mathcal{P}'}] \in \Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P}')$, we have

$$\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}')([(A_0, \Phi_0), F_{\mathcal{P}}, A_{\mathcal{P}}]) = \vec{0},$$

where $\vec{0}$ is the vector with all coordinates equal to zero. However, Lemma 6.7.3 then implies that $\vec{t}_f(\mathcal{P}, [\mathcal{P}'])(A_{\mathcal{P}}) = \vec{0}$, contradicting the assumption that $A_{\mathcal{P}} \in \bar{M}(\mathcal{P}, \varepsilon)$. \square

As we describe in the following lemma, the subspaces $\mathbf{BL}_{\mathfrak{t}, \mathfrak{s}}^{\mathrm{vir}}(\mathcal{P}_j)$ admit a fiber bundle structure similar to that provided by Proposition 8.2.1.

LEMMA 8.2.3. *Continue the hypotheses of Proposition 8.2.1. Then the space*

$$(8.2.4) \quad \mathbf{BL}_{\mathfrak{t}, \mathfrak{s}}^{\mathrm{vir}}(\mathcal{P}_j) = t_N^{-1}(0) \cap \bar{\mathbf{L}}_{\mathfrak{t}, \mathfrak{s}}^{\mathrm{vir}, i}(\mathcal{P}_j)$$

is the restriction of the fiber bundle (8.2.2) to the subspace

$$M_{\mathfrak{s}} \times K_j \subset N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) / S^1 \times K_j$$

of the base and is thus given by

$$(8.2.5) \quad \begin{array}{ccc} \bar{M}(\mathcal{P}_j, \varepsilon) & \longrightarrow & \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \mathrm{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j, g_{\mathcal{P}_j}) \times_{G(\mathcal{P}_j)} \bar{M}(\mathcal{P}, \varepsilon) \\ & & \downarrow \pi(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j) \\ & & M_{\mathfrak{s}} \times K_j \end{array}$$

PROOF. The conclusion of the lemma follows immediately from Proposition 8.2.1. \square

8.3. Boundaries of components of links of ideal Seiberg–Witten moduli spaces

In Chapter 9 we will show that the intersection number (8.1.19) can be written as a cohomological pairing and in Lemma 9.7.2 we will show that this pairing can be written as a pairing with a fundamental class defined by the subspace $\mathbf{BL}_{\mathfrak{t}, \mathfrak{s}}^{\mathrm{vir}}$. To apply the technique for computing such pairings described in Section 1.2.7, we will define quotients of the intersections of the subspaces $\mathbf{BL}_{\mathfrak{t}, \mathfrak{s}}^{\mathrm{vir}}(\mathcal{P}_j)$ in Chapter 10. To ensure these quotients satisfy the conditions described in Section 1.2.7, we now prove that these intersections are subbundles of the fiber bundles (8.2.5).

Define

$$(8.3.1) \quad \partial_{k_1} \cdots \partial_{k_p} \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_j) := \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_j) \cap \bigcap_{u=1}^p \vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_{k_u})^{-1}(\partial \bar{D}(\mathcal{P}_{k_u}, \varepsilon_{k_u})).$$

From the definitions (8.2.4) and (8.1.5), we have the equality,

$$(8.3.2) \quad \partial_k \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_j) = \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_j) \cap \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_k) = \partial_j \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_k).$$

More generally,

$$\bigcap_{j=0}^p \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_{i_j}) = \partial_{i_1} \cdots \partial_{i_p} \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_{i_0}).$$

If

$$\text{cl}(\Sigma(X^\ell, \mathcal{P}_j)) \cap \Sigma(X^\ell, \mathcal{P}_k) = \emptyset \quad \text{and} \quad \Sigma(X^\ell, \mathcal{P}_j) \cap \text{cl}(\Sigma(X^\ell, \mathcal{P}_k)) = \emptyset,$$

then the boundaries in (8.3.2) are empty.

To describe the intersection (8.3.1), we introduce the following notation. For $j < i_1 < i_2 < \cdots < i_v$, define

$$(8.3.3) \quad \partial_{i_1} \partial_{i_2} \cdots \partial_{i_v} \bar{M}(\mathcal{P}_j, \varepsilon) := \bar{M}(\mathcal{P}_j, \varepsilon) \cap \bigcap_{u=1}^v \vec{t}_f(\mathcal{P}_j, [\mathcal{P}_{i_u}])^{-1}(\partial \bar{D}(\mathcal{P}_{i_u}, \varepsilon_{i_u})).$$

For non-negative integers $k_1 < k_2 < \cdots < k_p < j \leq r$, we define

$$(8.3.4) \quad \partial_{k_1} \cdots \partial_{k_p} K_j := K_j \cap \bigcap_{u=1}^p \vec{t}(X^\ell, g_{\mathcal{P}_{k_u}})^{-1}(\partial \bar{D}(\mathcal{P}_{k_u}, \varepsilon_{k_u})) \subset \Sigma(X^\ell, \mathcal{P}_j),$$

where the function $\vec{t}(X^\ell, g_{\mathcal{P}_{k_u}})$ is defined in (4.7.1).

PROPOSITION 8.3.1. *For non-negative integers $k_1 < k_2 < \cdots < k_p < j$ and $j < i_1 < i_2 < \cdots < i_v \leq r$, the intersection,*

$$\partial_{k_1} \cdots \partial_{k_p} \partial_{i_1} \cdots \partial_{i_v} \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_j),$$

defined in (8.3.1) admits a description as a fiber bundle,

$$(8.3.5) \quad \begin{array}{ccc} \partial_{i_1} \partial_{i_2} \cdots \partial_{i_v} \bar{M}(\mathcal{P}_j, \varepsilon) & \longrightarrow & \partial_{k_1} \cdots \partial_{k_p} \partial_{i_1} \cdots \partial_{i_v} \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_j) \\ & & \pi(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j) \downarrow \\ & & M_{\mathfrak{s}} \times \partial_{k_1} \cdots \partial_{k_p} K_j \end{array}$$

arising from the equality,

$$\begin{aligned} & \partial_{k_1} \cdots \partial_{k_p} \partial_{i_1} \cdots \partial_{i_v} \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_j) \\ &= \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j) |_{\partial_{k_1} \cdots \partial_{k_p} K_j \times_{G(\mathcal{P}_j)} \partial_{i_1} \partial_{i_2} \cdots \partial_{i_v} \bar{M}(\mathcal{P}_j, \varepsilon)}. \end{aligned}$$

PROOF. For non-negative integers $j < i_u \leq r$, Lemma 6.7.3 implies that the intersection of $\bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_j)$ with $\vec{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_{i_u})^{-1}(\partial \bar{D}(\mathcal{P}_{i_u}, \varepsilon_{i_u}))$ is given by the intersection of the fiber $\bar{M}(\mathcal{P}_j, \varepsilon)$ with

$$\vec{t}_f(\mathcal{P}_j, [\mathcal{P}_{i_u}])^{-1}(\partial \bar{D}(\mathcal{P}_{i_u}, \varepsilon_{i_u})).$$

For non-negative integers $k < j \leq r$, we claim that the intersection of $\bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_j)$ with

$$\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)^{-1}(\partial\bar{D}(\mathcal{P}_k, \varepsilon_k))$$

is given by the inverse image, under the projection $\pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)$, of the intersection of $M_{\mathfrak{s}} \times K_j$ with

$$M_{\mathfrak{s}} \times \vec{t}(X^\ell, g_{\mathcal{P}_k})^{-1}(\partial\bar{D}(\mathcal{P}_k, \varepsilon_k)).$$

To prove this claim, we will show that for $\mathbf{A} \in \bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_j)$ and $\mathbf{A}' := \pi(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)(\mathbf{A})$ we have

$$(8.3.6) \quad \vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}) \in \partial\bar{D}(\mathcal{P}_k, \varepsilon_k) \quad \text{if and only if} \quad \vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}') \in \partial\bar{D}(\mathcal{P}_k, \varepsilon_k).$$

The latter condition is true if and only if $\mathbf{A}' \in M_{\mathfrak{s}} \times \partial_k K_j$ by Lemma 6.7.4 and the definition of $\partial_k K_j$. Thus it suffices to prove (8.3.6) in order to complete the proof of Proposition 8.3.1. To this end, we first collect the following observations.

Items (1) and (2) of Lemma 6.7.5 imply that for all $Q \in \mathcal{P}_k$ we have

$$(8.3.7) \quad \vec{t}^Q(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}') \leq \vec{t}^Q(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A})$$

and

$$(8.3.8) \quad \vec{t}^Q(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}) \neq \vec{t}^Q(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}') \quad \text{if and only if} \quad Q \in \mathcal{P}_j \cap \mathcal{P}_k$$

and in this case ($Q \in \mathcal{P}_j \cap \mathcal{P}_k$), we see that $\vec{t}^Q(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}') = 0$. We further note that because $\mathbf{A} \in \bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_j)$, we have

$$(8.3.9) \quad \vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)(\mathbf{A}) \in \bar{D}(\mathcal{P}_j, \varepsilon_j) \quad \text{and thus} \quad \vec{t}^Q(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)(\mathbf{A}) \leq \varepsilon_j, \quad \text{for all } Q \in \mathcal{P}_j.$$

Item (3) in Lemma 6.7.5 and (8.3.9) imply that

$$(8.3.10) \quad \vec{t}^Q(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}) = \vec{t}^Q(\mathbf{t}, \mathfrak{s}, \mathcal{P}_j)(\mathbf{A}) \leq \varepsilon_j < \varepsilon_k \quad \text{for } Q \in \mathcal{P}_j \cap \mathcal{P}_k.$$

If $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}) \in \partial\bar{D}(\mathcal{P}_k, \varepsilon_k)$, then $\vec{t}^Q(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}) \leq \varepsilon_k$ for all $Q \in \mathcal{P}_k$ and there is at least one $Q_0 \in \mathcal{P}_k$ with $\vec{t}^{Q_0}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}) = \varepsilon_k$. By (8.3.10), we have $Q_0 \notin \mathcal{P}_k \cap \mathcal{P}_j$ so (8.3.8) implies that $\vec{t}^{Q_0}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}') = \varepsilon_k$. This equality and (8.3.7) then imply that $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}') \in \partial\bar{D}(\mathcal{P}_k, \varepsilon_k)$.

If $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}') \in \partial\bar{D}(\mathcal{P}_k, \varepsilon_k)$, then $\vec{t}^Q(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}') \leq \varepsilon_k$ for all $Q \in \mathcal{P}_k$ and there is at least one $Q_0 \in \mathcal{P}_k$ with $\vec{t}^{Q_0}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}') = \varepsilon_k$. Because $\varepsilon_k \neq 0$, then (8.3.9) implies that $Q_0 \notin \mathcal{P}_k \cap \mathcal{P}_j$ and $\vec{t}^{Q_0}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}) = \varepsilon_k$. For all other $Q \in \mathcal{P}_k$, if $Q \notin \mathcal{P}_k \cap \mathcal{P}_j$, then (8.3.8) implies that

$$\vec{t}^Q(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}) = \vec{t}^Q(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}') \leq \varepsilon_k,$$

while for $Q \in \mathcal{P}_k \cap \mathcal{P}_j$, we see that (8.3.10) implies $\vec{t}^Q(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}) < \varepsilon_k$. Hence, $\vec{t}(\mathbf{t}, \mathfrak{s}, \mathcal{P}_k)(\mathbf{A}) \in \partial\bar{D}(\mathcal{P}_k, \varepsilon_k)$, completing the proof of (8.3.6). This completes the proof of Proposition 8.3.1. \square

Because $U_f(\mathcal{P}_i, [\mathcal{P}_j])$ is defined in (6.7.3) as the image of the injective (up to symmetric group action) splicing map $\rho_{f, \mathcal{P}_i, [\mathcal{P}_j]}^{\mathbf{t}, \mathfrak{s}, d}$, there is a homeomorphism between $U_f(\mathcal{P}_i, [\mathcal{P}_j])$ and an open subspace of

$$\bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \prod_{P \in \mathcal{P}_i} \left(\Delta^\circ(Z_P(\delta_P), \mathcal{P}_P'') \times \prod_{Q \in \mathcal{P}_P''} \bar{M}_{\text{SPL}, |Q|}^{\mathfrak{s}, \natural}(\delta_Q) \right) / \mathfrak{S}(\mathcal{P}_i).$$

Composing this homeomorphism with the obvious projection map defines a surjective map,

$$(8.3.11) \quad U_f(\mathcal{P}_i, [\mathcal{P}_j]) \rightarrow \bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \prod_{P \in \mathcal{P}_i} \Delta^\circ(Z_P(\delta_P), \mathcal{P}_P'') / \mathfrak{S}(\mathcal{P}_i).$$

In addition, because

$$\begin{aligned} \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \prod_{P \in \mathcal{P}_i} \prod_{Q \in \mathcal{P}''} \bar{M}_{\text{SPL}, |Q|}^{s, \natural}(\delta_Q) \right) / \mathfrak{S}(\mathcal{P}_i) &= \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \prod_{Q \in \mathcal{P}''} \bar{M}_{\text{SPL}, |Q|}^{s, \natural}(\delta_Q) \right) / \mathfrak{S}(\mathcal{P}_i) \\ &= \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \bar{M}(\mathcal{P}'') \right) / \mathfrak{S}(\mathcal{P}_i), \end{aligned}$$

where $\bar{M}(\mathcal{P}'')$ is defined in (6.2.5), there is a $G(\mathcal{P}_i)$ -equivariant map,

$$(8.3.12) \quad c_{j,i} : U_f(\mathcal{P}_i, [\mathcal{P}_j]) \rightarrow \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \bar{M}(\mathcal{P}'') \right) / \mathfrak{S}(\mathcal{P}_i),$$

which will be used in Chapter 10. In that chapter, we will also require the characterization of a group action on the image of $c_{j,i}$ as described in the following two lemmas.

LEMMA 8.3.2. *Let S^1 act on a topological space \mathcal{X} . Assume that there is a non-zero $N \in \mathbb{N}$ such that the order of every isotropy group of this action divides N . If there is an action of a finite group G on \mathcal{X} which commutes with the action of S^1 , then the S^1 action on \mathcal{X} descends to an S^1 action on \mathcal{X}/G such that if $\lambda[x] = [x]$ for $\lambda \in S^1$, $x \in \mathcal{X}$, and $[x] \in \mathcal{X}/G$, then λ is a $|G|N$ -th root of unity.*

PROOF. Denote the S^1 action on \mathcal{X}/G by $(\lambda, [x]) \mapsto [\lambda x]$. Because the S^1 and G actions on \mathcal{X} commute, this action is well-defined. If $\lambda[x] = [x]$, then there is a group element, $g \in G$, such that $\lambda x = gx$. Then $\lambda^{|G|}x = g^{|G|}x = ex = x$ so $\lambda^{|G|}$ is in the isotropy group of x . By the assumption on the order of the isotropy groups, $\lambda^{|G|N} = e$, as asserted. \square

We say that an action of a group H on a topological space \mathcal{X} has *finite isotropy* if there is a finite subgroup $K < H$ such that for all $x \in \mathcal{X}$, we have $H_x < K$, where $H_x \subset H$ denotes the isotropy subgroup of $x \in \mathcal{X}$.

LEMMA 8.3.3. *The restriction of the diagonal $\text{SO}(3)$ action on the frames in (8.2.3) to*

$$c_{j,i}(U_f(\mathcal{P}_i, [\mathcal{P}_j]) \cap \bar{M}(\mathcal{P}_i, \varepsilon)) \subset \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \bar{M}(\mathcal{P}'') \right) / \mathfrak{S}(\mathcal{P}_i)$$

has finite isotropy.

PROOF. For the product connection, Θ , on $S^4 \times \mathfrak{so}(3)$, we define

$$T(\mathcal{P}) := \{([A_P, F_P^s, \mathbf{x}_P])_{P \in \mathcal{P}} \in \bar{M}(\mathcal{P}) : A_P = \Theta \text{ for all } P \in \mathcal{P}\}.$$

We first claim that the image of the restriction of $c_{j,i}$ to $U_f(\mathcal{P}_i, [\mathcal{P}_j]) \cap \bar{M}(\mathcal{P}_i, \varepsilon)$ is disjoint from $T(\mathcal{P}'')$ for all $\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]$. The proof of Lemma 8.2.2 implies that $\bar{M}(\mathcal{P}_i, \varepsilon) \cap T(\mathcal{P}_i)$ is empty. If $A(\mathcal{P}_i) \in U_f(\mathcal{P}_i, [\mathcal{P}_j]) \cap \bar{M}(\mathcal{P}_i, \varepsilon)$ and $c_{j,i}(A(\mathcal{P}_i)) \in T(\mathcal{P}'')$, then the definition of $c_{j,i}$ in (8.3.12) and the definition of the map $\rho_{f, \mathcal{P}_i, [\mathcal{P}_j]}^{\natural, s, d}$ in (6.5.16) would imply that $A(\mathcal{P}_i) \in T(\mathcal{P}_i)$, a contradiction.

The proof of Lemma 8.2.2 implies that the action (8.2.3) is free on $\bar{M}(\mathcal{P}'') \setminus T(\mathcal{P}'')$. The action on the $\mathfrak{S}(\mathcal{P}_i)$ quotient then has finite isotropy by Lemma 8.3.2. This completes the proof of Lemma 8.3.3. \square

CHAPTER 9

Cohomology and duality

In this chapter, we will prove that the intersection number,

$$\# (\mathcal{V}(z) \cap \mathcal{W}^\eta \cap \bar{\mathbf{L}}_{t,s}),$$

appearing in (2.6.1), can be expressed as the pairing of a cohomology class with a homology class. Our aim in doing this is to rewrite the above intersection number in a form to which we can apply the pushforward-pullback argument described in Section 1.2.7.

9.1. Introduction

Explicitly, in this chapter we will prove the following analogue of [30, Proposition 5.2].

PROPOSITION 9.1.1. *For $\beta \in H_\bullet(X; \mathbb{R})$, let $\bar{\mu}_p(\beta), \bar{\mu}_c \in H^\bullet(\mathcal{M}_{t,s}^{\text{vir},*}/S^1; \mathbb{R})$ be the cohomology classes in Definition 9.4.8. Let $\bar{e}_s = e(\bar{\mathbf{Y}}_{t,s}^s/S^1)$ be the Euler class of the background obstruction bundle in (7.3.6) and let \bar{e}_I be the extension of the Euler class of the instanton obstruction bundle in Definition 9.4.8. If $d(t) = \dim \mathcal{M}_{t,s}^{*,0}$ and $\deg(z) + 2\eta = d(t) - 2$, then let $[\bar{\mathbf{L}}_{t,s}^{\text{vir}}] \in H_{d(t)-2}(\mathcal{M}_{t,s}^{\text{vir},*}/S^1; \mathbb{R})$ be the homology class defined in (9.3.3). Then*

$$(9.1.1) \quad \# (\mathcal{V}(z) \cap \mathcal{W}^\eta \cap \bar{\mathbf{L}}_{t,s}) = \langle \bar{\mu}_p(z) \smile \bar{\mu}_c^\eta \smile \bar{e}_I \smile \bar{e}_s, [\bar{\mathbf{L}}_{t,s}^{\text{vir}}] \rangle.$$

Let $\hat{\mathbf{L}}$ be any codimension-zero, compact submanifold of the top stratum of $\mathbf{L}_{t,s}^{\text{vir}}$ with the property that its boundary lies in a neighborhood U of the lower strata, where U retracts to the lower strata. The existence of such a neighborhood, U , follows immediately from Lemma 6.6.4. The fundamental class of $\hat{\mathbf{L}}$ is a relative class and we can define the fundamental class of $\mathbf{L}_{t,s}^{\text{vir}}$ through the exact sequence of the pair $(\mathbf{L}_{t,s}^{\text{vir}}, \mathbf{L}_{t,s}^{\text{vir}} \cap U)$.

A geometric representative \mathcal{V} on a stratified space \mathcal{M} , as described in Definition 9.4.1, naturally defines a relative cohomology class in $H^\bullet(\mathcal{M}, \mathcal{M} \setminus \mathcal{V}; \mathbb{R})$. If \mathcal{V} were a smooth submanifold of a manifold \mathcal{M} , then the relative cohomology class would be given by the Thom class of the normal bundle of \mathcal{V} . The geometric representatives $\mathcal{V}(z)$ and \mathcal{W} define such cohomology classes. In addition, the restriction of the zero locus of the obstruction section to the top stratum also defines such a relative cohomology class, namely the relative Euler class. We note that these classes are, initially, defined only on the top stratum of $\mathcal{M}_{t,s}^{\text{vir},*}/S^1$. The intersection number in (9.1.1) equals the pairing of the product of these relative cohomology classes with the relative fundamental class of the manifold with boundary $\hat{\mathbf{L}}$.

To rewrite this pairing of relative classes in a form to which we can apply the pushforward-pullback argument described in Section 1.2.7, we relate the relative cohomology classes mentioned above with absolute cohomology classes defined on $\mathcal{M}_{t,s}^{\text{vir},*}/S^1$ and compute these cohomology classes in terms of cohomology classes on $M_s \times \text{Sym}^\ell(X)$ and the Chern class defined by the S^1 action.

The relative cohomology classes defined by $\bar{\mathcal{V}}(z)$, $\bar{\mathcal{W}}$, and the obstruction section extend to relative cohomology classes on the complement of a small subspace of $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$. If an embedded surface $\Sigma \subset X$ satisfies $q[\Sigma] = h \in H_2(X; \mathbb{R})$ for $q \in \mathbb{R}$, then the relative cohomology class for the geometric representative $\bar{\mathcal{V}}(h)$ defines such a relative cohomology class in

$$H^\bullet \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(\Sigma)), \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus (\mathcal{I}(\nu(\Sigma)) \cup \bar{\mathcal{V}}(h)); \mathbb{R} \right),$$

where $\mathcal{I}(\nu(\Sigma))$ is the subspace of points $[A, \Phi, \mathbf{x}]$ where the support of \mathbf{x} intersects $\nu(\Sigma)$, a tubular neighborhood of Σ . The relative cohomology class for $\bar{\mathcal{W}}$ extends to a relative cohomology class on the complement of $\mathcal{I}(\nu(x))$. The relative Euler class of the obstruction section extends to the complement of the intersection of the zero locus of the obstruction section with the lower strata. We then compute that the image of these extended relative cohomology classes, in the exact sequence of the relevant pair, equals the restriction of the cohomology classes of the desired form from $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$ to the subspaces described above. With these equalities established, the proof of Proposition 9.1.1 is then largely a formal manipulation.

After defining subspaces of and cohomology classes on $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$ in Section 9.2, we define the relevant fundamental class in Section 9.3. In Section 9.4, we define the relative cohomology classes corresponding to the geometric representatives $\bar{\mathcal{V}}(z)$ and $\bar{\mathcal{W}}$, define their extensions, and compute the corresponding absolute cohomology classes in terms of the cohomology classes defined in Section 9.2. We discuss relative Euler classes and carry out a similar program for a geometric dual of $\bar{\chi}^{-1}(0)$ in Section 9.5. The proof of Proposition 9.1.1 appears in Section 9.6. Finally, in Section 9.7, we show how to replace the pairing in (9.1.1) with a pairing with the fundamental class of $\mathbf{BL}_{t,s}^{\text{vir}}$, the space defined in (8.1.16).

9.2. Definitions

9.2.1. Subspaces and maps. We begin by defining subspaces of $\bar{\mathcal{M}}_{t,s}^{\text{vir}}$. First, let

$$(9.2.1) \quad \iota : \mathcal{M}_{t,s}^{\text{vir}} \rightarrow \bar{\mathcal{M}}_{t,s}^{\text{vir},*}$$

be the inclusion of the top stratum, which is mapped to \mathcal{C}_t by the splicing and gluing maps $\gamma'_{\mathcal{M}}$ and $\gamma_{\mathcal{M}}$, and let

$$(9.2.2) \quad \bar{\mathcal{M}}_{t,s}^{\text{vir},*} = \bar{\mathcal{M}}_{t,s}^{\text{vir}} \setminus \left(M_s \times \text{Sym}^\ell(X) \right).$$

The S^1 action (6.6.8) is free on the subspace $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}$. The map

$$\pi_N : \bar{\mathcal{M}}_{t,s}^{\text{vir}} \rightarrow N_{t(\ell),s}(\delta)$$

was defined in (6.6.9) and we define

$$(9.2.3) \quad \pi_s : \bar{\mathcal{M}}_{t,s}^{\text{vir}} \rightarrow M_s$$

as the composition of π_N with the projection $N_{t(\ell),s}(\delta) \rightarrow M_s$. Recall that the projection

$$\pi_X : \bar{\mathcal{M}}_{t,s}^{\text{vir}} \rightarrow \text{Sym}^\ell(X)$$

was defined in (6.9.1). We will also write

$$(9.2.4) \quad \pi_{s,X} : \bar{\mathcal{M}}_{t,s}^{\text{vir}} \rightarrow M_s \times \text{Sym}^\ell(X),$$

$$(9.2.5) \quad \pi_{N,X} : \bar{\mathcal{M}}_{t,s}^{\text{vir}} \rightarrow N_{t(\ell),s}(\delta) \times \text{Sym}^\ell(X),$$

for the projection maps. We use the same notation to denote the projections from $\bar{\mathcal{M}}_{t,5}^{\text{vir}}/S^1$ (although we note that π_N maps this quotient to $N_{t(\ell),5}(\delta)/S^1$).

For any subspace $Y \subset X$, define

$$(9.2.6) \quad \mathcal{I}(Y) \subset \bar{\mathcal{M}}_{t,5}^{\text{vir},*}/S^1$$

to be the subset of triples $[A, \Phi, \mathbf{x}]$ where the support of \mathbf{x} contains a point of Y .

9.2.2. The incidence locus. We now define subspaces of $\text{Sym}^\ell(X) \times X$ and their Poincaré duals to be used in computing expressions for cohomology classes dual to the geometric representatives $\tilde{\mathcal{V}}(z)$ appearing in (9.1.1).

The incidence locus,

$$(9.2.7) \quad \mathcal{V}_\ell(\Delta) \subset \text{Sym}^\ell(X) \times X,$$

is defined to be the set of points $(\mathbf{x}, y) \in \text{Sym}^\ell(X) \times X$ such that y is in the support of \mathbf{x} . Alternately, one can describe $\mathcal{V}_\ell(\Delta)$ by first defining

$$(9.2.8) \quad \tilde{\mathcal{V}}_\ell(\Delta) = \left(\bigcup_i (\pi_i \times \text{id}_X)^{-1}(\Delta_2) \right) \subset X^\ell \times X,$$

where $\pi_i : X^\ell \rightarrow X$ is projection onto the i -th factor and $\Delta_2 \subset X \times X$ is the diagonal and observing that $\mathcal{V}_\ell(\Delta) = \tilde{\mathcal{V}}_\ell(\Delta)/\mathfrak{S}_\ell$. We will write

$$(9.2.9) \quad \begin{aligned} \tilde{\mathcal{V}}_{\mathcal{P}}(\Delta) &= (\Delta^\circ(X^\ell, \mathcal{P}) \times X) \cap \tilde{\mathcal{V}}_\ell(\Delta), \\ \mathcal{V}_{\mathcal{P}}(\Delta) &= (\Sigma(X^\ell, \mathcal{P}) \times X) \cap \mathcal{V}_\ell(\Delta). \end{aligned}$$

Observe that $\mathcal{V}_{\mathcal{P}}(\Delta) = \mathcal{V}_{\mathcal{P}' }(\Delta)$ if the partitions \mathcal{P} and \mathcal{P}' are in the same orbit of the \mathfrak{S}_ℓ action on partitions of N_ℓ .

For any smooth submanifold $Y \subseteq X$, we define analogous subspaces of $\text{Sym}^\ell(X) \times Y$,

$$(9.2.10) \quad \begin{aligned} \mathcal{V}_\ell(\Delta, Y) &= \mathcal{V}_\ell(\Delta) \cap (\text{Sym}^\ell(X) \times Y), \\ \tilde{\mathcal{V}}_\ell(\Delta, Y) &= \tilde{\mathcal{V}}_\ell(\Delta) \cap (X^\ell \times Y), \\ \mathcal{V}_{\mathcal{P}}(\Delta, Y) &= \mathcal{V}_{\mathcal{P}}(\Delta) \cap (\Sigma(X^\ell, \mathcal{P}) \times Y), \\ \tilde{\mathcal{V}}_{\mathcal{P}}(\Delta, Y) &= \tilde{\mathcal{V}}_{\mathcal{P}}(\Delta) \cap (\Delta^\circ(X^\ell, \mathcal{P}) \times Y). \end{aligned}$$

We have the following description of the preceding subspaces.

LEMMA 9.2.1. *Let $Y \subseteq X$ be an oriented, smooth submanifold with an oriented normal bundle and let \mathcal{P} be a partition of N_ℓ . The subspace $\tilde{\mathcal{V}}_{\mathcal{P}}(\Delta, Y)$ defined in (9.2.10) is then the disjoint union of components,*

$$\tilde{\mathcal{V}}_{\mathcal{P}}(\Delta, Y) = \bigsqcup_{P \in \mathcal{P}} \tilde{\mathcal{V}}_{\mathcal{P}}(\Delta, Y, P),$$

where

$$(9.2.11) \quad \tilde{\mathcal{V}}_{\mathcal{P}}(\Delta, Y, P) := \{(x_P)_{P \in \mathcal{P}}, y) \in \tilde{\mathcal{V}}_{\mathcal{P}}(\Delta, Y) : x_P = y\}.$$

Each component (9.2.11) is a codimension-four, oriented, smooth submanifold of $\Delta^\circ(X^\ell, \mathcal{P}) \times Y$ with an oriented normal bundle. Under the action of $\mathfrak{S}(\mathcal{P})$ on $\Delta^\circ(X^\ell, \mathcal{P}) \times T$, the components $\tilde{\mathcal{V}}_\mathcal{P}(\Delta, Y, P)$ and $\tilde{\mathcal{V}}_\mathcal{P}(\Delta, Y, P')$ are identified if $|P| = |P'|$.

PROOF. The fact that $\tilde{\mathcal{V}}_\mathcal{P}(\Delta, Y)$ is given by the union in the statement of the lemma is clear. The fact that the given union is disjoint follows by observing that for any $\mathbf{x} = (x_P)_{P \in \mathcal{P}} \in \Delta^\circ(X^\ell, \mathcal{P})$, the points x_P and $x_{P'}$ are always unequal for $P \neq P'$, so for $(\mathbf{x}, y) \in \tilde{\mathcal{V}}_\mathcal{P}(\Delta, Y)$, the equality $x_P = y = x_{P'}$ is impossible.

To see that each component (9.2.11) is a codimension-four, oriented, smooth submanifold we observe that $\Delta_2(Y) := \{(y, y) \in X \times Y : y \in Y\}$ is a smooth submanifold of $X \times Y$, diffeomorphic to Y , with orientable normal bundle given by the direct sum of orientable bundles,

$$\{(w_Y^\perp, 0) \in T(X \times Y)|_{\Delta_2(Y)} : w_Y^\perp \in \nu(Y)\} \oplus \{(w_Y, -w_Y)|_{\Delta_2(Y)} : w_Y \in TY\},$$

where $\nu(Y)$ is the normal bundle of Y in X . Because the restriction of the projection maps $\pi_i, \pi_j : X^\ell \rightarrow X$ to $\Delta^\circ(X^\ell, \mathcal{P})$ are equal for any $i, j \in P$, the component (9.2.11) is the pre-image of $\Delta_2(Y)$ under the map

$$\pi_i \times \text{id}_Y : \Delta^\circ(X^\ell, \mathcal{P}) \times Y \rightarrow X \times Y$$

for any $i \in P$. Because the map $\pi_i \times \text{id}_Y$ is a submersion, this pre-image is then also a codimension-four, smooth submanifold with normal bundle given by pullback of the normal bundle of $\Delta_2(Y)$. Because the normal bundle of $\Delta_2(Y)$ is orientable, its pullback by $\pi_i \times \text{id}_Y$ is also orientable, as required.

Lastly, the proof of the observation on the action of $\mathfrak{S}(\mathcal{P})$ is straightforward. \square

We will use the following computation of the cohomology of the complement of $\mathcal{V}_\mathcal{P}(\Delta, Y)$.

LEMMA 9.2.2. *Let \mathcal{P} be a partition of N_ℓ , let $Y \subset X$ be a smooth submanifold, and let $\mathcal{V}_\mathcal{P}(\Delta, Y) \subset \Sigma(X^\ell, \mathcal{P}) \times Y$ be the subspace defined in (9.2.10). Then,*

$$(9.2.12) \quad H_k \left(\Sigma(X^\ell, \mathcal{P}) \times Y, \left(\Sigma(X^\ell, \mathcal{P}) \times Y \right) \setminus \mathcal{V}_\mathcal{P}(\Delta, Y); \mathbb{R} \right) \cong \begin{cases} 0 & \text{if } k < 4, \\ (\oplus_{P \in \mathcal{P}} \mathbb{R}) / \mathfrak{S}(\mathcal{P}) & \text{if } k = 4. \end{cases}$$

The group $H_4(\Sigma(X^\ell, \mathcal{P}) \times Y, (\Sigma(X^\ell, \mathcal{P}) \times Y) \setminus \mathcal{V}_\mathcal{P}(\Delta, Y); \mathbb{R})$ is generated by $\phi_{\mathcal{P}, P} := (\tilde{\pi}_\ell \times \text{id}_Y) \circ \tilde{\phi}_{\mathcal{P}, P}$, where

$$(9.2.13) \quad \tilde{\phi}_{\mathcal{P}, P} : D^4 \rightarrow \Delta^\circ(X^\ell, \mathcal{P}) \times Y$$

is any map intersecting $\tilde{\mathcal{V}}_\mathcal{P}(\Delta, Y, P)$ transversely at the origin and $\tilde{\pi}_\ell : X^\ell \rightarrow \text{Sym}^\ell(X)$ is the projection.

PROOF. The conclusion follows immediately from the Thom isomorphism, [92, Theorem 5.7.10], [68, Theorem 10.4], or [11, Equation VIII.11.3], and the assertion in Lemma 9.2.1 that $\mathcal{V}_\mathcal{P}(\Delta, Y)$ is a disjoint union of smooth, codimension-four submanifolds with oriented normal bundles, enumerated by $\mathcal{P} / \mathfrak{S}(\mathcal{P})$, of $\Sigma(X^\ell, \mathcal{P}) \times Y$. \square

9.2.3. Cohomology classes. We now define some cohomology classes on $\tilde{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1$ to express our computations of the duals of the geometric representatives of $\tilde{\mathcal{V}}(z)$ and $\tilde{\mathcal{W}}$.

DEFINITION 9.2.3. Let $\nu \in H^2(\overline{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1; \mathbb{Z})$ be the first Chern class of the S^1 bundle,

$$(9.2.14) \quad \overline{\mathcal{M}}_{t,s}^{\text{vir},*} \rightarrow \overline{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1,$$

where the S^1 action is defined in (6.6.8). If

$$(9.2.15) \quad \mathbb{L}_\nu = \overline{\mathcal{M}}_{t,s}^{\text{vir},*} \times_{S^1} \mathbb{C},$$

is the complex line bundle associated to this S^1 action, then $c_1(\mathbb{L}_\nu) = \nu$.

DEFINITION 9.2.4. For $\beta \in H_\bullet(X; \mathbb{R})$, let $\mu_s(\beta) \in H^{2-\bullet}(M_s; \mathbb{Z})$ be the μ -class defined in (2.3.8). We will use the same notation for the pullback by the projection π_s of these classes to $\overline{\mathcal{M}}_{t,s}^{\text{vir}}/S^1$. We will also use the notation

$$\mathbb{L}_s = \tilde{M}_s \times_{\mathcal{G}_s} (X \times \mathbb{C}) \rightarrow M_s \times X$$

for the restriction of the bundle defined in (2.3.5).

By [7, Theorem II.19.2], there is an isomorphism,

$$(9.2.16) \quad H^\bullet(\text{Sym}^\ell(X); \mathbb{R}) \cong H^\bullet(X^\ell; \mathbb{R})^{\mathfrak{S}_\ell}.$$

We can thus make the following definition.

DEFINITION 9.2.5. If $\beta \in H_\bullet(X; \mathbb{R})$ and $\pi_i : X^\ell \rightarrow X$ is projection onto the i -th factor, define

$$\tilde{S}^\ell(\beta) := \sum_i \pi_i^* \text{PD}[\beta] \in H^{4-\bullet}(X^\ell; \mathbb{R})$$

and let $S^\ell(\beta) \in H^{4-\bullet}(\text{Sym}^\ell(X); \mathbb{R})$ be the cohomology class satisfying $\tilde{\pi}_\ell^* S^\ell(\beta) = \tilde{S}^\ell(\beta)$, where $\varpi : X^\ell \rightarrow \text{Sym}^\ell(X)$ is the projection, as specified by Equation (9.2.16), and PD denotes the Poincaré duality isomorphism.

For $\alpha \in H^\bullet(X; \mathbb{R})$, define

$$\tilde{S}^\ell(\alpha) := \sum_i \pi_i^* \alpha \in H^\bullet(X^\ell; \mathbb{R})$$

and let $S^\ell(\alpha) \in H^\bullet(\text{Sym}^\ell(X); \mathbb{R})$ be the cohomology class satisfying $\varpi^* S^\ell(\alpha) = \tilde{S}^\ell(\alpha)$.

Although the incidence locus $\mathcal{V}_\ell(\Delta)$ defined in (9.2.7) is not a submanifold of $\text{Sym}^\ell(X) \times X$, we can define a cohomology class which will serve the same role as a Poincaré dual for the purposes of counting intersections as follows. Recall that if Y is a smooth, closed oriented m -dimensional submanifold of an oriented, closed n -dimensional manifold Z then by [11, Proposition VIII.11.18], the Poincaré dual of the fundamental class of Y , namely $[Y] \in H_m(Z; \mathbb{Z})$, and the Thom class of the normal bundle of Y , namely $\text{Th}(Y) \in H^{n-m}(Z, Z \setminus Y; \mathbb{Z})$, are related by

$$(9.2.17) \quad \text{PD}_M([Y]) = j^* \text{Th}(Y),$$

where $j : (Z, \emptyset) \rightarrow (Z, Z \setminus Y)$ is the inclusion map.

Because the diagonal $\Delta_2 \subset X \times X$ is a smooth submanifold, it has a Thom class and a Poincaré dual,

$$\text{Th}(\Delta_2) \in H^4(X \times X, X \times X \setminus \Delta_2; \mathbb{Z}) \quad \text{and} \quad \text{PD}[\Delta_2] \in H^4(X \times X; \mathbb{Z}),$$

satisfying (9.2.17). The relation (9.2.8) between the incidence locus $\mathcal{V}_\ell(\Delta)$ and Δ_2 implies that it makes sense to define

$$(9.2.18a) \quad T_{\tilde{\mathcal{V}}} := \sum_i (\pi_i \times \text{id}_X)^* \text{Th}(\Delta_2) \in H^4(X^\ell \times X, X^\ell \times X \setminus \tilde{\mathcal{V}}_\ell(\Delta); \mathbb{R}),$$

$$(9.2.18b) \quad \text{PD}[\tilde{\mathcal{V}}] := \sum_i (\pi_i \times \text{id}_X)^* \text{PD}[\Delta_2] \in H^4(X^\ell \times X; \mathbb{R}),$$

where $\pi_i : X^\ell \rightarrow X$ is projection onto the i -th factor and we take images of integer cohomology classes in real cohomology.

The isomorphism (9.2.16) and the \mathfrak{S}_ℓ symmetry of $T_{\tilde{\mathcal{V}}}$ and $\text{PD}[\tilde{\mathcal{V}}]$ allow us to define

$$(9.2.19a) \quad \text{PD}[\mathcal{V}] \in H^4(\text{Sym}^\ell(X) \times X; \mathbb{R}),$$

$$(9.2.19b) \quad T_{\mathcal{V}} \in H^4(\text{Sym}^\ell(X) \times X, \text{Sym}^\ell(X) \times X \setminus \mathcal{V}_\ell(\Delta); \mathbb{R})$$

to be the unique cohomology classes satisfying

$$(9.2.20) \quad (\varpi_\ell \times \text{id}_X)^* \text{PD}[\mathcal{V}] = \text{PD}[\tilde{\mathcal{V}}] \quad \text{and} \quad (\varpi_\ell \times \text{id}_X)^* T_{\mathcal{V}} = T_{\tilde{\mathcal{V}}},$$

where $\varpi_\ell : X^\ell \rightarrow \text{Sym}^\ell(X)$ denotes the projection map. There is a commutative diagram

$$\begin{array}{ccc} (X^\ell \times X, \emptyset) & \xrightarrow{\tilde{j}} & (X^\ell \times X, X^\ell \times X \setminus \tilde{\mathcal{V}}_\ell(\Delta)) \\ \varpi_\ell \times \text{id}_X \downarrow & & \varpi_\ell \times \text{id}_X \downarrow \\ (\text{Sym}^\ell(X) \times X, \emptyset) & \xrightarrow{j} & (\text{Sym}^\ell(X) \times X, \text{Sym}^\ell(X) \times X \setminus \mathcal{V}_\ell(\Delta)) \end{array}$$

where the horizontal maps are inclusions. By the commutativity of this diagram, the definitions (9.2.19), and the relation between the Thom class and Poincaré dual (9.2.17) satisfied by $\text{PD}[\Delta_2]$ and $\text{Th}(\Delta_2)$, we have

$$(9.2.21) \quad \tilde{j}^* T_{\tilde{\mathcal{V}}} = \text{PD}[\tilde{\mathcal{V}}] \quad \text{and} \quad j^* T_{\mathcal{V}} = \text{PD}[\mathcal{V}].$$

For a closed, oriented, smooth submanifold, $Y \subseteq X$, let $\iota_{\mathcal{P}} : \Sigma(X^\ell, \mathcal{P}) \rightarrow \text{Sym}^\ell(X)$ and $\iota_Y : Y \rightarrow X$ denote the inclusion maps. There is an inclusion map of pairs,

$$\begin{array}{ccc} (\Sigma(X^\ell, \mathcal{P}) \times Y, \Sigma(X^\ell, \mathcal{P}) \times Y \setminus \mathcal{V}_\Sigma(\mathcal{P}, Y)) & & \\ \iota_{\mathcal{P}} \times \iota_Y \downarrow & & \\ (\text{Sym}^\ell(X) \times X, \text{Sym}^\ell(X) \times X \setminus \mathcal{V}_\ell(\Delta)) & & \end{array}$$

The following computation, together with Lemma 9.2.2, will be useful in characterizing $\text{PD}[\mathcal{V}]$.

LEMMA 9.2.6. *Let $Y \subseteq X$ be a closed, oriented, smooth submanifold, $T_{\mathcal{V}}$ be the cohomology class defined in (9.2.19b), and $[\phi_{\mathcal{P}, P}]$ be as defined prior to equation (9.2.13). Then,*

$$\langle (\iota_{\mathcal{P}} \times \iota_Y)^* T_{\mathcal{V}}, [\phi_{\mathcal{P}, P}] \rangle = |P|,$$

where $\iota_{\mathcal{P}}$ and ι_Y are the inclusion maps defined above.

PROOF. By construction, the map $\phi_{\mathcal{P}, P}$ is covered by the map $\tilde{\phi}_{\mathcal{P}, P} : D^4 \rightarrow \Delta^\circ(X^\ell, \mathcal{P}) \times Y$ defined in (9.2.13) in the sense that $(\tilde{\pi}_\ell \times \text{id}_Y) \circ \tilde{\phi}_{\mathcal{P}, P} = \phi_{\mathcal{P}, P}$. This gives the equality

$$(\tilde{\pi}_\ell \times \text{id}_Y)_* [\tilde{\phi}_{\mathcal{P}, P}] = [\phi_{\mathcal{P}, P}].$$

The inclusion $\iota_{\mathcal{D}} \times \iota_Y$ is thus covered by an inclusion map

$$\tilde{\iota}_{\mathcal{D}} \times \iota_Y : \Delta^\circ(X^\ell, \mathcal{D}) \times Y \rightarrow X^\ell \times X,$$

in the sense that $(\tilde{\pi}_\ell \times \text{id}_X) \circ (\tilde{\iota}_{\mathcal{D}} \times \iota_Y) = \iota_{\mathcal{D}} \times \iota_Y$. We can therefore compute that

$$\begin{aligned} \langle (\iota_{\mathcal{D}} \times \iota_Y)^* T_{\mathcal{V}}, [\phi_{\mathcal{D}, P}] \rangle &= \langle (\iota_{\mathcal{D}} \times \iota_Y)^* T_{\mathcal{V}}, (\tilde{\pi}_\ell \times \text{id}_Y)_* [\tilde{\phi}_{\mathcal{D}, P}] \rangle \\ &= \langle (\varpi_\ell \times \text{id}_Y)^* (\iota_{\mathcal{D}} \times \iota_Y)^* T_{\mathcal{V}}, [\tilde{\phi}_{\mathcal{D}, P}] \rangle \\ &= \langle (\tilde{\iota}_{\mathcal{D}} \times \iota_Y)^* (\tilde{\pi}_\ell \times \text{id}_X)^* T_{\mathcal{V}}, [\tilde{\phi}_{\mathcal{D}, P}] \rangle \\ &= \langle (\tilde{\iota}_{\mathcal{D}} \times \iota_Y)^* T_{\tilde{\mathcal{V}}}, [\tilde{\phi}_{\mathcal{D}, P}] \rangle \quad (\text{by (9.2.20)}) \\ &= \left\langle \sum_i (\pi_i \times \text{id}_X)^* \text{Th}(\Delta_2), (\tilde{\iota}_{\mathcal{D}} \times \iota_Y)_* [\tilde{\phi}_{\mathcal{D}, P}] \right\rangle \quad (\text{by (9.2.18a)}) \\ &= \sum_i \left\langle \text{Th}(\Delta_2), (\pi_i \circ \iota_{\mathcal{D}}) \times \iota_Y \right\rangle_* [\tilde{\phi}_{\mathcal{D}, P}]. \end{aligned}$$

By the argument in Lemma 9.2.1, the image of $\Delta^\circ(X^\ell, \mathcal{D}) \times Y$ under the map $\pi_i \times \iota_Y$ intersects the diagonal Δ_2 transversely. Hence, the image of the map

$$(\pi_i \circ \iota_{\mathcal{D}}) \times \iota_Y \circ \tilde{\phi}_{\mathcal{D}, P} : D^4 \rightarrow X \times X$$

intersects the diagonal transversely at one point if $i \in P$ and is disjoint from the diagonal if $i \notin P$. This yields the equality

$$\langle \text{Th}(\Delta_2), (\pi_i \circ \iota_{\mathcal{D}}) \times \iota_Y \right\rangle_* [\tilde{\phi}_{\mathcal{D}, P}] = \begin{cases} 1 & \text{if } i \in P, \\ 0 & \text{if } i \notin P, \end{cases}$$

and this in turn gives the equality in the statement of the lemma. \square

The equality on Poincaré duality given in [47, Theorem 30.6], the definition of $\text{PD}[\mathcal{V}]$ following (9.2.19a), the definition of $S^\ell(\beta)$ in Definition 9.2.5, and the isomorphism (9.2.16), yield the following

LEMMA 9.2.7. *For any $\beta \in H_\bullet(X; \mathbb{R})$,*

$$\text{PD}[\mathcal{V}]/\beta = S^\ell(\beta),$$

where $S^\ell(\beta)$ is defined in Definition 9.2.5 and $\text{PD}[\mathcal{V}]$ in (9.2.19a).

9.3. Fundamental class of the virtual link of the ideal moduli space of Seiberg–Witten monopoles

We now define the fundamental class of the virtual link appearing in (9.1.1),

$$[\bar{\mathbf{L}}_{t,s}^{\text{vir}}] \in H_{d(t)-2} \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1; \mathbb{R} \right),$$

where $d(t)$ is the dimension of $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}$. The union of the lower strata in $\bar{\mathbf{L}}_{t,s}^{\text{vir}}$ will be denoted $\mathbf{L}_{t,s}^{\text{sing}} \subset \bar{\mathbf{L}}_{t,s}^{\text{vir}}$.

LEMMA 9.3.1. *Let $\bar{\mathcal{V}}(z)$ and $\bar{\mathcal{W}}$ be the geometric representatives appearing in (9.1.1) and let $\bar{\chi}$ be the obstruction section appearing in Hypothesis 7.8.1. For any neighborhood V of $\mathbf{L}_{t,s}^{\text{sing}}$ in the lower strata of $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$ such that $\text{cl}(V) \cap \bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\chi}^{-1}(0)$ is empty, there is a neighborhood U of the singular strata in $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$ such that the following hold:*

- (1) *There is a deformation retraction r of U onto V which respects the level sets of the function t_N given in (8.1.1).*
- (2) *There is a compact topological manifold with boundary, $\hat{\mathbf{L}}$, given as the boundary of a smooth manifold with corners with $\mathbf{L}_{t,s}^{\text{vir}} \setminus U \subset \hat{\mathbf{L}} \subset \bar{\mathbf{L}}_{t,s}^{\text{vir}}$.*
- (3) *The intersection $\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\chi}^{-1}(0) \cap U$ is empty.*

PROOF. Let $\mathcal{M}_{t,s}^{\text{sing},*}/S^1 \subset \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$ be the union of the singular strata. The local Whitney stratified structure of the spliced-ends moduli space, specifically the property described in Lemma 5.9.2, implies that the union of the lower strata is locally a neighborhood deformation retraction and thus a neighborhood deformation retraction (NDR) (combine the results [40, Proposition 5] and [92, Corollary 3.3.11]). The fact that this deformation retraction respects the level sets of t_N follows by observing that the deformation retractions defined in Lemma 5.9.2 are invariant with respect to the frame action. The NDR structure gives a cofinal sequence of neighborhoods of $\mathcal{M}_{t,s}^{\text{sing},*}/S^1$ which deformation retract onto $\mathcal{M}_{t,s}^{\text{sing},*}/S^1$. Because $\text{cl}(V) \cap \bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\chi}^{-1}(0)$ is empty and $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$ is normal, we can find a neighborhood deformation retraction, $r : N \rightarrow \mathcal{M}_{t,s}^{\text{sing},*}/S^1$ onto $\mathcal{M}_{t,s}^{\text{sing},*}/S^1$, such that $U = r^{-1}(V)$ satisfies the conclusions (1) and (3) of the lemma. Define $\hat{\mathbf{L}}$ to be the complement in $\mathbf{L}_{t,s}^{\text{vir}}$ of tubular neighborhoods (in the Thom–Mather sense) of the local strata of $\mathcal{M}_{t,s}^{\text{sing},*}/S^1$. By choosing these tubular neighborhoods to be contained in N , the resulting topological manifold with boundary will satisfy the conclusion (2). \square

For the subspaces U and V appearing in Lemma 9.3.1, the manifold with boundary $\hat{\mathbf{L}}$ defined in Lemma 9.3.1 has a fundamental class,

$$(9.3.1) \quad [\hat{\mathbf{L}}, \partial\hat{\mathbf{L}}] \in H_{d(t)-2}(\mathbf{L}_{t,s}^{\text{vir}} \cup U^\circ, U^\circ; \mathbb{Z}),$$

where $U^\circ := U \setminus \mathcal{M}_{t,s}^{\text{sing},*}/S^1$. Note that there is an excision isomorphism,

$$\iota_* : H_\bullet(\mathbf{L}_{t,s}^{\text{vir}} \cup U^\circ, U^\circ; \mathbb{Z}) \cong H_\bullet(\bar{\mathbf{L}}_{t,s}^{\text{vir}} \cup U, U; \mathbb{Z}).$$

Because V is contained in the codimension-four subspace $\mathcal{M}_{t,s}^{\text{sing},*}/S^1$, the set V has dimension three less than $\mathbf{L}_{t,s}^{\text{vir}}$. Because U deformation retracts onto V , we have $H_k(U; \mathbb{Z}) \cong H_k(V; \mathbb{Z}) = 0$ for $k > \dim \hat{\mathbf{L}} - 3$. Hence, the inclusion map of pairs,

$$(9.3.2) \quad \bar{j} : (\bar{\mathbf{L}}_{t,s}^{\text{vir}} \cup U, \emptyset) \rightarrow (\bar{\mathbf{L}}_{t,s}^{\text{vir}} \cup U, U),$$

induces an isomorphism,

$$\bar{j}_* : H_{d(t)-2}(\bar{\mathbf{L}}_{t,s}^{\text{vir}} \cup U; \mathbb{Z}) \cong H_{d(t)-2}(\bar{\mathbf{L}}_{t,s}^{\text{vir}} \cup U, U; \mathbb{Z}).$$

We now define

$$(9.3.3) \quad [\bar{\mathbf{L}}_{t,s}^{\text{vir}}] := j_*^{-1} \iota_* [\hat{\mathbf{L}}, \partial\hat{\mathbf{L}}] \in H_{d(t)-2}(\bar{\mathbf{L}}_{t,s}^{\text{vir}} \cup U; \mathbb{Z})$$

to be the fundamental class of the virtual link.

9.4. Computation of the μ -classes

The geometric representatives $\bar{\mathcal{V}}(z)$ and $\bar{\mathcal{W}}$ are dual to the cohomology classes $\mu_p(z)$ and μ_c defined in Section 2.4. We now compute the pullbacks $\gamma_{\mathcal{M}}^* \mu_p(z)$ and $\gamma_{\mathcal{M}}^* \mu_c$ in terms of the cohomology classes defined in Section 9.2.3.

9.4.1. Geometric representatives and cocycles.

We first recall the

DEFINITION 9.4.1. (See Kronheimer and Mrowka [53, p. 588].) Let Z be a smooth manifold. A *geometric representative* for $\mu \in H^d(Z; \mathbb{R})$ is a closed, smoothly stratified subspace \mathcal{V} of Z together with a real coefficient, q , the *multiplicity*, with the following properties:

- (1) The top stratum \mathcal{V}_0 of \mathcal{V} is a codimension- d , smooth submanifold of Z with an oriented normal bundle.
- (2) All other strata of \mathcal{V} have codimension $d + 2$ or greater in Z .
- (3) The pairing of μ with a homology class h of dimension d is obtained by choosing a smooth singular cycle σ representing h , transverse to all strata of \mathcal{V} , then taking q times the count (with signs) of the intersection points between the cycle and the top stratum of \mathcal{V} ,

$$\langle \mu, h \rangle = q \cdot \#(\mathcal{V}_0 \cap \sigma).$$

A choice of a geometric representative \mathcal{V} for a cohomology class μ specifies a singular cocycle c representing μ , supported on \mathcal{V} , as described in the following lemma.

LEMMA 9.4.2. *Let Z be a smooth manifold and let (\mathcal{V}, q) be a geometric representative for a real cohomology class μ of dimension d on Z . Then there is a class $c \in H^d(Z, Z \setminus \mathcal{V}; \mathbb{R})$ such that $j_{\mathcal{V}}^* c = \mu$, where*

$$j_{\mathcal{V}} : (Z, \emptyset) \rightarrow (Z, Z \setminus \mathcal{V})$$

is the inclusion map of pairs.

PROOF. Let $\Delta_{\bullet}^{\infty}(Z)$ denote the chain complex of smooth singular chains [8, p. 291] with $Z_d^{\infty}(Z) \subset \Delta_d^{\infty}(Z)$ and $B_d^{\infty}(Z) \subset \Delta_d^{\infty}(Z)$ denoting the submodules of cycles and boundaries respectively. As we are considering real cohomology, we shall view $\Delta_{\bullet}^{\infty}(Z)$ as a real vector space rather than as a \mathbb{Z} -module. Property (3) in Definition 9.4.1 implies that a geometric representative (\mathcal{V}, q) for $\mu \in H^d(Z; \mathbb{R})$ defines an element of $\text{Hom}(Z_d^{\infty}(Z); \mathbb{R})$ which is a representative of the cohomology class μ . However, the intersection number $\#(\mathcal{V}_0 \cap \sigma)$ in Property (3) is well-defined if σ is a smooth, singular chain whose boundary lies in $Z \setminus \mathcal{V}$, that is, $\sigma \in \partial^{-1}(\Delta_{d-1}^{\infty}(Z \setminus \mathcal{V}))$. The intersection number $\#(\mathcal{V}_0 \cap \sigma)$ vanishes if σ is a boundary or an element of $\Delta_d^{\infty}(Z \setminus \mathcal{V})$. Hence, a geometric representative (\mathcal{V}, q) of $\mu \in H^d(Z; \mathbb{R})$ defines an element of

$$(9.4.1) \quad \text{Hom}(\partial^{-1}(\Delta_{d-1}^{\infty}(Z \setminus \mathcal{V})) / (B_d^{\infty}(Z) + \Delta_d^{\infty}(Z \setminus \mathcal{V})); \mathbb{R}).$$

We now argue that (9.4.1) is isomorphic to $H^d(Z, Z \setminus \mathcal{V}; \mathbb{R})$. By the de Rham Theorem (see the discussion in [8, p. 291]), there is a functorial isomorphism between $H^{\bullet}(Z; \mathbb{R})$ and the homology of the complex $\text{Hom}(\Delta_{\bullet}^{\infty}(Z); \mathbb{R})$. The Universal Coefficient Theorem then identifies $H_{\bullet}(Z; \mathbb{R})$ with the homology of the complex $\Delta_{\bullet}^{\infty}(Z)$. Thus, we have isomorphisms,

$$(9.4.2) \quad H_{\bullet}(Z; \mathbb{R}) \cong H_{\bullet}(\Delta_{\bullet}^{\infty}(Z); \mathbb{R}), \quad H_{\bullet}(Z \setminus \mathcal{V}; \mathbb{R}) \cong H_{\bullet}(\Delta_{\bullet}^{\infty}(Z \setminus \mathcal{V}); \mathbb{R}).$$

The Five Lemma then identifies $H_{\bullet}(Z, Z \setminus \mathcal{V}; \mathbb{R})$ with the homology of the complex

$$\Delta_{\bullet}^{\infty}(Z) / \Delta_{\bullet}^{\infty}(Z \setminus \mathcal{V}),$$

and thus,

$$H_d(Z, Z \setminus \mathcal{V}; \mathbb{R}) \cong \partial^{-1}(\Delta_{d-1}^\infty(Z \setminus \mathcal{V})) / (B_d^\infty(Z) + \Delta_d^\infty(Z \setminus \mathcal{V})).$$

The Universal Coefficient Theorem then yields the required isomorphism between (9.4.1) and $H^d(Z, Z \setminus \mathcal{V}; \mathbb{R})$. \square

9.4.2. Cocycles as pullbacks. For $z = \beta_1 \cdots \beta_r \in \mathbb{A}(X)$, where $\beta_i \in H_{\dim \beta_i}(X; \mathbb{R})$, we now review the definition of the cocycles defined by $\bar{\mathcal{V}}(\beta_i)$ and $\bar{\mathcal{W}}$.

Let $T \subset X$ be a smooth submanifold with $q[T] = \beta$ for $q \in \mathbb{R}$ and let $\nu(\beta) \subset X$ be a tubular neighborhood of T . Let $\mathcal{B}^*(\nu(\beta))$ be the space of irreducible connections on the bundle $\mathfrak{g}_{\mathfrak{t}}|_{\nu(\beta)}$ (see [53, Section 2 (ii)] or [29, Section 3.2.2]) and (abusing notation slightly) let

$$r_\beta : \mathcal{M}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}} / S^1 \ni [A, \Phi] \mapsto [\hat{A}|_{\nu(\beta)}] \in \mathcal{B}^*(\nu(\beta)),$$

be the composition of the gluing map $\gamma_{\mathcal{M}}$ and the restriction map. For $\mathcal{I}(\nu(\beta))$ as defined in (9.2.6), let

$$(9.4.3) \quad \bar{r}_\beta : \bar{\mathcal{M}}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}, *} / S^1 \setminus \mathcal{I}(\nu(\beta)) \ni [A, \Phi, \mathbf{x}] \mapsto [\hat{A}|_{\nu(\beta)}] \in \mathcal{B}^*(\nu(\beta))$$

be the extension of r_β . We will denote the image of r_β by \mathcal{M}_β and the image of \bar{r}_β by $\bar{\mathcal{M}}_\beta$.

Let $\mathcal{V}_T \subset \mathcal{B}^*(\nu(\beta))$ be the geometric representative for the cohomology class $\mu_p(\beta)$ defined in [53, Section 2 (ii)]. From the definition in [29, Section 3.2.3], it follows that

$$\bar{\mathcal{V}}(\beta) \cap \left(\bar{\mathcal{M}}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}, *} / S^1 \setminus \mathcal{I}(\nu(\beta)) \right) = \bar{r}_\beta^{-1}(\mathcal{V}_T).$$

Note that in [29, Section 3.2.3], a tubular neighborhood of the union of T and certain loops (a “suitable” neighborhood in the sense of [53]) were used instead of the tubular neighborhood $\nu(\beta)$ in defining $\bar{\mathcal{V}}(\beta)$. However, by [30, Lemma 5.5] the geometric representative $\bar{\mathcal{V}}(\beta)$ can be replaced with geometric representatives pulled back from tubular rather than suitable neighborhoods of T , when computing the intersection numbers in (9.1.1). Let

$$(9.4.4) \quad [c_{T, \beta}] \in H^{4-\dim \beta}(\bar{\mathcal{M}}_\beta, \bar{\mathcal{M}}_\beta \setminus \mathcal{V}_T; \mathbb{R})$$

be the cohomology class defined by the geometric representative \mathcal{V}_T as described in Lemma 9.4.2 and define

$$(9.4.5) \quad [c_\beta] := \bar{r}_\beta^*[c_{T, \beta}] \in H^{4-\bullet} \left(\bar{\mathcal{M}}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}, *} / S^1 \setminus \mathcal{I}(\nu(\beta)), \bar{\mathcal{M}}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}, *} / S^1 \setminus (\mathcal{I}(\nu(\beta)) \cup \bar{\mathcal{V}}(\beta)); \mathbb{R} \right).$$

A similar construction defines a cocycle representing $\bar{\mathcal{W}}$ of the cohomology class μ_c . For the configuration space $\mathcal{C}_{\mathfrak{t}}^{*, 0}(\nu(x))$ of pairs on the restriction of the spin^u structure \mathfrak{t} to $\nu(x)$ (see [29, Section 3.2.2]), we define

$$(9.4.6) \quad \bar{r}_x : \bar{\mathcal{M}}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}, *} / S^1 \setminus \mathcal{I}(\nu(x)) \rightarrow \mathcal{C}_{\mathfrak{t}}^{*, 0}(\nu(x)) / S^1$$

to be the composition of the gluing map $\gamma_{\mathcal{M}}$ with the restriction map. In [29, Equation (3.14)], the geometric representative $\bar{\mathcal{W}}$ is defined by

$$\bar{\mathcal{W}} \cap \left(\bar{\mathcal{M}}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}, *} / S^1 \setminus \mathcal{I}(\nu(x)) \right) = \bar{r}_x^{-1}(\mathcal{W}_T),$$

where \mathcal{W}_T is the zero-locus of a section of an appropriate line bundle. Define

$$(9.4.7) \quad [c_{T, W}] \in H^2(\mathcal{C}_{\mathfrak{t}}^{*, 0}(\nu(x)) / S^1, \mathcal{C}_{\mathfrak{t}}^{*, 0}(\nu(x)) / S^1 \setminus \mathcal{W}_T; \mathbb{R})$$

to be the cocycle obtained from the geometric representative \mathscr{W}_T by Lemma 9.4.2 and define

$$(9.4.8) \quad [c_{\mathscr{W}}] := \bar{r}_x^*[c_{T,W}] \in H^2(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(x)), \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \bar{\mathscr{W}}; \mathbb{R})$$

to be its pullback.

9.4.3. Computations of cocycles. In Section 9.6.1, we will write the intersection number appearing in (9.1.1) in terms of a cup product of $[c_{\mathscr{W}}]$ and $[c_{\beta_i}]$. We will then compute this product in terms of the cohomology classes defined in Section 9.2.3. For this purpose, we will need to relate the relative cohomology classes $[c_{\mathscr{W}}]$ and $[c_{\beta_i}]$ to absolute cohomology classes. Specifically, we will need to compute the cohomology class $\bar{j}_{\beta}^* \bar{r}_{\beta}^*[c_{T,\beta}]$ in terms of the cohomology classes defined in Section 9.2.3, where

$$(9.4.9) \quad \bar{j}_{\beta} : \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(\beta)), \emptyset \right) \rightarrow \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(\beta)), \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus (\mathcal{I}(\nu(\beta)) \cup \bar{\mathscr{V}}_{\beta}) \right)$$

is the inclusion of pairs. To that end, we consider the bundle

$$(9.4.10) \quad \mathbb{F}_T := \mathscr{A}_{\nu(T)}^* \times_{\mathscr{G}_{\nu(T)}} \mathfrak{g}_{\mathfrak{t}}|_T \rightarrow \mathscr{B}_{\nu(T)}^* \times T.$$

If $\bar{\mathcal{M}}_{\beta}$ is as defined in the paragraph following (9.4.3), then the construction of the geometric representative \mathscr{V}_T implies that

$$j_T^*[c_{T,\beta}] = p_1(\mathbb{F}_T)/[T],$$

where

$$j_T : (\bar{\mathcal{M}}_{\beta} \times T, \emptyset) \rightarrow (\bar{\mathcal{M}}_{\beta} \times T, \bar{\mathcal{M}}_{\beta} \times T \setminus \mathscr{V}_T)$$

is the inclusion of pairs. To compute $\bar{r}_{\beta}^* j_T^*[c_{T,\beta}]$, we shall compare the Pontrjagin classes of the bundles

$$(9.4.11) \quad (\bar{r}_{\beta} \times \text{id}_T)^* \mathbb{F}_T \quad \text{and} \quad (\text{id}_{\bar{\mathcal{M}}} \times \iota_T)^* \mathbb{F}_{t,s}^{\text{vir}},$$

where

$$\text{id}_{\bar{\mathcal{M}}} \times \iota_T : \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \times T \rightarrow \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \times X$$

is the inclusion map and $\mathbb{F}_{t,s}^{\text{vir}} \rightarrow \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \times X$ is the bundle defined by restricting

$$(9.4.12) \quad \mathbb{F}_{t,s}^{\text{vir}} = \bigcup_{\mathscr{P}} \left(\tilde{N}_{\mathfrak{t}(\ell),s}(\delta) \times \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathscr{P}) \times_{\mathscr{G}_{\mathfrak{s}} \times S^1} \mathfrak{g}_{\mathfrak{t}(\ell)} \right)$$

to $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \times X$. We begin with the computation of $p_1(\mathbb{F}_{t,s}^{\text{vir}})$.

LEMMA 9.4.3. *Assume the spin^u structure $\mathfrak{t}(\ell)$ admits a splitting $\mathfrak{t}(\ell) = \mathfrak{s} \oplus \mathfrak{s} \otimes L$ as described in Section 2.1.1, where $L \rightarrow X$ is a complex line bundle. Let $\mathbb{F}_{t,s}^{\text{vir}} \rightarrow \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \times X$ be the bundle defined in (9.4.12) and $\mathbb{L}_{\mathfrak{s}}$ the line bundle given in Definition 9.2.4. Let $\pi_{X,2} : \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \times X \rightarrow X$ be the projection onto the second factor. Then*

$$(9.4.13) \quad p_1(\mathbb{F}_{t,s}^{\text{vir}}) = (2(\pi_{\mathfrak{s}} \times \text{id}_X)^* c_1(\mathbb{L}_{\mathfrak{s}}) - \nu + \pi_{X,2}^* c_1(L))^2.$$

PROOF. Equation (9.4.13) follows from [30, Lemma 4.7]. \square

To compare the Pontrjagin classes in (9.4.11), we first describe a subspace on which the bundles are isomorphic. Let $\mathcal{V}_{\mathcal{P}}(\Delta, T) \subset \Sigma(X^\ell, \mathcal{P}) \times T$ be the incidence locus defined in (9.2.10) (with $Y = T$). Let

$$(9.4.14) \quad \pi(T, \mathcal{P}) : \bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \subseteq \tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times_{\mathcal{G}_s \times S^1} \bar{\mathrm{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \Sigma(X^\ell, \mathcal{P})$$

be the composition of the projection $\pi(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ defined in (6.7.2) and the projection

$$N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times \Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, g_{\mathcal{P}}) \cong N_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times \Sigma(X^\ell, \mathcal{P}) \rightarrow \Sigma(X^\ell, \mathcal{P}),$$

where the identification $\Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, g_{\mathcal{P}}) \cong \Sigma(X^\ell, \mathcal{P})$ is given in (6.2.8). Define

$$(9.4.15) \quad \mathcal{O}(T, \mathcal{P}) = (\pi(T, \mathcal{P}) \times \mathrm{id}_T)^{-1} \left(\Sigma(X^\ell, \mathcal{P}) \times T \setminus \mathcal{V}_{\mathcal{P}}(\Delta, T) \right).$$

We now have the following

LEMMA 9.4.4. *The restrictions of the bundles $(\bar{r}_\beta \times \mathrm{id}_T)^* \mathbb{F}_T$ and $(\mathrm{id}_{\mathcal{M}} \times \iota_T)^* \mathbb{F}_{\mathfrak{t}, \mathfrak{s}}^{\mathrm{vir}}$ to*

$$\mathcal{O}(T) := \bigcup_{\mathcal{P}} \mathcal{O}(T, \mathcal{P})$$

are isomorphic.

PROOF. The construction of the bundle $\mathfrak{g}_{\mathfrak{t}}$ by splicing together the background bundle $\mathfrak{g}_{\mathfrak{t}(\ell)}$ and the bundles $\mathfrak{g}_P \rightarrow S^4$ in (6.4.6) and the definition of $\mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ in (9.4.15) as the complement of the points where any of the splicing points lie on T give an isomorphism of the restriction of the bundles to each subspace $\mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$. These isomorphisms agree on the overlaps $\mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \cap \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}')$ and hence define the desired global isomorphism. \square

Lemma 9.4.4 implies that the difference,

$$(9.4.16) \quad (\bar{r}_\beta \times \mathrm{id}_T)^* p_1(\mathbb{F}_T) - (\mathrm{id}_{\mathcal{M}} \times \iota_T)^* \mathbb{F}_{\mathfrak{t}, \mathfrak{s}}^{\mathrm{vir}},$$

lies in the image of the map $\bar{\mathcal{J}}_{\mathcal{O}, \beta}^*$, where $\bar{\mathcal{J}}_{\mathcal{O}, \beta}$ is the inclusion map of pairs,

$$\begin{array}{c} \left(\left(\bar{\mathcal{M}}_{\mathfrak{t}, \mathfrak{s}}^{\mathrm{vir},*} / S^1 \setminus \mathcal{I}(\beta) \right) \times T, \emptyset \right) \\ \bar{\mathcal{J}}_{\mathcal{O}, \beta} \downarrow \\ \left(\left(\bar{\mathcal{M}}_{\mathfrak{t}, \mathfrak{s}}^{\mathrm{vir},*} / S^1 \setminus \mathcal{I}(\beta) \right) \times T, \mathcal{O}(T) \right) \end{array}$$

We can thus calculate generators of the image of $\bar{\mathcal{J}}_{\mathcal{O}, \beta}^*$ in the next lemma.

LEMMA 9.4.5. *Let T be a closed, connected, oriented, smooth submanifold of X with $[T] = \beta \in H_\bullet(X; \mathbb{R})$ and let $\mathcal{O}(T)$ be as defined in Lemma 9.4.4. Then the relative homology,*

$$H_k \left(\left(\bar{\mathcal{M}}_{\mathfrak{t}, \mathfrak{s}}^{\mathrm{vir},*} / S^1 \setminus \mathcal{I}(\beta) \right) \times T, \mathcal{O}(T); \mathbb{Z} \right),$$

is trivial for $k < 4$, while for $k = 4$ it is generated by the images of disks,

$$\psi_{\mathcal{P}, P} : D^4 \rightarrow \left(\bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \setminus \mathcal{I}(\beta) \right) \times T \subset \bar{\mathcal{M}}_{\mathfrak{t}, \mathfrak{s}}^{\mathrm{vir},*} / S^1 \times T,$$

for a partition \mathcal{P} of N_ℓ and $P \in \mathcal{P}$, satisfying

- (1) *The image of $\psi_{\mathcal{P}, P}$ is in the top stratum of $\bar{\mathcal{M}}_{\mathfrak{t}, \mathfrak{s}}^{\mathrm{vir},*} / S^1 \times T$,*

(2) For the map $\phi_{\mathcal{P},P}$ defined in Lemma 9.2.2 and the map $\pi(T, \mathcal{P})$ defined in (9.4.14),

$$(9.4.17) \quad (\pi(T, \mathcal{P}) \times \text{id}_T) \circ \psi_{\mathcal{P},P} = \phi_{\mathcal{P},P}.$$

PROOF. By the Whitney stratification of $\bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \times T$, we can require any relative homology class to be represented by a chain which intersects $(\pi(T, \mathcal{P}) \times \text{id}_T)^{-1}(\mathcal{V}_{\mathcal{P}}(\Delta, T))$ transversely (compare the proof of [35, Lemma 6.2]) and thus has the same image in the relative homology group as $\psi_{\mathcal{P},P}$. The conclusion follows by observing that the codimension of the intersection of $(\pi(T, \mathcal{P}) \times \text{id}_T)^{-1}\mathcal{V}_{\mathcal{P}}(\Delta, T)$ with all lower strata is greater than four. \square

We then compute.

LEMMA 9.4.6. *Continue the notation of Lemmas 9.4.5 and 9.4.4. For any partition \mathcal{P} of N_ℓ , the following equality holds:*

$$(9.4.18) \quad \langle (\bar{r}_\beta \times \text{id}_T)^* p_1(\mathbb{F}_T) - (\text{id}_{\mathcal{M}} \times \iota_T)^* p_1(\mathbb{F}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}), (\bar{\mathcal{O}}, \beta)_* [\psi_{\mathcal{P},P}] \rangle = -4|P|.$$

PROOF. We begin by giving an explicit construction of a map $\psi_{\mathcal{P},P} : D^4 \rightarrow \bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \times T$ generating the relative homology as in Lemma 9.4.5. Let $f : D^4 \rightarrow X$ be a smooth embedding with $f(0) = x_T \in T$. For $Q \in \mathcal{P}$ with $Q \neq P$, fix points $y_Q \in X$ with $y_Q \notin f(D^4)$. Define the map $\psi_{\mathcal{P},P}$, for $v \in D^4$, by

$$(9.4.19) \quad \psi_{\mathcal{P},P}(v) := \left(\left[(A_0, \Phi_0), \left[(F_Q^{\mathfrak{g}}(v), F_Q^T(v)), [A_Q, F_Q^{\mathfrak{s}}] \right]_{Q \in \mathcal{P}} \right], x_T \right),$$

where points in $\bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ are described in Notation 6.4.1 and

- (1) $(A_0, \Phi_0) \in \tilde{N}_{\mathfrak{t}(\ell)}(\delta)$, and $[A_Q, F_Q^{\mathfrak{s}}] \in \bar{M}_{\text{SPL}, |Q|}^{\mathfrak{s}, \mathfrak{h}}$, and $x_T \in T$ do not depend on v ,
- (2) For $Q \neq P$, then $F_Q^{\mathfrak{g}}(v) \in \text{Fr}(\mathfrak{g}_{\mathfrak{t}(\ell)})|_{y_Q}$ and $F_Q^T(v) \in \text{Fr}(TX)|_{y_Q}$ are fixed frames, independent of $v \in D^4$,
- (3) The map $D^4 \rightarrow \text{Fr}(\mathfrak{g}_{\mathfrak{t}(\ell)}) \times_X \text{Fr}(TX)$ defined by $v \mapsto (F_P^{\mathfrak{g}}(v), F_P^T(v))$ is a smooth lift of $f : D^4 \rightarrow X$.

Then the map $\phi_{\mathcal{P},P} = (\pi(T, \mathcal{P}) \times \text{id}_T) \circ \psi_{\mathcal{P},P}$ as in (9.4.17) is transverse to the incidence locus $\mathcal{V}_{\mathcal{P}}(\Delta, T)$ defined in (9.2.10) and thus is one of the generators of homology appearing in Lemma 9.2.2. Hence, $\psi_{\mathcal{P},P}$ satisfies the conditions of Lemma 9.4.5.

Let $\pi_{\mathcal{U}} : \bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \times T \rightarrow \bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ and $\pi_T : \bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \times T \rightarrow T$ be the projection maps. Define

$$\begin{aligned} \mathbb{F}_T(x_T) &:= \mathbb{F}_T|_{\mathcal{B}_{\nu(T)}^* \times \{x_T\}}, \\ \mathbb{F}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(x_T) &:= \bigcup_{\mathcal{P}} \left(\tilde{N}_{\mathfrak{t}(\ell), \mathfrak{s}}(\delta) \times \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \mathfrak{g}_{\mathfrak{t}(\ell)}|_{x_T} \right). \end{aligned}$$

Because $\pi_T \circ \psi_{\mathcal{P},P} = x_T$ from (9.4.19), then the definitions (9.4.10) and (9.4.12) yield

$$(9.4.20a) \quad \mathbb{F}_T(D^4) := \psi_{\mathcal{P},P}^* (\bar{r}_\beta \times \text{id}_T)^* \mathbb{F}_T = (\pi_{\mathcal{U}} \circ \psi_{\mathcal{P},P})^* \bar{r}_\beta^* \mathbb{F}_T(x_T),$$

$$(9.4.20b) \quad \mathbb{F}^{\text{vir}}(D^4) := \psi_{\mathcal{P},P}^* (\text{id}_{\mathcal{M}} \times \iota_T)^* \mathbb{F}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}} = (\pi_{\mathcal{U}} \circ \psi_{\mathcal{P},P})^* \mathbb{F}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(x_T).$$

Parallel translation with respect to the connection \hat{A}_0 of the frame $F_P^{\mathfrak{g}}(v)$ appearing in (9.4.19) from $f(v)$ to x_T then defines a trivialization of $\mathbb{F}^{\text{vir}}(D^4)$. If $\mathcal{O}(T, \mathcal{P})$ is as defined in (9.4.15), then $\psi_{\mathcal{P},P}(D^4 \setminus \{0\}) \subset \mathcal{O}(T, \mathcal{P})$ and so the isomorphism of Lemma 9.4.4 gives an isomorphism

$\mathbb{F}_T(D^4)|_{\partial D^4} \cong \mathbb{F}^{\text{vir}}(D^4)|_{\partial D^4}$. Therefore, to compute the difference (9.4.18), we only need to compute the Pontrjagin class of $\mathbb{F}_T(D^4)$ relative to the trivialization over ∂D^4 given by the trivialization of $\mathbb{F}^{\text{vir}}(D^4)$ and the isomorphism between the two bundles over ∂D^4 .

Let $\mathfrak{g}_t(v) \rightarrow X$ be the $\text{SO}(3)$ bundle constructed in (6.4.6) using the data $(\pi_{\mathcal{Q}} \circ \psi_{\mathcal{D},P})(v)$. The definition (9.4.19) implies that $\mathfrak{g}_t(v)$ is constructed by attaching bundles $\mathfrak{g}_Q \rightarrow S^4$ to $\mathfrak{g}_t(\ell)$ at points $y_{P'}$ for $Q = P' \neq P$ and at $f(v)$ for $Q = P$.

As in Section 6.4.1, let $\varphi_n(B(\lambda)) \subset S^4 \setminus \{s\}$ be the image of a ball of radius λ centered at the origin under the conformal diffeomorphism $\varphi_n : \mathbb{R}^4 \rightarrow S^4 \setminus \{s\}$ defined by inverse stereographic projection. Consider the map $F : D^4 \times B(\lambda) \rightarrow X$ defined by the property that $F(v, \cdot) : B(\lambda) \rightarrow X$ is the smooth embedding defined by the gluing data $\pi_{\mathcal{Q}} \circ \psi_{\mathcal{D},P}(v)$, identifying the ball $B(\lambda)$ with a ball around $f(v)$ using the exponential map around $f(v)$ used in the construction of the crude splicing map in Section 6.4.2 (given by the flattened metric and tangent frames specified by $\psi_{\mathcal{D},P}(v)$) and φ_n^{-1} . Hence, if $n \in S^4$ denotes the North Pole, then $F(v, n) = f(v)$ for all $v \in D^4$. By the Implicit Function Theorem, there is a smooth map $h : D^4 \rightarrow B(\lambda)$ with the property that $F(v, h(v)) = x_T$. The equality $F(v, n) = f(v)$ implies that $dF_{(0,n)}(\delta v, 0) = df_0(\delta v)$. The definition of $F(0, \cdot) : B(\lambda) \rightarrow X$ as the embedding given by composing φ_n^{-1} with the exponential map specified above implies that $dF_{(0,n)}(0, \cdot) : T_n S^4 \rightarrow T_{x_T} X$ is an orientation-preserving isomorphism. The definition $F(v, h(v)) = x_T$ implies that

$$0 = dF_{(0,n)}(\delta v, dh_0(\delta v)) = df_0(\delta v) + dF_{(0,n)}(0, dh_0(\delta v)),$$

and thus, because $T_0 D^4$ is even-dimensional, dh_0 is an orientation-preserving isomorphism. Hence, $h : (D^4, D^4 \setminus \{0\}) \rightarrow (B(\lambda), B(\lambda) \setminus \{n\})$ is a degree-one map.

By the construction of $\mathfrak{g}_t(v)$ and the isomorphism (9.4.20a), the fiber $\mathfrak{g}_t(v)|_{x_T}$ is identified with $\mathfrak{g}_P|_{h(v)}$, giving

$$(9.4.21) \quad \mathbb{F}_T(D^4) \cong h^* \mathfrak{g}_P.$$

Recall that we trivialized $\mathbb{F}^{\text{vir}}(D^4)$ by parallel translating $F_P^{\mathfrak{g}}(v)$ from $f(v)$ to x_T . In the construction of $\mathfrak{g}_t(v)$, this parallel translation of $F_P^{\mathfrak{g}}(v)$ is identified with the parallel translation of the frame $F^s \in \text{Fr}(\mathfrak{g}_P)|_s$, where $s \in S^4$ is the South Pole, from s to $h(v)$. Therefore, the trivialization of $\mathbb{F}_T(D^4)|_{\partial D^4}$ given by the trivialization of $\mathbb{F}^{\text{vir}}(D^4)$ and the isomorphism $\mathbb{F}_T(D^4)|_{\partial D^4} \cong \mathbb{F}^{\text{vir}}(D^4)|_{\partial D^4}$ equals the pullback by h of the trivialization of \mathfrak{g}_P given by parallel translations of the frame F^s . Consequently, the relative Pontrjagin class of $h^* \mathfrak{g}_P$ equals the pullback by h of the Pontrjagin class of \mathfrak{g}_P relative to the trivialization given by parallel translations of the frame F^s and this in turn equals the absolute Pontrjagin class of \mathfrak{g}_P . Because h is a degree-one map, the relative Pontrjagin class of the bundle (9.4.20a) equals $p_1(\mathfrak{g}_P) = -4|P|$, as required. \square

COROLLARY 9.4.7. *For any submanifold T of X , the equality,*

$$(9.4.22) \quad (\bar{r}_\beta \times \text{id}_T)^* p_1(\mathbb{F}_T) - (\text{id}_{\mathcal{M}} \times \iota_T)^* p_1(\mathbb{F}_{t,s}^{\text{vir}}) = -4(\pi_X \times \text{id}_T)^* \text{PD}[\mathcal{V}],$$

holds in real cohomology.

PROOF. By Lemmas 9.4.6 and 9.2.6, the cohomology classes appearing on the two sides of equation (9.4.22) agree on the generators of the relative homology described in Lemma 9.4.5. Hence, by the Universal Coefficient Theorem the equality holds in real cohomology. \square

Let $\gamma_1, \dots, \gamma_b \in H_1(X; \mathbb{Z})$ be a basis for $H_1(X; \mathbb{Z})/\text{Tor}$ with dual basis $\gamma_1^*, \dots, \gamma_b^* \in H^1(X; \mathbb{R})$ satisfying $\langle \gamma_i^*, \gamma_j \rangle = \delta^{ij}$ (the Kronecker delta). Let $\gamma_1^{J,*}, \dots, \gamma_b^{J,*} \in H^1(M_5; \mathbb{R})$ be the related basis

as defined in [28, Definition 2.21 and Corollary 2.22] (denoted there by $\mathbf{r}^*\gamma_i^{J,*}$). We now define expressions in the cohomology classes from Section 9.2.3 which will be related to the cocycles previously defined.

DEFINITION 9.4.8. For $x \in H_0(X; \mathbb{R})$, define $\bar{\mu}_p(x) \in H^4(\mathcal{M}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir},*}/S^1; \mathbb{R})$ by

$$(9.4.23) \quad \bar{\mu}_p(x) := -\frac{1}{4}(2\mu_{\mathfrak{s}} - \nu)^2 + S^\ell(x),$$

where ν is as in Definition 9.2.3. For $\gamma \in H_1(X; \mathbb{R})$ written as $\gamma = \sum_i q_i \gamma_i$, where $q_i \in \mathbb{R}$ and $\gamma_i \in H_1(X; \mathbb{Z})$ are the generators above, define

$$(9.4.24) \quad \bar{\mu}_p(\gamma) := -\frac{1}{2} \sum_i q_i (2\mu_{\mathfrak{s}} - \nu) \gamma_i^{J,*} + S^\ell(\gamma) \in H^3(\mathcal{M}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir},*}/S^1; \mathbb{R}).$$

For $\beta \in H_2(X; \mathbb{R})$, define $\bar{\mu}_p(\beta) \in H^2(\mathcal{M}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir},*}/S^1; \mathbb{R})$ by

$$(9.4.25) \quad \bar{\mu}_p(\beta) := -\frac{1}{2}(2\mu_{\mathfrak{s}} - \nu) \langle c_1(\mathfrak{t}) - c_1(\mathfrak{s}), \beta \rangle - 2 \sum_{i < j} \mu_{\mathfrak{s}}(\gamma_i \gamma_j) \langle \gamma_i^* \smile \gamma_j^*, \beta \rangle + S^\ell(\beta),$$

where $S^\ell(\beta)$ is as in Definition 9.2.5.

For $Y \in H_3(X; \mathbb{R})$, define $\bar{\mu}_p(Y) \in H^1(\mathcal{M}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir},*}/S^1; \mathbb{R})$ by

$$(9.4.26) \quad \bar{\mu}_p(Y) := - \sum_i \langle (c_1(\mathfrak{t}) - c_1(\mathfrak{s}) \smile \gamma_i^*, [Y]) \mu_{\mathfrak{s}}(\gamma_i) + S^\ell([Y]).$$

Define

$$(9.4.27) \quad \bar{\mu}_c := -\nu \in H^2(\mathcal{M}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir},*}/S^1; \mathbb{R}).$$

We define

$$(9.4.28) \quad \begin{aligned} \iota_{\beta,1} &: \mathcal{M}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir},*}/S^1 \rightarrow \mathcal{M}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(\beta)), \\ \iota_{\beta,2} &: \mathcal{M}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(\beta)) \rightarrow \mathcal{M}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir},*}/S^1, \end{aligned}$$

to be the inclusion maps. Standard computations (compare [30, Lemma 4.10] and [29, Corollary 4.7]), Lemma 9.4.3, Corollary 9.4.7, Lemma 9.2.7, and the equality,

$$c_1(\mathbb{L}_{\mathfrak{s}}) = \mu_{\mathfrak{s}} \times 1 + \sum_{i=1}^{b_1} \gamma_i^{J,*} \times \gamma_i^*,$$

from [28, Lemma 2.24], where $\mathbb{L}_{\mathfrak{s}}$ is as in Definition 9.2.4, then yield the

COROLLARY 9.4.9. *Assume that the spin^u structure $\mathfrak{t}(\ell)$ admits a splitting $\mathfrak{t}(\ell) = \mathfrak{s} \oplus \mathfrak{s} \otimes L$, so $c_1(L) = c_1(\mathfrak{t}) - c_1(\mathfrak{s})$. For ω equal to $x \in H_0(X; \mathbb{R})$, $\gamma \in H_1(X; \mathbb{R})$, $\beta \in H_2(X; \mathbb{R})$, or $[Y] \in H_3(X; \mathbb{R})$, let $[c_\omega]$ be the relative cohomology class defined in (9.4.5) and $\iota_{\omega,2}$ the inclusion defined in (9.4.28). Then*

$$(9.4.29) \quad \begin{aligned} \bar{j}_x^*[c_x] &= \iota_{x,2}^* \bar{\mu}_p(x), & \bar{j}_\gamma^*[c_\gamma] &= \iota_{\gamma,2}^* \bar{\mu}_p(\gamma), \\ \bar{j}_\beta^*[c_\beta] &= \iota_{\beta,2}^* \bar{\mu}_p(\beta), & \bar{j}_Y^*[c_Y] &= \iota_{Y,2}^* \bar{\mu}_p(Y), \end{aligned}$$

where \bar{j}_x , \bar{j}_β , \bar{j}_{γ_i} , and \bar{j}_Y are the inclusion maps of pairs defined in equation (9.4.9).

The analogue below of Corollary 9.4.9 for $[c_{\mathcal{W}}]$ follows immediately from the observation that $\bar{\mathcal{W}}$ is the zero locus of the complex line bundle L_{ν} defined in (9.2.15) and the computation in [30, Lemma 4.2].

COROLLARY 9.4.10. *If*

$$\bar{j}_{\nu} : \left(\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(x)), \emptyset \right) \rightarrow \left(\mathcal{M}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(x)), \bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},*}/S^1 \setminus \bar{\mathcal{W}} \right)$$

is the inclusion map of the pair, then

$$(9.4.30) \quad \bar{j}_{\nu}^*[c_{\mathcal{W}}] = -\iota_{x,2}^*\nu,$$

where $[c_{\mathcal{W}}]$ is defined in (9.4.8) and $\iota_{x,2}$ is defined in (9.4.28).

9.5. Relative Euler class of the obstruction pseudo-bundle

We now use the framework of relative Euler classes to construct a cocycle to represent the geometric representative defined by the zero locus of the obstruction section, $\chi^{-1}(0) \subset \bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},*}/S^1$, appearing in Hypothesis 7.8.1. The restriction of the obstruction pseudo-bundle to the top stratum of $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},*}/S^1$ is a bundle which we will refer to as the *obstruction bundle*. We begin by showing that the Euler class of this obstruction bundle is the restriction to the top stratum of $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},*}/S^1$ of a cohomology class

$$(9.5.1) \quad \bar{e}_I \smile e(\bar{\Upsilon}_{\mathfrak{t},\mathfrak{s}}^s/S^1) \in H^{\bullet}(\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},*}/S^1; \mathbb{R}),$$

given by an expression in the cohomology classes appearing in Section 9.2.3. We use the relative Euler class of the obstruction bundle to construct a cocycle corresponding to the geometric representative $\chi^{-1}(0)$ in the sense of Lemma 9.4.2 and prove that that cocycle extends over the subspace of $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},*}/S^1$ given by the complement of a neighborhood of the intersection of $\bar{\chi}^{-1}(0)$ with the lower strata of $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},*}/S^1$. Finally, we prove that this extension of the cocycle given by the relative Euler class is mapped to the cohomology class (9.5.1) in the exact sequence of the pair.

9.5.1. Euler class of the Seiberg–Witten component of the obstruction pseudo-bundle. The computation of the Euler class of the Seiberg–Witten, or background, component of the obstruction is identical to that given in [30, Lemma 4.11].

LEMMA 9.5.1. *Let r_{Ξ} be the complex rank of the background obstruction bundle, $\bar{\Upsilon}_{\mathfrak{t},\mathfrak{s}}^s \rightarrow \bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},*}$. Then*

$$(9.5.2) \quad e(\bar{\Upsilon}_{\mathfrak{t},\mathfrak{s}}^s/S^1) = \iota^*(-\nu)^{r_{\Xi}}.$$

Because the background obstruction bundle is defined over all of $\bar{\mathcal{M}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},*}$, there is no need to define an extension of $e(\bar{\Upsilon}_{\mathfrak{t},\mathfrak{s}}^s/S^1)$.

9.5.2. Local Euler class of the instanton component of the obstruction bundle. We now give a description of the Euler class of the bundle defined by the restriction of the instanton component of the obstruction pseudo-bundle, $\Upsilon_{\mathfrak{t},\mathfrak{s}}^i/S^1 \rightarrow \mathcal{M}_{\mathfrak{t},\mathfrak{s}}^{\text{vir},*}/S^1$, to the top stratum, $\mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$, of $\bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$.

This description requires the introduction of an additional cohomology class. Let $\pi_{\mathfrak{s},i} : M_{\mathfrak{s}} \times X^{\ell} \rightarrow M_{\mathfrak{s}} \times X$ be defined by the identity map on $M_{\mathfrak{s}}$ and projection map from X^{ℓ} onto the i -th

factor of X . Let $\mathbb{L}_{\mathfrak{s}} \rightarrow M_{\mathfrak{s}} \times X$ be the line bundle in Definition 9.2.4. The Chern classes of the bundle

$$\bigoplus_{i=1}^{\ell} \pi_{\mathfrak{s},i}^* \mathbb{L}_{\mathfrak{s}} \rightarrow M_{\mathfrak{s}} \times X^{\ell}$$

are invariant under the obvious action of the symmetric group \mathfrak{S}_{ℓ} and thus define cohomology classes,

$$(9.5.3) \quad c_{\mathfrak{s},\ell,j} \in H^{2j}(M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X); \mathbb{R}).$$

In addition, we let

$$\iota_{\mathcal{P}} : \Sigma(X^{\ell}, \mathcal{P}) \rightarrow \text{Sym}^{\ell}(X) \quad \text{and} \quad \tilde{\iota}_{\mathcal{P}} : \Delta^{\circ}(X^{\ell}, \mathcal{P}) \rightarrow X^{\ell}$$

denote the inclusion maps. Finally, we let

$$(9.5.4) \quad c(\mathfrak{t}) \in H^{\bullet}(X; \mathbb{R})$$

denote any real characteristic class of the bundle $\text{Fr}_{\mathbb{C}\ell(T^*X)}(V) \rightarrow X$; those characteristic classes include $c_1(\mathfrak{t})$, $p_1(\mathfrak{t})$, $p_1(X)$, and $e(X)$.

If $\omega \in H^{\bullet}(\cdot; \mathbb{Z})$, we refer to the image of ω in $H^{\bullet}(\cdot; \mathbb{R})$ under the functor $H^{\bullet}(\cdot; \mathbb{Z}) \rightarrow H^{\bullet}(\cdot; \mathbb{R})$ defined by the homomorphism $\mathbb{Z} \rightarrow \mathbb{R}$ as the image of ω in real cohomology.

PROPOSITION 9.5.2. *Let $\mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \subset \bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ be the top stratum. The image in real cohomology of the Euler class of the restriction of the obstruction bundle $\Upsilon_{\mathfrak{t},\mathfrak{s}}^i/S^1$ to $\mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ is given by a polynomial in the cohomology classes*

$$(9.5.5) \quad \nu, \quad \pi_X^* \iota_{\mathcal{P}}^* S^{\ell}(c(\mathfrak{t})), \quad \text{and} \quad \pi_{X,\mathfrak{s}}^* \iota_{\mathcal{P}}^* c_{\mathfrak{s},\ell,j},$$

with coefficients depending only on the partition $\ell = |P_1| + \dots + |P_r|$, where $\mathcal{P} = \{P_1, \dots, P_r\}$.

To prove Proposition 9.5.2, we first show that the restriction of the instanton obstruction bundle to $\mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ admits a direct sum decomposition in Lemma 9.5.3 and then in Lemmas 9.5.4 and 9.5.5 prove that the components of this direct sum satisfy the conclusions of Proposition 9.5.2.

Let $\text{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \subset \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ denote the top stratum,

$$\text{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) = \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \times_{G(\mathcal{P})} \prod_{P \in \mathcal{P}} M_{\text{SPL},|P|}^{\mathfrak{s},\natural}(\delta_P).$$

Because $\tilde{N}_{\mathfrak{t}(\ell),\mathfrak{s}}(\delta)$ retracts $\mathcal{G}_{\mathfrak{s}}$ -equivariantly onto $\tilde{M}_{\mathfrak{s}}$, the space $\mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ retracts onto the intersection of $\bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ with the subspace

$$(9.5.6) \quad \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \text{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}).$$

We compute the pullback of the restriction of the Euler class of the instanton obstruction to the subspace (9.5.6) by the map,

$$(9.5.7) \quad \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \text{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}),$$

defined by the projection

$$\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) := \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \times_{\tilde{G}(\mathcal{P})} \prod_{P \in \mathcal{P}} M_{\text{SPL},|P|}^{\mathfrak{s},\natural}(\delta_P) \rightarrow \text{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}),$$

where the group $\widetilde{G}(\mathcal{P}) \leq G(\mathcal{P})$ is defined in (6.2.3) by omitting the symmetric group factor in $G(\mathcal{P})$. Observe that the preceding map appears in the diagram,

$$\begin{array}{ccc} \widetilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \widetilde{\mathrm{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) & \longrightarrow & \widetilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \\ \widetilde{\pi}_{X, \mathfrak{s}} \downarrow & & \pi_{X, \mathfrak{s}} \downarrow \\ M_{\mathfrak{s}} \times \Delta^\circ(X^\ell, \mathcal{P}) & \longrightarrow & M_{\mathfrak{s}} \times \Sigma(X^\ell, \mathcal{P}) \end{array}$$

For each $P \in \mathcal{P}$, we define

$$(9.5.8) \quad \mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, P) := \mathrm{Fr}(\mathfrak{g}_{\mathfrak{t}(\ell)}) \times_X \mathrm{Fr}(TX) \times_{\mathrm{SO}(3) \times \mathrm{SO}(4)} M_{\mathrm{SPL}, |P|}^{s, \natural}(\delta_P),$$

and observe that the projection $\widetilde{\mathrm{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \rightarrow \Delta^\circ(X^\ell, \mathcal{P})$ fits into a diagram

$$(9.5.9) \quad \begin{array}{ccc} \widetilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \widetilde{\mathrm{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) & \xrightarrow{\pi_{\widetilde{\mathrm{Gl}}, P}} & \widetilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, P) \\ \downarrow & & \downarrow \\ M_{\mathfrak{s}} \times \Delta^\circ(X^\ell, \mathcal{P}) & \xrightarrow{\mathrm{id}_{M_{\mathfrak{s}}} \times \pi_P} & M_{\mathfrak{s}} \times X \end{array}$$

We then have the following

LEMMA 9.5.3. *Let $\mathfrak{t}(\ell) = (\rho, V_\ell)$ be the spin^u structure defined in (2.1.13). We abbreviate $\mathfrak{g}_\ell = \mathfrak{g}_{\mathfrak{t}(\ell)}$ and $V_\ell = V_{\mathfrak{t}(\ell)}$. The pullback of the restriction of $\Upsilon_{\mathfrak{t}, \mathfrak{s}}^i/S^1$ to $\mathcal{U}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ by the covering map (9.5.7) splits into a direct sum,*

$$\bigoplus_{P \in \mathcal{P}} \pi_{\widetilde{\mathrm{Gl}}, P}^* \Upsilon_{\mathfrak{t}, \mathfrak{s}}^i(P)/S^1,$$

where for $\mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, P)$ defined in (9.5.8), the bundle

$$(9.5.10) \quad \Upsilon_{\mathfrak{t}, \mathfrak{s}}^i(P)/S^1 \rightarrow \widetilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, P)$$

is defined by

$$\begin{array}{c} \widetilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V_\ell) \times_{\mathrm{Spin}^u(4)} \Upsilon_{\mathrm{SPL}, |P|}^i \\ \downarrow \\ \widetilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, P) \end{array}$$

and the map $\pi_{\widetilde{\mathrm{Gl}}, P}$ is defined in the diagram (9.5.9).

The S^1 action in (9.5.10) on the space

$$(9.5.11) \quad \widetilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V_\ell) \times_{\mathrm{Spin}^u(4)} \Upsilon_{\mathrm{SPL}, |P|}^i$$

is defined by the action of S^1 on the infinite-dimensional obstruction space described following (7.2.1). That is, for $(A_0, \Phi_0) \in \widetilde{M}_{\mathfrak{s}}$, and $\tilde{F}_u \in \mathrm{Fr}_{\mathbb{C}\ell(T^*X)}(V_\ell)|_x$, and $[A, F_s, \Psi] \in \Upsilon_{\mathrm{SPL}, |P|}^i$, the action is given by

$$(9.5.12) \quad \left(e^{i\theta}, \left[(A_0, \Phi_0), \left(\tilde{F}_u, [A, F_s, \Psi] \right) \right] \right) \mapsto \left[(A_0, \Phi_0 e^{i\theta}), \left(\tilde{F}_u, [A, F_s, \Psi e^{i\theta}] \right) \right].$$

Recall that $\varrho_{\mathfrak{s}}(e^{i\theta})$ is defined in (2.3.16). Because the action of scalar multiplication on V_ℓ differs from the action of $\varrho_{\mathfrak{s}}(e^{i\theta})$ by an element of the stabilizer of $(A_0, \Phi_0) \in \tilde{M}_{\mathfrak{s}}$, the action (9.5.12) is the same as the action

$$(9.5.13) \quad \left(e^{i\theta}, \left[(A_0, \Phi_0), \left(\tilde{F}_u, [A, F_s, \Psi] \right) \right] \right) \mapsto \left[(A_0, \Phi_0), \left(\varrho_{\mathfrak{s}}(e^{-i\theta}) \tilde{F}_u, [A, F_s, \Psi e^{i\theta}] \right) \right].$$

Lemma 9.5.3 implies that to compute the restriction of the instanton obstruction bundle to the subspace (9.5.6), it suffices to characterize the Euler class of the bundle (9.5.10). The base of the bundle $\Upsilon_{\mathfrak{t}, \mathfrak{s}}^i(P)/S^1$ admits a product decomposition,

$$\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, P) \cong M_{\mathfrak{s}} \times S^1 \setminus \mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, P).$$

Define

$$(9.5.14) \quad \Upsilon_{X, P}^i/S^1 := M_{\mathfrak{s}} \times S^1 \setminus \left(\mathrm{Fr}_{\mathrm{C}\ell(T^*X)}(V_\ell) \times_{\mathrm{Spin}^u(4)} \Upsilon_{\mathrm{SPL}, |P|}^i \right) \rightarrow M_{\mathfrak{s}} \times S^1 \setminus \mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, P).$$

As in the discussion of the S^1 action on (9.5.11), the S^1 action in (9.5.14) is given by the diagonal action of $\varrho_{\mathfrak{s}}(e^{-i\theta})$ on $\mathrm{Fr}_{\mathrm{C}\ell(T^*X)}(V_\ell)$ and scalar multiplication on the fibers of $\Upsilon_{\mathrm{SPL}, |P|}^i$.

LEMMA 9.5.4. *Let $\mathbb{L}_{\mathfrak{s}} \rightarrow M_{\mathfrak{s}} \times X$ be the restriction of the universal Seiberg–Witten line bundle defined in (2.3.5) and $\pi_{X, \mathfrak{s}} : M_{\mathfrak{s}} \times S^1 \setminus \mathrm{Gl}(\mathfrak{t}, \mathfrak{s}, P) \rightarrow M_{\mathfrak{s}} \times X$ be the projection map. Then*

$$(9.5.15) \quad \Upsilon_{\mathfrak{t}, \mathfrak{s}}^i(P)/S^1 \cong \pi_{X, \mathfrak{s}}^* \mathbb{L}_{\mathfrak{s}}^* \otimes \Upsilon_{X, P}^i/S^1.$$

PROOF. By [28, Lemma 3.27], the tensor product in (9.5.15) can be described by the fibered product,

$$(9.5.16) \quad \begin{aligned} & \left(\left(\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} X \times S^1 \right) \times_{M_{\mathfrak{s}} \times X} \Upsilon_{X, P}^i/S^1 \right) / S^1 \\ &= \left(\left(\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} X \times S^1 \right) \times_{M_{\mathfrak{s}} \times X} \left(M_{\mathfrak{s}} \times S^1 \setminus \mathrm{Fr}_{\mathrm{C}\ell(T^*X)}(V_\ell) \times_{\mathrm{Spin}^u(4)} \Upsilon_{\mathrm{SPL}, |P|}^i \right) \right) / S^1, \end{aligned}$$

where the final S^1 acts diagonally on the factor of S^1 in $\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} X \times S^1$ and on the fiber of $\Upsilon_{X, P}^i/S^1$. The fibered product (9.5.16) admits a simplification as

$$(9.5.17) \quad \begin{aligned} & \left(\left(\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} X \times S^1 \right) \times_{M_{\mathfrak{s}} \times X} \left(M_{\mathfrak{s}} \times S^1 \setminus \mathrm{Fr}_{\mathrm{C}\ell(T^*X)}(V_\ell) \times_{\mathrm{Spin}^u(4)} \Upsilon_{\mathrm{SPL}, |P|}^i \right) \right) / S^1 \\ & \cong \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} S^1 \setminus \left(\mathrm{Fr}_{\mathrm{C}\ell(T^*X)}(V_\ell) \times_{\mathrm{Spin}^u(4)} \Upsilon_{\mathrm{SPL}, |P|}^i \right), \end{aligned}$$

given by the map (using the same notation as in (9.5.13)),

$$\left(\left((A_0, \Phi_0), x, e^{i\theta} \right), \left[[A_0, \Phi_0], \left[\tilde{F}_u, [A, F_s, \Psi] \right] \right] \right) \mapsto \left((A_0, \Phi_0), \left[\varrho_{\mathfrak{s}}(e^{-i\theta}) \tilde{F}_u, [A, F_s, \Psi] \right] \right).$$

This map is S^1 -equivariant with respect to the S^1 action on the domain given by the diagonal action on the factor S^1 and scalar multiplication on the fibers of $\Upsilon_{\mathrm{SPL}, |P|}^i$, if S^1 acts on the image by $\varrho_{\mathfrak{s}}(e^{-i\theta})$ on $\mathrm{Fr}_{\mathrm{C}\ell(T^*X)}(V_\ell)$ and by scalar multiplication on the fibers of $\Upsilon_{\mathrm{SPL}, |P|}^i$. This is the same S^1 action as that appearing in (9.5.13). The conclusion of the lemma then follows by observing that the presentation for the S^1 group action in (9.5.13) identifies the bundle on the right-hand-side of (9.5.17) with $\Upsilon_{\mathfrak{t}, \mathfrak{s}}^i(P)/S^1$ as required. \square

Next, we characterize the (rational) Euler class of $\Upsilon_{X,P}^i/S^1$. To this end, we define the cohomology class,

$$(9.5.18) \quad \nu_{\text{Gl}} \in H^2(M_{\mathfrak{s}} \times S^1 \setminus \text{Gl}(\mathfrak{t}, \mathfrak{s}, P); \mathbb{R}),$$

(denoted by $\nu_{\mathfrak{t}}$ in [30, Definition 5.22]) to be the first Chern class of the S^1 bundle,

$$M_{\mathfrak{s}} \times \text{Gl}(\mathfrak{t}, \mathfrak{s}, P) \rightarrow M_{\mathfrak{s}} \times S^1 \setminus \text{Gl}(\mathfrak{t}, \mathfrak{s}, P).$$

The equality,

$$(9.5.19) \quad \nu = \nu_{\text{Gl}} + 2\pi_{X,\mathfrak{s}}^* c_1(\mathbb{L}_{\mathfrak{s}}),$$

follows from [30, Lemma 5.23]. The advantage of the bundle (9.5.18) over the S^1 bundle defining ν in (9.2.15) is that the former pulls back by the projection $M_{\mathfrak{s}} \times S^1 \setminus \text{Gl}(\mathfrak{t}, \mathfrak{s}, P) \rightarrow S^1 \setminus \text{Gl}(\mathfrak{t}, \mathfrak{s}, P)$.

LEMMA 9.5.5. *The images of the Chern classes of $\Upsilon_{X,P}^i/S^1$ in real cohomology are given by polynomials in $\pi_X^* c(\mathfrak{t})$ and ν_{Gl} with coefficients depending only on $|P|$.*

PROOF. Observe that the bundle $\Upsilon_{X,P}^i/S^1$ is the pullback by the projection

$$M_{\mathfrak{s}} \times S^1 \setminus \text{Gl}(\mathfrak{t}, \mathfrak{s}, P) \rightarrow S^1 \setminus \text{Gl}(\mathfrak{t}, \mathfrak{s}, P)$$

of the bundle

$$(9.5.20) \quad S^1 \setminus \text{Fr}_{\mathbb{C}\ell(T^*X)}(V_{\ell}) \times_{\text{Spin}^u(4)} \Upsilon_{\text{SPL},|P|}^i \rightarrow S^1 \setminus \text{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}).$$

Thus, it suffices to prove that the Chern class of the bundle (9.5.20) satisfies the conclusion of the lemma.

The kernel of the projection

$$(9.5.21) \quad (\text{Ad}_{\text{SO}(3)}^u, \text{Ad}_{\text{SO}(4)}^u) : \text{Spin}^u(4) \rightarrow \text{SO}(3) \times \text{SO}(4)$$

is the central S^1 in $\text{Spin}^u(4)$. By the identity (see [30, Equation (3.14)])

$$\text{Fr}_{\mathbb{C}\ell(T^*X)}(V_{\ell})/S^1 = \text{Fr}(\mathfrak{g}_{\ell}) \times_X \text{Fr}(TX),$$

(where the S^1 is the kernel of the homomorphism (9.5.21)), we can rewrite the space $\text{Gl}(\mathfrak{t}, \mathfrak{s}, P)$ as

$$\text{Gl}(\mathfrak{t}, \mathfrak{s}, P) = \text{Fr}_{\mathbb{C}\ell(T^*X)}(V_{\ell}) \times_{\text{Spin}^u(4)} M_{\text{SPL},|P|}^{s,\natural}(\delta_P),$$

where $\text{Spin}^u(4)$ acts on $M_{\text{SPL},|P|}^{s,\natural}(\delta_P)$ via the projection (9.5.21) and the action of $\text{SO}(3) \times \text{SO}(4)$ on $M_{\text{SPL},|P|}^{s,\natural}(\delta_P)$. Let $\tilde{\iota}_u : \text{Fr}_{\mathbb{C}\ell(T^*X)}(V_{\ell}) \rightarrow \text{ESpin}^u(4)$ be the classifying map appearing in the diagram,

$$\begin{array}{ccc} \text{Fr}_{\mathbb{C}\ell(T^*X)}(V_{\ell}) & \xrightarrow{\tilde{\iota}_u} & \text{ESpin}^u(4) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\iota_u} & \text{BSpin}^u(4) \end{array}$$

Recall from [3, Section 4], that the map $[A] \rightarrow [D_A]$, where D_A is the Dirac operator defined by the connection A , defines a continuous map,

$$f_{\kappa} : \mathcal{B}_{\kappa}^s \rightarrow \mathcal{F}_{\kappa},$$

where \mathcal{F}_{κ} is the space of Fredholm operators of index κ . The space \mathcal{F}_{κ} is homotopic to BU , the classifying space for the stable unitary group $\text{U} := \lim_{n \rightarrow \infty} \text{U}(n)$ (see [3, Equation (4.7)]. Although

there is no index bundle defined for the Dirac operators parameterized by \mathcal{B}_κ^s , because \mathcal{B}_κ^s is not compact, the restrictions of the pullback of the universal Chern classes in BU by f_κ to the subspace $\mathcal{K}_\kappa^s \subset \mathcal{B}_\kappa^s$, where the kernel of the Dirac operator vanishes, are equal to the Chern class of the vector bundle defined by $\text{Index}(\mathbf{D}_\kappa^*)$ in (7.4.2) on this subspace. By Property (5) in the conclusion of Theorem 7.6.1, the Chern classes of the bundle,

$$\Upsilon_{\text{SPL},|P|}^i \rightarrow M_{\text{SPL},|P|}^{s,\natural},$$

are given by the pullbacks of the universal Chern classes in $H^\bullet(\text{BU})$ by the map $f_{|P|}$. By the $\text{Spin}^u(4)$ -equivariance of the Dirac operator, the map f_κ extends to a map

$$F_\kappa : \text{ESpin}^u(4) \times_{\text{Spin}^u(4)} \mathcal{B}_\kappa^s \rightarrow \mathcal{F}_\kappa,$$

such that the restriction of F_κ to

$$\text{ESpin}^u(4) \times_{\text{Spin}^u(4)} \mathcal{K}_\kappa^s$$

is the composition of a classifying map for the index bundle with the inclusion $\text{BU}(\kappa) \rightarrow \text{BU}$. We conclude that the classifying map for the bundle

$$\Upsilon_{X,P}^i = \text{Fr}_{\mathcal{C}\ell(T^*X)}(V_\ell) \times_{\text{Spin}^u(4)} \Upsilon_{\text{SPL},|P|}^i \rightarrow \text{Fr}_{\mathcal{C}\ell(T^*X)}(V_\ell) \times_{\text{Spin}^u(4)} M_{\text{SPL},|P|}^{s,\natural}(\delta_P)$$

factors through the composition

$$(9.5.22) \quad \text{Fr}_{\mathcal{C}\ell(T^*X)}(V_\ell) \times_{\text{Spin}^u(4)} M_{\text{SPL},|P|}^{s,\natural}(\delta_P) \xrightarrow{\tilde{\iota}_u \times \iota_M} \text{ESpin}^u(4) \times_{\text{Spin}^u(4)} \mathcal{B}_{|P|}^s \xrightarrow{F_{|P|}} \mathcal{F}_{|P|},$$

where $\iota_M : M_{\text{SPL},|P|}^{s,\natural}(\delta_P) \rightarrow \mathcal{B}_{|P|}^s$ is the inclusion map. The bundle map $\tilde{\iota}_u \times \iota_M$ is S^1 -equivariant and thus descends to the S^1 quotients,

$$\tilde{\iota}_u \times \iota_M : S^1 \setminus \left(\text{Fr}_{\mathcal{C}\ell(T^*X)}(V_\ell) \times_{\text{Spin}^u(4)} M_{\text{SPL},|P|}^{s,\natural}(\delta_P) \right) \rightarrow S^1 \setminus \left(\text{ESpin}^u(4) \times_{\text{Spin}^u(4)} \mathcal{B}_{|P|}^s \right).$$

Because the rational cohomology of $\mathcal{B}_{|P|}^s$ is trivial (see [15, Lemma 5.1.14]), the rational cohomology of the space

$$S^1 \setminus \left(\text{ESpin}^u(4) \times_{\text{Spin}^u(4)} \mathcal{B}_{|P|}^s \right)$$

is generated by the first Chern class of the S^1 action and cohomology classes pulled back by the projection

$$S^1 \setminus \left(\text{ESpin}^u(4) \times_{\text{Spin}^u(4)} \mathcal{B}_{|P|}^s \right) \rightarrow \text{BSpin}^u(4).$$

Under the map $\tilde{\iota}_u \times \iota_M$, the first Chern class of the S^1 action pulls back to ν_{Gl} , while the cohomology classes pulled back from $\text{BSpin}^u(4)$ pull back to characteristic classes of $\text{Fr}_{\mathcal{C}\ell(T^*X)}(V_\ell)$, which we are denoting by $c(\mathfrak{t})$. The conclusion of the lemma then follows from the factorization of the classifying map for the bundle (9.5.20) given in (9.5.22). \square

REMARK 9.5.6. Unlike the case $\ell = 1$, the action of \mathcal{G}_s on $S^1 \setminus \text{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ is not trivial because there can be more than one frame in the definition of $\text{Gl}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$. That is, if F_1 and F_2 are frames over separate points x_1 and x_2 in X , the action of S^1 identifies (F_1, F_2) with $(e^{i\theta} F_1, e^{i\theta} F_2)$. But there are gauge transformations, $u \in \mathcal{G}_s$, with $u(x_1) \neq u(x_2)$ and for such $u \in \mathcal{G}_s$, the pair (uF_1, uF_2) would not be identified with (F_1, F_2) by the S^1 action.

PROOF OF PROPOSITION 9.5.2. By the Splitting Principle, we can assume that there are line bundles $N_{P,i}$ such that $\Upsilon_{X,P}^i/S^1 \cong \bigoplus_{i \in P} N_{P,i}$. By Lemmas 9.5.3, 9.5.4, and 9.5.5, we see that the pullback by the map (9.5.7) of the restriction of $\Upsilon_{t,s}^i/S^1$ to the space (9.5.6) is given by

$$\begin{aligned} \bigoplus_{P \in \mathcal{P}} \pi_{\widetilde{\text{Gl}},P}^* \Upsilon_{t,s}^i(P)/S^1 &= \bigoplus_{P \in \mathcal{P}} \pi_{\widetilde{\text{Gl}},P}^* (\pi_{X,s}^* \mathbb{L}_s^* \otimes \Upsilon_{X,P}^i/S^1) \\ &= \bigoplus_{P \in \mathcal{P}} \bigoplus_{i \in P} \pi_{\widetilde{\text{Gl}},P}^* (\pi_{X,s}^* \mathbb{L}_s^* \otimes N_{P,i}). \end{aligned}$$

The Euler class of the bundle

$$\bigoplus_{P \in \mathcal{P}} \pi_{\widetilde{\text{Gl}},P}^* \Upsilon_{t,s}^i(P)/S^1$$

on the left-hand side of the preceding sequence of equalities is thus

$$\prod_{P \in \mathcal{P}} \prod_{i \in P} \pi_{\widetilde{\text{Gl}},P}^* (-\pi_{X,s}^* c_1(\mathbb{L}_s) + c_1(N_{P,i})).$$

Observe that $\pi_{X,s} \circ \pi_{\widetilde{\text{Gl}},P} = \pi_{s,i} \tilde{\pi}_{X,s}$ (where $\pi_{s,i}$ is defined prior to (9.5.3)). Hence, the preceding cohomology class can be expressed in terms of symmetric polynomials in $\tilde{\pi}_{X,s}^* \pi_{s,i}^* c_1(\mathbb{L}_s)$ and $\pi_{\widetilde{\text{Gl}},P}^* c_1(N_{P,i})$. Symmetric polynomials in $\tilde{\pi}_{X,s}^* \pi_{s,i}^* c_1(\mathbb{L}_s)$ are given by the Chern classes $c_{s,\ell,j}$ defined in (9.5.3), while symmetric polynomials in $\pi_{\widetilde{\text{Gl}},P}^* c_1(N_{P,i})$ are given by the Chern classes of the bundle

$$\bigoplus_{P \in \mathcal{P}} \pi_{\widetilde{\text{Gl}},P}^* \Upsilon_{X,P}^i/S^1.$$

By Lemma 9.5.5, the Chern classes of the preceding bundle are given by polynomials in ν and $\pi_X^* c(t)$, with coefficients depending only on the partition $\ell = |P_1| + \dots + |P_r|$. \square

9.5.3. Global Euler class of the instanton component of the obstruction bundle. We now piece together the local computations of Proposition 9.5.2 to give a global characterization of the instanton obstruction bundle, $\Upsilon_{t,s}^i/S^1 \rightarrow \mathcal{M}_{t,s}^{\text{vir}}/S^1$.

PROPOSITION 9.5.7. *The Euler class of the instanton obstruction bundle, $\Upsilon_{t,s}^i/S^1 \rightarrow \mathcal{M}_{t,s}^{\text{vir}}/S^1$, is given by an element in $H^{2\ell}(\mathcal{M}_{t,s}^{\text{vir}}/S^1; \mathbb{R})$ that is a polynomial in $\iota^* \nu$, and $\iota^* \pi_X^* S^\ell(c(t))$, and $\iota^* \pi_{X,s}^* c_{s,\ell,i}$, with coefficients which are independent of X .*

PROOF. The construction of the bundle $\Upsilon_{\text{SPL},\kappa}^i \rightarrow M_{\text{SPL},\kappa}^{s,1}(\delta)$ and the $\text{Spin}^u(4)$ equivariance of the diagram (7.6.9) imply that the particular classifying maps of the local instanton obstruction bundles which admit the factorization (9.5.22) can be chosen to be equal on the overlap given in Lemma 7.7.1. Hence, the local equality of the cohomology classes given in Proposition 9.5.2 is an equality of cocycles and thus an equality of global cohomology classes. \square

DEFINITION 9.5.8. Let

$$(9.5.23) \quad \bar{e}_I \in H^{2\ell}(\mathcal{M}_{t,s}^{\text{vir},*}/S^1; \mathbb{R})$$

be the extension of $e(\Upsilon_{t,s}^i/S^1) \in H^{2\ell}(\mathcal{M}_{t,s}^{\text{vir},*}/S^1; \mathbb{R})$ obtained by omitting the pullback ι^* from the expression in Proposition 9.5.7.

REMARK 9.5.9. Note that the vector bundle $\Upsilon_{t,s}^i/S^1$ does not extend from $\mathcal{M}_{t,s}^{\text{vir},*}/S^1$ to $\overline{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$. Indeed, the cohomology class u_i from [15, Definition 8.3.16] can be seen as an obstruction to such an extension.

The following lemma describing the extension \bar{e}_I will be used in Section 9.6 to relate intersection numbers with a pairing of cohomology and homology classes.

LEMMA 9.5.10. *Let $i_\Sigma : \Sigma \subset \overline{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$ be the inclusion map of any smooth stratum. Let $e_I(\Sigma)$ be the Euler class of the vector bundle defined by the restriction of the pseudo-vector bundle $\tilde{\Upsilon}_{t,s}^i$ to Σ . For \bar{e}_I as defined in (9.5.23), then*

$$i_\Sigma^*(\bar{e}_I \smile e(\tilde{\Upsilon}_{t,s}^s/S^1)) = e_I(\Sigma)_{\mathbb{R}} \smile \omega_i(\Sigma) \smile i_\Sigma^*(e(\tilde{\Upsilon}_{t,s}^s/S^1)),$$

where $e_I(\Sigma)_{\mathbb{R}}$ is the image of the integral cohomology class $e_I(\Sigma)$ in real cohomology and $\omega_i(\Sigma)$ is a real cohomology class.

PROOF. If N is a sufficiently small neighborhood of Σ with a deformation retraction $\pi : N \rightarrow \Sigma$, then the description of $\tilde{\Upsilon}_{t,s}^i$ in Theorem 7.6.1 and [15, Proposition 7.2.32] imply that there is an inclusion of vector bundles,

$$\pi^*(\tilde{\Upsilon}_{t,s}^i|_\Sigma)|_{N \cap \mathcal{M}_{t,s}^{\text{vir},*}/S^1} \subset \tilde{\Upsilon}_{t,s}^i|_{N \cap \mathcal{M}_{t,s}^{\text{vir},*}/S^1}.$$

Therefore, the restriction of \bar{e}_I to $N \cap \mathcal{M}_{t,s}^{\text{vir},*}/S^1$ admits the factorization asserted in the lemma. \square

9.5.4. Relative Euler classes. An extensive discussion of relative characteristic classes appears in [50] but we shall only use a small and self-contained portion of that theory here. For any oriented vector bundle, $V \rightarrow Y$, and section, $s : Y \rightarrow V$, a *relative Euler class* can be defined as follows. The section s defines a map of pairs,

$$s : (Y, Y \setminus s^{-1}(0)) \rightarrow (V, V \setminus Y).$$

If $\text{Th}(V) \in H^r(V, V \setminus Y)$ is the Thom class of V , where $r = \text{rank}_{\mathbb{R}}(V)$, then the relative Euler class of the section $s : Y \rightarrow V$ is defined to be

$$(9.5.24) \quad e(V, s) := s^* \text{Th}(V) \in H^r(Y, Y \setminus s^{-1}(0); \mathbb{R}).$$

We note the following properties of the relative Euler class.

LEMMA 9.5.11. *Let $V \rightarrow Y$ be an oriented, real vector bundle of rank r and $s : Y \rightarrow V$ be a section. Then the following hold.*

(1) *If $j_s : (Y, \emptyset) \rightarrow (Y, Y \setminus s^{-1}(0))$ is the inclusion map, then*

$$(9.5.25) \quad j_s^* e(V, s) = e(V),$$

where $e(V) \in H^r(Y; \mathbb{Z})$ is the Euler class of V .

(2) *If $s = s_1 \oplus s_2$ with respect to a decomposition $V = W_1 \oplus W_2$ as a direct sum of oriented real vector bundles, then*

$$e(V, s) = e(W_1, s_1) \smile e(W_2, s_2).$$

PROOF. Item (1) follows from [50, Equation (11.2)] or [68, p. 98]. Item (2) follows from the product formula for Thom classes (see [11, Proposition VIII.11.26]). \square

REMARK 9.5.12. If $s : Y \rightarrow V$ is the zero section, then Item (1) of Lemma 9.5.11, reduces to the well-known result that $s^* \text{Th}(V) \in H^r(Y, \emptyset; \mathbb{Z})$ is the absolute Euler class. If one of the sections s_i in Item (2) of Lemma 9.5.11 is the zero section, then the resulting product formula decomposes $e(V, s)$ as a cup product of a relative and an absolute class.

Recalling that $\mathcal{M}_{t,s}^{\text{vir}} \subset \bar{\mathcal{M}}_{t,s}^{\text{vir}}$ is the top stratum, we denote

$$\bar{\Upsilon}_{t,s}/S^1 := \bar{\Upsilon}_{t,s}^i/S^1 \oplus \bar{\Upsilon}_{t,s}^s/S^1 \quad \text{and} \quad \Upsilon_{t,s}/S^1 := \bar{\Upsilon}_{t,s}/S^1|_{\mathcal{M}_{t,s}^{\text{vir}}/S^1}.$$

For any $\varepsilon > 0$, let $D_\varepsilon := D_\varepsilon(\Upsilon_{t,s}/S^1) \subset \Upsilon_{t,s}/S^1$ be the open ε -disk subbundle defined by the L^2 -norm on the fibers of $\Upsilon_{t,s}/S^1$. Let r be the real rank of $\Upsilon_{t,s}/S^1$. For the Eilenberg-MacLane space $K(\mathbb{Q}, r)$, let $V(\mathbb{Q}, r)$ be a contractible neighborhood of a basepoint $* \in K(\mathbb{Q}, r)$. Let the cocycle

$$(9.5.26) \quad \iota(\mathbb{Q}, r) \in Z^r(K(\mathbb{Q}, r), V(\mathbb{Q}, r); \mathbb{Q})$$

represent the universal class. There is a map of triples

$$(9.5.27) \quad k_\varepsilon : (\Upsilon_{t,s}/S^1, \Upsilon_{t,s}^\circ/S^1, \Upsilon_{t,s}/S^1 \setminus D_\varepsilon) \rightarrow (K(\mathbb{Q}, r), V(\mathbb{Q}, r), *),$$

where $\Upsilon_{t,s}^\circ/S^1$ is the complement of the zero-section, such that $[k_\varepsilon^* \iota(\mathbb{Q}, r)]$ is the Thom class of $\Upsilon_{t,s}/S^1$.

The following lemma constructs an extension of the relative Euler class $e(\Upsilon_{t,s}/S^1, \chi)$ to a larger subspace of $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$.

LEMMA 9.5.13. *Let $\bar{\chi}$ be the obstruction section of $\bar{\Upsilon}_{t,s}/S^1$ appearing in Hypothesis 7.8.1. For any open neighborhood \mathcal{U} of $\bar{\chi}^{-1}(0)$ in $\bar{\mathcal{M}}_{t,s}^{\text{vir}}/S^1$, there is a constant $c > 0$ such that $\bar{\chi}' := c\bar{\chi}$ obeys the following:*

- (1) $\bar{\chi}'(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{U}) \subset \Upsilon_{t,s}/S^1 \setminus D_\varepsilon$.
- (2) For the map k_ε defined in (9.5.27), $k_\varepsilon \circ \bar{\chi}'(\bar{\mathcal{M}}_{t,s}^{\text{sing}}/S^1 \setminus \mathcal{U}) = *$.
- (3) The composition $k_\varepsilon \circ \bar{\chi}'$ extends as a map of pairs,

$$(9.5.28) \quad \bar{k}_\chi : \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \bar{\mathcal{M}}_{t,s}^{\text{sing}} \cap \mathcal{U}, \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{U} \right) \rightarrow (K(\mathbb{Q}, r), *)$$

which is constant on a neighborhood of $\bar{\mathcal{M}}_{t,s}^{\text{sing}}/S^1 \setminus \mathcal{U}$.

PROOF. By Hypothesis 7.8.1, we have $M_s \times \text{Sym}^\ell(X) \subset \bar{\chi}^{-1}(0)$. Hence, $\bar{\mathcal{M}}_{t,s}^{\text{vir}}/S^1 \setminus \mathcal{U} = \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{U}$ and we can assume that $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{U}$ is compact. By the lower semi-continuity of the L^2 norm of $\bar{\chi}$ given by (7.2.3) on the fibers of $\bar{\Upsilon}_{t,s}/S^1$ (see Hypothesis 7.8.1), there is a minimum value, ε_0 , of the L^2 norm of $\bar{\chi}$ on $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{U}$. Therefore, if $\bar{\chi}' = (\varepsilon/\varepsilon_0)\bar{\chi}$, then $\bar{\chi}'(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{U}) \subset \Upsilon_{t,s}/S^1 \setminus D_\varepsilon$, proving Property (1). Properties (2) and (3) follow from Property (1) and the definition (9.5.27) of k_ε . \square

Lemma 9.5.13 gives an extension of the relative Euler class $e(\Upsilon_{t,s}/S^1, \chi)$ to $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \bar{\mathcal{M}}_{t,s}^{\text{sing}} \cap \mathcal{U}$. We now compare this extension with the extension of the Euler class defined in (9.5.23).

LEMMA 9.5.14. *Continue the notation of Lemma 9.5.13. Define*

$$(9.5.29) \quad \bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\chi}') := [\bar{k}_\chi^* \iota(\mathbb{Q}, r)],$$

where the map \bar{k}_χ is defined in (9.5.28) and the cocycle $\iota(\mathbb{Q}, r)$ is defined in (9.5.26). If we abbreviate $\bar{\mathcal{M}}(\mathcal{U}) := \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus (\mathcal{M}^{\text{sing}} \cap \mathcal{U})$ then

$$\bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\chi}') \in H^r(\bar{\mathcal{M}}(\mathcal{U}), \bar{\mathcal{M}}(\mathcal{U}) \setminus \bar{\chi}^{-1}(0); \mathbb{R}).$$

As an element of real cohomology, $\bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\chi}')$ satisfies

$$(9.5.30) \quad \iota_{\chi,1}^* \bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\chi}') = e(\Upsilon_{t,s}/S^1, \bar{\chi}) \quad \text{and} \quad J_{\chi,2}^* \bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\chi}') = \iota_{\chi,2}^*(\bar{e}_I \smile \bar{e}_s),$$

where $\iota_{\chi,1}$, $\iota_{\chi,2}$, and $J_{\chi,2}$ are the inclusion maps,

$$\mathcal{M}_{t,s}^{\text{vir}}/S^1 \xrightarrow{\iota_{\chi,1}} \bar{\mathcal{M}}(\mathcal{U}) \xrightarrow{\iota_{\chi,2}} \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$$

and

$$J_{\chi,2} : (\bar{\mathcal{M}}(\mathcal{U}), \emptyset) \rightarrow (\bar{\mathcal{M}}(\mathcal{U}), \bar{\mathcal{M}}(\mathcal{U}) \setminus \bar{\chi}^{-1}(0)),$$

and \bar{e}_I is the extension of the real Euler class of $\Upsilon_{t,s}/S^1$ from (9.5.23) and $\bar{e}_s := e(\Upsilon_{t,s}^s/S^1)$.

PROOF. The defining property of k_ε in (9.5.27), namely, $[k_\varepsilon^* \iota(\mathbb{Q}, r)] = \text{Th}(\Upsilon_{t,s}/S^1)$, the definition of the relative Euler class in (9.5.24), and the homotopy between $\bar{\chi}$ and $\bar{\chi}'$ give the first equality in (9.5.30).

By [92, Theorem 8.1.15], the cohomology classes $J_{\chi,2}^* \bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\chi}')$ and $\iota_{\chi,2}^*(\bar{e}_I \smile \bar{e}_s)$ are determined by pointed homotopy classes $[k_\chi]$ and $[k_2]$, respectively, in $[\bar{\mathcal{M}}(\mathcal{U}), K(\mathbb{Q}, r)]$. Let $\mathcal{M}^{\text{sing}}(U) \subset \bar{\mathcal{M}}(\mathcal{U})$ be the complement of the top stratum and let $N^{\text{sing}} \subset \bar{\mathcal{M}}(\mathcal{U})$ be a neighborhood of $\mathcal{M}^{\text{sing}}(\mathcal{U})$. Because the inclusion $\mathcal{M}^{\text{sing}}(\mathcal{U}) \subset \bar{\mathcal{M}}(\mathcal{U})$ is a cofibration, we can assume that N^{sing} deformation retracts onto $\mathcal{M}^{\text{sing}}(\mathcal{U})$.

If N^{sing} is sufficiently small, the construction of \bar{k}_χ in Lemma 9.5.13 implies that $k_\chi(N^{\text{sing}}) = *$, where $*$ is the basepoint in $K(\mathbb{Q}, r)$. The description of the restriction of \bar{e}_I to a lower stratum $\Sigma \subset \bar{\mathcal{M}}(\mathcal{U})$ in Lemma 9.5.10,

$$\bar{e}_I|_\Sigma = e_I(\Sigma)_\mathbb{Q} \smile \omega_i(\Sigma),$$

and the argument of Lemma 9.5.13 imply that the homotopy class $[k_2]$ also admits a representative k_2 satisfying $k_2(\mathcal{M}^{\text{sing}}(U)) = *$. Because N^{sing} retracts to $\mathcal{M}^{\text{sing}}(U)$, we can assume that $k_2(N^{\text{sing}}) = *$. Thus,

$$(9.5.31) \quad \bar{e}_I \smile \bar{e}_s \in \text{Im}(H^r(\bar{\mathcal{M}}(\mathcal{U}), N^{\text{sing}}; \mathbb{Q}) \rightarrow H^r(\bar{\mathcal{M}}(\mathcal{U}); \mathbb{Q})).$$

By the first equality in (9.5.30) and the relation between relative and absolute Euler classes in (9.5.25),

$$\iota_{\chi,1}^* (J_{\chi,2}^* \bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\chi}') - \iota_{\chi,2}^*(\bar{e}_I \smile \bar{e}_s)) = 0.$$

Thus,

$$(9.5.32) \quad J_{\chi,2}^* \bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\chi}') - \iota_{\chi,2}^*(\bar{e}_I \smile \bar{e}_s) \in \text{Im}(H^r(\bar{\mathcal{M}}(\mathcal{U}), \mathcal{M}_{t,s}^{\text{vir}}/S^1; \mathbb{Q}) \rightarrow H^r(\bar{\mathcal{M}}(\mathcal{U}); \mathbb{Q})).$$

By excision,

$$H^r(\bar{\mathcal{M}}(\mathcal{U}), \mathcal{M}_{t,s}^{\text{vir}}/S^1; \mathbb{Q}) \cong H^r(N^{\text{sing}}, N^{\text{sing}} - \mathcal{M}^{\text{sing}}(\mathcal{U}); \mathbb{Q}),$$

and by the construction of k_χ and by (9.5.31), both $\bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\chi}')$ and $\bar{e}_I \smile \bar{e}_s$ vanish on cycles in N^{sing} . Hence, the difference (9.5.32) vanishes, completing the proof of the second equality in (9.5.30). \square

9.6. Duality and the link of an ideal Seiberg–Witten moduli space

We now perform the computation proving the equality (9.1.1).

9.6.1. The initial duality. We begin by showing how the intersection number in (9.1.1) can be written as a pairing of relative cohomology classes with the fundamental class $[\hat{\mathbf{L}}, \partial\hat{\mathbf{L}}]$.

The relative cohomology classes describing the geometric representatives are given by the cocycles defined in Section 9.4.2. For $z = \beta_1 \cdots \beta_s \in \mathbb{A}(X)$ and $\beta_i \in H_\bullet(X; \mathbb{R})$, denote

$$(9.6.1) \quad \mathcal{K}(z, \eta) := \mathcal{V}(z) \cap \mathcal{W}^\eta \quad \text{and} \quad \bar{\mathcal{K}}(z, \eta) := \bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta.$$

The inclusion map $\iota_{\beta,1}$ defined in (9.4.28) also defines an inclusion of pairs,

$$\begin{array}{c} \left(\mathcal{M}_{t,s}^{\text{vir},*}/S^1, \mathcal{M}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{V}(\beta) \right) \\ \downarrow \iota_{\beta,1} \\ \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(\beta)), \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(\beta)) \setminus \bar{\mathcal{V}}(\beta) \right) \end{array}$$

A similar inclusion of pairs also holds for $\bar{\mathcal{W}}$ in place of $\bar{\mathcal{V}}(\beta)$. We then define

$$(9.6.2) \quad [c(z, \eta)] := \iota_{\beta_1,1}^*[c_{\beta_1}] \smile \cdots \smile \iota_{\beta_r,1}^*[c_{\beta_r}] \smile \iota_{x,1}^*[c_{\mathcal{W}}]^\eta,$$

where $[c_{\beta_i}]$ is as defined in (9.4.5) and $[c_{\mathcal{W}}]$ is as defined in (9.4.8).

With these relative cohomology classes defined, we observe that dimension-counting arguments yield

$$(9.6.3) \quad \bar{\mathcal{K}}(z, \eta) \cap \bar{\chi}^{-1}(0) \cap \mathcal{M}_{t,s}^{\text{sing},*}/S^1 = \emptyset.$$

Hence, for a sufficiently small neighborhood U of $\mathcal{M}_{t,s}^{\text{sing}}/S^1$ as constructed in Lemma 9.3.1, we have

$$(9.6.4) \quad \bar{\mathcal{K}}(z, \eta) \cap \bar{\chi}^{-1}(0) \cap U = \emptyset.$$

For $\beta \in H_\bullet(X; \mathbb{R})$, let $\nu(\beta)$ be the tubular neighborhood described in Section 9.4.2 and let $\iota_{\beta,1}$ and $\iota_{\beta,2}$ be the inclusion maps defined in (9.4.28). For the complement $U^\circ \subset U$ of the lower strata in U , as defined following (9.3.1), the cup product

$$(9.6.5) \quad [c(z, \eta)] \smile e(\Upsilon_{t,s}/S^1, \chi) \in H^d(\mathbf{L}_{t,s}^{\text{vir}} \cup U^\circ, \mathbf{L}_{t,s}^{\text{vir}} \cup U^\circ \setminus \bar{\mathcal{K}}(z, \eta) \cap \chi^{-1}(0); \mathbb{R})$$

can be paired with the homology class $[\hat{\mathbf{L}}, \partial\hat{\mathbf{L}}]$ from (9.3.1). By the definition of the cocycles $[c_\beta]$ and $[c_{\mathcal{W}}]$ as dual to the geometric representatives $\mathcal{V}(\beta)$ and \mathcal{W} , the definition of $e(\Upsilon_{t,s}/S^1, \chi)$ in terms of a Thom class, and because χ vanishes transversely (see Hypothesis 7.8.1), we can write the intersection number as

$$(9.6.6) \quad \begin{aligned} & \# (\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}_{t,s}) \\ &= \left\langle \iota_{\beta_1,1}^*[c_{\beta_1}] \smile \cdots \smile \iota_{\beta_r,1}^*[c_{\beta_r}] \smile \iota_{x,1}^*[c_{\mathcal{W}}]^\eta \smile e(\Upsilon_{t,s}/S^1, \chi), [\hat{\mathbf{L}}, \partial\hat{\mathbf{L}}] \right\rangle. \end{aligned}$$

The first equality in Equation (9.5.30) allows us to rewrite Equation (9.6.6) as

$$(9.6.7) \quad \begin{aligned} & \# (\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}_{t,s}) \\ &= \left\langle \iota_{\beta_1,1}^*[c_{\beta_1}] \smile \cdots \smile \iota_{\beta_r,1}^*[c_{\beta_r}] \smile \iota_{x,1}^*[c_{\mathcal{W}}]^\eta \smile \iota_{\chi,1}^* \bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\chi}'), [\hat{\mathbf{L}}, \partial\hat{\mathbf{L}}] \right\rangle. \end{aligned}$$

To replace the pairing of the relative classes in (9.6.6) by pairings of absolute classes with the fundamental class of $\mathbf{L}_{t,s}^{\text{vir}}$, we need to write the cohomology class $[c(z, \eta)] \smile e(\Upsilon_{t,s}, \chi)$ as

$$[c(z, \eta)] \smile e(\Upsilon_{t,s}, \chi) = \iota^*[\bar{c}(z, \eta)] \smile \bar{e}(\Upsilon_{t,s}, \chi),$$

where

$$j^*[\bar{c}(z, \eta)] \smile \bar{e}(\Upsilon_{t,s}, \chi) = \bar{\mu}_p(z) \smile \bar{\mu}_c^\eta \smile \bar{e}_I \smile \bar{e}_s.$$

We will show how to accomplish this in the following sections.

9.6.2. Extension of the cocycles. The relative cohomology classes,

$$[c_\beta] \in H^\bullet(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(\beta)), \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus (\mathcal{I}(\nu(\beta)) \cup \bar{\mathcal{V}}(\beta)); \mathbb{R}),$$

$$[c_{\mathcal{W}}] \in H^\bullet(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(x)), \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus (\mathcal{I}(\nu(x)) \cup \bar{\mathcal{V}}(\beta)); \mathbb{R}),$$

$$\bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\chi}') \in H^\bullet\left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus (\mathcal{M}^{\text{sing}} \cap U'), \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus (\mathcal{M}^{\text{sing}} \cap U' \cup \chi^{-1}(0)); \mathbb{R}\right),$$

defined in (9.4.5), (9.4.8), and Lemma 9.5.14, respectively, are extensions only over subspaces of $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$. We now discuss how these cohomology classes can be extended to relative cohomology classes on $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$.

For any pair of spaces (A, B) , let $S^p(A, B; \mathbb{R}) = \text{Hom}(S_p(A)/S_p(B); \mathbb{R})$ denote the complex of real-valued relative simplicial cochains and let $Z^p(A, B; \mathbb{R})$ be the cycles of this complex. We can consider $S^p(A, B; \mathbb{R})$ to be the set of real-valued cochains on A which vanish when restricted to B . If \mathfrak{U} is an open cover of A , then a chain is *small of order* \mathfrak{U} (see [47, Remark 15.8]) if it is a sum of simplices each of which has image contained in some element of \mathfrak{U} . By [47, Theorem 15.9], any element of $H_p(A, B; \mathbb{R})$ can be represented by a cycle which is small of order \mathfrak{U} . By the Universal Coefficient Theorem, to define an element of $H^p(A, B; \mathbb{R})$ it then suffices to define it on cycles which are small of order \mathfrak{U} .

Lemma 9.6.1 below shows that we can change the cocycle c_β to one that extends over $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$ by adding an exact cocycle supported near $\mathcal{I}(\nu(\beta))$. In the proof of Proposition 9.1.1, we will show that such a change does not change the pairing (9.6.7).

LEMMA 9.6.1. *For $\beta \in H_\bullet(X; \mathbb{R})$, let $\mathcal{U}(\beta) \subset \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$ be any open neighborhood of $\mathcal{I}(\nu(\beta))$ satisfying $\mathcal{I}(\nu(\beta)) \Subset \mathcal{U}(\beta)$. Let c_β be the relative cocycle defined in (9.4.5). Then there are a cochain,*

$$\theta_\beta \in S^{\deg(\beta)-1}\left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(\beta)), \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{U}(\beta); \mathbb{R}\right)$$

and a cocycle

$$\bar{c}_\beta \in Z^{\deg(\beta)}\left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1, \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \bar{\mathcal{V}}(\beta) \setminus \mathcal{U}(\beta); \mathbb{R}\right)$$

satisfying

$$j_{\mathcal{U}(\beta)}^*[\bar{c}_\beta] = \bar{\mu}_p(\beta) \quad \text{and} \quad \iota_{\beta,2}^* \bar{c}_\beta = c_\beta + \delta^* \theta_\beta,$$

where the map $\iota_{\beta,2}$ is defined in (9.4.28) and

$$j_{\mathcal{U}(\beta)} : \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1, \emptyset\right) \rightarrow \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1, \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \bar{\mathcal{V}}(\beta) \setminus \mathcal{U}(\beta)\right)$$

is the inclusion map.

PROOF. The conclusion of the lemma follows from an argument similar to the proof of [30, Lemma 5.14]. If

$$(9.6.8) \quad j_{\beta,2} : \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(\beta)), \emptyset\right) \rightarrow \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(\beta)), \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(\beta)) \setminus \bar{\mathcal{V}}(\beta)\right)$$

is the inclusion map, then the equality,

$$\iota_{\beta,2}^* \bar{\mu}_p(\beta) = j_{\beta,2}^* [c_\beta],$$

provided by Corollary 9.4.9 implies that there are a cocycle $\bar{c}'_\beta \in Z^{\deg(\beta)} \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1; \mathbb{R} \right)$ and a cochain

$$\theta_0 \in S^{\deg(\beta)-1} \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1 \setminus \mathcal{I}(\nu(\beta)); \mathbb{R} \right),$$

satisfying

$$\bar{\mu}_p(\beta) = [\bar{c}'_\beta] \quad \text{and} \quad \iota_{\beta,2}^* \bar{c}'_\beta = j_{\beta,2}^* c_\beta + \delta^* \theta_0.$$

Because $\mathcal{I}(\nu(\beta)) \Subset \mathcal{U}(\beta)$, there is a closed neighborhood, $\mathcal{U}'(\beta)$, of $\mathcal{I}(\nu(\beta))$ satisfying $\mathcal{I}(\nu(\beta)) \Subset \mathcal{U}'(\beta) \subset \mathcal{U}(\beta)$. Then

$$\mathfrak{U} = \left\{ \bar{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1 \setminus \mathcal{U}'(\beta), \mathcal{U}(\beta) \right\}$$

is an open cover of $\bar{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1$ and using the argument of the proof of the Excision Lemma (see [48, Proposition 2.21]), we can assume that all cochains in this discussion are defined only on chains which are small with respect to this open cover.

Because the pairs appearing in the diagram are excisive couples (see [92, Theorem 4.6.3] and [92, p. 218]), the map

$$\begin{array}{c} S^p \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1 \setminus \mathcal{I}(\nu(\beta)), \bar{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1 \setminus \mathcal{U}(\beta); \mathbb{R} \right) \oplus S^p \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1 \setminus \mathcal{I}(\nu(\beta)), \mathcal{U}'(\beta); \mathbb{R} \right) \\ \downarrow \\ S^p \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1 \setminus \mathcal{I}(\nu(\beta)), \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1 \setminus \mathcal{U}(\beta) \right) \cap \mathcal{U}'(\beta); \mathbb{R} \right) \end{array}$$

is surjective. In addition, because the intersection

$$\left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1 \setminus \mathcal{U}(\beta) \right) \cap \mathcal{U}'(\beta)$$

is empty, the map is actually a map to the space of absolute cochains,

$$S^p \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1 \setminus \mathcal{I}(\nu(\beta)), \emptyset; \mathbb{R} \right).$$

Thus we can write $\theta_0 = \theta_\beta + \theta_p$, where

$$\begin{aligned} \theta_\beta &\in S^{\deg(\beta)-1} \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1 \setminus \mathcal{I}(\nu(\beta)), \bar{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1 \setminus \mathcal{U}(\beta); \mathbb{R} \right), \\ \theta_p &\in S^{\deg(\beta)-1} \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1 \setminus \mathcal{I}(\nu(\beta)), \mathcal{U}'(\beta); \mathbb{R} \right). \end{aligned}$$

By definition, θ_p vanishes on all simplices in $\mathcal{U}'(\beta)$. Because we have assumed that we only need to define cochains on chains which are small with respect to the open cover \mathfrak{U} and because $\mathcal{I}(\nu(\beta)) \Subset \mathcal{U}'(\beta)$, then θ_p defines a cochain,

$$\bar{\theta}_p \in S^{\deg(\beta)-1} \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*} / S^1, \mathcal{U}'(\beta); \mathbb{R} \right),$$

with $\iota_{\beta,2}^* \bar{\theta}_p = \theta_p$. Define $\bar{c}_\beta := \bar{c}'_\beta - \delta^* \bar{\theta}_p$ and observe that

$$\iota_{\beta,2}^* \bar{c}_\beta = \iota_{\beta,2}^* \bar{c}'_\beta - \iota_{\beta,2}^* \delta^* \bar{\theta}_p = \iota_{\beta,2}^* \bar{c}'_\beta - \delta^* \theta_p = j_{\beta,2}^* c_\beta + \delta^* \theta_\beta,$$

as required. The preceding equation implies that the cocycle \bar{c}_β vanishes when restricted to $\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus (\bar{\mathcal{V}}(\beta) \cup \mathcal{U})$ and thus defines the relative cohomology class asserted by the lemma. Finally, we compute that

$$J_{\mathcal{U}(\beta)}^*[\bar{c}_\beta] = [\bar{c}'_\beta - \delta^* \bar{\theta}_p] = [\bar{c}'_\beta] = \bar{\mu}_p(\beta),$$

completing the proof of Lemma 9.6.1. \square

The argument of Lemma 9.6.1 gives corresponding extension results for $[c_{\mathcal{W}}]$ and $e(\Upsilon, \hat{\chi})$. The extension result for $[c_{\mathcal{W}}]$ is given by

LEMMA 9.6.2. *Let $c_{\mathcal{W}}$ be the cochain defined in (9.4.8). For any neighborhood $\mathcal{U}(x) \subset \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$ of $\mathcal{I}(\nu(x))$ satisfying $\mathcal{I}(\nu(x)) \Subset \mathcal{U}(x)$, there are a cochain,*

$$\theta_{\mathcal{W}} \in S^1 \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{I}(\nu(x)), \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{U}(x); \mathbb{R} \right),$$

and a cocycle,

$$\bar{c}_{\mathcal{W}} \in Z^2(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1, \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \bar{\mathcal{W}} \setminus \mathcal{U}(x); \mathbb{R}),$$

satisfying

$$J_{\mathcal{U}(x)}^*[\bar{c}_{\mathcal{W}}] = \bar{\mu}_c \quad \text{and} \quad \iota_{x,2}^* \bar{c}_{\mathcal{W}} = c_{\mathcal{W}} + \delta^* \theta_{\mathcal{W}},$$

where

$$J_{\mathcal{U}(x)} : \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1, \emptyset \right) \rightarrow \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1, \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \bar{\mathcal{W}} \setminus \mathcal{U}(x) \right)$$

is the inclusion map and $\iota_{x,2}$ is defined in (9.4.28).

Finally, we record the extension result for $\bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\chi}')$.

LEMMA 9.6.3. *Continue the notation of Lemma 9.5.14. For any neighborhood $U_{\mathcal{X}} \subset \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$ of $\mathcal{M}^{\text{sing}} \cap \mathcal{U}$ with $\mathcal{M}^{\text{sing}} \cap \mathcal{U} \Subset U_{\mathcal{X}}$, there are a cochain,*

$$\theta_{\mathcal{X}} \in S^{r-1} \left(\bar{\mathcal{M}}(\mathcal{U}), \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus U_{\mathcal{X}} \right),$$

and a cocycle,

$$\bar{e}_{\mathcal{X}} \in Z^r \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1, \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{X}^{-1}(0) \setminus U_{\mathcal{X}}; \mathbb{R} \right),$$

satisfying

$$J_{\mathcal{X}}^*[\bar{e}_{\mathcal{X}}] = \bar{e}_I \smile \bar{e}_s \quad \text{and} \quad \iota_{\mathcal{X},2}^* \bar{e}_{\mathcal{X}} = \bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\chi}') + \delta^* \theta_{\mathcal{X}},$$

where

$$J_{\mathcal{X}} : \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1, \emptyset \right) \rightarrow \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1, \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus \mathcal{X}^{-1}(0) \setminus U_{\mathcal{X}} \right)$$

is the inclusion map.

We can now give the

PROOF OF PROPOSITION 9.1.1. The proof is analogous to that of [30, Proposition 5.2], using relative Euler classes of the obstruction sections and representatives of the cohomology with compact support along the geometric representatives.

Let \mathcal{U} be the neighborhood of $\bar{\mathcal{X}}^{-1}(0)$ appearing in Lemma 9.5.13. Because the intersection,

$$(9.6.9) \quad \mathcal{I}(\nu(\beta_1)) \cap \cdots \cap \mathcal{I}(\nu(\beta_k)) \cap \mathcal{I}(\nu(x_1)) \cap \cdots \cap \mathcal{I}(\nu(x_\eta)) \cap \mathcal{U} \cap \mathcal{M}^{\text{sing}},$$

is empty, we can find open neighborhoods $U(\beta_i)$ of $\mathcal{I}(\nu(\beta_i))$, and $U_{\mathcal{W},j}$ of $\mathcal{I}(\nu(x_j))$, and $U_{\mathcal{X}}$ of \mathcal{U} , and U of $\mathcal{M}^{\text{sing}}$, such that the intersection obtained by replacing any of $\mathcal{I}(\nu(\beta_i))$ with $U(\beta_i)$, or

$\mathcal{S}(\nu(x_j))$ with $U_{\mathcal{W},j}$, or \mathcal{U} with $U_{\mathcal{X}}$, or $\mathcal{M}^{\text{sing}}$ with U in (9.6.9) is still empty. For pairs of spaces and subspaces, (Y, A) and (Y, B) , cup product defines a map of relative cohomology,

$$H^p(Y, A; \mathbb{R}) \otimes H^q(Y, B; \mathbb{R}) \rightarrow H^{p+q}(Y, A \cup B; \mathbb{R}).$$

Therefore, if we let θ_{β_i} , θ_ν , and $\theta_{\mathcal{X}}$ be the cochains defined in Lemmas 9.6.1, 9.6.2, and 9.6.3 respectively, for the neighborhoods $U(\beta_i)$, $U_{\mathcal{W},j}$ and $U_{\mathcal{X}}$, and we replace one or more of $[c_{\beta_1}]$ with $\delta^*\theta_{\beta_i}$, or $[c_{\mathcal{W}}]$ with $\delta^*\theta_\nu$, or $\bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\mathcal{X}}')$ with $\delta^*\theta_{\mathcal{X}}$, in

$$\iota_{\beta_1,1}^*[c_{\beta_1}] \smile \cdots \smile \iota_{\beta_r,1}^*[c_{\beta_r}] \smile \iota_{x,1}^*[c_{\mathcal{W}}]^\eta \smile \iota_{\mathcal{X},1}^*\bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\mathcal{X}}'),$$

then the resulting cup product will vanish. Hence, we obtain the following equality for the pairing (9.6.7),

$$\begin{aligned} & \left\langle \iota_{\beta_1,1}^*[c_{\beta_1}] \smile \cdots \smile \iota_{\beta_r,1}^*[c_{\beta_r}] \smile \iota_{x,1}^*[c_{\mathcal{W}}]^\eta \smile \iota_{\mathcal{X},1}^*\bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\mathcal{X}}'), [\hat{\mathbf{L}}, \partial\hat{\mathbf{L}}] \right\rangle \\ (9.6.10) \quad & = \left\langle \iota_{\beta_1,1}^*[c_{\beta_1} + \delta^*\theta_{\beta_1}] \smile \cdots \smile \iota_{\beta_r,1}^*[c_{\beta_s} + \delta^*\theta_{\beta_s}] \right. \\ & \quad \left. \smile \iota_{x,1}^*[c_{\mathcal{W}} + \delta^*\theta_\nu]^\eta \smile \iota_{\mathcal{X},1}^*(\bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\mathcal{X}}') + \delta^*\theta_{\mathcal{X}}), [\hat{\mathbf{L}}, \partial\hat{\mathbf{L}}] \right\rangle. \end{aligned}$$

Recall that for $j = 1, 2$, the inclusion maps $\iota_{\beta,j}$ and $\iota_{\mathcal{X},j}$ are defined in (9.4.28) and Lemma 9.5.14, respectively. Applying the equalities

$$\iota = \iota_{\beta,2} \circ \iota_{\beta,1} = \iota_{\mathcal{X},2} \circ \iota_{\mathcal{X},1}$$

and Lemmas 9.6.1, 9.6.2, and 9.6.3 to equations (9.6.7) and (9.6.10) then implies that for the cocycle \bar{c}_{β_1} defined in Lemma 9.6.1, the cocycle $\bar{c}_{\mathcal{W}}$ defined in Lemma 9.6.2, and the cocycle $\bar{e}_{\mathcal{X}}$ defined in Lemma 9.6.3, we obtain the equalities,

$$\begin{aligned} (9.6.11) \quad \#(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}_{t,s}) & = \left\langle \iota^*([\bar{c}_{\beta_1}] \smile \cdots \smile [\bar{c}_{\beta_s}] \smile [\bar{c}_{\mathcal{W}}]^\eta \smile [\bar{e}_{\mathcal{X}}]), [\hat{\mathbf{L}}, \partial\hat{\mathbf{L}}] \right\rangle \\ & = \left\langle [\bar{c}_{\beta_1}] \smile \cdots \smile [\bar{c}_{\beta_s}] \smile [\bar{c}_{\mathcal{W}}]^\eta \smile [\bar{e}_{\mathcal{X}}], \bar{j}_*[\bar{\mathbf{L}}_{t,s}^{\text{vir}}] \right\rangle, \end{aligned}$$

where the map \bar{j} is defined in (9.3.2) and we have used the definition of the homology class $[\bar{\mathbf{L}}_{t,s}^{\text{vir}}]$ given in (9.3.3). If $J_{\mathcal{W}(\beta_i)}$, $J_{\mathcal{W}(x)}$, and $J_{\mathcal{X}}$ are the inclusion maps defined in Lemmas 9.6.1, 9.6.2, and 9.6.3, then

$$\begin{aligned} & \#(\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}_{t,s}) \\ & = \left\langle \bar{j}^*([\bar{c}_{\beta_1}] \smile \cdots \smile [\bar{c}_{\beta_s}] \smile [\bar{c}_{\mathcal{W}}]^\eta \smile [\bar{e}_{\mathcal{X}}]), [\bar{\mathbf{L}}_{t,s}^{\text{vir}}] \right\rangle \\ & = \left\langle J_{\mathcal{W}(\beta_1)}^*[\bar{c}_{\beta_1}] \smile \cdots \smile J_{\mathcal{W}(\beta_s)}^*[\bar{c}_{\beta_s}] \smile J_{\mathcal{W}(x)}^*[\bar{c}_{\mathcal{W}}]^\eta \smile J_{\mathcal{X}}^*[\bar{e}_{\mathcal{X}}], [\bar{\mathbf{L}}_{t,s}^{\text{vir}}] \right\rangle \\ & \quad (\text{by [92, Theorem 5.6.8]}) \\ & = \left\langle \bar{\mu}_p(\beta_1) \smile \cdots \smile \bar{\mu}_p(\beta_s) \smile \bar{\mu}_c^\eta \smile \bar{e}_I \smile \bar{e}_s, [\bar{\mathbf{L}}_{t,s}^{\text{vir}}] \right\rangle, \end{aligned}$$

where the final equality follows from Lemmas 9.6.1, 9.6.2, and 9.6.3. This completes the proof of Proposition 9.1.1. \square

9.7. Reduction to the subspace $\bar{\mathbf{B}}\bar{\mathbf{L}}_{t,s}^{\text{vir}}$

We now show how to rewrite pairings with the homology class $[\bar{\mathbf{L}}_{t,s}^{\text{vir}}]$ defined in (9.3.3) as pairings with the homology class of a subspace and thus eliminate the topology of the normal bundle $N_{t(\ell),s}(\delta) \rightarrow M_5$.

We first give a global description of the relation between the space $\overline{\mathbf{BL}}_{t,s}^{\text{vir}}$ defined in (8.1.16) and the space $\overline{\mathbf{L}}_{t,s}^{\text{vir},i} \subset \overline{\mathbf{L}}_{t,s}^{\text{vir}}$ defined in (8.1.2), (8.1.4), and (8.1.5). Let $\tilde{\iota}_\nu$ and ι_ν be the maps in the classifying diagram for the cohomology class ν of Definition 9.2.3,

$$(9.7.1) \quad \begin{array}{ccc} \overline{\mathcal{M}}_{t,s}^{\text{vir},*} & \xrightarrow{\tilde{\iota}_\nu} & \text{ES}^1 \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 & \xrightarrow{\iota_\nu} & \text{BS}^1 \end{array}$$

The proof of the following lemma is identical to that of [30, Lemma 5.19].

LEMMA 9.7.1. *Let S^1 act on the fibers of $N_{t(\ell),s}(\delta) \rightarrow M_s$ by scalar multiplication. Then the deformation retraction, $\overline{\mathbf{L}}_{t,s}^{\text{vir},i} \rightarrow \overline{\mathbf{BL}}_{t,s}^{\text{vir}}$, is the pullback of the disk bundle,*

$$N_{t(\ell),s}(\delta) \times_{S^1} \text{ES}^1 \rightarrow M_s \times \text{BS}^1,$$

by the map

$$\pi_N \times \tilde{\iota}_\nu : \overline{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \rightarrow N_{t(\ell),s}(\delta) \times_{S^1} \text{ES}^1,$$

where $\overline{\mathcal{M}}_{t,s}^{\text{vir},*}$ is defined in (9.2.2), the map π_N is defined in (6.6.9), and the map $\tilde{\iota}_\nu$ in (9.7.1).

Let r_N denote the complex rank of $N_{t(\ell),s} \rightarrow M_s$ and define

$$(9.7.2) \quad \text{Th}(t_N) \in H^{2r_N} \left(\overline{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1, \overline{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus t_N^{-1}(0); \mathbb{R} \right)$$

to be the pullback of the Thom class of the bundle

$$N_{t(\ell),s}(\delta) \times_{S^1} \text{ES}^1 \rightarrow M_s \times \text{BS}^1$$

by the map $\pi_N \times \tilde{\iota}_\nu$ appearing in Lemma 9.7.1. A fundamental class,

$$(9.7.3) \quad [\overline{\mathbf{BL}}_{t,s}^{\text{vir}}] \in H_{d(t)-2r_N-2} \left(\overline{\mathbf{L}}_{t,s}^{\text{vir}} \cap t_N^{-1}(0); \mathbb{R} \right),$$

is defined exactly as in (9.3.3). Namely, let

$$(9.7.4) \quad [\widehat{\mathbf{BL}}, \partial \mathbf{BL}]$$

be the homology class defined by the fundamental class of the manifold with corners given by the intersection of $t_N^{-1}(0)$ with the manifold defining the homology class $[\hat{\mathbf{L}}, \partial \hat{\mathbf{L}}]$ from (9.3.1). Now define $[\overline{\mathbf{BL}}_{t,s}^{\text{vir}}]$ to be the homology class satisfying

$$(9.7.5) \quad \iota_* [\widehat{\mathbf{BL}}, \partial \mathbf{BL}] = \bar{j}_* [\overline{\mathbf{BL}}_{t,s}^{\text{vir}}],$$

where ι and \bar{j} are the maps defined in Equations (9.2.1) and (9.3.2), respectively. Note that the restriction of \bar{j} to $t_N^{-1}(0)$ also defines an isomorphism on homology of the appropriate dimension because the deformation retraction r in Lemma 9.3.1 preserves the level sets of t_N .

Let U be the neighborhood of $\overline{\mathcal{M}}_{t,s}^{\text{sing}}/S^1$ used in (9.3.1) to define the homology class $[\hat{\mathbf{L}}, \partial \hat{\mathbf{L}}]$. Because we can choose the boundaries of the space U to be generic, we can assume that t_N vanishes transversely on these boundaries. Hence, we have the equality,

$$(9.7.6) \quad [\widehat{\mathbf{BL}}, \partial \mathbf{BL}] = \iota^* \text{Th}(t_N) \cap [\hat{\mathbf{L}}, \partial \hat{\mathbf{L}}].$$

We then have

LEMMA 9.7.2. *If $b := \dim \mathbf{BL}_{t,s}^{\text{vir}}$, then for any $\omega \in H^b(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1)$ we have*

$$\langle \iota_{\mathbf{B}}^* \omega, [\bar{\mathbf{B}}\mathbf{L}_{t,s}^{\text{vir}}] \rangle = \langle \omega \smile j_{\mathbf{B}}^* \text{Th}(t_N), [\mathbf{L}_{t,s}^{\text{vir}}] \rangle,$$

where $\text{Th}(t_N)$ is defined in (9.7.2) and

$$j_{\mathbf{B}} : \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1, \emptyset \right) \rightarrow \left(\bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1, \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1 \setminus t_N^{-1}(0) \right)$$

and

$$\iota_{\mathbf{B}} : t_N^{-1}(0) \rightarrow \bar{\mathcal{M}}_{t,s}^{\text{vir},*}/S^1$$

are the inclusion maps.

PROOF. The proof is similar to that of [30, Equation (5.59)].

Let U be the neighborhood of $\bar{\mathcal{M}}_{t,s}^{\text{sing}}/S^1$ as described prior to the statement of the lemma. Abbreviate $\bar{\mathbf{L}}_{t,s}^{\text{vir}} \cup U$ by $\mathbf{L}(U)$ and write $t_N^{-1}(0) = \mathbf{B}$. Then for $d(\mathfrak{t}) = \dim \bar{\mathcal{M}}_{t,s}^{\text{vir}}$ and $d(\mathfrak{t}) - 2 = b + 2r_N$, where r_N is the complex rank of the vector bundle $N_{\mathfrak{t}(\ell),s} \rightarrow M_s$, we have

$$\omega \smile \text{Th}(t_N) \in H^{b+2r_N}(\mathbf{L}(U), \mathbf{L}(U) \setminus \mathbf{B}; \mathbb{R}).$$

Consider the following commutative diagram of inclusions:

$$(9.7.7) \quad \begin{array}{ccc} (\mathbf{L}(U), \emptyset) & \xrightarrow{j_{\mathbf{B}}} & (\mathbf{L}(U), \mathbf{L}(U) \setminus \mathbf{B}) \\ \bar{j} \downarrow & & j_{\mathbf{B},U} \downarrow \\ (\mathbf{L}(U), U) & \xrightarrow{j_{U,\mathbf{B}}} & (\mathbf{L}(U), U \cup (\mathbf{L}(U) \setminus \mathbf{B})) \end{array}$$

Define $U^{\text{sing}} := U \cap \bar{\mathcal{M}}_{t,s}^{\text{sing}}/S^1$. Because U retracts onto U^{sing} and $\dim U^{\text{sing}} \leq d(\mathfrak{t}) - 1 - 4$, the inclusion map $j_{\mathbf{B},U}$ defines an isomorphism,

$$j_{\mathbf{B},U}^* : H^{b+2r_N}(\mathbf{L}(U), U \cup (\mathbf{L}(U) \setminus \mathbf{B}); \mathbb{R}) \cong H^{b+2r_N}(\mathbf{L}(U), \mathbf{L}(U) \setminus \mathbf{B}; \mathbb{R}).$$

Consequently, there is a unique class $\Omega \in H^{b+2r_N}(\mathbf{L}(U), U \cup (\mathbf{L}(U) \setminus \mathbf{B}); \mathbb{R})$ such that

$$(9.7.8) \quad j_{\mathbf{B},U}^* \Omega = \omega \smile \text{Th}(t_N).$$

We now calculate that

$$\begin{aligned} \langle j_{\mathbf{B}}^*(\omega \smile \text{Th}(t_N)), [\bar{\mathbf{L}}_{t,s}^{\text{vir}}] \rangle &= \langle j_{\mathbf{B}}^* j_{\mathbf{B},U}^* \Omega, [\bar{\mathbf{L}}_{t,s}^{\text{vir}}] \rangle \quad (\text{by (9.7.8)}) \\ &= \langle \bar{j}^* j_{U,\mathbf{B}}^* \Omega, [\bar{\mathbf{L}}_{t,s}^{\text{vir}}] \rangle \quad (\text{by commutativity of (9.7.7)}) \\ &= \langle j_{U,\mathbf{B}}^* \Omega, \bar{j}_* [\bar{\mathbf{L}}_{t,s}^{\text{vir}}] \rangle \\ &= \langle j_{U,\mathbf{B}}^* \Omega, \iota_* [\mathbf{L}, \partial \mathbf{L}] \rangle \quad (\text{by (9.3.3)}), \end{aligned}$$

that is,

$$(9.7.9) \quad \langle j_{\mathbf{B}}^*(\omega \smile \text{Th}(t_N)), [\bar{\mathbf{L}}_{t,s}^{\text{vir}}] \rangle = \langle j_{U,\mathbf{B}}^* \Omega, \iota_* [\mathbf{L}, \partial \mathbf{L}] \rangle.$$

To get a better description of Ω , we now reduce the preceding pairing to one on the subspace,

$$\mathbf{L}^i(U) := \mathbf{L}(U) \setminus \text{int}(\bar{\mathbf{L}}_{t,s}^{\text{vir},s}),$$

where $\bar{\mathbf{L}}_{t,s}^{\text{vir},s}$ is defined in (8.1.3). Observe that the top stratum, $\mathbf{L}^i(U) \cap \mathbf{L}_{t,s}^{\text{vir}}$, is an oriented topological manifold with boundary by the argument giving Item (3) of Lemma 8.1.2 and thus has a relative fundamental class (see, for example, [48, p. 253]) which we denote by

$$[\mathbf{L}^i, \partial\mathbf{L}^i] \in H_{d(t)-2}(\mathbf{L}^i(U), U \cup (\mathbf{L}^i \setminus \mathbf{B}); \mathbb{R}).$$

Because $\bar{\mathbf{L}}_{t,s}^{\text{vir},s} \subset \mathbf{L}(U) \setminus \mathbf{B}$, the inclusion of pairs,

$$\iota_{U,i} : (\mathbf{L}^i(U), U \cup (\mathbf{L}^i \setminus \mathbf{B})) \rightarrow (\mathbf{L}(U), U \cup (\mathbf{L}(U) \setminus \mathbf{B})),$$

is an excision map and thus induces an isomorphism on cohomology. Consequently,

$$(9.7.10) \quad (\iota_{U,i})_*[\mathbf{L}^i, \partial\mathbf{L}^i] = (j_{U,\mathbf{B}})_* \iota_*[\mathbf{L}, \partial\mathbf{L}],$$

because both $[\mathbf{L}^i, \partial\mathbf{L}^i]$ and $[\mathbf{L}, \partial\mathbf{L}]$ are fundamental classes. We compute that the right-hand side of Equation (9.7.9) is given by

$$\begin{aligned} \langle j_{U,\mathbf{B}}^* \Omega, \iota_*[\mathbf{L}, \partial\mathbf{L}] \rangle &= \langle \Omega, (j_{U,\mathbf{B}})_* \iota_*[\mathbf{L}, \partial\mathbf{L}] \rangle \\ &= \langle \Omega, (\iota_{U,i})_*[\mathbf{L}^i, \partial\mathbf{L}^i] \rangle \quad (\text{by (9.7.10)}) \\ &= \langle \iota_{U,i}^* \Omega, [\mathbf{L}^i, \partial\mathbf{L}^i] \rangle, \end{aligned}$$

that is,

$$(9.7.11) \quad \langle j_{U,\mathbf{B}}^* \Omega, \iota_*[\mathbf{L}, \partial\mathbf{L}] \rangle = \langle \iota_{U,i}^* \Omega, [\mathbf{L}^i, \partial\mathbf{L}^i] \rangle.$$

We now compute $\iota_{U,i}^* \Omega$ in terms of similar restrictions of ω and $\text{Th}(t_N)$. The $(\mathcal{G}_s \times S^1)$ -equivariant deformation retraction, $\tilde{N}_{t(\ell),s} \rightarrow \tilde{M}_s$, defines a deformation retraction

$$r_B : \mathbf{L}^i(U) \rightarrow \mathbf{B}.$$

Observe that r_B is a homotopy equivalence and also defines a homotopy equivalence of pairs,

$$r_{\mathbf{B}} : (\mathbf{L}^i(U), U) \rightarrow (\mathbf{B}, \mathbf{B} \cap U).$$

Because the retraction of U in Lemma 9.3.1 onto the lower strata respects the level sets of t_N , the subspace $\mathbf{B} \cap U$ retracts onto the lower strata of \mathbf{B} which have codimension greater than or equal to four in \mathbf{B} . Therefore, the map

$$\bar{j}^* : H^b(\mathbf{B}, \mathbf{B} \cap U; \mathbb{R}) \rightarrow H^b(\mathbf{B}; \mathbb{R})$$

is an isomorphism. Because the map $r_{\mathbf{B}}$ induces isomorphisms $r_{\mathbf{B}}^* : H^b(\mathbf{B}, \mathbf{B} \cap U; \mathbb{R}) \rightarrow H^b(\mathbf{L}^i(U), U; \mathbb{R})$ and $r_{\mathbf{B}}^* : H^b(\mathbf{B}; \mathbb{R}) \rightarrow H^b(\mathbf{L}^i(U); \mathbb{R})$, there is a commutative diagram,

$$\begin{array}{ccc} H^b(\mathbf{B}, \mathbf{B} \cap U; \mathbb{R}) & \xrightarrow{\bar{j}^*} & H^b(\mathbf{B}; \mathbb{R}) \\ r_{\mathbf{B}}^* \downarrow & & \downarrow r_{\mathbf{B}}^* \\ H^b(\mathbf{L}^i(U), \mathbf{L}^i(U) \cap U; \mathbb{R}) & \xrightarrow{\bar{j}^*} & H^b(\mathbf{L}^i(U); \mathbb{R}) \end{array}$$

Hence, there is a unique $y_{\mathbf{B}} \in H^b(\mathbf{B}, U \cap \mathbf{B}; \mathbb{R})$ such that

$$(9.7.12) \quad \iota_i^* \omega = \bar{j}^* r_{\mathbf{B}}^* y_{\mathbf{B}} \quad \text{and} \quad \iota_{\mathbf{B}}^* \omega = \bar{j}^* y_{\mathbf{B}},$$

where $\iota_i : \mathbf{L}^i(U) \rightarrow \mathbf{L}(U)$ is the inclusion map. We now claim that

$$(9.7.13) \quad \iota_{U,i}^* \Omega = (r_{\mathbf{B}}^* y_{\mathbf{B}}) \smile \iota_i^* \text{Th}(t_N).$$

To see this, we consider the following commutative diagram:

$$\begin{array}{ccc}
H^b(\mathbf{L}^i(U), U; \mathbb{R}) \otimes H^{2r_N}(\mathbf{L}^i, \mathbf{L}^i \setminus \mathbf{B}; \mathbb{R}) & \xrightarrow{j_{\mathbf{B},U}^* \otimes \text{id}} & H^b(\mathbf{L}^i(U); \mathbb{R}) \otimes H^{2r_N}(\mathbf{L}^i(U), \mathbf{L}^i \setminus \mathbf{B}; \mathbb{R}) \\
\downarrow \smile & & \downarrow \smile \\
H^{b+2r_N}(\mathbf{L}^i(U), U \cup \mathbf{L}^i \setminus \mathbf{B}; \mathbb{R}) & \xrightarrow{j_{\mathbf{B},U}^*} & H^{b+2r_N}(\mathbf{L}^i(U), \mathbf{L}^i \setminus \mathbf{B}; \mathbb{R}) \\
\iota_i^* \uparrow & & \iota_i^* \uparrow \\
H^{b+2r_N}(\mathbf{L}(U), U \cup (\mathbf{L}(U) \setminus B); \mathbb{R}) & \xrightarrow{j_{\mathbf{B},U}^*} & H^{b+2r_N}(\mathbf{L}(U), \mathbf{L}(U) \setminus B; \mathbb{R})
\end{array}$$

The class $r_{\mathbf{B}}^*(y_{\mathbf{B}}) \otimes \iota_i^* \text{Th}(t_N)$ is in the upper-left entry of the preceding diagram, while Ω is in the lower-left entry of the preceding diagram. To prove that (9.7.13) holds, we must show that these two classes have the same image in the center-left entry. By (9.7.8), the class $\iota_i^* \Omega$ is mapped to $\iota_i^* \omega \smile \iota_i^* \text{Th}(t_N)$ in the center-right entry. By (9.7.12), the class $r_{\mathbf{B}}^*(y_{\mathbf{B}}) \smile \iota_i^* \text{Th}(t_N)$ is also mapped to $\iota_i^* \omega \smile \iota_i^* \text{Th}(t_N)$ in the center-right entry, so

$$j_{\mathbf{B},U}^* \iota_i^* \Omega = j_{\mathbf{B},U}^*(r_{\mathbf{B}}^*(y_{\mathbf{B}}) \smile \iota_i^* \text{Th}(t_N)).$$

Because U retracts onto a set of codimension greater than or equal to two and $b + 2r_N = d(\mathbf{t}) - 2$ equals the dimension of the top stratum of $\mathbf{L}(U)$, we observe that both arrows labeled $j_{\mathbf{B},U}^*$ in the preceding diagram are isomorphisms. Hence, the preceding equality yields Equation (9.7.13), as claimed.

By applying Equation (9.7.13) to the right-hand side of Equation (9.7.11), we obtain

$$\begin{aligned}
\langle \iota_{U,i}^* \Omega, [\mathbf{L}^i, \partial \mathbf{L}^i] \rangle &= \langle r_{\mathbf{B}}^*(y_{\mathbf{B}}) \smile \iota_i^* \text{Th}(t_N), [\mathbf{L}^i, \partial \mathbf{L}^i] \rangle \\
&= \langle r_{\mathbf{B}}^* y_{\mathbf{B}}, \iota_i^*(\text{Th}(t_N)) \cap [\mathbf{L}^i, \partial \mathbf{L}^i] \rangle,
\end{aligned}$$

that is,

$$(9.7.14) \quad \langle \iota_{U,i}^* \Omega, [\mathbf{L}^i, \partial \mathbf{L}^i] \rangle = \langle r_{\mathbf{B}}^* y_{\mathbf{B}}, \iota_i^*(\text{Th}(t_N)) \cap [\mathbf{L}^i, \partial \mathbf{L}^i] \rangle.$$

From [8, p. 371, Equation (1)], we have the equality

$$(9.7.15) \quad \iota_i^* \text{Th}(t_N) \cap [\mathbf{L}^i, \partial \mathbf{L}^i] = (\iota_{\mathbf{B},i})_* \iota_* [\mathbf{B}, \partial \mathbf{B}],$$

where $\iota_{\mathbf{B},i} : (\mathbf{B}, \mathbf{B} \cap U) \rightarrow (\mathbf{L}^i(U), \mathbf{L}^i \setminus B)$ is the inclusion map. The right-hand side of Equation (9.7.14) is then equal to

$$\begin{aligned}
\langle r_{\mathbf{B}}^* y_{\mathbf{B}}, \iota_i^*(\text{Th}(t_N)) \cap [\mathbf{L}^i, \partial \mathbf{L}^i] \rangle &= \langle r_{\mathbf{B}}^* y_{\mathbf{B}}, (\iota_{\mathbf{B},i})_* \iota_* [\mathbf{B}, \partial \mathbf{B}] \rangle \\
&= \langle (\iota_{\mathbf{B},i})^* r_{\mathbf{B}}^* y_{\mathbf{B}}, \iota_* [\mathbf{B}, \partial \mathbf{B}] \rangle \\
&= \langle y_{\mathbf{B}}, \iota_* [\mathbf{B}, \partial \mathbf{B}] \rangle \quad (\text{because } r_{\mathbf{B}} \text{ is a retraction, so } r_{\mathbf{B}} \circ \iota_{\mathbf{B},i} = \text{id}) \\
&= \langle y_{\mathbf{B}}, j_* [\bar{\mathbf{B}}\mathbf{L}_{t,s}^{\text{vir}}] \rangle \quad (\text{by (9.7.5)}) \\
&= \langle j^* y_{\mathbf{B}}, [\bar{\mathbf{B}}\mathbf{L}_{t,s}^{\text{vir}}] \rangle \\
&= \langle \iota_{\mathbf{B}}^* \omega, [\bar{\mathbf{B}}\mathbf{L}_{t,s}^{\text{vir}}] \rangle \quad (\text{by (9.7.12)}).
\end{aligned}$$

This completes the proof of Lemma 9.7.2. □

Lemma 9.7.2 can be used to reduce the pairing in (9.1.1) to one with $[\bar{\mathbf{B}}\mathbf{L}_{t,s}^{\text{vir}}]$ when we have an expression for division by the Thom class $\text{Th}(t_N)$. The following proposition gives that result.

PROPOSITION 9.7.3. *Continue to denote $d_s = \dim M_s$. For $0 \leq j \leq [d_s/2]$, let $s_j(N) \in H^{2j}(M_s; \mathbb{R})$ be the Segre classes of the complex-rank- r_N vector bundle $N_{t(\ell),s} \rightarrow M_s$. Let k and m be non-negative integers satisfying $k + 2m = \dim \mathbf{L}_{t,s}^{\text{vir}}$. For any $\alpha \in H^k(M_s \times \text{Sym}^\ell(X); \mathbb{R})$ and the Chern class $\nu \in H^2(\bar{\mathbf{L}}_{t,s}^{\text{vir}}; \mathbb{R})$ defined in Definition 9.2.3, we have the equality*

$$(9.7.16) \quad \begin{aligned} & \langle \nu^m \smile \pi_{X,s}^* \alpha, [\bar{\mathbf{L}}_{t,s}^{\text{vir}}] \rangle \\ &= \sum_{j=0}^{d_s/2} (-1)^{r_N+j} \left\langle \nu^{m-r_N-j} \smile \pi_s^* s_j(N) \smile \pi_{X,s}^* \alpha, [\bar{\mathbf{B}}\mathbf{L}_{t,s}^{\text{vir}}] \right\rangle. \end{aligned}$$

PROOF. See [30, Section 5.2]. □

REMARK 9.7.4. The Segre classes $s_i(N)$ have been computed under some assumptions on $H^1(X; \mathbb{R})$ in [28, Lemma 4.11]. From [28, Theorem 3.29], one can see that in general, these Segre classes will be given by a universal polynomial in $\mu_s(x)$ and $\mu_s(\gamma_i)$ with coefficients depending only on the indices n'_s and n''_s appearing in (2.3.24) and which in turn only depend on $p_1(t(\ell))$ and $c_1(t(\ell))$.

Computation of the intersection numbers

10.1. Introduction

In this chapter, we perform the computation leading to a proof of the following theorems which form the technical heart of this monograph.

THEOREM 10.1.1. *Let X be a closed, connected, oriented, smooth, Riemannian four-manifold with $b_1(X) = 0$ and \mathfrak{t} be a spin^u structure on X . Suppose that $\ell \geq 0$ is an integer and \mathfrak{s} is a spin^c structure on X such that $M_{\mathfrak{s}} \times \text{Sym}^{\ell}(X)$ is a subset of the space of gauge-equivalence classes of ideal monopoles $I\mathcal{M}_{\mathfrak{t}}$ defined in (2.1.14). Let $\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}$ be the link of the stratum of gauge-equivalence classes of reducible $\text{SO}(3)$ monopoles determined by $\mathfrak{s} \in \text{Spin}^c(X)$ as in Definition 8.1.3. For non-negative integers m, δ, η satisfying*

$$\delta - 2m + 2\eta = \dim \mathcal{M}_{\mathfrak{t}} - 2,$$

a class $h \in H_2(X; \mathbb{R})$, and a generator $x \in H_0(X; \mathbb{Z})$, let $z = h^{\delta-2m}x^m \in \mathbb{A}(X)$ and $\bar{\mathcal{V}}(z)$ and $\bar{\mathcal{W}}^{\eta}$ be the geometric representatives defined in Section 2.4. Then

$$(10.1.1) \quad \begin{aligned} & \# (\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\eta} \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}) \\ &= SW_X(\mathfrak{s}) \sum_{i=0}^{\min(\ell, \lceil \frac{\delta-2m}{2} \rceil)} (q_{\delta,\ell,m,i}(c_1(\mathfrak{s}) - c_1(\mathfrak{t}), c_1(\mathfrak{t})) Q_X^i)(h), \end{aligned}$$

where $q_{\delta,\ell,m,i}$ are degree $\delta - 2m - 2i$ homogeneous polynomials which are universal functions of the constants given in Theorem 1.

We note that a similar result can also be achieved by the methods of this monograph without the assumptions that $b_1(X) = 0$ and $z = h^{\delta-2m}x^m$, but the resulting expression (10.1.1) becomes considerably more complicated. While an explicit formula for the intersection number in (10.1.1) is still unknown in general, many special cases have been computed in [29, 30, 33, 25, 23] under additional hypotheses. However, the following important result, referred to as the *Multiplicity Conjecture* in [29, 20, 30], holds without any additional hypotheses. When $b_1(X) > 0$, we write

$$SW_{X,\mathfrak{s}}(\omega) := \langle \mu_{\mathfrak{s}}(\omega), [M_{\mathfrak{s}}] \rangle,$$

for all $\omega \in \mathbb{A}_2(X)$.

THEOREM 10.1.2. *Assume the hypotheses of Theorem 10.1.1, but allow $b_1(X) > 0$. If $SW_{X,\mathfrak{s}}(\omega)$ vanishes for all $\omega \in \mathbb{A}_2(X)$, then*

$$\# (\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^{\eta} \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}) = 0.$$

To prove Theorems 10.1.1 and 10.1.2, we begin by observing that Proposition 9.1.1 gives the equality,

$$(10.1.2) \quad \# (\bar{\mathcal{V}}(z) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}_{t,s}) = \langle \bar{\mu}_p(z) \smile \bar{\mu}_c^\eta \smile \bar{e}_I \smile \bar{e}_s, [\bar{\mathbf{L}}_{t,s}^{\text{vir}}] \rangle.$$

The expression for the μ -classes in Corollary 9.4.9, the expression for \bar{e}_s in Lemma 9.5.1, the expression for \bar{e}_I in Proposition 9.5.7, and the equality between the homology classes $[\bar{\mathbf{L}}_{t,s}^{\text{vir}}]$ and $[\mathbf{BL}_{t,s}^{\text{vir}}]$ in Proposition 9.7.3 implies that the intersection number in (10.1.2) can be reduced to a linear combination of pairings of the form

$$(10.1.3) \quad \langle \nu^a \smile \pi_X^* S^\ell(\beta) \smile \pi_s^* \mu_s(z_2), [\mathbf{BL}_{t,s}^{\text{vir}}] \rangle$$

where $\beta \in H_\bullet(X; \mathbb{R})$ and $z_2 \in \mathbb{A}_2(X)$.

We wish to compute the pairings (10.1.3) by writing them as sums over pairings with the subspaces $\bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)$ defined in (8.1.16) and then applying a pushforward-pullback argument to the fiber-bundle structure in Lemma 8.2.3 of each of these subspaces. To do this, we replace the cohomology classes in (10.1.3) by cohomology classes with compact support away from the boundaries $\partial_j \bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)$ defined in (8.3.1). Such a replacement requires a choice of cocycle in the cohomology class and this choice must be done consistently on each subspace. We encode these choices in the following geometric data. We will define a quotient, $\mathbf{QL}_{t,s}^{\text{vir}}$, of $\bar{\mathbf{BL}}_{t,s}^{\text{vir}}$ by replacing the boundaries $\partial_j \bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)$ with spaces of codimension greater than or equal to two. The cohomology classes in (10.1.3) pull back from cohomology classes on this quotient. Thus, we may choose cocycles representing the cohomology classes in (10.1.3) which pull back from the quotient. Because the image of the boundaries in the quotient has codimension greater than or equal to two, the cocycles pulled back from the quotient will have compact support away from the boundaries. Using these cocycles is equivalent to computing (10.1.3) using the fundamental class of $\mathbf{QL}_{t,s}^{\text{vir}}$ which, because the image of the boundaries has codimension two, can be written as a sum of fundamental classes of subspaces.

One can compare this method of computation to the method of “adding caps” employed by Ozsváth in [79] as follows. In the case where there are only two subspaces, if one adds the cap to each subspace defined by the mapping cone of the restriction of the quotient map to the boundary of that subspace, the resulting compactification of each subspace $\bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)$ would be homotopic to the image of $\bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)$ in the quotient. Thus our method differs from that of Ozsváth in that there are more than two open subspaces in this computation and in that there are no correction terms arising from using different compactifications of the subspaces. Ozsváth has informed us that he has an extension of his method to the case of arbitrarily many open sets, so our method offers no advantage in that regard. Correction terms arise in [79] because the natural compactifications to each subspace define different quotients of the common boundary. No correction terms appear in this computation because on each boundary, $\partial_j \bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)$, the natural quotient for the top stratum (defined by extending the fiber bundle from Σ_j to $\text{cl}(\Sigma_j)$) respects the fiber-bundle structure of the lower stratum. Hence, the absence of correction terms in this computation is not an indication of a better method but merely an exploitation of a simpler situation.

In Section 10.2, we construct the quotient space $\mathbf{QL}_{t,s}^{\text{vir}}$. This construction shows that the image of each subspace $\bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)$ in the quotient has a fiber-bundle structure identical to that of $\bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)$ but with a compact fiber and base. In Section 10.3, we show how to use the quotient

to obtain homology classes with the desired properties. In Section 10.4, we describe the fiber-bundle properties of the images of the subspaces in the quotient. In Section 10.5, we perform the computation giving the needed characterization of the intersection pairing (10.1.3). Finally in Section 10.6, we give the proofs of Theorems 10.1.1 and 10.1.2.

10.2. Quotient space of $\overline{\mathbf{BL}}_{t,s}^{\text{vir}}$

We wish to use the fiber bundle structure,

$$(10.2.1) \quad \overline{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_j) \rightarrow M_s \times K_j \subset M_s \times \Sigma(X^\ell, \mathcal{P}_j),$$

given in Lemma 8.2.3 to compute cohomological pairings with $\overline{\mathbf{BL}}_{t,s}^{\text{vir}}$. To do this, we must write the pairing in (10.1.3) with the homology class $[\overline{\mathbf{BL}}_{t,s}^{\text{vir}}]$ as a sum over pairings with homology classes representing the subspaces $\overline{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_j)$ defined in (8.1.16). However, the subspaces $\overline{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_j)$ have boundaries as described in Section 8.3 and so would only define relative homology classes. To overcome this difficulty, we construct a quotient of $\overline{\mathbf{BL}}_{t,s}^{\text{vir}}$ obtained by leaving the interiors of the subspaces $\overline{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_j)$ unchanged and replacing the boundaries $\partial_k \overline{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_j)$ with quotients obtained by deleting part of the gluing data as described in the following

PROPOSITION 10.2.1. *There is a surjective, continuous map,*

$$Q : \overline{\mathbf{BL}}_{t,s}^{\text{vir}} \rightarrow \mathbf{QL}_{t,s}^{\text{vir}}$$

onto a smoothly-stratified space $\mathbf{QL}_{t,s}^{\text{vir}}$ with the following properties:

- (1) *The map Q is injective on the complement of $\cup_{j \neq i} \partial_j \overline{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)$.*
- (2) *The image of each intersection, $Q(\partial_j \overline{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i))$, has codimension greater than two in $\mathbf{QL}_{t,s}^{\text{vir}}$.*
- (3) *For the space $\overline{M}(\mathcal{P}_i, \varepsilon_i)$ defined in (8.2.1), there is a continuous, surjective, $G(\mathcal{P}_i)$ -equivariant map, $q_i : \overline{M}(\mathcal{P}_i, \varepsilon_i) \rightarrow \check{M}(\mathcal{P}_i, \varepsilon_i)$, onto the smoothly-stratified space $\check{M}(\mathcal{P}_i, \varepsilon_i)$ given in Definition 10.2.9 such that*

$$Q(\overline{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)) \cong M_s \times_{\mathcal{G}_s \times S^1} \overline{\text{Fr}}(t, \mathfrak{s}, \mathcal{P}_i) \times_{G(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon_i),$$

where $\overline{\text{Fr}}(t, \mathfrak{s}, \mathcal{P}_i)$ is defined in (10.2.2) and the restriction of the map Q to $\overline{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)$ is given, for $(A_0, \Phi_0) \in \check{M}_s$, and $F \in \text{Fr}(t, \mathfrak{s}, \mathcal{P}_i)$, and $\vec{A} \in \overline{M}(\mathcal{P}_i, \varepsilon_i)$ by

$$Q \left(\left[(A_0, \Phi_0), F, [\vec{A}] \right] \right) := \left[(A_0, \Phi_0), R_i(F), q_i([\vec{A}]) \right],$$

where R_i is the map defined in Lemma 10.2.4.

- (4) *There is a positive integer b such that the restriction of the bundle $\mathbb{L}_\nu^{\otimes b}$, where \mathbb{L}_ν was defined in (9.2.15), to $\overline{\mathbf{BL}}_{t,s}^{\text{vir}}$ is pulled back from an S^1 bundle over $\mathbf{QL}_{t,s}^{\text{vir}}$.*
- (5) *There is a map, $\pi_{Q,X} : \mathbf{QL}_{t,s}^{\text{vir}} \rightarrow \text{Sym}^\ell(X)$, satisfying $\pi_X = \pi_{Q,X} \circ Q$, where π_X is the map defined in Lemma 6.9.2.*
- (6) *There is a map,*

$$\tilde{\pi}_s : \mathbf{QL}_{t,s}^{\text{vir}} \rightarrow M_s,$$

such that $\tilde{\pi}_s \circ Q = \pi_s$, where π_s is the map defined in (9.2.3).

Because the images of the boundaries, $Q(\partial_j \bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i))$, have codimension greater than two, the fundamental class of $\mathbf{Q}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}$ can be written as a sum of the fundamental classes of the images of the subspaces. This allows us to apply a pushforward-pullback argument to the pairings of the cohomology classes with these subspaces.

We will construct the quotient $\mathbf{Q}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}$ as follows. Let

$$(10.2.2) \quad \bar{\text{Fr}}(\mathbf{t}, \mathbf{s}, \mathcal{P}_j) \rightarrow \Delta(X^\ell, \mathcal{P}_j)$$

be the extension of the fiber bundle $\text{Fr}(\mathbf{t}, \mathbf{s}, \mathcal{P}_j) \rightarrow \Delta^\circ(X^\ell, \mathcal{P}_j)$ defined in (6.2.2) from the open diagonal $\Delta^\circ(X^\ell, \mathcal{P}_j)$ to its closure, $\Delta(X^\ell, \mathcal{P}_j)$. In Lemma 10.2.4, we define a surjective, $G(\mathcal{P}_j)$ -equivariant map,

$$R_j : \text{Fr}(\mathbf{t}, \mathbf{s}, \mathcal{P}_j)|_{K_j} \rightarrow \bar{\text{Fr}}(\mathbf{t}, \mathbf{s}, \mathcal{P}_j),$$

which is injective over the complement of the boundaries $\cup_{i < j} \partial_i K_j$ defined in (8.3.4). We will define R_j as a composition of maps $R_{j,i}$, defined in Lemma 10.2.2.

Each of the maps $R_{j,i}$ defines a quotient $Q_{j,i}$ of the boundary $\partial_i \bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_j)$. In Lemma 10.2.7, we show that for $j < k$ the quotient $Q_{k,j}$ is defined by a quotient of the fiber of $\bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_j) \rightarrow M_{\mathbf{s}} \times K_j$. For $j < k < r$, the fibers of the quotient $Q_{r,j}$ contain those of $Q_{k,j}$ in the corner $\partial_r \partial_k \bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_j)$ and hence one can ‘compose’ these quotients. The result of applying the quotients $Q_{j+1,j}, Q_{j+2,j}, \dots, Q_{n,j}$ successively to $\bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_j)$ will be to replace the fiber $\bar{M}(\mathcal{P}_j, \varepsilon)$ with the quotient $\check{M}(\mathcal{P}_j, \varepsilon_j)$ appearing in Proposition 10.2.1. For $i < j$, the quotient map $Q_{j,i}$ respects the fibers of the bundle $\bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_j) \rightarrow M_{\mathbf{s}} \times K_j$ and thus can be applied to the quotient obtained by replacing the fiber $\bar{M}(\mathcal{P}_j, \varepsilon)$ with the quotient $\check{M}(\mathcal{P}_j, \varepsilon_j)$. In Lemma 10.2.6, we show that one can take the quotients $Q_{j,j-1}, Q_{j,j-2}, \dots, Q_{j,0}$ successively and obtain the quotient of $\bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_j)$ described in Proposition 10.2.1.

Finally, we show that the quotient maps $Q_{j,i}$ on $\partial_i \partial_j \bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_k)$ only identify points already identified by quotient maps $Q_{k,r}$ or $Q_{r,k}$ with $r = i$ or $r = j$. Thus, applying all the quotient maps $Q_{j,i}$ to the space $\bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}$ defines the quotient space $\mathbf{Q}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}$ described in the proposition.

10.2.1. Quotient maps. We construct the quotient maps $Q_{j,i}$ by defining a quotient of the frame bundle $\text{Fr}(\mathbf{t}, \mathbf{s}, \mathcal{P}_k)|_{K_k}$ in (6.2.2). As described in the Introduction to Section 10.2, this quotient will be given by the extension (10.2.2) of the fiber bundle $\text{Fr}(\mathbf{t}, \mathbf{s}, \mathcal{P}) \rightarrow \Delta^\circ(X^\ell, \mathcal{P})$ to the closed diagonal, $\Delta(X^\ell, \mathcal{P})$. Note that the structure group $G(\mathcal{P})/\Gamma(\mathcal{P})$ does not act freely on the extension $\bar{\text{Fr}}(\mathbf{t}, \mathbf{s}, \mathcal{P})$ due to the presence of diagonals corresponding to cruder partitions in $\Delta(X^\ell, \mathcal{P})$. For this reason, we will discuss $G(\mathcal{P})$ -equivariant maps rather than bundle maps.

For partitions $\mathcal{P}_i < \mathcal{P}_k$ of N_ℓ , let $\nu(X^\ell, \mathcal{P}_i \rightarrow [\mathcal{P}_k])$ be the normal bundle of $\Sigma(X^\ell, \mathcal{P}_i)$ in $\Sigma(X^\ell, \mathcal{P}_k)$ defined in (3.3.4). Let $\mathcal{O}(X^\ell, \mathcal{P}_i \rightarrow [\mathcal{P}_k], g_{\mathcal{P}_i}) \subset \nu(X^\ell, \mathcal{P}_i \rightarrow [\mathcal{P}_k])$ be the neighborhood of $\Sigma(X^\ell, \mathcal{P}_i)$ on which the exponential map $e(X^\ell, g_{\mathcal{P}_i})$ of (4.3.1) is defined. Let $\mathcal{U}(X^\ell, \mathcal{P}_i \rightarrow [\mathcal{P}_k], g_{\mathcal{P}_i}) \subset \Sigma(X^\ell, \mathcal{P}_k)$ be the image of $\mathcal{O}(X^\ell, \mathcal{P}_i \rightarrow [\mathcal{P}_k], g_{\mathcal{P}_i})$ under $e(X^\ell, g_{\mathcal{P}_i})$. For each $(A_0, \Phi_0) \in \check{N}_{\mathbf{t}(\ell), \mathbf{s}}(\delta)$, there is a $G(\mathcal{P}_k)$ -equivariant homeomorphism,

$$(10.2.3) \quad \begin{array}{c} \bar{\text{Fr}}(\mathbf{t}, \mathbf{s}, \mathcal{P}_k) \times_{\Sigma(X^\ell, \mathcal{P}_i)} \check{\mathcal{O}}(X^\ell, \mathcal{P}_i \rightarrow [\mathcal{P}_k], g_{\mathcal{P}_i}) \\ \downarrow \\ T(\mathcal{P}_i, \mathcal{P}_k)(A_0) \\ \downarrow \\ \bar{\text{Fr}}(\mathbf{t}, \mathbf{s}, \mathcal{P}_k)|_{\mathcal{U}(X^\ell, \mathcal{P}_i \rightarrow [\mathcal{P}_k], g_{\mathcal{P}_i})} \end{array}$$

defined by parallel translation of frames with respect to the locally flattened connection A'_0 , where $(A'_0, \Phi'_0) = \Theta_{\mathcal{P}_i}(A_0, \Phi_0)$, and the locally flattened metric $g_{\mathcal{P}_i, \mathbf{x}}$, where the frames in $\bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)$ lie over $\mathbf{x} \in \Sigma(X^\ell, \mathcal{P}_i)$. This is the same parallel translation used in the definition of the upwards transition map in (6.5.12).

LEMMA 10.2.2. *Let $\mathcal{P}_i < \mathcal{P}_k$ be partitions of N_ℓ . Let $\bar{t}(X^\ell, g_{\mathcal{P}_i})$ be the tubular distance function defined in (4.7.1) and let $\bar{D}(\mathcal{P}_i, \varepsilon_i)$ be the cube defined in (4.7.2). Then there is a $G(\mathcal{P}_k)$ -equivariant map, $R_{k,i} : \bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k) \rightarrow \bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)$, with the following properties:*

- (1) $R_{k,i}$ is injective on the restriction of $\bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)$ to the complement of the neighborhood $\bar{t}(X^\ell, g_{\mathcal{P}_i})^{-1}(\bar{D}(\mathcal{P}_i, \varepsilon_i))$.
- (2) For $j < i$, $R_{k,i}$ maps the restriction of $\bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i)$ to $\bar{t}(X^\ell, g_{\mathcal{P}_j})^{-1}(\bar{D}(\mathcal{P}_j, \varepsilon_j))$ to itself.
- (3) If $K_{k,i} = K_k \cup (\cup_{j=k-1}^i T_{k,j})$, where $T_{k,j}$ is defined in Lemma 4.8.2, then $R_{k,i}$ maps $\bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{K_{k,i+1}}$ onto $\bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{K_{k,i}}$.
- (4) If $F_1, F_2 \in \bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)$, then $R_{k,i}(F_1) = R_{k,i}(F_2)$ if and only if there are pairs (F_3, v_a) , for $a = 1, 2$, in the domain of the map $T(\mathcal{P}_i, \mathcal{P}_k)$ defined in (10.2.3) with $v_a \in \mathcal{O}_1(X^\ell, \mathcal{P}_i \rightarrow [\mathcal{P}_k])$ and $T(\mathcal{P}_i, \mathcal{P}_k)(F_3, v_a) = F_a$.
- (5) $R_{k,i}$ covers the restriction of the map r_i defined in Lemma 4.8.3 to $\Sigma(X^\ell, \mathcal{P}_k)$.

PROOF. For a partition \mathcal{P}_i of N_ℓ with $\mathcal{P}_i < \mathcal{P}_k$, Item (3) of Lemma 4.8.3 implies that the map $r_i : \text{Sym}^\ell(X) \rightarrow \text{Sym}^\ell(X)$ pulls back, by $e(X^\ell, g_{\mathcal{P}_i})$, to a bundle map,

$$r_{k,i} : \mathcal{O}(X^\ell, \mathcal{P}_i \rightarrow [\mathcal{P}_k], g_{\mathcal{P}_i}) \rightarrow \mathcal{O}(X^\ell, \mathcal{P}_i \rightarrow [\mathcal{P}_k], g_{\mathcal{P}_i}).$$

The map $r_{k,i}$ and the $G(\mathcal{P}_k)$ -equivariant homeomorphism (10.2.3) define a $G(\mathcal{P}_k)$ -equivariant map of $\bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{\mathcal{U}(X^\ell, \mathcal{P}_i \rightarrow [\mathcal{P}_k], g_{\mathcal{P}_i})}$ to itself. Because r_i is the identity on the complement of the subspace $\mathcal{U}(X^\ell, g_{\mathcal{P}_i})$, we can extend this map as the identity on the complement of the restriction $\bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{\mathcal{U}(X^\ell, g_{\mathcal{P}_i})}$. Let $R_{k,i}$ be this extension. Item (5) follows immediately from this definition.

Item (1) follows from this definition of $R_{k,i}$ and Item (9) of Lemma 4.8.3. Item (2) follows from Item (6) of Lemma 4.8.3. Item (3) follows from Item (8) of Lemma 4.8.3.

The $G(\mathcal{P}_k)$ -equivariance of $R_{k,i}$ implies that $R_{k,i}(F_1) = R_{k,i}(F_2)$ only if F_1 and F_2 lie over distinct points identified by $r_{k,i}$. Item (4) then follows from the construction of $R_{k,i}$. \square

The following lemma will be used to compare the fibers of the quotient maps $Q_{k,i}$.

LEMMA 10.2.3. *Let $\mathcal{P}_i < \mathcal{P}_j < \mathcal{P}_k$ be partitions of N_ℓ . Let $\partial_i \partial_j K_k$ be the corner of K_k defined in (8.3.4). Let $R_{k,i}$ and $R_{k,j}$ be the $G(\mathcal{P}_k)$ -equivariant maps defined in Lemma 10.2.2. If $F_1, F_2 \in \bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{\partial_i \partial_j K_k}$ and $R_{k,j}(F_1) = R_{k,j}(F_2)$, then $R_{k,i}(F_1) = R_{k,i}(F_2)$.*

PROOF. By Item (4) of Lemma 10.2.2, and the assumption that $R_{k,j}(F_1) = R_{k,j}(F_2)$, there are pairs (F_3, v_a) , for $a = 1, 2$, in the domain of $T(\mathcal{P}_j, \mathcal{P}_k)(A_0)$ with $T(\mathcal{P}_j, \mathcal{P}_k)(A_0)(F_3, v_a) = F_a$. The frame F_3 is itself a parallel translation of a frame $F_4 \in \bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{\Sigma(X^\ell, \mathcal{P}_i)}$. Because the parallel translations defining these maps are performed with respect to a locally flattened connection and metric which have no holonomy along these paths, there are points $w_a \in \tilde{\nu}(X^\ell, \mathcal{P}_i \rightarrow [\mathcal{P}_k])$ with $T(\mathcal{P}_i, \mathcal{P}_k)(A_0)(F_4, w_a) = F_a$. Hence, Item (3) of Lemma 10.2.2 implies that $R_{k,i}(F_1) = R_{k,i}(F_2)$. \square

LEMMA 10.2.4. *Let \mathcal{P}_k be a partition of N_ℓ and let K_k° be the interior of K_k . Then there is a surjective, $G(\mathcal{P})$ -equivariant map,*

$$R_k : \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{K_k} \rightarrow \bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k),$$

which is injective on $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{K_k^\circ}$. For $F_1, F_2 \in \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{\partial_j K_k - \cup_{i < j} \partial_i K_k}$, we have $R_k(F_1) = R_k(F_2)$ if and only if there are pairs (F_3, v_a) , for $a = 1, 2$, in the domain of the map $T(\mathcal{P}_j, \mathcal{P}_k)$ defined in (10.2.3) satisfying $T(\mathcal{P}_j, \mathcal{P}_k)(F_3, v_a) = F_a$. In addition, R_k covers the map m_{k-1} defined in the proof of Lemma 4.8.4.

PROOF. For the map $R_{k,i}$ defined in Lemma 10.2.2, define R_0 to be the identity map and for $k > 0$, set

$$(10.2.4) \quad R_k := R_{k,0} \circ R_{k,1} \circ \cdots \circ R_{k,k-1}.$$

Because $R_{k,i}$ covers r_i by Item (5) of Lemma 10.2.2, the definition (10.2.4) implies that R_k covers $m_{k-1} = r_0 \circ r_1 \circ \cdots \circ r_{k-1}$ as required. Because it is the composition of $G(\mathcal{P}_k)$ -equivariant maps, R_k is $G(\mathcal{P}_k)$ -equivariant. Let $T_{k,i}$ be as defined in Lemma 4.8.2. Item (3) from Lemma 10.2.2 and the equality,

$$\text{cl} \left(\Sigma(X^\ell, \mathcal{P}_k) \right) = K_k \cup \bigcup_{i < k} T_{k,i},$$

from Item (1) of Lemma 4.8.2 imply that R_k maps $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{K_k}$ onto $\bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)$. It remains only to check the injectivity properties.

Let $F_1, F_2 \in \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{K_k}$ satisfy $R_k(F_1) = R_k(F_2)$. There is an index u such that if $\tilde{F}_a := (R_{k,u+1} \circ \cdots \circ R_{k,k-1})(F_a)$ for $a = 1, 2$, then $\tilde{F}_1 \neq \tilde{F}_2$ while $R_{k,u}(\tilde{F}_1) = R_{k,u}(\tilde{F}_2)$. By Item (1) of Lemma 10.2.2, \tilde{F}_1 and \tilde{F}_2 lie over $\tilde{t}(X^\ell, g_{\mathcal{P}_u})^{-1}(\bar{D}(\mathcal{P}_u, \varepsilon_u))$. By Item (2) of Lemma 10.2.2, F_1 and F_2 also lie over $\tilde{t}(X^\ell, g_{\mathcal{P}_u})^{-1}(\bar{D}(\mathcal{P}_u, \varepsilon_u))$. Thus, F_1 and F_2 lie over $K_k \cap \tilde{t}(X^\ell, g_{\mathcal{P}_u})^{-1}(\bar{D}(\mathcal{P}_u, \varepsilon_u)) = \partial_u K_k$ and the restriction of R_k to $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{K_k^\circ}$ is injective.

If F_1 and F_2 lie over $\partial_j K_k - \cup_{i < j} \partial_i K_k$, and u and \tilde{F}_a are as in the preceding paragraph, then because the frames F_a lie over $\partial_u K_k$, we must have $u \geq j$. By Lemma 10.2.3, the equality $R_{k,u}(\tilde{F}_1) = R_{k,u}(\tilde{F}_2)$ would imply that $R_{k,j}(\tilde{F}_1) = R_{k,j}(\tilde{F}_2)$. By Item (4) of Lemma 10.2.2, there are pairs (F_3, v_a) in the domain of $T(\mathcal{P}_j, \mathcal{P}_k)$ with $T(\mathcal{P}_j, \mathcal{P}_k)(F_3, v_a) = \tilde{F}_a$ for $a = 1, 2$. By construction of the maps $R_{k,i}$, the frames F_a are parallel translations along piecewise continuous paths of the frames \tilde{F}_a . Because the maps $T(\mathcal{P}_i, \mathcal{P}_k)$ are defined by parallel translation with respect to a locally-flattened connection and metric which have no holonomy along these paths, we therefore obtain pairs (F_3, w_a) in the domain of $T(\mathcal{P}_j, \mathcal{P}_k)$ with $T(\mathcal{P}_j, \mathcal{P}_k)(F_3, w_a) = F_a$ for $a = 1, 2$, completing the proof of the lemma. \square

DEFINITION 10.2.5. For $\mathcal{P}_i < \mathcal{P}_k$ and the subspace $\partial_i \bar{\mathbf{BL}}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_k)$ defined in (8.1.16), define a quotient map $Q_{k,i}$ on $\bar{\mathbf{BL}}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_k)$ by identifying points,

$$[(A_a, \Phi_a), F_a, [\vec{A}_a]] \in \partial_i \bar{\mathbf{BL}}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_k) \quad \text{for } a = 1, 2,$$

where $(A_a, \Phi_a) \in \tilde{M}_s$, and $F_a \in \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)$, and $[\vec{A}_a] \in \bar{M}(\mathcal{P}_k, \varepsilon)$ (the fiber of $\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_k)$ defined in (8.2.1)), if $R_{k,i}(F_1) = R_{k,i}(F_2)$, where $R_{k,i}$ is the map defined in Lemma 10.2.2.

Therefore, by Item (1) of Lemma 10.2.2, the quotient map $Q_{k,i}$ is injective on the complement of the boundary $\partial_i \bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_k)$ and on that boundary is defined by the projection map,

$$\text{Fr}(\mathbf{t}, \mathbf{s}, \mathcal{P}_k)|_{\partial_i K_k} \rightarrow \bar{\text{Fr}}(\mathbf{t}, \mathbf{s}, \mathcal{P}_k)|_{K_i},$$

defined by the parallel translation map in (10.2.3) and the projection map,

$$\bar{\text{Fr}}(\mathbf{t}, \mathbf{s}, \mathcal{P}_k) \times_{\Sigma(X^\ell, \mathcal{P}_i)} \tilde{\mathcal{O}}(X^\ell, \mathcal{P}_i \rightarrow [\mathcal{P}_k], g_{\mathcal{P}_i}) \rightarrow \bar{\text{Fr}}(\mathbf{t}, \mathbf{s}, \mathcal{P}_k)|_{\Sigma(X^\ell, \mathcal{P}_i)}.$$

We note the following relation between quotient maps $Q_{k,i}$ and $Q_{k,j}$.

LEMMA 10.2.6. *Let $\mathcal{P}_i < \mathcal{P}_j < \mathcal{P}_k$ be partitions of N_ℓ . If $\mathbf{A}_1, \mathbf{A}_2 \in \partial_i \partial_j \bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_k)$, the subspace defined in (8.1.16), and $Q_{k,j}(\mathbf{A}_1) = Q_{k,j}(\mathbf{A}_2)$ then $Q_{k,i}(\mathbf{A}_1) = Q_{k,i}(\mathbf{A}_2)$.*

PROOF. This follows immediately from the corresponding properties of $R_{k,i}$ and $R_{k,j}$ described in Lemma 10.2.3. \square

Lemma 10.2.6 implies that for each k , one can apply the quotient maps $Q_{k,i}$ in order of descending i as was done for the maps $R_{k,i}$ in Equation (10.2.4). The resulting quotient map is Q_k .

10.2.2. Construction of the local quotient. We now construct a quotient of the subspace $\bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i)$ using the maps $Q_{j,i}$. We have described the quotient map $Q_{j,i}$ on $\partial_i \bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_j)$ when $\mathcal{P}_i < \mathcal{P}_j$. By (8.3.2), we have $\partial_j \bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i) = \partial_i \bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_j)$ so $Q_{j,i}$ also defines a quotient of the boundary $\partial_j \bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i)$. We begin by showing that for $\mathcal{P}_i < \mathcal{P}_j$, the quotient map $Q_{j,i}$ on $\partial_j \bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i)$ is defined by a quotient of the fiber of $\bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i) \rightarrow M_{\mathbf{s}} \times K_i$.

LEMMA 10.2.7. *If $\mathcal{P}_i < \mathcal{P}_j$ are partitions of N_ℓ and $\partial_j \bar{M}(\mathcal{P}_i, \varepsilon_i)$ is the boundary defined in (8.3.3), then the restriction of the map $Q_{j,i}$ to $\partial_j \bar{\mathbf{B}}\mathbf{L}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i)$ is given by the map,*

$$\begin{aligned} M_{\mathbf{s}} \times_{\mathcal{G}_{\mathbf{s}} \times S^1} \text{Fr}(\mathbf{t}, \mathbf{s}, \mathcal{P}_i) \times_{G(\mathcal{P}_i)} \partial_j \bar{M}(\mathcal{P}_i, \varepsilon_i) \\ \downarrow \\ M_{\mathbf{s}} \times_{\mathcal{G}_{\mathbf{s}} \times S^1} \text{Fr}(\mathbf{t}, \mathbf{s}, \mathcal{P}_i) \times_{G(\mathcal{P}_i)} \bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \bar{M}(\mathcal{P}'') / \mathfrak{S}(\mathcal{P}_i) \end{aligned}$$

determined by the map $c_{j,i}$ in (8.3.12) on the fibers.

PROOF. The upwards overlap map,

$$\begin{aligned} \tilde{M}_{\mathbf{s}} \times_{\mathcal{G}_{\mathbf{s}}} \text{Fr}(\mathbf{t}, \mathbf{s}, \mathcal{P}_i) \times_{G(\mathcal{P}_i)} \bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \prod_{P \in \mathcal{P}_j} (\Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}_P'') \times \bar{M}(\mathcal{P}_P'')) \\ \downarrow \rho_{\mathcal{P}_i, [\mathcal{P}_i]}^{\mathbf{t}, \mathbf{s}, u} \\ \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \left(\tilde{M}_{\mathbf{s}} \times_{\mathcal{G}_{\mathbf{s}}} \text{Fr}(\mathbf{t}, \mathbf{s}, \mathcal{P}'') \times_{G(\mathcal{P}'')} \bar{M}(\mathcal{P}'') \right) \right) / \mathfrak{S}(\mathcal{P}_i) \\ \cong \downarrow \\ \tilde{M}_{\mathbf{s}} \times_{\mathcal{G}_{\mathbf{s}}} \text{Fr}(\mathbf{t}, \mathbf{s}, \mathcal{P}_j) \times_{G(\mathcal{P}_j)} \bar{M}(\mathcal{P}_j) \end{aligned}$$

is defined in (6.5.13) by the exponential map and parallel translation with respect to the same connections used to define $R_{j,i}$. Hence, pullback of the quotient $Q_{j,i}$ by the map $\rho_{\mathcal{P}_i, [\mathcal{P}_i]}^{\mathbf{t}, \mathbf{s}, u}$ is defined

by the map on the fibers of the domain of $\rho_{\mathcal{P}_i, [\mathcal{P}_i]}^{\mathfrak{t}, \mathfrak{s}, u}$ given by the projection,

$$\begin{aligned} & \bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \prod_{P \in \mathcal{P}_j} (\Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}''_P) \times \bar{M}(\mathcal{P}''_P)) \\ & \quad \downarrow \\ & \bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \prod_{P \in \mathcal{P}_j} \bar{M}(\mathcal{P}''_P) \\ & \quad = \downarrow \\ & \bigsqcup_{\mathcal{P}'' \in [\mathcal{P}_i < \mathcal{P}_j]} \bar{M}(\mathcal{P}'') \end{aligned}$$

The overlap map $\rho_{\mathcal{P}_i, [\mathcal{P}_i]}^{\mathfrak{t}, \mathfrak{s}, d}$ is defined by the $G(\mathcal{P}_i)$ -equivariant map $\rho_{f, \mathcal{P}_i, [\mathcal{P}_i]}^{\mathfrak{t}, \mathfrak{s}, d}$ of fibers from (6.5.16) and thus pushes the preceding projection forward to the projection described in the statement of the lemma. \square

To compare the quotient maps $Q_{i,k}$ and $Q_{j,k}$, for $k < i < j$, Lemma 10.2.7 shows that it suffices to compare the maps $c_{i,k}$ and $c_{j,k}$ and which we now do.

LEMMA 10.2.8. *Let $\mathcal{P}_k < \mathcal{P}_i < \mathcal{P}_j$ be partitions of N_ℓ . If $[\vec{A}_1], [\vec{A}_2] \in \partial_i \partial_j \bar{M}(\mathcal{P}_k, \varepsilon)$, the intersection defined in (8.3.3), satisfy $c_{i,k}([\vec{A}_1]) = c_{i,k}([\vec{A}_2])$, then $c_{j,k}([\vec{A}_1]) = c_{j,k}([\vec{A}_2])$.*

PROOF. The assumption that $[\vec{A}_a] \in \partial_i \partial_j \bar{M}(\mathcal{P}_k, \varepsilon)$ for $a = 1, 2$ implies that $[\vec{A}_a]$ lies in the images of the maps of the fibers $\rho_{f, \mathcal{P}_k, [\mathcal{P}_i]}^{\mathfrak{t}, \mathfrak{s}, d}$ and $\rho_{f, \mathcal{P}_k, [\mathcal{P}_j]}^{\mathfrak{t}, \mathfrak{s}, d}$ defined in (6.5.17). Thus, by the definition of the maps $\rho_{f, \mathcal{P}_k, [\mathcal{P}_i]}^{\mathfrak{t}, \mathfrak{s}, d}$ and $\rho_{f, \mathcal{P}_k, [\mathcal{P}_j]}^{\mathfrak{t}, \mathfrak{s}, d}$, if we write $[\vec{A}_a] = ([A_{P,a}]_{P \in \mathcal{P}_k})$, where $[A_{P,a}] \in \bar{M}_{\text{SPL}, |P|}^{\mathfrak{s}, \mathfrak{t}}(\delta_P)$, then

$$[A_{P,a}] \in \text{Im}(\gamma'_{\Theta, \mathcal{P}'_P}) \cap \text{Im}(\gamma'_{\Theta, \mathcal{P}''_P}) \quad \text{for all } P \in \mathcal{P}_k,$$

for some $\mathcal{P}' \in [\mathcal{P}_k < \mathcal{P}_i]$ and $\mathcal{P}'' \in [\mathcal{P}_k < \mathcal{P}_j]$, where $\gamma'_{\Theta, \mathcal{P}'_P}$ is the splicing map defined in (5.4.3). Because the preceding intersection is non-empty, we have $\mathcal{P}_k < \mathcal{P}' < \mathcal{P}''$. Lemma 5.6.5 and the commutativity of the diagram (5.5.27) imply that

$$A_{P,a} = \left(\gamma'_{\Theta, \mathcal{P}'_P} \circ \rho_{\mathcal{P}'_P, [\mathcal{P}'_P]}^{\Theta, d} \right) (B_P) = \left(\gamma'_{\Theta, \mathcal{P}''_P} \circ \rho_{\mathcal{P}''_P, [\mathcal{P}''_P]}^{\Theta, u} \right) (B_P),$$

for some B_P in the following overlap space defined in (5.6.3) (omitting the symmetric group as it is not relevant to this discussion),

$$\begin{aligned} & \tilde{\mathcal{O}}^{\text{ASD}}(\Theta, \mathcal{P}'_P, [\mathcal{P}''_P], \delta) \\ & \subseteq \Delta^\circ(Z_P(\delta_P), \mathcal{P}'_P) \times \prod_{Q' \in \mathcal{P}'_P} \left(\Delta^\circ(Z_{Q'}(\delta_{Q'}), \mathcal{P}''_{Q'}) \times \prod_{Q'' \in \mathcal{P}''_{Q'}} \bar{M}_{\text{SPL}, |Q''|}^{\mathfrak{s}, \mathfrak{t}}(\delta_{Q''}) \right). \end{aligned}$$

The map,

$$c_{j,k} \circ \left(\prod_{P \in \mathcal{P}_k} \gamma'_{\Theta, \mathcal{P}''_P} \circ \rho_{\mathcal{P}''_P, [\mathcal{P}''_P]}^{\Theta, u} \right),$$

is given by the projection,

$$\begin{aligned} & \prod_{P \in \mathcal{P}_k} \left(\Delta^\circ(Z_P(\delta_P), \mathcal{P}'_P) \times \prod_{Q' \in \mathcal{P}'_P} \left(\Delta^\circ(Z_{Q'}(\delta_{Q'}), \mathcal{P}''_{Q'}) \times \prod_{Q'' \in \mathcal{P}''_{Q'}} \bar{M}_{\text{SPL}, |Q''|}^{s, \natural}(\delta_{Q''}) \right) \right) \\ & \quad \downarrow \\ & \prod_{P \in \mathcal{P}_k} \prod_{Q' \in \mathcal{P}'_P} \prod_{Q'' \in \mathcal{P}''_{Q'}} \bar{M}_{\text{SPL}, |Q''|}^{s, \natural}(\delta_{Q''}) \end{aligned}$$

Moreover, the map,

$$c_{i,k} \circ \left(\prod_{P \in \mathcal{P}_k} \gamma'_{\Theta, \mathcal{P}'_P} \circ \rho_{\mathcal{P}'_P, [\mathcal{P}''_P]}^{\Theta, d} \right),$$

is given by the projection,

$$\begin{aligned} & \prod_{P \in \mathcal{P}_k} \left(\Delta^\circ(Z_P(\delta_P), \mathcal{P}'_P) \times \prod_{Q' \in \mathcal{P}'_P} \left(\Delta^\circ(Z_{Q'}(\delta_{Q'}), \mathcal{P}''_{Q'}) \times \prod_{Q'' \in \mathcal{P}''_{Q'}} \bar{M}_{\text{SPL}, |Q''|}^{s, \natural}(\delta_{Q''}) \right) \right) \\ & \quad \downarrow \\ & \prod_{P \in \mathcal{P}_k} \prod_{Q' \in \mathcal{P}'_P} \left(\Delta^\circ(Z_{Q'}(\delta_{Q'}), \mathcal{P}''_{Q'}) \times \prod_{Q'' \in \mathcal{P}''_{Q'}} \bar{M}_{\text{SPL}, |Q''|}^{s, \natural}(\delta_{Q''}) \right) \end{aligned}$$

The desired equality, $c_{j,k}(A_1) = c_{j,k}(A_2)$, follows by comparing the preceding projections and using the commutativity of the diagram (5.5.27). \square

Lemma 10.2.8 allows us to make the following

DEFINITION 10.2.9. Define $\check{M}(\mathcal{P}_k, \varepsilon)$ to be the quotient of the fiber $\bar{M}(\mathcal{P}_k, \varepsilon)$ of $\bar{\mathbf{L}}_{t,s}^{\text{vir}}(\mathcal{P}_k)$ defined in (8.2.1) by applying the quotient maps $c_{k+1,k}, c_{k+2,k}, \dots, c_{n,k}$, defined in (8.3.12), successively. Let $q_k : \bar{M}(\mathcal{P}_k, \varepsilon) \rightarrow \check{M}(\mathcal{P}_k, \varepsilon)$ denote the resulting quotient map.

LEMMA 10.2.10. *Let $q_k : \bar{M}(\mathcal{P}_k, \varepsilon) \rightarrow \check{M}(\mathcal{P}_k, \varepsilon)$ be the quotient map in Definition 10.2.9. Then $\check{M}(\mathcal{P}_k, \varepsilon)$ is a smoothly-stratified space, with a smoothly-stratified $G(\mathcal{P}_k)$ action satisfying the following properties:*

- (1) *For all $i > k$, the image $q_k(\partial_i \bar{M}(\mathcal{P}_k, \varepsilon))$ has codimension greater than or equal to two in $\check{M}(\mathcal{P}_k, \varepsilon)$.*
- (2) *The action of $S^1 < \text{SO}(3)$ on*

$$\bar{M}(\mathcal{P}_k, \varepsilon) \subseteq \prod_{P \in \mathcal{P}_k} \bar{M}_{\text{SPL}, |P|}^{s, \natural}(\delta_P)$$

given by (8.2.3) pulls back from an action of $\text{SO}(3)$ on $\check{M}(\mathcal{P}_k, \varepsilon)$ with finite isotropy as defined prior to Lemma 8.3.3.

- (3) *The map q_k is $G(\mathcal{P}_k)$ -equivariant.*

PROOF. Recall from (8.2.1) that $\bar{M}(\mathcal{P}_k, \varepsilon)$ is the subspace of $\bar{M}(\mathcal{P}_k)$ where the maps $\vec{t}_f(\mathcal{P}_k, [\mathcal{P}_j])$ with $k \leq j$ take values in smoothly-stratified subspaces of $[0, 1]^{|\mathcal{P}_j|}$. Hence, the smooth strata of $\bar{M}(\mathcal{P}_k, \varepsilon)$ are given by taking the intersection of smooth strata of $\bar{M}(\mathcal{P}_k)$ with the pre-image under $\vec{t}_f(\mathcal{P}_k, [\mathcal{P}_j])$ of these smooth strata of $[0, 1]^{|\mathcal{P}_j|}$. The quotients given by the maps $c_{j,k}$ are defined by replacing the union of strata $\partial_j \bar{M}(\mathcal{P}_k, \varepsilon)$ with the smoothly-stratified space given by the image of $c_{j,k}$. Hence, $\check{M}(\mathcal{P}_k, \varepsilon)$ is a smoothly-stratified space. The maps $\vec{t}_f(\mathcal{P}_k, [\mathcal{P}_j])$ are invariant under the smooth, stratum-preserving $G(\mathcal{P}_k)$ action on $\bar{M}(\mathcal{P}_k)$, so the $G(\mathcal{P}_k)$ action

also preserves the pre-images of the strata of $[0, 1]^{|\mathcal{P}_j|}$ under the map $\vec{t}_f(\mathcal{P}_k, [\mathcal{P}_j])$. The quotient maps $c_{j,k}$ are $G(\mathcal{P}_k)$ -equivariant and so the action of $G(\mathcal{P}_k)$ on $\bar{M}(\mathcal{P}_k, \varepsilon)$ descends to define an action on $\check{M}(\mathcal{P}_k, \varepsilon)$. The equivariance of q_k in Item (3) follows immediately from this construction.

The assertion regarding the codimensions of the images of the boundaries follows immediately from the construction. The assertion regarding the $\text{SO}(3)$ action follows from the equivariance of each map $c_{i,k}$ with respect to this action and from the fact that $\text{SO}(3)$ acts with finite isotropy on the image of each map $c_{i,k}$ by Lemma 8.3.3. That is, if $e^{i\theta} \in S^1 < \text{SO}(3)$ and $[[\vec{A}]] \in \check{M}(\mathcal{P}_k, \varepsilon)$ is the image of $[\vec{A}] \in \bar{M}(\mathcal{P}_k, \varepsilon)$ with $e^{i\theta}[[\vec{A}]] = [e^{i\theta}[\vec{A}]] = [[\vec{A}]]$, then there is a smallest integer j such that $c_{j,k}(e^{i\theta}[\vec{A}]) = c_{j,k}([\vec{A}])$. Lemma 8.3.2 implies that $e^{i\theta}$ is therefore a root of unity, of order o_j determined by j . Hence, if $[\vec{A}] \in \bar{M}(\mathcal{P}_k, \varepsilon)$ and $e^{i\theta} \in S^1 < \text{SO}(3)$ satisfy $e^{i\theta}[[\vec{A}]] = [[\vec{A}]]$, then $e^{i\theta}$ must be a root of unity whose order divides the product $o_{k+1}o_{k+2} \dots o_n$ of those orders. \square

By Lemma 10.2.7, the result of applying the quotient maps

$$Q_{k+1,k}, Q_{k+2,k}, \dots, Q_{n,k}$$

to $\bar{\mathbf{BL}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_k)$ is the same as that of applying the quotient maps

$$c_{k+1,k}, c_{k+2,k}, \dots, c_{n,k}$$

to the fiber $\bar{M}(\mathcal{P}_k, \varepsilon)$. From Lemma 10.2.10, we see that the quotient of $\bar{\mathbf{BL}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_k)$ by the quotient maps $Q_{k+1,k}, Q_{k+2,k}, \dots, Q_{n,k}$ is equal to

$$(10.2.5) \quad \widehat{\mathbf{BL}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_k) := \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{K_k} \times_{G(\mathcal{P}_k)} \check{M}(\mathcal{P}_k, \varepsilon).$$

For integers $j < k$, the quotient maps $Q_{k,j}$ preserve the fibers of $\bar{\mathbf{BL}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_k) \rightarrow M_{\mathfrak{s}} \times K_k$ and hence define quotients of $\widehat{\mathbf{BL}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_k)$. We characterize the result of taking these further quotients as follows.

LEMMA 10.2.11. *Let \mathcal{P}_k be a partition of N_{ℓ} . Then after applying the quotient maps*

$$Q_{k+1,k}, Q_{k+2,k}, \dots, Q_{n,k}, Q_{k,k-1}, Q_{k,k-2}, \dots, Q_{k,1}$$

successively, we obtain the quotient map,

$$\begin{array}{c} \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{K_k} \times_{G(\mathcal{P}_k)} \bar{M}(\mathcal{P}_k, \varepsilon) \\ \downarrow Q_k \\ \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k) \times_{G(\mathcal{P}_k)} \check{M}(\mathcal{P}_k, \varepsilon) \end{array}$$

defined for $(A_0, \Phi_0) \in \tilde{M}_{\mathfrak{s}}$, and $F \in \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{K_k}$, and $[\vec{A}] \in \bar{M}(\mathcal{P}_k, \varepsilon)$ by

$$Q_k \left([(A_0, \Phi_0), F, [\vec{A}]] \right) := \left[(A_0, \Phi_0), R_k(F), q_k([\vec{A}]) \right],$$

where R_k is defined in Lemma 10.2.4 and q_k is defined in Lemma 10.2.10.

PROOF. By applying the quotient maps $Q_{k+1,k}, Q_{k+2,k}, \dots, Q_{n,k}$, we obtain the space $\widehat{\mathbf{BL}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_k)$ as described above in (10.2.5). From the definition of the quotient maps $Q_{k,j}$ in Definition 10.2.5, the result of applying the quotient maps $Q_{k,k-1}, Q_{k,k-2}, \dots, Q_{k,1}$ to $\widehat{\mathbf{BL}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_k)$ is the same as that

of applying the quotient maps $R_{k,k-1}, R_{k,k-2}, \dots, R_{k,1}$ to $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{K_k}$. By Lemma 10.2.4, applying the quotient maps $R_{k,k-1}, R_{k,k-2}, \dots, R_{k,1}$ to $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{K_k}$ gives the quotient of $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{K_k}$ defined by the surjective map, $R_k : \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)|_{K_k} \rightarrow \bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k)$. \square

We note the following

LEMMA 10.2.12. *Continue the notation of Lemma 10.2.11. The pullback by Q_k of the projection,*

$$(10.2.6) \quad \tilde{\pi}_{X,k} : \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_k) \times_{G(\mathcal{P}_k)} \check{M}(\mathcal{P}_k, \varepsilon), \rightarrow \Sigma(X^\ell, \mathcal{P}_k)$$

is equal to the restriction of the map π_X to $\bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_k)$.

PROOF. The conclusion follows immediately from the assertion that R_k covers the map m_{k-1} in Lemma 10.2.4 and the construction of the map π_X in Lemmas 4.8.4 and 6.9.2. \square

10.2.3. Global quotient of $\bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$. Lemma 10.2.11 describes a quotient of each subspace $\bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_j)$ of $\bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$. We have used Lemma 10.2.7 to ensure that we are applying the same quotient to both boundaries in the equality (8.3.2),

$$\partial_i \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_j) = \partial_j \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_i).$$

We will define the global quotient appearing in the statement of Proposition 10.2.1 by applying all the quotient maps $Q_{j,i}$ in an appropriate order. To ensure that the resulting global quotient satisfies Proposition 10.2.1, we need the following lemma to show that no identifications, other than those already in the statement of Lemma 10.2.11, appear in the global quotient.

LEMMA 10.2.13. *For $i < j$, if $\mathbf{A}_1, \mathbf{A}_2 \in \partial_i \partial_j \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_r)$ satisfy $Q_{j,i}(\mathbf{A}_1) = Q_{j,i}(\mathbf{A}_2)$, then the following hold:*

- (1) *If $i < j < r$, then $Q_{r,i}(\mathbf{A}_1) = Q_{r,i}(\mathbf{A}_2)$.*
- (2) *If $r < i < j$, then $Q_{j,r}(\mathbf{A}_1) = Q_{j,r}(\mathbf{A}_2)$.*
- (3) *If $i < r < j$, then there is an element $\mathbf{A}' \in \partial_i \partial_j \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_r)$ such that $Q_{r,i}(\mathbf{A}_1) = Q_{r,i}(\mathbf{A}')$, and $Q_{j,r}(\mathbf{A}_1) = Q_{j,r}(\mathbf{A}')$.*

PROOF. Items (1) and (2) follow immediately from Lemmas 10.2.8 and 10.2.6, respectively. We now prove Item (3). Because $\partial_i \partial_j \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_r) = \partial_i \partial_r \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_j)$ by (8.3.2), we can consider the points \mathbf{A}_1 and \mathbf{A}_2 to be elements of $\bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_j)$. By Definition 10.2.5, points $\mathbf{A}_1, \mathbf{A}_2 \in \partial_i \bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_j)$ are identified by $Q_{j,i}$ if they can be expressed as

$$\mathbf{A}_a = [(A_0, \Phi_0), T(\mathcal{P}_i, \mathcal{P}_j)(A_0)(F_3, v_a), ([A_Q])_{Q \in \mathcal{P}_j}], \quad a = 1, 2,$$

where $(A_0, \Phi_0) \in \tilde{M}_{\mathfrak{s}}$, and (F_3, v_a) is in the domain of $T(\mathcal{P}_i, \mathcal{P}_j)$ given in (10.2.3), and $([A_Q])_{Q \in \mathcal{P}_j} \in \bar{M}(\mathcal{P}_j, \varepsilon)$. Note that the frames $T(\mathcal{P}_i, \mathcal{P}_j)(A_0)(F_3, v_a)$ lie over points in $\partial_i \partial_r K_j$. Since

$$(10.2.7) \quad \bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j)|_{\Sigma(X^\ell, \mathcal{P}_i)} \cong \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i) \times_{\tilde{G}(\mathcal{P}_i)} \tilde{G}(\mathcal{P}_j),$$

we can assume that $F_3 \in \bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_i)$. Because the parallel translations used to define the maps $T(\mathcal{P}_i, \mathcal{P}_j)(A_0)$ are performed with the locally flattened connection, A'_0 , we can write the parallel translation of F_3 given by $T(\mathcal{P}_i, \mathcal{P}_j)(A_0)(F_3, v_a)$ as a composition of parallel translations, first the parallel translation $T(\mathcal{P}_i, \mathcal{P}_r)(A_0)(\cdot, u_a)$ to a point in $\partial_i K_r$, and then the parallel translation $T(\mathcal{P}_r, \mathcal{P}_j)(A_0)(\cdot, w_a)$ from that point to the point in $\partial_i \partial_r K_j$:

$$(10.2.8) \quad T(\mathcal{P}_i, \mathcal{P}_j)(A_0)(F_3, v_a) = T(\mathcal{P}_j, \mathcal{P}_r)(A_0) (T(\mathcal{P}_i, \mathcal{P}_r)(A_0) (F_3, v'_a), w_a).$$

By the identification in (6.5.6), we can write the elements (F_3, v'_a) of the domain of $T(\mathcal{P}_i, \mathcal{P}_r)(A_0)$ as $(F_3, (v'_{P,a})_{P \in \mathcal{P}''})$, where $\mathcal{P}'' \in [\mathcal{P}_r < \mathcal{P}_j]$ and $v'_{P,a} \in \Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}''_P)$. Then the definition of the upwards transition map in (6.5.14) implies that $\mathbf{A}_a = \rho_{\mathcal{P}_r, [\mathcal{P}_j]}^{\mathfrak{t}, \mathfrak{s}, u}(\mathbf{A}_a^r)$, where

$$\mathbf{A}_a^r = \left[(A_0, \Phi_0), T(\mathcal{P}_i, \mathcal{P}_r)(A_0)(F_3, w_a), (v'_{P,a}, (A_Q)_{Q \in \mathcal{P}''_P})_{P \in \mathcal{P}_r} \right],$$

and $T(\mathcal{P}_i, \mathcal{P}_r)(A_0)(F_3, w_a) \in \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_r)$, and $\mathcal{P}'' \in [\mathcal{P}_r < \mathcal{P}_j]$, $v'_{P,a}$ and A_Q are as above.

If we define

$$\mathbf{A}' := \rho_{\mathcal{P}_r, [\mathcal{P}_j]}^{\mathfrak{t}, \mathfrak{s}, u} \left(\left[(A_0, \Phi_0), T(\mathcal{P}_i, \mathcal{P}_r)(A_0)(F_3, w_2), (v'_{P,1}, (A_Q)_{Q \in \mathcal{P}''_P})_{P \in \mathcal{P}_r} \right] \right),$$

then $Q_{j,r}(\mathbf{A}_2) = Q_{j,r}(\mathbf{A}')$. The latter equality holds because if we write $F'_3 = T(\mathcal{P}_i, \mathcal{P}_r)(A_0)(F_3, w_a)$, then by (10.2.8) we can express \mathbf{A}_2 and \mathbf{A}' as

$$\begin{aligned} \mathbf{A}_2 &= \left[(A_0, \Phi_0), T(\mathcal{P}_j, \mathcal{P}_r)(A_0)(F'_3, v'_{P,2}), (A_Q)_{Q \in \mathcal{P}_j} \right], \\ \mathbf{A}' &= \left[(A_0, \Phi_0), T(\mathcal{P}_j, \mathcal{P}_r)(A_0)(F'_3, v'_{P,1}), (A_Q)_{Q \in \mathcal{P}_j} \right]. \end{aligned}$$

To see that $Q_{r,i}(\mathbf{A}_1) = Q_{r,i}(\mathbf{A}')$, we apply the downwards overlap map $\rho_{\mathcal{P}_r, [\mathcal{P}_j]}^{\mathfrak{t}, \mathfrak{s}, d}$ defined in (6.5.15), so that

$$\begin{aligned} \mathbf{A}_1 &= \rho_{\mathcal{P}_r, [\mathcal{P}_j]}^{\mathfrak{t}, \mathfrak{s}, d} \left(\left[(A_0, \Phi_0), T(\mathcal{P}_i, \mathcal{P}_r)(A_0)(F_3, w_1), (v'_{P,1}, (A_Q)_{Q \in \mathcal{P}''_P})_{P \in \mathcal{P}_r} \right] \right) \\ &= \left[(A_0, \Phi_0), T(\mathcal{P}_i, \mathcal{P}_r)(A_0)(F_3, w_1), (A'_P)_{P \in \mathcal{P}_r} \right], \\ \mathbf{A}' &= \rho_{\mathcal{P}_r, [\mathcal{P}_j]}^{\mathfrak{t}, \mathfrak{s}, d} \left(\left[(A_0, \Phi_0), T(\mathcal{P}_i, \mathcal{P}_r)(A_0)(F_3, w_2), (v'_{P,1}, (A_Q)_{Q \in \mathcal{P}''_P})_{P \in \mathcal{P}_r} \right] \right) \\ &= \left[(A_0, \Phi_0), T(\mathcal{P}_i, \mathcal{P}_r)(A_0)(F_3, w_2), (A'_P)_{P \in \mathcal{P}_r} \right], \end{aligned}$$

where, for $P \in \mathcal{P}_r$ and $\mathcal{P}'' \in [\mathcal{P}_r < \mathcal{P}_j]$, we denote

$$A'_P := \gamma'_{\Theta, \mathcal{P}''_P} \left((v'_{P,1}, (A_Q)_{Q \in \mathcal{P}''_P}) \right).$$

Because the preceding expressions for \mathbf{A}_1 and \mathbf{A}' differ only in the “ w_a ” coordinates, the desired equality $Q_{r,i}(\mathbf{A}_1) = Q_{r,i}(\mathbf{A}')$ follows. \square

We can now give the

PROOF OF PROPOSITION 10.2.1. Let $\mathcal{P}_0, \dots, \mathcal{P}_n$ be the partitions of N_ℓ chosen, as in §3.4, to enumerate the strata of $\text{Sym}^\ell(X)$. We perform the quotients $Q_{j,i}$ in the following order:

$$\begin{aligned} &Q_{n,n-1}, \\ &Q_{n-1,n-2}, Q_{n,n-2}, \\ &\vdots \\ &Q_{j+1,j}, Q_{j+2,j}, \dots, Q_{n,j}, \\ &\vdots \\ &Q_{1,0}, Q_{2,0}, \dots, Q_{n,0}. \end{aligned}$$

Lemma 10.2.13 implies that the application of these quotients to the subspace $\bar{\mathbf{B}}\mathbf{L}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_r)$ yields the same quotient as that defined by Lemma 10.2.11. This establishes Items (1)–(3).

The restriction of the bundle \mathbb{L}_ν to the subspace $\overline{\mathbf{BL}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_r)$ is the complex line bundle associated to the S^1 bundle,

$$(10.2.9) \quad \begin{array}{c} \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_r)|_{K_r} \times_{G(\mathcal{P}_r)} \bar{M}(\mathcal{P}_r, \varepsilon) \\ \downarrow \\ \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_r)|_{K_r} \times_{G(\mathcal{P}_r) \times S^1} \bar{M}(\mathcal{P}_r, \varepsilon) \end{array}$$

where the S^1 action is defined in (6.2.13). This S^1 action is trivial on $\tilde{M}_{\mathfrak{s}}$ and acts diagonally on the frames in $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_r)$ as described in Section 6.2.4. Because of the action of $G(\mathcal{P}_r)$ on the base of (10.2.9), this S^1 action equals the diagonal action of S^1 on the frames in $\bar{M}(\mathcal{P}_r, \varepsilon)$ as described in (8.2.3). By Item (2) of Lemma 10.2.10, this action on $\bar{M}(\mathcal{P}_r, \varepsilon)$ pulls back from an action with finite isotropy group, Γ_r , on $\check{M}(\mathcal{P}_r, \varepsilon)$. By Lemma 8.3.2, the orders of the isotropy groups of the S^1 action on

$$(10.2.10) \quad \tilde{M}_{\mathfrak{s}} \times \bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_r) \times_{G(\mathcal{P}_r)} \check{M}(\mathcal{P}_r, \varepsilon),$$

given by this S^1 action on $\check{M}(\mathcal{P}_r, \varepsilon)$ will divide $b_r := |\Gamma_r| |\mathfrak{S}(\mathcal{P}_r)|$. If we define $b := b_0 \cdots b_N$, then there is a complex line bundle over $\mathbf{QL}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$, with restriction to $Q(\overline{\mathbf{BL}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_r))$ given by

$$\tilde{M}_{\mathfrak{s}} \times \bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_r) \times_{G(\mathcal{P}_r)} \check{M}(\mathcal{P}_r, \varepsilon) \times_{(S^1, \times b)} \mathbb{C},$$

whose pullback to $\overline{\mathbf{BL}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ is $\mathbb{L}_\nu^{\otimes b}$. By $(S^1, \times b)$ in the preceding expression, we mean that S^1 acts with multiplicity b on \mathbb{C} to ensure that the stabilizers act trivially on \mathbb{C} . This completes the proof of Item (4).

Define the map $\pi_{X,Q}$ by the projection (10.2.6) on $Q(\overline{\mathbf{BL}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_j))$. Then Item (5) follows from Lemma 10.2.12. On each subspace $Q(\overline{\mathbf{BL}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_j))$, the projection map

$$\tilde{\pi}_{\mathfrak{s},j} : \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \bar{\text{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}_j) \times_{G(\mathcal{P}_j)} \check{M}(\mathcal{P}_j, \varepsilon) \rightarrow M_{\mathfrak{s}}$$

satisfies $\pi_{\mathfrak{s}} = \tilde{\pi}_{\mathfrak{s},j} \circ Q$. By the definition of the quotient map Q and by the proof of Corollary 6.6.5, we see that $\tilde{\pi}_{\mathfrak{s},i} = \tilde{\pi}_{\mathfrak{s},j}$ on the appropriate intersection of images of the subspaces $Q(\overline{\mathbf{BL}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_i))$ in the quotient $\mathbf{QL}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$. Hence, the maps $\tilde{\pi}_{\mathfrak{s},i}$ fit together to define a global map $\tilde{\pi}_{\mathfrak{s}}$ on $\mathbf{QL}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ satisfying Item (6). \square

10.3. Homology and cohomology classes of the quotient

To use the quotient space $\mathbf{QL}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ in Proposition 10.2.1 to compute the pairing (10.1.3), we must verify that cohomology classes appearing in (10.1.3) pull back from the quotient.

LEMMA 10.3.1. *Continue the notation of Proposition 10.2.1. Then there are a positive integer n and a class,*

$$(10.3.1) \quad \check{\nu} \in H^2(\mathbf{QL}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}; \mathbb{R}),$$

such that $Q^* \check{\nu} = n\nu$, where ν is the cohomology class appearing in (10.1.3).

PROOF. Choose $\check{\nu}$ to be the first Chern class of the line bundle over $\mathbf{QL}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}$ defined in Item (6) of Proposition 10.2.1. \square

LEMMA 10.3.2. *Continue the notation of Proposition 10.2.1. Define*

$$\mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i) := Q(\bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)).$$

Then there are homology classes,

$$(10.3.2) \quad [\mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i)] \in H_\bullet(\mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i); \mathbb{Z}),$$

satisfying

$$(10.3.3) \quad Q_*[\bar{\mathbf{BL}}_{t,s}^{\text{vir}}] = \sum_i (\check{\iota}_i)_*[\mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i)],$$

where $\check{\iota}_i : \bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i) \rightarrow \mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i)$ is the inclusion map.

PROOF. From the definition of $\bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)$ in (8.1.16) and (8.1.5) and the definition of $\partial_j \bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)$ in (8.3.1), we see that the boundary of $\bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)$ is given by

$$\partial \bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i) = \bigcup_{j \neq i} \partial_j \bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i).$$

We then define

$$b\mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i) := Q(\partial \bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)).$$

Then there is an equality of cycles,

$$(10.3.4) \quad [\bar{\mathbf{BL}}_{t,s}^{\text{vir}}] = \sum_i (\iota_i)_*[\bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i), \partial \bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)],$$

where $\iota_i : \bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i) \rightarrow \bar{\mathbf{BL}}_{t,s}^{\text{vir}}$ is the inclusion map and

$$[\bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i), \partial \bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)] \in H_\bullet(\bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i), \partial \bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i); \mathbb{R})$$

is a relative fundamental class. Item (2) of Proposition 10.2.1 implies that the map in the exact sequence of the pair $(\mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i), b\mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i))$,

$$(\check{j}_i)_* : H_{\max}(\mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i); \mathbb{R}) \rightarrow H_{\max}(\mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i), b\mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i); \mathbb{R}),$$

is an isomorphism. Hence, there is a homology class,

$$[\mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i)] \in H_\bullet(\mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i); \mathbb{R}),$$

as in (10.3.2) which maps to $Q_*([\bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i), \partial \bar{\mathbf{BL}}_{t,s}^{\text{vir}}(\mathcal{P}_i)])$ under $(\check{j}_i)_*$. Applying Q_* to the equality (10.3.4) yields the conclusion. \square

10.4. Fiber bundles and pushforwards

We now show how to use the fiber bundle structure of $\mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i)$ in cohomological pairings. The subspaces of the quotient $\mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i)$ are not fiber bundles over $M_s \times \text{cl}(\Sigma(X^\ell, \mathcal{P}_i))$ but do pull back to a fiber bundle over $\Delta(X^\ell, \mathcal{P}_i)$ as we record in the following lemma.

LEMMA 10.4.1. For the subgroup $\tilde{G}(\mathcal{P}_i)$ of the structure group $G(\mathcal{P}_i)$ defined in (6.2.3a), define

$$\widetilde{\mathbf{QL}}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i) := M_{\mathbf{s}} \times_{\mathcal{G}_{\mathbf{s}} \times S^1} \bar{\text{Fr}}(\mathbf{t}, \mathbf{s}, \mathcal{P}_i) \times_{\tilde{G}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon).$$

Then there are a branched covering, $c_i : \widetilde{\mathbf{QL}}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i) \rightarrow \mathbf{QL}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i)$, and a homology class,

$$[\widetilde{\mathbf{QL}}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i)] \in H_{\bullet}(\widetilde{\mathbf{QL}}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i); \mathbb{R}),$$

satisfying $(c_i)_*[\widetilde{\mathbf{QL}}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i)] = e_i[\mathbf{QL}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i)]$ for some $e_i \in \mathbb{Z}$ and a commutative diagram,

$$\begin{array}{ccccc} \check{M}(\mathcal{P}_i, \varepsilon) & \longrightarrow & \widetilde{\mathbf{QL}}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i) & \xrightarrow{c_i} & \mathbf{QL}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i) \\ & & \tilde{\pi}_{\mathbf{Q},i} \downarrow & & \tilde{\pi}_i \downarrow \\ & & M_{\mathbf{s}} \times \Delta(X^\ell, \mathcal{P}_i) & \longrightarrow & M_{\mathbf{s}} \times \text{cl}(\Sigma(X^\ell, \mathcal{P}_i)) \end{array}$$

where the map $\widetilde{\mathbf{QL}}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i) \rightarrow M_{\mathbf{s}} \times \Delta(X^\ell, \mathcal{P}_i)$ is a fiber bundle with structure group $\mathcal{G}_{\mathbf{s}} \times \tilde{G}(\mathcal{P}_i)$.

We record the behavior of the cohomology class $\check{\nu}$ under the branched cover c_i .

LEMMA 10.4.2. Continue the notation of Lemma 10.4.1. Let $\check{\nu}$ be the cohomology class defined in (10.3.1). Then $c_i^* l_i^* \check{\nu} = \tilde{\nu}_i$, where $\tilde{\nu}_i$ is the first Chern class of the S^1 -bundle given by

$$(10.4.1) \quad M_{\mathbf{s}} \times_{\mathcal{G}_{\mathbf{s}}} \bar{\text{Fr}}(\mathbf{t}, \mathbf{s}, \mathcal{P}_i) \times_{\tilde{G}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon) \rightarrow M_{\mathbf{s}} \times_{\mathcal{G}_{\mathbf{s}} \times S^1} \bar{\text{Fr}}(\mathbf{t}, \mathbf{s}, \mathcal{P}_i) \times_{\tilde{G}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon).$$

PROOF. The conclusion follows immediately from the definitions $\check{\nu}$ and of the map c_i . \square

We write

$$(10.4.2) \quad \tilde{\pi}_{X,i} : \widetilde{\mathbf{QL}}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i) \rightarrow \Delta(X^\ell, \mathcal{P}_i) \quad \text{and} \quad \tilde{\pi}_{\mathbf{s},i} : \widetilde{\mathbf{QL}}_{\mathbf{t},\mathbf{s}}^{\text{vir}}(\mathcal{P}_i) \rightarrow M_{\mathbf{s}}$$

for the composition of $\tilde{\pi}_{\mathbf{Q},i}$ with the obvious projection maps.

We now analyze the structure group of the bundle defined in Lemma 10.4.1. The equality $\Delta(X^\ell, \mathcal{P}) = \prod_{P \in \mathcal{P}} X$, implies that we can write the bundle $\bar{\text{Fr}}(\mathbf{t}, \mathbf{s}, \mathcal{P}) \rightarrow \Delta(X^\ell, \mathcal{P})$ as

$$\bar{\text{Fr}}(\mathbf{t}, \mathbf{s}, \mathcal{P}) = \prod_{P \in \mathcal{P}} \text{Fr}(\mathfrak{g}_{\mathbf{t}(P)}) \times_X \text{Fr}(TX).$$

The preceding identity and the diagonal inclusion map, $\tilde{M}_{\mathbf{s}} \rightarrow \prod_{P \in \mathcal{P}} \tilde{M}_{\mathbf{s}}$, define an inclusion of S^1 bundles,

$$(10.4.3) \quad \begin{array}{c} \tilde{M}_{\mathbf{s}} \times_{\mathcal{G}_{\mathbf{s}}} \bar{\text{Fr}}(\mathbf{t}, \mathbf{s}, \mathcal{P}) \times_{\tilde{G}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon) \\ \downarrow \\ \prod_{P \in \mathcal{P}} \left(\tilde{M}_{\mathbf{s}} \times_{\mathcal{G}_{\mathbf{s}}} \text{Fr}(\mathfrak{g}_{\mathbf{t}(P)}) \times_X \text{Fr}(TX) \right) \times_{\tilde{G}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon) \end{array}$$

covering the inclusion of base spaces,

$$\begin{array}{c} \tilde{M}_{\mathbf{s}} \times_{\mathcal{G}_{\mathbf{s}} \times S^1} \bar{\text{Fr}}(\mathbf{t}, \mathbf{s}, \mathcal{P}) \times_{\tilde{G}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon) \\ \downarrow \\ \prod_{P \in \mathcal{P}} \left(\tilde{M}_{\mathbf{s}} \times_{\mathcal{G}_{\mathbf{s}}} \text{Fr}(\mathfrak{g}_{\mathbf{t}(P)}) \times_X \text{Fr}(TX) \right) / S^1 \times_{\tilde{G}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon) \end{array}$$

We now show how to reduce the structure group of the preceding bundle. The $\mathrm{SO}(3)$ bundle, $\mathrm{Fr}(\mathfrak{g}_{\mathfrak{t}(\ell)}) \rightarrow X$, admits a reduction to an S^1 bundle,

$$(10.4.4) \quad \mathrm{Fr}(\mathfrak{g}_{\mathfrak{t}(\ell)}) \cong Q_{\mathfrak{t},\mathfrak{s}} \times_{S^1} \mathrm{SO}(3),$$

where $c_1(Q_{\mathfrak{t},\mathfrak{s}}) = c_1(\mathfrak{t}) - c_1(\mathfrak{s})$, as described in the beginning of Section 2.3 (see also [28, Lemma 2.9]).

LEMMA 10.4.3. *Define an S^1 -bundle $P_{\mathfrak{t},\mathfrak{s}}$ by*

$$P_{\mathfrak{t},\mathfrak{s}} := \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} Q_{\mathfrak{t},\mathfrak{s}} \rightarrow M_{\mathfrak{s}} \times X,$$

where $\mathcal{G}_{\mathfrak{s}}$ acts on $Q_{\mathfrak{t},\mathfrak{s}}$ by $(q, u) \mapsto qu(p_Q(q))^2$, for $q \in Q_{\mathfrak{t},\mathfrak{s}}$ and $u \in \mathcal{G}_{\mathfrak{s}}$. Then

$$\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} Q_{\mathfrak{t},\mathfrak{s}} \rightarrow M_{\mathfrak{s}} \times X$$

is a principal S^1 -bundle with first Chern class equal to $2c_1(\mathbb{L}_{\mathfrak{s}}) + \pi_2^* c_1(Q_{\mathfrak{t},\mathfrak{s}})$, where $\pi_2 : M_{\mathfrak{s}} \times X \rightarrow X$ is the projection.

PROOF. The conclusion is given by [28, Equation (3.68)]. \square

We define

$$\tilde{H}(\mathcal{P}_i) := \prod_{P \in \mathcal{P}} S^1 \times \mathrm{SO}(4).$$

We can thus rewrite the bundle

$$\begin{aligned} & \prod_{P \in \mathcal{P}_i} \left(\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \mathrm{Fr}(\mathfrak{g}_{\mathfrak{t}(\ell)}) \times_X \mathrm{Fr}(TX) \right) \times_{\tilde{G}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon) \\ & \cong \prod_{P \in \mathcal{P}_i} \left(\tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} Q_{\mathfrak{t},\mathfrak{s}} \times_X \mathrm{Fr}(TX) \right) \times_{\tilde{H}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon) \\ & \cong \prod_{P \in \mathcal{P}_i} (P_{\mathfrak{t},\mathfrak{s}} \times_X \mathrm{Fr}(TX)) \times_{\tilde{H}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon). \end{aligned}$$

If we denote $\mathrm{Fr}_{\tilde{H}}(\mathcal{P}_i) := \prod_{P \in \mathcal{P}} P_{\mathfrak{t},\mathfrak{s}} \times_X \mathrm{Fr}(TX)$, then composing the preceding reduction of the structure group with the inclusion (10.4.3) yields an inclusion of S^1 bundles,

$$(10.4.5) \quad \begin{array}{ccc} \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}}} \bar{\mathrm{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \times_{\tilde{G}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon) & \xrightarrow{\Delta_{\mathfrak{t},\mathfrak{s}}^i} & \mathrm{Fr}_{\tilde{H}}(\mathcal{P}_i) \times_{\tilde{H}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon) \\ \downarrow & & \downarrow \\ \tilde{M}_{\mathfrak{s}} \times_{\mathcal{G}_{\mathfrak{s}} \times S^1} \bar{\mathrm{Fr}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \times_{\tilde{G}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon) & \xrightarrow{\Delta_{\mathfrak{t},\mathfrak{s}}^i} & \mathrm{Fr}_{\tilde{H}}(\mathcal{P}_i) \times_{\tilde{H}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon) / S^1 \end{array}$$

where the S^1 action on $\check{M}(\mathcal{P}_i, \varepsilon)$ is given by the diagonal S^1 action on frames defined in (8.2.3).

Let $f_{\mathfrak{t},\mathfrak{s}} : M_{\mathfrak{s}} \times X \rightarrow \mathrm{BS}^1 \times \mathrm{BSO}(4)$ be the classifying map for the bundle

$$F_{\mathfrak{t},\mathfrak{s}} := P_{\mathfrak{t},\mathfrak{s}} \times_X \mathrm{Fr}(TX) \rightarrow M_{\mathfrak{s}} \times X.$$

By the identities,

$$\mathrm{E}\tilde{H}(\mathcal{P}_i) = \prod_{P \in \mathcal{P}_i} \mathrm{ES}^1 \times \mathrm{ESO}(4) \quad \text{and} \quad \mathrm{B}\tilde{H}(\mathcal{P}_i) = \prod_{P \in \mathcal{P}_i} \mathrm{BS}^1 \times \mathrm{BSO}(4),$$

if we define $f_{t,s}(\mathcal{P}) := \prod_{P \in \mathcal{P}} f_{t,s}$ and $\tilde{f}_{t,s}(\mathcal{P})$ to be the bundle maps in the diagram,

$$(10.4.6) \quad \begin{array}{ccc} \prod_{P \in \mathcal{P}} F_{t,s} & \xrightarrow{\tilde{f}_{t,s}(\mathcal{P})} & \text{EH}(\mathcal{P}) \\ \downarrow & & \downarrow \\ \prod_{P \in \mathcal{P}} M_s \times X & \xrightarrow{f_{t,s}(\mathcal{P})} & \text{BH}(\mathcal{P}) \end{array}$$

and if we write id_M and id_{M/S^1} for the identify maps on $\check{M}(\mathcal{P}_i, \varepsilon)$ and $\check{M}(\mathcal{P}_i, \varepsilon)/S^1$ respectively, then we have a commutative diagram of S^1 bundles and $\tilde{H}(\mathcal{P}_i)$ bundles:

$$(10.4.7) \quad \begin{array}{ccc} \prod_{P \in \mathcal{P}_i} F_{t,s} \times_{\tilde{H}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon) & \xrightarrow{\tilde{f}_{t,s}(\mathcal{P}_i) \times \text{id}_M} & \text{EH}(\mathcal{P}_i) \times_{\tilde{H}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon) \\ \downarrow & & \downarrow \\ \prod_{P \in \mathcal{P}_i} F_{t,s} \times_{\tilde{H}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon)/S^1 & \xrightarrow{\tilde{f}_{t,s}(\mathcal{P}_i) \times \text{id}_{M/S^1}} & \text{EH}(\mathcal{P}_i) \times_{\tilde{H}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon)/S^1 \\ \downarrow & & \downarrow p_i \\ \prod_{P \in \mathcal{P}_i} (M_s \times X) & \xrightarrow{f_{t,s}(\mathcal{P}_i)} & \text{BH}(\mathcal{P}_i) \end{array}$$

We then have the following lemma.

LEMMA 10.4.4. *Define*

$$\nu_{H,i} \in H^2(\text{EH}(\mathcal{P}_i) \times_{\tilde{H}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon)/S^1; \mathbb{R})$$

to be the first Chern class of the bundle

$$\text{EH}(\mathcal{P}_i) \times_{\tilde{H}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon) \rightarrow \text{EH}(\mathcal{P}_i) \times_{\tilde{H}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon)/S^1.$$

If $\tilde{\nu}_i$ is the cohomology class defined in (10.4.1), $\Delta_{t,s}^i$ the map defined in (10.4.5), and we define

$$(10.4.8) \quad \tilde{g}_i := (\tilde{f}_{t,s}(\mathcal{P}_i) \times \text{id}_{M/S^1}) \circ \Delta_{t,s}^i,$$

then $\tilde{g}_i^* \nu_{H,i} = \tilde{\nu}_i$.

PROOF. The lemma follows immediately from the definitions of $\nu_{H,i}$ and $\tilde{\nu}_i$ and the commutativity of the diagrams (10.4.5) and (10.4.7). \square

We will require the following computations of pushforwards.

LEMMA 10.4.5. *Continue the notation of Lemma 10.4.4. For $\mathcal{P} = \{P_1, \dots, P_r\}$, define $k(\mathcal{P}_i) = \sum_{i=1}^r (8|P_i| - 4) - 2$. Let $p_i : \text{EH}(\mathcal{P}_i) \times_{\tilde{H}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon)/S^1 \rightarrow \text{BH}(\mathcal{P}_i)$ be the projection map appearing in the diagram (10.4.7). Then there is a class,*

$$m_{|\mathcal{P}_i|, \delta} \in H^{2\delta - k(\mathcal{P}_i)}(\text{BH}(\mathcal{P}_i); \mathbb{R}),$$

such that

$$(p_i)_*(\nu_{H,i}^\delta) = m_{|\mathcal{P}_i|, \delta}.$$

PROOF. The conclusion follows by observing that $k(\mathcal{P}_i)$ is the dimension of the fiber $\check{M}(\mathcal{P}_i, \varepsilon)/S^1$. \square

10.5. Computations of intersection numbers on $\bar{\mathbf{L}}_{t,s}$

We can now begin to compute the pairings appearing in (10.1.3). First, we show how to reduce such pairings to pairings with the subspaces of the quotient space.

LEMMA 10.5.1. *For any $\mu_s(z_2) \in H^\bullet(M_s; \mathbb{R})$, and $S^\ell(\beta) \in H^\bullet(\text{Sym}^\ell(X); \mathbb{R})$, and $z_2 \in \mathbb{A}_2(X)$, the equality,*

$$(10.5.1) \quad \begin{aligned} & \left\langle \nu^\delta \smile \pi_s^* \mu_s(z_2) \smile \pi_X^* S^\ell(\beta), [\bar{\mathbf{B}}\mathbf{L}_{t,s}^{\text{vir}}] \right\rangle \\ &= \sum_i n^{-\delta} \left\langle \iota_i^* \left(\check{\nu}^\delta \smile \check{\pi}_s^* \mu_s(z_2) \smile \check{\pi}_{X,i}^* S^\ell(\beta) \right), [\mathbf{Q}\mathbf{L}_{t,s}^{\text{vir}}(\mathcal{P}_i)] \right\rangle, \end{aligned}$$

holds, where we defined n and $\check{\nu}$ in Lemma 10.3.1, and $\check{\pi}_{X,i}$ in (10.2.6), and $\check{\pi}_s$ in Item (6) of Proposition 10.2.1.

PROOF. From Items (4), (5), and (6) of Proposition 10.2.1 and from Lemma 10.3.1 we have

$$\begin{aligned} & \left\langle \nu^\delta \smile \pi_s^* \mu_s(z_2) \smile \pi_X^* S^\ell(\beta), [\bar{\mathbf{B}}\mathbf{L}_{t,s}^{\text{vir}}] \right\rangle \\ &= n^{-\delta} \left\langle (Q^* \check{\nu})^\delta \smile Q^* \check{\pi}_s^* \mu_s \smile Q^* \pi_{Q,X}^* S^\ell(\beta), [\bar{\mathbf{B}}\mathbf{L}_{t,s}^{\text{vir}}] \right\rangle \\ &= n^{-\delta} \left\langle Q^* \left(\check{\nu}^\delta \smile \check{\pi}_s^* \mu_s \smile \check{\pi}_{Q,X}^* S^\ell(\beta) \right), [\bar{\mathbf{B}}\mathbf{L}_{t,s}^{\text{vir}}] \right\rangle \\ &= n^{-\delta} \left\langle \check{\nu}^\delta \smile \check{\pi}_s^* \mu_s \smile \check{\pi}_{Q,X}^* S^\ell(\beta), Q_* [\bar{\mathbf{B}}\mathbf{L}_{t,s}^{\text{vir}}] \right\rangle \\ &= n^{-\delta} \sum_i \left\langle \check{\nu}^\delta \smile \check{\pi}_s^* \mu_s \smile \check{\pi}_{X,i}^* S^\ell(\beta), (\iota_i)_* [\mathbf{Q}\mathbf{L}_{t,s}^{\text{vir}}(\mathcal{P}_i)] \right\rangle, \end{aligned}$$

where the final equality follows from Lemma 10.3.2 and the definition of $\check{\pi}_{Q,X}$ in terms of $\check{\pi}_{X,i}$. \square

The description of the fiber-bundle structures in the preceding section leads to the following expression for the pairing on the right-hand side of equation (10.5.1).

LEMMA 10.5.2. *Continue the notation of Lemma 10.5.1. For $\mathcal{P}_i = \{P_1, \dots, P_r\}$, let $k(\mathcal{P}_i)$ be the constant defined in Lemma 10.4.5. Write $\mu_s(z_2)$ and $\tilde{S}^\ell(\beta)$ for $\mu_s(z_2) \times 1$ and $1 \times \tilde{S}^\ell(\beta)$ respectively. Then there are universal polynomials,*

$$m_{|\mathcal{P}_i|, \delta}(\mathbf{t}, \mathbf{s}) \in H^{2\delta - k(\mathcal{P}_i)}(M_s \times \Delta(X^\ell, \mathcal{P}_i); \mathbb{R}),$$

in the cohomology classes $c_1(\mathbb{L}_s)$ and $\tilde{S}^\ell(\alpha)$, where α may denote $c_1(\mathbf{t}) - c_1(\mathbf{s})$, $e(X)$, or $p_1(X)$, with coefficients depending only on δ and the partition $\ell = |P_1| + \dots + |P_r|$, such that

$$\begin{aligned} & \left\langle \iota_i^* \left(\check{\nu}^\delta \smile \check{\pi}_s^* \mu_s(z_2) \smile \check{\pi}_{X,i}^* S^\ell(\beta) \right), [\mathbf{Q}\mathbf{L}_{t,s}^{\text{vir}}(\mathcal{P}_i)] \right\rangle \\ &= \left\langle m_{|\mathcal{P}_i|, \delta}(\mathbf{t}, \mathbf{s}) \smile \mu_s(z_2) \smile \tilde{S}^\ell(\beta), [M_s \times \Delta(X^\ell, \mathcal{P}_i)] \right\rangle. \end{aligned}$$

PROOF. Lemmas 10.4.1 and 10.4.2 allow us to write

$$\begin{aligned}
& \left\langle l_i^* \left(\check{\nu}^\delta \smile \check{\pi}_s^* \mu_s(z_2) \smile \check{\pi}_{X,i}^* S^\ell(\beta) \right), [\mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i)] \right\rangle \\
&= e_i^{-1} \left\langle l_i^* \left(\check{\nu}^\delta \smile \check{\pi}_s^* \mu_s(z_2) \smile \check{\pi}_{X,i}^* S^\ell(\beta) \right), (c_i)_* \left[\widetilde{\mathbf{QL}}_{t,s}^{\text{vir}}(\mathcal{P}_i) \right] \right\rangle \\
&= e_i^{-1} \left\langle c_i^* l_i^* \left(\check{\nu}^\delta \smile \check{\pi}_s^* \mu_s(z_2) \smile \check{\pi}_{X,i}^* S^\ell(\beta) \right), \left[\widetilde{\mathbf{QL}}_{t,s}^{\text{vir}}(\mathcal{P}_i) \right] \right\rangle \\
&= e_i^{-1} \left\langle \check{\nu}_i^\delta \smile \check{\pi}_{\mathbf{Q},i}^* \left(\mu_s(z_2) \smile \check{S}^\ell(\beta) \right), \left[\widetilde{\mathbf{QL}}_{t,s}^{\text{vir}}(\mathcal{P}_i) \right] \right\rangle,
\end{aligned}$$

where in the final step we have adopted the convention of writing $\mu_s(z_2)$ for $\mu_s(z_2) \times 1$ and $\check{S}^\ell(\beta)$ for $1 \times \check{S}^\ell(\beta)$ mentioned in the statement of the lemma; we record the conclusion as

$$\begin{aligned}
(10.5.2) \quad & \left\langle l_i^* \left(\check{\nu}^\delta \smile \check{\pi}_s^* \mu_s(z_2) \smile \check{\pi}_{X,i}^* S^\ell(\beta) \right), [\mathbf{QL}_{t,s}^{\text{vir}}(\mathcal{P}_i)] \right\rangle \\
&= e_i^{-1} \left\langle \check{\nu}_i^\delta \smile \check{\pi}_{\mathbf{Q},i}^* \left(\mu_s(z_2) \smile \check{S}^\ell(\beta) \right), \left[\widetilde{\mathbf{QL}}_{t,s}^{\text{vir}}(\mathcal{P}_i) \right] \right\rangle.
\end{aligned}$$

We then use a pushforward-pullback argument to compute the pairing in the right-hand side of (10.5.2). Define

$$\Delta_M : M_s \times \prod_{P \in \mathcal{P}} X \rightarrow \prod_{P \in \mathcal{P}} M_s \times \Delta(X^\ell, \mathcal{P}),$$

by the diagonal inclusion, $M_s \rightarrow \prod_{P \in \mathcal{P}} M_s$, and the identification $\Delta(X^\ell, \mathcal{P}) = \prod_{P \in \mathcal{P}} X$ (where $\Delta(X^\ell, \mathcal{P}) \subset X^\ell$ is defined in (3.1.4)). Composing the diagrams (10.4.5) and (10.4.7) gives the diagram,

$$\begin{array}{ccc}
\widetilde{\mathbf{QL}}_{t,s}^{\text{vir}}(\mathcal{P}_i) & \xrightarrow{\tilde{g}_i} & \text{EH}(\mathcal{P}_i) \times_{\tilde{G}(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon) / S^1 \\
\tilde{\pi}_{\mathbf{Q},i} \downarrow & & p_i \downarrow \\
M_s \times \Delta(X^\ell, \mathcal{P}_i) & \xrightarrow{g_i} & \text{BH}(\mathcal{P}_i)
\end{array}
\tag{10.5.3}$$

where \tilde{g}_i is defined in (10.4.8) and $g_i := (\prod_{P \in \mathcal{P}} f_{t,s}) \circ \Delta_M$ (and $f_{t,s}$ is defined prior to (10.4.6)). Lemma 10.4.4 then implies that we can rewrite the right-hand side of (10.5.2) as

$$\begin{aligned}
(10.5.4) \quad & e_i^{-1} \left\langle \check{\nu}_i^\delta \smile \check{\pi}_{\mathbf{Q},i}^* \left(\mu_s(z_2) \smile \check{S}^\ell(\beta) \right), \left[\widetilde{\mathbf{QL}}_{t,s}^{\text{vir}}(\mathcal{P}_i) \right] \right\rangle \\
&= e_i^{-1} \left\langle \tilde{g}_i^* \nu_{H,i}^\delta \smile \check{\pi}_{\mathbf{Q},i}^* \left(\mu_s(z_2) \smile \check{S}^\ell(\beta) \right), \left[\widetilde{\mathbf{QL}}_{t,s}^{\text{vir}}(\mathcal{P}_i) \right] \right\rangle.
\end{aligned}$$

We then apply a pushforward-pullback argument to the diagram (10.5.3) to rewrite the right-hand side of (10.5.4) as:

$$\begin{aligned}
& e_i^{-1} \left\langle \tilde{g}_i^* \nu_{H,i}^\delta \smile \tilde{\pi}_{\mathbf{Q},i}^* \left(\mu_{\mathfrak{s}}(z_2) \smile \tilde{S}^\ell(\beta) \right), \left[\widetilde{\mathbf{QL}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}(\mathcal{P}_i) \right] \right\rangle \\
&= e_i^{-1} \left\langle (\tilde{\pi}_{\mathbf{Q},i})_* \left(\tilde{g}_i^* \nu_{H,i}^\delta \smile \tilde{\pi}_{\mathbf{Q},i}^* \left(\mu_{\mathfrak{s}}(z_2) \smile \tilde{S}^\ell(\beta) \right) \right), [M_{\mathfrak{s}} \times \Delta(X^\ell, \mathcal{P}_i)] \right\rangle \\
&= e_i^{-1} \left\langle (\tilde{\pi}_{\mathbf{Q},i})_* \left(\tilde{g}_i^* \nu_{H,i}^\delta \right) \smile \mu_{\mathfrak{s}}(z_2) \smile \tilde{S}^\ell(\beta), [M_{\mathfrak{s}} \times \Delta(X^\ell, \mathcal{P}_i)] \right\rangle \\
&= e_i^{-1} \left\langle \left(\tilde{g}_i^*(p_i)_* \nu_{H,i}^\delta \right) \smile \mu_{\mathfrak{s}}(z_2) \smile \tilde{S}^\ell(\beta), [M_{\mathfrak{s}} \times \Delta(X^\ell, \mathcal{P}_i)] \right\rangle \\
&= e_i^{-1} \left\langle \tilde{g}_i^* m_{|\mathcal{P}_i|,\delta} \smile \mu_{\mathfrak{s}}(z_2) \smile \tilde{S}^\ell(\beta), [M_{\mathfrak{s}} \times \Delta(X^\ell, \mathcal{P}_i)] \right\rangle,
\end{aligned}$$

where the final equality follows from Lemma 10.4.5. By the definition of g_i in terms of the maps $f_{\mathfrak{t},\mathfrak{s}}$ and by the definition of the maps $f_{\mathfrak{t},\mathfrak{s}}$ as classifying maps for the bundles $P_{\mathfrak{t},\mathfrak{s}}$ (defined in Lemma 10.4.3) and $\text{Fr}(TX)$, the pullback $g_i^* m_{|\mathcal{P}_i|,\delta}$ is given by a universal polynomial in the characteristic classes of $P_{\mathfrak{t},\mathfrak{s}}$ and $\text{Fr}(TX)$. By Lemma 10.4.3, these characteristic classes are $2c_1(\mathbb{L}_{\mathfrak{s}}) + c_1(\mathfrak{t}) - c_1(\mathfrak{s})$, and $e(X)$, and $p_1(X)$. This completes the proof of Lemma 10.5.2. \square

10.6. Proofs of the main theorems

We now prove Theorems 10.1.1 and 10.1.2 and Theorem 1.

PROOF OF THEOREM 10.1.1. Recall Equation (10.1.2):

$$(10.6.1) \quad \# \left(\bar{\mathcal{V}}(h^{\delta-2m} x^m) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}} \right) = \left\langle \bar{\mu}_p(h^{\delta-2m} x^m) \smile \bar{\mu}_c^\eta \smile \bar{e}_I \smile \bar{e}_s, [\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}] \right\rangle.$$

For convenience, we abbreviate $L_h := -\frac{1}{2}\langle c_1(\mathfrak{t}) - c_1(\mathfrak{s}), h \rangle$ and $\alpha := 2\mu_{\mathfrak{s}} - \nu$. By Definition 9.4.8 we can then rewrite (10.6.1) as

$$\begin{aligned}
& \# \left(\bar{\mathcal{V}}(h^{\delta-2m} x^m) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}} \right) \\
&= \left\langle \left(\alpha L_h + \pi_X^* S^\ell(h) \right)^{\delta-2m} \smile \left(-\frac{1}{4} \alpha^2 + S^\ell(x) \right)^m \smile \nu^\eta \smile (-\nu)^{r\Xi} \smile \bar{e}_I, [\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}] \right\rangle \\
&= \sum_{r=0}^{\delta-m} \sum_{i+2j=r} (-1)^{r\Xi} C_{\delta,m}(i,j) L_h^i \left\langle \alpha^r \smile S^\ell(h)^{\delta-2m-i} \smile S^\ell(x)^{m-j} \smile \nu^{\eta+r\Xi} \smile \bar{e}_I, [\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}] \right\rangle,
\end{aligned}$$

that is,

$$\begin{aligned}
(10.6.2) \quad & \# \left(\bar{\mathcal{V}}(h^{\delta-2m} x^m) \cap \bar{\mathcal{W}}^\eta \cap \bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}} \right) \\
&= \sum_{r=0}^{\delta-m} \sum_{i+2j=r} (-1)^{r\Xi} C_{\delta,m}(i,j) L_h^i \left\langle \alpha^r \smile S^\ell(h)^{\delta-2m-i} \smile S^\ell(x)^{m-j} \smile \nu^{\eta+r\Xi} \smile \bar{e}_I, [\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}] \right\rangle,
\end{aligned}$$

where

$$C_{\delta,m}(i,j) := \binom{\delta-2m}{i} \binom{m}{j} (-1)^j 2^{-r}.$$

The expression for \bar{e}_I in Proposition 9.5.7 implies that we can write the terms in the right-hand side of (10.6.2) as

$$(10.6.3) \quad \begin{aligned} & L_h^i \left\langle \alpha^r \smile S^\ell(h)^{\delta-2m-i} \smile S^\ell(x)^{m-j} \smile \nu^{\eta+r\Xi} \smile \bar{e}_I, [\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}] \right\rangle \\ &= \sum_{j=0}^{\ell} L_h^i \left\langle \alpha^r \smile S^\ell(h)^{\delta-2m-i} \smile S^\ell(x)^{m-j} \smile \nu^{\eta+r\Xi+j} \smile n_{\ell,j}, [\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}] \right\rangle, \end{aligned}$$

where $n_{\ell,j} \in H^{2\ell-2j}(M_{\mathfrak{s}} \times \text{Sym}^\ell(X); \mathbb{R})$ is a polynomial in $\mu_{\mathfrak{s}}(\cdot)$ and $S^\ell(c(\mathfrak{t}))$ with real coefficients depending only on ℓ and j . Proposition 9.7.3 and the expression for the Segre classes $s_i(N)$ in [29, Lemma 4.11] imply that the terms in the sum on the left-hand side of (10.6.3) can be rewritten as

$$(10.6.4) \quad \begin{aligned} & L_h^i \left\langle \alpha^r \smile S^\ell(h)^{\delta-2m-i} \smile S^\ell(x)^{m-j} \smile \nu^{\eta+r\Xi+j} \smile n_{\ell,j}, [\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}] \right\rangle \\ &= L_h^i \left\langle S^\ell(h)^{\delta-2m-i} \smile S^\ell(x)^{m-j} \smile s_{\mathfrak{t},\mathfrak{s},r}(\nu, \mu_{\mathfrak{s}}, \mathfrak{t}, X), [\bar{\mathbf{B}}\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}] \right\rangle, \end{aligned}$$

where $s_{\mathfrak{t},\mathfrak{s},r}(\nu, \mu_{\mathfrak{s}}, \mathfrak{t}, X)$ is a polynomial in ν , $\pi_{\mathfrak{s}}^* \mu_{\mathfrak{s}}(\cdot)$, and $\pi_X^* S^\ell(\alpha)$ where α is $c_1(\mathfrak{t})$, $c_1(\mathfrak{s})$, $e(X)$, or $p_1(X)$ and the coefficients of $s_{\mathfrak{t},\mathfrak{s},r}(\nu, \mu_{\mathfrak{s}}, \mathfrak{t}, X)$ depend only on the homotopy type of \mathfrak{t} , \mathfrak{s} , and X . Applying Lemmas 10.5.1 and 10.5.2 to (10.6.4) yields the equality,

$$(10.6.5) \quad \begin{aligned} & \left\langle S^\ell(h)^{\delta-2m-i} \smile S^\ell(x)^{m-j} \smile s_{\mathfrak{t},\mathfrak{s},r}(\nu, \mu_{\mathfrak{s}}, \mathfrak{t}, X), [\bar{\mathbf{B}}\bar{\mathbf{L}}_{\mathfrak{t},\mathfrak{s}}^{\text{vir}}] \right\rangle \\ &= \sum_{k=0}^r L_h^i \left\langle q(\mathcal{P}_k) \smile S^\ell(h)^{\delta-2m-i} \smile S^\ell(x)^{m-j}, [M_{\mathfrak{s}} \times \Delta(X^\ell, \mathcal{P}_k)] \right\rangle, \end{aligned}$$

where $q(\mathcal{P}_k) \in H^\bullet(M_{\mathfrak{s}} \times \Delta(X^\ell, \mathcal{P}_k))$ is a polynomial in $\mu_{\mathfrak{s}}(\cdot)$ and $S^\ell(\alpha)$, and where α can equal $c_1(\mathfrak{t})$, $c_1(\mathfrak{s})$, $(c_1(\mathfrak{t}) - c_1(\mathfrak{s}))^2$, $e(X)$, or $p_1(X)$. Because $b_1(X) = 0$, the equality

$$[M_{\mathfrak{s}} \times \Delta(X^\ell, \mathcal{P}_k)] = [M_{\mathfrak{s}}] \times [\Delta(X^\ell, \mathcal{P}_k)]$$

implies that the expressions in (10.6.5) will vanish unless they contain a factor $\mu_{\mathfrak{s}}(x)^{d_{\mathfrak{s}}(\mathfrak{s})/2}$. Hence, the terms in the sum on the right-hand side of (10.6.5) can be rewritten as

$$(10.6.6) \quad \begin{aligned} & L_h^i \left\langle q(\mathcal{P}_k) \smile S^\ell(h)^{\delta-2m-i} \smile S^\ell(x)^{m-j}, [M_{\mathfrak{s}} \times \Delta(X^\ell, \mathcal{P}_k)] \right\rangle \\ &= SW_X(\mathfrak{s}) \lambda_h^i \left\langle S^\ell(h)^{\delta-2m-i} \smile S^\ell(x)^{m-j} \smile \alpha_{\mathfrak{t},\mathfrak{s}}(\delta, m, j, k), [\Delta(X^\ell, \mathcal{P}_k)] \right\rangle, \end{aligned}$$

where $\alpha_{\mathfrak{t},\mathfrak{s}}(\delta, m, j, k) \in H^\bullet(\Delta(X^\ell, \mathcal{P}_k))$ is a polynomial in $S^\ell(\alpha)$ for α equal to $c_1(\mathfrak{t})$, $c_1(\mathfrak{s})$, $(c_1(\mathfrak{t}) - c_1(\mathfrak{s}))^2$, $p_1(X)$, or $e(X)$. Because $\Delta(X^\ell, \mathcal{P}_k)$ is a product of $|\mathcal{P}_k|$ copies of the four-dimensional space X and because $S^\ell(h)$ is a two-dimensional cohomology class, terms in (10.6.6) will vanish unless each factor of $S^\ell(h)$ is paired either with another factor of $S^\ell(h)$ or a factor $S^\ell(\alpha_2)$, where $\alpha_2 = c_1(\mathfrak{t})$ or $c_1(\mathfrak{t}) - c_1(\mathfrak{s})$. Pairing $S^\ell(h)$ with itself will give powers of $Q_X(h)$, pairing $S^\ell(h)$ with $S^\ell(\alpha_2)$ will give powers of $\langle c_1(\mathfrak{t}), h \rangle$ or $\langle c_1(\mathfrak{t}) - c_1(\mathfrak{s}), h \rangle$. Thus, recalling the definition of L_h from before (10.6.2), we see that the expression on the right-hand side of (10.6.6) must be a sum over terms of the form

$$\langle c_1(\mathfrak{t}) - c_1(\mathfrak{s}), h \rangle^i \langle c_1(\mathfrak{t}), h \rangle^j Q_X(h)^a,$$

where $i + j + 2a = \delta - 2m$. We note that the power a appearing in $Q_X(h)^a$ must be less than or equal to the length of the partition \mathcal{P}_k , so $a \leq \ell$. This completes the proof of Theorem 10.1.1. \square

We record a particular case of Theorem 10.1.1 in a form which is used in the proof of Witten’s Conjecture [25, 23].

PROPOSITION 10.6.1. *Continue the notation and hypotheses of Theorem 10.1.1. In addition, assume that $\dim M_{\mathfrak{s}} = 0$ and*

$$2(\delta + \eta) = \dim \mathcal{M}_t - 2.$$

Writing $\mathfrak{s}_h := \langle c_1(\mathfrak{s}), h \rangle$ and $\lambda_h := \langle c_1(\mathfrak{t}), h \rangle$, then

$$(10.6.7) \quad \left\langle \nu^{\delta+\eta-2m-i} \smile \pi_X^* S^\ell(h)^i \smile \bar{\mu}_p(x)^m \smile \bar{e}_I \smile e_s, [\bar{\mathbf{L}}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}] \right\rangle \\ = SW_X(\mathfrak{s}) \sum_{j=0}^{\min(\ell, [i/2])} f_{\delta, \ell, m, \eta, i, j}(\mathfrak{s}_h, \lambda_h) Q_X(h)^j,$$

where

$$f_{\delta, \ell, m, \eta, i, j}(\mathfrak{s}_h, \lambda_h) = \sum_{k=0}^{i-2j} a_{\delta, \ell, m, \eta, i, j, k} \mathfrak{s}_h^{i-2j-k} \lambda_h^k,$$

and the coefficients $a_{\delta, \ell, m, \eta, i, j, k}$ depend on $\delta, \ell, m, \eta, i, j, k, \chi, \sigma, c_1(\mathfrak{t})^2, c_1(\mathfrak{s}) \cdot c_1(\mathfrak{t}),$ and $c_1(\mathfrak{s})^2$.

PROOF. Under the assumption $\dim M_{\mathfrak{s}} = 0$, the left-hand side of (10.6.3) reduces to the expression on the left-hand side of (10.6.7). Hence, the conclusion of Proposition 10.6.1 follows from the proof of Theorem 10.1.1 from (10.6.3) to the end. \square

We now give the

PROOF OF THEOREM 10.1.2. The proof of Theorem 10.1.1 shows that the intersection number in the statement of the theorem equals a sum of expressions of the form on the left-hand-side of (10.6.6). Without the assumption that $b_1(X) = 0$, the left-hand side of (10.6.6) equals a sum over terms that each contain a factor of the form $\langle \mu_{\mathfrak{s}}(\omega), [M_{\mathfrak{s}}] \rangle$, for some $\omega \in \mathbb{A}_2(X)$. These vanish by hypothesis and hence the intersection number vanishes. \square

The definition of the Donaldson invariant for X in (2.5.2) is given in terms of the blow-up, $\tilde{X} = X \# \bar{\mathbb{C}P}^2$, of X . Hence, before commencing the proof of Theorem 1 we will need to restate Theorem 10.1.1 in a form useful for computations on \tilde{X} .

Let $e \in H_2(\tilde{X}; \mathbb{Z})$ be the fundamental class of the exceptional curve and define $e^* := \text{PD}[e]$. We will identify elements of the homology and cohomology of X with elements of the homology and cohomology of \tilde{X} by means of the obvious inclusions.

By the discussions in [87, Section 12.4] or [78, Section 4.6], given $\mathfrak{s} \in \text{Spin}^c(X)$, there is a unique spin^c structure $\tilde{\mathfrak{s}}_k \in \text{Spin}^c(\tilde{X})$ such that $c_1(\tilde{\mathfrak{s}}_k) = c_1(\mathfrak{s}) + (2k - 1)e^*$ and all such spin^c structures on \tilde{X} are so obtained. The dimensions of the corresponding Seiberg–Witten moduli space satisfy

$$d_s(\tilde{\mathfrak{s}}_k) = d_s(\mathfrak{s}) - k(k - 1).$$

We then have the

LEMMA 10.6.2. *Continue the notation and hypotheses of Theorem 10.1.1. Let $\tilde{X} = X \# \bar{\mathbb{C}P}^2$ be the blow-up of X and let $e \in H_2(\tilde{X}; \mathbb{Z})$ be the homology class of the exceptional curve. Let $\tilde{\mathfrak{t}}$ be a spin^u structure over \tilde{X} satisfying*

$$p_1(\tilde{\mathfrak{t}}) = p_1(\mathfrak{t}) - 1, \quad c_1(\tilde{\mathfrak{t}}) = c_1(\mathfrak{t}), \quad \text{and} \quad w_2(\tilde{\mathfrak{t}}) \equiv w_2(\mathfrak{t}) + \text{PD}[e] \pmod{2}.$$

Then

$$(10.6.8) \quad \begin{aligned} & \# \left(\mathcal{V}(ze) \cap \mathcal{W}^\eta \cap \bar{\mathbf{L}}_{\tilde{\mathbf{t}}, \tilde{\mathbf{s}}_k} \right) \\ &= SW_X(\mathfrak{s}) \sum_{i=0}^{\min(\ell, \lfloor \frac{\delta-2m}{2} \rfloor)} (\tilde{q}_{\delta, \ell, m, i, k}(c_1(\mathfrak{s}) - c_1(\tilde{\mathbf{t}}), c_1(\tilde{\mathbf{t}})) Q_X^i)(h), \end{aligned}$$

where $\tilde{q}_{\delta, \ell, m, i, k}$ are degree $\delta - 2m - 2i$ homogeneous polynomials whose coefficients are universal functions of the constants given in Theorem 1.

PROOF. The proof follows that of Theorem 10.1.1 once one has made the following observations. By (2.3.14), the levels of \mathfrak{s} and $\tilde{\mathbf{s}}_k$ are related by

$$\begin{aligned} 4\ell(\tilde{\mathbf{t}}, \tilde{\mathbf{s}}_k) &= (c_1(\tilde{\mathbf{s}}_k) - c_1(\tilde{\mathbf{t}}))^2 - p_1(\tilde{\mathbf{t}}) \\ &= (c_1(\mathfrak{s}) - c_1(\tilde{\mathbf{t}}))^2 - (2k-1)^2 - p_1(\tilde{\mathbf{t}}) + 1 \\ &= 4\ell(\mathbf{t}, \mathfrak{s}) - 4k(k-1). \end{aligned}$$

Hence, $\ell(\tilde{\mathbf{t}}, \tilde{\mathbf{s}}_k) \leq \ell(\mathbf{t}, \mathfrak{s})$. Next, by [78, Theorem 4.6.8],

$$SW_{\tilde{X}}(\tilde{\mathbf{s}}_k) = \begin{cases} \pm SW_X(\mathfrak{s}) & \text{if } d(\mathfrak{s}) \geq k(k-1), \\ 0 & \text{if } d(\mathfrak{s}) < k(k-1). \end{cases}$$

The conclusion of the lemma now follows from the proof of Theorem 10.1.1. \square

PROOF OF THEOREM 1. To compute the Donaldson polynomial appearing on the left-hand side of (1.1.1), we select a spin^u structure $\tilde{\mathbf{t}}$ over \tilde{X} with $\Lambda = c_1(\tilde{\mathbf{t}})$ and $p_1(\tilde{\mathbf{t}})$ determined by

$$\delta + 1 = -p_1(\tilde{\mathbf{t}}) - \frac{3}{4}(\chi(X) + \sigma(X)),$$

where δ is the constant appearing in (1.1.1). To apply (2.6.2), we need to verify that $n_a(\tilde{\mathbf{t}}) > 0$. From the definition of $n_a(\tilde{\mathbf{t}})$ in (2.1.12) and for $i(\Lambda) = \Lambda^2 - \frac{1}{4}(3\chi + 7\sigma)$ as in the statement of Theorem 1, we have

$$\begin{aligned} 4n_a(\tilde{\mathbf{t}}) &= p_1(\tilde{\mathbf{t}}) + \Lambda^2 - i(\tilde{X}) \\ &= -\delta - 1 - \frac{3}{4}(\chi(X) + \sigma(X)) + \Lambda^2 - i(\tilde{X}) + 1 \\ &= -\delta + \Lambda^2 - \frac{1}{4}(3\chi(X) + 7\sigma(X)) \\ &= i(\Lambda) - \delta. \end{aligned}$$

Hence, the hypothesis of the theorem that $\delta < i(\Lambda)$ implies that $n_a(\tilde{\mathbf{t}}) > 0$. Equation (2.6.2) and the definition of the Donaldson invariant in (2.5.2) then imply that for $z = h^{\delta-2m}x^m$, and $n_a = n_a(\tilde{\mathbf{t}})$, and $\tilde{w} = w + e^*$, and $\tilde{\kappa} = -(1/4)p_1(\tilde{\mathbf{t}})$ we have

$$(10.6.9) \quad \begin{aligned} D_X^w(z) &= \# \left(\mathcal{V}(ze) \cap \bar{M}_{\tilde{\kappa}}^{\tilde{w}} \right) \\ &= 2^{1-n_a} \# \left(\mathcal{V}(ze) \cap \mathcal{W}^{n_a-1} \cap \bar{\mathbf{L}}_{\tilde{\mathbf{t}}, \tilde{\kappa}}^{\tilde{w}} \right). \end{aligned}$$

The cobordism formula (8.1.21) implies that

$$(10.6.10) \quad \# \left(\bar{\mathcal{V}}(ze) \cap \bar{\mathcal{W}}^{n_a-1} \cap \bar{\mathbf{L}}_{\tilde{\mathbf{t}}, \tilde{\kappa}}^{\tilde{w}} \right) = - \sum_{\tilde{\mathbf{s}} \in \text{Spin}^c(\tilde{X})} (-1)^{\sigma_{\tilde{\mathbf{t}}}(\tilde{w}, \tilde{\mathbf{s}})} \# \left(\bar{\mathcal{V}}(ze) \cap \bar{\mathcal{W}}^{n_a-1} \cap \bar{\mathbf{L}}_{\tilde{\mathbf{t}}, \tilde{\mathbf{s}}} \right).$$

For any four-manifold Y with $b_1(Y) = 0$, let $B(Y)$ denote the set of Seiberg–Witten basic classes,

$$B(Y) := \{c_1(\mathfrak{s}) : \mathfrak{s} \in \text{Spin}^c(Y) \text{ and } SW_Y(\mathfrak{s}) \neq 0\}.$$

By [78, Theorem 4.6.8], the Seiberg–Witten basic classes of \tilde{X} and X are related by

$$B(\tilde{X}) = \{c_1(\tilde{\mathfrak{s}}_k) : c_1(\mathfrak{s}) \in B(X) \text{ and } d_s(\mathfrak{s}) - k(k-1) \geq 0\}.$$

Theorem 10.1.2 then implies that the right-hand side of (10.6.10) equals

$$(10.6.11) \quad \begin{aligned} & - \sum_{\tilde{\mathbf{s}} \in B(\tilde{X})} (-1)^{\sigma_{\tilde{\mathbf{t}}}(\tilde{w}, \tilde{\mathbf{s}})} \# \left(\bar{\mathcal{V}}(ze) \cap \bar{\mathcal{W}}^{n_a-1} \cap \bar{\mathbf{L}}_{\tilde{\mathbf{t}}, \tilde{\mathbf{s}}} \right) \\ & = - \sum_{\mathfrak{s} \in B(X)} \sum_{\substack{k^2 - k \leq d_s(\mathfrak{s}), \\ k \geq 0}} (-1)^{\sigma_{\tilde{\mathbf{t}}}(\tilde{w}, \tilde{\mathbf{s}}_k)} \# \left(\bar{\mathcal{V}}(ze) \cap \bar{\mathcal{W}}^{n_a-1} \cap \bar{\mathbf{L}}_{\tilde{\mathbf{t}}, \tilde{\mathbf{s}}_k} \right). \end{aligned}$$

The definition of $\sigma_{\tilde{\mathbf{t}}}(\tilde{w}, \tilde{\mathbf{s}}_k)$ in (8.1.18) implies that

$$\sigma_{\tilde{\mathbf{t}}}(\tilde{w}, \tilde{\mathbf{s}}_k) = \frac{1}{4} (w + e^* - \Lambda + c_1(\mathfrak{s}) + (2k-1)e^*)^2 = \sigma_{\tilde{\mathbf{t}}}(w, \mathfrak{s}) - k.$$

Thus, we can rewrite the sum on the right-hand side of (10.6.11) as

$$(10.6.12) \quad - \sum_{\mathfrak{s} \in B(X)} (-1)^{\sigma_{\tilde{\mathbf{t}}}(w, \mathfrak{s})} \sum_{\substack{k^2 - k \leq d_s(\mathfrak{s}), \\ k \geq 0}} (-1)^k \# \left(\bar{\mathcal{V}}(ze) \cap \bar{\mathcal{W}}^{n_a-1} \cap \bar{\mathbf{L}}_{\tilde{\mathbf{t}}, \tilde{\mathbf{s}}_k} \right).$$

Theorem 1 then follows from Equations (10.6.9)–(10.6.12) and Lemma 10.6.2. \square

Kotschick–Morgan Conjecture

We now show how the proof of the Kotschick–Morgan Conjecture, stated here as Conjecture 11.2.1 and described in Section 1.3, can be reduced to the proof of a gluing theorem analogous to Hypothesis 7.8.1. The methods for this reduction are exactly those that lead to the proof of Theorem 10.1.1. Hence, there will be little new work in this section; instead, we will reference the analogous discussion from earlier sections. Throughout this chapter, we will assume that X is a smooth, closed, and oriented four-manifold with $b^+(X) = 1$ and $b_1(X) = 0$.

11.1. Cobordisms and reducible connections

As described in Section 1.3, the Donaldson invariant of a manifold with $b^+ = 1$ depends on the Riemannian metric. The Kotschick–Morgan Conjecture 11.2.1, describes how the Donaldson invariant changes with the Riemannian metric. These changes are given by intersection numbers on the link of gauge-equivalence classes of reducible connections in the cobordism (11.1.1). We therefore now review how such gauge-equivalence classes appear in the cobordism.

For a complex, rank-two Hermitian vector bundle $E \rightarrow X$ with $c_1(E) = w$, denote $\mathfrak{g}_\kappa^w := \mathfrak{su}(E)$, where $-4\kappa = p_1(\mathfrak{g}_\kappa^w)$. Let g_I denote a smooth path of Riemannian metrics on X given by $I = [-1, 1] \ni t \mapsto g_t$. Let $M_\kappa^w(g_t)$ be the moduli space of $\mathrm{SO}(3)$ connections on \mathfrak{g}_κ^w which are anti-self-dual with respect to the metric g_t , as defined in (2.2.1), and let $\bar{M}_\kappa^w(g_t)$ be its Uhlenbeck compactification. The smooth path of Riemannian metrics, g_I , on X defines a cobordism of moduli spaces,

$$(11.1.1) \quad \bar{M}_\kappa^w(g_I) = \{[A, \mathbf{x}, t] : [A, \mathbf{x}] \in \bar{M}_\kappa^w(g_t) \text{ and } t \in [-1, 1]\}.$$

To simplify the discussion, we shall assume that w is good in the sense of Definition 2.2.1. By the Morgan–Mrowka blow-up trick [73], this assumption can always be satisfied and so there is no loss of generality.

A reducible connection A on $\mathfrak{g}_\kappa^w \rightarrow X$ can be written as $A = \Theta_{\mathbb{R}} \oplus A_L$ with respect to a splitting $\mathfrak{g}_\kappa^w \cong i\mathbb{R} \oplus L$, where $\Theta_{\mathbb{R}}$ is the product connection on the product bundle $\mathbb{R} = X \times \mathbb{R}$ and A_L is a unitary connection on a complex line bundle L with $c_1(L) \equiv w \pmod{2}$, and $c_1(L)^2 = p_1(\mathfrak{g}_\kappa^w) = -4\kappa$ (see [15, Proposition 4.2.15]). The connection $\Theta_{\mathbb{R}} \oplus A_L$ will be anti-self-dual if and only if A_L is anti-self-dual. Let $\omega^+(g)$ denote the unique (once an orientation for $H^+(X)$ is specified) unit-length self-dual harmonic two-form for the metric g . Then, by [15, p. 147], the connection A_L will be anti-self-dual if and only if

$$(11.1.2) \quad \omega^+(g) \smile F_{A_L} = 0.$$

Hence, $M_\kappa^w(g)$ will contain a gauge-equivalence class of a reducible connection of the form $\Theta_{\mathbb{R}} \oplus A_L$ if and only if $\omega^+(g)$ lies on the codimension-one ‘wall’ in the positive cone,

$$\Omega_X := \{\beta \in H^2(X; \mathbb{R}) : \beta^2 > 0\} / \mathbb{R}^+,$$

of $H^2(X; \mathbb{R})/\mathbb{R}^+$ defined by the zero-locus of the map $\Omega \ni [h] \mapsto h \smile c_1(L)$. If (11.1.2) holds and $b^1(X) = 0$, then by [15, Proposition 2.2.6] there is a unique gauge-equivalence class in $M_\kappa^w(g)$ of a connection giving the splitting $\mathfrak{g}_\kappa^w \cong i\mathbb{R} \oplus L$. We summarize the above discussion in the following

LEMMA 11.1.1. *The compactification $\bar{M}_\kappa^w(g_0)$ contains a gauge-equivalence class of an ideal reducible connection A in the sense that*

$$[A] \times \text{Sym}^\ell(X) \subset \bar{M}_\kappa^w(g_0),$$

for $A = \Theta_\mathbb{R} \oplus A_L$ with respect to a reduction $\mathfrak{g}_{\kappa-\ell}^w \cong \mathbb{R} \oplus L$ for a complex line bundle L with $\ell \geq 0$ if and only if

$$(11.1.3) \quad c_1(L)^2 = -4(\kappa - \ell) \geq 0 \quad \text{and} \quad c_1(L) \equiv w \pmod{2},$$

and (11.1.2) holds with $g = g_0$.

The wall defined by $\alpha \in H^2(X; \mathbb{Z})$ is a (w, κ) -wall if there is a complex line bundle $L \rightarrow X$ satisfying (11.1.3) with $\alpha = c_1(L)$.

11.2. Cohomology classes on the cobordism

Let \mathcal{A}_κ^w be the space of L_k^2 ($k \geq 2$) $\text{SO}(3)$ connections on \mathfrak{g}_κ^w and for a rank-two, complex Hermitian bundle $E \rightarrow X$ with $\mathfrak{su}(E) \cong \mathfrak{g}_\kappa^w$, let \mathcal{G}_E be the Hilbert Lie group of L_{k+1}^2 , determinant-one, unitary gauge transformations of E . The group \mathcal{G}_E acts through the adjoint representation on $\mathfrak{su}(E) \cong \mathfrak{g}_\kappa^w$. Let $\mathcal{B}_\kappa^w := \mathcal{A}_\kappa^w / \mathcal{G}_E$ denote the quotient space of connections and define

$$\bar{\mathcal{B}}_\kappa^w := \bigsqcup_{\ell=0}^{[\kappa]} \mathcal{B}_{\kappa-\ell}^w \times \text{Sym}^\ell(X),$$

with topology defined by Uhlenbeck convergence as in Definition 2.1.2. Then there is an embedding, ι_M , defined by the composition,

$$\iota_M : \bar{M}_\kappa^w(g_I) \rightarrow \bar{\mathcal{B}}_\kappa^w \times [-1, 1] \rightarrow \bar{\mathcal{B}}_\kappa^w,$$

of the inclusion with the projection. The set of gauge-equivalence classes of reducible connections in $\bar{\mathcal{B}}_\kappa^w$ are given by

$$\bar{R}_\kappa^w = \{[A, \mathbf{x}] \in \bar{\mathcal{B}}_\kappa^w : A \text{ is reducible}\},$$

and we define

$$\mathcal{B}_\kappa^{w,*} := \mathcal{B}_\kappa^w \setminus \bar{R}_\kappa^w, \quad \bar{\mathcal{B}}_\kappa^{w,*} := \bar{\mathcal{B}}_\kappa^w \setminus \bar{R}_\kappa^w, \quad \text{and} \quad \bar{M}_\kappa^{w,*}(g_I) := \bar{M}_\kappa^w(g_I) \setminus \iota_M^{-1}(\bar{R}_\kappa^w).$$

For $z \in \mathbb{A}(X)$, let $\mu(z) \in H^{\deg(z)}(\mathcal{B}_\kappa^{w,*}; \mathbb{R})$ be the cohomology class described in [15, Definition 5.1.12] or [53, Section 2.2]. We note that the cohomology class $\mu_p(z)$ defined in (2.4.4) is the pullback of $\mu(z)$ by the projection map $\mathcal{C}_\iota^* \rightarrow \mathcal{B}_\kappa^{w,*}$. In [52, Lemma 4.1.2], Kotschick and Morgan define a cohomology class,

$$\bar{\mu}(z) \in H^{\deg(z)}(\bar{\mathcal{B}}_\kappa^{w,*}; \mathbb{R}),$$

whose restriction to $\mathcal{B}_\kappa^{w,*}$ equals $\mu(z)$. For Riemannian metrics g with $\bar{M}_\kappa^w(g) \cap \bar{R}_\kappa^w$ empty, the Donaldson invariant for g is given by

$$D_X^w(g)(z) := \langle \bar{\mu}(z), [\bar{M}_\kappa^w(g)] \rangle.$$

From [52, Theorem 3.0.1], this invariant only depends on the ‘chamber’ in which the self-dual harmonic two-form $\omega^+(g)$ appearing in (11.1.2) lies. By *chamber*, we mean a connected component

of the complement in the positive cone Ω_X of all the walls of type (w, κ) . As described in Section 1.3, if g_I is a smooth path of Riemannian metrics, the failure of the cobordism given by $\bar{M}_\kappa^w(g_I)$ to give the equality $D_X^w(g_{-1})(z) = D_X^w(g_1)(z)$ as in (1.3.1) arises from the presence of gauge-equivalence classes of ideal reducible connections, $[A, \mathbf{x}] \in \iota_M(\bar{M}_\kappa^w(g_I))$.

For each complex line bundle $L \rightarrow X$ with $c_1(L)$ defining a wall of type (w, κ) , we will construct in Section 11.3.4 (see (11.3.7)) a closed neighborhood $\bar{U}_\kappa^w(L)$ of $[A] \times \text{Sym}^\ell(X)$ in $\iota_M(\bar{M}_\kappa^w(g_I))$. The *difference term* corresponding to this gauge-equivalence class of a reducible connection is

$$(11.2.1) \quad \delta_{L,\kappa}^w(z) := \langle \iota_M^* \bar{\mu}(z), [\partial \bar{U}_\kappa^w(L)] \rangle.$$

Computations of the difference term for $\ell = 0$ appeared in [13, 51], for $\ell = 1$ in [104, 59], and for $\ell = 2$ in [59], where ℓ is the level of the reducible connection as it appears in Lemma 11.1.1. Details of the construction of $\bar{U}_\kappa^w(L)$ appearing in the following sections show that the boundary of this neighborhood admits a fundamental class. The difference between the Donaldson invariants for the Riemannian metrics g_1 and g_{-1} is given by

$$D_X^w(g_1)(z) - D_X^w(g_{-1})(z) = \sum_L \delta_{L,\kappa}^w(z),$$

where the sum is over all complex line bundles L for which $c_1(L)$ defines a wall of type (w, κ) which $\omega^+(g_t)$ crosses as t moves from -1 to 1 . We can now state the *Kotschick–Morgan Conjecture*.

CONJECTURE 11.2.1. (See Kotschick and Morgan [52, Conjectures 6.2.1 and 6.2.2].) Let X be a smooth, connected, oriented four-manifold with $b^+(x) = 1$ and $b_1(X) = 0$. Let $\kappa \in \frac{1}{4}\mathbb{Z}$ with $\kappa < 0$ and let $w \in H^2(X; \mathbb{Z})$ satisfy $w^2 = -4\kappa \pmod{4}$. Let δ and m be non-negative integers satisfying $2\delta = 8\kappa - (3/2)(\chi(X) + \sigma(X))$ and $2m \leq \delta$. For a complex line bundle $L \rightarrow X$ with $c_1(L)$ defining a wall of type (w, κ) , let $\ell = (c_1(L)^2 + 4\kappa)/4$. Then, for $h \in H_2(X; \mathbb{R})$ and a generator $x \in H_0(X; \mathbb{Z})$,

$$\delta_{L,\kappa}^w(h^{\delta-2m} x^m) = \sum_{i=0}^{\min(\ell, [(\delta-2m)/2])} a_i \langle c_1(L), h \rangle^{\delta-2m-2i} Q_X(h)^i,$$

where the coefficients a_i depend only on $e(X)$, $\sigma(X)$, m , δ , and ℓ .

11.3. Neighborhoods of gauge-equivalence classes of ideal reducible connections

To compute the difference term (11.2.1), we show how to define and parameterize the neighborhood $\bar{U}_\kappa^w(L)$ of $\{[A]\} \times \text{Sym}^\ell(X)$ in $\bar{M}_\kappa^w(g_I)$, where $[A]$ is the gauge-equivalence class of a reducible anti-self-dual connection.

11.3.1. Kuranishi model for a neighborhood of a reducible connection. We first record a description of a neighborhood of a gauge-equivalence class of a reducible connection in the parameterized moduli space $M_{\kappa-\ell}^w(g_I)$.

LEMMA 11.3.1. *Let $[A] \in M_{\kappa-\ell}^w(g)$ be a gauge-equivalence class of a reducible connection, $A = \Theta_{\mathbb{R}} \oplus A_L$, with respect to a splitting $\mathfrak{g}_{\kappa-\ell}^w \cong \mathbb{R} \oplus L$. Then for generic smooth paths of metrics g_I with $g = g_0$ and for integers $n, r \in \mathbb{Z}$ satisfying $2n = \dim M_{\kappa-\ell}^w + 1$ and $n + r \geq 0$, there are*

- (1) *An open ball, $B(\delta) \subset \mathbb{C}^{n+r}$, with center at the origin and radius δ ,*
- (2) *A smooth, S^1 -equivariant map $\chi_A : B(\delta) \rightarrow \mathbb{C}^r$ (where S^1 acts by scalar multiplication on the domain and range), and*

(3) An S^1 -equivariant embedding $\tilde{\varphi}_A : B(\delta) \rightarrow \mathcal{A}_{\kappa-\ell}^w$ (where S^1 acts on $\mathcal{A}_{\kappa-\ell}^w$ as the stabilizer of A),

such that $\tilde{\varphi}_A$ covers a smoothly-stratified embedding,

$$\varphi_A : B(\delta)/S^1 \rightarrow \mathcal{B}_{\kappa-\ell}^w,$$

with the property that $\varphi_A(0) = [A]$ and $\varphi_A(\chi_A^{-1}(0)/S^1)$ is a neighborhood of $[A]$ in $M_{\kappa-\ell}^w(g_I)$.

PROOF. The conclusion (compare [15, Proposition 4.2.23]) follows from a standard argument using the Kuranishi Lemma [15, Proposition 4.2.19] and the transversality of the period map $g \mapsto \omega^+(g)$ from [15, Proposition 4.3.14]. An extended discussion of this model appears in [21, Section 2.6]. \square

REMARK 11.3.2. If the index n in Lemma 11.3.1 is strictly positive, then by adapting [34, Theorem 4.19] to the case $b^+(X) = 1$, one can assume that for a generic path of metrics, there is no obstruction in the Kuranishi model, that is, $r = 0$. The index n will be negative when $c_1(L)^2 = -1$ and as this case can be handled by the techniques developed in this monograph, we include it for completeness.

11.3.2. Crude splicing maps. We now combine the description of a neighborhood of a gauge-equivalence class of a reducible connection, $[A]$ in $\iota_M(M_{\kappa-\ell}^w(g_I))$, from Lemma 11.3.1 with the construction of the space of global splicing data from Chapter 6 to parameterize the neighborhood $\bar{U}_{\kappa}^w(g_I)$ appearing in (11.2.1) of the level $[A] \times \text{Sym}^{\ell}(X)$ in $\iota_M(\bar{M}_{\kappa}^w(g_I))$.

Let \mathcal{P} be a partition of N_{ℓ} with $\Sigma = \Sigma(X^{\ell}, \mathcal{P})$. Although spin^u structures play no direct role in this construction, we introduce them here to allow us to use the results from earlier parts of this monograph. Let $\mathfrak{t} = (\rho, V_{\mathfrak{t}})$ be a spin^u structure on X with $\mathfrak{g}_{\mathfrak{t}} \cong \mathfrak{g}_{\kappa}^w$ and let $\mathfrak{s} = (\rho_{\mathfrak{s}}, W)$ be a spin^c structure on X so that the spin^u structure $\mathfrak{t}(\ell) = (\rho_{\ell}, V_{\mathfrak{t}(\ell)})$ admits a splitting $V_{\mathfrak{t}(\ell)} = W \oplus W \otimes L$ as in Section 2.3.3 and $\mathfrak{g}_{\mathfrak{t}(\ell)} \cong i\mathbb{R} \oplus L$ as in (2.3.17). We note the equality,

$$c_1(L) = c_1(\mathfrak{t}) - c_1(\mathfrak{s}),$$

and define

$$\text{Fr}(L, \mathcal{P}) := \text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}), \quad \bar{\text{Gl}}(L, \mathcal{P}) := \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}), \quad \mathcal{O}(L, \mathcal{P}) := \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}),$$

where the frame bundle $\text{Fr}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ is defined in (6.2.2), the bundle of gluing data $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ is defined in (6.2.6), and the subspace $\mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}) \subset \bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ is defined in (6.4.2). The reduction $\mathfrak{g}_{\kappa-\ell}^w \cong \mathbb{R} \oplus L$ and the S^1 action on L defined by scalar multiplication define an S^1 action on $\text{Fr}(\mathfrak{g}_{\kappa-\ell}^w)$. This S^1 action in turn defines an S^1 action on $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$,

$$(11.3.1) \quad S^1 \times \bar{\text{Gl}}(L, \mathcal{P}) \rightarrow \bar{\text{Gl}}(L, \mathcal{P}),$$

given by the diagonal S^1 action on the factors of $\text{Fr}(\mathfrak{g}_{\kappa-\ell}^w)$ making up $\bar{\text{Gl}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$. We will define the maps parameterizing a neighborhood of $[A] \times \Sigma(X^{\ell}, \mathcal{P})$ in $\bar{M}_{\kappa}^w(g_I)$ on a subspace of

$$B(\delta) \times_{S^1} \bar{\text{Gl}}(L, \mathcal{P}),$$

where $B(\delta)$ is as defined in Lemma 11.3.1 and where S^1 acts diagonally by the action in Lemma 11.3.1 on $B(\delta)$ and by the action (11.3.1) on $\bar{\text{Gl}}(L, \mathcal{P})$.

For $\zeta \in B(\delta)$, let $g(\zeta)$ be the Riemannian metric with respect to which $\tilde{\varphi}_A(\zeta)$ is anti-self-dual, where $\tilde{\varphi}_A$ is the map given in Lemma 11.3.1. By applying the flattening construction of Lemma 4.5.2 to the metrics $g(\zeta)$ with $\zeta \in B(\delta)$, we obtain a family of locally-flattened metrics parameterized

by $B(\delta)/S^1 \times \Sigma$. Define $\mathcal{O}(L, \mathcal{P}) := \mathcal{O}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$ as in (6.4.2). The crude splicing map from (6.4.11), applied with this family of locally-flattened Riemannian metrics and with the background section Φ_0 set equal to zero, defines a smoothly-stratified embedding,

$$(11.3.2) \quad \gamma''_{L, \mathcal{P}} : B(\delta) \times_{S^1} \mathcal{O}(L, \mathcal{P}) \rightarrow \bar{\mathcal{B}}_\kappa^w.$$

We will construct the space $\bar{U}_\kappa^w(L)$ as the union, over partitions \mathcal{P} of N_ℓ , of the images of the zero locus of an obstruction section under these maps.

11.3.3. Overlap spaces and maps. The next step in constructing the link $\partial \bar{U}_\kappa^w(L)$ is to define a space of overlap data and upwards and downwards transition maps as in Section 6.5. Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_ℓ . Following (6.5.1), we define

$$\mathrm{Gl}(L, \mathcal{P}, [\mathcal{P}']) = \mathrm{Fr}(L, \mathcal{P}) \times_{G(\mathcal{P})} \bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \prod_{P \in \mathcal{P}} (\Delta^\circ(Z_{|P|}(\delta_P), \mathcal{P}''_P) \times \bar{M}(\mathcal{P}''_P)).$$

To define the upwards overlap map, we follow (6.5.10) and introduce the space

$$\mathrm{Gl}(L, [\mathcal{P} < \mathcal{P}']) = \left(\bigsqcup_{\mathcal{P}'' \in [\mathcal{P} < \mathcal{P}']} \mathrm{Fr}(L, \mathcal{P}'') \times_{\tilde{G}(\mathcal{P}'')} \prod_{Q \in \mathcal{P}''} \bar{M}_{\mathrm{SPL}, |Q|}^{s, \natural} \right) / \mathfrak{S}(\mathcal{P}).$$

If we define an open subspace $\mathcal{O}(L, \mathcal{P}, [\mathcal{P}']) \subset \mathrm{Gl}(L, \mathcal{P}, [\mathcal{P}'])$ by condition (6.5.11), then we can define an S^1 -equivariant upwards transition map,

$$\rho_{\mathcal{P}, [\mathcal{P}']}^{u, L} : B(\delta) \times \mathcal{O}(L, \mathcal{P}, [\mathcal{P}']) \rightarrow B(\delta) \times \bar{\mathrm{Gl}}(L, [\mathcal{P} < \mathcal{P}']),$$

exactly as was done for the upwards transition map $\rho_{\mathcal{P}, [\mathcal{P}']}^{t, s, u}$ in (6.5.14). By S^1 -equivariant above, we mean equivariant with respect to the S^1 action on the domain given by the diagonal action on $B(\delta)$ and on $\mathrm{Fr}(L, \mathcal{P})$ and the same action on the image.

If we further assume that the open subspace $\mathcal{O}(L, \mathcal{P}, [\mathcal{P}'])$ satisfies the condition in (6.5.18), we can define an S^1 -equivariant downward transition map,

$$\rho_{\mathcal{P}, [\mathcal{P}']}^{d, L} : B(\delta) \times \mathcal{O}(L, \mathcal{P}, [\mathcal{P}']) \rightarrow B(\delta) \times \mathrm{Gl}(L, \mathcal{P}),$$

by the construction in (6.5.15) and (6.5.16).

As described in the proof of Proposition 6.5.3, if the neighborhoods $\mathcal{O}(L, \mathcal{P})$ and $\mathcal{O}(L, \mathcal{P}'')$ are sufficiently small, then we can find an open set $\mathcal{O}(L, \mathcal{P}, [\mathcal{P}'])$ such that the overlap of the images of the crude splicing maps is “controlled” by the overlap space in the sense that

$$(11.3.3) \quad \begin{aligned} & \gamma''_{L, \mathcal{P}}(B(\delta) \times_{S^1} \mathcal{O}(L, \mathcal{P})) \cap \gamma''_{L, \mathcal{P}'}(B(\delta) \times_{S^1} \mathcal{O}(L, \mathcal{P}')) \\ &= \left(\gamma''_{L, \mathcal{P}} \circ \rho_{\mathcal{P}, [\mathcal{P}']}^{d, L} \right) (B(\delta) \times_{S^1} \mathcal{O}(L, \mathcal{P}, [\mathcal{P}'])). \end{aligned}$$

More precisely, we have the following

PROPOSITION 11.3.3. *Let $\mathcal{P} < \mathcal{P}'$ be partitions of N_ℓ . Assume that the families of metrics $g_{\mathcal{P}''}$ and $g_{\mathcal{P}}$ used to define the crude splicing maps $\gamma''_{L, \mathcal{P}}$ and $\gamma''_{L, \mathcal{P}'}$ satisfy the conditions in Section 6.2.2. Then there are open neighborhoods $\mathcal{O}(L, \mathcal{P})$ of $\Sigma(X^\ell, \mathcal{P})$ in $\bar{\mathrm{Gl}}(L, \mathcal{P})$ and an $\mathfrak{S}(\mathcal{P})$ -invariant family of neighborhoods $\mathcal{O}(L, \mathcal{P}'')$ in $\bar{\mathrm{Gl}}(L, \mathcal{P}'')$ of $\Sigma(X^\ell, \mathcal{P}'')$ such that if the open*

set $\mathcal{O}(L, \mathcal{P}, [\mathcal{P}'])$ satisfies the analogues of the conditions (6.5.11), (6.5.18), and those following (6.5.21), then the following diagram commutes and (11.3.3) holds:

$$(11.3.4) \quad \begin{array}{ccc} B(\delta) \times_{S^1} \mathcal{O}(L, \mathcal{P}, [\mathcal{P}']) & \xrightarrow{\rho_{\mathcal{P}, [\mathcal{P}']}^{L,u}} & B(\delta) \times_{S^1} \mathcal{O}(L, [\mathcal{P} < \mathcal{P}']) \\ \rho_{\mathcal{P}, [\mathcal{P}']}^{L,d} \downarrow & & \gamma''_{L, [\mathcal{P} < \mathcal{P}']} \downarrow \\ \tilde{B}(\delta) \times_{S^1} \mathcal{O}(L, \mathcal{P}) & \xrightarrow{\gamma''_{L, \mathcal{P}}} & \bar{\mathcal{B}}_{\kappa}^w \end{array}$$

PROOF. The proof is identical to that of Proposition 6.5.3. \square

11.3.4. Definition of the neighborhood of a gauge-equivalence class of an ideal reducible connection. We define the space of global splicing data by

$$(11.3.5) \quad \tilde{\mathcal{U}}_{\kappa}^w(L) := \left(\bigsqcup_{\mathcal{P}} B(\delta) \times \mathcal{O}(L, \mathcal{P}) \right) / \sim,$$

where the relation \sim is defined as follows. If $\mathcal{P}_1 < \mathcal{P}_2$ are partitions of N_{ℓ} , $B(\delta) \subset \mathbb{C}^n$ is the ball appearing in Lemma 11.3.1, $\mathcal{O}(L, \mathcal{P}_i)$ is the space of gluing data defined prior to (11.3.2), and $(z_i, \mathbf{A}_i) \in B(\delta) \times \mathcal{O}(L, \mathcal{P}_i)$ for $i = 1, 2$, then $(z_1, \mathbf{A}_1) \sim (z_2, \mathbf{A}_2)$ if $z_1 = z_2$ and there is a point $\mathbf{A}_{12} \in \mathcal{O}(L, \mathcal{P}_i, [\mathcal{P}_j])$ with $\rho_{\mathcal{P}_1, [\mathcal{P}_2]}^{L,u}(z_2, \mathbf{A}_{12}) = (z_2, \mathbf{A}_2)$ and $\rho_{\mathcal{P}_1, [\mathcal{P}_2]}^{L,d}(z_1, \mathbf{A}_{12}) = (z_1, \mathbf{A}_1)$. While Proposition 11.3.3 implies that this relation satisfies the transitivity property up to S^1 quotients, the proof without the S^1 quotient requires the following.

LEMMA 11.3.4. *The relation \sim defined above is transitive.*

PROOF. Assume that $(z_1, \mathbf{A}_1) \sim (z_2, \mathbf{A}_2)$ and $(z_2, \mathbf{A}_2) \sim (z_3, \mathbf{A}_3)$. Thus, $z_1 = z_2 = z_3$. By Proposition 11.3.3, $\gamma''_{L, \mathcal{P}_1}([z_1, \mathbf{A}_1]) = \gamma''_{L, \mathcal{P}_2}([z_2, \mathbf{A}_2])$ and $\gamma''_{L, \mathcal{P}_2}([z_2, \mathbf{A}_2]) = \gamma''_{L, \mathcal{P}_3}([z_3, \mathbf{A}_3])$, so $\gamma''_{L, \mathcal{P}_1}([z_1, \mathbf{A}_1]) = \gamma''_{L, \mathcal{P}_3}([z_3, \mathbf{A}_3])$, where $[z_i, \mathbf{A}_i] \in B(\delta) \times_{S^1} \mathcal{O}(L, \mathcal{P}_i)$ denotes the image of $(z_i, \mathbf{A}_i) \in B(\delta) \times \mathcal{O}(L, \mathcal{P}_i)$ in the S^1 quotient. Assume that $\mathcal{P}_1 < \mathcal{P}_2 < \mathcal{P}_3$ (the proofs for the other cases are identical). Proposition 11.3.3, the equality $\gamma''_{L, \mathcal{P}_1}([z_1, \mathbf{A}_1]) = \gamma''_{L, \mathcal{P}_3}([z_3, \mathbf{A}_3])$, and the equalities $z_1 = z_2 = z_3$ imply that there is a point $\mathbf{A}_{13} \in \mathcal{O}(L, \mathcal{P}_1, [\mathcal{P}_3])$ with $[\rho_{\mathcal{P}_1, [\mathcal{P}_3]}^{L,u}(z_1, \mathbf{A}_{13})] = [z_1, \mathbf{A}_3] = [z_3, \mathbf{A}_3]$ and $[\rho_{\mathcal{P}_1, [\mathcal{P}_2]}^{L,d}(z_1, \mathbf{A}_{13})] = [z_1, \mathbf{A}_1]$. Consequently, there is a $\lambda \in S^1$ such that $(z_1, \mathbf{A}_1) \sim \lambda(z_1, \mathbf{A}_3)$. Hence, $z_1 = \lambda z_1$ and so if $z_1 \neq 0$, then $\lambda = 1$ and $(z_1, \mathbf{A}_1) \sim (z_1, \mathbf{A}_3)$.

If $z_1 = 0$, then we still have $\mathbf{A}_{13} \in \mathcal{O}(L, \mathcal{P}_1, [\mathcal{P}_3])$ with $[\rho_{\mathcal{P}_1, [\mathcal{P}_3]}^{L,u}(0, \mathbf{A}_{13})] = [0, \mathbf{A}_3]$ and $[\rho_{\mathcal{P}_1, [\mathcal{P}_2]}^{L,d}(0, \mathbf{A}_{13})] = [0, \mathbf{A}_1]$. Because $\rho_{\mathcal{P}_1, [\mathcal{P}_2]}^{L,d}$ and $\rho_{\mathcal{P}_1, [\mathcal{P}_2]}^{L,u}$ are open embeddings, we can find a sequence $\{w_{\alpha}\} \subset B(\delta) \setminus \{0\}$ converging to 0 and a sequence $\{\mathbf{A}_{13}(\alpha)\} \subset \mathcal{O}(L, \mathcal{P}_1, [\mathcal{P}_3])$ with $[\rho_{\mathcal{P}_1, [\mathcal{P}_3]}^{L,u}(w_{\alpha}, \mathbf{A}_{13}(\alpha))] = [w_{\alpha}, \mathbf{A}_3]$ and $[\rho_{\mathcal{P}_1, [\mathcal{P}_2]}^{L,d}(w_{\alpha}, \mathbf{A}_{13}(\alpha))] = [w_{\alpha}, \mathbf{A}_1]$. Since $w_{\alpha} \neq 0$, the preceding argument implies that $\rho_{\mathcal{P}_1, [\mathcal{P}_3]}^{L,u}(w_{\alpha}, \mathbf{A}_{13}(\alpha)) = (w_{\alpha}, \mathbf{A}_3)$ and $\rho_{\mathcal{P}_1, [\mathcal{P}_3]}^{L,d}(w_{\alpha}, \mathbf{A}_{13}(\alpha)) = (w_{\alpha}, \mathbf{A}_1)$. Because $\rho_{\mathcal{P}_1, [\mathcal{P}_2]}^{L,d}$ and $\rho_{\mathcal{P}_1, [\mathcal{P}_2]}^{L,u}$ are open embeddings, $\mathbf{A}_{13}(\alpha)$ converges to \mathbf{A}_{13} and so the continuity of these maps implies that $\rho_{\mathcal{P}_1, [\mathcal{P}_3]}^{L,u}(0, \mathbf{A}_{13}) = (0, \mathbf{A}_3)$ and $\rho_{\mathcal{P}_1, [\mathcal{P}_3]}^{L,d}(0, \mathbf{A}_{13}) = (0, \mathbf{A}_1)$. Hence $(0, \mathbf{A}_1) \sim (0, \mathbf{A}_3)$ as required. \square

For partitions $\mathcal{P}_1 < \mathcal{P}_2$ of N_{ℓ} , the overlap maps $\rho_{\mathcal{P}_1, [\mathcal{P}_2]}^{L,u}$ and $\rho_{\mathcal{P}_1, [\mathcal{P}_2]}^{L,d}$ are S^1 equivariant, so the S^1 actions on $B(\delta) \times \mathcal{O}(L, \mathcal{P}_1)$ and $B(\delta) \times \mathcal{O}(L, \mathcal{P}_2)$ respect the equivalence relation \sim . Hence,

these S^1 actions define an S^1 action on $\widetilde{\mathcal{U}}_\kappa^w(L)$. By Proposition 11.3.3, we then obtain

$$\gamma''_{\mathcal{U},L} : \widetilde{\mathcal{U}}_\kappa^w(L)/S^1 \rightarrow \widetilde{\mathcal{B}}_\kappa^w,$$

a global crude splicing map.

11.3.5. Thom–Mather structures on $\widetilde{\mathcal{U}}_\kappa^w(L)/S^1$. We consider $\mathcal{U}(L, \mathcal{P}) = B(\delta) \times \mathcal{O}(L, \mathcal{P})$ as a subspace of $\widetilde{\mathcal{U}}_\kappa^w(L)$ and define maps

$$\pi(L, \mathcal{P}) : \mathcal{U}(L, \mathcal{P}) \rightarrow B(\delta) \times \Sigma(X^\ell, \mathcal{P}), \quad \vec{t}(L, \mathcal{P}) : \mathcal{U}(L, \mathcal{P}) \rightarrow [0, 1]^\mathcal{P},$$

as was done in (6.7.2) and (6.7.8).

The proof of Lemma 6.7.2 then implies that for $\mathcal{P} < \mathcal{P}'$ on $\mathcal{U}(L, \mathcal{P}) \cap \mathcal{U}(L, \mathcal{P}')$, we have the equality

$$\pi(L, \mathcal{P}') \circ \pi(L, \mathcal{P}) = \pi(L, \mathcal{P}).$$

The analogues of Lemmas 6.7.5 and 6.7.6 hold for $\vec{t}(L, \mathcal{P})$.

11.3.6. Global projection map for $\widetilde{\mathcal{U}}_\kappa^w(L)/S^1$. By the arguments of Lemma 6.9.2, we can define a map,

$$\pi_{L,X} : \widetilde{\mathcal{U}}_\kappa^w(L)/S^1 \rightarrow \text{Sym}^\ell(X),$$

whose restriction to $\mathcal{U}(L, \mathcal{P})$ is homotopic to $\pi(L, \mathcal{P})$.

11.3.7. Global splicing map on $\widetilde{\mathcal{U}}_\kappa^w(L)/S^1$. The construction of the global splicing map $\gamma'_{\mathcal{M}}$ of (6.8.5) translates to give a global splicing map,

$$\gamma'_L : \widetilde{\mathcal{U}}_\kappa^w(L)/S^1 \rightarrow \widetilde{\mathcal{B}}_\kappa^w,$$

whose restriction to $\mathcal{U}(L, \mathcal{P})$ is homotopic to the crude splicing map $\gamma''_{L,\mathcal{P}}$.

11.3.8. Obstruction bundle on $\widetilde{\mathcal{U}}_\kappa^w(L)/S^1$. Let n and r be the non-negative integers appearing in Lemma 11.3.1. Define a complex-rank- r vector bundle, $\widetilde{\Xi}_\kappa^w(L) \rightarrow \widetilde{\mathcal{U}}_\kappa^w(L)$, by

$$\widetilde{\Xi}_\kappa^w(L) := \left(\bigsqcup_{\mathcal{P}} (B(\delta) \times \mathbb{C}^r \times \mathcal{O}(L, \mathcal{P})) \right) / \sim_{\Xi}$$

where the equivalence relation \sim_{Ξ} is defined, for $(z_i, \zeta_i, \mathbf{A}_i) \in B(\delta) \times \mathbb{C}^r \times \mathcal{O}(L, \mathcal{P}_i)$, $i = 1, 2$, by

$$(z_1, \zeta_1, \mathbf{A}_1) \sim_{\Xi} (z_2, \zeta_2, \mathbf{A}_2)$$

if $\zeta_1 = \zeta_2$ and $(z_1, \mathbf{A}_1) \sim (z_2, \mathbf{A}_2)$, where \sim is the equivalence relation used in the definition of $\widetilde{\mathcal{U}}_\kappa^w(L)$ in (11.3.5). The diagonal action of S^1 on $B(\delta) \times \mathbb{C}^r \times \mathcal{O}(L, \mathcal{P})$ given by S^1 acting on $B(\delta)$ and \mathbb{C}^r by scalar multiplication and on $\mathcal{O}(L, \mathcal{P})$ as in (11.3.1) is compatible with the relation \sim_{Ξ} and so $\widetilde{\Xi}_\kappa^w(L)$ is an S^1 -equivariant bundle over $\widetilde{\mathcal{U}}_\kappa^w(L)$. If we define

$$\widetilde{\mathcal{U}}_\kappa^{w,*}(L) := \widetilde{\mathcal{U}}_\kappa^w(L) \setminus ([A] \times \text{Sym}^\ell(X)),$$

then the S^1 action on $\widetilde{\mathcal{U}}_\kappa^w(L)$ is free on $\widetilde{\mathcal{U}}_\kappa^{w,*}(L)$ and the S^1 -quotients define a vector bundle,

$$(11.3.6) \quad \widetilde{\Xi}_\kappa^w(L)/S^1 \rightarrow \widetilde{\mathcal{U}}_\kappa^{w,*}(L)/S^1.$$

This is the desired obstruction bundle.

11.3.9. Gluing hypothesis. We can now state the analogue for parameterized anti-self-dual connections of the gluing hypothesis for $\mathrm{SO}(3)$ monopoles.

HYPOTHESIS 11.3.5. There is a smoothly-stratified, continuous embedding,

$$\gamma_L : \widetilde{\mathcal{U}}_\kappa^w(L)/S^1 \rightarrow \widetilde{\mathcal{B}}_\kappa^w,$$

and a smoothly-stratified section χ_L of the vector bundle $\widetilde{\Xi}_\kappa^w(L)/S^1 \rightarrow \widetilde{\mathcal{U}}_\kappa^w(L)/S^1$ defined in (11.3.6) with the following properties:

- (1) The restriction of γ_L to $[A] \times \mathrm{Sym}^\ell(X)$ is the identity map.
- (2) There is a homotopy, through smoothly-stratified embeddings, between γ'_L and γ_L .
- (3) The image $\gamma_L(\chi_L^{-1}(0))$ is an open neighborhood of $[A] \times \mathrm{Sym}^\ell(X)$ in $\widetilde{M}_\kappa^w(g_I)$.
- (4) The restriction of the section χ_L to each stratum of $\widetilde{\mathcal{U}}_\kappa^{w,*}(L)/S^1$ vanishes transversely.

We then set

$$(11.3.7) \quad \bar{U}_\kappa^w(L) := \gamma_L \left(\chi_L^{-1}(0) \cap \widetilde{\mathcal{U}}_\kappa^w(L)/S^1 \right),$$

to define the neighborhood $\bar{U}_\kappa^w(L)$ appearing in (11.2.1).

11.4. Cohomology classes on the space of global splicing data

Define

$$\nu \in H^2(\widetilde{\mathcal{U}}_\kappa^{w,*}(L)/S^1; \mathbb{Z})$$

to be the first Chern class of the S^1 bundle $\widetilde{\mathcal{U}}_\kappa^{w,*}(L) \rightarrow \widetilde{\mathcal{U}}_\kappa^{w,*}(L)/S^1$. From [52, Lemma 4.7.4], we have the following computation of the pullback of the μ -classes by the gluing map.

LEMMA 11.4.1. *For $h \in H_2(X; \mathbb{R})$ and a generator $x \in H_0(X; \mathbb{Z})$, then the following hold:*

- (1) $\gamma_L^* \bar{\mu}(h) = \frac{1}{2} \langle c_1(L), h \rangle \nu + \pi_{L,X}^* S^\ell(h)$,
- (2) $\gamma_L^* \bar{\mu}(x) = -\frac{1}{4} \nu^2 + \pi_{L,X}^* S^\ell(x)$,

where the cohomology classes $S^\ell(h)$ and $S^\ell(x)$ are defined in Definition 9.2.5.

LEMMA 11.4.2. *The Euler class of the obstruction bundle (11.3.6) is given by*

$$e \left(\widetilde{\Xi}_\kappa^w(L)/S^1 \right) = (-\nu)^r.$$

PROOF. The result follows immediately from [28, Lemma 3.27] and the observation that the S^1 action on $\widetilde{\Xi}_\kappa^w(L)$ is diagonal, so the obstruction bundle is the direct sum of r copies of the line bundle given by the negative of the S^1 action defining the cohomology class ν . \square

11.5. Definition of the link of a gauge-equivalence class of an ideal reducible connection

We define a virtual link $\partial \widetilde{\mathcal{U}}_\kappa^w(L)/S^1$ following the construction of Section 8.1.1. The standard norm on \mathbb{C}^n defines an S^1 -invariant norm on $B(\delta)$ and thus a map,

$$t_B : \widetilde{\mathcal{U}}_\kappa^{w,*}(L)/S^1 \rightarrow [0, \delta].$$

Enumerate the strata of $\text{Sym}^\ell(X)$ by partitions $\mathcal{P}_0, \dots, \mathcal{P}_n$ in the manner described in Section 3.4. Choose a small, generic constant ε_i for each stratum as was done in Section 8.1.1 (with $\varepsilon_i > \varepsilon_j$ for $i < j$). Following (8.1.5), define

$$\partial\mathcal{U}(L, \mathcal{P}_i) := \vec{t}(L, \mathcal{P}_i)^{-1}(\partial\bar{D}(\mathcal{P}_i, \varepsilon_i)) \setminus \bigcup_{j \neq i} \vec{t}(L, \mathcal{P}_j)^{-1}(D(\mathcal{P}_j, \varepsilon_j)).$$

Then, following (8.1.2), (8.1.3), and (8.1.4), we define the virtual link of $[A_L] \times \text{Sym}^\ell(X)$ by

$$\partial\widetilde{\mathcal{U}}_\kappa^w(L) := t_B^{-1}(\delta) \cup \left(\bigcup_i \partial\mathcal{U}(L, \mathcal{P}_i) \right).$$

The link of $[A_L] \times \text{Sym}^\ell(X)$ is then defined by

$$\partial\bar{U}_\kappa^w(L) := \gamma_L \left(\chi_L^{-1}(0) \cap \partial\widetilde{\mathcal{U}}_\kappa^w(L)/S^1 \right).$$

The existence of a fundamental class for the virtual link $\partial\widetilde{\mathcal{U}}_\kappa^w(L)$ follows from the discussion in Section 9.3. Up to a sign depending on the orientation discussed in [52, p. 450–451],

$$(11.5.1) \quad \begin{aligned} \delta_{L,\kappa}^w(h^{\delta-2m}x^m) = & \left\langle \left(\frac{1}{2} \langle c_1(L), h \rangle \nu + \pi_{L,X}^* S^\ell(h) \right)^{\delta-2m} \right. \\ & \left. \smile \left(-\frac{1}{4} \nu^2 + \pi_{L,X}^* S^\ell(x) \right)^m, e \left(\widetilde{\Xi}_\kappa^w(L)/S^1 \right) \smile \left[\partial\widetilde{\mathcal{U}}_\kappa^w(L)/S^1 \right] \right\rangle \end{aligned}$$

is the difference term.

11.6. Computations of the difference term

We can now prove

THEOREM 11.6.1. *Conjecture 11.2.1 is true.*

PROOF. Using the argument in Lemma 9.7.2 and Proposition 9.7.3, the pairing in (11.5.1) can be reduced to a pairing with $t_B^{-1}(0) \cap \partial\widetilde{\mathcal{U}}_\kappa^w(L)$. Observe that

$$t_B^{-1}(0) \cap \partial\mathcal{U}(L, \mathcal{P}_i) \subset S^1 \setminus \mathcal{C}(L, \mathcal{P}_i),$$

that is, $t_B^{-1}(0)$ is given by $\{0\} \in B(\delta)$. Following Proposition 10.2.1, we construct a quotient map

$$Q : t_B^{-1}(0) \cap \partial\widetilde{\mathcal{U}}_\kappa^w(L)/S^1 \rightarrow Q_\kappa^w(L)$$

satisfying the following properties:

- (1) There is a map $\tilde{\pi}_X : Q_\kappa^w(L) \rightarrow \text{Sym}^\ell(X)$ satisfying $\tilde{\pi}_X \circ Q = \pi_{L,X}$.
- (2) There is a class $\tilde{\nu} \in H^2(Q_\kappa^w(L))$ and a non-zero integer r with $Q^* \tilde{\nu} = r\nu$.
- (3) If $Q_\kappa^w(L, \mathcal{P}_i) = Q(t_B^{-1}(0) \cap \partial\mathcal{U}(L, \mathcal{P}_i))$, then

$$Q_\kappa^w(L, \mathcal{P}_i) \cong S^1 \setminus \bar{\text{Fr}}(L, \mathcal{P}_i) \times_{G(\mathcal{P}_i)} \check{M}(\mathcal{P}_i, \varepsilon_i).$$

We can write the pairing (11.5.1) in terms of a sum over pairings of the form

$$(11.6.1) \quad \left\langle \tilde{\nu}^i \smile \tilde{\pi}_X^*(S^\ell(h)^j \smile S^\ell(x)^k), [Q_\kappa^w(L, \mathcal{P}_u)] \right\rangle.$$

The conclusion now follows by an argument analogous to that of the proofs of Lemma 10.5.2 and Theorem 10.1.1. \square

REMARK 11.6.2. If $b^1(X) > 0$, then the gauge-equivalence classes of reducible, anti-self-dual, ideal connections in $\bar{M}_\kappa^w(g_I)$ would no longer appear in families of the form $[A] \times \text{Sym}^\ell(X)$ but rather in families of the form

$$(H^1(X; i\mathbb{R})/H^1(X; 2i\mathbb{Z})) \times \text{Sym}^\ell(X).$$

The techniques of this chapter extend to construct a link $\partial\bar{U}_\kappa^w(L)$ of such families but one must replace the spaces $B(\delta) \times_{S^1} \mathcal{O}(L, \mathcal{P})$ appearing in (11.3.2) with spaces of the form

$$B(\delta) \times H^1(X; i\mathbb{R}) \times_{S^1 \times H^1(X; 2i\mathbb{Z})} \mathcal{O}(L, \mathcal{P}),$$

where $S^1 \times H^1(X; 2i\mathbb{Z})$ acts on $\text{Gl}(L, \mathcal{P})$ through the inclusion $\mathbf{h}_p : S^1 \times H^1(X; i\mathbb{Z}) \rightarrow \text{Map}(X, S^1)$ given by the harmonic gauge transformations described in [28, Section 2.4.1]. The construction of the space $\tilde{\mathcal{U}}_\kappa^w(L)$ for manifolds with $b^1 = 0$ works when $b^1 > 0$ as does the construction of the quotient space $Q_\kappa^w(L)$ given in Section 11.6. However, the expression for the difference term $\delta_{L, \kappa}^w$ will be more complicated when $b^1(X) > 0$ because of the presence of the Chern class of the line bundle,

$$(11.6.2) \quad H^1(X; i\mathbb{R}) \times X \times_{H^1(X; i\mathbb{Z})} \mathbb{C} \rightarrow (H^1(X; i\mathbb{R})/H^1(X; i\mathbb{Z})) \times X,$$

computed in [28, Lemma 2.23] among the characteristic classes appearing in the analogue of Lemma 10.5.2 for pairings of the form (11.6.1). In addition, the Chern class of the line bundle (11.6.2) will appear in the computations of the μ -classes in Lemma 11.4.1 and change those expressions to a form similar to that of the cohomology classes appearing in Definition 9.4.8.

For this reason, the difference term $\delta_{L, \kappa}^w$, although still only depending on the homotopy type of X , \mathfrak{g}_κ^w , ℓ , and L , will not be a polynomial only in $c_1(L)$ and Q_X but will also contain terms given by elements of $H^1(X; \mathbb{Z})$.

Glossary of Notation

\mathbf{A}, \mathbf{A}'	Eq. (6.5.2)	$D(\mathcal{P}, \varepsilon), \bar{D}(\mathcal{P}, \varepsilon), \partial\bar{D}(\mathcal{P}, \varepsilon)$	Eq. (4.7.2)
$[\bar{A}]$	Lemma 10.2.5	$d(\mathbf{t}), d_a(\mathbf{t})$	Eq. (2.1.12)
\mathcal{A}_t	§2.1.2	$\Delta^\circ(X^\ell, \mathcal{P}), \Delta(X^\ell, \mathcal{P})$	Eq. (3.1.4)
\mathcal{A}_s	§2.3.1	$\Delta^\circ(Z_P, \mathcal{P}'_P)$	Eq. (4.2.5)
$\mathbb{A}_2(X)$	Eq. (2.3.7)	$\Delta^\circ(Z_\kappa(\delta), \mathcal{P})$	Eq. (5.3.2)
$\mathbb{A}(X)$	Eq. (2.4.3)	$d_s(\mathfrak{s})$	Eq. (2.3.10)
$\text{Ad}_{\text{SO}(3)}^u, \text{Ad}_{\text{SO}(4)}^u, \text{Ad}^u$	Eq. (7.5.1)	D_X^w	§2.5
$\mathcal{B}_\kappa, \mathcal{B}_\kappa^s$	before Eq. (5.2.1)	\bar{e}_I	Eq. (9.5.23)
$\bar{\mathcal{P}}_{ P }^s(\varepsilon), \bar{\mathcal{P}}_{ P }^{s, \sharp}(\varepsilon)$	Eq. (5.2.4)	$\bar{e}(\bar{\Upsilon}_{t,s}/S^1, \bar{\chi}')$	Eq. (9.5.29)
$\bar{\mathcal{B}}_\kappa$	Eq. (5.2.1)	\bar{e}_χ	Lemma 9.6.3
$\mathbf{BL}_{t,s}^{\text{vir}}$	Eq. (8.1.16)	$e(Z_P, \mathcal{P}'_P)$	Eq. (4.2.10)
$\bar{\mathbf{B}}\mathbf{L}_{t,s}^{\text{vir}}(\mathcal{P}_j)$	Eq. (8.2.4)	$e(X^\ell, g_{\mathcal{P}})$	Eq. (4.3.1)
$[\bar{\mathbf{B}}\mathbf{L}_{t,s}^{\text{vir}}]$	Eq. (9.7.3)	\mathbb{F}_t	Eq. (2.4.1)
$\widehat{\mathbf{B}}\mathbf{L}_{t,s}^{\text{vir}}(\mathcal{P}_k)$	Eq. (10.2.5)	\mathbb{F}_T	Eq. (9.4.10)
$\partial_{k_1} \dots \partial_{k_r} \mathbf{BL}_{t,s}^{\text{vir}}(\mathcal{P}_j)$	Eq. (8.3.1)	$\mathbb{F}_{t,s}^{\text{vir}}$	Eq. (9.4.12)
c_P	Eq. (4.2.2)	$\text{Fr}(t, \mathfrak{s}, \mathcal{P})$	Eq. (6.2.2)
$\mathcal{C}_s, \tilde{\mathcal{C}}_s$	Eq. (2.3.2)	$\bar{\text{F}}\mathbf{r}(t, \mathfrak{s}, \mathcal{P}_j)$	Eq. (10.2.2)
\mathcal{C}_t	Eq. (2.1.8)	$\text{Fr}(V, \mathcal{P}, g_{\mathcal{P}})$	Eq. (7.7.1)
\mathcal{C}_t	Eq. (2.1.8)	\mathfrak{g}_t	Eq. (2.1.7)
$\mathcal{C}_t^*, \mathcal{C}_t^0, \mathcal{C}_t^{*,0}$	following Eq. (2.1.8)	\mathcal{G}_s	§2.3.1
$c_1(\mathfrak{s})$	Eq. (2.1.3)	\mathcal{G}_t	§2.1.2
$c_1(\mathbf{t})$	Eq. (2.1.6)	$\bar{\text{G}}\mathbf{l}(t, \mathfrak{s}, \mathcal{P})$	Eq. (6.2.6)
$[c_{T,\beta}]$	Eq. (9.4.4)	$\text{G}\mathbf{l}(t, \mathfrak{s}, P)$	Eq. (9.5.8)
$[c_\beta]$	Eq. (9.4.5)	$\bar{\text{G}}\mathbf{l}(t, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$	Eq. (6.5.1)
$[c_{T,W}]$	Eq. (9.4.7)	$\bar{\text{G}}\mathbf{l}(t, \mathfrak{s}, [\mathcal{P} < \mathcal{P}'])$	Eq. (6.5.10)
$[c_{\mathcal{W}}]$	Eq. (9.4.8)	$\tilde{G}(T, \mathcal{P})$	Eq. (4.3.3)
\bar{c}_β	Lemma 9.6.1	$\tilde{G}(\mathcal{P}), G(\mathcal{P})$	Eq. (6.2.3)
$[c(z, \eta)]$	Eq. (9.6.2)	$g_{\mathcal{P}}, g_{\mathcal{P}, \mathbf{x}}$	Lemma 4.3.1
$\bar{c}_{\mathcal{W}}$	Lemma 9.6.2	$\gamma'_{\Theta, \mathcal{P}}$	Eqs. (5.4.2) & (5.4.3)
$c_{s,\ell,j}$	Eq. (9.5.3)	$\gamma'_{\Theta, \mathcal{P}, [\mathcal{P}']}$	Eq. (5.5.26)
$c(\mathbf{t})$	Eq. (9.5.4)	$\gamma''_{t,s, \mathcal{P}}$	Eqs. (6.4.1) & (6.4.11)
$c_{j,i}$	Eq. (8.3.12)	$\gamma''_{t,s, [\mathcal{P} < \mathcal{P}']}$	Eq. (6.5.20)
$\bar{\chi}, \chi_s, \chi_i$	Hypothesis 7.8.1	$\gamma''_{\mathcal{M}}$	Eq. (6.6.10)
		$\gamma'_{\mathcal{M}}$	Eq. (6.8.5)
		$\gamma_{\mathcal{M}}$	Eq. (7.8.1)
		ι	Eq. (9.2.1)
		$\iota_{\beta,1}, \iota_{\beta,2}$	Eq. (9.4.28)
		$\text{Index}(\mathbf{D}_\kappa^*)$	Eq. (7.4.2)

$\mathcal{I}(Y)$	Eq. (9.2.6)	$\tilde{\nu}(X^\ell, \mathcal{P})$	Eq. (3.3.2)
\bar{j}_β	Eq. (9.4.9)	$\nu(X^\ell, \mathcal{P})$	Lemma 3.3.2
\bar{K}_k	Lemma 4.8.2	$\tilde{\nu}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}')$	Eq. (3.3.3)
$\partial_{k_1} \dots \partial_{k_r} K_j$	Eq. (8.3.4)	$\nu(X^\ell, \mathcal{P} \rightarrow \mathcal{P}')$	Eq. (3.3.4)
\mathbb{L}_s	Eq. (2.3.5)	$\tilde{\nu}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}')$	Eq. (4.4.1)
\mathbb{L}_ν	Eq. (9.2.15)	$\tilde{\nu}(X^\ell, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}})$	Eq. (4.6.1)
$\bar{\mathbb{L}}_{t,\kappa}^w$	Eq. (2.2.3)	$\nu(X^\ell, \mathcal{P} \rightarrow [\mathcal{P}'])$	Eq. (3.3.4)
$\bar{\mathbb{L}}_{t,s}$	Definition 8.1.3	$\tilde{\nu}(Z_\kappa(\delta), \mathcal{P})$	Eq. (5.3.3)
$\bar{\mathbb{L}}_{t,s}^{\text{vir}}$	Eq. (8.1.2)	$\tilde{\nu}(Z_\kappa(\delta), \mathcal{P}, \mathcal{P}')$	Eq. (5.3.5)
$\bar{\mathbb{L}}_{t,s}^{\text{vir},s}$	Eq. (8.1.3)	$\tilde{\nu}(Z_\kappa, [\mathcal{P} < \mathcal{P}'])$	Eq. (5.3.8)
$\bar{\mathbb{L}}_{t,s}^{\text{vir},i}(\mathcal{P}_j)$	Eq. (8.1.5)	$\tilde{\nu}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$	Eq. (5.5.2)
$[\bar{\mathbb{L}}_{t,s}^{\text{vir}}]$	Eq. (9.3.3)	$\nu(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$	Eq. (5.5.21)
$\ell(\mathbf{t}, \mathfrak{s})$	Eq. (2.3.14)	$\tilde{\mathcal{O}}(X^\ell, \mathcal{P} \rightarrow \mathcal{P}', g_{\mathcal{P}})$	Lemma 3.3.5
$\lambda[A, \mathbf{x}]$	Eq. (5.2.3)	$\tilde{\mathcal{O}}(X^\ell, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}})$	Eq. (4.6.3)
$\tilde{\lambda}_\kappa$	Eq. (5.9.1)	$\mathcal{O}(X^\ell, \mathcal{P}, [\mathcal{P}'], g_{\mathcal{P}})$	Eq. (4.6.3)
\mathcal{M}_t	before Eq.(2.1.11)	$\mathcal{O}(X^\ell, [\mathcal{P} < \mathcal{P}'], g_{\mathcal{P}''})$	Eq. (4.6.5)
$\tilde{\mathcal{M}}_t$	before Eq.(2.1.14)	$\tilde{\mathcal{O}}(Z_\kappa(\delta), \mathcal{P} \rightarrow \mathcal{P}')$,	Eq. (5.3.6)
$\mathcal{M}_t^{\text{red}}$	Eq. (2.2.2)	$\mathcal{O}(Z_P, \mathcal{P}'_P)$	Lemma 4.2.1
$\mathcal{M}_t^{*,0}$	Eq. (2.1.11)	$\tilde{\mathcal{O}}(X^\ell, g_{\mathcal{P}})$	Lemma 4.3.1
$\tilde{\mathcal{M}}_t^{*,0}$	following Eq.(2.2.2)	$\tilde{\mathcal{O}}(Z_\kappa(\delta), \mathcal{P})$	Lemma 5.3.2
M_κ^w	Eq. (2.2.1)	$\mathcal{O}(Z_\kappa(\delta), [\mathcal{P} < \mathcal{P}'])$	Lemma 5.3.4
M_s	§2.3.1	$\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \delta)$	Eq. (5.4.1)
$\bar{M}_{\text{SPL},\kappa}^{s,\natural}(\delta)$	Theorem 5.1.1	$\mathcal{O}(\Theta, \mathcal{P}, \delta)$	Eq. (5.4.2)
$\bar{M}(\mathcal{P})$	Eq. (6.2.5)	$\tilde{\mathcal{O}}(\Theta, \mathcal{P}, \mathcal{P}', \delta)$	Eqs. (5.5.6) & (5.5.10)
$\tilde{M}(\mathcal{P}_k, \varepsilon)$	Definition 10.2.9	$\tilde{\mathcal{O}}^{\text{ASD}}(\Theta, \mathcal{P}, \delta), \mathcal{O}^{\text{ASD}}(\Theta, \mathcal{P}, \delta)$	Eq. (5.6.1)
$\tilde{\mathcal{M}}_{t,s}^{\text{vir}}$	Eq. (6.6.3)	$\tilde{\mathcal{O}}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$	Eq. (5.6.3)
$\mathcal{M}_{t,s}^{\text{vir}}$	Eq. (9.2.1)	$\mathcal{O}(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$	Eq. (5.6.3)
$\tilde{\mathcal{M}}_{t,s}^{\text{vir},*}$	Eq. (9.2.2)	$\mathcal{O}^{\text{ASD}}(\Theta, [\mathcal{P} < \mathcal{P}'], \delta)$	Eq. (5.6.5)
$\partial_{i_1} \partial_{i_2} \dots \partial_{i_v} \bar{M}(\mathcal{P}_j, \varepsilon)$	Eq. (8.3.3)	$\mathcal{O}_d(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$	Eq. (6.5.18)
μ_p	Eq. (2.4.4)	$\text{Ob}(\mathbf{t}, \mathfrak{s}, \mathcal{P})$	Eq. (7.7.5)
μ_s	Eq. (2.3.9)	$\text{Ob}(\mathbf{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$	Eq. (7.7.9)
μ_c	Eq. (2.4.7)	$p_1(\mathbf{t})$	Eq. (2.1.6)
$\bar{\mu}_c$	Eq. (9.4.27)	\mathcal{P}	§3.1.1
$\bar{\mu}_p(\beta)$	Definition 9.4.8	$[\mathcal{P}]$	Lemma 3.1.4
N_ℓ	§3.1.1	\mathcal{P}'_P	Eq. (4.2.4)
$n_a(\mathbf{t})$	Eq. (2.1.12)	$P_{t,s}$	Lemma 10.4.3
$n_s(\mathbf{t}, \mathfrak{s})$	Eq. (2.3.24)	$[\mathcal{P} < \mathcal{P}']$	Eq. (3.2.1)
$N_{t,s}$	Eq. (2.3.19)	$\text{PD}[\mathcal{V}]$	Eq. (9.2.19a)
ν	Eq. (9.2.3)	$\pi(Z_P, \mathcal{P}'_P)$	Eq. (4.2.13)
ν_{Gl}	Eq. (9.5.18)	$\pi(X^\ell, g_{\mathcal{P}})$	Eq. (4.3.2)
$\check{\nu}$	Eq. (10.3.1)	$\pi(\Theta, \mathcal{P})$	Eq. (5.7.1)
$\tilde{\nu}_i$	Lemma 10.4.2	π_N	Eq. (6.6.9)
ν_{Gl}	Eq. (9.5.18)	$\pi(\mathbf{t}, \mathfrak{s}, \mathcal{P})$	Eq. (6.7.2)
		π_X	Eq. (6.9.1)

$\pi_{\mathfrak{s}}$	Eq. (9.2.3)	$\Sigma(Z_{\kappa}(\delta), \mathcal{P})$	Eq. (5.3.2)
$\pi_{\mathfrak{s}, X}$	Eq. (9.2.4)	$\Sigma(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$	Eq. (6.2.7)
$\pi_{N, X}$	Eq. (9.2.5)	$\mathfrak{S}(\mathcal{P}), \Gamma(\mathcal{P}), W(\mathcal{P})$	§3.1.1
$\varphi'_{\Theta, \mathcal{P}}$	Eq. (7.6.1)	$SW_X(\mathfrak{s})$	Eq. (2.3.11)
$\varphi''_{\mathfrak{s}}(\mathcal{P})$	Eq. (7.3.2)	$S^{\ell}(\beta), \tilde{S}^{\ell}(\beta)$	Definition 9.2.5
$\varphi''_{\mathfrak{s}}$	Eq. (7.3.7)	\mathfrak{t}	§2.1.1
$\varphi'_{\mathfrak{s}}$	Eq. (7.3.8)	$\mathfrak{t}(\ell)$	Eq. (2.1.13)
$\varphi_{\mathfrak{t}, \mathfrak{s}, \mathcal{P}}$	Eq. (7.7.6)	$T\Delta^{\circ}(X^{\ell}, \mathcal{P})$	Eq. (3.3.1)
$\varphi_{\mathfrak{t}, \mathfrak{s}, [\mathcal{P} < \mathcal{P}']}$	Eq. (7.7.13)	$T(\Theta, \mathcal{P}, \delta)$	Eq. (5.4.7)
φ'_i	Eq. (7.7.16)	$T(\Theta, \mathcal{P}, \mathcal{P}', \delta)$	Eq. (5.5.12)
$Q, \mathbf{QL}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}$	Proposition 10.2.1	$T(\Theta, \mathcal{P}, [\mathcal{P}'], \delta)$	Eq. (5.5.23)
$Q_{k, i}$	Definition 10.2.5	$T(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$	Eq. (6.2.9)
q_k	Lemma 10.2.10	$T(\mathfrak{t}, \mathfrak{s}, \mathcal{P}, [\mathcal{P}'])$	Eq. (6.5.4)
$[\mathbf{QL}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_i)]$	Eq. (10.3.2)	$\Theta_{\mathcal{P}}$	Eq. (6.3.6)
$\widetilde{\mathbf{QL}}_{\mathfrak{t}, \mathfrak{s}}^{\text{vir}}(\mathcal{P}_i)$	Lemma 10.4.1	$t_{P, 0}$	Eq. (4.2.7)
r_P	Eq. (4.2.8)	$t(Z_P)$	Lemma 4.2.5
$r(X^{\ell}, \mathcal{P})$	Lemma 4.7.1	$\tilde{t}(X^{\ell}, g_{\mathcal{P}})$	Eq. (4.7.1)
r_i	Lemma 4.8.3	$\tilde{t}(\mathfrak{t}, \mathfrak{s}, \mathcal{P}), \tilde{t}^P(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$	Eq. (6.7.8)
\bar{r}_x	Eq. (9.4.6)	$\tilde{t}_f(\mathcal{P}, [\mathcal{P}'])$	Lemma 6.7.3
$\rho_{\mathcal{P}, \mathcal{P}'}^{X, d}$	Eq. (4.4.3)	t_N	Eq. (8.1.1)
$\rho_{\mathcal{P}, \mathcal{P}'}^{X, u}$	Eq. (4.4.5)	T_i	Lemma 4.8.1
$\rho_{\mathcal{P}, [\mathcal{P}']}^d$	Eq. (4.6.4)	$T_{k, j}$	Lemma 4.8.2
$\rho_{\mathcal{P}, [\mathcal{P}']}^u$	Eq. (4.6.6)	$T(\mathcal{P}_i, \mathcal{P}_k)(A_0)$	Eq. (10.2.3)
$\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, u}$	Eq. (5.5.5)	$\mathcal{U}(Z_P, \mathcal{P}'_P)$	Lemma 4.2.1
$\rho_{\mathcal{P}, \mathcal{P}'}^{\Theta, d}$	Eq. (5.5.8)	$\tilde{\mathcal{U}}(X^{\ell}, g_{\mathcal{P}})$	Lemma 4.3.1
$\rho_{\mathcal{P}, [\mathcal{P}']}^{\Theta, u}$	Eq. (5.5.24)	$\tilde{\mathcal{U}}(Z_{\kappa}(\delta), \mathcal{P})$	following Lemma 5.3.2
$\rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, d}$	Eq. (7.6.5)	$\mathcal{U}(\Theta, \mathcal{P})$	Lemma 5.7.1
$\rho_{\mathcal{P}, \mathcal{P}'}^{\Psi, u}$	Eq. (7.6.7)	$\bar{\mathcal{U}}(\mathfrak{t}, \mathfrak{s}, \mathcal{P})$	Eq. (6.7.1)
$\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, u}$	Eq. (6.5.12)	$U_f(\mathcal{P}, [\mathcal{P}'])$	Eq. (6.7.3)
$\rho_{\mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, d}$	Eq. (6.5.15)	$\tilde{\Upsilon}^i$	Eq. (7.6.2)
$\rho_{f, \mathcal{P}, [\mathcal{P}']}^{\mathfrak{t}, \mathfrak{s}, d}$	Eq. (6.5.16)	$\tilde{\Upsilon}_{\text{SPL}, \kappa}^i$	Eq. (7.6.4)
$\rho_{\mathcal{P}, [\mathcal{P}']}^{\Xi, \mathfrak{t}, \mathfrak{s}, u}, \rho_{\mathcal{P}, [\mathcal{P}']}^{\Xi, \mathfrak{t}, \mathfrak{s}, d}$	Eq. (7.3.4)	$\tilde{\Upsilon}_{\mathfrak{t}, \mathfrak{s}}^s$	Eq. (7.3.6)
$\rho_{\mathcal{P}, \mathcal{P}'}^{V, d}$	Eq. (7.7.10)	$\Upsilon_{X, P}^i/S^1$	Eq. (9.5.14)
$\rho_{\mathcal{P}, \mathcal{P}'}^{V, u}$	Eq. (7.7.12)	$\Upsilon_{\mathfrak{t}, \mathfrak{s}}^i(P)/S^1$	Eq. (9.5.10)
\bar{r}_{β}	Eq. (9.4.3)	V, V^+, V^-	§2.1.1
$R_{k, i}$	Lemma 10.2.2	$\mathcal{V}(z), \tilde{\mathcal{V}}(z)$	§2.4
R_k	Eq. (10.2.4)	\mathfrak{V}_{κ}	Eq. (7.4.3)
$\mathfrak{s}, \mathfrak{s} \otimes L$	§2.1.1	$\bar{\mathfrak{V}}_{\kappa}$	Eq. (7.4.4)
$\Sigma(X^{\ell}, \mathcal{P})$	Eq. (3.1.8)	$\mathfrak{V}_{\mathfrak{t}}$	Eq. (7.2.1)
		$\tilde{\mathfrak{V}}_{\mathfrak{t}}$	Eq. (7.2.2)
		$\mathcal{V}_{\ell}(\Delta)$	Eq. (9.2.7)
		$\mathcal{V}_{\mathcal{P}}(\Delta), \mathcal{V}_{\mathcal{P}'}(\Delta)$	Eq. (9.2.9)
		$\mathcal{V}_{\ell}(\Delta, T), \mathcal{V}_{\mathcal{P}}(\Delta, T)$	Eq. (9.2.10)
		$\mathcal{W}, \tilde{\mathcal{W}}$	following Eq.(2.4.7)

W_κ	Eq. (5.6.2)
w	before Eq. (2.2.1)
$\Xi_{t,s}$	Eq. (2.3.19)
$\Xi_s(\mathbf{t}, \mathcal{P})$	Eq. (7.3.1)
Z_P	Eq. (4.2.1)
$Z(\mathcal{P})$	Eq. (4.2.3)
$Z(\mathcal{P}, \mathcal{P}')$	Eq. (4.2.6)
$z[A, \mathbf{x}]$	Eq. (5.2.2)

Index

- Anti-self-dual connection, 16
- Associativity of splicing maps, 7
- Circle action on space of $SO(3)$ pairs, 14, 19
- Clifford module, 13
- Cohomology classes on configuration space of pairs, 21
 - Cocycles and their pullbacks, 158
 - Computation of cocycles, 159
 - Extension of cocycles, 175
 - Cohomology classes on the virtual moduli space of $SO(3)$ monopoles, 152
 - Computation, 157
 - Dual geometric representatives, 157, 173
- Computation of intersection numbers, 185
- Configuration space of $SO(3)$ pairs, 14
- Connections over the four-dimensional sphere, 53
 - Center of mass, 53
 - Composition of splicing maps, 60
 - Definition of the overlap data, 61
 - Equality of splicing maps, 63
 - Scale, 54
 - Splicing onto the product connection, 57
 - Strata containing the product connection, 54
- Crude splicing map, 5, 79, 86, 88
 - Properties, 89
- Dirac operator, 14
- Donaldson invariants, 22
- Donaldson series, 22
- Gauge transformations on space of $SO(3)$ pairs, 14
- Geometric representatives for cohomology classes, 21
 - Cocycles, 157
- Global splicing data space, 9, 79, 80, 81, 96
 - Background pairs, 80
 - Flattening map on pairs, 82
 - Gluing data bundle, 80
 - Overlap space, 90
 - Downwards overlap map, 93
 - Equality of splicing maps, 94
 - Upwards overlap map, 92
 - Projections onto symmetric products, 106
 - Thom–Mather structures, 99
- Global splicing map, 105
- Instanton moduli space with spliced ends, 8, 51, 67
 - As a Whitney stratified space, 53
 - Isotopy, 76
 - Properties, 78
 - Tubular neighborhoods, 74
- Kotschick–Morgan conjecture, 209, 1, 10
- Kronheimer–Mrowka basic classes, 22
- Kronheimer–Mrowka simple type, 22
- Kronheimer–Mrowka structure theorem for Donaldson invariants, 22
- Link of ideal Seiberg–Witten moduli space, 135, 139
 - Boundaries of components, 143
 - Instanton component, 136
 - Fiber bundle structure, 141
 - Orientation of link of ideal Seiberg–Witten moduli space, 140
 - Virtual link of ideal Seiberg–Witten moduli space, 135
- Locally flattened Riemannian metric, 39, 40
- Moduli space
 - of ideal $SO(3)$ monopoles, 15
 - of $SO(3)$ monopoles, 3, 15
 - of Seiberg–Witten monopoles, 3
- Obstruction bundle, 109
 - Background obstruction bundle, 19, 112
 - Equivariant Dirac index bundle, 114
 - Infinite-dimensional obstruction pseudo-bundle, 111
 - Instanton obstruction pseudo-bundle, 120
 - Instanton component of the local obstruction bundle
 - Euler class, 164
 - Frame bundles, 120
 - Instanton component of the global obstruction bundle
 - Euler class, 170
 - Local gluing hypothesis for $SO(3)$ monopoles, 124

- Obstruction pseudo-bundle over instanton moduli space with spliced ends, 115
- Overlap obstruction pseudo-bundle, 116
- Overlap space and overlap maps, 6, 122
- Splicing map, 121
- Relative Euler class of the obstruction bundle, 164
- Seiberg–Witten obstruction bundle, 19
- Seiberg–Witten component of the obstruction bundle
 - Euler class, 164
- Problem of overlaps, 5
- Pidstrigach–Tyurin $SO(3)$ -monopole cobordism program, 1
- Pseudo-bundle, 109
- Partial tubular neighborhood structure, 31
 - Vector-valued tubular distance function, 31
- Perturbation parameters for $SO(3)$ monopole equations, 15
- Property P for knots, 2
- Reducible $SO(3)$ monopoles, 18
- Relative Euler classes, 171
- Seiberg–Witten invariant, 17
- Seiberg–Witten monopole, 17
 - Seiberg–Witten monopole equations, 17
- Smoothly stratified space, 31
 - Compatibility condition, 31
 - Condition of the frontier, 31
 - Control condition, 31
 - Thom–Mather stratification, 31
 - Whitney stratification, 31
- $SO(3)$ monopole, 14
 - $SO(3)$ monopole equations, 14
- $SO(3)$ -monopole cobordism formula, 1
 - As equality of intersection pairings with links, 23, 141
- spin^c structure, 3, 13
- spin^u structure, 3, 13
- Standard splicing map, 87
- Stratum of anti-self-dual (zero-section) $SO(3)$ -monopoles, 16
- Stratum of Seiberg–Witten (reducible) $SO(3)$ -monopoles, 16
- Superconformal simple type conjecture, 2
- Symmetric group, 25
 - Partition, 25
 - Partition length, 25
 - Partition refinement, 27
- Symmetric products of manifolds, 25
 - Big diagonal, 26
 - Center of mass map, 33
 - Cone parameter function, 33
 - Cone point, 32
 - Diagonal of symmetric product of manifolds, 26
 - Incidence relations among diagonals and strata, 27
 - Normal bundle of diagonal, 28
 - Normal bundle of stratum, 29, 42
 - Overlap maps, 37
 - Commuting overlap maps, 39
 - Downwards overlap map, 38
 - Upwards overlap map, 38
 - Partial ordering of strata, 30
 - Small diagonal, 26
 - Stratum of symmetric product defined by partition, 27
 - Tubular distance function, 31, 44
 - Tubular neighborhood projection, 31
 - Tubular neighborhood structure, 31
 - Transversality for the moduli space of $SO(3)$ monopoles, 15
 - Uhlenbeck compactification for the moduli space of $SO(3)$ monopoles, 3, 16
 - Uhlenbeck topology for the moduli space of $SO(3)$ monopoles, 15
 - Universal Seiberg–Witten line bundle, 17
 - Virtual fundamental class of moduli space of $SO(3)$ monopoles, 19
 - Virtual link of the ideal moduli space of Seiberg–Witten monopoles,
 - Fundamental class, 155
 - Virtual moduli space of Seiberg–Witten monopoles, 19
 - Virtual normal bundle of Seiberg–Witten moduli space, 19
 - Witten’s conjecture, 2

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