

Relating the curvature of De Sitter Universe to Open Quantum Lamb Shift Spectroscopy

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ABSTRACT: In this paper, our prime objective is to connect the curvature of our observable De Sitter Universe with the spectroscopic study of entanglement of two atoms in an open quantum system (OQS). The OQS considered in our work is made up of two atoms which are represented by Pauli spin tensor operators projected along any arbitrary direction. They mimic the role of a pair of freely falling *Unruh De-Witt detectors*, which are allowed to non-adiabatically interact with a conformally coupled massless probe scalar field in the De Sitter background. The effective dynamics of the atomic detectors are actually an outcome of their non-adiabatic interaction, which is commonly known as the *Resonant Casimir Polder Interaction (RCPI)* with the thermal bath. We find from our analysis that the **RCPI** of two stable entangled atoms in the quantum vacuum states in OQS depends on the De Sitter space-time curvature relevant to the temperature of the thermal bath felt by the static observer. We also find that, in OQS, **RCPI** produces a new significant contribution appearing in the effective Hamiltonian of the total system and thermal bath under consideration. This will finally give rise to *Lamb Spectroscopic Shift*, as appearing in the context of atomic and molecular physics. This analysis actually plays a pivotal role to make the bridge between the geometry of our observed Universe to the entanglement in OQS through *Lamb Shift atomic spectroscopy*. In two atomic OQS, *Lamb Shift spectra* is characterised by a L^{-2} decreasing inverse square power law behaviour when inter atomic Euclidean distance (L) is much larger than a characteristic length scale (k) associated with the system, which quantifies the breakdown of a local inertial description within OQS. On the other hand, the **RCPI** of this two atomic OQS immersed in a thermal bath in the background of Minkowski flat Universe is completely characterised by a temperature independent L^{-1} decreasing inverse power law. This mimics exactly the same situation where the characteristic length scale k is sufficiently large compared to the interatomic Euclidean distance between the two atoms. Thus, we are strongly aiming to connect the curvature of the background space-time of our Universe to open quantum *Lamb Shift* spectroscopy by measuring the quantum properties of a two entangled OQS in the atomic experiment.

KEYWORDS: Open Quantum Systems, Quantum Dissipative Systems, Many Body Quantum entanglement, Quantum Field Theory of De Sitter space, Cosmology beyond the Standard Model, Lamb Shift Spectroscopy.

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1 Introduction

Quantum theory of closed systems is a very popular topic in theoretical physics which have been studied thoroughly in many many decades. But to study the complete time evolutionary description of a quantum mechanical system requires incorporation of their detail of the thermal environment, which compels us to look towards the study of Open Quantum System (OQS), where the physical system is interacting in a weakly with its environment. Some examples would be vibrational relaxation of molecules in liquids, coupling to a photon bath in the context of cavity quantum electrodynamics, etc. In general, these type of interactions significantly control the time evolutionary dynamics of the quantum mechanical system and induce *quantum mechanical dissipation*, where the quantum information contained in the system is completely disappeared to its thermal environment or bath degrees of freedom. Because in the practical situations there is never exists any quantum mechanical system in nature which is completely isolated from its thermal environment or bath in true sense, it is important to develop a theoretical framework for treating these non-adiabatic interactions in order to obtain an accurate understanding of quantum mechanical systems. Theoretical tools and techniques developed in the context of OQS have provided significant applications in the context of quantum description of optical systems, quantum mechanical aspects of measurement theory, quantum description of statistical mechanics, quantum description of information theory, quantum version of thermodynamics, quantum aspects of cosmology, quantum mechanical description of biological systems, and also in the context of physics governed by semi-classical approximations. In a most generalised prescription, the time evolution of OQS is described by the non-adiabatic interactions between the physical system and its thermal environment.

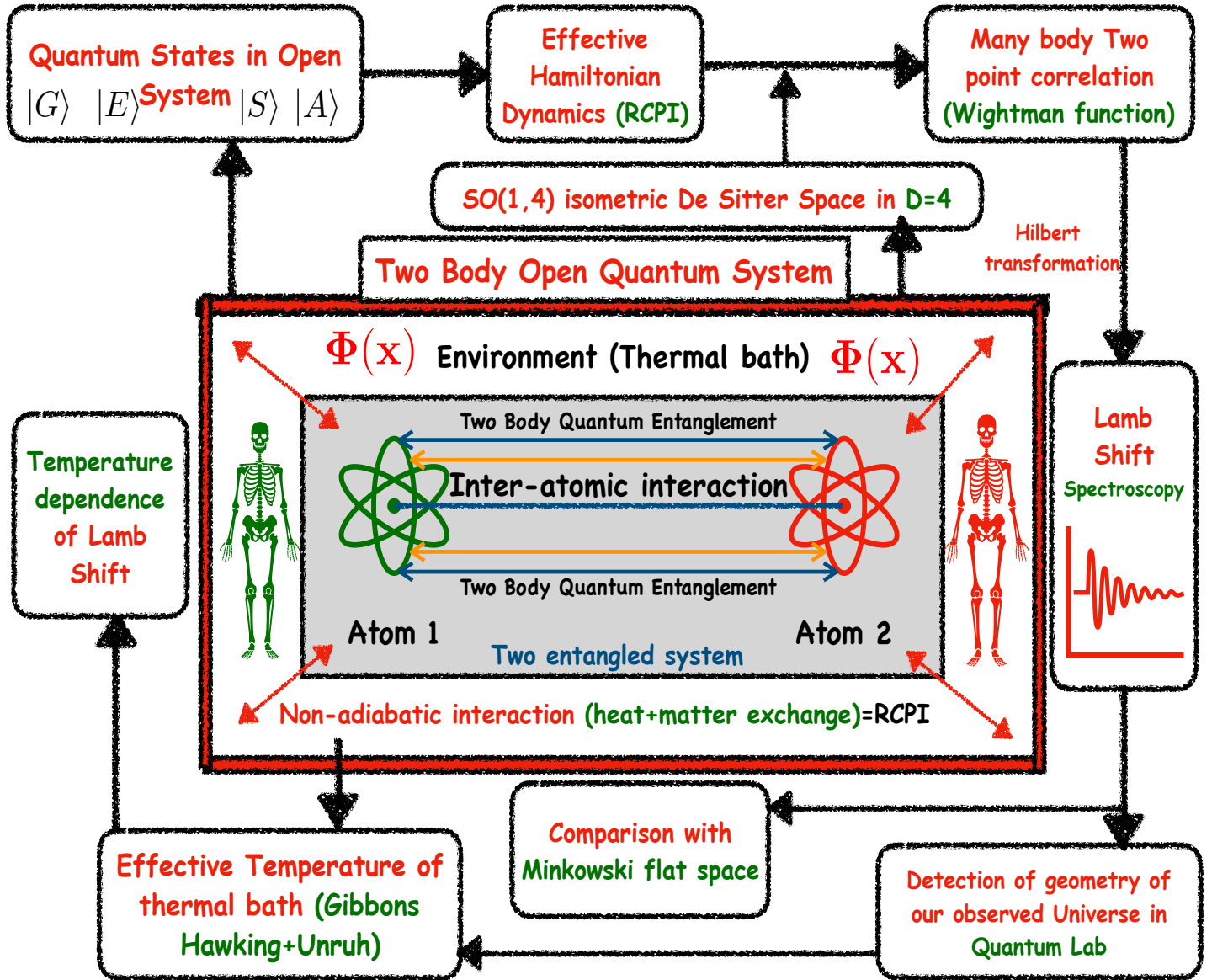


Figure 1. Schematic diagram of detecting the geometry of our Universe from the Lamb Shift spectroscopy of OQS with entanglement of two atoms. This two entangled atomic system is free to exchange information in the form of energy, with its environment. As a result, no observables are conserved during the non unitary time evolution of the system under consideration.

Consequently, the dynamical behaviour of the OQS cannot be accurately described using unitary time evolution operators alone after integrating out the bath degrees of freedom from the environment. The time evolution of OQS can be explicitly determined by solving the effective master equations of motion, also known as *GoriniKossakowskiSudarshanLindblad* (GSKL) master equations, from which one can understand the non-unitary time evolution of the reduced density matrix of the system. In this context, the time evolution actually describes the time dependent behaviour of the system at different stages over time and also the dynamical behaviour of the observables that are associated with the OQS. The theory of OQS treats the system with dependent degrees of freedom as a subsystem in a much larger thermal bath [1]. Due to the complicated structure of the environmental degrees of freedom finding exact analytical solution of *GoriniKossakowskiSudarshanLindblad* (GSKL) master equations are extremely difficult for practical purposes. Due to the difficulty of determining the exact solutions to the master equations for a particular OQS and environment, a variety of approaches have been developed in the present context. In this connection, a common objective is to derive the reduced time dependent description of the OQS wherein the dynamical behaviour of the quantum system are considered

explicitly and the corresponding bath dynamics are described implicitly to know about the underlying physics of OQS under consideration. The background physical prescription of OQS provides an economical theoretical treatment of the dynamics of the quantum system and its associated observables. Here it is important to note that, typical physical observables of interest include energy as well as the robustness of quantum mechanical coherence. The seemingly obvious approach to deal with such systems, which is simulating evolution of both system and environment, would be naive as its complexity evolves exponentially. The OQS can no longer be defined by a pure state and to study the modified time evolutionary dynamics in presence of weak interaction between the system and thermal bath, density matrix formalism is explicitly required. We can study smaller subsystems to get across this problem by incorporating probabilistic description, where the quantum state of the subsystem is described by density operator. In an OQS, the effects of dissipation and decoherence is introduced by the environment degrees of freedom. The induced effects of decoherence and dissipation owing to the system is introduced by an operator more famously known as the Lindbladian [43]. While dealing with OQS, following crucial assumptions play significant role to describe the underlying physics:

1. **Assumption I:**

The combination of the system and the thermal environment is treated as a *large closed system*. Therefore we can assume that *the time evolution is given by a unitary transformation generated by a global Hamiltonian*.

2. **Assumption II:**

The interaction between system and environment is considered as *Markovian*, which describes *the state of the quantum mechanical system in the next instant and dependent only on the current moment, not in the past*. In short the interaction between the system and thermal bath describes a phenomena without memory. This approximation is justified when the OQS under consideration has enough time for the system to relax to achieve equilibrium state before being quantum mechanically perturbed again by non-adiabatic interactions with its thermal environment ¹.

3. **Assumption III:**

The interaction between the system and thermal bath is considered to be *weak* in nature, which implies that *the only change over time we see originates in the open system*. This helps us to treat the time evolution of the system in a perturbative manner.

4. **Assumption IV:**

In the present context, additionally we have assumed that *system and thermal bath are completely uncorrelated at initial times*.

In this paper, we have attempted to connect the curvature of our De Sitter universe with the two atomic spectroscopy using the framework of OQS. To construct the theoretical set up we use a two body entangled system, which is described by two identical atoms in the present context precisely. In our set up this two entangled OQS is considered as our theoretical probe which mimic the role of a pair of freely falling *Unruh de Witt detectors*. In this connection, it is important to note that, the concept of quantum entanglement describes a physical phenomenon which deals with strong correlations between the two atomic quantum states of the entangled particles [2, 3] in OQS. Since this non-local property of quantum mechanics seemed very puzzling, Einstein, Podolsky and Rosen (EPR) argued for the existence of hidden variables in the context of quantum theory [4]. Later Bell proposed a set of inequalities to test their existence which when untrue, support the non-locality of quantum theory [5]. Bell's inequalities have to be violated in inflationary era to get persistent long range quantum correlation in the early universe cosmology and for this the concept of quantum entanglement is commonly used in this context. The violation of the Bell-inequality in De Sitter space has been addressed with axionic Bell pair to a great extent in several works [7, 8]. Bell-inequality violation in various cosmological scenario with quenched time dependent mass profiles in De Sitter space has also been extensively studied in [9, 10].

In the present context, we use mainly the concept of two body quantum entanglement in OQS which currently serves as one of the leading candidates to study the long range quantum correlation and many other unknown physics to know about the physical implications of De Sitter space. The idea being that two or more entangled atoms which mimics the role of particle detectors in OQS can be used to measure the spacetime curvature of our observed Universe and the consequent thermal effects, some of which we will discuss in detail in the following sections. In refs. [6, 23, 45, 49], the authors have explored many more aspects of quantum entanglement in the background of a fluctuating scalar field in De Sitter space ². In OQS the interaction of single particle (i.e. atom in the present context) with conformally coupled massless scalar field in De Sitter space is identical to the interaction of the same with thermal bath environment in Minkowski space-time, making it difficult to distinguish between the two frameworks.

¹For OQS which have very fast or very frequent perturbations from their coupling to their environmental degrees of freedom, this specific approximation not works well in the present context.

²De Sitter space describes a universe with constant positive curvature and having the same degree of symmetry as a Minkowski space-time, which also fits in with the picture of our current universe quite well.

This problem has led to further investigations with two body entangled OQS in order to shed light on the quantum structure and nature of the space-time of our universe.

In a more general prescription, it is important to note that, when an accelerated particle detector moves through an external field in its vacuum state, spontaneous excitation may occur. The particle behaves as in the presence of a thermal bath, giving rise to an *Unruh Temperature*, which is proportional to its acceleration in this context [14, 15]. The excitation rate, of such a particle detector vanishes when it is at rest, but is found to be non-zero when uniformly accelerated. Same result is achieved if we analyse the vacuum state of the external conformally coupled scalar field in the frame of the accelerating particle detector. This is known as *thermalising theorem* and shows that these accelerating detectors act as an OQS in the context of two body physics. Very recently, it has been explicitly shown that scalar fields coupled conformally to a De Sitter space-time can be treated as an out-of-equilibrium system with a fluctuating background that introduced the thermal excitations in De Sitter space [11]. Such a detector-system combined set up sees the above thermalisation phenomena as a manifestation of the decoherence and dissipation due to the interactions with the surrounding. Furthermore, the approach followed in this work is also used to investigate the *Gibbons-Hawking effect* [28], which can be treated as a consequence of the thermalisation effect of the vacuum fluctuation of the external field in the De Sitter space-time. These two phenomena encode the thermal behaviour of De Sitter space-time.

Another very important outcome of vacuum fluctuations of quantised fields is *Casimir effect* which is heavily influenced by the curvature of our observed space-time [16, 17, 19]. *Casimir effect* and the associated *Casimir-Polder interaction* (CPI) has been verified experimentally in multitudes of systems at microscopic and macroscopic levels. It has been used to study properties of quantum entanglement, long range effect of field correlations, Unruh effect etc. The space-time curvature can alter the *Casimir-Polder interaction* [22, 34, 35] between the atoms [36] in two body OQS. The *Casimir-Polder interaction* in the De Sitter space-time [55] has been investigated in detail in refs [23, 24, 39, 40]. Apart from that, using the concept of quantum entanglement as a probe to determine the two different space-time scenarios have been investigated in [41]. In this paper, we aim to distinguish the curvature of two different space-times using *Casimir-Polder interaction* as a theoretical probe in order to distinguish a conformally coupled massless scalar field with a De Sitter background to that of a Minkowski space-time interacting with a thermal bath in which the excited *Bell-states* of the pair of atoms interacts with the background field placed at thermal bath via the exchange of real photons [26, 27, 37, 58, 59]. Additionally, in ref. [51] the authors have explicitly shown the use of *Resonance Casimir-Polder interaction* (RCPI) to detect space-time curvature. In this paper, we use this methodology to know about the curvature for our universe by analysing the *Lamb shift spectra* computed from the two entangled atomic OQS.

The various discussions in this paper can be summarised as follows:

In [section 2](#), we investigate the reduced dynamics of a pair of *Unruh-De-Witt detectors* or atoms entangled with each other conformally coupled to a scalar field in a thermal bath in De Sitter universe. The background metric has a hyperbolic geometry with a cosmological constant Λ as a solution of Einstein's field equations. In this section our objective is to observe how the entanglement between the pair of detectors manifests the thermal nature of the space-time background [47] and the shift in energy levels of the entangled states paves a way to distinguish between the nature of space-time. In this section, we tend to study the following problem with a very general structure of model Hamiltonian in order to get a clear picture of how to distinguish the curvature of space-time hides into it the deepest mysteries of the two body quantum entanglement giving us a strong information theoretic measure of the emergence of the quantum phenomena in the thermal perspective of the space-time curvature. In this section we employ the techniques of OQS to measure the manifestation of curvature of space-time due as a result of two body quantum entanglement between the two atoms which mimics the role of *Unruh-De-Witt detectors*. Specifically, we have discussed the construction as well as various properties of the two atomic two body OQS under consideration in this paper. Further, we have explicitly studied the reduced dynamics of the OQS two atomic model by constructing the effective Hamiltonian and the Lindbladian in presence of non-adiabatic interaction between the two atomic system and external conformally coupled massless scalar field which mimics the role of environmental thermal bath in the present context.

Further, in [section 3](#), we have derived the expression for the energy shift to explicitly study the role of *Lamb Shift spectroscopy* in the context of the two atomic entangled OQS under consideration. To serve this purpose, first of all we actually construct all possible two atomic entangled states out of the individual ground and excited states of the two atoms. By using the tensor product we have constructed four possible combination of the two body entangled states, which are ground, excited, symmetric and antisymmetric states. Further, using these two body entangled quantum states we compute the expectation values of the most relevant part of the effective Hamiltonian, known as the *Lamb Shift Hamiltonian*. Here we can able to write all of these expectation values of the energy shifts in terms of the interaction strengths in the two atomic entangled OQS. Once we know about all such possible two body interaction strengths explicitly, we can easily get to know about the complete description of *Lamb Shift Spectroscopy* and its role of determining the geometry of our observed Universe.

Finally, in [section 4](#), we explicitly compute the strength of all possible interaction strengths appearing in the two atomic entangled OQS. To compute this, we first compute the two body two point Wightman functions for the external conformally coupled massless scalar field which is the bath degrees of freedom in our problem. Further we take the Fourier transform of the two body Wightman function in the frequency space and using those results we perform Hilbert transform on that. Finally, using

these Hilbert transformed results one can explicitly write down the expressions for the interaction strengths which will completely specify the structure of the all possible energy shifts computed from the two body entangled two atomic ground, excited, symmetric and antisymmetric quantum states. Further, we compute the meaningful finite contribution of the *Lamb Shift* using the *Bethe regularisation* technique in the spectroscopic integral from the present OQS. Here we have actually expressed the expressions for spectroscopic energy shift in terms of the Euclidean distance between the two atoms separated in the space time and the surface gravity of our observed De Sitter space. Further, we have expressed the temperature of the thermal bath in terms of the surface gravity of De Sitter space. This allows us to connect macroscopic description of the environment in terms of the microscopic quantum mechanical observables of the two entangled OQS under consideration as well the geometry of our Universe. Here particularly the microscopic observables can be characterised through few sets of spectroscopic integrals which is in principle divergent in nature in QFT. To collect only the finite contributions out of these integrals we apply the *Bethe regularisation technique*. At the end one can express the surface gravity in terms of the curvature of De Sitter space, where curvature is related to the positive cosmological constant. This helps us to measure the curvature of De Sitter space from microscopic quantum spectroscopy.

2 Open quantum system (OQS) of two entangled atoms

2.1 Two atomic OQS model

In this section, we investigate a model of OQS with entanglement. For simplicity, we consider a pair of entangled atoms each with two sets of internal energy levels represented by:

$$\text{Internal Energy States of two atoms : } \{|g_\alpha\rangle, |e_\alpha\rangle\} \quad \forall \alpha = (1, 2). \quad (2.1)$$

These are the ground-states and the excited-states for two atoms respectively. To avoid any further confusion here we note that, by the word "atom" we actually represented a very simplest spin bath open quantum model where two spin is immersed in the thermal bath. Here such spin pairs are characterised by the Pauli spin matrices. However, for a more general situation where we are interested in very complicated two atomic models of Hamiltonian the present analysis holds good, but solving that two body OQS problem itself extremely complicated. So for our better understanding we restrict ourself to a pair of spins which we are treating as a pair of atom, using which we will study various underlying physics of OQS. In this discussion, such pair of atoms are conformally coupled to a massless scalar field in the De Sitter background. The scalar field acts as a thermal bath for the pair of entangled atoms. In our discussion, the above two atomic entangled system are represented by a pair of *Unruh-De-Witt detectors*. These two identical atoms interact weakly with a quantised conformally coupled massless probe scalar field in its quantum mechanical vacuum state. Consequently, the corresponding two energy levels of the two atoms are identified as:

$$\text{Energy levels of two atoms : } E_\alpha^{(\pm)} = \pm \frac{1}{2}\omega \quad \forall \alpha = (1, 2). \quad (2.2)$$

Here, ω represents the renormalized energy level for two atoms, given by:

$$\text{Renormalized energy : } \omega = \begin{cases} \left(\omega_0 + i[\mathcal{K}^{(11)}(-\omega_0) - \mathcal{K}^{(11)}(\omega_0)] \right) & \text{Atom 1} \\ \left(\omega_0 + i[\mathcal{K}^{(22)}(-\omega_0) - \mathcal{K}^{(22)}(\omega_0)] \right) & \text{Atom 2.} \end{cases} \quad (2.3)$$

Here $\mathcal{K}^{\alpha\alpha}(\pm\omega_0)$ for $\alpha \in \{1, 2\}$ are Hilbert transformations of two-point Wightmann function, which will be defined explicitly in later section of this paper. Also, ω_0 represents the natural frequency of the two identical atoms. In this context, the atoms are characterised by the label $\alpha \in [1, 2]$ and $\sigma_i^\alpha \forall i \in [1, 2, 3]$ are the Pauli matrices.

The Hilbert space of such a system is bipartite in nature, i. e.

$$\text{Total Hilbert Space : } \mathcal{H}_{\text{Total}} := \mathcal{H}_{\text{System}} \otimes \mathcal{H}_{\text{Bath}}, \quad (2.4)$$

where $\mathcal{H}_{\text{System}}$ and $\mathcal{H}_{\text{Bath}}$ are the corresponding Hilbert spaces of the system and bath. Also, $\mathcal{H}_{\text{Total}}$ represents the Hilbert space corresponding to the combined configuration of the system and the bath. This entangled two atomic OQS can be represented by the following total Hamiltonian:

$$\text{Total Hamiltonian : } H_{\text{total}}(\tau) = H_{\text{System}}(\tau) \otimes \mathbf{I}_{\text{Bath}} + \mathbf{I}_{\text{System}} \otimes H_{\text{Bath}}(\tau) + H_{\text{Int}}(\tau), \quad (2.5)$$

where $\mathbf{I}_{\text{System}}$ and \mathbf{I}_{Bath} are the identity operators defined for the system and bath. When we are accessing the system we don't see anything from the bath and converse is also true. Here, the identity operators as appearing in the system and bath corresponds

to the no access. More precisely, if one observer is sitting on the system, made by **atom 1** and **atom 2**, then that observer will not feel any further effect from the thermal bath environment. On the other hand, once the observer is sitting at the reference frame of thermal bath, that observer will not see any further effect from the two entangled OQS. However, in the interaction term both the system and thermal bath explicitly contribute and due to the entanglement one cannot separate their contribution from this term in the Hamiltonian. More details of each of the terms of the total Hamiltonian is as follows:

1. **System:**

The system Hamiltonian of the two entangled atoms are described by the linear combination of the individual contributions coming for each atom ³:

$$\text{System Hamiltonian : } H_{\text{System}}(\tau) = \sum_{\alpha=1}^2 \frac{\omega}{2} n^{\hat{\alpha}} \cdot \vec{\sigma}^{\hat{\alpha}} = \frac{\omega}{2} \sum_{\alpha=1}^2 [n_1^{\alpha} \sigma_1^{\alpha} \cos \alpha^{\alpha} + n_2^{\alpha} \sigma_2^{\alpha} \cos \beta^{\alpha} + n_3^{\alpha} \sigma_3^{\alpha} \cos \gamma^{\alpha}] , \quad (2.7)$$

where, the sum is taken over two individual contribution appearing from the Hamiltonians of the two atoms.

2. **Thermal bath:**

The thermal bath Hamiltonian in the present context is characterised by a free rescaled field $\Phi(x) = a(\tau)\chi(x)$, where the original massless scalar field $\chi(x)$ is conformally coupled with De Sitter background with scale factor $a(\tau)$. The Hamiltonian in the De Sitter background can be written as:

$$\text{Bath Hamiltonian : } H_{\text{Bath}}(\tau) = \int d^3x \left[\frac{1}{2} \Pi^2(x) + \frac{1}{2} (\nabla \Phi(x))^2 + m_{\text{eff}}^2(\tau) (\Phi(x))^2 \right] , \quad (2.8)$$

where $\Pi(x) = \partial_{\tau} \Phi(x)$ is the canonically conjugate momentum of the scalar field $\Phi(x)$. Here the conformal time dependent effective mass of the redefined scalar field $\Phi(x)$ can be written as:

$$\text{Effective Mass : } m_{\text{eff}}^2(\tau) = -\frac{1}{a(\tau)} \frac{d^2 a(\tau)}{d\tau^2} . \quad (2.9)$$

Further expression the bath Hamiltonian in terms of the creation (a_k^{\dagger}) and annihilation (a_k) operators one can further express it in the following normal ordered simplified form:

$$\text{Normal ordered bath Hamiltonian : } H_{\text{Bath}}(\tau) = \int \frac{d^3k}{(2\pi)^3} \frac{\omega_k(\tau)}{2} a_k^{\dagger} a_k , \quad (2.10)$$

where the conformal time dependent frequency of the each mode can be written as:

$$\text{Time dependent frequency : } \omega_k^2(\tau) = (k^2 + m_{\text{eff}}^2(\tau)) = \left(k^2 - \frac{1}{a(\tau)} \frac{d^2 a(\tau)}{d\tau^2} \right) . \quad (2.11)$$

3. **System-thermal bath interaction:**

The interaction Hamiltonian is characterised by the following expression:

$$\text{Interaction Hamiltonian : } H_{\text{Int}}(\tau) = \mu \sum_{\alpha=1}^2 n^{\hat{\alpha}} \cdot \vec{\sigma}^{\hat{\alpha}} \phi(x^{\alpha}) = \mu \sum_{\alpha=1}^2 [n_1^{\alpha} \sigma_1^{\alpha} \cos \alpha^{\alpha} + n_2^{\alpha} \sigma_2^{\alpha} \cos \beta^{\alpha} + n_3^{\alpha} \sigma_3^{\alpha} \cos \gamma^{\alpha}] \phi(x^{\alpha}) , \quad (2.12)$$

where, μ represents the coupling between the pair of atoms and the external massless scalar field placed at the thermal bath. While deriving the reduced dynamics of the two atomic detectors, we consider the weak coupling limiting approximation between the pair of atoms and the external massless probe scalar field. We assume that the coupling parameter μ is sufficiently small so that perturbation theory is applicable in the open quantum mechanical system under consideration. Also in the above set of expressions we have considered the normal vectors for two atoms represented by $n_i^{\alpha} \forall \alpha = 1, 2, \& i = 1, 2, 3$ and the corresponding projection of Pauli-matrices are characterised by the direction cosine of Euler angles α, β and γ respectively.

³ Here we define the following direction cosines for two entangled atoms:

$$\cos(\alpha_1^{\alpha}) = \cos(\alpha^{\alpha}), \quad \cos(\alpha_2^{\alpha}) = \cos(\beta^{\alpha}), \quad \cos(\alpha_3^{\alpha}) = \cos(\gamma^{\alpha}) \quad \alpha = (1 \Rightarrow \text{atom 1}, 2 \Rightarrow \text{atom 2}). \quad (2.6)$$

2.2 Time dynamics of two atomic OQS

To construct the effective Hamiltonian from the time dynamics of the present system, we first consider that there is no correlation between the pair of atoms with the the external probe free scalar field. Hence, the thermal density-matrix of the combined system and thermal environment can be expressed in the following form:

$$\text{Initial total thermal density matrix : } \rho_{\text{Total}}(0) = \underbrace{\rho_{\text{System}}(0)}_{\text{System}} \otimes \underbrace{\rho_{\text{Bath}}(0)}_{\text{Thermal bath}} , \quad (2.13)$$

where $\rho_{\text{System}}(0)$ and $\rho_{\text{Bath}} = |0\rangle\langle 0|$ represent the initial density matrix for the pair of atoms (system) and environment (bath) degrees of freedom. Here, $|0\rangle$ characterise the vacuum state of the external free scalar field.

Now it is important to note that, in the interaction picture of OQS, the time evolution of the total density matrix can be written in the following form:

$$\text{Density matrix evolution for total system : } \frac{d\rho_{\text{Total}}(\tau)}{d\tau} = 0 \implies \frac{\partial\rho_{\text{Total}}(\tau)}{\partial\tau} = -i[H_{\text{Total}}(\tau), \rho_{\text{Total}}(\tau)] , \quad (2.14)$$

where τ represents the proper time in the co-moving frame of two atoms under consideration.

One can further write the most general structure of the time evolved version of the total density matrix of the combined system and thermal bath, as given by:

$$\text{Time evolved density matrix : } \rho_{\text{Total}}(\tau) = \underbrace{\rho_{\text{System}}(\tau)}_{\text{System}} \otimes \underbrace{\rho_{\text{Bath}}(\tau)}_{\text{Thermal bath}} + \underbrace{\rho_{\text{correlation}}(\tau)}_{\text{Interaction}} . \quad (2.15)$$

Now, we have already started with the assumption that the non-adiabatic interaction between the system and thermal bath is switched on at $t = 0$ (from initial structure of the total density matrix) and prior to that the interaction between the system and the thermal bath is extremely weak or absent for which there is no correlation exists in the OQS under consideration i.e.

$$\text{Initial absence of correlation : } \rho_{\text{correlation}}(0) = 0. \quad (2.16)$$

The idea of having no correlation at the initial stage is not very restrictive in nature, since for the computational purpose we can always find a time prior to which the system and environment did not interact. In the weak coupling limiting situation, the time evolved density matrix can be further simplified as:

$$\text{Time evolved approx density matrix : } \rho_{\text{Total}}(\tau) \approx \underbrace{\rho_{\text{System}}(\tau)}_{\text{System}} \otimes \underbrace{\rho_{\text{Bath}}(\tau)}_{\text{Thermal bath}} , \quad (2.17)$$

which is valid through a time scale on which the perturbation theory in OQS is valid. Furthermore, we also assume that the time scale corresponding to the correlation, τ_{Bath} , which is sometimes identified to be the *relaxation time scale* of the thermal bath, assumed to be extremely weak. Here one can write:

$$\text{Time evolution of the bath for time scale } \tau \gg \tau_{\text{Bath}} : \rho_{\text{Bath}}(\tau) \approx \rho_{\text{Bath}}(0) . \quad (2.18)$$

In this context, the time evolved total density matrix can be represented as:

$$\text{Time dependent density matrix : } \rho_{\text{Total}}(\tau) = U_{\text{Total}}\rho_{\text{Total}}(0)U_{\text{Total}}^\dagger = U_{\text{Total}} \left(\underbrace{\rho_{\text{System}}(0)}_{\text{System}} \otimes \underbrace{\rho_{\text{Bath}}(0)}_{\text{Thermal bath}} \right) U_{\text{Total}}^\dagger , \quad (2.19)$$

where U_{Total} represents the time evolution operator of OQS under consideration. But, since, we are only interested in the reduced dynamics of the density matrix of the entangled atoms only, therefore, we trace out the external field (thermal bath) degrees of freedom from the total combined open quantum two atomic system.

Since, we are only interested in the reduced dynamics of the density matrix of the entangled atoms only, therefore, we trace out the external field (thermal bath) degrees of freedom from the total combined open quantum two atomic system. Consequently, the reduced density matrix of such a two atomic system can be expressed as:

$$\text{Reduced density matrix of the atomic system : } \rho_{\text{System}}(\tau) = \text{Tr}_{\text{Bath}}[\rho_{\text{Total}}(\tau)] . \quad (2.20)$$

Further, doing simplifications one can write the reduced density matrix of the system after performing the partial trace operation over the bath degrees of freedom in the following simplified form:

$$\rho_{\text{System}}(\tau) = \sum_k \langle k|U_{\text{Total}}\rho_{\text{Total}}(0)U_{\text{Total}}^\dagger|k\rangle = \sum_k \langle k|U_{\text{Total}}|0\rangle \underbrace{\rho_{\text{System}}(0)}_{\text{System}} \langle 0|U_{\text{Total}}^\dagger|k\rangle = \sum_k \mathcal{M}_k \underbrace{\rho_{\text{System}}(0)}_{\text{System}} \mathcal{M}_k^\dagger, \quad (2.21)$$

where $|k\rangle$ represents all possible orthonormal basis states of the thermal bath defined in the Hilbert space $\mathcal{H}_{\text{Bath}}$. Here, additionally we define the operator \mathcal{M}_k as:

$$\text{New operator : } \mathcal{M}_k = \langle k|U_{\text{Total}}|0\rangle = \text{Tr}_{\text{Bath}} [|0\rangle\langle k|U_{\text{Total}}], \quad (2.22)$$

which is defined on the Hilbert space corresponding to the system under consideration i.e. $\mathcal{H}_{\text{System}}$.

Since in the present context, the time evolution operator U_{Total} is unitary i.e. $U_{\text{Total}}^\dagger U_{\text{Total}} = \mathbf{I}_{\text{Total}}$, which further implies that:

$$\sum_k \mathcal{M}_k^\dagger \mathcal{M}_k = \sum_k \langle 0|U_{\text{Total}}^\dagger|k\rangle \langle k|U_{\text{Total}}|0\rangle = \langle 0|U_{\text{Total}}^\dagger \left(\underbrace{\sum_k |k\rangle\langle k|}_{\equiv \mathbf{I}_{\text{Bath}}} \right) U_{\text{Total}}|0\rangle = \langle 0|U_{\text{Total}}^\dagger U_{\text{Total}}|0\rangle = \langle 0|0\rangle = \mathbf{I}_{\text{System}}. \quad (2.23)$$

Now, we explicitly mention about the properties of the system density matrix:

1. **Hermiticity:**

The system density matrix is hermitian in nature which can be tested as:

$$\rho_{\text{System}}^\dagger(\tau) = \left(\sum_k \mathcal{M}_k \underbrace{\rho_{\text{System}}(0)}_{\text{System}} \mathcal{M}_k^\dagger \right)^\dagger = \sum_k \mathcal{M}_k \underbrace{\rho_{\text{System}}^\dagger(0)}_{\text{System}} \mathcal{M}_k^\dagger = \sum_k \mathcal{M}_k \underbrace{\rho_{\text{System}}(0)}_{\text{System}} \mathcal{M}_k^\dagger = \rho_{\text{System}}(\tau). \quad (2.24)$$

2. **Unit trace:**

The system density matrix satisfies the unit trace property which can be tested as:

$$\text{Tr}_{\text{Bath}}[\rho_{\text{System}}(\tau)] = \text{Tr}_{\text{Bath}}\left[\sum_k \mathcal{M}_k \underbrace{\rho_{\text{System}}(0)}_{\text{System}} \mathcal{M}_k^\dagger\right] = \text{Tr}_{\text{Bath}}\left[\underbrace{\rho_{\text{System}}(0)}_{\text{System}} \sum_k \mathcal{M}_k \mathcal{M}_k^\dagger\right] = \text{Tr}_{\text{Bath}}\left[\underbrace{\rho_{\text{System}}(0)}_{\text{System}} \mathbf{I}_{\text{System}}\right]. \quad (2.25)$$

3. **Positivity:**

The system density matrix also satisfies positivity property.

Now, the non-unitary time evolution of the reduced density matrix in the weak coupling limiting situation can be expressed in terms of the *GoriniKossakowskiSudarshanLindblad* (GKSL) master equation in OQS, as given by ⁴:

$$\text{Reduced density matrix evolution of the atomic system : } \frac{\partial \rho_{\text{System}}(\tau)}{\partial \tau} = -i[H_{\text{eff}}, \rho_{\text{System}}(\tau)] + \mathcal{L}[\rho_{\text{System}}(\tau)], \quad (2.26)$$

where H_{eff} is the effective Hamiltonian of the two atomic system under consideration, which incorporates the effect of inter atomic interaction *aka Resonant Casimir Polder Interaction (RCPI)*. Also, the last term in the above mentioned evolution is known as the *Lindbladian*, which describes the dissipative contribution due to the influence of the thermal bath on the two entangled atomic system. The other contributions describe the possible transitions that the system may undergo due to interactions with the reservoir. The quantum operators which can be expressed in terms of the Pauli spin operators, commonly known as the *Lindblad quantum operators* or *quantum jump operators*. Using the idea of infinitesimal time evolution of the reduced density matrix of the system described by the *Kraus sum representation* one can explicitly derive the *GoriniKossakowskiSudarshanLindblad (GKSL) master equation*. In the first term of the *GKSL master equation*, the usual Schrödinger term which generates unitary time evolution. In the following subsections we discuss about these contributions in detail.

⁴In the *GoriniKossakowskiSudarshanLindblad* (GKSL) master equation in OQS, the contribution from the Lindbladian operator is known as the *Dissipator* which captures the effect of dissipation within the framework of quantum mechanics.

2.2.1 Effective Hamiltonian construction

where in the present context the effective Hamiltonian can be expressed as:

$$\text{Effective Hamiltonian : } H_{\text{eff}} = H_{\text{System}} + H_{\text{Lamb Shift}} = \underbrace{\sum_{\alpha=1}^2 \frac{\omega}{2} \hat{n}^{\alpha} \cdot \sigma^{\alpha}}_{\text{System}} - \underbrace{\frac{i}{2} \sum_{\alpha, \beta=1}^2 \sum_{i, j=1}^3 H_{ij}^{(\alpha\beta)} (n_i^{\alpha} \cdot \sigma_i^{\alpha}) (n_j^{\beta} \cdot \sigma_j^{\beta})}_{\text{Lamb shift=Heisenberg spin chain}}, \quad (2.27)$$

where the first term in the effective Hamiltonian represents the system Hamiltonian of the two atomic system, the detail of which we have already mentioned earlier. On the other hand, the second-part of the above effective Hamiltonian characterise the *atomic Lamb Shift*, which arises due to the inter atomic interaction between the conformally coupled external free probe scalar field with the atomic system under consideration and for usual practice this specific part of the Hamiltonian is known as *Lamb shift Hamiltonian*. It actually determines the shift in energy level as a consequence of the inter atomic interaction in the two body system (here it is two atomic system) with the external probe free scalar field in its vacuum state.

Also, it is important to note that, the *Lamb shift Hamiltonian*, is equivalent to a *Heisenberg spin chain* in condensed matter system. Here, $\hat{n}^{\alpha} \forall \alpha = (1, 2)$ and $\hat{n}^{\beta} \forall \beta = (1, 2)$ are the the normal unit vectors for two atoms under consideration in our OQS. Also the angles between normal vector and the Pauli matrices are characterised by the three Euler angles, α , β and γ . Therefore, the Lamb Shift Hamiltonian [51, 53, 55] can be simplified in terms of the Eulerian angles as:

$$\text{Lamb shift : } H_{\text{Lamb shift}} = -\frac{i}{2} \sum_{\alpha, \beta=1}^2 \sum_{i, j=1}^3 H_{ij}^{(\alpha\beta)} (n_i^{\alpha} \cdot \sigma_i^{\alpha}) (n_j^{\beta} \cdot \sigma_j^{\beta}) = -\frac{i}{2} \sum_{\alpha, \beta=1}^2 \sum_{i, j=1}^3 H_{ij}^{(\alpha\beta)} \cos(\alpha_i^{\alpha}) \cos(\alpha_j^{\beta}) \sigma_i^{\alpha} \sigma_j^{\beta}. \quad (2.28)$$

Now, to know the explicit contribution in the effective Hamiltonian in the present two entangled atomic set up we define the following set of Pauli operators:

$$\text{Pauli (Tensor) operators : } \quad \text{Atom 1} := \sigma_i^1 = \sigma_i \otimes \sigma_0, \quad \text{Atom 2} := \sigma_i^2 = \sigma_0 \otimes \sigma_i, \quad (2.29)$$

which is actually expressed in terms of the tensor product of 2×2 identity matrix σ_0 and the three 2×2 Pauli matrices σ_i , $\forall i = 1, 2, 3$, which satisfy the following conditions:

$$\text{Pauli matrix algebra : } \quad [\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\sigma_0. \quad (2.30)$$

Here also it is obvious from the mathematical structures of these Pauli operators that they satisfy the following criteria:

$$\text{Pauli (Tensor) operator algebra : } \quad [\sigma_i^{\alpha}, \sigma_j^{\beta}] = 2i\delta^{\alpha\beta}\epsilon_{ijk}\sigma_k^{\alpha}, \quad \{\sigma_i^1, \sigma_j^1\} = \{\sigma_i^2, \sigma_j^2\} = 2\delta_{ij}\sigma_0 \otimes \sigma_0, \quad \{\sigma_i^1, \sigma_j^2\} = 2\sigma_i \otimes \sigma_j, \quad (2.31)$$

where $(i, j) = 1, 2, 3$ and $(\alpha, \beta) = 1, 2$. With these set of definition of the Pauli matrices the Pauli operators can be expressed in terms of the tensor product as:

$$\sigma_1^1 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \sigma_1^2 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad (2.32)$$

$$\sigma_2^1 = \begin{pmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & 0 \end{pmatrix}, \quad \sigma_2^2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad (2.33)$$

$$\sigma_3^1 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad \sigma_3^2 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \quad (2.34)$$

Set of eigenstates ($|g_1\rangle, |e_1\rangle$ and $|g_2\rangle, |e_2\rangle$) of the two atomic OQS is given as:

$$\text{For atom 1 : } \quad \text{Ground state} \Rightarrow |g_1\rangle = (0 \ 1)^T, \quad \text{Excited state} \Rightarrow |e_1\rangle = (1 \ 0)^T \quad (2.35)$$

$$\text{For atom 2 : } \quad \text{Ground state} \Rightarrow |g_2\rangle = (0 \ 1)^T, \quad \text{Excited state} \Rightarrow |e_2\rangle = (1 \ 0)^T \quad (2.36)$$

In this context the matrix elements, $H^{\alpha\beta}$, can be determined from the Hilbert transformation (as defined later) from the vacuum field correlation function which is the forward two point propagator and can be expressed as:

$$\text{Forward two atomic Wightman function : } G^{\alpha\beta}(\Delta\tau = \tau - \tau') = \langle \Phi(\mathbf{x}_\alpha, \tau) \Phi(\mathbf{x}_\beta, \tau') \rangle . \quad (2.37)$$

The Fourier-transform of the above two point propagator in frequency (ω_0) space can be written as:

$$\text{Fourier - transform : } \mathcal{G}^{\alpha\beta}(\pm\omega_0) = \int_{-\infty}^{\infty} d\Delta\tau e^{\pm i\omega_0\Delta\tau} G^{\alpha\beta}(\Delta\tau) , \quad (2.38)$$

where ω_0 represents the difference between the energy levels of the ground and the excited states of the atoms respectively. In turn, the elements of the effective Hamiltonian matrix $H_{ij}^{\alpha\beta}$ are given by the following Hilbert transform of the Wightman function (two point correlator) as given by:

$$\text{Hilbert transform : } \mathcal{K}^{\alpha\beta}(\pm\omega_0) = \frac{P}{\pi i} \int_{-\infty}^{\infty} d\omega \frac{\mathcal{G}^{\alpha\beta}(\pm\omega)}{\omega \pm \omega_0} = \frac{P}{\pi i} \int_{-\infty}^{\infty} d\omega \frac{1}{\omega \pm \omega_0} \int_{-\infty}^{\infty} d\Delta\tau e^{\pm i\omega\Delta\tau} G^{\alpha\beta}(\Delta\tau) , \quad (2.39)$$

where P is the Principal value of the integral.

Now, the elements of co-efficient matrix $H_{ij}^{(\alpha\beta)}$ of the effective Hamiltonian can be explicitly represented by the following expression:

$$\text{Effective Hamiltonian co - efficient matrix : } H_{ij}^{(\alpha\beta)} = \mathcal{A}^{(\alpha\beta)}\delta_{ij} - i\mathcal{B}^{(\alpha\beta)}\epsilon_{ijk}\delta_{3k} - \mathcal{A}^{(\alpha\beta)}\delta_{3i}\delta_{3j} , \quad (2.40)$$

where, the quantities $\mathcal{A}^{(\alpha\beta)}$ and $\mathcal{B}^{(\alpha\beta)}$ for the two atomic system are defined as:

$$\begin{aligned} \mathcal{A}^{(\alpha\beta)} &= \frac{\mu^2}{4} [\mathcal{K}^{(\alpha\beta)}(\omega_0) + \mathcal{K}^{(\alpha\beta)}(-\omega_0)] = \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \left[\frac{\mathcal{G}^{\alpha\beta}(\omega)}{\omega + \omega_0} + \frac{\mathcal{G}^{\alpha\beta}(-\omega)}{\omega - \omega_0} \right] \\ &= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\Delta\tau G^{\alpha\beta}(\Delta\tau) \left[\frac{e^{i\omega\Delta\tau}}{\omega + \omega_0} + \frac{e^{-i\omega\Delta\tau}}{\omega - \omega_0} \right] , \end{aligned} \quad (2.41)$$

$$\begin{aligned} \mathcal{B}^{(\alpha\beta)} &= \frac{\mu^2}{4} [\mathcal{K}^{(\alpha\beta)}(\omega_0) - \mathcal{K}^{(\alpha\beta)}(-\omega_0)] = \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \left[\frac{\mathcal{G}^{\alpha\beta}(\omega)}{\omega + \omega_0} - \frac{\mathcal{G}^{\alpha\beta}(-\omega)}{\omega - \omega_0} \right] \\ &= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\Delta\tau G^{\alpha\beta}(\Delta\tau) \left[\frac{e^{i\omega\Delta\tau}}{\omega + \omega_0} - \frac{e^{-i\omega\Delta\tau}}{\omega - \omega_0} \right] . \end{aligned} \quad (2.42)$$

Further, substituting the explicit form of the $\mathcal{A}^{(\alpha\beta)}$ and $\mathcal{B}^{(\alpha\beta)}$ for the two atomic system we get the following simplified expression for the co-efficient matrix $H_{ij}^{(\alpha\beta)}$ of the effective Hamiltonian:

$$\begin{aligned} H_{ij}^{(\alpha\beta)} &= \frac{\mu^2}{4} \left[(\delta_{ij} - \delta_{3i}\delta_{3j}) \left\{ \mathcal{K}^{(\alpha\beta)}(\omega_0) + \mathcal{K}^{(\alpha\beta)}(-\omega_0) \right\} - i\epsilon_{ijk}\delta_{3k} \left\{ \mathcal{K}^{(\alpha\beta)}(\omega_0) - \mathcal{K}^{(\alpha\beta)}(-\omega_0) \right\} \right] \\ &= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \left[(\delta_{ij} - \delta_{3i}\delta_{3j}) \left\{ \frac{\mathcal{G}^{\alpha\beta}(\omega)}{\omega + \omega_0} + \frac{\mathcal{G}^{\alpha\beta}(-\omega)}{\omega - \omega_0} \right\} - i\epsilon_{ijk}\delta_{3k} \left\{ \frac{\mathcal{G}^{\alpha\beta}(\omega)}{\omega + \omega_0} - \frac{\mathcal{G}^{\alpha\beta}(-\omega)}{\omega - \omega_0} \right\} \right] \\ &= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\Delta\tau G^{\alpha\beta}(\Delta\tau) \left[(\delta_{ij} - \delta_{3i}\delta_{3j}) \left\{ \frac{e^{i\omega\Delta\tau}}{\omega + \omega_0} + \frac{e^{-i\omega\Delta\tau}}{\omega - \omega_0} \right\} - i\epsilon_{ijk}\delta_{3k} \left\{ \frac{e^{i\omega\Delta\tau}}{\omega + \omega_0} - \frac{e^{-i\omega\Delta\tau}}{\omega - \omega_0} \right\} \right] . \end{aligned} \quad (2.43)$$

Here, μ is the coupling parameter which represents the interaction strength between the system and the external thermal bath (conformally coupled scalar field) degrees of freedom. Determining the structure of the elements of the co-efficient matrix $H_{ij}^{(\alpha\beta)}$ in terms of the two atomic two point correlation function (Wightman function) of external free conformally coupled massless scalar field in De Sitter background actually fix the structure of the effective Hamiltonian in the present context. Here, we have expressed each co-efficient matrix elements in terms of the entries of the two atomic Wightman functions. The explicit mathematical structures of these two body two point correlation function for function external free conformally coupled massless scalar field in De Sitter background we will compute in the later part of this paper.

In this paper, one of our prime objectives are to find out the expectation value of the of the Lamb Shift Hamiltonian of the OQS described by two atoms, which are entangled with each other. Fixing the co-efficient matrix $H_{ij}^{(\alpha\beta)}$ in terms of the two atomic two point correlation function (Wightman function) of external free conformally coupled massless scalar field in De Sitter background helps us to compute the analytical expression for the energy shift explicitly. To compute this expression the main ingredient is the all possible quantum mechanical states which we have to construct in the present context from the ground and excited states of the two atoms. Using these atomic states we construct four possible quantum mechanical two atomic entangled states for the combined system (two atomic system+thermal bath), which are ground state, excited state, symmetric state and antisymmetric state respectively. We explicitly do this computation in the later part of the paper.

2.2.2 Linbladian construction

The fluctuation-dissipation into the system is introduced by the additional contribution in the time-evolution equation of the reduced density-matrix often known as the *Lindbladian* [43] or the *Lindblad operator* in OQS. The second significant term in the *GoriniKossakowskiSudarshanLindblad* (GKSL) *Master equation* is actually characterised as the *Lindbladian* or *Quantum Dissipator* of the OQS, which is for the present two atomic model can be written as:

$$\text{Lindbladian : } \mathcal{L}[\rho_{\text{System}}(\tau)] = \frac{1}{2} \sum_{i,j=1}^3 \sum_{\alpha,\beta=1}^2 C_{ij}^{\alpha\beta} \left[2(n_j^\beta \cdot \sigma_j^\beta) \rho_{\text{System}}(\tau) (n_i^\alpha \cdot \sigma_i^\alpha) - \left\{ (n_i^\alpha \cdot \sigma_i^\alpha) (n_j^\beta \cdot \sigma_j^\beta), \rho_{\text{System}}(\tau) \right\} \right], \quad (2.44)$$

where, $\rho_{\text{System}}(\tau)$ is the reduced density matrix of the two entangled atomic system where we have trace over the external bath scalar field degrees of freedom. The co-efficient matrix $C_{ij}^{\alpha\beta}$ is known as the *GoriniKossakowskiSudarshanLindblad matrix*, which is constructed under the weak coupling limiting approximation on the coupling parameter μ , as appearing in the interaction Hamiltonian⁵. In the context of any OQS, *Lindbladian* captures the effect of dissipation implicitly describing by the system operator $(n_i^\alpha \cdot \sigma_i^\alpha)$, which one can treat as an influence of the thermal bath on the two entangled atomic system under consideration.

Following the same procedure performed in the previous section to compute the co-efficient matrix element, $H_{ij}^{(\alpha\beta)}$, in the present context the elements of the *GoriniKossakowskiSudarshanLindblad matrix*, $C_{ij}^{(\alpha\beta)}$, as appearing in the expression for the *Lindbladian* can be expressed as:

$$\text{GoriniKossakowskiSudarshanLindblad matrix : } C_{ij}^{(\alpha\beta)} = \tilde{\mathcal{A}}^{(\alpha\beta)} \delta_{ij} - i \tilde{\mathcal{B}}^{(\alpha\beta)} \epsilon_{ijk} \delta_{3k} - \tilde{\mathcal{A}}^{(\alpha\beta)} \delta_{3i} \delta_{3j}, \quad (2.45)$$

where, the quantities $\tilde{\mathcal{A}}^{(\alpha\beta)}$ and $\tilde{\mathcal{B}}^{(\alpha\beta)}$ for the two atomic system are defined as:

$$\tilde{\mathcal{A}}^{(\alpha\beta)} = \frac{\mu^2}{4} [\mathcal{G}^{(\alpha\beta)}(\omega_0) + \mathcal{G}^{(\alpha\beta)}(-\omega_0)] = \frac{\mu^2}{4} \int_{-\infty}^{\infty} d\Delta\tau G^{\alpha\beta}(\Delta\tau) [e^{i\omega_0\Delta\tau} + e^{-i\omega_0\Delta\tau}], \quad (2.46)$$

$$\tilde{\mathcal{B}}^{(\alpha\beta)} = \frac{\mu^2}{4} [\mathcal{G}^{(\alpha\beta)}(\omega_0) - \mathcal{G}^{(\alpha\beta)}(-\omega_0)] = \frac{\mu^2}{4} \int_{-\infty}^{\infty} d\Delta\tau G^{\alpha\beta}(\Delta\tau) [e^{i\omega_0\Delta\tau} - e^{-i\omega_0\Delta\tau}]. \quad (2.47)$$

In the later half of the paper we will explicitly compute the entries of the *GoriniKossakowskiSudarshanLindblad matrix*, $C_{ij}^{(\alpha\beta)}$ to fix the mathematical structure of the *Lindbladian* operator in the present two entangled atomic OQS under consideration. This can be done once we compute the all of the possible combinations of two body Wightman (two point) correlation function for the external probe scalar field placed in the thermal bath.

3 Lamb Shift spectroscopy from OQS of two entangled atoms

In the collective state representation of OQS the ground state ($|G\rangle$), excited state ($|E\rangle$), symmetric state ($|S\rangle$) and the antisymmetric state ($|A\rangle$) of the two-entangled atomic OQS [61] can be expressed as:

$$\text{Ground state : } \Rightarrow |G\rangle = |g_1\rangle \otimes |g_2\rangle = (0 \ 0 \ 0 \ 1)^{\mathbf{T}}, \quad (3.1)$$

$$\text{Excited state : } \Rightarrow |E\rangle = |e_1\rangle \otimes |e_2\rangle = (1 \ 0 \ 0 \ 0)^{\mathbf{T}}, \quad (3.2)$$

$$\text{Symmetric state : } \Rightarrow |S\rangle = \frac{1}{\sqrt{2}} [|e_1\rangle \otimes |g_2\rangle + |g_1\rangle \otimes |e_2\rangle] = \frac{1}{\sqrt{2}} (0 \ 1 \ 1 \ 0)^{\mathbf{T}}, \quad (3.3)$$

$$\text{Antisymmetric state : } \Rightarrow |A\rangle = \frac{1}{\sqrt{2}} [|e_1\rangle \otimes |g_2\rangle - |g_1\rangle \otimes |e_2\rangle] = \frac{1}{\sqrt{2}} (0 \ 1 \ -1 \ 0)^{\mathbf{T}}. \quad (3.4)$$

⁵In the case of closed quantum mechanical system all the entries of the *GoriniKossakowskiSudarshanLindblad matrix* is zero. Consequently, the time evolution of the reduced density matrix is described by the quantum *Liouville equation*, which is the quantum mechanical analog of the classical *Liouville equation*.

Now, we explicitly calculate the expectation value of the Lamb shift Hamiltonian H_{LS} with respect to the ground state ($|G\rangle$), excited state ($|E\rangle$), symmetric state ($|S\rangle$) and the antisymmetric state ($|A\rangle$) which are given by the following expression:

I. Spectral Shift from the Ground state :

$$\begin{aligned}\delta E_G &= \langle G|H_{LS}|G\rangle \\ &= -\frac{i}{2}[H_{11}^{(11)} \cos^2(\alpha^1) + H_{11}^{(22)} \cos^2(\alpha^2) + H_{22}^{(11)} \cos^2(\beta^1) + H_{22}^{(22)} \cos^2(\beta^2) + H_{33}^{(11)} \cos^2(\gamma^1) + H_{33}^{(22)} \cos^2(\gamma^2) \\ &\quad - i \left(H_{12}^{(11)} - H_{21}^{(11)} \right) \cos(\alpha^1) \cos(\beta^1) - i \left(H_{12}^{(22)} - H_{21}^{(22)} \right) \cos(\alpha^2) \cos(\beta^2) \\ &\quad + \left(H_{33}^{(12)} + H_{33}^{(21)} \right) \cos(\gamma^1) \cos(\gamma^2)],\end{aligned}\quad (3.5)$$

II. Spectral Shift from the Excited state :

$$\begin{aligned}\delta E_E &= \langle E|H_{LS}|E\rangle \\ &= -\frac{i}{2}[H_{11}^{(11)} \cos^2(\alpha^1) + H_{11}^{(22)} \cos^2(\alpha^2) + H_{22}^{(11)} \cos^2(\beta^1) + H_{22}^{(22)} \cos^2(\beta^2) + H_{33}^{(11)} \cos^2(\gamma^1) + H_{33}^{(22)} \cos^2(\gamma^2) \\ &\quad + i \left(H_{12}^{(11)} - H_{21}^{(11)} \right) \cos(\alpha^1) \cos(\beta^1) + i \left(H_{12}^{(22)} - H_{21}^{(22)} \right) \cos(\alpha^2) \cos(\beta^2) \\ &\quad + \left(H_{33}^{(12)} + H_{33}^{(21)} \right) \cos(\gamma^1) \cos(\gamma^2)],\end{aligned}\quad (3.6)$$

III. Spectral Shift from the Symmetric state :

$$\begin{aligned}\delta E_S &= \langle S|H_{LS}|S\rangle \\ &= -\frac{i}{2}[H_{11}^{(11)} \cos^2(\alpha^1) + H_{11}^{(22)} \cos^2(\alpha^2) + H_{22}^{(11)} \cos^2(\beta^1) + H_{22}^{(22)} \cos^2(\beta^2) + H_{33}^{(11)} \cos^2(\gamma^1) + H_{33}^{(22)} \cos^2(\gamma^2) \\ &\quad + \left(H_{12}^{(11)} + H_{21}^{(11)} \right) \cos(\alpha^1) \cos(\alpha^2) + \left(H_{12}^{(22)} + H_{21}^{(22)} \right) \cos(\beta^1) \cos(\beta^2) \\ &\quad + \left(H_{33}^{(12)} + H_{33}^{(21)} \right) \cos(\gamma^1) \cos(\gamma^2)],\end{aligned}\quad (3.7)$$

IV. Spectral Shift from the Antisymmetric state :

$$\begin{aligned}\delta E_A &= \langle A|H_{LS}|A\rangle \\ &= -\frac{i}{2}[H_{11}^{(11)} \cos^2(\alpha^1) + H_{11}^{(22)} \cos^2(\alpha^2) + H_{22}^{(11)} \cos^2(\beta^1) + H_{22}^{(22)} \cos^2(\beta^2) + H_{33}^{(11)} \cos^2(\gamma^1) + H_{33}^{(22)} \cos^2(\gamma^2) \\ &\quad - \left(H_{12}^{(11)} + H_{21}^{(11)} \right) \cos(\alpha^1) \cos(\alpha^2) - \left(H_{12}^{(22)} + H_{21}^{(22)} \right) \cos(\beta^1) \cos(\beta^2) \\ &\quad - \left(H_{33}^{(12)} + H_{33}^{(21)} \right) \cos(\gamma^1) \cos(\gamma^2)].\end{aligned}\quad (3.8)$$

Now, using the Hilbert transformations mentioned in the earlier section one can easily fix the elements of the effective Hamiltonian matrix $H_{ij}^{(\alpha\beta)}$ as:

$$H_{ij}^{(11)} = H_{ij}^{(22)} = \mathcal{A}_1 \delta_{ij} - i\mathcal{B}_1 \epsilon_{ijk} \delta_{3k} - \mathcal{A}_1 \delta_{3i} \delta_{3j}, \quad (3.9)$$

$$H_{ij}^{(12)} = H_{ij}^{(21)} = \mathcal{A}_2 \delta_{ij} - i\mathcal{B}_2 \epsilon_{ijk} \delta_{3k} - \mathcal{A}_2 \delta_{3i} \delta_{3j}. \quad (3.10)$$

In the above set of equations, the coefficients \mathcal{A}_1 , \mathcal{B}_1 , \mathcal{A}_2 and \mathcal{B}_2 are defined in the Appendix.

From this Hilbert transformation, it implies that the following contributions of the effective Hamiltonian matrix trivially vanishes:

$$H_{33}^{(11)} = H_{33}^{(22)} = 0, \quad H_{33}^{(12)} = H_{33}^{(21)} = 0, \quad (3.11)$$

and the rest of the non vanishing components are given by:

$$H_{11}^{(11)} = H_{11}^{(22)} = \mathcal{A}_1, \quad H_{22}^{(11)} = H_{22}^{(22)} = \mathcal{A}_1, \quad (3.12)$$

$$H_{11}^{(12)} = H_{11}^{(21)} = \mathcal{A}_2, \quad H_{22}^{(12)} = H_{22}^{(21)} = \mathcal{A}_2, \quad (3.13)$$

$$H_{12}^{(11)} = H_{12}^{(22)} = -i\mathcal{B}_1, \quad H_{21}^{(11)} = H_{21}^{(22)} = i\mathcal{B}_1. \quad (3.14)$$

and they will explicitly contribute to the final expression for the expectation values of the Lamb shift Hamiltonian.

Finally, we get the following simplified expressions for the expectation value of the Lamb Shift Hamiltonian:

I. Spectral Shift from the Ground state :

$$\begin{aligned}\delta E_G &= \langle G|H_{LS}|G\rangle \\ &= -\frac{i}{2}[H_{11}^{(11)} \cos^2(\alpha^1) + H_{11}^{(22)} \cos^2(\alpha^2) + H_{22}^{(11)} \cos^2(\beta^1) + H_{22}^{(22)} \cos^2(\beta^2) \\ &\quad - i(H_{12}^{(11)} - H_{21}^{(11)}) \cos(\alpha^1) \cos(\beta^1) - i(H_{12}^{(22)} - H_{21}^{(22)}) \cos(\alpha^2) \cos(\beta^2)],\end{aligned}\quad (3.15)$$

II. Spectral Shift from the Excited state :

$$\begin{aligned}\delta E_E &= -\frac{i}{2}[H_{11}^{(11)} \cos^2(\alpha^1) + H_{11}^{(22)} \cos^2(\alpha^2) + H_{22}^{(11)} \cos^2(\beta^1) + H_{22}^{(22)} \cos^2(\beta^2) \\ &\quad + i(H_{12}^{(11)} - H_{21}^{(11)}) \cos(\alpha^1) \cos(\beta^1) + i(H_{12}^{(22)} - H_{21}^{(22)}) \cos(\alpha^2) \cos(\beta^2)],\end{aligned}\quad (3.16)$$

III. Spectral Shift from the Symmetric state :

$$\begin{aligned}\delta E_S &= -\frac{i}{2}[H_{11}^{(11)} \cos^2(\alpha^1) + H_{11}^{(22)} \cos^2(\alpha^2) + H_{22}^{(11)} \cos^2(\beta^1) + H_{22}^{(22)} \cos^2(\beta^2) \\ &\quad + (H_{12}^{(11)} + H_{21}^{(11)}) \cos(\alpha^1) \cos(\alpha^2) + (H_{12}^{(22)} + H_{21}^{(22)}) \cos(\beta^1) \cos(\beta^2)],\end{aligned}\quad (3.17)$$

IV. Spectral Shift from the Antisymmetric state :

$$\begin{aligned}\delta E_A &= -\frac{i}{2}[H_{11}^{(11)} \cos^2(\alpha^1) + H_{11}^{(22)} \cos^2(\alpha^2) + H_{22}^{(11)} \cos^2(\beta^1) + H_{22}^{(22)} \cos^2(\beta^2) \\ &\quad - (H_{12}^{(11)} + H_{21}^{(11)}) \cos(\alpha^1) \cos(\alpha^2) - (H_{12}^{(22)} + H_{21}^{(22)}) \cos(\beta^1) \cos(\beta^2)].\end{aligned}\quad (3.18)$$

In the next section, we explicitly compute the elements of the co-efficient of $H_{ij}^{(\alpha\beta)}$, from the two point two body Wightman function of the external probe scalar field placed in the thermal bath, which will fix the final behaviour of the *Lamb Shift* in terms of the Euclidean distance between two atoms and the temperature of the external thermal bath and curvature of De Sitter space.

4 Geometry of our De Sitter Universe from Lamb Shift Spectroscopy

4.1 Geometry of De Sitter space

Here our prime objective is to compute the two body two point (Wightman) correlation function between two atom conformally coupled with a massless probe external scalar field in De Sitter space. To serve this purpose we start with the background metric which is represented by the surface of the following *hyperboloid*:

$$\text{Equation for hyperboloid :} \quad (z_0^2 - z_1^2 - z_2^2 - z_3^2 - z_4^2) = -\alpha^2 = -\frac{3}{\Lambda}, \quad (4.1)$$

which describes a solution of the *Einstein's field equations* with the following radius of the *hyperboloid*:

$$\text{Radius of Hyperboloid :} \quad \alpha = \sqrt{\frac{3}{\Lambda}} > 0. \quad (4.2)$$

Here, Λ is the cosmological constant with positive signature in De Sitter space and the corresponding embedded metric in the five dimensional Minkowski space is given by [29, 30]:

$$\text{Five dimensional Minkowski flat metric :} \quad ds^2 = \left(dz_0^2 - \sum_{p=1}^4 dz_p^2 \right) = (dz_0^2 - dz_1^2 - dz_2^2 - dz_3^2 - dz_4^2). \quad (4.3)$$

By applying the following appropriate parametrisation one can express the five dimensional Minkowski flat metric in terms of a static four dimensional De Sitter metric, as given by:

$$z_0 = \sqrt{\alpha^2 - r^2} \sinh\left(\frac{t}{\alpha}\right), \quad (4.4)$$

$$z_1 = \sqrt{\alpha^2 - r^2} \cosh\left(\frac{t}{\alpha}\right), \quad (4.5)$$

Coordinate re – parametrisation :
$$z_2 = r \cos \theta, \quad (4.6)$$

$$z_3 = r \sin \theta \cos \phi, \quad (4.7)$$

$$z_4 = r \sin \theta \sin \phi, \quad (4.8)$$

which are consistent with the equation of the surface of the hyperboloid in five dimension as expressed in Eq (4.1).

With the above parametrisation the static De Sitter metric in four dimensions can be expressed as:

Static four dimensional De Sitter metric :
$$ds^2 = \left(1 - \frac{r^2}{\alpha^2}\right) dt^2 - \left(1 - \frac{r^2}{\alpha^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.9)$$

which is actually characterised by (t, r, θ, ϕ) in spherical polar coordinate in four dimensions. In the present context, to compute the explicit contributions of the two body Wight function of the probe scalar field present in the external thermal bath the geometry of the re-parametrised four dimensional De Sitter space play the most crucial role. In the next subsection, we actually compute these two body Wightman function in detail.

4.2 Two atomic Wightman functions for probe external field in De Sitter space

To compute the expression for the each of the entries of the two body Wight function of the probe scalar field present in the external thermal bath we use the four dimensional static De Sitter geometry of our space-time. In this set of coordinate system in four dimension, the Klein-Gordon field equation for the massless conformally coupled external probe scalar field for the non-adiabatic environment can be expressed as:

Massless probe :
$$\left[\frac{1}{\cosh^3\left(\frac{t}{\alpha}\right)} \frac{\partial}{\partial t} \left(\cosh^3\left(\frac{t}{\alpha}\right) \frac{\partial}{\partial t} \right) - \frac{1}{\alpha^2 \cosh^2\left(\frac{t}{\alpha}\right)} \mathbf{L}^2 \right] \Phi(t, \chi, \theta, \phi) = 0, \quad (4.10)$$

where \mathbf{L}^2 is the *Laplacian differential operator* in the three dimensions characterised by the coordinate (χ, θ, ϕ) , which is explicitly defined as:

Laplacian operator :
$$\mathbf{L}^2 = \frac{1}{\sin^2 \chi} \left[\frac{\partial}{\partial \chi} \left(\sin^2 \chi \frac{\partial}{\partial \chi} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right], \quad (4.11)$$

where we introduce a new coordinate χ which is related to the radial coordinate r as:

$$r = \sin \chi. \quad (4.12)$$

Now, in the present context of discussion we are not exactly interested to solve the wave function for the probe scalar field. Since our prime objective is to compute the two body two point correlation function for the probe external scalar field we construct the two body Green's function.

From the geometrical structure of the four dimensional static De Sitter metric it is obvious that the coordinate singularity, $(1 - r^2/\alpha^2)^{-2} \rightarrow \infty$, which appears at the point $r = \alpha$, and this is identified to be the *cosmological horizon* in the present context. Here it is important to note that, in flat space there is no problem to define the vacuum state (*i.e.* Minkowski vacuum) of the quantum field in open quantum mechanical system. But for curved space the definition of the vacuum state is more complicated in OQS than the result obtained in the context of flat space. For the computation of Wightman function from the present two entangled atomic open quantum set up, in the curved space we choose specifically the De Sitter invariant isometric $SO(1, 4)$ group to connect the idea of spectroscopic energy shift with the geometry of our observed Universe. In this computation the isometric vacuum state (*i.e.* Bunch Davies and α -vacua) is actually identified with the quantum mechanical state of open system described by the conformally coupled massless probe scalar field. Now, the corresponding two point correlation function, often known as the Wightman function for massless probe scalar field takes the following form [29]:

Massless Wightman function :
$$G^+(x, x') = -\frac{1}{4\pi^2} \frac{1}{(z_0 - z'_0)^2 - \Delta z^2 - i\epsilon}, \quad (4.13)$$

where, ϵ represents an infinitesimal constant, which is appearing in the representation of Wightman function in the $i\epsilon$ prescription. Also, we define the distance between two static atoms localized at the coordinates (r, θ, ϕ) and (r, θ', ϕ) , appearing in this computation as:

$$\begin{aligned}
\Delta z^2 &= (z_1 - z'_1)^2 + (z_2 - z'_2)^2 + (z_3 - z'_3)^2 + (z_4 - z'_4)^2 \\
&= (\alpha^2 - r^2) \left[\cosh\left(\frac{t}{\alpha}\right) - \cosh\left(\frac{t'}{\alpha}\right) \right]^2 + r^2 \left[(\cos\theta - \cos\theta')^2 + (\sin\theta - \sin\theta')^2 \right] \\
&= (\alpha^2 - r^2) \left[\cosh\left(\frac{t}{\alpha}\right) - \cosh\left(\frac{t'}{\alpha}\right) \right]^2 + 4r^2 \sin^2\left(\frac{\Delta\theta}{2}\right) \\
&= (\alpha^2 - r^2) \left[\cosh\left(\frac{t}{\alpha}\right) - \cosh\left(\frac{t'}{\alpha}\right) \right]^2 + L^2,
\end{aligned} \tag{4.14}$$

Here L represents the Euclidean distance between the coordinates (r, θ, ϕ) and (r, θ', ϕ) , which is defined as:

$$L = 2r \sin\left(\frac{\Delta\theta}{2}\right), \tag{4.15}$$

where, the angular difference $\Delta\theta$ is defined as, $\Delta\theta = \theta - \theta'$. Here both the signatures of the $\Delta\theta$ is allowed in the present context depending on the relative ordering of the angular coordinates θ and θ' . Additionally, the parameter ϵ in the present context represents an infinitesimal constant. The corresponding two body Wightman function between two space-time points for massless probe scalar field can be expressed as ⁶:

$$G(x, x') = \begin{pmatrix} G^{11}(x, x') & G^{12}(x, x') \\ G^{21}(x, x') & G^{22}(x, x') \end{pmatrix} = \begin{pmatrix} \langle \Phi(\mathbf{x}_1, \tau) \Phi(\mathbf{x}_1, \tau') \rangle & \langle \Phi(\mathbf{x}_1, \tau) \Phi(\mathbf{x}_2, \tau') \rangle \\ \langle \Phi(\mathbf{x}_2, \tau) \Phi(\mathbf{x}_1, \tau') \rangle & \langle \Phi(\mathbf{x}_2, \tau) \Phi(\mathbf{x}_2, \tau') \rangle \end{pmatrix}, \tag{4.16}$$

where the components of the two body Wightman function can be expressed as:

Two atomic Wightman function representing autocorrelation :

$$\begin{aligned}
G^{11}(x, x') &= G^{22}(x, x') = \langle \Phi(\mathbf{x}_1, \tau) \Phi(\mathbf{x}_1, \tau') \rangle \\
&= \langle \Phi(\mathbf{x}_2, \tau) \Phi(\mathbf{x}_2, \tau') \rangle \\
&= \frac{1}{4\pi^2} \frac{1}{\{(z_0 - z'_0)^2 - (z_1 - z'_1)^2 - i\epsilon\}} \\
&= \frac{1}{4\pi^2} \frac{1}{\left\{ (\alpha^2 - r^2) \left[\left\{ \sinh\left(\frac{t}{\alpha}\right) - \sinh\left(\frac{t'}{\alpha}\right) \right\}^2 - \left\{ \cosh\left(\frac{t}{\alpha}\right) - \cosh\left(\frac{t'}{\alpha}\right) \right\}^2 \right] - i\epsilon \right\}} \\
&= \frac{1}{4\pi^2} \frac{1}{\left\{ 2(\alpha^2 - r^2) \left[\cosh\left(\frac{t-t'}{\alpha}\right) - 1 \right] - i\epsilon \right\}} \\
&= \frac{1}{4\pi^2} \frac{1}{\left\{ 4(\alpha^2 - r^2) \sinh^2\left(\frac{t-t'}{2\alpha}\right) - i\epsilon \right\}} \\
&= \frac{1}{4\pi^2} \frac{1}{\left\{ 4(\alpha^2 - r^2) \sinh^2\left(\frac{\Delta\tau}{2\sqrt{g_{00}}\alpha}\right) - i\epsilon \right\}} \\
&= \frac{1}{4\pi^2} \frac{1}{\left\{ 4k^2 \sinh^2\left(\frac{\Delta\tau}{2k}\right) - i\epsilon \right\}} \\
&= \frac{1}{16\pi^2 k^2} \frac{1}{\sinh^2\left(\frac{\Delta\tau}{2k} - i\epsilon\right)},
\end{aligned} \tag{4.17}$$

⁶From the two atomic two body system we get four Wightman functions in the present context due to the interaction between the two atoms with the external probe conformally coupled massless scalar field. The diagonal entries of the two body Wightman function represents the auto correlation function of the atom 1 and atom 2 with the external probe scalar field present in the thermal bath respectively. Also it is import to note that, these diagonal entries are same as we have considered two identical in our computation. On the other hand, the off diagonal entries of the Wightman function represent the cross correlation between the atom 1 and atom 2 with the external probe scalar field. Since in the present context the Wightman function is symmetric and constructed due to the interaction of two identical atoms with the external probe scalar field, the contributions appearing from the off-diagonal entries are exactly same.

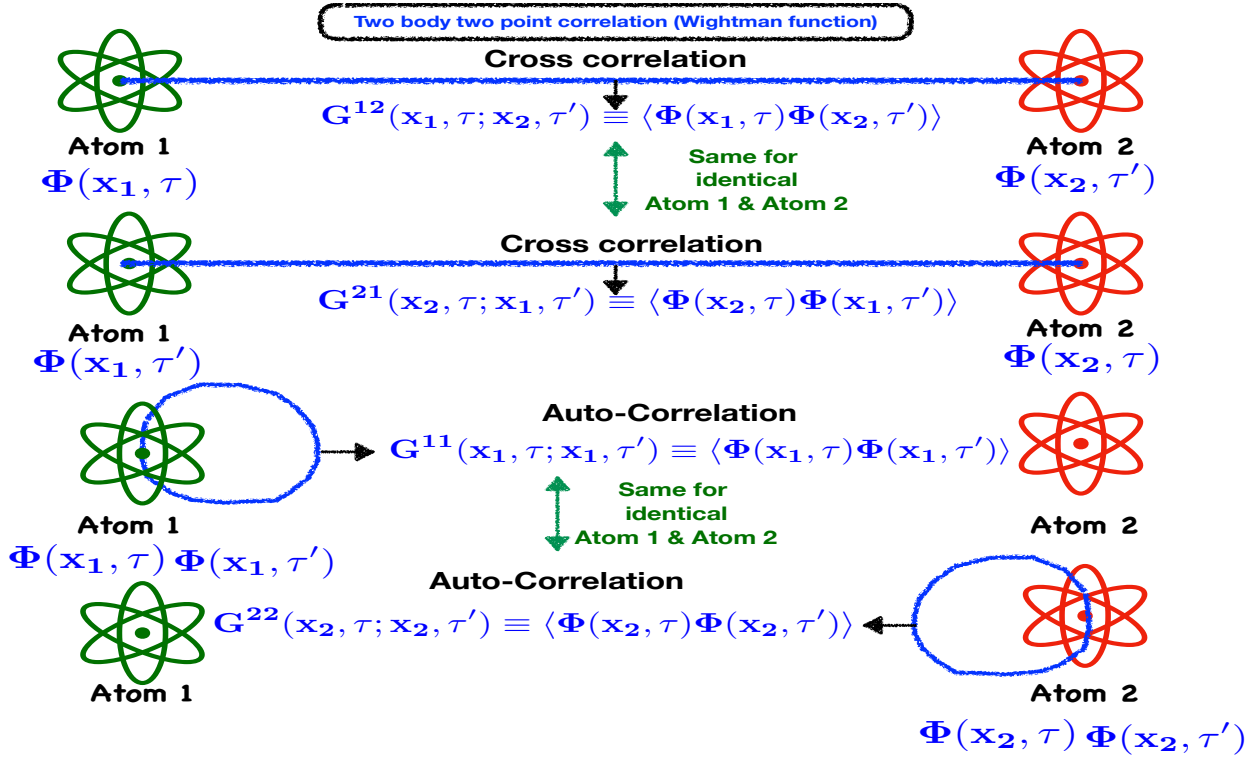


Figure 2. Schematic diagram of the two body quantum correlation between the two atoms in De Sitter space.

Two atomic Wightman function representing crosscorrelation :

$$\begin{aligned}
G^{12}(x, x') &= G^{21}(x, x') = \langle \Phi(\mathbf{x}_1, \tau) \Phi(\mathbf{x}_2, \tau') \rangle \\
&= \langle \Phi(\mathbf{x}_2, \tau) \Phi(\mathbf{x}_1, \tau') \rangle \\
&= -\frac{1}{4\pi^2} \frac{1}{(z_0 - z'_0)^2 - \Delta z^2 - i\epsilon} \\
&= -\frac{1}{4\pi^2} \left\{ (\alpha^2 - r^2) \left[\left(\sinh\left(\frac{t}{\alpha}\right) - \sinh\left(\frac{t'}{\alpha}\right) \right)^2 - \left(\cosh\left(\frac{t}{\alpha}\right) - \cosh\left(\frac{t'}{\alpha}\right) \right)^2 \right] - i\epsilon \right. \\
&\quad \left. - r^2 [(\cos\theta - \cos\theta')^2 - (\sin\theta - \sin\theta')^2] \right\}^{-1} \\
&= -\frac{1}{4\pi^2} \frac{1}{\{2(\alpha^2 - r^2) [\cosh\left(\frac{t-t'}{\alpha}\right) - 1] + 2r^2 [\cos(\theta - \theta') - 1] - i\epsilon\}} \\
&= -\frac{1}{4\pi^2} \frac{1}{\{2(\alpha^2 - r^2) [\cosh\left(\frac{t-t'}{\alpha}\right) - 1] - 4r^2 \sin^2\left(\frac{\theta-\theta'}{2}\right) - i\epsilon\}} \\
&= -\frac{1}{4\pi^2} \frac{1}{\{4(\alpha^2 - r^2) [\sinh^2\left(\frac{t-t'}{2\alpha}\right) - i\epsilon] - 4r^2 \sin^2\left(\frac{\theta-\theta'}{2}\right)\}} \\
&= -\frac{1}{16\pi^2 k^2} \frac{1}{\left\{ \sinh^2\left(\frac{\Delta\tau}{2k} - i\epsilon\right) - \frac{r^2}{k^2} \sin^2\left(\frac{\Delta\theta}{2}\right) \right\}}, \tag{4.18}
\end{aligned}$$

where we use the following identity:

$$\sinh\left(\frac{\Delta\tau}{2k} - i\epsilon\right) = \sinh\left(\frac{\Delta\tau}{2k}\right) \cosh(i\epsilon) - \cosh\left(\frac{\Delta\tau}{2k}\right) \sinh(i\epsilon), \tag{4.19}$$

where ϵ being an infinitesimal constant which we have already mentioned earlier. Furthermore, we have used the fact that since ϵ

is an infinitesimal constant then one can approximate:

$$\sinh(i\epsilon) = i \sin(\epsilon) \sim i\epsilon, \quad (4.20)$$

$$\cosh(i\epsilon) = \cos(\epsilon) \sim 1. \quad (4.21)$$

As a result, we get the following simplified result:

$$\sinh\left(\frac{\Delta\tau}{2k} - i\epsilon\right) \sim \sinh\left(\frac{\Delta\tau}{2k}\right) - i\epsilon \cosh\left(\frac{\Delta\tau}{2k}\right). \quad (4.22)$$

Additionally, we have used the following definitions:

$$k = \sqrt{g_{00}}\alpha = \sqrt{1 - \left(\frac{r}{\alpha}\right)^2} \alpha = \sqrt{\alpha^2 - r^2}, \quad (4.23)$$

$$\Delta\tau = \tau - \tau' = \sqrt{g_{00}}(t - t') = \sqrt{1 - \left(\frac{r}{\alpha}\right)^2} (t - t') = \sqrt{\alpha^2 - r^2} \left(\frac{t - t'}{\alpha}\right) = k \left(\frac{t - t'}{\alpha}\right), \quad (4.24)$$

with τ being the proper-time in the co-moving frame of the pair of atoms and k represents the surface gravity in the present context.

4.3 Computing regularised Lamb Shift Spectra in De Sitter space

4.3.1 Fixing Lamb Shift Spectra from Hilbert transformations

In this section, we want to evaluate the each of the entries of the co-efficient matrix $H_{ij}^{(\alpha\beta)}$ to fix the expressions for the energy shift from ground, excited, symmetric and antisymmetric quantum state for the two atomic entangled OQS under consideration. In the earlier section of this we have already expressed the general mathematical form of the co-efficient matrix $H_{ij}^{(\alpha\beta)}$ in terms of the two atomic Wightman functions through successive Fourier and Hilbert transformations. Now we substitute the explicit mathematical forms of the two atomic Wightman functions computed in the previous section. This will finally fix the expressions for the energy shift computed from this open system under consideration.

Now, the Fourier transform of the two point Wightman functions representing the two point field correlation functions in frequency (ω) space for external massless probe scalar field can be written as:

$$\begin{aligned} \text{Fourier transform of auto correlation : } \quad \mathcal{G}^{11}(\omega) = \mathcal{G}^{22}(\omega) &= - \int_{-\infty}^{\infty} d\Delta\tau \frac{e^{i\omega\Delta\tau}}{16\pi^2 k^2 \sinh^2\left(\frac{\Delta\tau}{2k} - i\epsilon\right)} \\ &= \frac{1}{2\pi} \frac{\omega}{1 - e^{-2\pi k\omega}}, \end{aligned} \quad (4.25)$$

$$\begin{aligned} \text{Fourier transform of cross correlation : } \quad \mathcal{G}^{12}(\omega) = \mathcal{G}^{21}(\omega) &= - \int_{-\infty}^{\infty} d\Delta\tau \frac{1}{16\pi^2 k^2} \frac{e^{i\omega\Delta\tau}}{\sinh^2\left(\frac{\Delta\tau}{2k} - i\epsilon\right) - \frac{r^2}{k^2} \sin^2\left(\frac{\Delta\theta}{2}\right)} \\ &= \frac{1}{2\pi} \frac{\omega}{1 - e^{-2\pi k\omega}} f(\omega, L/2), \end{aligned} \quad (4.26)$$

where, we define the spectral function $f(\omega, L/2)$ as:

$$\text{Spectral function : } \quad f(\omega, L/2) = \frac{1}{L\omega \sqrt{1 + \left(\frac{L}{2k}\right)^2}} \sin\left(2k\omega \sinh^{-1}\left(\frac{L}{2k}\right)\right), \quad (4.27)$$

which is actually the outcome of the Fourier transformation of the two atomic Wightman function representing symmetric and identical cross correlation functions. In this expression the Euclidean distance L between the coordinates (r, θ, ϕ) and (r, θ', ϕ) is already defined earlier.

In the present context the elements of the *GoriniKossakowskiSudarshanLindblad matrix*, $C_{ij}^{(\alpha\beta)}$, as appearing in the expression for the *Linbladian* can be expressed as:

$$C_{ij}^{(11)} = C_{ij}^{(22)} = \tilde{A}_1 \delta_{ij} - i\tilde{B}_1 \epsilon_{ijk} \delta_{3k} - \tilde{A}_1 \delta_{3i} \delta_{3j}, \quad (4.28)$$

$$C_{ij}^{(12)} = C_{ij}^{(21)} = \tilde{A}_2 \delta_{ij} - i\tilde{B}_2 \epsilon_{ijk} \delta_{3k} - \tilde{A}_2 \delta_{3i} \delta_{3j}. \quad (4.29)$$

From this Hilbert transformation, it implies that the following contributions of the *GoriniKossakowskiSudarshanLindblad matrix*, trivially vanishes:

$$C_{33}^{(11)} = C_{33}^{(22)} = 0, \quad C_{33}^{(12)} = C_{33}^{(21)} = 0, \quad (4.30)$$

and the rest of the non vanishing components are given by:

$$C_{11}^{(11)} = C_{11}^{(22)} = \tilde{\mathcal{A}}_1, \quad C_{22}^{(11)} = C_{22}^{(22)} = \tilde{\mathcal{A}}_1, \quad (4.31)$$

$$C_{11}^{(12)} = C_{11}^{(21)} = \tilde{\mathcal{A}}_2, \quad C_{22}^{(12)} = C_{22}^{(21)} = \tilde{\mathcal{A}}_2, \quad (4.32)$$

$$C_{12}^{(11)} = C_{12}^{(22)} = -i\tilde{\mathcal{B}}_1, \quad C_{21}^{(11)} = C_{21}^{(22)} = i\tilde{\mathcal{B}}_1. \quad (4.33)$$

and they will explicitly contribute to the final expression for the *Lindbladian* operator.

Here the quantities $\tilde{\mathcal{A}}_1$, $\tilde{\mathcal{B}}_1$, $\tilde{\mathcal{A}}_2$ and $\tilde{\mathcal{B}}_2$ for the two atomic system are defined as:

$$\tilde{\mathcal{A}}_1 = \frac{\mu^2}{8\pi}\omega_0 \left[\frac{1}{1 - e^{-2\pi k\omega_0}} - \frac{1}{1 - e^{2\pi k\omega_0}} \right], \quad (4.34)$$

$$\tilde{\mathcal{B}}_1 = \frac{\mu^2}{8\pi}\omega_0 \left[\frac{1}{1 - e^{-2\pi k\omega_0}} + \frac{1}{1 - e^{2\pi k\omega_0}} \right], \quad (4.35)$$

$$\tilde{\mathcal{A}}_2 = \frac{\mu^2}{8\pi}\omega_0 \left[\frac{f(\omega_0, L/2)}{1 - e^{-2\pi k\omega_0}} - \frac{f(-\omega_0, L/2)}{1 - e^{2\pi k\omega_0}} \right], \quad (4.36)$$

$$\tilde{\mathcal{B}}_2 = \frac{\mu^2}{8\pi}\omega_0 \left[\frac{f(\omega_0, L/2)}{1 - e^{-2\pi k\omega_0}} + \frac{f(-\omega_0, L/2)}{1 - e^{2\pi k\omega_0}} \right]. \quad (4.37)$$

Now, the corresponding Hilbert transformations of the Fourier transformed Wightman functions, which we have used throughout our rest of the computation are expressed as:

$$\textbf{Hilbert transformed auto correlation: } \mathcal{K}^{11}(\omega_0) = \mathcal{K}^{22}(\omega_0) = \frac{P}{2\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{1}{\omega - \omega_0} \frac{\omega}{1 - e^{2\pi k\omega}}, \quad (4.38)$$

$$\textbf{Hilbert transformed cross correlation: } \mathcal{K}^{12}(\omega_0) = \mathcal{K}^{21}(\omega_0) = \frac{P}{2\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{1}{\omega - \omega_0} \frac{\omega}{1 - e^{2\pi k\omega}} f(\omega, L/2). \quad (4.39)$$

Further using Eq (4.38) and Eq (4.39) we get the following simplified expression for \mathcal{A}_1 , \mathcal{B}_1 , \mathcal{A}_2 and \mathcal{B}_2 :

$$\mathcal{A}_1 = = \frac{\mu^2 P}{4\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{\omega^2}{(\omega + \omega_0)(\omega - \omega_0)(1 - e^{-2\pi k\omega})}, \quad (4.40)$$

$$\mathcal{B}_1 = = \frac{\mu^2 P}{4\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{\omega_0 \omega}{(\omega + \omega_0)(\omega - \omega_0)(1 - e^{-2\pi k\omega})}, \quad (4.41)$$

$$\mathcal{A}_2 = = \frac{\mu^2 P}{4\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{\omega^2 f(\omega, L/2)}{(\omega + \omega_0)(\omega - \omega_0)(1 - e^{-2\pi k\omega})}, \quad (4.42)$$

$$\mathcal{B}_2 = = \frac{\mu^2 P}{4\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{\omega_0 \omega f(\omega, L/2)}{(\omega + \omega_0)(\omega - \omega_0)(1 - e^{-2\pi k\omega})}. \quad (4.43)$$

Now, using these above sets of Hilbert transformations one can easily fix the elements of the co-efficient matrix $H_{ij}^{(\alpha\beta)}$, which are given by:

$$\textbf{Diagonal matrix elements: } H_{ij}^{(11)} = H_{ij}^{(22)} = \frac{\mu^2 P}{4\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{\omega \{(\delta_{ij} - \delta_{3i}\delta_{3j})\omega - i\epsilon_{ijk}\delta_{3k}\omega_0\}}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)}, \quad (4.44)$$

$$\textbf{Off - diagonal matrix elements: } H_{ij}^{(12)} = H_{ij}^{(21)} = \frac{\mu^2 P}{4\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{\omega \{(\delta_{ij} - \delta_{3i}\delta_{3j})\omega - i\epsilon_{ijk}\delta_{3k}\omega_0\} f(\omega, L/2)}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)}. \quad (4.45)$$

Therefore, the shift in energy levels for ground, excited, symmetric and antisymmetric quantum state can be further simplified as:

I. Spectral Shift from the Ground state :

$$\delta E_G = \frac{\mu^2 P}{8\pi^2} \int_{-\infty}^{\infty} d\omega \frac{\omega}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)} [2\omega_0 \{ \cos(\alpha^1) \cos(\beta^1) + \cos(\alpha^2) \cos(\beta^2) \} - \omega \{ \cos^2(\alpha^1) + \cos^2(\beta^1) + \cos^2(\alpha^2) + \cos^2(\beta^2) \}], \quad (4.46)$$

II. Spectral Shift from the Excited state :

$$\delta E_E = -\frac{\mu^2 P}{8\pi^2} \int_{-\infty}^{\infty} d\omega \frac{\omega}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)} [2\omega_0 \{ \cos(\alpha^1) \cos(\beta^1) + \cos(\alpha^2) \cos(\beta^2) \} + \omega \{ \cos^2(\alpha^1) + \cos^2(\beta^1) + \cos^2(\alpha^2) + \cos^2(\beta^2) \}], \quad (4.47)$$

III. Spectral Shift from the Symmetric state :

$$\delta E_S = -\frac{\mu^2 P}{8\pi^2} \int_{-\infty}^{\infty} d\omega \frac{\omega^2}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)} [2f(\omega, L/2) \{ \cos(\alpha^1) \cos(\alpha^2) + \cos(\beta^1) \cos(\beta^2) \} + \{ \cos^2(\alpha^1) + \cos^2(\alpha^2) + \cos^2(\beta^1) + \cos^2(\beta^2) \}], \quad (4.48)$$

IV. Spectral Shift from the Antisymmetric state :

$$\delta E_A = \frac{\mu^2 P}{8\pi^2} \int_{-\infty}^{\infty} d\omega \frac{\omega^2}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)} [2f(\omega, L/2) \{ \cos(\alpha^1) \cos(\alpha^2) + \cos(\beta^1) \cos(\beta^2) \} - \{ \cos^2(\alpha^1) + \cos^2(\alpha^2) + \cos^2(\beta^1) + \cos^2(\beta^2) \}]. \quad (4.49)$$

Now from the above mentioned results of energy (spectral) shift obtained for four different entangled state of two atoms we get the following overall features:

1. In the frequency range, $-\infty < \omega < \infty$, all the following representative integrals are divergent in nature:

(a) **Spectroscopic Integral I:**

$$\Delta_1 := \int_{-\infty}^{\infty} d\omega \frac{2\omega_0 \omega}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)}. \quad (4.50)$$

(b) **Spectroscopic Integral II:**

$$\Delta_2 := \int_{-\infty}^{\infty} d\omega \frac{\omega^2}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)}. \quad (4.51)$$

(c) **Spectroscopic Integral III:**

$$\Delta_3 := \int_{-\infty}^{\infty} d\omega \frac{2 \omega^2 f(\omega, L/2)}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)}. \quad (4.52)$$

2. However, to remove the unwanted divergences from this spectroscopic calculation we introduce a cut-off regulator by following *Bethe regularisation* technique. It actually very useful to extract the required physical information from the energy shift for all possible entangled quantum states for the two atomic open system under consideration.

Finally, writhing the energy shifts in terms of these defined integrals Δ_1 , Δ_2 and Δ_3 we get the following simplified results:

I. Spectral Shift from the Ground state :

$$\delta E_G = \frac{\mu^2 P}{8\pi^2} [\{ \cos(\alpha^1) \cos(\beta^1) + \cos(\alpha^2) \cos(\beta^2) \} \Delta_1 - \{ \cos^2(\alpha^1) + \cos^2(\beta^1) + \cos^2(\alpha^2) + \cos^2(\beta^2) \} \Delta_2], \quad (4.53)$$

II. Spectral Shift from the Excited state :

$$\delta E_E = -\frac{\mu^2 P}{8\pi^2} [\{ \cos(\alpha^1) \cos(\beta^1) + \cos(\alpha^2) \cos(\beta^2) \} \Delta_1 + \{ \cos^2(\alpha^1) + \cos^2(\beta^1) + \cos^2(\alpha^2) + \cos^2(\beta^2) \} \Delta_2], \quad (4.54)$$

III. Spectral Shift from the Symmetric state :

$$\delta E_S = -\frac{\mu^2 P}{8\pi^2} [\{\cos(\alpha^1) \cos(\alpha^2) + \cos(\beta^1) \cos(\beta^2)\} \Delta_3 + \{\cos^2(\alpha^1) + \cos^2(\alpha^2) + \cos^2(\beta^1) + \cos^2(\beta^2)\} \Delta_2], \quad (4.55)$$

IV. Spectral Shift from the Antisymmetric state :

$$\delta E_A = \frac{\mu^2 P}{8\pi^2} [\{\cos(\alpha^1) \cos(\alpha^2) + \cos(\beta^1) \cos(\beta^2)\} \Delta_3 - \{\cos^2(\alpha^1) + \cos^2(\alpha^2) + \cos^2(\beta^1) + \cos^2(\beta^2)\} \Delta_2]. \quad (4.56)$$

4.3.2 Bethe regularised Lamb Shift Spectra

1. Bethe regularised energy shift from Ground & Excited states:

In this context, the ground and excited state of this two entangled atom will not finally contribute to the **RCPI**. To show this in detail for completeness we evaluate the contributions coming from the ground state and excited state expectation values, as given by:

$$\delta E_Y = \{\cos(\alpha^1) \cos(\beta^1) + \cos(\alpha^2) \cos(\beta^2)\} \mathcal{J}_Y^{(1)} + \{\cos^2(\alpha^1) + \cos^2(\alpha^2) + \cos^2(\beta^1) + \cos^2(\beta^2)\} \mathcal{J}_Y^{(2)} \quad \forall Y = (G, E), \quad (4.57)$$

where $\mathcal{J}_Y^{(1)}(L)$ and $\mathcal{J}_Y^{(2)} \forall Y = (G, E)$, are two integrals representing the ground and excited state contributions defined as:

$$\mathcal{J}_G^{(1)} = -\mathcal{J}_E^{(1)} = \frac{\mu^2 P}{8\pi^2} \Delta_1, \quad (4.58)$$

$$\mathcal{J}_G^{(2)} = \mathcal{J}_E^{(2)} = -\frac{\mu^2 P}{8\pi^2} \Delta_2. \quad (4.59)$$

Here, the integrals $\mathcal{J}_G^{(2)}$ and $\mathcal{J}_E^{(2)}$ are divergent. Additionally, in the limit $\omega \rightarrow \infty$, one can approximate the factor appearing in the denominator of both the integrands as, $(1 - e^{-2\pi k\omega}) \sim 1$, where we have considered only the leading order contribution in the Taylor expansion. Consequently, the integrals $\mathcal{J}_G^{(2)}$ and $\mathcal{J}_E^{(2)}$ can be recast as:

$$\mathcal{J}_G^{(1)} = -\mathcal{J}_E^{(1)} = \frac{\mu^2 P}{8\pi^2} \int_{-\infty}^{\infty} d\omega \frac{2\omega_0 \omega}{(\omega + \omega_0)(\omega - \omega_0)}, \quad (4.60)$$

$$\mathcal{J}_G^{(2)} = \mathcal{J}_E^{(2)} = -\frac{\mu^2 P}{8\pi^2} \int_{-\infty}^{\infty} d\omega \frac{\omega^2}{(\omega + \omega_0)(\omega - \omega_0)}. \quad (4.61)$$

However, after simplification both the integrals $\mathcal{J}_G^{(2)}$ and $\mathcal{J}_E^{(2)}$ are till now giving infinite contributions. But introducing a cut-off regulator ω_c following the Bethe's cut-off regularisation method [31, 62, 63] we get:

$$\mathcal{J}_G^{(2)}(\omega_c) = \mathcal{J}_A^{(2)}(\omega_c) = -\frac{\mu^2 P}{8\pi^2} \int_{-\omega_c}^{\omega_c} d\omega \frac{\omega^2}{(\omega + \omega_0)(\omega - \omega_0)} = -\frac{\mu^2 P}{4\pi^2} \left[\omega_c - \omega_0 \tanh^{-1} \left(\frac{\omega_c}{\omega_0} \right) \right]. \quad (4.62)$$

Similarly after introducing the cut-off for the other sets of integrals $\mathcal{J}_G^{(1)}$ and $\mathcal{J}_E^{(1)}$ we get the following finite contribution:

$$\mathcal{J}_G^{(1)}(\omega_c) = -\mathcal{J}_E^{(1)}(\omega_c) = \frac{\mu^2 P}{8\pi^2} \int_{-\omega_c}^{\omega_c} d\omega \frac{2\omega_0 \omega}{(\omega + \omega_0)(\omega - \omega_0)} = 0, \quad (4.63)$$

Consequently, the analytical expressions for the energy shift computed from the ground and excited state wave function of the two entangled atoms can be expressed as ⁷:

$$\delta E_G(\omega_c) = -\frac{\mu^2}{8\pi^2} \{\cos^2(\alpha^1) + \cos^2(\alpha^2) + \cos^2(\beta^1) + \cos^2(\beta^2)\} \left[\omega_c - \omega_0 \tanh^{-1} \left(\frac{\omega_c}{\omega_0} \right) \right], \quad (4.64)$$

$$\delta E_E(\omega_c) = -\frac{\mu^2}{8\pi^2} \{\cos^2(\alpha^1) + \cos^2(\alpha^2) + \cos^2(\beta^1) + \cos^2(\beta^2)\} \left[\omega_c - \omega_0 \tanh^{-1} \left(\frac{\omega_c}{\omega_0} \right) \right]. \quad (4.65)$$

Finally, to show the cut-off independence in the final result we take a limit $\omega_c \ll \omega_0$ for which we get:

$$\tanh^{-1} \left(\frac{\omega_c}{\omega_0} \right) = \left(\frac{\omega_c}{\omega_0} \right) + \underbrace{\frac{1}{3} \left(\frac{\omega_c}{\omega_0} \right)^3 + \frac{1}{5} \left(\frac{\omega_c}{\omega_0} \right)^5 + \dots}_{\text{Negligible contribution for } \omega_c \ll \omega_0} \approx \left(\frac{\omega_c}{\omega_0} \right). \quad (4.66)$$

Negligible contribution for $\omega_c \ll \omega_0$

⁷The principal value of the Integrals appearing here is $P = 1/2$.

As a result, the spectral shift from the ground and excited shift can be written as:

$$\delta E_G = 0 = \delta E_E . \quad (4.67)$$

This directly implies that in Lamb Shift spectroscopy ground and excited states for two entangled atoms in OQS will not contribute.

2. Bethe regularised energy shift from Symmetric & Antisymmetric states:

Now, from the present analysis we observe that the **RCPI** between the two entangled atoms in the De Sitter background is being contributed only by the symmetric and antisymmetric part of the Hamiltonian H_{LS} as it consists of the term $f(\omega, L/2)$, which contains a measure of the Euclidean distance L between the two entangled atoms. This actually contributes in the shift in the energy levels or more precisely the inter atomic interaction energy computed from the symmetric ($|S\rangle$) and antisymmetric ($|A\rangle$) quantum state constructed out of two entangled atoms in De Sitter space. On the other hand, we have already seen that there is no such term present in the shift in the energy levels between the ground state and excited state constructed solely from the quantum states $|G\rangle$ and $|E\rangle$ for two atoms. This is appearing due to the non inter atomic interaction appearing between the uncorrelated two atomic quantum states in the second order of perturbation theory of OQS under consideration in this work. The presence of the Euclidean distance dependent term in the expression of the symmetric and the antisymmetric part of the shift in the energy level defines the gradient of a potential between the two atoms in the curved De Sitter space and is the manifestation of the **RCPI** between them. Therefore, only the terms which contributes towards the **RCPI** between the two entangled atoms are given by the following expression:

$$\delta E_X(L) = \{ \cos(\alpha^1) \cos(\alpha^2) + \cos(\beta^1) \cos(\beta^2) \} \mathcal{I}_X^{(1)}(L) + \{ \cos^2(\alpha^1) + \cos^2(\alpha^2) + \cos^2(\beta^1) + \cos^2(\beta^2) \} \mathcal{I}_X^{(2)} \quad \forall X = (S, A), \quad (4.68)$$

where $\mathcal{I}_X^{(1)}(L)$ and $\mathcal{I}_X^{(2)} \quad \forall X = (S, A)$, are two integrals representing the symmetric and antisymmetric contributions in **RCPI** are defined as:

$$\mathcal{I}_S^{(1)} = -\mathcal{I}_A^{(1)} = -\frac{\mu^2 P}{8\pi^2} \Delta_3, \quad (4.69)$$

$$\mathcal{I}_S^{(2)} = \mathcal{I}_A^{(2)} = -\frac{\mu^2 P}{8\pi^2} \Delta_2. \quad (4.70)$$

Here, the integrals $\mathcal{I}_S^{(2)}$ and $\mathcal{I}_A^{(2)}$ are divergent, which are explicitly appearing in the expression which contributes in the shift in the energy level computed from the symmetric and antisymmetric quantum states of two entangled atoms. Additionally, we consider the limiting situation where $\omega \rightarrow \infty$, which is physically consistent with the present model of open quantum two entangled atomic system. As a result, one can approximate the factor appearing in the denominator of both the integrands as, $(1 - e^{-2\pi k\omega}) \sim 1$. Consequently, the integrals $\mathcal{I}_S^{(2)}$ and $\mathcal{I}_A^{(2)}$ can be recast as:

$$\mathcal{I}_S^{(1)}(L) = -\mathcal{I}_A^{(1)}(L) = -\frac{\mu^2 P}{4\pi^2} \int_0^\infty d\omega \frac{2\omega^2 f(\omega, L/2)}{(\omega + \omega_0)(\omega - \omega_0)}, \quad (4.71)$$

$$\mathcal{I}_S^{(2)} = \mathcal{I}_A^{(2)} = -\frac{\mu^2 P}{4\pi^2} \int_0^\infty d\omega \frac{\omega^2}{(\omega + \omega_0)(\omega - \omega_0)}. \quad (4.72)$$

Now even after simplification both the integrals $\mathcal{I}_S^{(2)}$ and $\mathcal{I}_A^{(2)}$ are still divergent. However, by introducing a cut-off regulator ω_c in the upper limit of the integral following the Bethe's cut-off regularisation method [31, 62, 63] we get:

$$\mathcal{I}_S^{(2)}(\omega_c) = \mathcal{I}_A^{(2)}(\omega_c) = -\frac{\mu^2 P}{4\pi^2} \int_0^{\omega_c} d\omega \frac{\omega^2}{(\omega + \omega_0)(\omega - \omega_0)} = -\frac{\mu^2 P}{4\pi^2} \left[\omega_c - \omega_0 \tanh^{-1} \left(\frac{\omega_c}{\omega_0} \right) \right]. \quad (4.73)$$

On the other hand, the analytical expression for the integrals $\mathcal{I}_S^{(1)}$ and $\mathcal{I}_A^{(1)}$ can be written as:

$$\mathcal{I}_S^{(1)}(L) = -\mathcal{I}_A^{(1)}(L) = -\frac{\mu^2 P}{4\pi^2} \int_0^\infty d\omega \frac{2\omega^2 f(\omega, L/2)}{(\omega + \omega_0)(\omega - \omega_0)} = -\frac{\mu^2 P}{4\pi} \frac{1}{L\sqrt{1 + \left(\frac{L}{2k}\right)^2}} \cos \left(2\omega_0 k \sinh^{-1} \left(\frac{L}{2k} \right) \right), \quad (4.74)$$

Consequently, the analytical expressions for the interatomic interaction energy shift appearing due to symmetric and anti-symmetric **RCPI** interaction are given by ⁸:

$$\delta E_S(L) = -\frac{\mu^2}{8\pi} \left\{ \frac{\{\cos(\alpha^1)\cos(\alpha^2) + \cos(\beta^1)\cos(\beta^2)\}}{L\sqrt{1 + \left(\frac{L}{2k}\right)^2}} \cos\left(2\omega_0 k \sinh^{-1}\left(\frac{L}{2k}\right)\right) + \frac{\{\cos^2(\alpha^1) + \cos^2(\alpha^2) + \cos^2(\beta^1) + \cos^2(\beta^2)\}}{\pi} \left[\omega_c - \omega_0 \tanh^{-1}\left(\frac{\omega_c}{\omega_0}\right)\right] \right\}, \quad (4.75)$$

$$\delta E_A(L) = \frac{\mu^2}{8\pi} \left\{ \frac{\{\cos(\alpha^1)\cos(\alpha^2) + \cos(\beta^1)\cos(\beta^2)\}}{L\sqrt{1 + \left(\frac{L}{2k}\right)^2}} \cos\left(2\omega_0 k \sinh^{-1}\left(\frac{L}{2k}\right)\right) - \frac{\{\cos^2(\alpha^1) + \cos^2(\alpha^2) + \cos^2(\beta^1) + \cos^2(\beta^2)\}}{\pi} \left[\omega_c - \omega_0 \tanh^{-1}\left(\frac{\omega_c}{\omega_0}\right)\right] \right\}. \quad (4.76)$$

Finally, to show the cut-off independence in the final result we take a limit $\omega_c \ll \omega_0$ for which we get:

$$\tanh^{-1}\left(\frac{\omega_c}{\omega_0}\right) = \left(\frac{\omega_c}{\omega_0}\right) + \underbrace{\frac{1}{3}\left(\frac{\omega_c}{\omega_0}\right)^3 + \frac{1}{5}\left(\frac{\omega_c}{\omega_0}\right)^5 + \dots}_{\text{Negligible contribution for } \omega_c \ll \omega_0} \approx \left(\frac{\omega_c}{\omega_0}\right). \quad (4.77)$$

As a result, we get the following simplified result for the inter atomic interaction energy shift:

$$\delta E_S(L) = -\delta E_A(L) = -\frac{\mu^2}{8\pi} \frac{\{\cos(\alpha^1)\cos(\alpha^2) + \cos(\beta^1)\cos(\beta^2)\}}{L\sqrt{1 + \left(\frac{L}{2k}\right)^2}} \cos\left(2\omega_0 k \sinh^{-1}\left(\frac{L}{2k}\right)\right). \quad (4.78)$$

In fig. (3) and fig. (4), we have explicitly shown the behaviour of Lamb Shift computed from the symmetric and antisymmetric wave function with respect to the surface gravity of our De Sitter Universe. The characteristics of these figures are appended below:

- **Set-I (For $L = 1$):**

In fig (3(a)), if we increase the surface gravity then the Lamb Shift computed from the symmetric wave function for two entangled atoms decrease and after a certain point it saturates. Here it is important to note that, we have fixed the Euclidean distance at $L = 1$ and also we have fixed the Bethe regularisation cut-off at $\omega_c = 0.01$ and also the reference frequency at $\omega_0 = 0.1$. Here for this plot $\omega_c/\omega_0 = 0.1 < 1$ approximation perfectly holds good. Additionally, we observe that in the region $L \geq k$ the energy shift decrease very fast with respect to the surface gravity. After that, in the region $L < k$ the Lamb Shift saturates with respect to the surface gravity of De Sitter space. For clear demonstration we fix the position of the two detectors at fixed angles, which are actually characterised by the direction cosines (α^1, β^1) and (α^2, β^2) . For our analysis, we fix all the angles at $0, \pi/8, \pi/4, 3\pi/8$ respectively for all the plots in this paper. In fig (3(b)), the behaviour of the Lamb Shift computed from the antisymmetric wave function for two entangled atoms increase with respect to the surface gravity of the De Sitter space up to a certain point when $L \geq k$ and after that the behaviour saturates in the region $L < k$. Comparing both the plots obtained from the symmetric and antisymmetric wave functions we observe that the behaviour of the Lamb Shift spectra exactly opposite with respect to the surface gravity. From the plot we found that for the symmetric case the energy shift remains negative after saturation. On the other hand, for the antisymmetric case after saturation energy shift remains positive.

- **Set-II (For $L = 100$):**

In fig (3(c)), if we increase the surface gravity then the Lamb Shift computed from the symmetric wave function for two entangled atoms then it shows oscillating behaviour with irregular period initially upto a certain region when $L > k$. After that it shows gradual increment and then it saturates to a certain value. Here it is important to note that, we have fixed the Euclidean distance at $L = 100$ and also we have fixed the other parameters at the same values as mentioned in the

⁸The principal value of the Integrals appearing here ids $P = 1/2$.

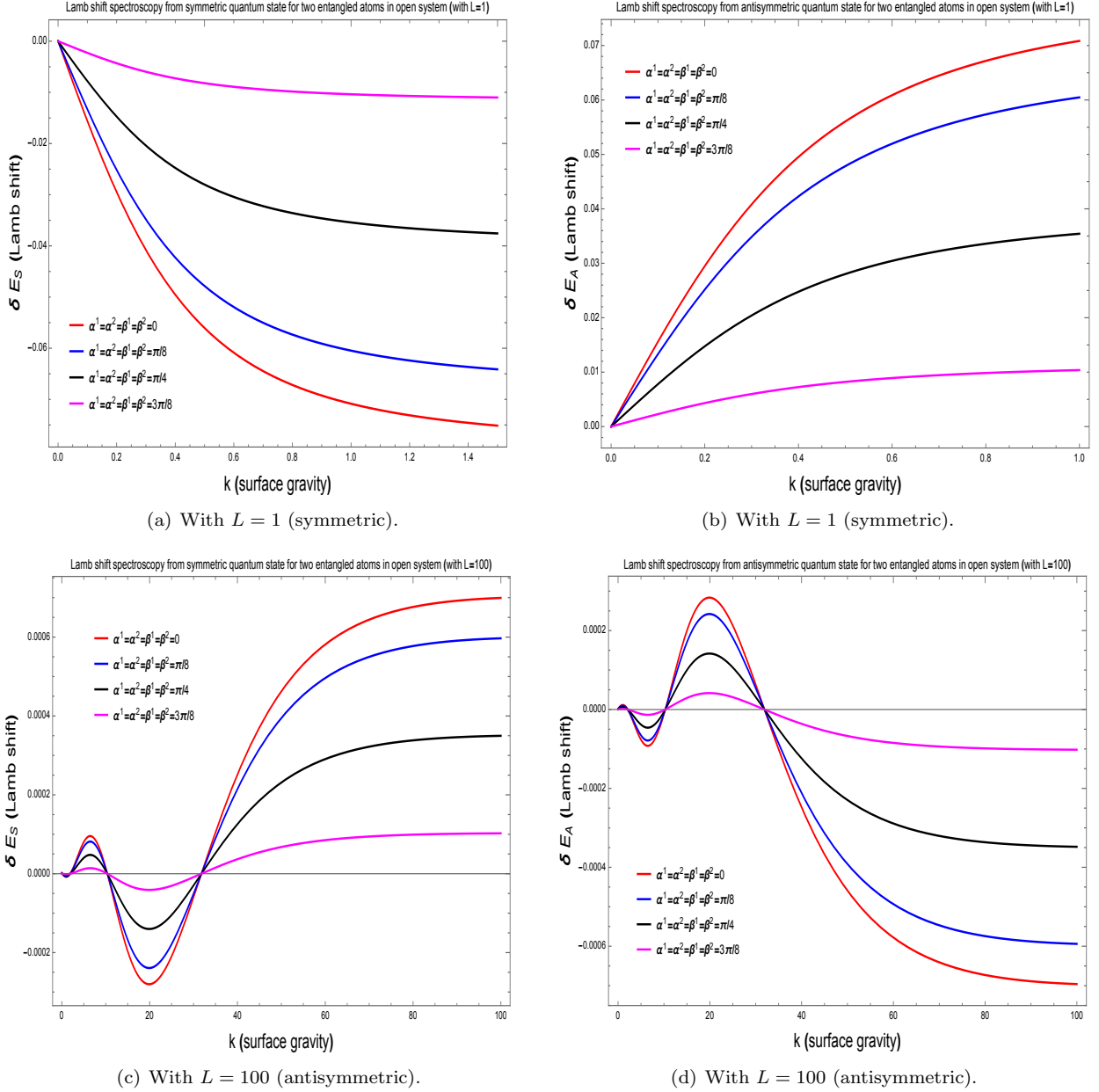
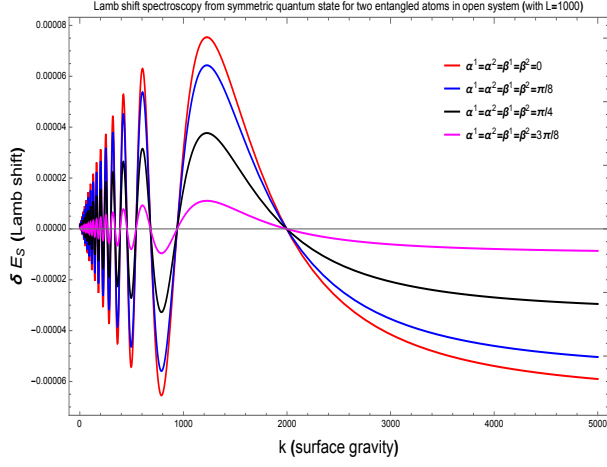


Figure 3. Lamb shift spectroscopy from symmetric (δE_S) and antisymmetric (δE_A) quantum states with respect to surface gravity of universe (k) in an OQS of two entangled atoms.

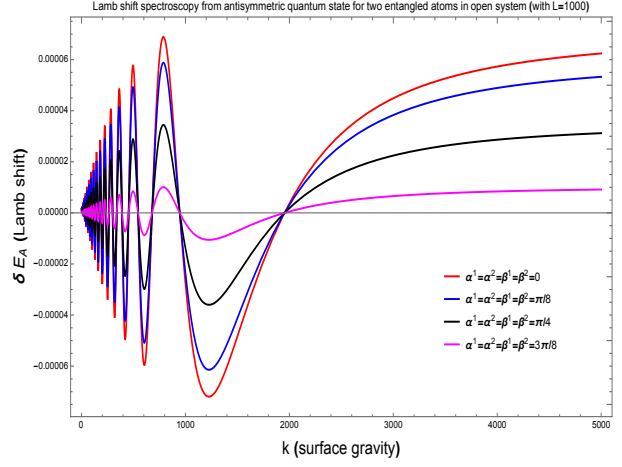
previous case and will keep these numbers fixed throughout this paper. In fig (3(d)), we did the similar analysis with the antisymmetric wave function and we get the exactly opposite behaviour in this case compared to to behaviour obtained for the symmetric case. The main characteristics we found that for the symmetric case the spectral shift initially a positive value with very small magnitude and then it takes negative magnitude for a certain period. After that it saturates to a positive value. On the other hand, for the antisymmetric case after saturation spectral shift becomes negative and takes the same magnitude that we have achieved for the symmetric case.

- **Set-III (For $L = 1000$):**

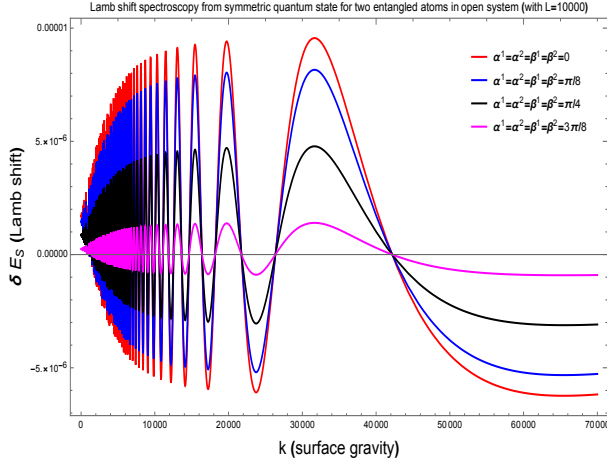
In fig (4(a)), if we increase the surface gravity then the Lamb Shift computed from the symmetric wave function for two entangled atoms then it shows highly oscillating behaviour with irregular period and increase in magnitude up to a certain region when $L \leq k$. After that the oscillation decrease with the increase of the surface gravity and for $L > k$ it saturates to a



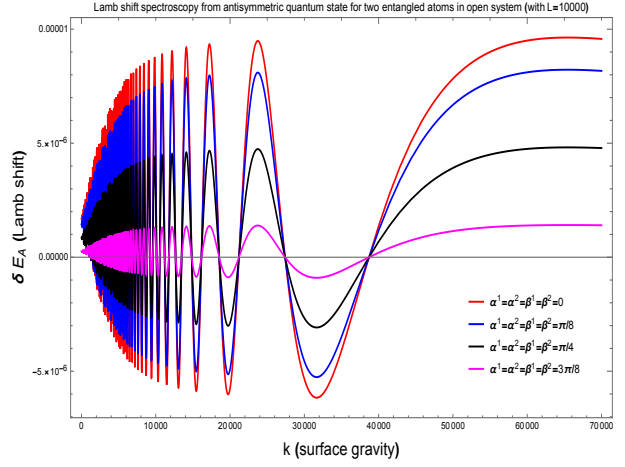
(a) With $L = 1000$ (symmetric).



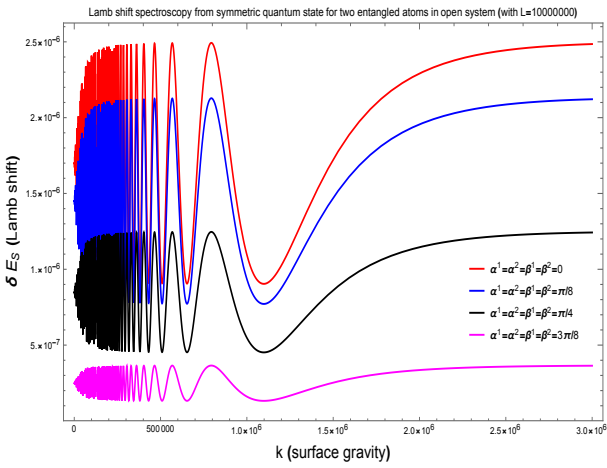
(b) With $L = 1000$ (antisymmetric).



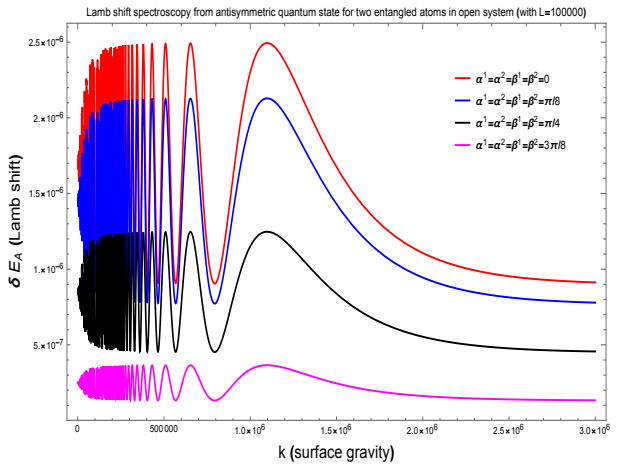
(c) With $L = 10000$ (symmetric).



(d) With $L = 10000$ (antisymmetric).



(e) With $L = 100000$ (symmetric).



(f) With $L = 100000$ (antisymmetric).

Figure 4. Lamb shift spectroscopy from symmetric (δE_S) and antisymmetric (δE_A) quantum states with respect to surface gravity of universe (k) in an OQS of two entangled atoms.

certain negative value. Here it is important to note that, we have fixed the Euclidean distance at $L = 1000$. In fig (4(b)), we did exactly similar analysis with the antisymmetric wave function where we get the opposite behaviour in this case compared to the behaviour obtained for the symmetric case. In this case for $L > k$ we found that the magnitude of the energy shift saturates to positive value.

- **Set-IV (For $L = 10000$):**

In fig (4(c)) and fig (4(d)), we found the similar behaviour as obtained for $L = 1000$. In the region $L \leq k$ only the significant difference we found that the amplitude and the frequency of the oscillations are larger for all the positions of the two detectors compared to the previous case. Also we found that in both the plots the saturation in the energy shift appears at larger values of the surface gravity.

- **Set-V (For $L = 100000$):**

In fig (4(e)) and fig (4(f)), we observe that initially in the region $L \leq k$ the highly oscillating behaviour are exactly same for both symmetric and antisymmetric case. In both the situations the saturation value of the energy shift achieve positive value.

After studying all of these plots we get the following characteristic features:

1. If we increase the value of the Euclidean distance (L), then the oscillation in the Lamb Shift increase with respect to the surface gravity in the region $L \leq k$.
2. We also found that the magnitude of the saturation value of the Lamb Shift decrease with increasing value of the Euclidean distance (L) in the region $L > k$.

Further, it is important to note that, the final results of the inter atomic energy shifts depend on the background De Sitter metric through the following relation:

$$k = \sqrt{g_{00}}\alpha = \sqrt{\alpha^2 - r^2} = \sqrt{\frac{3}{\Lambda} - r^2}. \quad (4.79)$$

This directly implies that the parameter k is directly related to the positive cosmological constant of De Sitter space. Consequently, one can theoretically probe De Sitter space using a pair of entangled atoms in OQS in presence of **RCPI**. Here it is important to note that the result obtained for inter atomic energy level shift for two detectors (two entangled atoms) can be interpreted as the energy level shift obtained for a single detector immersed in a thermal bath with temperature:

$$T = \frac{1}{2\pi k} = \frac{1}{2\pi\sqrt{\alpha^2 - r^2}} = \frac{1}{2\pi\sqrt{\frac{3}{\Lambda} - r^2}}, \quad (4.80)$$

which is interpreted as the *Unruh Temperature*. A freely falling observer under a steady acceleration observes this temperature in de-Sitter space. In this case, the inter atomic interaction exhibits non thermal behaviour and carrying non thermal fluctuation.

Now, to understand the detailed physical features of the obtained result for energy level shift from **RCPI** in De Sitter space we consider two limiting situations, as given by:

1. **Case I:**

When the inter atomic distance is much larger than characteristic length scale k i.e. $L \gg k$. In this case the two entangled atomic system is placed near to the cosmological horizon. In this limit, the energy-level shift from **RCPI** can be simplified as:

$$\delta E_S(L \gg k) = -\delta E_A(L \gg k) = -\frac{\mu^2 k P \{ \cos(\alpha^1) \cos(\alpha^2) + \cos(\beta^1) \cos(\beta^2) \}}{4\pi L^2} \cos\left(2\omega_0 k \sinh^{-1}\left(\frac{L}{2k}\right)\right). \quad (4.81)$$

This result shows that the energy level shift is non trivially dependent on the parameter k because of that fact that in this case curvature of the De Sitter space is significant. Here we observe that the **RCPI** in the limit $L \gg k$ falls as $1/L^2$, which shows that the first order correction to the energy maintains the inverse square law in De Sitter space.

2. **Case II:**

When the inter atomic distance is of the order of the characteristic length scale k i.e. $L \sim k$. In this case the two entangled atomic system is placed exactly at the cosmological horizon. In this limit, the energy-level shift from **RCPI** can be simplified as:

$$\delta E_S(L \sim k) = -\delta E_A(L \sim k) = -\frac{\mu^2 P \{ \cos(\alpha^1) \cos(\alpha^2) + \cos(\beta^1) \cos(\beta^2) \}}{4\sqrt{5}\pi L} \cos\left(2\omega_0 L \sinh^{-1}\left(\frac{1}{2}\right)\right). \quad (4.82)$$

This result shows that the energy level shift is dependent on the parameter k because of that fact that in this case curvature of the De Sitter space is comparable to the atomic distance, which will give rise to the following constraint condition:

$$L = 2r \sin\left(\frac{\theta - \theta'}{2}\right) \sim k. \quad (4.83)$$

Here we observe that the **RCPI** in the limit $L \sim k$ falls as $1/L$, which shows that the first order correction to the energy maintains the inverse law in Minkowski space.

3. **Case III:**

When the inter atomic distance is much smaller than characteristic length scale k i.e. $L \ll k$. In this case the two entangled atomic system is placed far from the cosmological horizon. In this case it is possible to find a local inertial frame of reference where all physical principles coincides with that in Minkowski space. In this limit, the energy-level shift from **RCPI** can be simplified as:

$$\delta E_S(L \ll k) = -\delta E_A(L \ll k) = -\frac{\mu^2 P \{\cos(\alpha^1) \cos(\alpha^2) + \cos(\beta^1) \cos(\beta^2)\}}{8\pi L} \cos(\omega_0 L). \quad (4.84)$$

This result shows that the energy level shift is independent on the parameter k because of that fact that in this case curvature of the De Sitter space is negligibly small. This result exactly matches with the result obtained for Minkowski space. Here we observe that the **RCPI** in the limit $L \sim k$ falls as $1/L$, which shows that the first order correction to the energy maintains the inverse law in De Sitter space and exact Minkowski space.

Additionally, in all of these physical limits we found the following features:

- **Feature I:**

We also observe that the **RCPI** contains the Eulerian angles within it in both the limiting results. This suggests that the shift in energy is described by the Eulerian angles which means that the **RCPI** is a function of the direction along which the spin of the atoms is directed along. In De Sitter space the **RCPI** is dependent on how the spin of the two atoms are oriented. The orientation of the spins of the two atoms which is determined by the Euler angles of rotation α^i , β^i and $\gamma^i \forall i = 1, 2, 3$, quantifies the **RCPI** along arbitrary direction of spin projection.

- **Feature II:**

The Euler angles of rotation determines the manifestation of the thermal environment that the atoms see in their comoving frame. Furthermore, we observe that the pre-factors in the energy shifts determines the parameter k associated with the temperature [28, 45, 49, 52, 54] of the thermal bath:

$$T = \frac{1}{2\pi k} = \sqrt{T_{\text{GH}}^2 + T_{\text{Unruh}}^2} = \frac{1}{2\pi\alpha} \sqrt{1 + \frac{r^2}{(\alpha^2 - r^2)}} = \frac{1}{2\pi} \sqrt{\frac{\Lambda}{3}} \sqrt{1 + \frac{r^2}{(\frac{3}{\Lambda} - r^2)}}, \quad (4.85)$$

where, the **Gibbons-Hawking temperature** and **Unruh temperature** are defined through the following expressions:

$$T_{\text{GH}} = \frac{1}{2\pi\alpha} = \frac{1}{2\pi} \sqrt{\frac{\Lambda}{3}}, \quad (4.86)$$

$$T_{\text{Unruh}} = \frac{a}{2\pi} = \frac{1}{2\pi\alpha} \frac{r}{\sqrt{\alpha^2 - r^2}} = \frac{1}{2\pi} \sqrt{\frac{\Lambda}{3}} \frac{r}{\sqrt{\frac{3}{\Lambda} - r^2}}. \quad (4.87)$$

with the proper acceleration given by:

$$a = \frac{1}{\alpha} \frac{r}{\sqrt{\alpha^2 - r^2}} = \sqrt{\frac{\Lambda}{3}} \frac{r}{\sqrt{\frac{3}{\Lambda} - r^2}}, \quad (4.88)$$

which is defined in the co-moving frame of the two entangled atoms for the given OQS under consideration.

- **Feature III:**

Now, it is important to note that in De Sitter space the curvature can be quantified though the Ricci scalar, which can be further expressed in terms of the cosmological constant as:

$$R_{\text{DS}} = \frac{12}{\alpha} = 12\sqrt{\frac{\Lambda}{3}}. \quad (4.89)$$

As a result, the **Gibbons-Hawking temperature** and **Unruh temperature** can be expressed in terms of the Curvature of De Sitter space as:

$$T_{\text{GH}} = \frac{R_{\text{DS}}}{24\pi}, \quad (4.90)$$

$$T_{\text{Unruh}} = \frac{R_{\text{DS}}}{24\pi} \frac{Rr}{\sqrt{144 - (R_{\text{DS}}r)^2}}. \quad (4.91)$$

Consequently, the temperature of thermal bath can be expressed in terms of the curvature of De Sitter space as:

$$T = \frac{1}{2\pi k} = \frac{R_{\text{DS}}}{24\pi} \frac{1}{\sqrt{1 - \left(\frac{R_{\text{DS}}r}{12}\right)^2}}. \quad (4.92)$$

In this case, the **RCPI** can be expressed in terms of the curvature of the De Sitter space as:

$$\begin{aligned} \delta E_S(L) = -\delta E_A(L) = & -\frac{\mu^2 P}{8\pi} \frac{\{\cos(\alpha^1) \cos(\alpha^2) + \cos(\beta^1) \cos(\beta^2)\}}{L \sqrt{1 + \left(\frac{LR_{\text{DS}}}{24\sqrt{1 - \left(\frac{R_{\text{DS}}r}{12}\right)^2}}\right)^2}} \\ & \times \cos\left(\frac{24\omega_0}{R_{\text{DS}}} \sqrt{1 - \left(\frac{Rr}{12}\right)^2} \sinh^{-1}\left(\frac{LR_{\text{DS}}}{24\sqrt{1 - \left(\frac{Rr}{12}\right)^2}}\right)\right). \end{aligned} \quad (4.93)$$

In the limiting situation, $L \gg k = 12\sqrt{1 - \left(\frac{Rr}{12}\right)^2}/R_{\text{DS}}$, the **RCPI** can be expressed in terms of the curvature of the De Sitter space as:

$$\begin{aligned} \delta E_S \left(L \gg k = 12\sqrt{1 - \left(\frac{R_{\text{DS}}r}{12}\right)^2}/R_{\text{DS}} \right) & = -\delta E_A \left(L \gg k = 12\sqrt{1 - \left(\frac{R_{\text{DS}}r}{12}\right)^2}/R_{\text{DS}} \right) \\ & = -\frac{3\mu^2 P}{\pi} \sqrt{1 - \left(\frac{R_{\text{DS}}r}{12}\right)^2} \times \frac{\{\cos(\alpha^1) \cos(\alpha^2) + \cos(\beta^1) \cos(\beta^2)\}}{R_{\text{DS}}L^2} \\ & \times \cos\left(\frac{24\omega_0}{R_{\text{DS}}} \sqrt{1 - \left(\frac{R_{\text{DS}}r}{12}\right)^2} \sinh^{-1}\left(\frac{LR_{\text{DS}}}{24\sqrt{1 - \left(\frac{R_{\text{DS}}r}{12}\right)^2}}\right)\right). \end{aligned} \quad (4.94)$$

Here one can consider another limiting situation, where $L \sim k = 12\sqrt{1 - \left(\frac{Rr}{12}\right)^2}/R_{\text{DS}}$. For this case, **RCPI** can be expressed in terms of the curvature of the De Sitter space as:

$$\begin{aligned} \delta E_S \left(L \sim k = 12\sqrt{1 - \left(\frac{R_{\text{DS}}r}{12}\right)^2}/R \right) & = -\delta E_A \left(L \sim k = 12\sqrt{1 - \left(\frac{R_{\text{DS}}r}{12}\right)^2}/R \right) \\ & = -\frac{R_{\text{DS}}\mu^2 P \{\cos(\alpha^1) \cos(\alpha^2) + \cos(\beta^1) \cos(\beta^2)\}}{48\sqrt{5}\pi \sqrt{1 - \left(\frac{R_{\text{DS}}r}{12}\right)^2}} \cos\left(\frac{24\omega_0}{R_{\text{DS}}} \sqrt{1 - \left(\frac{R_{\text{DS}}r}{12}\right)^2} \sinh^{-1}\left(\frac{1}{2}\right)\right). \end{aligned} \quad (4.95)$$

- **Feature IV:**

Now, if we take the limit $R_{\text{DS}} \rightarrow 0$ (i.e. $\Lambda \rightarrow 0$ or $\alpha \rightarrow \infty$), then we get:

$$\lim_{R_{\text{DS}} \rightarrow 0} T_{\text{GH}} = 0, \quad \lim_{R_{\text{DS}} \rightarrow 0} T_{\text{Unruh}} = 0 \implies \lim_{R_{\text{DS}} \rightarrow 0} T = 0 \implies k \rightarrow \infty. \quad (4.96)$$

In this case the **RCPI** will be reduced to the result obtained in the limiting situation $L \ll k$, which is exactly same result as obtained for the Minkowski space.

- **Feature V:**

Here additionally it is important to note that, only if the **Unruh temperature** vanishes, the corresponding proper acceleration of static atom vanishes i.e. $a = 0$ and this can be obtained when the atoms are localised at $r = 0$. As a result, kinematic contribution will not appear in the expression for **RCPI**. However, in this case, the **RCPI** is still can be expressed in terms of the curvature of the De Sitter space, as in this case temperature of the thermal bath is quantified by the non vanishing **Gibbons-Hawking temperature** i.e.

$$T = \frac{1}{2\pi k} = T_{\text{GH}} = \frac{R}{24\pi}. \quad (4.97)$$

In this case, the **RCPI** can be expressed in terms of the curvature of the De Sitter space as:

$$\delta E_S(L) = -\delta E_A(L) = -\frac{\mu^2 P}{8\pi} \frac{\{\cos(\alpha^1) \cos(\alpha^2) + \cos(\beta^1) \cos(\beta^2)\}}{L\sqrt{1 + \left(\frac{LR_{\text{DS}}}{24}\right)^2}} \cos\left(\frac{24\omega_0}{R_{\text{DS}}} \sinh^{-1}\left(\frac{LR_{\text{DS}}}{24}\right)\right). \quad (4.98)$$

In the limiting situation, $L \gg k = 12/R_{\text{DS}}$, the **RCPI** can be expressed in terms of the curvature of the De Sitter space as:

$$\begin{aligned} \delta E_S(L \gg k = 12/R_{\text{DS}}) &= -\delta E_A(L \gg k = 12/R_{\text{DS}}) \\ &= -\frac{3\mu^2 P}{\pi} \frac{\{\cos(\alpha^1) \cos(\alpha^2) + \cos(\beta^1) \cos(\beta^2)\}}{R_{\text{DS}}L^2} \cos\left(\frac{24\omega_0}{R_{\text{DS}}} \sinh^{-1}\left(\frac{LR_{\text{DS}}}{24}\right)\right). \end{aligned} \quad (4.99)$$

Here one can consider another limiting situation, where $L \sim k = 12/R_{\text{DS}}$. For this case, **RCPI** can be expressed in terms of the curvature of the De Sitter space as:

$$\begin{aligned} \delta E_S(L \sim k = 12/R_{\text{DS}}) &= -\delta E_A(L \sim k = 12/R_{\text{DS}}) \\ &= -\frac{R_{\text{DS}} \mu^2 P}{48\sqrt{5}\pi} \frac{\{\cos(\alpha^1) \cos(\alpha^2) + \cos(\beta^1) \cos(\beta^2)\}}{R_{\text{DS}}} \cos\left(\frac{24\omega_0}{R_{\text{DS}}} \sinh^{-1}\left(\frac{1}{2}\right)\right). \end{aligned} \quad (4.100)$$

- **Feature VI:**

Now, we compare the obtained results for Lamb Shift in De Sitter space with the result corresponding to the Minkowski space. For this purpose we consider a specific situation where two static atoms are interacting with the environment, where it is represented by the massless scalar field in OQS. In this system, the two point field correlation can be expressed in terms of the Wightman function given by:

$$\begin{aligned} G^{11}(x, x') = G^{22}(x, x') &= -\frac{1}{4\pi^2} \sum_{q=-\infty}^{\infty} \frac{1}{(\tau - \tau' - i\{\frac{q}{T} + \epsilon\})^2} = -\frac{1}{4\pi^2} \sum_{q=-\infty}^{\infty} \frac{1}{(\tau - \tau' - i\{2\pi kq + \epsilon\})^2} \\ &= \frac{1}{16\pi^2 k^2} \operatorname{cosec}^2\left(\frac{\epsilon + i(\tau - \tau')}{2k}\right), \end{aligned} \quad (4.101)$$

$$\begin{aligned} G^{12}(x, x') = G^{21}(x, x') &= -\frac{1}{4\pi^2} \sum_{q=-\infty}^{\infty} \frac{1}{(\tau - \tau' - i\{\frac{q}{T} + \epsilon\})^2 - L^2} = -\frac{1}{4\pi^2} \sum_{q=-\infty}^{\infty} \frac{1}{(\tau - \tau' - i\{2\pi kq + \epsilon\})^2 - L^2} \\ &= \frac{1}{16\pi^2 kL} \left[2 \left\{ \operatorname{Floor}\left(\frac{\arg\left(\frac{\epsilon + i(t-t'+L)}{k}\right)}{2\pi}\right) - \operatorname{Floor}\left(\frac{\arg\left(\frac{\epsilon + i(t-t'-L)}{k}\right)}{2\pi}\right) \right\} \right. \\ &\quad \left. + i \left\{ \cot\left(\frac{\epsilon + i(t-t'+L)}{2k}\right) - \cot\left(\frac{\epsilon + i(t-t'-L)}{2k}\right) \right\} \right], \end{aligned} \quad (4.102)$$

where L is the Euclidean distance between two atoms in OQS. Using this Wightman function we can carry forward the similar calculation for Lamb Shift from the **RCPI** in Minkowski space, which will finally give rise to the following expression:

$$\delta E_S^{(M)} = -\delta E_A^{(M)} = -\frac{\mu^2 P \{\cos(\alpha^1) \cos(\alpha^2) + \cos(\beta^1) \cos(\beta^2)\}}{8\pi L} \cos(\omega_0 L). \quad (4.103)$$

From the above mentioned result it is clearly observed that the Lamb Shift obtained from **RCPI** in Minkowski space not containing any contribution from the temperature of the thermal bath, $T = 1/2\pi k$ and only depends on Eulerian angles and

the Euclidean distance L . Also we found that this result exactly matches with the result obtained for the case $L \ll k$ with two inertial atoms in De Sitter space. This result additionally implies that the interatomic interaction between two atoms behave differently in Minkowski space and De Sitter space. But instead of using two atoms if we use only one single atom in OQS. Then the behaviour of **RCPI** is completely indistinguishable from the perspective of computation of Lamb Shift. So two or more atomic OQS is the only feasible option using which one can precisely distinguish the behaviour of **RCPI** in Minkowski space and De Sitter space.

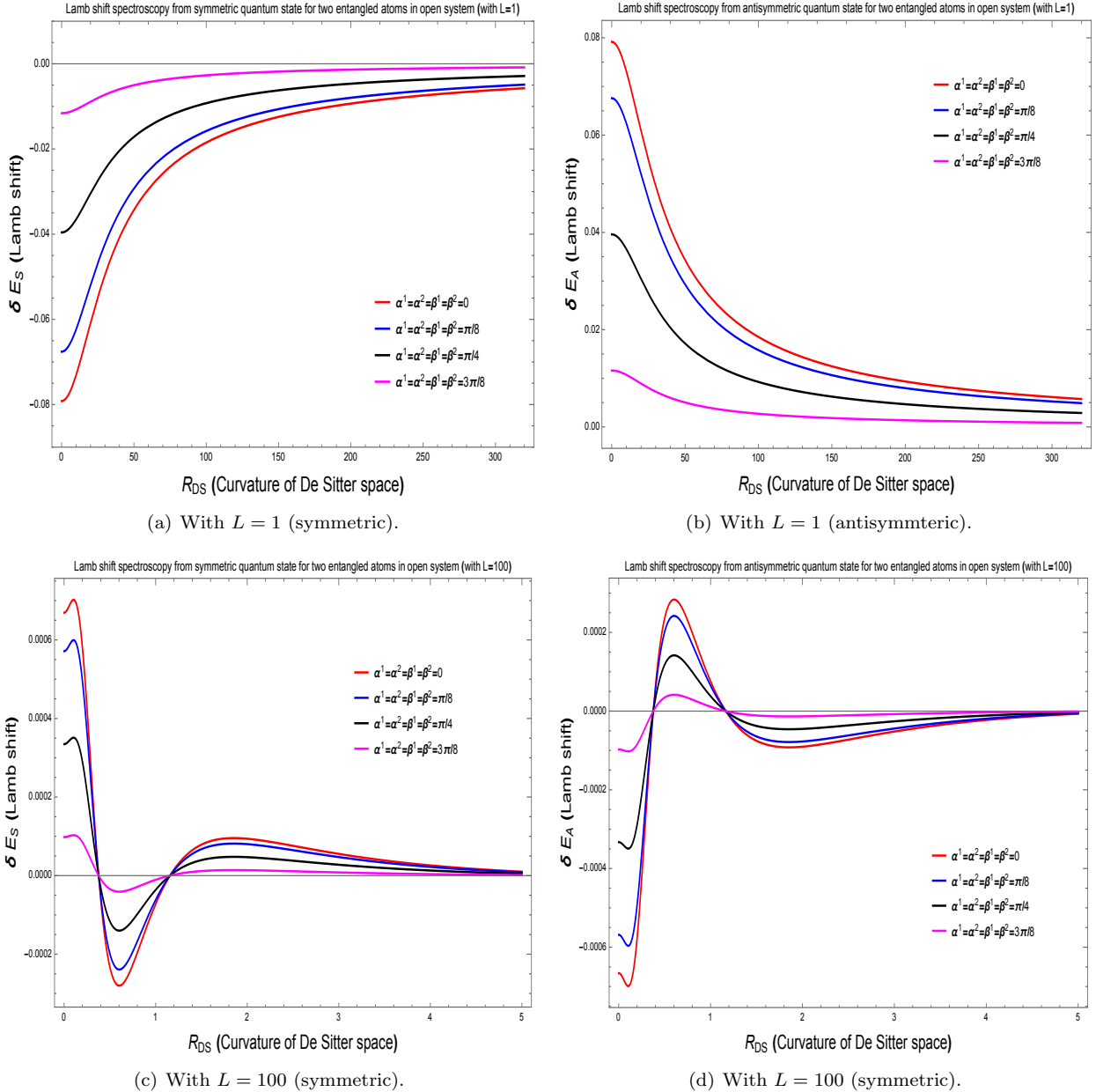
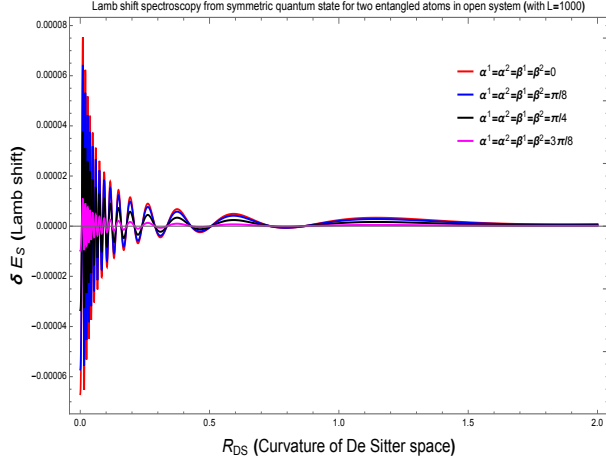
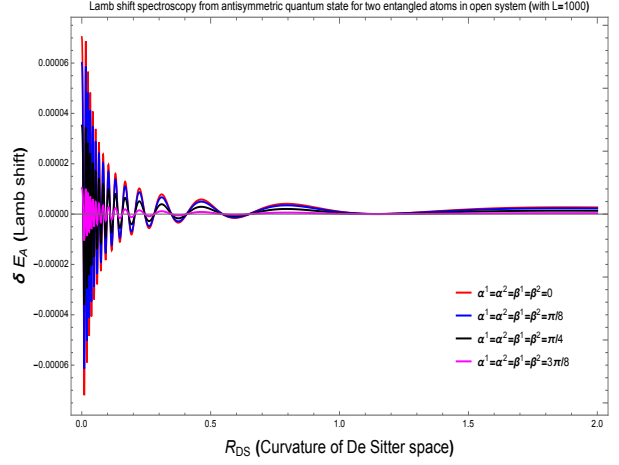


Figure 5. Lamb shift spectroscopy from symmetric (δE_S) and antisymmetric (δE_A) quantum states with respect to the curvature of De Sitter space (Ricci scalar) (R_{DS}) for OQS with respect to two entangled atoms.

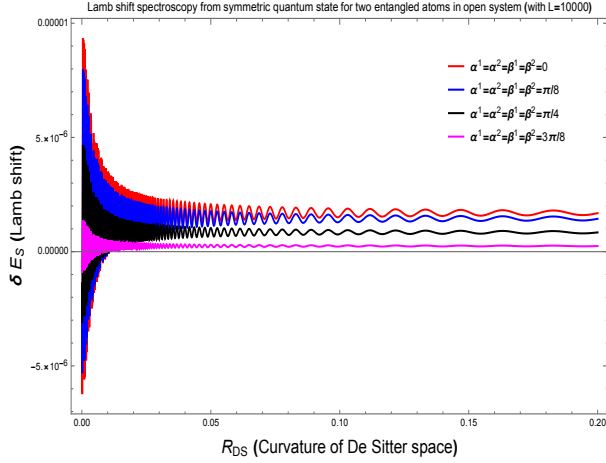
In fig. (5) and fig. (6), we have shown the behaviour of Lamb Shift computed from the symmetric and antisymmetric wave function with respect to the curvature of De Sitter Universe.



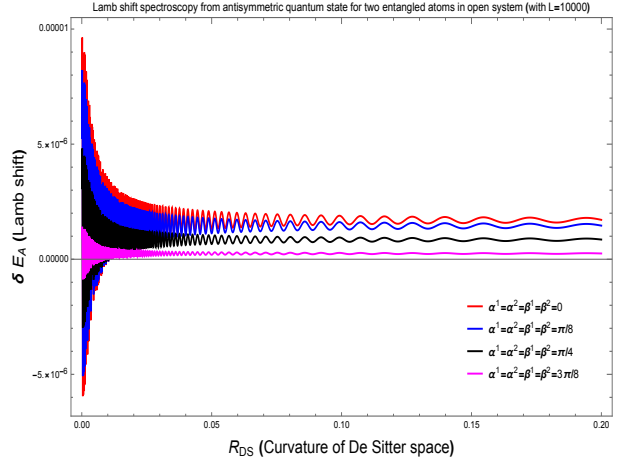
(a) With $L = 1000$ (symmetric).



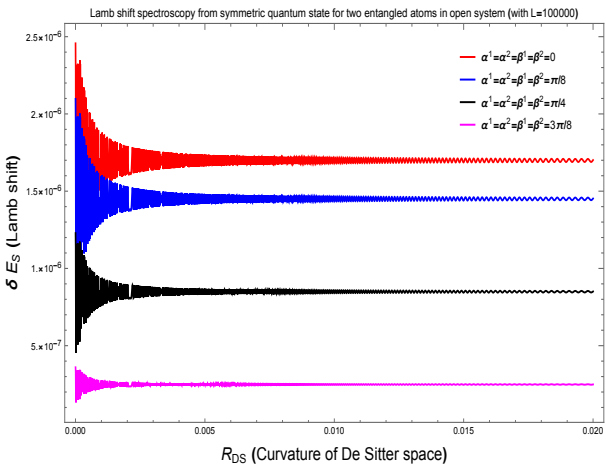
(b) With $L = 1000$ (antisymmetric).



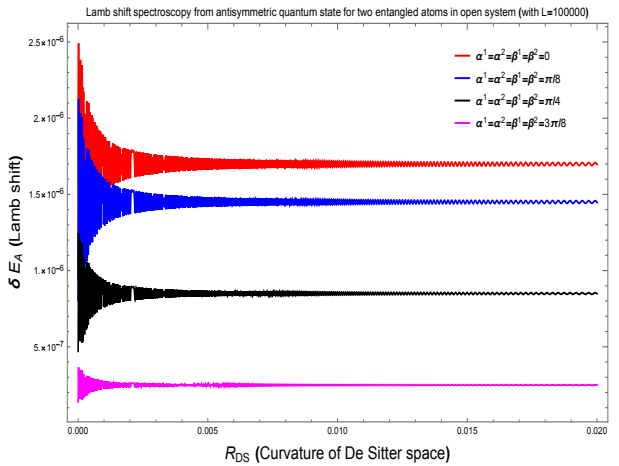
(c) With $L = 10000$ (symmetric).



(d) With $L = 10000$ (antisymmetric).



(e) With $L = 100000$ (symmetric).



(f) With $L = 100000$ (antisymmetric).

Figure 6. Lamb shift spectroscopy from symmetric (δE_S) and antisymmetric (δE_A) quantum states with respect to the curvature of De Sitter space (Ricci scalar) (R_{DS}) for OQS with respect to two entangled atoms.

5 Conclusion

In this paper, we have developed a direct connection between the geometry of our observed Universe and atomic spectroscopy for an OQS described by two entangled atoms. This fundamental idea of joining the dots between the geometry of our observed Universe (De Sitter space) and a quantum mechanical laboratory experiment helps us to know about many more unknown physical facts of our Universe. More precisely, by following this idea it is possible to quantify geometry in terms of the quantum entanglement in two body open system. Due to the present development it is possible to know about the physics of the gravitational sector of our Universe from the quantum mechanics of matter. For this purpose, without doing any terrestrial and space observations designing a laboratory atomic experiment is very useful to establish the connection between space time geometry and quantum mechanics. To summarise, in this work, we have addressed the following issues to implement the above mentioned idea:

- To begin with, we have started our discussion with an OQS characterised by two entangled atoms. We have considered the two body quantum entanglement as in this situation it is allowed to exchange energy through Lamb Shift in terms of the geometry of our background space-time. In this theoretical construction these two atomic pair represents *Unruh-De-Witt detectors*, which are considered to be conformally coupled to a background scalar field in thermal bath.
- The non-adiabatic interaction between the detectors and the thermal bath in OQS is characterised by Resonant Casimir Polder Interaction (**RCPI**), which is affected by the curvature of the background space-time hosting a fluctuating test scalar field. In this work, **RCPI** in OQS plays the key ingredient to determine the curvature of De Sitter space from the Lamb Shift atomic spectroscopy.
- In order to study the full dynamics of the two entangled atoms in OQS for any arbitrary position of two detectors, we have used a generalised Hamiltonian described by Pauli operators which include contributions from the Euler rotation angles due to projection in any arbitrary direction. This direction of projection actually playing the role of direction of observation of the atoms in the atomic detectors. In this discussion, the Lamb Shift Hamiltonian includes a term that arises from the interaction between the atomic detectors with the background test scalar field. This is the most significant term which can be experimentally probed using atomic spectroscopy to detect to geometry of our Universe. In the time dynamic of the reduced density matrix a generalised expression for this contributions actually sourced by the interaction between the atomic detectors with the test scalar field, encapsulated by the Lindbladian operator in two body quantum entangled open system. This study actually helps us to study the time evolution of the open quantum two body entangled system from the perspective of experimentalist in the atomic detector's side.
- Apart from solving this prime issue, another significant motivation of our work is to quantify the two point correlation function (i.e Wightman function) between the two entangled atoms for OQS in the De Sitter background. This results in expressions for four possibilities of the Wightman function that would now directly relate the quantum fluctuations in the background geometry of De Sitter space to the Lamb Shift in atomic spectroscopy.
- To compare between the geometrical features of De Sitter and Minkowski flat space we have also computed the Lamb Shift from the two body entangled OQS set up in the thermal state for the flat case. In the case of De Sitter space the Lamb Shift is described by inverse square power law dependence on the Euclidean distance (L), which is characterised by the length scale associated with the breaking of local inertial description of the two entangled atomic OQS. On the other hand, in the Minkowski flat case we do not get any temperature dependence and in this case the the spectroscopic Lamb Shift is described by the the inverse power law dependence on the Euclidean distance (L). From this discussion it is evident that, even both thermal Minkowski space and De Sitter space satisfy similar kind of properties and cannot explicitly discriminated by a single external probe field, by explicitly studying non adiabatic **RCPI** in the context of two atomic entangled OQS it is possible to differentiate between these two geometrical space-times.
- In this context, it is possible to interpret the underlying physics and the consequences of two atomic quantum entanglement in a open system in detail. To serve this purpose two atomic detectors are placed far from their cosmological horizon for the De Sitter case. In this case effect of two body quantum entanglement is extremely small. On the contrary, in our present methodology instead of using the idea of extracting vacuum entanglement we have actually used the concept of inter-atomic interaction in two body quantum entanglement. Additionally, it is important to note that, to implement this methodology the knowledge of the appropriate locations of two atoms are not extremely significant as the different locations of the detectors will change the Euler angles within the range:

$$0 \leq \alpha^i < \pi/2, 0 \leq \beta^i < \pi/2 \forall i = 1, 2$$

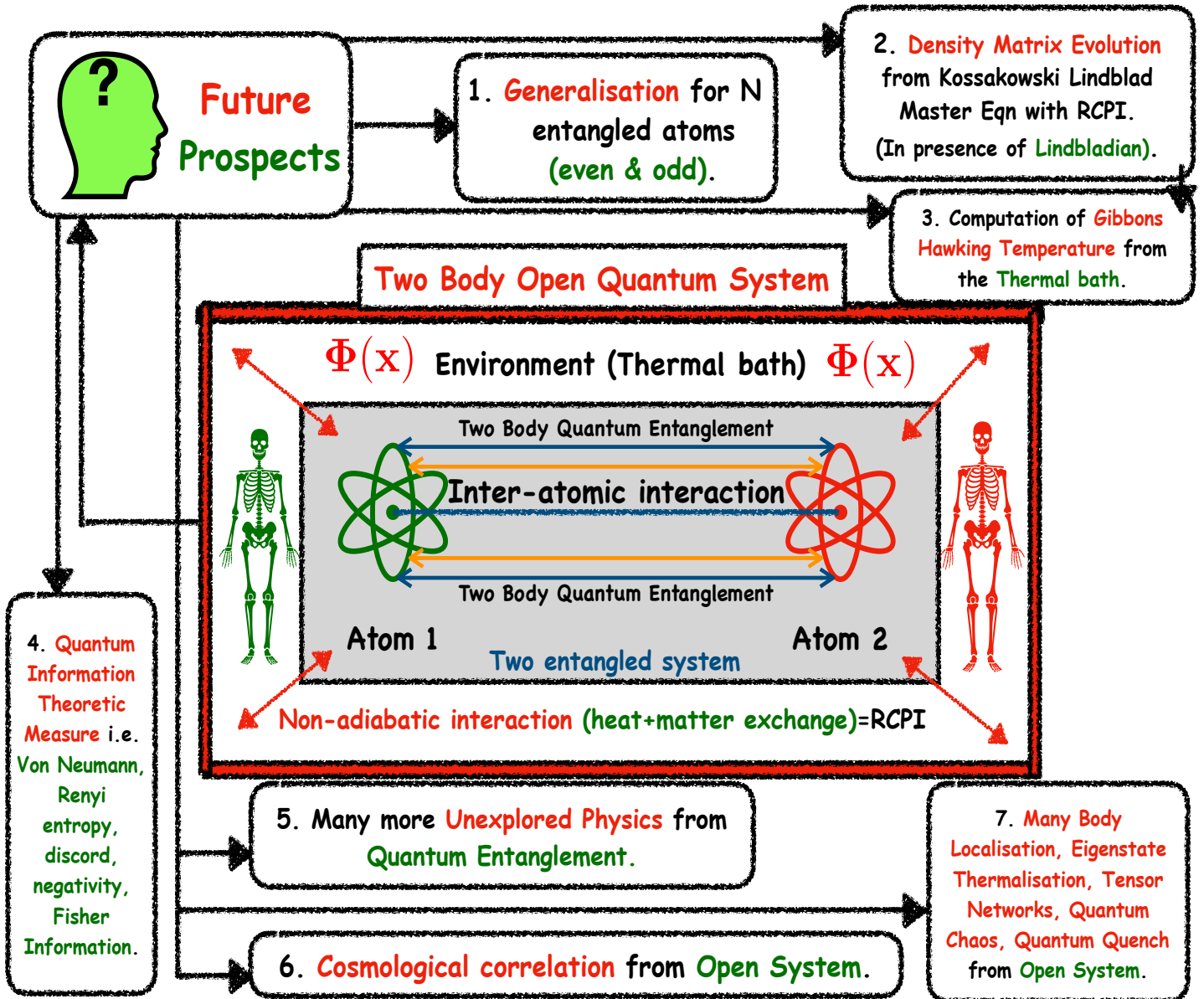


Figure 7. Schematic diagram of the future prospects of studying two body OQS.

and this will just change the overall amplitude of the Lamb Shift in atomic spectroscopy due to slight modulation in the amplitude of direction cosines lying within the following range:

$$0 < \cos(\alpha^i) < 1, \quad 0 < \cos(\beta^i) < 1 \quad \forall i = 1, 2 .$$

For this purpose the only restriction we follow that we don't put the atomic detectors on the Euler angular position $(\alpha^i, \beta^i) = (\pi/2, \pi/2) \forall i = 1, 2$ to avoid getting zero magnitude of the Lamb Shift due to getting perfectly vanishing contribution from the direction cosines.

- From the obtained result for the spectroscopic Lamb Shift it is evident that, if the geometry of the space-time is curved, particularly if it is our observed De Sitter Universe then in the context of open quantum two body entangled system the non-adiabatic inter-atomic RCPI is purely characterised by two important contributions which are appended bellow:

1. The amplitude of the Lamb Shift is mainly characterised by the L^{-2} factor, which indicates the inverse square power

law decay in the limiting situation $L \gg k$, where the inter atomic distance is larger than the characteristic length scale k .

2. Also the amplitude of the Lamb Shift is slightly controlled by an angular modulation factor, given by the following expression lying within the following window:

$$0 < \mathcal{D} = (\cos(\alpha^1) \cos(\alpha^2) + \cos(\beta^1) \cos(\beta^2)) < 2.$$

3. In the amplitude of the Lamb Shift another angular modulation factor contribute, given by the following expression lying within the following window:

$$0 < \mathcal{B} = (\cos^2(\alpha^1) + \cos^2(\alpha^2) + \cos^2(\beta^1) + \cos^2(\beta^2)) < 4.$$

This factor is appearing in the coefficient of the cut-off dependent contribution after applying Bethe regularisation procedure. However, in the limiting situation where the Bethe cut-off frequency is smaller than the natural frequency of the two entangled atomic system i.e. $\omega_c \ll \omega_0$, such contribution will not contribute in the amplitude of Lamb Shift.

- On the other hand, in case of flat space-time the amplitude of the Lamb Shift is proportional to L^{-1} , and the angular modulation factor \mathcal{D} lying within the same window as mentioned above. Most importantly, in the final expression for the Lamb Shift as appearing in the case of flat space no signature of the actual origin of quantum state i.e. whether it is non-thermal or thermal, can be observable.
- Finally, we have added a short discussion regarding the *Gibbons Hawking temperature* and *Unruh temperature* that these detectors would measure when accelerating through the De Sitter background space-time. This temperature is a manifestation of the thermal open quantum state of the conformally coupled test scalar field to the De Sitter background space-time. We have included this discussion regarding the temperature of the thermal bath as it can be directly expressed in terms of the curvature of the De Sitter space and consequently the energy shift can be expressed in terms of these fundamental quantities.

Thus, not only do we mathematically express how the dynamics of the detectors are affected by the background space-time curvature of our Universe, but also give a speculation on how this affects the detectable quantities in a laboratory spectroscopic experiment on OQS of two body quantum entanglement. This directly implies that the space-time geometry of our Universe, that is an attribute of gravitational sector can be linked to energy shifts in entangled atoms, which is an outcome of two body quantum mechanics.

The future prospects of our work are appended below:

1. In this paper we have restricted our analysis up to two entangled atoms to make a connection between curvature of De Sitter space and Lamb Shift spectroscopy. In future we will extend our discussion for multi entangled (even and odd number of atoms) OQS to understand this connection and many more unexplored physics in a more better way.
2. In this paper we have not studied explicitly the equilibrium behaviour of the reduced density matrix at very late time scale by solving the time evolution equation in presence of the Lindbladian of the particular open system under consideration. By explicitly studying this time dynamics once can actually compute the expression for the *Gibbons Hawking* temperature from open quantum field theory side and will consistency check about the expression for the temperature from gravity sector. We have a future plan to execute this work from two as well as multi entangled atoms.
3. One can also study various quantum information theoretic measure i.e. Von Newman entropy, Renyi entropy, quantum discord, logarithmic negativity, quantum Fisher information etc. from the present multi entangled open quantum theory set up.
4. Once can study many other unexplored physical outcomes of many other open quantum field theory set up in the context of two body quantum entanglement. Particularly in the context of De sitter and anti-De Sitter space how the two body interpretation of quantum entanglement in OQS is related to the physics of bulk (gravitational sector) and the boundary (CFT) one can explicitly study from the present framework.
5. Study of quantum fluctuations and related inflationary perturbations from the open quantum field theory set up is not well established yet in the cosmology literature. For this reason it is good to study the cosmological consequences from the correlation functions and comparison with the various observables from OQS to know about many more unknown physical facts.
6. Many body localisation and eigenstate thermalisation, study of tensor networks [64–66] and physics of quantum chaos [11, 13] from the open quantum set up are also unexplored issues which one can study in detail from the present set up.

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A Calculation of Lamb Shift spectroscopy from two entangled atomic OQS

In this appendix, we explicitly compute the expectation value of Lamb Shift Hamiltonian which will contribute to the atomic spectroscopy. To serve this purpose let us first express the Lamb Shift Hamiltonian in terms of the Pauli operators (defined earlier):

$$\text{Lamb shift Hamiltonian : } H_{\text{Lamb shift}} \equiv H_{LS} = \sum_{\alpha, \beta=1}^2 H_{LS}^{(\alpha\beta)} = H_{LS}^{(11)} + H_{LS}^{(22)} + H_{LS}^{(12)} + H_{LS}^{(21)}, \quad (\text{A.1})$$

where the Lamb Shift Hamiltonian corresponding to all possible allowed interaction between two atoms are represented by the following expressions:

$$\begin{aligned} H_{LS}^{(11)} = & -\frac{i}{2} [H_{11}^{(11)} \sigma_1^1 \sigma_1^1 \cos^2(\alpha^1) + H_{12}^{(11)} \sigma_1^1 \sigma_2^1 \cos(\alpha^1) \cos(\beta^1) + H_{13}^{(11)} \sigma_1^1 \sigma_3^1 \cos(\alpha^1) \cos(\gamma^1) \\ & + H_{21}^{(11)} \sigma_2^1 \sigma_1^1 \cos(\beta^1) \cos(\alpha^1) + H_{22}^{(11)} \sigma_2^1 \sigma_2^1 \cos^2(\beta^1) + H_{23}^{(11)} \sigma_2^1 \sigma_3^1 \cos(\beta^1) \cos(\gamma^1) \\ & + H_{31}^{(11)} \sigma_3^1 \sigma_1^1 \cos(\gamma^1) \cos(\alpha^1) + H_{32}^{(11)} \sigma_3^1 \sigma_2^1 \cos(\gamma^1) \cos(\beta^1) + H_{33}^{(11)} \sigma_3^1 \sigma_3^1 \cos^2(\gamma^1)], \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} H_{LS}^{(22)} = & -\frac{i}{2} [H_{11}^{(22)} \sigma_1^2 \sigma_1^2 \cos^2(\alpha^2) + H_{12}^{(22)} \sigma_1^2 \sigma_2^2 \cos(\alpha^2) \cos(\beta^2) + H_{13}^{(22)} \sigma_1^2 \sigma_3^2 \cos(\alpha^2) \cos(\gamma^2) \\ & + H_{21}^{(22)} \sigma_2^2 \sigma_1^2 \cos(\beta^2) \cos(\alpha^2) + H_{22}^{(22)} \sigma_2^2 \sigma_2^2 \cos^2(\beta^2) + H_{23}^{(22)} \sigma_2^2 \sigma_3^2 \cos(\beta^2) \cos(\gamma^2) \\ & + H_{31}^{(22)} \sigma_3^2 \sigma_1^2 \cos(\gamma^2) \cos(\alpha^2) + H_{32}^{(22)} \sigma_3^2 \sigma_2^2 \cos(\gamma^2) \cos(\beta^2) + H_{33}^{(22)} \sigma_3^2 \sigma_3^2 \cos^2(\gamma^2)], \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} H_{LS}^{(12)} = & -\frac{i}{2} [H_{11}^{(12)} \sigma_1^1 \sigma_1^2 \cos(\alpha^1) \cos(\alpha^2) + H_{12}^{(12)} \sigma_1^1 \sigma_2^2 \cos(\alpha^1) \cos(\beta^2) + H_{13}^{(12)} \sigma_1^1 \sigma_3^2 \cos(\alpha^1) \cos(\gamma^2) \\ & + H_{21}^{(12)} \sigma_2^1 \sigma_1^2 \cos(\beta^1) \cos(\alpha^2) + H_{22}^{(12)} \sigma_2^1 \sigma_2^2 \cos(\beta^1) \cos(\beta^2) + H_{23}^{(12)} \sigma_2^1 \sigma_3^2 \cos(\beta^1) \cos(\gamma^2) \\ & + H_{31}^{(12)} \sigma_3^1 \sigma_1^2 \cos(\gamma^1) \cos(\alpha^2) + H_{32}^{(12)} \sigma_3^1 \sigma_2^2 \cos(\gamma^1) \cos(\beta^2) + H_{33}^{(12)} \sigma_3^1 \sigma_3^2 \cos(\gamma^1) \cos(\gamma^2)], \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} H_{LS}^{(21)} = & -\frac{i}{2} [H_{11}^{(21)} \sigma_1^2 \sigma_1^1 \cos(\alpha^2) \cos(\alpha^1) + H_{12}^{(21)} \sigma_1^2 \sigma_2^1 \cos(\alpha^2) \cos(\beta^1) + H_{13}^{(21)} \sigma_1^2 \sigma_3^1 \cos(\alpha^2) \cos(\gamma^1) \\ & + H_{21}^{(21)} \sigma_2^2 \sigma_1^1 \cos(\beta^2) \cos(\alpha^1) + H_{22}^{(21)} \sigma_2^2 \sigma_2^1 \cos(\beta^2) \cos(\beta^1) + H_{23}^{(21)} \sigma_2^2 \sigma_3^1 \cos(\beta^2) \cos(\gamma^1) \\ & + H_{31}^{(21)} \sigma_3^2 \sigma_1^1 \cos(\gamma^2) \cos(\alpha^1) + H_{32}^{(21)} \sigma_3^2 \sigma_2^1 \cos(\gamma^2) \cos(\beta^1) + H_{33}^{(21)} \sigma_3^2 \sigma_3^1 \cos(\gamma^2) \cos(\gamma^1)]. \end{aligned} \quad (\text{A.5})$$

Further, to compute the expectation value of the Lamb Shift Hamiltonian let us start with the following collective quantum state representations in terms of two qubit entangled system:

$$\begin{aligned} \text{Ground state : } & \Rightarrow |G\rangle = |g_1\rangle \otimes |g_2\rangle = (0 \ 0 \ 0 \ 1)^{\mathbf{T}}, \\ \text{Excited state : } & \Rightarrow |E\rangle = |e_1\rangle \otimes |e_2\rangle = (1 \ 0 \ 0 \ 0)^{\mathbf{T}}, \\ \text{Symmetric state : } & \Rightarrow |S\rangle = \frac{1}{\sqrt{2}} [|e_1\rangle \otimes |g_2\rangle + |g_1\rangle \otimes |e_2\rangle] = \frac{1}{\sqrt{2}} (0 \ 1 \ 1 \ 0)^{\mathbf{T}}, \\ \text{Antisymmetric state : } & \Rightarrow |A\rangle = \frac{1}{\sqrt{2}} [|e_1\rangle \otimes |g_2\rangle - |g_1\rangle \otimes |e_2\rangle] = \frac{1}{\sqrt{2}} (0 \ 1 \ -1 \ 0)^{\mathbf{T}} \end{aligned} \quad (\text{A.6})$$

As we can see here all the entangled states are constructed out of all possible quantum states of two atoms $|g_1\rangle$, $|g_2\rangle$, $|e_1\rangle$ and $|e_2\rangle$. Now in the next subsections we derive the explicit contributions of the ground state, excited state, symmetric and antisymmetric state to the expectation value of Lamb Shift Hamiltonian.

A.1 For Ground state

For the ground state of two entangled atoms ($|G\rangle$) the expectation values of all the possible Pauli tensor operators, which are explicitly contributing in the part of the Lamb Shift Hamiltonian ($H_{LS}^{(11)}$) are given by:

$$\begin{aligned}\langle G|\sigma_1^1\sigma_1^1|G\rangle &= 1 & \langle G|\sigma_1^1\sigma_2^1|G\rangle &= -i & \langle G|\sigma_1^1\sigma_3^1|G\rangle &= 0 \\ \langle G|\sigma_2^1\sigma_1^1|G\rangle &= i & \langle G|\sigma_2^1\sigma_2^1|G\rangle &= 1 & \langle G|\sigma_2^1\sigma_3^1|G\rangle &= 0 \\ \langle G|\sigma_3^1\sigma_1^1|G\rangle &= 0 & \langle G|\sigma_3^1\sigma_2^1|G\rangle &= 0 & \langle G|\sigma_3^1\sigma_3^1|G\rangle &= 1\end{aligned}\quad (\text{A.7})$$

Consequently, the expectation value of the part of the Lamb Shift Hamiltonian ($H_{LS}^{(11)}$) with respect to the ground state can be written as:

$$\begin{aligned}\delta E_G^{(11)} &= \langle G|H_{LS}^{(11)}|G\rangle = -\frac{i}{2}[H_{11}^{(11)}\langle G|\sigma_1^1\sigma_1^1|G\rangle \cos^2(\alpha^1) + H_{12}^{(11)}\langle G|\sigma_1^1\sigma_2^1|G\rangle \cos(\alpha^1) \cos(\beta^1) \\ &\quad + H_{13}^{(11)}\langle G|\sigma_1^1\sigma_3^1|G\rangle \cos(\alpha^1) \cos(\gamma^1) + H_{21}^{(11)}\langle G|\sigma_2^1\sigma_1^1|G\rangle \cos(\beta^1) \cos(\alpha^1) \\ &\quad + H_{22}^{(11)}\langle G|\sigma_2^1\sigma_2^1|G\rangle \cos^2(\beta^1) + H_{23}^{(11)}\langle G|\sigma_2^1\sigma_3^1|G\rangle \cos(\beta^1) \cos(\gamma^1) \\ &\quad + H_{31}^{(11)}\langle G|\sigma_3^1\sigma_1^1|G\rangle \cos(\gamma^1) \cos(\alpha^1) + H_{32}^{(11)}\langle G|\sigma_3^1\sigma_2^1|G\rangle \cos(\gamma^1) \cos(\beta^1) \\ &\quad + H_{33}^{(11)}\langle G|\sigma_3^1\sigma_3^1|G\rangle \cos^2(\gamma^1)] \\ &= -\frac{i}{2}[H_{11}^{(11)} \cos^2(\alpha^1) + H_{22}^{(11)} \cos^2(\beta^1) + H_{33}^{(11)} \cos^2(\gamma^1) \\ &\quad - i(H_{12}^{(11)} - H_{21}^{(11)}) \cos(\alpha^1) \cos(\beta^1)].\end{aligned}\quad (\text{A.8})$$

For the ground state of two entangled atoms ($|G\rangle$) the expectation values of all the possible Pauli tensor operators, which are explicitly contributing in the part of the Lamb Shift Hamiltonian ($H_{LS}^{(12)}$) are given by:

$$\begin{aligned}\langle G|\sigma_1^1\sigma_1^2|G\rangle &= 0 & \langle G|\sigma_1^1\sigma_2^2|G\rangle &= 0 & \langle G|\sigma_1^1\sigma_3^2|G\rangle &= 0 \\ \langle G|\sigma_2^1\sigma_1^2|G\rangle &= 0 & \langle G|\sigma_2^1\sigma_2^2|G\rangle &= 0 & \langle G|\sigma_2^1\sigma_3^2|G\rangle &= 0 \\ \langle G|\sigma_3^1\sigma_1^2|G\rangle &= 0 & \langle G|\sigma_3^1\sigma_2^2|G\rangle &= 0 & \langle G|\sigma_3^1\sigma_3^2|G\rangle &= 1\end{aligned}\quad (\text{A.9})$$

Consequently, the expectation value of the part of the Lamb Shift Hamiltonian ($H_{LS}^{(12)}$) with respect to the ground state can be written as:

$$\begin{aligned}\delta E_G^{(12)} &= \langle G|H_{LS}^{(12)}|G\rangle = -\frac{i}{2}[H_{11}^{(12)}\langle G|\sigma_1^1\sigma_1^2|G\rangle \cos(\alpha^1) \cos(\alpha^2) + H_{12}^{(12)}\langle G|\sigma_1^1\sigma_2^2|G\rangle \cos(\alpha^1) \cos(\beta^2) \\ &\quad + H_{13}^{(12)}\langle G|\sigma_1^1\sigma_3^2|G\rangle \cos(\alpha^1) \cos(\gamma^2) + H_{21}^{(12)}\langle G|\sigma_2^1\sigma_1^2|G\rangle \cos(\beta^1) \cos(\alpha^2) \\ &\quad + H_{22}^{(12)}\langle G|\sigma_2^1\sigma_2^2|G\rangle \cos(\beta^1) \cos(\beta^2) + H_{23}^{(12)}\langle G|\sigma_2^1\sigma_3^2|G\rangle \cos(\beta^1) \cos(\gamma^2) \\ &\quad + H_{31}^{(12)}\langle G|\sigma_3^1\sigma_1^2|G\rangle \cos(\gamma^1) \cos(\alpha^2) + H_{32}^{(12)}\langle G|\sigma_3^1\sigma_2^2|G\rangle \cos(\gamma^1) \cos(\beta^2) \\ &\quad + H_{33}^{(12)}\langle G|\sigma_3^1\sigma_3^2|G\rangle \cos(\gamma^1) \cos(\gamma^2)] \\ &= -\frac{i}{2}H_{33}^{(12)} \cos(\gamma^1) \cos(\gamma^2).\end{aligned}\quad (\text{A.10})$$

For the ground state of two entangled atoms ($|G\rangle$) the expectation values of all the possible Pauli tensor operators, which are explicitly contributing in the part of the Lamb Shift Hamiltonian ($H_{LS}^{(21)}$) are given by:

$$\begin{aligned}\langle G|\sigma_1^2\sigma_1^1|G\rangle &= 0 & \langle G|\sigma_1^2\sigma_2^1|G\rangle &= 0 & \langle G|\sigma_1^2\sigma_3^1|G\rangle &= 0 \\ \langle G|\sigma_2^2\sigma_1^1|G\rangle &= 0 & \langle G|\sigma_2^2\sigma_2^1|G\rangle &= 0 & \langle G|\sigma_2^2\sigma_3^1|G\rangle &= 0 \\ \langle G|\sigma_3^2\sigma_1^1|G\rangle &= 0 & \langle G|\sigma_3^2\sigma_2^1|G\rangle &= 0 & \langle G|\sigma_3^2\sigma_3^1|G\rangle &= 1\end{aligned}\quad (\text{A.11})$$

Consequently, the expectation value of the part of the Lamb Shift Hamiltonian ($H_{LS}^{(21)}$) with respect to the ground state can be written as:

$$\begin{aligned}
\delta E_G^{(21)} = \langle G|H_{LS}^{(21)}|G\rangle &= -\frac{i}{2}[H_{11}^{(21)}\langle G|\sigma_1^2\sigma_1^1|G\rangle \cos(\alpha^2) \cos(\alpha^1) + H_{12}^{(21)}\langle G|\sigma_1^2\sigma_2^1|G\rangle \cos(\alpha^2) \cos(\beta^1) \\
&\quad + H_{13}^{(21)}\langle G|\sigma_1^2\sigma_3^1|G\rangle \cos(\alpha^2) \cos(\gamma^1) + H_{21}^{(21)}\langle G|\sigma_2^2\sigma_1^1|G\rangle \cos(\beta^2) \cos(\alpha^1) \\
&\quad + H_{22}^{(21)}\langle G|\sigma_2^2\sigma_2^1|G\rangle \cos(\beta^2) \cos(\beta^1) + H_{23}^{(21)}\langle G|\sigma_2^2\sigma_3^1|G\rangle \cos(\beta^2) \cos(\gamma^1) \\
&\quad + H_{31}^{(21)}\langle G|\sigma_3^2\sigma_1^1|G\rangle \cos(\gamma^2) \cos(\alpha^1) + H_{32}^{(21)}\langle G|\sigma_3^2\sigma_2^1|G\rangle \cos(\gamma^2) \cos(\beta^1) \\
&\quad + H_{33}^{(21)}\langle G|\sigma_3^2\sigma_3^1|G\rangle \cos(\gamma^2) \cos(\gamma^1)] \\
&= -\frac{i}{2}H_{33}^{(21)} \cos(\gamma^1) \cos(\gamma^2). \tag{A.12}
\end{aligned}$$

For the ground state of two entangled atoms ($|G\rangle$) the expectation values of all the possible Pauli tensor operators, which are explicitly contributing in the part of the Lamb Shift Hamiltonian ($H_{LS}^{(22)}$) are given by:

$$\begin{aligned}
\langle G|\sigma_1^2\sigma_1^2|G\rangle &= 1 & \langle G|\sigma_1^2\sigma_2^2|G\rangle &= -i & \langle G|\sigma_1^2\sigma_3^2|G\rangle &= 0 \\
\langle G|\sigma_2^2\sigma_1^2|G\rangle &= i & \langle G|\sigma_2^2\sigma_2^2|G\rangle &= 1 & \langle G|\sigma_2^2\sigma_3^2|G\rangle &= 0 \\
\langle G|\sigma_3^2\sigma_1^2|G\rangle &= 0 & \langle G|\sigma_3^2\sigma_2^2|G\rangle &= 0 & \langle G|\sigma_3^2\sigma_3^2|G\rangle &= 1
\end{aligned} \tag{A.13}$$

Consequently, the expectation value of the part of the Lamb Shift Hamiltonian ($H_{LS}^{(22)}$) with respect to the ground state can be written as:

$$\begin{aligned}
\delta E_G^{(22)} = \langle G|H_{LS}^{(22)}|G\rangle &= -\frac{i}{2}[H_{11}^{(22)}\langle G|\sigma_1^2\sigma_1^2|G\rangle \cos^2(\alpha^2) + H_{12}^{(22)}\langle G|\sigma_1^2\sigma_2^2|G\rangle \cos(\alpha^2) \cos(\beta^2) \\
&\quad + H_{13}^{(22)}\langle G|\sigma_1^2\sigma_3^2|G\rangle \cos(\alpha^2) \cos(\gamma^2) + H_{21}^{(22)}\langle G|\sigma_2^2\sigma_1^2|G\rangle \cos(\beta^2) \cos(\alpha^2) \\
&\quad + H_{22}^{(22)}\langle G|\sigma_2^2\sigma_2^2|G\rangle \cos^2(\beta^2) + H_{23}^{(22)}\langle G|\sigma_2^2\sigma_3^2|G\rangle \cos(\beta^2) \cos(\gamma^2) \\
&\quad + H_{31}^{(22)}\langle G|\sigma_3^2\sigma_1^2|G\rangle \cos(\gamma^2) \cos(\alpha^2) + H_{32}^{(22)}\langle G|\sigma_3^2\sigma_2^2|G\rangle \cos(\gamma^2) \cos(\beta^2) \\
&\quad + H_{33}^{(22)}\langle G|\sigma_3^2\sigma_3^2|G\rangle \cos^2(\gamma^2)] \\
&= -\frac{i}{2}[H_{11}^{(22)} \cos^2(\alpha^2) + H_{22}^{(22)} \cos^2(\beta^2) + H_{33}^{(22)} \cos^2(\gamma^2) \\
&\quad - i(H_{12}^{(22)} - H_{21}^{(22)}) \cos(\alpha^2) \cos(\beta^2)]. \tag{A.14}
\end{aligned}$$

After that, summing over all the possible contributions obtained for the ground state of two entangled atoms ($|G\rangle$) the expectation value of the Lamd Shift Hamiltonian can be expressed as:

$$\begin{aligned}
\delta E_G &= \sum_{i=1}^2 \sum_{j=1}^2 \delta E_G^{(ij)} = \sum_{i=1}^2 \sum_{j=1}^2 \langle G|H_{LS}^{(ij)}|G\rangle \\
&= -\frac{i}{2}[H_{11}^{(11)} \cos^2(\alpha^1) + H_{11}^{(22)} \cos^2(\alpha^2) + H_{22}^{(11)} \cos^2(\beta^1) + H_{22}^{(22)} \cos^2(\beta^2) + H_{33}^{(11)} \cos^2(\gamma^1) + H_{33}^{(22)} \cos^2(\gamma^2) \\
&\quad - i(H_{12}^{(11)} - H_{21}^{(11)}) \cos(\alpha^1) \cos(\beta^1) - i(H_{12}^{(22)} - H_{21}^{(22)}) \cos(\alpha^2) \cos(\beta^2) \\
&\quad + (H_{33}^{(12)} + H_{33}^{(21)}) \cos(\gamma^1) \cos(\gamma^2)]. \tag{A.15}
\end{aligned}$$

A.2 For Excited state

For the excited state of two entangled atoms ($|E\rangle$) the expectation values of all the possible Pauli tensor operators, which are explicitly contributing in the part of the Lamb Shift Hamiltonian ($H_{LS}^{(11)}$) are given by:

$$\begin{aligned}
\langle E|\sigma_1^1\sigma_1^1|E\rangle &= 1 & \langle G|\sigma_1^1\sigma_2^1|E\rangle &= i & \langle E|\sigma_1^1\sigma_3^1|E\rangle &= 0 \\
\langle E|\sigma_2^1\sigma_1^1|E\rangle &= -i & \langle G|\sigma_2^1\sigma_2^1|E\rangle &= 1 & \langle E|\sigma_2^1\sigma_3^1|E\rangle &= 0 \\
\langle E|\sigma_3^1\sigma_1^1|E\rangle &= 0 & \langle G|\sigma_3^1\sigma_2^1|E\rangle &= 0 & \langle E|\sigma_3^1\sigma_3^1|E\rangle &= 1
\end{aligned} \tag{A.16}$$

Consequently, the expectation value of the part of the Lamb Shift Hamiltonian ($H_{LS}^{(11)}$) with respect to the excited state can be written as:

$$\begin{aligned}
\delta E_E^{(11)} &= \langle E|H_{LS}^{(11)}|E\rangle = -\frac{i}{2}[H_{11}^{(11)}\langle E|\sigma_1^1\sigma_1^1|E\rangle \cos^2(\alpha^1) + H_{12}^{(11)}\langle E|\sigma_1^1\sigma_2^1|E\rangle \cos(\alpha^1) \cos(\beta^1) \\
&\quad + H_{13}^{(11)}\langle E|\sigma_1^1\sigma_3^1|E\rangle \cos(\alpha^1) \cos(\gamma^1) + H_{21}^{(11)}\langle E|\sigma_2^1\sigma_1^1|E\rangle \cos(\beta^1) \cos(\alpha^1) \\
&\quad + H_{22}^{(11)}\langle E|\sigma_2^1\sigma_2^1|E\rangle \cos^2(\beta^1) + H_{23}^{(11)}\langle E|\sigma_2^1\sigma_3^1|E\rangle \cos(\beta^1) \cos(\gamma^1) \\
&\quad + H_{31}^{(11)}\langle E|\sigma_3^1\sigma_1^1|E\rangle \cos(\gamma^1) \cos(\alpha^1) + H_{32}^{(11)}\langle E|\sigma_3^1\sigma_2^1|E\rangle \cos(\gamma^1) \cos(\beta^1) \\
&\quad + H_{33}^{(11)}\langle E|\sigma_3^1\sigma_3^1|E\rangle \cos^2(\gamma^1)] \\
&= -\frac{i}{2}[H_{11}^{(11)} \cos^2(\alpha^1) + H_{22}^{(11)} \cos^2(\beta^1) + H_{33}^{(11)} \cos^2(\gamma^1) \\
&\quad + i(H_{12}^{(11)} - H_{21}^{(11)}) \cos(\alpha^1) \cos(\beta^1)]. \tag{A.17}
\end{aligned}$$

For the excited state of two entangled atoms ($|E\rangle$) the expectation values of all the possible Pauli tensor operators, which are explicitly contributing in the part of the Lamb Shift Hamiltonian ($H_{LS}^{(12)}$) are given by:

$$\begin{aligned}
\langle E|\sigma_1^1\sigma_1^2|E\rangle &= 0 & \langle E|\sigma_1^1\sigma_2^2|E\rangle &= 0 & \langle E|\sigma_1^1\sigma_3^2|E\rangle &= 0 \\
\langle E|\sigma_2^1\sigma_1^2|E\rangle &= 0 & \langle E|\sigma_2^1\sigma_2^2|E\rangle &= 0 & \langle E|\sigma_2^1\sigma_3^2|E\rangle &= 0 \\
\langle E|\sigma_3^1\sigma_1^2|E\rangle &= 0 & \langle E|\sigma_3^1\sigma_2^2|E\rangle &= 0 & \langle E|\sigma_3^1\sigma_3^2|E\rangle &= 1
\end{aligned} \tag{A.18}$$

Consequently, the expectation value of the part of the Lamb Shift Hamiltonian ($H_{LS}^{(12)}$) with respect to the excited state can be written as:

$$\begin{aligned}
\delta E_E^{(12)} &= \langle E|H_{LS}^{(12)}|E\rangle = -\frac{i}{2}[H_{11}^{(12)}\langle E|\sigma_1^1\sigma_1^2|E\rangle \cos(\alpha^1) \cos(\alpha^2) + H_{12}^{(12)}\langle E|\sigma_1^1\sigma_2^2|E\rangle \cos(\alpha^1) \cos(\beta^2) \\
&\quad + H_{13}^{(12)}\langle E|\sigma_1^1\sigma_3^2|E\rangle \cos(\alpha^1) \cos(\gamma^2) + H_{21}^{(12)}\langle E|\sigma_2^1\sigma_1^2|E\rangle \cos(\beta^1) \cos(\alpha^2) \\
&\quad + H_{22}^{(12)}\langle E|\sigma_2^1\sigma_2^2|E\rangle \cos(\beta^1) \cos(\beta^2) + H_{23}^{(12)}\langle E|\sigma_2^1\sigma_3^2|E\rangle \cos(\beta^1) \cos(\gamma^2) \\
&\quad + H_{31}^{(12)}\langle E|\sigma_3^1\sigma_1^2|E\rangle \cos(\gamma^1) \cos(\alpha^2) + H_{32}^{(12)}\langle E|\sigma_3^1\sigma_2^2|E\rangle \cos(\gamma^1) \cos(\beta^2) \\
&\quad + H_{33}^{(12)}\langle E|\sigma_3^1\sigma_3^2|E\rangle \cos(\gamma^1) \cos(\gamma^2)] \\
&= -\frac{i}{2}H_{33}^{(12)} \cos(\gamma^1) \cos(\gamma^2). \tag{A.19}
\end{aligned}$$

For the excited state of two entangled atoms ($|E\rangle$) the expectation values of all the possible Pauli tensor operators, which are explicitly contributing in the part of the Lamb Shift Hamiltonian ($H_{LS}^{(21)}$) are given by:

$$\begin{aligned}
\langle E|\sigma_1^2\sigma_1^1|E\rangle &= 0 & \langle E|\sigma_1^2\sigma_2^1|E\rangle &= 0 & \langle E|\sigma_1^2\sigma_3^1|E\rangle &= 0 \\
\langle E|\sigma_2^2\sigma_1^1|E\rangle &= 0 & \langle E|\sigma_2^2\sigma_2^1|E\rangle &= 0 & \langle E|\sigma_2^2\sigma_3^1|E\rangle &= 0 \\
\langle E|\sigma_3^2\sigma_1^1|E\rangle &= 0 & \langle E|\sigma_3^2\sigma_2^1|E\rangle &= 0 & \langle E|\sigma_3^2\sigma_3^1|E\rangle &= 1
\end{aligned} \tag{A.20}$$

Consequently, the expectation value of the part of the Lamb Shift Hamiltonian ($H_{LS}^{(21)}$) with respect to the excited state can be written as:

$$\begin{aligned}
\delta E_E^{(21)} &= \langle E|H_{LS}^{(21)}|E\rangle = -\frac{i}{2}[H_{11}^{(21)}\langle E|\sigma_1^2\sigma_1^1|E\rangle \cos(\alpha^2) \cos(\alpha^1) + H_{12}^{(21)}\langle E|\sigma_1^2\sigma_2^1|E\rangle \cos(\alpha^2) \cos(\beta^1) \\
&\quad + H_{13}^{(21)}\langle E|\sigma_1^2\sigma_3^1|E\rangle \cos(\alpha^2) \cos(\gamma^1) + H_{21}^{(21)}\langle E|\sigma_2^2\sigma_1^1|E\rangle \cos(\beta^2) \cos(\alpha^1) \\
&\quad + H_{22}^{(21)}\langle E|\sigma_2^2\sigma_2^1|E\rangle \cos(\beta^2) \cos(\beta^1) + H_{23}^{(21)}\langle E|\sigma_2^2\sigma_3^1|E\rangle \cos(\beta^2) \cos(\gamma^1) \\
&\quad + H_{31}^{(21)}\langle E|\sigma_3^2\sigma_1^1|E\rangle \cos(\gamma^2) \cos(\alpha^1) + H_{32}^{(21)}\langle E|\sigma_3^2\sigma_2^1|E\rangle \cos(\gamma^2) \cos(\beta^1) \\
&\quad + H_{33}^{(21)}\langle E|\sigma_3^2\sigma_3^1|E\rangle \cos(\gamma^2) \cos(\gamma^1)] \\
&= -\frac{i}{2}H_{33}^{(21)} \cos(\gamma^1) \cos(\gamma^2). \tag{A.21}
\end{aligned}$$

For the excited state of two entangled atoms ($|E\rangle$) the expectation values of all the possible Pauli tensor operators, which are explicitly contributing in the part of the Lamb Shift Hamiltonian ($H_{LS}^{(22)}$) are given by:

$$\begin{aligned}\langle E|\sigma_1^2\sigma_1^2|E\rangle &= 1 & \langle E|\sigma_1^2\sigma_2^2|E\rangle &= i & \langle E|\sigma_1^2\sigma_3^2|E\rangle &= 0 \\ \langle E|\sigma_2^2\sigma_1^2|E\rangle &= -i & \langle E|\sigma_2^2\sigma_2^2|E\rangle &= 1 & \langle E|\sigma_2^2\sigma_3^2|E\rangle &= 0 \\ \langle E|\sigma_3^2\sigma_1^2|E\rangle &= 0 & \langle E|\sigma_3^2\sigma_2^2|E\rangle &= 0 & \langle E|\sigma_3^2\sigma_3^2|E\rangle &= 1\end{aligned}\quad (\text{A.22})$$

Consequently, the expectation value of the part of the Lamb Shift Hamiltonian ($H_{LS}^{(22)}$) with respect to the excited state can be written as:

$$\begin{aligned}\delta E_E^{(22)} &= \langle E|H_{LS}^{(22)}|E\rangle = -\frac{i}{2}[H_{11}^{(22)}\langle E|\sigma_1^2\sigma_1^2|E\rangle \cos^2(\alpha^2) + H_{12}^{(22)}\langle E|\sigma_1^2\sigma_2^2|E\rangle \cos(\alpha^2) \cos(\beta^2) \\ &\quad + H_{13}^{(22)}\langle E|\sigma_1^2\sigma_3^2|E\rangle \cos(\alpha^2) \cos(\gamma^2) + H_{21}^{(22)}\langle E|\sigma_2^2\sigma_1^2|E\rangle \cos(\beta^2) \cos(\alpha^2) \\ &\quad + H_{22}^{(22)}\langle E|\sigma_2^2\sigma_2^2|E\rangle \cos^2(\beta^2) + H_{23}^{(22)}\langle E|\sigma_2^2\sigma_3^2|E\rangle \cos(\beta^2) \cos(\gamma^2) \\ &\quad + H_{31}^{(22)}\langle E|\sigma_3^2\sigma_1^2|E\rangle \cos(\gamma^2) \cos(\alpha^2) + H_{32}^{(22)}\langle E|\sigma_3^2\sigma_2^2|E\rangle \cos(\gamma^2) \cos(\beta^2) \\ &\quad + H_{33}^{(22)}\langle E|\sigma_3^2\sigma_3^2|E\rangle \cos^2(\gamma^2)] \\ &= -\frac{i}{2}[H_{11}^{(22)} \cos^2(\alpha^2) + H_{22}^{(22)} \cos^2(\beta^2) + H_{33}^{(22)} \cos^2(\gamma^2) \\ &\quad + i(H_{12}^{(22)} - H_{21}^{(22)}) \cos(\alpha^2) \cos(\beta^2)].\end{aligned}\quad (\text{A.23})$$

After that, summing over all the possible contributions obtained for the excited state of two entangled atoms ($|E\rangle$) the expectation value of the Lamb Shift Hamiltonian can be expressed as:

$$\begin{aligned}\delta E_E &= \sum_{i=1}^2 \sum_{j=1}^2 \delta E_E^{(ij)} = \sum_{i=1}^2 \sum_{j=1}^2 \langle E|H_{LS}^{(ij)}|E\rangle \\ &= -\frac{i}{2}[H_{11}^{(11)} \cos^2(\alpha^1) + H_{11}^{(22)} \cos^2(\alpha^2) + H_{22}^{(11)} \cos^2(\beta^1) + H_{22}^{(22)} \cos^2(\beta^2) + H_{33}^{(11)} \cos^2(\gamma^1) + H_{33}^{(22)} \cos^2(\gamma^2) \\ &\quad + i(H_{12}^{(11)} - H_{21}^{(11)}) \cos(\alpha^1) \cos(\beta^1) + i(H_{12}^{(22)} - H_{21}^{(22)}) \cos(\alpha^2) \cos(\beta^2) \\ &\quad + (H_{33}^{(12)} + H_{33}^{(21)}) \cos(\gamma^1) \cos(\gamma^2)].\end{aligned}\quad (\text{A.24})$$

A.3 For Symmetric state

For the symmetric state of two entangled atoms ($|S\rangle$) the expectation values of all the possible Pauli tensor operators, which are explicitly contributing in the part of the Lamb Shift Hamiltonian ($H_{LS}^{(11)}$) are given by:

$$\begin{aligned}\langle S|\sigma_1^1\sigma_1^1|S\rangle &= 1 & \langle S|\sigma_1^1\sigma_2^1|S\rangle &= 0 & \langle S|\sigma_1^1\sigma_3^1|S\rangle &= 0 \\ \langle S|\sigma_2^1\sigma_1^1|S\rangle &= 0 & \langle S|\sigma_2^1\sigma_2^1|S\rangle &= 1 & \langle S|\sigma_2^1\sigma_3^1|S\rangle &= 0 \\ \langle S|\sigma_3^1\sigma_1^1|S\rangle &= 0 & \langle S|\sigma_3^1\sigma_2^1|S\rangle &= 0 & \langle S|\sigma_3^1\sigma_3^1|S\rangle &= 1\end{aligned}\quad (\text{A.25})$$

Consequently, the expectation value of the part of the Lamb Shift Hamiltonian ($H_{LS}^{(11)}$) with respect to the symmetric state can be written as:

$$\begin{aligned}\delta E_S^{(11)} &= \langle S|H_{LS}^{(11)}|S\rangle = -\frac{i}{2}[H_{11}^{(11)}\langle S|\sigma_1^1\sigma_1^1|S\rangle \cos^2(\alpha^1) + H_{12}^{(11)}\langle S|\sigma_1^1\sigma_2^1|S\rangle \cos(\alpha^1) \cos(\beta^1) \\ &\quad + H_{13}^{(11)}\langle S|\sigma_1^1\sigma_3^1|S\rangle \cos(\alpha^1) \cos(\gamma^1) + H_{21}^{(11)}\langle S|\sigma_2^1\sigma_1^1|S\rangle \cos(\beta^1) \cos(\alpha^1) \\ &\quad + H_{22}^{(11)}\langle S|\sigma_2^1\sigma_2^1|S\rangle \cos^2(\beta^1) + H_{23}^{(11)}\langle S|\sigma_2^1\sigma_3^1|S\rangle \cos(\beta^1) \cos(\gamma^1) \\ &\quad + H_{31}^{(11)}\langle S|\sigma_3^1\sigma_1^1|S\rangle \cos(\gamma^1) \cos(\alpha^1) + H_{32}^{(11)}\langle S|\sigma_3^1\sigma_2^1|S\rangle \cos(\gamma^1) \cos(\beta^1) \\ &\quad + H_{33}^{(11)}\langle S|\sigma_3^1\sigma_3^1|S\rangle \cos^2(\gamma^1)] \\ &= -\frac{i}{2}[H_{11}^{(11)} \cos^2(\alpha^1) + H_{22}^{(11)} \cos^2(\beta^1) + H_{33}^{(11)} \cos^2(\gamma^1)].\end{aligned}\quad (\text{A.26})$$

For the symmetric state of two entangled atoms ($|S\rangle$) the expectation values of all the possible Pauli tensor operators, which are explicitly contributing in the part of the Lamb Shift Hamiltonian ($H_{LS}^{(12)}$) are given by:

$$\begin{aligned}
\langle S|\sigma_1^1\sigma_1^2|S\rangle &= 1 & \langle S|\sigma_1^1\sigma_2^2|S\rangle &= 0 & \langle S|\sigma_1^1\sigma_3^2|S\rangle &= 0 \\
\langle S|\sigma_2^2\sigma_1^2|S\rangle &= 0 & \langle S|\sigma_2^1\sigma_2^2|S\rangle &= 1 & \langle S|\sigma_2^1\sigma_3^2|S\rangle &= 0 \\
\langle S|\sigma_3^1\sigma_1^2|S\rangle &= 0 & \langle S|\sigma_3^1\sigma_2^2|S\rangle &= 0 & \langle S|\sigma_3^1\sigma_3^2|S\rangle &= -1
\end{aligned} \tag{A.27}$$

Consequently, the expectation value of the part of the Lamb Shift Hamiltonian ($H_{LS}^{(12)}$) with respect to the symmetric state can be written as:

$$\begin{aligned}
\delta E_S^{(12)} = \langle S|H_{LS}^{(12)}|S\rangle &= -\frac{i}{2}[H_{11}^{(12)}\langle S|\sigma_1^1\sigma_1^2|S\rangle \cos(\alpha^1)\cos(\alpha^2) + H_{12}^{(12)}\langle S|\sigma_1^1\sigma_2^2|S\rangle \cos(\alpha^1)\cos(\beta^2) \\
&\quad + H_{13}^{(12)}\langle S|\sigma_1^1\sigma_3^2|S\rangle \cos(\alpha^1)\cos(\gamma^2) + H_{21}^{(12)}\langle S|\sigma_2^1\sigma_1^2|S\rangle \cos(\beta^1)\cos(\alpha^2) \\
&\quad + H_{22}^{(12)}\langle S|\sigma_2^1\sigma_2^2|S\rangle \cos(\beta^1)\cos(\beta^2) + H_{23}^{(12)}\langle S|\sigma_2^1\sigma_3^2|S\rangle \cos(\beta^1)\cos(\gamma^2) \\
&\quad + H_{31}^{(12)}\langle S|\sigma_3^1\sigma_1^2|S\rangle \cos(\gamma^1)\cos(\alpha^2) + H_{32}^{(12)}\langle S|\sigma_3^1\sigma_2^2|S\rangle \cos(\gamma^1)\cos(\beta^2) \\
&\quad + H_{33}^{(12)}\langle S|\sigma_3^1\sigma_3^2|S\rangle \cos(\gamma^1)\cos(\gamma^2)] \\
&= -\frac{i}{2}[H_{11}^{(12)}\cos(\alpha^1)\cos(\alpha^2) + H_{22}^{(12)}\cos(\beta^1)\cos(\beta^2) - H_{33}^{(12)}\cos(\gamma^1)\cos(\gamma^2)].
\end{aligned} \tag{A.28}$$

For the symmetric state of two entangled atoms ($|E\rangle$) the expectation values of all the possible Pauli tensor operators, which are explicitly contributing in the part of the Lamb Shift Hamiltonian ($H_{LS}^{(21)}$) are given by:

$$\begin{aligned}
\langle S|\sigma_1^2\sigma_1^1|S\rangle &= 1 & \langle S|\sigma_1^2\sigma_2^1|S\rangle &= 0 & \langle S|\sigma_1^2\sigma_3^1|S\rangle &= 0 \\
\langle S|\sigma_2^2\sigma_1^1|S\rangle &= 0 & \langle S|\sigma_2^2\sigma_2^1|S\rangle &= 1 & \langle S|\sigma_2^2\sigma_3^1|S\rangle &= 0 \\
\langle S|\sigma_3^2\sigma_1^1|S\rangle &= 0 & \langle S|\sigma_3^2\sigma_2^1|S\rangle &= 0 & \langle S|\sigma_3^2\sigma_3^1|S\rangle &= -1
\end{aligned} \tag{A.29}$$

Consequently, the expectation value of the part of the Lamb Shift Hamiltonian ($H_{LS}^{(21)}$) with respect to the symmetric state can be written as:

$$\begin{aligned}
\delta E_S^{(21)} = \langle S|H_{LS}^{(21)}|S\rangle &= -\frac{i}{2}[H_{11}^{(21)}\langle S|\sigma_1^2\sigma_1^1|S\rangle \cos(\alpha^2)\cos(\alpha^1) + H_{12}^{(21)}\langle S|\sigma_1^2\sigma_2^1|S\rangle \cos(\alpha^2)\cos(\beta^1) \\
&\quad + H_{13}^{(21)}\langle S|\sigma_1^2\sigma_3^1|S\rangle \cos(\alpha^2)\cos(\gamma^1) + H_{21}^{(21)}\langle S|\sigma_2^2\sigma_1^1|S\rangle \cos(\beta^2)\cos(\alpha^1) \\
&\quad + H_{22}^{(21)}\langle S|\sigma_2^2\sigma_2^1|S\rangle \cos(\beta^2)\cos(\beta^1) + H_{23}^{(21)}\langle S|\sigma_2^2\sigma_3^1|S\rangle \cos(\beta^2)\cos(\gamma^1) \\
&\quad + H_{31}^{(21)}\langle S|\sigma_3^2\sigma_1^1|S\rangle \cos(\gamma^2)\cos(\alpha^1) + H_{32}^{(21)}\langle S|\sigma_3^2\sigma_2^1|S\rangle \cos(\gamma^2)\cos(\beta^1) \\
&\quad + H_{33}^{(21)}\langle S|\sigma_3^2\sigma_3^1|S\rangle \cos(\gamma^2)\cos(\gamma^1)] \\
&= -\frac{i}{2}[H_{11}^{(21)}\cos(\alpha^2)\cos(\alpha^1) + H_{22}^{(21)}\cos(\beta^2)\cos(\beta^1) - H_{33}^{(21)}\cos(\gamma^2)\cos(\gamma^1)].
\end{aligned} \tag{A.30}$$

For the symmetric state of two entangled atoms ($|E\rangle$) the expectation values of all the possible Pauli tensor operators, which are explicitly contributing in the part of the Lamb Shift Hamiltonian ($H_{LS}^{(22)}$) are given by:

$$\begin{aligned}
\langle S|\sigma_1^2\sigma_1^2|S\rangle &= 1 & \langle S|\sigma_1^2\sigma_2^2|S\rangle &= 0 & \langle S|\sigma_1^2\sigma_3^2|S\rangle &= 0 \\
\langle S|\sigma_2^2\sigma_1^2|S\rangle &= 0 & \langle S|\sigma_2^2\sigma_2^2|S\rangle &= 1 & \langle S|\sigma_2^2\sigma_3^2|S\rangle &= 0 \\
\langle S|\sigma_3^2\sigma_1^2|S\rangle &= 0 & \langle S|\sigma_3^2\sigma_2^2|S\rangle &= 0 & \langle S|\sigma_3^2\sigma_3^2|S\rangle &= 1
\end{aligned} \tag{A.31}$$

Consequently, the expectation value of the part of the Lamb Shift Hamiltonian ($H_{LS}^{(22)}$) with respect to the symmetric state can be written as:

$$\begin{aligned}
\delta E_S^{(22)} &= \langle S | H_{LS}^{(22)} | S \rangle = -\frac{i}{2} [H_{11}^{(22)} \langle S | \sigma_1^2 \sigma_1^2 | S \rangle \cos^2(\alpha^2) + H_{12}^{(22)} \langle S | \sigma_1^2 \sigma_2^2 | S \rangle \cos(\alpha^2) \cos(\beta^2) \\
&\quad + H_{13}^{(22)} \langle S | \sigma_1^2 \sigma_3^2 | S \rangle \cos(\alpha^2) \cos(\gamma^2) + H_{21}^{(22)} \langle S | \sigma_2^2 \sigma_1^2 | S \rangle \cos(\beta^2) \cos(\alpha^2) \\
&\quad + H_{22}^{(22)} \langle S | \sigma_2^2 \sigma_2^2 | S \rangle \cos^2(\beta^2) + H_{23}^{(22)} \langle S | \sigma_2^2 \sigma_3^2 | S \rangle \cos(\beta^2) \cos(\gamma^2) \\
&\quad + H_{31}^{(22)} \langle S | \sigma_3^2 \sigma_1^2 | S \rangle \cos(\gamma^2) \cos(\alpha^2) + H_{32}^{(22)} \langle S | \sigma_3^2 \sigma_2^2 | S \rangle \cos(\gamma^2) \cos(\beta^2) \\
&\quad + H_{33}^{(22)} \langle S | \sigma_3^2 \sigma_3^2 | S \rangle \cos^2(\gamma^2)] \\
&= -\frac{i}{2} [H_{11}^{(22)} \cos^2(\alpha^2) + H_{22}^{(22)} \cos^2(\beta^2) + H_{33}^{(22)} \cos^2(\gamma^2)]. \tag{A.32}
\end{aligned}$$

After that, summing over all the possible contributions obtained for the symmetric state of two entangled atoms ($|S\rangle$) the expectation value of the Lamd Shift Hamiltonian can be expressed as:

$$\begin{aligned}
\delta E_S &= \sum_{i=1}^2 \sum_{j=1}^2 \delta E_S^{(ij)} = \sum_{i=1}^2 \sum_{j=1}^2 \langle S | H_{LS}^{(ij)} | S \rangle \\
&= -\frac{i}{2} [H_{11}^{(11)} \cos^2(\alpha^1) + H_{11}^{(22)} \cos^2(\alpha^2) + H_{22}^{(11)} \cos^2(\beta^1) + H_{22}^{(22)} \cos^2(\beta^2) + H_{33}^{(11)} \cos^2(\gamma^1) + H_{33}^{(22)} \cos^2(\gamma^2) \\
&\quad + (H_{12}^{(11)} + H_{21}^{(11)}) \cos(\alpha^1) \cos(\alpha^2) + (H_{12}^{(22)} + H_{21}^{(22)}) \cos(\beta^1) \cos(\beta^2) \\
&\quad + (H_{33}^{(12)} + H_{33}^{(21)}) \cos(\gamma^1) \cos(\gamma^2)]. \tag{A.33}
\end{aligned}$$

A.4 For Antisymmetric state

For the antisymmetric state of two entangled atoms ($|A\rangle$) the expectation values of all the possible Pauli tensor operators, which are explicitly contributing in the part of the Lamb Shift Hamiltonian ($H_{LS}^{(11)}$) are given by:

$$\begin{aligned}
\langle A | \sigma_1^1 \sigma_1^1 | A \rangle &= 1 & \langle A | \sigma_1^1 \sigma_2^1 | A \rangle &= 0 & \langle A | \sigma_1^1 \sigma_3^1 | A \rangle &= 0 \\
\langle A | \sigma_2^1 \sigma_1^1 | A \rangle &= 0 & \langle A | \sigma_2^1 \sigma_2^1 | A \rangle &= 1 & \langle A | \sigma_2^1 \sigma_3^1 | A \rangle &= 0 \\
\langle A | \sigma_3^1 \sigma_1^1 | A \rangle &= 0 & \langle A | \sigma_3^1 \sigma_2^1 | A \rangle &= 0 & \langle A | \sigma_3^1 \sigma_3^1 | A \rangle &= 1
\end{aligned} \tag{A.34}$$

Consequently, the expectation value of the part of the Lamb Shift Hamiltonian ($H_{LS}^{(11)}$) with respect to the antisymmetric state can be written as:

$$\begin{aligned}
\delta E_A^{(11)} &= \langle A | H_{LS}^{(11)} | A \rangle = -\frac{i}{2} [H_{11}^{(11)} \langle A | \sigma_1^1 \sigma_1^1 | A \rangle \cos^2(\alpha^1) + H_{12}^{(11)} \langle A | \sigma_1^1 \sigma_2^1 | A \rangle \cos(\alpha^1) \cos(\beta^1) \\
&\quad + H_{13}^{(11)} \langle A | \sigma_1^1 \sigma_3^1 | A \rangle \cos(\alpha^1) \cos(\gamma^1) + H_{21}^{(11)} \langle A | \sigma_2^1 \sigma_1^1 | A \rangle \cos(\beta^1) \cos(\alpha^1) \\
&\quad + H_{22}^{(11)} \langle A | \sigma_2^1 \sigma_2^1 | A \rangle \cos^2(\beta^1) + H_{23}^{(11)} \langle A | \sigma_2^1 \sigma_3^1 | A \rangle \cos(\beta^1) \cos(\gamma^1) \\
&\quad + H_{31}^{(11)} \langle A | \sigma_3^1 \sigma_1^1 | A \rangle \cos(\gamma^1) \cos(\alpha^1) + H_{32}^{(11)} \langle A | \sigma_3^1 \sigma_2^1 | A \rangle \cos(\gamma^1) \cos(\beta^1) \\
&\quad + H_{33}^{(11)} \langle A | \sigma_3^1 \sigma_3^1 | A \rangle \cos^2(\gamma^1)] \\
&= -\frac{i}{2} [H_{11}^{(11)} \cos^2(\alpha^1) + H_{22}^{(11)} \cos^2(\beta^1) + H_{33}^{(11)} \cos^2(\gamma^1)]. \tag{A.35}
\end{aligned}$$

For the antisymmetric state of two entangled atoms ($|A\rangle$) the expectation values of all the possible Pauli tensor operators, which are explicitly contributing in the part of the Lamb Shift Hamiltonian ($H_{LS}^{(12)}$) are given by:

$$\begin{aligned}
\langle A | \sigma_1^1 \sigma_1^2 | A \rangle &= -1 & \langle A | \sigma_1^1 \sigma_2^2 | A \rangle &= 0 & \langle A | \sigma_1^1 \sigma_3^2 | A \rangle &= 0 \\
\langle A | \sigma_2^1 \sigma_1^2 | A \rangle &= 0 & \langle A | \sigma_2^1 \sigma_2^2 | A \rangle &= -1 & \langle A | \sigma_2^1 \sigma_3^2 | A \rangle &= 0 \\
\langle A | \sigma_3^1 \sigma_1^2 | A \rangle &= 0 & \langle A | \sigma_3^1 \sigma_2^2 | A \rangle &= 0 & \langle A | \sigma_3^1 \sigma_3^2 | A \rangle &= -1
\end{aligned} \tag{A.36}$$

Consequently, the expectation value of the part of the Lamb Shift Hamiltonian ($H_{LS}^{(12)}$) with respect to the symmetric state can be written as:

$$\begin{aligned}
\delta E_S^{(12)} = \langle S | H_{LS}^{(12)} | S \rangle &= -\frac{i}{2} [H_{11}^{(12)} \langle S | \sigma_1^1 \sigma_1^2 | S \rangle \cos(\alpha^1) \cos(\alpha^2) + H_{12}^{(12)} \langle S | \sigma_1^1 \sigma_2^2 | S \rangle \cos(\alpha^1) \cos(\beta^2) \\
&\quad + H_{13}^{(12)} \langle S | \sigma_1^1 \sigma_3^2 | S \rangle \cos(\alpha^1) \cos(\gamma^2) + H_{21}^{(12)} \langle S | \sigma_2^1 \sigma_1^2 | S \rangle \cos(\beta^1) \cos(\alpha^2) \\
&\quad + H_{22}^{(12)} \langle S | \sigma_2^1 \sigma_2^2 | S \rangle \cos(\beta^1) \cos(\beta^2) + H_{23}^{(12)} \langle S | \sigma_2^1 \sigma_3^2 | S \rangle \cos(\beta^1) \cos(\gamma^2) \\
&\quad + H_{31}^{(12)} \langle S | \sigma_3^1 \sigma_1^2 | S \rangle \cos(\gamma^1) \cos(\alpha^2) + H_{32}^{(12)} \langle S | \sigma_3^1 \sigma_2^2 | S \rangle \cos(\gamma^1) \cos(\beta^2) \\
&\quad + H_{33}^{(12)} \langle S | \sigma_3^1 \sigma_3^2 | S \rangle \cos(\gamma^1) \cos(\gamma^2)] \\
&= \frac{i}{2} [H_{11}^{(12)} \cos(\alpha^1) \cos(\alpha^2) + H_{22}^{(12)} \cos(\beta^1) \cos(\beta^2) + H_{33}^{(12)} \cos(\gamma^1) \cos(\gamma^2)]. \tag{A.37}
\end{aligned}$$

For the antisymmetric state of two entangled atoms ($|A\rangle$) the expectation values of all the possible Pauli tensor operators, which are explicitly contributing in the part of the Lamb Shift Hamiltonian ($H_{LS}^{(21)}$) are given by:

$$\begin{aligned}
\langle A | \sigma_1^2 \sigma_1^1 | A \rangle &= -1 & \langle A | \sigma_1^2 \sigma_2^1 | A \rangle &= 0 & \langle A | \sigma_1^2 \sigma_3^1 | A \rangle &= 0 \\
\langle A | \sigma_2^2 \sigma_1^1 | A \rangle &= 0 & \langle A | \sigma_2^2 \sigma_2^1 | A \rangle &= -1 & \langle A | \sigma_2^2 \sigma_3^1 | A \rangle &= 0 \\
\langle A | \sigma_3^2 \sigma_1^1 | A \rangle &= 0 & \langle A | \sigma_3^2 \sigma_2^1 | A \rangle &= 0 & \langle A | \sigma_3^2 \sigma_3^1 | A \rangle &= -1
\end{aligned} \tag{A.38}$$

Consequently, the expectation value of the part of the Lamb Shift Hamiltonian ($H_{LS}^{(21)}$) with respect to the antisymmetric state can be written as:

$$\begin{aligned}
\delta E_A^{(21)} = \langle A | H_{LS}^{(21)} | A \rangle &= -\frac{i}{2} [H_{11}^{(21)} \langle A | \sigma_1^2 \sigma_1^1 | A \rangle \cos(\alpha^2) \cos(\alpha^1) + H_{12}^{(21)} \langle A | \sigma_1^2 \sigma_2^1 | A \rangle \cos(\alpha^2) \cos(\beta^1) \\
&\quad + H_{13}^{(21)} \langle A | \sigma_1^2 \sigma_3^1 | A \rangle \cos(\alpha^2) \cos(\gamma^1) + H_{21}^{(21)} \langle A | \sigma_2^2 \sigma_1^1 | A \rangle \cos(\beta^2) \cos(\alpha^1) \\
&\quad + H_{22}^{(21)} \langle A | \sigma_2^2 \sigma_2^1 | A \rangle \cos(\beta^2) \cos(\beta^1) + H_{23}^{(21)} \langle A | \sigma_2^2 \sigma_3^1 | A \rangle \cos(\beta^2) \cos(\gamma^1) \\
&\quad + H_{31}^{(21)} \langle A | \sigma_3^2 \sigma_1^1 | A \rangle \cos(\gamma^2) \cos(\alpha^1) + H_{32}^{(21)} \langle A | \sigma_3^2 \sigma_2^1 | A \rangle \cos(\gamma^2) \cos(\beta^1) \\
&\quad + H_{33}^{(21)} \langle A | \sigma_3^2 \sigma_3^1 | A \rangle \cos(\gamma^2) \cos(\gamma^1)] \\
&= \frac{i}{2} [H_{11}^{(21)} \cos(\alpha^2) \cos(\alpha^1) + H_{22}^{(21)} \cos(\beta^2) \cos(\beta^1) + H_{33}^{(21)} \cos(\gamma^2) \cos(\gamma^1)]. \tag{A.39}
\end{aligned}$$

For the antisymmetric state of two entangled atoms ($|A\rangle$) the expectation values of all the possible Pauli tensor operators, which are explicitly contributing in the part of the Lamb Shift Hamiltonian ($H_{LS}^{(22)}$) are given by:

$$\begin{aligned}
\langle A | \sigma_1^2 \sigma_1^2 | A \rangle &= 1 & \langle A | \sigma_1^2 \sigma_2^2 | A \rangle &= 0 & \langle A | \sigma_1^2 \sigma_3^2 | A \rangle &= 0 \\
\langle A | \sigma_2^2 \sigma_1^2 | A \rangle &= 0 & \langle A | \sigma_2^2 \sigma_2^2 | A \rangle &= 1 & \langle A | \sigma_2^2 \sigma_3^2 | A \rangle &= 0 \\
\langle A | \sigma_3^2 \sigma_1^2 | A \rangle &= 0 & \langle A | \sigma_3^2 \sigma_2^2 | A \rangle &= 0 & \langle A | \sigma_3^2 \sigma_3^2 | A \rangle &= 1
\end{aligned} \tag{A.40}$$

Consequently, the expectation value of the part of the Lamb Shift Hamiltonian ($H_{LS}^{(22)}$) with respect to the antisymmetric state can be written as:

$$\begin{aligned}
\delta E_A^{(22)} = \langle A | H_{LS}^{(22)} | A \rangle &= -\frac{i}{2} [H_{11}^{(22)} \langle A | \sigma_1^2 \sigma_1^2 | A \rangle \cos^2(\alpha^2) + H_{12}^{(22)} \langle A | \sigma_1^2 \sigma_2^2 | A \rangle \cos(\alpha^2) \cos(\beta^2) \\
&\quad + H_{13}^{(22)} \langle A | \sigma_1^2 \sigma_3^2 | A \rangle \cos(\alpha^2) \cos(\gamma^2) + H_{21}^{(22)} \langle A | \sigma_2^2 \sigma_1^2 | A \rangle \cos(\beta^2) \cos(\alpha^2) \\
&\quad + H_{22}^{(22)} \langle A | \sigma_2^2 \sigma_2^2 | A \rangle \cos^2(\beta^2) + H_{23}^{(22)} \langle A | \sigma_2^2 \sigma_3^2 | A \rangle \cos(\beta^2) \cos(\gamma^2) \\
&\quad + H_{31}^{(22)} \langle A | \sigma_3^2 \sigma_1^2 | A \rangle \cos(\gamma^2) \cos(\alpha^2) + H_{32}^{(22)} \langle A | \sigma_3^2 \sigma_2^2 | A \rangle \cos(\gamma^2) \cos(\beta^2) \\
&\quad + H_{33}^{(22)} \langle A | \sigma_3^2 \sigma_3^2 | A \rangle \cos^2(\gamma^2)] \\
&= -\frac{i}{2} [H_{11}^{(22)} \cos^2(\alpha^2) + H_{22}^{(22)} \cos^2(\beta^2) + H_{33}^{(22)} \cos^2(\gamma^2)]. \tag{A.41}
\end{aligned}$$

After that, summing over all the possible contributions obtained for the antisymmetric state of two entangled atoms ($|A\rangle$) the expectation value of the Lamd Shift Hamiltonian can be expressed as:

$$\begin{aligned}
\delta E_A &= \sum_{i=1}^2 \sum_{j=1}^2 \delta E_A^{(ij)} = \sum_{i=1}^2 \sum_{j=1}^2 \langle A | H_{LS}^{(ij)} | A \rangle \\
&= -\frac{i}{2} [H_{11}^{(11)} \cos^2(\alpha^1) + H_{11}^{(22)} \cos^2(\alpha^2) + H_{22}^{(11)} \cos^2(\beta^1) + H_{22}^{(22)} \cos^2(\beta^2) + H_{33}^{(11)} \cos^2(\gamma^1) + H_{33}^{(22)} \cos^2(\gamma^2) \\
&\quad - (H_{12}^{(11)} + H_{21}^{(11)}) \cos(\alpha^1) \cos(\alpha^2) - (H_{12}^{(22)} + H_{21}^{(22)}) \cos(\beta^1) \cos(\beta^2) \\
&\quad - (H_{33}^{(12)} + H_{33}^{(21)}) \cos(\gamma^1) \cos(\gamma^2)]. \tag{A.42}
\end{aligned}$$

B Coefficients of the Hilbert transformation of Wightman function of probe massless scalar field

Now, using the Hilbert transformations mentioned in the earlier section one can easily fix the elements of the effective Hamiltonian matrix $H_{ij}^{(\alpha\beta)}$ as:

$$H_{ij}^{(11)} = H_{ij}^{(22)} = \mathcal{A}_1 \delta_{ij} - i\mathcal{B}_1 \epsilon_{ijk} \delta_{3k} - \mathcal{A}_1 \delta_{3i} \delta_{3j}, \tag{B.1}$$

$$H_{ij}^{(12)} = H_{ij}^{(21)} = \mathcal{A}_2 \delta_{ij} - i\mathcal{B}_2 \epsilon_{ijk} \delta_{3k} - \mathcal{A}_2 \delta_{3i} \delta_{3j}. \tag{B.2}$$

In the above set of equations, \mathcal{A}_1 , \mathcal{B}_1 , \mathcal{A}_2 and \mathcal{B}_2 are defined as:

$$\begin{aligned}
\mathcal{A}_1 &= \frac{\mu^2}{4} [\mathcal{K}^{(11)}(\omega_0) + \mathcal{K}^{(11)}(-\omega_0)] \\
&= \frac{\mu^2}{4} [\mathcal{K}^{(22)}(\omega_0) + \mathcal{K}^{(22)}(-\omega_0)] \\
&= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \left[\frac{\mathcal{G}^{11}(\omega)}{\omega + \omega_0} + \frac{\mathcal{G}^{11}(-\omega)}{\omega - \omega_0} \right] \\
&= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \left[\frac{\mathcal{G}^{22}(\omega)}{\omega + \omega_0} + \frac{\mathcal{G}^{22}(-\omega)}{\omega - \omega_0} \right] \\
&= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\Delta\tau G^{11}(\Delta\tau) \left[\frac{e^{i\omega\Delta\tau}}{\omega + \omega_0} + \frac{e^{-i\omega\Delta\tau}}{\omega - \omega_0} \right] \\
&= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\Delta\tau G^{22}(\Delta\tau) \left[\frac{e^{i\omega\Delta\tau}}{\omega + \omega_0} + \frac{e^{-i\omega\Delta\tau}}{\omega - \omega_0} \right], \tag{B.3}
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_1 &= \frac{\mu^2}{4} [\mathcal{K}^{(11)}(\omega_0) - \mathcal{K}^{(11)}(-\omega_0)] \\
&= \frac{\mu^2}{4} [\mathcal{K}^{(22)}(\omega_0) - \mathcal{K}^{(22)}(-\omega_0)] \\
&= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \left[\frac{\mathcal{G}^{11}(\omega)}{\omega + \omega_0} - \frac{\mathcal{G}^{11}(-\omega)}{\omega - \omega_0} \right] \\
&= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \left[\frac{\mathcal{G}^{22}(\omega)}{\omega + \omega_0} - \frac{\mathcal{G}^{22}(-\omega)}{\omega - \omega_0} \right] \\
&= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\Delta\tau G^{11}(\Delta\tau) \left[\frac{e^{i\omega\Delta\tau}}{\omega + \omega_0} - \frac{e^{-i\omega\Delta\tau}}{\omega - \omega_0} \right] \\
&= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\Delta\tau G^{22}(\Delta\tau) \left[\frac{e^{i\omega\Delta\tau}}{\omega + \omega_0} - \frac{e^{-i\omega\Delta\tau}}{\omega - \omega_0} \right], \tag{B.4}
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_2 &= \frac{\mu^2}{4} \left[\mathcal{K}^{(12)}(\omega_0) + \mathcal{K}^{(12)}(-\omega_0) \right] \\
&= \frac{\mu^2}{4} \left[\mathcal{K}^{(21)}(\omega_0) + \mathcal{K}^{(21)}(-\omega_0) \right] \\
&= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \left[\frac{\mathcal{G}^{12}(\omega)}{\omega + \omega_0} + \frac{\mathcal{G}^{12}(-\omega)}{\omega - \omega_0} \right] \\
&= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \left[\frac{\mathcal{G}^{12}(\omega)}{\omega + \omega_0} + \frac{\mathcal{G}^{12}(-\omega)}{\omega - \omega_0} \right] \\
&= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\Delta\tau G^{12}(\Delta\tau) \left[\frac{e^{i\omega\Delta\tau}}{\omega + \omega_0} + \frac{e^{-i\omega\Delta\tau}}{\omega - \omega_0} \right] \\
&= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\Delta\tau G^{22}(\Delta\tau) \left[\frac{e^{i\omega\Delta\tau}}{\omega + \omega_0} + \frac{e^{-i\omega\Delta\tau}}{\omega - \omega_0} \right], \tag{B.5}
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_2 &= \frac{\mu^2}{4} \left[\mathcal{K}^{(12)}(\omega_0) - \mathcal{K}^{(12)}(-\omega_0) \right] \\
&= \frac{\mu^2}{4} \left[\mathcal{K}^{(21)}(\omega_0) - \mathcal{K}^{(21)}(-\omega_0) \right] \\
&= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \left[\frac{\mathcal{G}^{12}(\omega)}{\omega + \omega_0} - \frac{\mathcal{G}^{12}(-\omega)}{\omega - \omega_0} \right] \\
&= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \left[\frac{\mathcal{G}^{12}(\omega)}{\omega + \omega_0} - \frac{\mathcal{G}^{12}(-\omega)}{\omega - \omega_0} \right] \\
&= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\Delta\tau G^{12}(\Delta\tau) \left[\frac{e^{i\omega\Delta\tau}}{\omega + \omega_0} - \frac{e^{-i\omega\Delta\tau}}{\omega - \omega_0} \right] \\
&= \frac{\mu^2 P}{4\pi i} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\Delta\tau G^{22}(\Delta\tau) \left[\frac{e^{i\omega\Delta\tau}}{\omega + \omega_0} - \frac{e^{-i\omega\Delta\tau}}{\omega - \omega_0} \right], \tag{B.6}
\end{aligned}$$

where $\mathcal{K}^{\alpha\beta}(\pm\omega_0)\forall(\alpha, \beta = 1, 2)$ represents the Hilbert transform of the two point function in Fourier space, which we have defined earlier. In the next subsection we will explicitly compute the mathematical structure of all possible two point functions and the corresponding Hilbert transformations in de Sitter space. Now, the corresponding Hilbert transformations of the Fourier transformed Wightman functions, which we have used throughout our rest of the computation are expressed as:

$$\textbf{Hilbert transformed auto correlation : } \mathcal{K}^{11}(\omega_0) = \mathcal{K}^{22}(\omega_0) = \frac{P}{2\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{1}{\omega - \omega_0} \frac{\omega}{1 - e^{2\pi k\omega}}, \tag{B.7}$$

$$\textbf{Hilbert transformed cross correlation : } \mathcal{K}^{12}(\omega_0) = \mathcal{K}^{21}(\omega_0) = \frac{P}{2\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{1}{\omega - \omega_0} \frac{\omega}{1 - e^{2\pi k\omega}} f(\omega, L/2). \tag{B.8}$$

Further using Eq (B.7) and Eq (B.8), in Eq (B.3), Eq (B.4), Eq (B.5) and Eq (B.6) we get the following simplified expression for \mathcal{A}_1 , \mathcal{B}_1 , \mathcal{A}_2 and \mathcal{B}_2 :

$$\mathcal{A}_1 = \frac{\mu^2 P}{4\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{\omega^2}{(\omega + \omega_0)(\omega - \omega_0)(1 - e^{-2\pi k\omega})}, \tag{B.9}$$

$$\mathcal{B}_1 = \frac{\mu^2 P}{4\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{\omega_0 \omega}{(\omega + \omega_0)(\omega - \omega_0)(1 - e^{-2\pi k\omega})}, \tag{B.10}$$

$$\mathcal{A}_2 = \frac{\mu^2 P}{4\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{\omega^2 f(\omega, L/2)}{(\omega + \omega_0)(\omega - \omega_0)(1 - e^{-2\pi k\omega})}, \tag{B.11}$$

$$\mathcal{B}_2 = \frac{\mu^2 P}{4\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{\omega_0 \omega f(\omega, L/2)}{(\omega + \omega_0)(\omega - \omega_0)(1 - e^{-2\pi k\omega})}. \tag{B.12}$$

Finally, using these above sets of Hilbert transformations one can easily fix the elements of the co-efficient matrix $H_{ij}^{(\alpha\beta)}$, which are given by:

$$\text{Diagonal matrix elements : } H_{ij}^{(11)} = H_{ij}^{(22)} = \frac{\mu^2 P}{4\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{\omega \{(\delta_{ij} - \delta_{3i}\delta_{3j})\omega - i\epsilon_{ijk}\delta_{3k}\omega_0\}}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)}, \quad (\text{B.13})$$

$$\text{Off - diagonal matrix elements : } H_{ij}^{(12)} = H_{ij}^{(21)} = \frac{\mu^2 P}{4\pi^2 i} \int_{-\infty}^{\infty} d\omega \frac{\omega \{(\delta_{ij} - \delta_{3i}\delta_{3j})\omega - i\epsilon_{ijk}\delta_{3k}\omega_0\} f(\omega, L/2)}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)}. \quad (\text{B.14})$$

Similarly, in the present context the elements of the *GoriniKossakowskiSudarshanLindblad matrix*, $C_{ij}^{(\alpha\beta)}$, as appearing in the expression for the *Linbladian* can be expressed as:

$$C_{ij}^{(11)} = C_{ij}^{(22)} = \tilde{\mathcal{A}}_1 \delta_{ij} - i\tilde{\mathcal{B}}_1 \epsilon_{ijk} \delta_{3k} - \tilde{\mathcal{A}}_1 \delta_{3i} \delta_{3j}, \quad (\text{B.15})$$

$$C_{ij}^{(12)} = C_{ij}^{(21)} = \tilde{\mathcal{A}}_2 \delta_{ij} - i\tilde{\mathcal{B}}_2 \epsilon_{ijk} \delta_{3k} - \tilde{\mathcal{A}}_2 \delta_{3i} \delta_{3j}. \quad (\text{B.16})$$

In the above set of equations $\tilde{\mathcal{A}}_1$, $\tilde{\mathcal{B}}_1$, $\tilde{\mathcal{A}}_2$ and $\tilde{\mathcal{B}}_2$ for the two atomic system are defined as:

$$\begin{aligned} \tilde{\mathcal{A}}_1 &= \frac{\mu^2}{4} \left[\mathcal{G}^{(11)}(\omega_0) + \mathcal{G}^{(11)}(-\omega_0) \right] \\ &= \frac{\mu^2}{4} \left[\mathcal{G}^{(22)}(\omega_0) + \mathcal{G}^{(22)}(-\omega_0) \right] \\ &= \frac{\mu^2}{8\pi} \omega_0 \left[\frac{1}{1 - e^{-2\pi k\omega_0}} - \frac{1}{1 - e^{2\pi k\omega_0}} \right], \end{aligned} \quad (\text{B.17})$$

$$\begin{aligned} \tilde{\mathcal{B}}_1 &= \frac{\mu^2}{4} \left[\mathcal{G}^{(11)}(\omega_0) - \mathcal{G}^{(11)}(-\omega_0) \right] \\ &= \frac{\mu^2}{4} \left[\mathcal{G}^{(22)}(\omega_0) - \mathcal{G}^{(22)}(-\omega_0) \right] \\ &= \frac{\mu^2}{8\pi} \omega_0 \left[\frac{1}{1 - e^{-2\pi k\omega_0}} + \frac{1}{1 - e^{2\pi k\omega_0}} \right], \end{aligned} \quad (\text{B.18})$$

$$\begin{aligned} \tilde{\mathcal{A}}_2 &= \frac{\mu^2}{4} \left[\mathcal{G}^{(12)}(\omega_0) + \mathcal{G}^{(12)}(-\omega_0) \right] \\ &= \frac{\mu^2}{4} \left[\mathcal{G}^{(21)}(\omega_0) + \mathcal{G}^{(21)}(-\omega_0) \right] \\ &= \frac{\mu^2}{8\pi} \omega_0 \left[\frac{f(\omega_0, L/2)}{1 - e^{-2\pi k\omega_0}} - \frac{f(-\omega_0, L/2)}{1 - e^{2\pi k\omega_0}} \right], \end{aligned} \quad (\text{B.19})$$

$$\begin{aligned} \tilde{\mathcal{B}}_2 &= \frac{\mu^2}{4} \left[\mathcal{G}^{(12)}(\omega_0) - \mathcal{G}^{(12)}(-\omega_0) \right] \\ &= \frac{\mu^2}{4} \left[\mathcal{G}^{(21)}(\omega_0) - \mathcal{G}^{(21)}(-\omega_0) \right] \\ &= \frac{\mu^2}{8\pi} \omega_0 \left[\frac{f(\omega_0, L/2)}{1 - e^{-2\pi k\omega_0}} + \frac{f(-\omega_0, L/2)}{1 - e^{2\pi k\omega_0}} \right]. \end{aligned} \quad (\text{B.20})$$

C Calculation of Bethe regularised spectroscopic integrals

In the following subsections we explicitly compute the *Bethe regularised* integrals, which are very useful to compute the expressions for the energy shift from ground, excited, symmetric and antisymmetric state respectively. These integrals are appended below:

$$\text{Integral I : } \Delta_1 : = \int_{-\infty}^{\infty} d\omega \frac{2\omega_0 \omega}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)}. \quad (\text{C.1})$$

$$\text{Integral II : } \Delta_2 : = \int_{-\infty}^{\infty} d\omega \frac{\omega^2}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)}. \quad (\text{C.2})$$

$$\text{Integral III : } \Delta_3 : = \int_{-\infty}^{\infty} d\omega \frac{2\omega^2 f(\omega, L/2)}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)}. \quad (\text{C.3})$$

C.1 Spectroscopic Integral I

In this subsection we explicitly compute the finite contribution from the following integral:

$$\Delta_1 := \int_{-\infty}^{\infty} d\omega \frac{2\omega_0 \omega}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)}. \quad (\text{C.4})$$

It is important to note that in the large frequency range, $-\infty < \omega < \infty$, one can further expand the integrand by taking large ω approximation as:

$$\mathcal{F}(\omega_0, \omega, k) := \frac{2\omega_0 \omega}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)} \xrightarrow{\omega \rightarrow \infty} \frac{2\omega_0 \omega}{(\omega + \omega_0)(\omega - \omega_0)} := \mathcal{F}(\omega_0, \omega). \quad (\text{C.5})$$

This implies that, after taking large ω approximation the integrand of Δ_1 becomes independent of the parameter k , which representing the surface gravity.

Now, further using this approximation the integral Δ_1 can be further simplified as:

$$\Delta_1 \approx \int_{-\infty}^{\infty} d\omega \mathcal{F}(\omega_0, \omega) = \underbrace{\int_{-\infty}^0 d\omega \mathcal{F}(\omega_0, \omega)}_{\equiv \mathcal{U}_1(\omega_0)} + \underbrace{\int_0^{\infty} d\omega \mathcal{F}(\omega_0, \omega)}_{\equiv \mathcal{U}_2(\omega_0)}, \quad (\text{C.6})$$

where we have decomposed the integrals into two parts, indicated by $\mathcal{U}_1(\omega_0)$ and $\mathcal{U}_2(\omega_0)$ in the parenthesis symbol. Now, here we see that in the large frequency range, $-\infty < \omega < \infty$, we get:

$$\mathcal{U}_1(\omega_0) = \int_{-\infty}^0 d\omega \mathcal{F}(\omega_0, \omega) = - \int_0^{\infty} d\omega \mathcal{F}(\omega_0, \omega) = -\mathcal{U}_2(\omega_0). \quad (\text{C.7})$$

Now, we see here that both $\mathcal{U}_1(\omega_0)$ and $\mathcal{U}_2(\omega_0)$ gives divergent contributions in the frequency range, $-\infty < \omega < 0$ and $0 < \omega < \infty$. To get the finite contributions from these integrals we introduce a cut-off regulator ω_c , by following *Bethe regularisation* technique. After introducing this cut-off we get:

$$\mathcal{U}_1(\omega_0, \omega_c) = \int_{-\omega_c}^0 d\omega \mathcal{F}(\omega_0, \omega) = - \int_0^{\omega_c} d\omega \mathcal{F}(\omega_0, \omega) = -\mathcal{U}_2(\omega_0, \omega_c) = -\omega_0 \ln \left[1 - \left(\frac{\omega_c}{\omega_0} \right)^2 \right]. \quad (\text{C.8})$$

Consequently, we get the following expression for the integral Δ_1 , as given by:

$$\text{Integral I: } \Delta_1 = \mathcal{U}_1(\omega_0, \omega_c) + \mathcal{U}_2(\omega_0, \omega_c) = \omega_0 \ln \left[1 - \left(\frac{\omega_c}{\omega_0} \right)^2 \right] - \omega_0 \ln \left[1 - \left(\frac{\omega_c}{\omega_0} \right)^2 \right] = 0. \quad (\text{C.9})$$

C.2 Spectroscopic Integral II

In this subsection we explicitly compute the finite contribution from the following integral:

$$\Delta_2 := \int_{-\infty}^{\infty} d\omega \frac{\omega^2}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)}. \quad (\text{C.10})$$

It is important to note that in the large frequency range, $-\infty < \omega < \infty$, one can further expand the integrand by taking large ω approximation as:

$$\mathcal{E}(\omega_0, \omega, k) := \frac{\omega^2}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)} \xrightarrow{\omega \rightarrow \infty} \frac{\omega^2}{(\omega + \omega_0)(\omega - \omega_0)} := \mathcal{E}(\omega_0, \omega). \quad (\text{C.11})$$

This implies that, after taking large ω approximation the integrand of Δ_2 becomes independent of the parameter k , which representing the surface gravity.

Now, further using this approximation the integral Δ_2 can be further simplified as:

$$\Delta_2 \approx \int_{-\infty}^{\infty} d\omega \mathcal{E}(\omega_0, \omega) = \underbrace{\int_{-\infty}^0 d\omega \mathcal{E}(\omega_0, \omega)}_{\equiv \mathcal{W}_1(\omega_0)} + \underbrace{\int_0^{\infty} d\omega \mathcal{E}(\omega_0, \omega)}_{\equiv \mathcal{W}_2(\omega_0)}, \quad (\text{C.12})$$

where we have decomposed the integrals into two parts, indicated by $\mathcal{W}_1(\omega_0)$ and $\mathcal{W}_2(\omega_0)$ in the parenthesis symbol. Now, here we see that in the large frequency range, $-\infty < \omega < \infty$, we get:

$$\mathcal{W}_1(\omega_0) = \int_{-\infty}^0 d\omega \mathcal{E}(\omega_0, \omega) = - \int_0^{\infty} d\omega \mathcal{E}(\omega_0, \omega) = -\mathcal{W}_2(\omega_0). \quad (\text{C.13})$$

Now, we see here that both $\mathcal{W}_1(\omega_0)$ and $\mathcal{W}_2(\omega_0)$ gives divergent contributions in the frequency range, $-\infty < \omega < 0$ and $0 < \omega < \infty$. To get the finite contributions from these integrals we introduce a cut-off regulator ω_c , by following *Bethe regularisation* technique. After introducing this cut-off we get:

$$\mathcal{W}_1(\omega_0, \omega_c) = \int_{-\omega_c}^0 d\omega \mathcal{E}(\omega_0, \omega) = \int_0^{\omega_c} d\omega \mathcal{E}(\omega_0, \omega) = \mathcal{W}_2(\omega_0, \omega_c) = \omega_c - \omega_0 \tanh^{-1} \left(\frac{\omega_c}{\omega_0} \right). \quad (\text{C.14})$$

Consequently, we get the following expression for the integral Δ_2 , as given by:

$$\Delta_2 = \mathcal{U}_1(\omega_0, \omega_c) + \mathcal{U}_2(\omega_0, \omega_c) = 2 \left\{ \omega_c - \omega_0 \tanh^{-1} \left(\frac{\omega_c}{\omega_0} \right) \right\}. \quad (\text{C.15})$$

Now, if we further use the approximation that, $\omega_c \ll \omega_0$ i.e. the *Bethe regularised* cut-off is smaller than the natural frequency of the two entangled atomic system under consideration, then we get ⁹:

$$\textbf{Spectroscopic Integral II : } \Delta_2 = \mathcal{U}_1(\omega_0, \omega_c) + \mathcal{U}_2(\omega_0, \omega_c) = 2 \left\{ \omega_c - \omega_0 \left(\frac{\omega_c}{\omega_0} \right) \right\} = 0. \quad (\text{C.17})$$

C.3 Spectroscopic Integral III

In this subsection we explicitly compute the finite contribution from the following integral:

$$\Delta_3 := \int_{-\infty}^{\infty} d\omega \frac{2\omega^2 f(\omega, L/2)}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)}, \quad (\text{C.18})$$

where, we define the spectral function $f(\omega, L/2)$ as:

$$\textbf{Spectral function : } f(\omega, L/2) = \frac{1}{L\omega \sqrt{1 + \left(\frac{L}{2k}\right)^2}} \sin \left(2k\omega \sinh^{-1} \left(\frac{L}{2k} \right) \right). \quad (\text{C.19})$$

It is important to note that in the large frequency range, $-\infty < \omega < \infty$, one can further expand the integrand by taking large ω approximation as:

$$\mathcal{O}(\omega_0, \omega, k) := \frac{2\omega^2 f(\omega, L/2)}{(1 - e^{-2\pi k\omega})(\omega + \omega_0)(\omega - \omega_0)} \xrightarrow{\omega \rightarrow \infty} \frac{2\omega^2 f(\omega, L/2)}{(\omega + \omega_0)(\omega - \omega_0)} := \mathcal{O}(\widetilde{\omega_0}, \omega, k). \quad (\text{C.20})$$

This implies that, after taking large ω approximation the integrand of Δ_3 becomes not independent of the parameter k , which representing the surface gravity.

Now, further using this approximation the integral Δ_3 can be further simplified as:

$$\Delta_3 \approx \int_{-\infty}^{\infty} d\omega \mathcal{O}(\widetilde{\omega_0}, \omega, k) = \underbrace{\int_{-\infty}^0 d\omega \mathcal{O}(\widetilde{\omega_0}, \omega, k)}_{\equiv \mathcal{Q}_1(\omega_0, k)} + \underbrace{\int_0^{\infty} d\omega \mathcal{O}(\widetilde{\omega_0}, \omega, k)}_{\equiv \mathcal{Q}_2(\omega_0, k)}, \quad (\text{C.21})$$

where we have decomposed the integrals into two parts, indicated by $\mathcal{Q}_1(\omega_0, k)$ and $\mathcal{Q}_2(\omega_0, k)$ in the parenthesis symbol. Now, here we see that in the large frequency range, $-\infty < \omega < \infty$, we get:

$$\mathcal{Q}_1(\omega_0, k) = \int_{-\infty}^0 d\omega \mathcal{O}(\widetilde{\omega_0}, \omega, k) = \int_0^{\infty} d\omega \mathcal{O}(\widetilde{\omega_0}, \omega, k) = \mathcal{Q}_2(\omega_0, k) = \frac{\pi}{L\sqrt{1 + \left(\frac{L}{2k}\right)^2}} \cos \left(2k\omega_0 \sinh^{-1} \left(\frac{L}{2k} \right) \right). \quad (\text{C.22})$$

Consequently, we get the following expression for the integral Δ_3 , as given by:

$$\textbf{Spectroscopic Integral III : } \Delta_3 = \mathcal{Q}_1(\omega_0, k) + \mathcal{Q}_2(\omega_0, k) = \frac{2\pi}{L\sqrt{1 + \left(\frac{L}{2k}\right)^2}} \cos \left(2k\omega_0 \sinh^{-1} \left(\frac{L}{2k} \right) \right). \quad (\text{C.23})$$

⁹In the limit, $\omega_c \ll \omega_0$ we can approximate the Taylor series expansion of the following function as:

$$\tanh^{-1} \left(\frac{\omega_c}{\omega_0} \right) = \left(\frac{\omega_c}{\omega_0} \right) + \frac{1}{3} \left(\frac{\omega_c}{\omega_0} \right)^3 + \dots \approx \left(\frac{\omega_c}{\omega_0} \right). \quad (\text{C.16})$$

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