# Vertex operator algebras, minimal models, and modular linear differential equations of order 4

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**Abstract.** In this paper we classify vertex operator algebras with three conditions which arise from Virasoro minimal models: (A) the central charge and conformal weights are rational numbers, (B) the space spanned by characters of all simple modules of a vertex operator algebra coincides with the space of solutions of a modular linear differential equation of order 4 and (C) the dimensions of first three weight subspaces of a VOA is 1, 0 and 1, respectively. It is shown that vertex operator algebras which we concern have central charges $c = -46/3, -3/5, -114/7, 4/5$, and are isomorphic to minimal models for $c = -46/3, -3/5$ and $\mathbb{Z}_2$-graded simple current extensions of minimal models for $c = -114/7, 4/5$.

## Introduction

In the theory of vertex operator algebras (VOAs), one of the most important properties is the modular invariance of the space spanned by characters of all simple modules. This is proved by Y. Zhu [27, Theorem 5.3.3] and C. Dong, G. Mason and H. Li [4, Theorem 1.3] under two conditions–$C_2$-cofiniteness and rationality. In the proof of the modular invariance of the space spanned by characters, particularly, convergence of characters, it is shown that there exists a linear differential equation with regular singularity only at $q = 0$ whose space of solutions contains the space spanned by characters of all simple modules of a given VOA. This differential equation is now called a **monic modular linear differential equation** (monic MLDE) and is studied together with the theory of vector-valued modular forms (cf. [4], [16], [17]). A monic MLDE is defined by means of the Serre derivation ([16], [17]). The Serre derivation $\vartheta_k = qd/dq - (k/12)E_2$ of weight $k$ is a differential operator acting on meromorphic functions on the upper half plane, where $E_2$ is the "quasimodular" Eisenstein series. The $i$th iterated Serre derivation is defined by $\vartheta_k^i = \vartheta_{k+2(i-1)} \circ \cdots \circ \vartheta_{k+2} \circ \vartheta_k$. A linear differential equation $\vartheta_0^p(f) + \sum_{i=0}^{p-1} P_i \vartheta_0^i(f) = 0$ is called a monic MLDE (of weight 0) if $P_i$ is a holomorphic modular form of weight $2(p-i)$ for each $i$ (see [16], [17]).

One of problems on the study of relationships between VOAs and monic MLDEs is to decide if the space spanned by characters of all simple modules of a VOA coincides with the space of all solutions of a monic MLDE.

One of the most studied examples of VOAs is the simple Virasoro VOA $L(c_{p,q}, 0)$ (cf. [9], [25]) associated with the irreducible highest weight module of the Virasoro algebra, where the central charge $c_{p,q}$ is $1 - 6(p - q)^2/pq$ for coprime positive integers $p$ and $q$. The VOA $L(c_{p,q}, 0)$ is often called the **minimal model** in physics literature. It was shown in [20] and [21] that there is a monic MLDE $L_{p,q}(\vartheta)f = 0$ such that the set of characters of simple $L(c_{p,q}, 0)$-modules forms a fundamental system of solutions of the monic MLDE, where $L_{p,q}(\vartheta)$ is a differential operator of the Serre derivation $\vartheta_0$ (We also give a simple proof of this fact in Proposition 6 in §2.).

Another example of VOAs is the affine VOA $L_{\mathfrak{g}}(k, 0)$ associated with the irreducible highest weight module of a finite-dimensional simple Lie algebra $\mathfrak{g}$ of positive integral level $k$ ([9]). We showed in [1] that if the dimension of the space $\mathcal{X}_{\mathfrak{g},k}$ spanned by characters of all simple $L_{\mathfrak{g}}(k, 0)$-modules is between 1 and 6, then $\mathcal{X}_{\mathfrak{g},k}$ coincides with the space of solutions of a monic MLDE. However, we also found several affine VOAs, $L_{A_2}(3, 0)$, etc., whose spaces of characters are not the spaces of solutions of any monic MLDEs.

The problem which is intensively studied in this paper is a classification of VOAs whose spaces spanned by characters of all simple modules coincide with the spaces of solutions of monic MLDEs of given orders. Since a monic MLDE of order 1 is $(qd/dq)f = 0$, any solution is a constant function. Therefore, the corresponding VOA is trivial. In [19], S. D. Mathur, S. Mukhi and A. Sen classified the rational conformal field theories whose partition functions (characters) satisfy monic MLDEs of order 2. In the language of VOAs their classification is interpreted as such VOAs are isomorphic to affine VOAs of level 1 associated with (the Deligne exceptional series) $A_1$, $A_2$, $G_2$, $D_4$, $F_4$, $E_6$, $E_7$, and the minimal model $L(c_{2,5}, 0)$. The 3rd order case is studied in [2], [10] and [24]. Since there are infinitely many VOAs whose spaces of characters of simple modules span the spaces of solutions of monic MLDEs of order 3 ([1]), we need an extra condition to obtain finite number of corresponding VOAs. In [2], [10] and [24], it is also assumed that the weight 1 subspace of a VOA is trivial. Then we obtain 8 candidate central charges and show that 6 candidate central charges have corresponding VOAs and the other central charges do not have corresponding $C_2$-cofinite and rational VOAs ([3]). However, it is not known if each of 6 central charges has the unique corresponding VOA.

Our main purpose of this paper is to classify the VOAs whose spaces spanned by characters of all simple modules coincide with spaces of solutions of monic MLDEs of **order** 4 under several conditions on VOAs, which may characterize the minimal models.

We first give several results on relations between (extensions of) minimal models and monic MLDEs, which are necessary to prove one of our main theorems. Let $p$ and $q$ be coprime positive integers such that $L(c_{p,q}, 0)$ does not have positive integral conformal weights. We can prove that if the space of solutions of the monic MLDE $L_{p,q}(\vartheta)f = 0$ contains the space spanned by characters of all simple modules of a VOA $V$ with central charge $c_{p,q}$, then $V$ is isomorphic to the minimal model $L(c_{p,q}, 0)$ (Theorem 8 in §3). This result can be considered as a characterization of the minimal models $L(c_{p,q}, 0)$ for specific pairs $(p, q)$ by means of the monic MLDEs $L_{p,q}(\vartheta)f = 0$.

We also study an extension of a minimal model by its simple module, which is

called a $\mathbb{Z}_2$-graded *simple current extension* (SCE). It is known that a $\mathbb{Z}_2$-graded SCE of $L(c_{p,q}, 0)$ uniquely exists if $(p, q) = (2t + 1, 4u + 2)$ for some positive integers $t$ and $u$. We also characterize $\mathbb{Z}_2$-graded SCEs of minimal models by monic MLDEs. More precisely, we prove that there is a unique monic MLDE $\hat{L}_{p,q}(\vartheta)f = 0$ whose space of solutions coincides with the space spanned by characters of all simple modules of the $\mathbb{Z}_2$-graded SCE of $L(c_{p,q}, 0)$ (Theorem 11 in §4). Then we show that if the space spanned by characters of all simple modules of a VOA with central charge $c_{p,q}$ is realized by the monic MLDE $\hat{L}_{p,q}(\vartheta)f = 0$, then this VOA is isomorphic to the $\mathbb{Z}_2$-graded SCE (Theorem 13 in §4).

We next prove our main result in this paper. Taking into account of the properties of the minimal models we suppose that a central charge and conformal weights of a VOA $V$ are *rational numbers* and that $\dim V_0 = 1$, $\dim V_1 = 0$, and $\dim V_2 = 1$. Then it is shown in [5, Lemma 5.3] that this VOA has no nonzero negative weight spaces, and therefore, the character $\mathrm{ch}_V$ of $V$ is expressed as $\mathrm{ch}_V = q^{-c/24}(1 + q^2 + mq^3 + O(q^4))$, where $c$ denotes the central charge of $V$ and $m$ is the dimension of $V_3$. By substituting $\mathrm{ch}_V$ to the general form of monic MLDEs of order 4, we can determine monic MLDEs uniquely (see §5). There are 2 monic MLDEs with generic parameters and 6 monic MLDEs.

The list of monic MLDEs that we obtain contains the monic MLDEs $L_{2,9}(\vartheta)f = 0$, $L_{3,5}(\vartheta)f = 0$, $\hat{L}_{5,6}(\vartheta)f = 0$ and $\hat{L}_{3,14}(\vartheta)f = 0$. It follows from our characterization that VOAs corresponding to these monic MLDEs are isomorphic to the minimal models $L(c_{2,9}, 0)$, $L(c_{3,5}, 0)$ ($c_{2,9} = -46/3$, $c_{3,5} = -3/5$), and the $\mathbb{Z}_2$-graded SCEs of minimal models with central charges $c_{5,6} = 4/5$, $c_{3,14} = -114/7$ by their simple modules with conformal weight 3, respectively. By solving other monic MLDEs which appear in the paper, we verify that there does not exist VOAs associated with these monic MLDEs and complete a proof of our main result (Theorem 19 in §6).

The paper is organized as follows. In §1 we review the definitions and the basic properties of vector-valued modular functions, monic MLDEs, and several results obtained by G. Mason in [17]. The Frobenius method of solving linear differential equations with regular singularities is also explained here. Several important facts and results of the representation theory of the minimal models are illustrated in §2. In §3 we describe the relations of monic MLDEs and characters of minimal models. The main issue of this section is to give a characterization of minimal models by means of monic MLDEs. In §4 we discuss $\mathbb{Z}_2$-graded SCEs of minimal models and prove that there is a monic MLDE whose space of solutions coincides with the space spanned by characters of all simple modules of a $\mathbb{Z}_2$-graded SCE of a minimal model. A characterization of $\mathbb{Z}_2$-graded SCEs of minimal models by means of monic MLDEs and central charges also proved here. In §5 we give recursive relations of Fourier coefficients of solutions ($q$-series) of monic MLDEs of order 4. We also determine rational numbers which can be central charges and monic MLDEs under our conditions. Finally, we complete a proof of one of our main results in §6 by excluding objects which do not fit our conditions. In Appendix, all solutions of the Diophantine equation appeared in §5 are determined, which is due to D. Zagier.

## 1. Monic modular linear differential equations and vector-valued modular functions

Throughout the paper we denote the complex upper half-plane by $\mathbb{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\}$. For a positive integer $k \geq 2$, let $E_{2k}$ be the *normalized Eisenstein series* of weight $2k$

$$\text{(1)} \qquad E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n, \quad q = e^{2\pi\sqrt{-1}\tau} \quad (\tau \in \mathbb{H}),$$

where $\sigma_m(n)$ is the division function and $B_n$ is the $n$th Bernoulli number. The quasi-modular form $E_2(\tau)$ is defined by setting $k = 1$ in (1).

DEFINITION. A **vector-valued modular function** (VVMF) on $SL_2(\mathbb{Z})$ is a (column) vector-valued holomorphic function $\mathbb{F} = {}^t\!\left(f_1, \ldots, f_p\right)$ on $\mathbb{H}$ with a $p$-dimensional complex representation $\rho : SL_2(\mathbb{Z}) \to GL_p(\mathbb{C})$ subject to (a) $\mathbb{F}(\gamma(\tau)) = \rho(\gamma)\mathbb{F}(\tau)$ for all $\gamma \in SL_2(\mathbb{Z})$ and (b) each component function $f_j$ has a $q$-expansion $f_i = q^{\lambda_i} \sum_{n=0}^{\infty} a_n^i q^n \, (a_0^i \neq 0, \, \lambda_i \in \mathbb{C})$ on $\mathbb{H}$.

A vector-valued modular function $\mathbb{F}$ is called *normalized* if $f_i = q^{\lambda_i}(1 + \sum_{n=1}^{\infty} a_n^i q^n) \, (1 \leq i \leq m)$, $f_{m+1}, \ldots, f_p = 0$ and $\lambda_1, \ldots, \lambda_m \, f_j = 0$ are mutually distinct. For a VVMF $\mathbb{F}$, there is an invertible matrix $A$ such that $A\mathbb{F}$ is normalized.

The Serre derivation for $k \in \mathbb{Z}_{\geq 0}$ is defined by

$$\vartheta_k(f) = D(f) - \frac{k}{12}E_2 f, \quad D = q\frac{d}{dq},$$

and the $i$th iterated Serre derivation is defined by $\vartheta_k^i = \vartheta_{k+2(i-1)} \circ \cdots \circ \vartheta_{k+2} \circ \vartheta_k$ with $\vartheta_k^0 = 1$. A linear differential equation

$$\text{(2)} \qquad \vartheta_0^p(f) + \sum_{i=0}^{p-1} P_i \vartheta_0^i(f) = 0$$

is called a **monic modular linear differential equation** (monic MLDE for short) if $P_i$ is a **holomorphic modular form** of weight $2(p-i)$. Since any holomorphic modular form of weight 2 is 0, we see that $P_i = 0$ for $i = p - 1$. It was shown by G. Mason in [17, Theorem 4.1] that the space of solutions is invariant under the usual slash action of $SL_2(\mathbb{Z})$ and that $q = 0$ is a unique regular singular point of (2).

THEOREM 1. *Let $\mathbb{F} = {}^t\!(f_1, \ldots, f_p)$ be a normalized vector-valued modular function. Suppose that $\{f_j\}$ is linearly independent and $f_i = q^{\lambda_i}(1 + \sum_{n=1}^{\infty} a_n^i q^n)$. Then the set of component functions $\{f_j\}$ forms a fundamental system of a monic modular linear differential equation of order $p$ if and only if $p(p-1) - 12\sum_{i=1}^{p} \lambda_j = 0$.*

REMARK. In this paper we restrict our attention to vector-valued modular functions (vector-valued modular forms of weight 0). In [17] all results cited in this section are proved for vector-valued modular forms of *non-negative integral weights*.

Since any monic MLDE has a regular singularity at $q = 0$, we can use the Frobenius

method ([11, Chapter XVI]) to obtain a fundamental system of solutions of a monic MLDE. Suppose that $f = q^\rho(1 + \sum_{n=0}^\infty a_n q^n)$ is a solution of (2). By substituting $f$ to (2) and taking the coefficient of $q^\rho$, we see that $\rho$ is a root of the polynomial

$$\Psi(\rho) = \prod_{k=0}^{p-1}\left(\rho - \frac{k}{6}\right) + \sum_{i=0}^{p-2} P_{i,0} \prod_{k=0}^{i-1}\left(\rho - \frac{k}{6}\right), \quad P_i = \sum_{n=0}^\infty P_{i,n} q^n.$$

This polynomial is usually called the **indicial polynomial** (characteristic polynomial) of the monic MLDE (2) and the roots of the indicial equation, that is, $\Psi(\rho) = 0$, are called the **indicial roots** (indices) of (2).

LEMMA 2 ([11, Chapter XVI, pp. 397]). *If $\lambda$ is an indicial root and there are no indicial roots in $\lambda + \mathbb{Z}_{>0}$, then a solution of the form $q^\lambda(1 + \sum_{n=1}^\infty a_n q^n)$ is uniquely determined.*

The coefficients $a_n$ are determined by recursive relations of Fourier coefficients (see §6 for monic MLDEs of order 4).

## 2. Virasoro vertex operator algebras

In this section we review fundamental properties of *Virasoro vertex operator algebras.*

The Virasoro algebra denoted here by **Vir** is a Lie algebra generated by $L_n$ $(n \in \mathbb{Z})$ and a central element $C$ with the commutation relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n} C.$$

A **vertex operator algebra** (VOA) is a quadruple $(V, Y, \mathbf{1}, \omega)$ consisting of a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ with distinguished vectors $\mathbf{1} \in V_0$ and $\omega \in V_2$, called the vacuum vector and the Virasoro element, respectively, and a linear map $Y : V \to \mathrm{End}_\mathbb{C}(V)[[z, z^{-1}]]$ $(a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1})$ which satisfies a number of axioms (cf. [15], [18]). Define $L_n$ for $n \in \mathbb{Z}$ by $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$. Then the set $\{L_n, \mathrm{id}_V \mid n \in \mathbb{Z}\}$ gives a module of the Virasoro algebra on $V$. Particularly, the central element $C$ acts as a scalar $c_V \in \mathbb{C}$ and $c_V$ is called the **central charge** of $V$. For each $n \in \mathbb{Z}$, a homogeneous subspace $V_n$ is a finite dimensional weight space of weight $n$ with respect to $L_0$. A VOA $V$ is said to be **of CFT type** if $V = \bigoplus_{n=0}^\infty V_n$ and $V_0 = \mathbb{C}\mathbf{1}$.

PROPOSITION 3 ([5, Lemma 5.2]). *Let $V$ be a vertex operator algebra. If $V_0 = \mathbb{C}\mathbf{1}$ then $V$ has no nonzero negative weight spaces, that is, $V$ is of CFT type.*

Let $V$ be a VOA with central charge $c_V$. A **weak $V$-module** is a pair $(M, Y_M)$ of a vector space $M$ and a linear map $Y_M : V \to \mathrm{End}_\mathbb{C}(M)[[z, z^{-1}]]$ with naturally required conditions for a module. A weak $V$-module $M$ is also a module of the Virasoro algebra of the central charge $c_V$ by letting $Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$. An **admissible $V$-module** is a weak $V$-module $M$ which carries a $\mathbb{Z}_{\geq 0}$-grading $M = \bigoplus_{n=0}^\infty M(n)$ satisfying $a_m M(m) \subseteq M(k + n - m - 1)$ for $a \in V_k$, $m, k \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 0}$. A weak $V$-module is said to be a **$V$-module** if there are finitely many complex numbers $h_i$ such that $M = \bigoplus_i \bigoplus_{n=0}^\infty M_{h_i+n}$ where $M_h$ is a finite-dimensional eigenspace of $L_0$ with eigenvalue $h$. If $M$ is a simple $V$-module there exists a complex number $h$ such that $M = \bigoplus_{n=0}^\infty M_{h+n}$

and $M_h \neq 0$. This complex number $h$ is called the **conformal weight** of a simple $V$-module $M$. The (formal) character of a simple $V$-module $M$ is defined to be

$$\mathrm{ch}_M(\tau) = \mathrm{tr}_M\, q^{L_0 - c_V/24} = \sum_{n=0}^{\infty}(\dim M_{h+n})q^{h-c_V/24+n}.$$

We can also define the character of a module such as $M = \bigoplus_{n=0}^{\infty} M_{h+n}$ which is not simple. We denote by $\mathcal{X}(V)$ the space spanned by characters of all simple $V$-modules.

It is called that a VOA $V$ is **rational** if any admissible $V$-module is completely reducible and $C_2$-**cofinite** if the subspace of $V$ spanned by elements of the form $a_{-2}b\,(a, b \in V)$ has finite-codimension in $V$. It is shown in [27, Theorem 4.4.1] that if $V$ is $C_2$-cofinite then the number of simple modules is finite up to isomorphisms and each character $\mathrm{ch}_M$ of the simple $V$-module $M$ converges on $\mathbb{H}$. Moreover, if $V$ is rational and $C_2$-cofinite then the space $\mathcal{X}(V)$ is invariant under the usual slash action of $SL_2(\mathbb{Z})$. In other words, the set of characters of all simple $V$-modules forms a finite-dimensional vector-valued modular function.

One of the most important examples of rational and $C_2$-cofinite VOAs is the concept of the minimal models. Let $\mathbf{Vir}_{\geq 0} = \bigoplus_{n=0}^{\infty}\mathbb{C}L_n$, and $c$, $h$ be complex numbers. Let $\mathbb{C}v_{c,h}$ be a 1-dimensional $U(\mathbf{Vir}_{\geq 0})$-module which is defined by $L_n v_{c,h} = h\delta_{n,0}v_{c,h}$ and $Cv_{c,h} = cv_{c,h}$, where $U(\mathfrak{g})$ denotes the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. Then the induced module $M(c,h) = U(\mathbf{Vir}) \otimes_{U(\mathbf{Vir}_{\geq 0})} \mathbb{C}v_{c,h}$ is called the **Verma module** of the Virasoro algebra with central charge $c$ and highest weight $h$, which has the unique maximal proper submodule $J(c,h)$. We denote by $L(c,h) = M(c,h)/J(c,h)$ the irreducible quotient.

Let $\langle L_{-1}v_{c,0}\rangle$ be the submodule of $M(c,0)$, which is generated by $L_{-1}v_{c,0}$. It is obvious that $\langle L_{-1}v_{c,0}\rangle$ is a proper submodule of $M(c,0)$. We denote the quotient module $M(c,0)/\langle L_{-1}v_{c,0}\rangle$ by $V(c,0)$ and by $\mathbf{1}$ the image of $v_{c,0}$ under the natural surjection. It is shown in [9, Theorem 4.3 and Remark (pp. 163)] that $V(c,0)$ and $L(c,0)$ are VOAs with the vacuum vectors $\mathbf{1}$ and the Virasoro elements $\omega = L_{-2}\mathbf{1}$. By Poincaré-Birkhoff-Witt theorem applied to $U(\mathbf{Vir})$ the space $V(c,0)$ has a basis

$$L_{-n_1}\cdots L_{-n_r}\mathbf{1} \quad (r \geq 0,\ n_i \in \mathbb{Z},\ n_1 \geq \cdots \geq n_r > 1)$$

whose weight is defined by $L_0$, that is, $\sum_{i=1}^{r}n_i$. Thus the character of $V(c,0)$ is given by

$$q^{-c/24}\prod_{n>1}(1-q^n)^{-1} = q^{-c/24}\left(1 + q^2 + q^3 + 2q^4 + O(q^5)\right).$$

A homogeneous vector $v$ of a highest weight $\mathbf{Vir}$-module is called **singular** if $L_n v = 0$ for all positive integer $n$ ($n = 1, 2$ are enough by the commutation relations). If $V(c,0)$ contains a singular vector of positive integral weight then $V(c,0)$ is not simple. It was shown in [6] and [7] that $V(c,0)$ contains a singular vector of positive weight if and only if

$$(3) \qquad\qquad\qquad c = c_{p,q} = 1 - \frac{6(p-q)^2}{pq}$$

for coprime positive integers $p$ and $q$. We call the simple Virasoro VOA $L(c_{p,q}, 0)$ the **minimal model**. Let $r$ and $s$ be integers, and write

(4) $$h_{r,s} = \frac{(rq - sp)^2 - (p-q)^2}{4pq}.$$

THEOREM 4 ([25, Theorem 4.2]).    *Let $p, q \in \mathbb{Z}_{>1}$ be coprime integers.*
(a) *The vertex operator algebra $L(c_{p,q}, 0)$ is rational and $C_2$-cofinite.*
(b) *For any simple $L(c_{p,q}, 0)$-module $M$, there exist $1 \leq r \leq p-1$ and $1 \leq s \leq q-1$ such that $M$ is isomorphic to $L(c_{p,q}, h_{r,s})$.*
(c) *Two $L(c_{p,q}, 0)$-modules $L(c_{p,q}, h_1)$ and $L(c_{p,q}, h_2)$ are isomorphic if and only if $h_1 = h_2$.*
(d) *As an $L(c_{p,q}, 0)$-module the rational number $h_{r,s}$ is the conformal weight of $L(c_{p,q}, h_{r,s})$.*

The number of distinct elements of the set $\{h_{r,s} \mid 1 \leq r \leq p-1,\ 1 \leq s \leq q-1\}$, which is equal to the number of inequivalent simple $L(c_{p,q}, 0)$-modules, is $(p-1)(q-1)/2$ partly since $h_{r,s} = h_{p-r, q-s}$.

DEFINITION.    Let $V$ be a vertex operator algebra and $\omega$ its Virasoro element. We denote by $V^\omega$ the vertex operator subalgebra of $V$ generated by $\omega$.

In order to study the subalgebra $V^\omega$ of a VOA $V$ with central charge $c_{p,q}$, we collect several results from the theory of Verma modules of the Virasoro algebra. It is known ([6], [7] and [12, Theorem 6.5]) that any proper submodule of $M(c_{p,q}, 0)$ is generated by at most two singular vectors. In particular, the maximal proper submodule $J(c_{p,q}, 0)$ is generated by $L_{-1}v_{c_{p,q},0}$ and a singular vector $w_{p,q}$ of weight $(p-1)(q-1)$. It is also known that any singular vector except $v_{c,0}$ and $w_{p,q}$ is contained in $\langle L_{-1}v_{c_{p,q},0}\rangle$.

THEOREM 5.    *Let $V$ be a vertex operator algebra with central charge $c_{p,q}$. Then the vertex operator subalgebra $V^\omega$ generated by the Virasoro element $\omega$ of $V$ is isomorphic to either $V(c_{p,q}, 0)$ or $L(c_{p,q}, 0)$.*

PROOF. Because the VOA $V^\omega$ is a highest weight **Vir**-module with central charge $c_{p,q}$, there is a non-zero surjective **Vir**-homomorphism $\phi : M(c_{p,q}, 0) \to V^\omega$. Since $L_{-1}\mathbf{1} = 0$ in $V$, $\ker\phi$ contains $L_{-1}v_{c_{p,q},0}$. If $\ker\phi - \langle L_{-1}v_{c_{p,q},0}\rangle$ does not contain a singular vector and is not empty, then $\ker\phi$ cannot be generated by singular vectors, which contradicts to the fact that every non-zero submodule is generated by singular vectors. Therefore, $\ker\phi - \langle L_{-1}v_{c_{p,q},0}\rangle$ contains a singular vector, otherwise it is empty.

If $\ker\phi - \langle L_{-1}v_{c_{p,q},0}\rangle$ contains a singular vector, then it must be $w_{p,q}$. Since the unique maximal submodule is generated by $L_{-1}v_{c_{p,q},0}$ and $w_{p,q}$, the submodule $\ker\phi$ is maximal and then $V^\omega \cong L(c_{p,q}, 0)$.

Suppose that $\ker\phi - \langle L_{-1}v_{c_{p,q},0}\rangle$ is empty. Then $\ker\phi = \langle L_{-1}v_{c_{p,q},0}\rangle$ and $V^\omega \cong V(c_{p,q}, 0)$ follows. $\square$

COROLLARY 1.    *Let $V$ be a vertex operator algebra with central charge $c_{p,q}$. Suppose that $\mathrm{ch}_{V^\omega} \neq \mathrm{ch}_{V(c_{p,q},0)}$. Then $V^\omega$ is isomorphic to $L(c_{p,q}, 0)$.*

PROOF. Since $\mathrm{ch}_{V^\omega} \neq \mathrm{ch}_{V(c_{p,q},0)}$, we have $V^\omega \not\cong V(c_{p,q},0)$. It then follows from Theorem 5 that $V^\omega$ is isomorphic to $L(c_{p,q},0)$. □

COROLLARY 2. *Let $V$ be a vertex operator algebra. Suppose that $\mathrm{ch}_V = \mathrm{ch}_{L(c_{p,q},0)}$. Then $V$ is isomorphic to $L(c_{p,q},0)$.*

PROOF. By Corollary 1, the VOA $V^\omega$ is isomorphic to $L(c_{p,q},0)$ since $\mathrm{ch}_{V^\omega} \neq ch_{V(c_{p,q},0)}$. Therefore $\dim V_n = \dim V_n^\omega$ for each $n$, which proves the corollary since $V^\omega \subseteq V$. □

## 3. Minimal models and monic modular linear differential equations

In this section we prove that there exists a monic MLDE whose space of solutions coincides with the space spanned by characters of all simple $L(c_{p,q},0)$-modules. This fact is already known in [21, Theorem 7.1]. Here we give a proof in the context of the theory of VVMFs.

The character $\chi_{r,s}$ of $L(c_{p,q},0)$-module $L(c_{p,q},h_{r,s})$ is given by (cf. [12, Theorem 6.13])

$$\chi_{r,s}(q) = \frac{q^{h_{r,s}-c_{p,q}/24}\sum_{n\in\mathbb{Z}}\left(q^{rsn^2+n(qr-ps)} - q^{rsn^2+n(qr+ps)+pq}\right)}{\prod_{n=1}^\infty(1-q^n)}$$

$$= \begin{cases} q^{-c_{p,q}/24}(1+q^2+q^3+\cdots) & h_{r,s}=0, \\ q^{h_{r,s}-c_{p,q}/24}(1+q+q^2+\cdots) & \text{otherwise,} \end{cases}$$

which is rewritten as

$$\chi_{r,s} = \eta(q)^{-1}\left(\theta_{pq,qr-ps} - \theta_{pq,qr+ps}\right),$$

where $\eta(q) = q^{-1/24}\prod_{n=1}^\infty(1-q^n)$ is the Dedekind eta function and $\theta_{a,b} = \sum_{n\in\mathbb{Z}}q^{a(n+b/2a)^2}$ $\left(a,b\in\frac{1}{2}\mathbb{Z}_{\geq0}\right)$ is the theta series (see [12, Corollary 6.1]). Since conformal weights of the simple $L(c_{p,q},0)$-modules are mutually distinct, the characters of simple $L(c_{p,q},0)$-modules give an $n=(p-1)(q-1)/2$-dimensional normalized VVMF.

PROPOSITION 6 ([21, Theorem 7.1]). *There exists a unique monic modular linear differential equation $L_{p,q}(\vartheta)f=0$ of order $n=(p-1)(q-1)/2$ whose space of solutions coincides with the space spanned by characters of all simple $L(c_{p,q},0)$-modules.*

PROOF. Since the lowest power in $q$ of the character $\chi_{r,s}$ is $h_{r,s}-c_{p,q}/24$ by Theorem 4 (d), it follows from (3) and (4) that

$$\sum_{r=1}^{p-1}\sum_{s=1}^{q-1}\left(h_{r,s}-\frac{c_{p,q}}{24}\right) = \frac{(p-1)(q-1)(pq-p-q-1)}{24}.$$

Because each term $h_{r,s}$ appears exactly twice in the left-hand side as $h_{r,s}=h_{p-r,q-s}$, we have

$$\frac{1}{2}\sum_{r=1}^{p-1}\sum_{s=1}^{q-1}\left(h_{r,s}-\frac{c_{p,q}}{24}\right) = \frac{1}{12}\frac{(p-1)(q-1)}{2}\left(\frac{(p-1)(q-1)}{2}-1\right) = \frac{1}{2}n(n-1).$$

Hence the proposition follows from Theorem 1. □

The central charges and conformal weights of $L(c_{p,q}, 0)$ with $(p-1)(q-1)/2 < 5$ are listed in Table 1 (that is, the number of simple modules is not greater than 4). It follows

| $(p,q)$ | Central charge | Conformal weights |
|---------|----------------|-------------------|
| $(2,3)$ | $0$ | $0$ |
| $(2,5)$ | $-22/5$ | $0, -1/5$ |
| $(2,7)$ | $-68/7$ | $0, -2/7, 3/7$ |
| $(3,4)$ | $1/2$ | $0, 1/2, 1/16$ |
| $(2,9)$ | $-46/3$ | $0, -1/3, -5/9, -2/3$ |
| $(3,5)$ | $-3/5$ | $0, -1/20, 1/5, 3/4$ |

Table 1. Central charges and conformal weights of $L(c_{p,q}, 0)$ with $(p-1)(q-1)/2 < 5$

from Theorem 4 and Proposition 6 that the set of indicial roots of $L_{p,q}(\vartheta)f = 0$ is $\{h_{r,s} - c_{p,q}/24 \,|\, 1 \leq r \leq p-1, 1 \leq s \leq q-1\}$. Since the differential operator $L_{p,q}(\vartheta)$ with $(p-1)(q-1)/2 < 5$ has order less than 5 and any holomorphic modular forms of weight $k < 10$ is a scalar multiple of $E_k$, we see that $L_{p,q}(\vartheta) = \vartheta_0^n + \sum_{i=0}^{n-2} \alpha_i E_{2(n-i)}\vartheta_0^i$, where $n = (p-1)(q-1)/2$. Since a monic MLDE of the form $\vartheta_0^n(f) + \sum_{i=0}^{n-2} \alpha_i E_{2(n-i)}\vartheta_0^i(f) = 0$ is uniquely determined by indicial roots ([16, Lemma 2.3]), it follows from Table 1 that the differential operators $L_{p,q}(\vartheta)$ with $(p-1)(q-1)/2 < 5$ are given by

(5)
$$L_{2,3}(\vartheta) = \vartheta_0,$$
$$L_{2,5}(\vartheta) = \vartheta_0^2 - \frac{11}{3600}E_4,$$
$$L_{2,7}(\vartheta) = \vartheta_0^3 - \frac{5}{252}E_4\vartheta_0 + \frac{85}{74088}E_6,$$
$$L_{3,4}(\vartheta) = \vartheta_0^3 - \frac{107}{2304}E_4\vartheta_0 + \frac{23}{55296}E_6,$$
$$L_{2,9}(\vartheta) = \vartheta_0^4 - \frac{13}{216}E_4\vartheta_0^2 + \frac{53}{5832}E_6\vartheta_0 - \frac{253}{559872}E_8,$$
$$L_{3,5}(\vartheta) = \vartheta_0^4 - \frac{949}{7200}E_4\vartheta_0^2 + \frac{139}{21600}E_6\vartheta_0 - \frac{279}{256000}E_8.$$

We next give a characterization of a VOA with central charge $c_{p,q}$ whose space spanned by characters of all simple modules is *contained* in (does not necessarily coincides with) the space of solutions of the monic MLDE $L_{p,q}(\vartheta)f = 0$. This result will be used to prove Theorem 19 in §6. We first show the following proposition.

PROPOSITION 7. *Let $V$ be a vertex operator algebra and $M = \bigoplus_{n=0}^{\infty} M_{h+n}$ a $V$-module (not necessarily simple). Suppose that the space $\mathcal{X}(V)$ spanned by characters of all simple $V$-modules is contained in the space of solutions of a monic modular linear differential equation. Then the space of solutions contains the character of $M$.*

PROOF. Since the number of linearly independent characters of simple $V$-modules is finite and $M_{h+n}$ is finite-dimensional for each $n$, the character of $M$ is a finite sum of characters of simple $V$-modules. $\square$

For any pair $(p,q)$ such that the set of conformal weights of simple $L(c_{p,q}, 0)$-modules

does not contain positive integers, we show that the monic MLDE $L_{p,q}(\vartheta)f = 0$ characterizes a VOA with central charge $c_{p,q}$.

THEOREM 8.    *Let $p$ and $q$ be coprime positive integers such that the set of conformal weights of simple $L(c_{p,q}, 0)$-modules does not contain positive integers, and let $V$ be a (not necessarily simple) vertex operator algebra with central charge $c_{p,q}$. Suppose that the space of solutions of $L_{p,q}(\vartheta)f = 0$ given in Proposition 6 contains the space spanned by characters of all simple $V$-modules. Then $V$ is isomorphic to $L(c_{p,q}, 0)$.*

PROOF.  It follows from Proposition 7 that the character of $V$ sits in the space of solutions of $L_{p,q}(\vartheta)f = 0$. Because $L(c_{p,q}, 0)$ does not have a positive integral conformal weight and any indicial root is written as $h_{r,s} - c_{p,q}/24$, the indicial roots does not exist in the set $-c_{p,q}/24 + \mathbb{Z}_{>0}$. Since the character $\chi_{1,1}$ of $L(c_{p,q}, 0)$ is a solution of $L_{p,q}(\vartheta)f = 0$, it follows from Lemma 2 that any solution with index $-c_{p,q}/24$ is a scalar multiple of $\chi_{1,1}$. Therefore, the character of $V$ is $\chi_{1,1}$ since $\dim V_0 = 1$ and the central charge of $V$ is $c_{p,q}$. It follows from Corollary 2 of Theorem 5 that $V$ is isomorphic to $L(c_{p,q}, 0)$.         □

If $(p-1)(q-1)/2 < 9$, then $L(c_{p,q}, 0)$ does not have a positive integral conformal weight. However, if $(p, q) = (3, 10)$, then $h_{1,9} = 2$.

COROLLARY.    *Let $p$ and $q$ be coprime positive integers such that $(p-1)(q-1)/2 < 9$ and $V$ a vertex operator algebra with central charge $c_{p,q}$. If the space spanned by characters of all simple modules is contained in the space of solutions of $L_{p,q}(\vartheta)f = 0$, then $V \cong L(c_{p,q}, 0)$.*

## 4.    Extensions of minimal models and monic modular linear differential equations

In this section we discuss the relations between extensions of minimal models and monic MLDEs. We first prove that there exists a monic MLDE $\hat{L}_{p,q}(\vartheta)f = 0$ whose space of solutions coincides with the space spanned by characters of all simple modules of the $\mathbb{Z}_2$-graded simple current extension of a minimal model with central charge $c_{p,q}$. Our main results in this section is that the monic MLDE $\hat{L}_{p,q}(\vartheta)f = 0$ for the $\mathbb{Z}_2$-graded simple current extension determines a VOA with central charge $c_{p,q}$.

DEFINITION.    Let $V$ be a vertex operator algebra and $U$ a vertex operator subalgebra of $V$ which has the same Virasoro element of $V$. Then $V$ is called an **extension** of $U$. It is called that an extension $V$ of $U$ is called **nontrivial** if $a_n b \neq 0$ for some $a, b \in V \setminus U$ and $n$.

Let $V$ be an extension of a rational VOA $U$. Then $V$ is a finite direct sum of simple $U$-modules with *integral conformal weights* since $U$ is the $\mathbb{Z}_{\geq 0}$-graded **Vir**-submodule of $V$.

Let $V$ be a rational and $C_2$-cofinite VOA. Suppose that $L$, $M$ and $N$ are $V$-modules. The **fusion rule** $N_{MN}^L (\in \mathbb{Z}_{\geq 0})$ of type $\binom{L}{M\ N}$ is the dimension of the space of intertwining operators of type $\binom{L}{M\ N}$ (see e.g. [8] for the definition of intertwining operators). The

relations among $L$, $M$ and $N$ are usually denoted by

$$M \times N = \sum_{L: \text{ simple } V\text{-modules}} N_{MN}^L \, L,$$

and a $V$-module $M$ is called **simple current** if for any simple $V$-module $N$ there is a simple $V$-module $L$ such that $M \times N = L$.

DEFINITION.    Let $V^0$ be a vertex operator algebra and $\{V^\alpha \,|\, \alpha \in A\}$ a set of simple current $V^0$-modules indexed by a finite abelian group $A$ such that $V^\alpha \times V^\beta = V^{\alpha+\beta}$. Then $V = \bigoplus_{\alpha \in A} V^\alpha$ is called a $A$-graded **simple current extension** (SCE) of $V^0$ if $(V, Y)$ is a *nontrivial* extension of $V^0$.

REMARK.    Let $V = \bigoplus_{\alpha \in A} V^\alpha$ be a $A$-graded simple current extension and $Y$ the vertex operator on $V$. Then the restriction of the vertex operator

$$Y : V^\alpha \to \bigoplus_{\gamma \in A} \mathrm{Hom}_\mathbb{C}(V^\beta, V^\gamma)[[z, z^{-1}]]$$

defines an intertwining operator. Since $V^\alpha$ $(\alpha \in A)$ is simple current, that is, $N_{V^\alpha, V^\beta}^{V^\gamma} = \delta_{\alpha+\beta,\gamma}$, we see that $Y(v^\alpha, z)v^\beta \in V^{\alpha+\beta}((z))$ for all $v^\alpha \in V^\alpha$ and $v^\beta \in V^\beta$. This is one of the reasons why we call this extension $A$-graded.

THEOREM 9 (CF. [12, Theorem 9.3]).    *Let $p, q > 1$ be coprime integers. For any integers $0 < r_1, r_2 < p$ and $0 < s_1, s_2 < q$, we have the fusion rules*

(6)   $L(c_{p,q}, h_{r_1,s_1}) \times L(c_{p,q}, h_{r_2,s_2})$
$$= \sum_{\substack{r_3 = |r_1 - r_2| + 1 \\ r_3 - 1 \equiv r_1 + r_2 \bmod 2}}^{\min\{r_1+r_2, 2p-(r_1+r_2)\}-1} \sum_{\substack{s_3 = |s_1 - s_2| + 1 \\ s_3 - 1 \equiv s_1 + s_2 \bmod 2}}^{\min\{s_1+s_2, 2q-(s_1+s_2)\}-1} L(c_{p,q}, h_{r_3,s_3}).$$

*Let $\mathcal{A}_{p,q}$ be a free abelian group generated by symbols $[L(c_{p,q}, h_{r,s})]$ labeled by all simple $L(c_{p,q}, 0)$-modules. Then $\mathcal{A}_{p,q}$ becomes a commutative associative $\mathbb{Z}$-algebra by (6).*

It follows from the theorem that

$$L(c_{p,q}, h_{r,s}) \times L(c_{p,q}, h_{1,q-1}) = L(c_{p,q}, h_{r,q-s})$$

which shows that $L(c_{p,q}, h_{1,q-1})$ is a simple current module, and that the set $\{L(c_{p,q}, 0), L(c_{p,q}, h_{1,q-1})\}$ with the fusion product $\times$ forms the abelian group $\mathbb{Z}_2$.

PROPOSITION 10 ([26, Lemma 2.6, Theorem 2.4],[14, Proposition 2.6]).    *Suppose that $h_{1,q-1}$ is a positive integer. Then $V = L(c_{p,q}, 0) \oplus L(c_{p,q}, h_{1,q-1})$ has a unique simple vertex operator algebra structure such that $V$ is a $\mathbb{Z}_2$-graded simple current extension of $L(c_{p,q}, 0)$. Moreover, this extension is rational and $C_2$-cofinite.*

REMARK.    Suppose that $h_{1,q-1}$ is a positive integer. We denote by $\hat{L}(c_{p,q}, h_{1,q-1})$ the (unique) $\mathbb{Z}_2$-graded simple current extension of $L(c_{p,q}, 0)$ by $L(c_{p,q}, h_{1,q-1})$ given in Proposition 10.

We next prove that there is a monic MLDE whose space of solutions coincides with the space spanned by characters of all simple $\hat{L}(c_{p,q}, h_{1,q-1})$-modules. Since $L(c_{p,q}, 0)$ is rational and $C_2$-cofinite, any simple module of the extension $\hat{L}(c_{p,q}, h_{1,q-1})$ is a direct sum of simple $L(c_{p,q}, 0)$-modules. Thus the character of a simple $\hat{L}(c_{p,q}, h_{1,q-1})$-module is obtained by the irreducible decomposition of a simple $\hat{L}(c_{p,q}, h_{1,q-1})$-module as an $L(c_{p,q}, 0)$-module.

Suppose that $h_{1,q-1}$ is a positive integer. Since $h_{1,q-1} = (p-2)(q-2)/4$ and $(p, q) = 1$, there exist positive integers $t$ and $u$ such that $(p, q) = (2t+1, 4u+2)$ or $(4u+2, 2t+1)$ (we can suppose $(p, q) = (2t+1, 4u+2)$ since $c_{p,q} = c_{q,p}$). For $1 \le r \le p-1$ and $1 \le s \le q-1$, we write

$$(7) \qquad \mathbb{Z}_2^{r,s} = \begin{cases} \mathbb{Z}_2 & s = 2u+1, \\ 0 & \text{otherwise} \end{cases}$$

and $S_{r,s} = \mathbb{Z}_2 / \mathbb{Z}_2^{r,s}$. For every $a \in S_{r,s}$ we introduce the notation

$$(8) \qquad L(c_{p,q}, h_{r,s})^a = \begin{cases} L(c_{p,q}, h_{r,s}) & \text{if } a = \bar{0} + \mathbb{Z}_2^{r,s}, \\ L(c_{p,q}, h_{r,q-s}) & \text{if } a = \bar{1} + \mathbb{Z}_2^{r,s}. \end{cases}$$

Then it follows from [26, Theorem 2.14] that a simple $\hat{L}(c_{p,q}, h_{1,q-1})$-module, as an $L(c_{p,q}, 0)$-module, is isomorphic to $\bigoplus_{a \in S_{r,s}} L(c_{p,q}, h_{r,s})^a \otimes U^a$ for some $r$ and $s$, where $U^a$ is a simple module of the twisted group algebra $\mathbb{C}^{\lambda_a}[\mathbb{Z}_2^{r,s}]$ with 2-cocycle $\lambda_a \in Z^2(\mathbb{Z}_2^{r,s}, \mathbb{C}^\times)$.

The twisted group algebra $\mathbb{C}^{\lambda_a}[\mathbb{Z}_2^{r,s}]$ is defined by the group algebra $\mathbb{C}[\mathbb{Z}_2^{r,s}] = \bigoplus_{\alpha \in \mathbb{Z}_2^{r,s}} \mathbb{C}e^\alpha$ with the deformed multiplication $*$ defined by $e^\alpha * e^\beta = \lambda_a(\alpha, \beta) e^{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}_2^{r,s}$. Since $\mathbb{Z}_2^{r,s}$ is either $0$ or $\mathbb{Z}_2$, each 2-cocycle $\lambda$ satisfies $\lambda_a(\alpha, \beta) = \lambda_a(\alpha, \beta)$ for all $\alpha, \beta \in \mathbb{Z}_2^{r,s}$, which shows that $\mathbb{C}^{\lambda_a}[\mathbb{Z}_2^{r,s}]$ is commutative. Thus any simple $\mathbb{C}^{\lambda_a}[\mathbb{Z}_2^{r,s}]$-module is 1-dimensional. Therefore, it follows from (7), (8) and $h_{r,s} = h_{p-r,q-s}$ that a simple $\hat{L}(c_{p,q}, h_{1,q-1})$-module is isomorphic to one of $L(c_{p,q}, 0)$-modules

$$(9) \quad L(c_{p,q}, h_{r,s}) \oplus L(c_{p,q}, h_{r,q-s}) \, (1 \le r \le t, \, 1 \le s \le 2u), \quad L(c_{p,q}, h_{r,2u+1}) \, (1 \le r \le t).$$

We remark that these $L(c_{p,q}, 0)$-modules are not always $\hat{L}(c_{p,q}, h_{1,q-1})$-modules.

It follows from [26, Theorem 3.2] that $L(c_{p,q}, h_{r,2u+1})$ is a simple $\hat{L}(c_{p,q}, h_{1,q-1})$-module, and that $L(c_{p,q}, h_{r,s}) \oplus L(c_{p,q}, h_{r,q-s})$ is a simple $\hat{L}(c_{p,q}, h_{1,q-1})$-module if $h_{r,s} - h_{r,q-s}$ is an integer. Since $h_{r,s} - h_{r,q-s} = -(2t+1-2r)(2u+1-s)/2 < 0$ if $1 \le s \le 2u$, we see that $s$ must be odd ($s = 1, 3, 5, \ldots, 2u-1$) and that $h_{r,s} < h_{r,q-s}$ for $1 \le r \le t$, $1 \le s \le 2u$. Therefore, every simple $\hat{L}(c_{p,q}, h_{1,q-1})$-module as an $L(c_{p,q}, 0)$-module is isomorphic to one of $L(c_{p,q}, h_{r,2u+1})$ and $L(c_{p,q}, h_{r,2v-1}) \oplus L(c_{p,q}, h_{r,4u+3-2v})$ for $1 \le r \le t$ and $1 \le v \le u$. This shows that the set of conformal weights of all simple $\hat{L}(c_{p,q}, h_{1,q-1})$-modules is given by

$$(10) \qquad \{h_{r,2u+1} \,|\, 1 \le r \le t\} \cup \{h_{r,2v-1} \,|\, 1 \le r \le t, \, 1 \le v \le u\},$$

and the set of characters of simple $\hat{L}(c_{p,q}, h_{1,q-1})$-modules is

$$(11) \qquad \{\chi_{r,2u+1} \,|\, 1 \le r \le t\} \cup \{\chi_{r,2v-1} + \chi_{r,q-2v+1} \,|\, 1 \le r \le t, \, 1 \le v \le u\}.$$

THEOREM 11. *Let* $(p,q) = (2t+1, 4u+2)$ *be a pair of coprime positive integers. Then there is a unique monic modular linear differential equation* $\hat{L}_{p,q}(\vartheta)f = 0$ *of order* $t(u+1)$ *whose space of solutions coincides with the space spanned by characters of all simple modules of the* $\mathbb{Z}_2$-*graded simple current extension* $\hat{L}(c_{p,q}, h_{1,q-1})$.

PROOF. It follows from (10) with a help of (3) and (4) that

$$\sum_{r=1}^{t} h_{r,2u+1} + \sum_{r=1}^{t}\sum_{v=1}^{u} h_{r,2v-1} - \frac{t(u+1)c_{p,q}}{24} = \frac{t(u+1)(tu+t-1)}{12}.$$

Hence the theorem is proved by Theorem 1. $\square$

REMARK. In [24], it is shown that the character of $\hat{L}(c_{p,q}, h_{1,q-1})$ is a solution of a monic MLDE of order $h_{1,q-1} + 1$. However, our theorem shows that there is a monic MLDE of order strictly smaller than $h_{1,q-1}+1$. The order of the monic MLDE $\hat{L}_{p,q}(\vartheta)f = 0$ is $h_{1,q-1}+1$ if and only if one of $p$ and $q$ is 3 and 6, respectively. For example, the monic MLDE for the extension $\hat{L}(c_{5,14}, 9)$ obtained in [24] is of order 10, whereas $\hat{L}_{5,14}(\vartheta)f = 0$ is of order 8.

We next show that any VOA $V$ with central charge $c_{p,q}$ whose space spanned by characters of all simple modules is contained in the space of solutions of the monic MLDE $\hat{L}_{p,q}(\vartheta)f = 0$ given in Theorem 11 is isomorphic to $\hat{L}(c_{p,q}, h_{1,q-1})$ for specific pairs $(p,q)$ (see Theorem 13 at the end of this section), which will be used to prove Theorem 19 given in §6. The main part of a proof of Theorem 13 is to show that $\dim V(c_{p,q}, 0)_n - \dim \hat{L}(c_{p,q}, h_{1,q-1})_n > 0$ for sufficiently large $n$. In order to prove this, we use asymptotic behavior of Fourier coefficients of characters of simple $L(c_{p,q}, 0)$-modules.

For series $\{a_n\}$ and $\{b_n\}$, we write $a_n \sim b_n$ if $\lim_{n\to\infty}(a_n/b_n) = 1$. It is known in [12, Lemma 9.14] that

$$(12) \qquad \dim L(c_{p,q}, h_{r,s})_{h_{r,s}+n} \sim \frac{1}{\sqrt{2}} S_{(r,s),(r_0,s_0)} \left(\frac{1-6/pq}{24}\right)^{1/4} n^{-3/4} e^{\pi\sqrt{\frac{2}{3}\left(1-\frac{6}{pq}\right)n}},$$

where

$$S_{(r,s),(r',s')} = (-1)^{(r+s)(r'+s')}\sqrt{\frac{8}{pq}}\sin\left(\frac{\pi rr'}{p}(p-q)\right)\sin\left(\frac{\pi ss'}{q}(p-q)\right)$$

and $(r_0, s_0)$ is a specific pair of positive integers satisfying $|r_0 q - s_0 p| = 1$. Suppose that $(p,q) = (2t+1, 4u+2)$ for some positive integers $t$ and $u$. It follows that $p-q$ and $s_0$

are odd since $|2r_0(2u+1) - 2ts_0 - s_0| = 1$. Then we have

(13)

$$
\begin{aligned}
S_{(1,q-1),\,(r_0,s_0)} &= (-1)^{q(r_0+s_0)} \sqrt{\frac{8}{pq}} \sin\left(\frac{\pi r_0}{p}(p-q)\right) \sin\left(\frac{\pi(q-1)s_0}{q}(p-q)\right) \\
&= (-1)^{q(r_0+s_0)} \sqrt{\frac{8}{pq}} \sin\left(\frac{\pi r_0}{p}(p-q)\right) \sin\left(\pi s_0(p-q) - \frac{\pi s_0}{q}(p-q)\right) \\
&= \sqrt{\frac{8}{pq}} \sin\left(\frac{\pi r_0}{p}(p-q)\right) \sin\left(\frac{\pi s_0}{q}(p-q)\right) \\
&= S_{(1,1),\,(r_0,s_0)}
\end{aligned}
$$

and

$$
\begin{aligned}
S_{(1,1),\,(r_0,s_0)} &= (-1)^{2(r_0+s_0)} \sqrt{\frac{8}{pq}} \sin\left(\pi(r_0-s_0) \mp \frac{\pi}{p}\right) \sin\left(\pi(r_0-s_0) \mp \frac{\pi}{q}\right) \\
&= \sqrt{\frac{8}{pq}} \sin\left(\frac{\pi}{p}\right) \sin\left(\frac{\pi}{q}\right) > 0
\end{aligned}
$$

since $|r_0q - s_0p| = 1$ and $(p,q) = 1$. By (12) and (13), the Fourier coefficients of $\chi_{1,1}$ and $\chi_{1,q-1}$ have the same asymptotic behavior at $n \to \infty$, which implies that

(14)     $\dim \hat{L}(c_{p,q}, h_{1,q-1})_n \sim \dfrac{2}{\sqrt{2}} S_{(1,1),\,(r_0,s_0)} \left(\dfrac{1-6/pq}{24}\right)^{1/4} n^{-3/4} e^{\pi\sqrt{\frac{2}{3}\left(1-\frac{6}{pq}\right)n}} > 0.$

PROPOSITION 12. *For sufficiently large $n$, we have* $\dim V(c_{p,q},0)_n - \dim \hat{L}(c_{p,q}, h_{1,q-1})_n > 0$.

PROOF. Let $T = U(\bigoplus_{n=3}^{\infty} \mathbb{C}L_{-n})v_{c_{p,q},0} \subset M(c_{p,q},0)$. It is shown in [12, Lemma 9.14] that

(15)                    $\dim T_n \sim \dfrac{\pi^2}{12\sqrt{3}n^2} e^{\pi\sqrt{\frac{2}{3}n}},$

where $T_n$ is a graded subspace of $T$ with weight $n$. Since the VOA $V(c_{p,q},0)$ is isomorphic to $S = U(\bigoplus_{n=2}^{\infty} \mathbb{C}L_{-n})v_{c_{p,q},0}$ as a vector space, we have $\dim V(c_{p,q},0)_n \geq \dim T_n$ for all $n$. Thus it suffices to prove $\dim T_n - \hat{L}(c_{p,q}, h_{1,q-1})_n > 0$ for large enough $n$. Let $A(n)$ and $B(n)$ be the right-hand sides of (14) and (15), respectively. Then it follows from (14) and (15) that there exists a small enough real number $\varepsilon > 0$ such that

$$
\begin{aligned}
0 < (1-\varepsilon)A(n) < \dim \hat{L}(c_{p,q}, h_{1,q-1})_n < (1+\varepsilon)A(n), \\
0 < (1-\varepsilon)B(n) < \dim T_n < (1+\varepsilon)B(n)
\end{aligned}
$$

for sufficiently large $n$. Thus we have

$$
\frac{\dim T_n}{\dim \hat{L}(c_{p,q}, h_{1,q-1})_n} > \left(\frac{1-\varepsilon}{1+\varepsilon}\right) \frac{B(n)}{A(n)},
$$

where

$$\frac{B(n)}{A(n)} = \frac{\frac{\pi^2}{12\sqrt{3}n^2}e^{\pi\sqrt{\frac{2}{3}n}}}{\frac{2}{\sqrt{2}}S_{(1,1),\,(r_0,s_0)}\left(\frac{1-6/pq}{24}\right)^{1/4}n^{-3/4}e^{\pi\sqrt{\frac{2}{3}(1-\frac{6}{pq})n}}}$$

$$= \frac{e^{\pi\sqrt{\frac{2}{3}n}-\pi\sqrt{\frac{2}{3}(1-\frac{6}{pq})n}}\frac{\pi^2}{12\sqrt{3}}}{n^{5/4}\frac{2}{\sqrt{2}}S_{(1,1),\,(r_0,s_0)}\left(\frac{1-6/pq}{24}\right)^{1/4}}.$$

It follows that $\lim_{n\to\infty} B(n)/A(n) = \infty$ since $0 < 1 - 6/pq < 1$ and $S_{(1,1),\,(r_0,s_0)} > 0$. Therefore, $\dim T_n/\dim \hat{L}(c_{p,q}, h_{1,q-1})_n \to \infty$ as $n \to \infty$, which proves the proposition. $\square$

THEOREM 13.    *Let $(p,q) = (2t+1, 4u+2)$ be a pair of coprime positive integers and $V$ a vertex operator algebra with central charge $c_{p,q}$ whose space spanned by characters of all simple $V$-modules is contained in the space of solutions of $\hat{L}_{p,q}(\vartheta)f = 0$ given in Theorem 11. Suppose that the set of conformal weights of simple $\hat{L}(c_{p,q}, h_{1,q-1})$ do not contain positive integers. Then $V$ is isomorphic to the $\mathbb{Z}_2$-graded simple current extension $\hat{L}(c_{p,q}, h_{1,q-1})$.*

PROOF. We first show that $V^\omega \cong L(c_{p,q}, 0)$. Since there are no positive integral conformal weights, it follows from Theorem 11 and Lemma 2 that the character $\chi_{1,1} + \chi_{1,q-1}$ of $\hat{L}(c_{p,q}, h_{1,q-1})$ is a unique solution of $\hat{L}_{p,q}(\vartheta)f = 0$ of the form $q^{-c_{p,q}/24}(1 + O(q))$. Therefore, the character of $V$ coincides with $\chi_{1,1} + \chi_{1,q-1}$, which shows that $\dim V_n = \dim \hat{L}(c_{p,q}, h_{1,q-1})_n$ for all $n$. It follows from Proposition 12 that $\dim V(c_{p,q}, 0)_n - \dim V_n > 0$ for sufficiently large $n$. Since $\dim V_n \geq \dim(V^\omega)_n$, the character of $V^\omega$ is not equal to the character of $V(c_{p,q}, 0)$. Then Corollary 1 of Proposition 4 implies that $V^\omega$ is isomorphic to $L(c_{p,q}, 0)$. Since $L(c_{p,q}, 0)$ is rational and $C_2$-cofinite and the character of $V$ is $\chi_{1,1} + \chi_{1,q-1}$, it follows that $V \cong L(c_{p,q}, 0) \oplus L(c_{p,q}, h_{1,q-1})$ as an $L(c_{p,q}, 0)$-module.

If $Y(u, z)v = 0$ for all $u, v \in L(c_{p,q}, h_{1,q-1})$, then the subspace $L(c_{p,q}, h_{1,q-1})$ is a $V$-submodule. Since $L(c_{p,q}, h_{1,q-1})$ is simple as an $L(c_{p,q}, 0)$-module, $L(c_{p,q}, h_{1,q-1})$ is also a simple $V$-module with positive integral conformal weight $h_{1,q-1}$. Thus $\chi_{1,q-1}$ is a solution of $\hat{L}_{p,q}(\vartheta)f = 0$, whereas $\chi_{1,q-1}$ is not a solution of $\hat{L}_{p,q}(\vartheta)f = 0$ by (11). This is a contradiction. It then follows from Proposition 10 that $V$ is isomorphic to $\hat{L}(c_{p,q}, h_{1,q-1})$. $\square$

At the end of the section we give the list of pairs $(p, q)$ such that the differential operator $\hat{L}_{p,q}(\vartheta)$ has order 3 or 4. By the same argument used to obtain (5) together

| $(p, q)$ | $h_{1,q}$ | Central charge | Conformal weights |
|---|---|---|---|
| $(3, 10)$ | 2 | $-44/5$ | $0, -1/5, -2/5$ |
| $(3, 14)$ | 3 | $-114/7$ | $0, -3/7, -4/7, -5/7$ |
| $(5, 6)$ | 3 | $4/5$ | $0, 1/15, 2/5, 2/3$ |

Table 2.   Central charges and conformal weights of $\hat{L}(c_{p,q}, h_{1,q-1})$ with $h_{1,q-1} < 4$

with Table 2, the differential operators $\hat{L}_{p,q}(\vartheta)$ for $(p,q)$ in Table 2 are given by

(16)
$$\hat{L}_{3,10}(\vartheta) = \vartheta_0^3 - \frac{11}{900} E_4 \vartheta_0 + \frac{11}{5400} E_6,$$
$$\hat{L}_{3,14}(\vartheta) = \vartheta_0^4 - \frac{37}{504} E_4 \vartheta_0^2 + \frac{517}{74088} E_6 \vartheta_0 - \frac{57}{87808} E_8,$$
$$\hat{L}_{5,6}(\vartheta) = \vartheta_0^4 - \frac{67}{900} E_4 \vartheta_0^2 + \frac{89}{5400} E_6 \vartheta_0 - \frac{209}{810000} E_8.$$

REMARK.    For any pair $(p,q)$ the order of $\hat{L}_{p,q}(\vartheta)$ is strictly greater than 2.

## 5.    Monic modular linear differential equations of order 4 and possible central charges

In this section we determine candidate central charges of VOAs satisfying the conditions (A) the central charges and conformal weights are rational numbers, (B) the spaces spanned by characters of all simple modules coincide with the spaces of solutions of monic MLDEs of order 4 and (C) $\dim V_0 = 1$, $\dim V_1 = 0$ and $\dim V_2 = 1$.

**Remarks.** (1) Every minimal model $L(c_{p,q}, 0)$ satisfies all conditions by the definition of $L(c_{p,q}, 0)$ and Proposition 6.
(2) The condition (C) is the same as the assertion that the first 3 coefficients of the character of a VOA coincide with those of the character $q^{-c/24} \prod_{n>1}(1-q^n)^{-1}$ of $V(c,0)$

It is known that a monic MLDE of order 4 can be written as (see [23])

(17)
$$D^4(f) - E_2 D^3(f) + (3E_2' + \alpha_1 E_4) D^2(f) - \left( E_2'' + \frac{\alpha_1}{2} E_4' - \alpha_2 E_6 \right) D(f) + \alpha_3 E_8 f = 0,$$

where $D = q(d/dq)$ and $E_k' = D(E_k)$, $E_k'' = D^2(E_k)$. This monic MLDE is in fact rewritten as

(18)
$$\vartheta_0^4(f) + \left( \alpha_1 - \frac{11}{36} \right) E_4 \vartheta_0^2(f) + \left( \frac{\alpha_1}{6} + \alpha_2 - \frac{5}{216} \right) E_6 \vartheta_0(f) + \alpha_3 E_8 f = 0,$$

where we have used the Ramanujan identities:

(19)
$$12E_2' = E_2^2 - E_4, \quad 3E_4' = E_2 E_4 - E_6, \quad 2E_6' = E_2 E_4 - E_8.$$

Let $V$ be a (not necessarily simple) VOA with central charge $c$, which satisfies the conditions (A)–(C). By Propositions 3 and 7, the character $\mathrm{ch}_V = q^{-c/24} \left( 1 + q^2 + mq^3 + O(q^4) \right)$ is a solution of (17), where $m$ is a non-negative integer (the dimension of $V_3$). By substituting $\mathrm{ch}_V$ into (17) and taking the Fourier coefficients of $q^{-c/24}$, $q^{1-c/24}$, $q^{2-c/24}$ and $q^{3-c/24}$, we have

(20)
$$576c^2\alpha_1 - 13824c\alpha_2 + 331776\alpha_3 = -c^4 - 24c^3,$$
$$240c(c+12)\alpha_1 + 12096c\alpha_2 + 276480\alpha_3 = c^3 + 72c^2 + 576c,$$
$$\left( 1244736c^2 + 29804544c + 1327104 \right) \alpha_1 + (229906944c + 663552)\alpha_2 + 20543901696\alpha_3$$
$$= -c^4 + 1896c^3 + 238464c^2 + 4257792c - 2654208.$$

and

$$\begin{aligned}
&\left(576c^2m + 4008960c^2 - 82944cm + 127733760c + 2985984m + 238878720\right)\alpha_1 \\
&\quad - (13824cm - 1706987520c - 995328m + 334430208)\alpha_2 \\
&\qquad\qquad + (331776m + 348603678720)\alpha_3 \\
&= -c^4m + 24c^3(11m + 120) - 5184c^2(5m - 88) \\
&\qquad\qquad + 41472c(27m + 296) - 1990656(9m - 8).
\end{aligned}$$

(21)

Since the determinant of the matrix associated with the simultaneous equation (20) is a non-zero scalar multiple of $c(578c - 7)(5c + 22)$, this equation has a unique solution

(22)
$$\begin{aligned}
\alpha_1 &= -\frac{56c^3 + 993c^2 - 11660c - 1440}{96(578c - 7)}, \\
\alpha_2 &= -\frac{-25c^4 - 829c^3 - 7347c^2 + 1008c + 3456}{1728(578c - 7)}, \\
\alpha_3 &= -\frac{14c^5 + 425c^4 + 3672c^3 + 5568c^2 + 9216c}{110592(578c - 7)}
\end{aligned}$$

if and only if $c \neq 0, -22/5, 7/578$.

If $c = 7/578$, then the system (20) is not compatible. Therefore, there does not exist any VOA with central charge $c = 7/578$ satisfying the conditions (B) and (C). If $c$ is either $0$ or $-22/5$ (rank $A = 2$), then the system (20) implies $(\alpha_2, \alpha_3) = (-2\alpha_1 - 4, 0)$ and $(17/675 - \alpha_1/6, 1793/4320000 - 11\alpha_1/3600)$, respectively. These cases are studied later in §6.

By (21) and (22), the rational number $c$ satisfies

(23) $\quad 1050c^5 + (5m + 31020)c^4 + (275600 - 703m)c^3$
$$\qquad + (32992m + 673104)c^2a + (504352 - 517172m)c + 3984m - 210432 = 0.$$

We confirmed that this equation does not have rational solutions $c$ for any $3 \leq m \leq 2^{16}$ (according to a private communication with D. Zagier by the 2nd author, this quintic equation has rational roots even when $m > 2^{16}$). It is proved by D. Zagier that the rational solutions $(c, m)$ with $m \in \mathbb{Z}_{\geq 0}$ of (23) are given in Table 3. The proof is given in the appendix which is written by him.

| $m$ | $c$ |
|---|---|
| 1 | $-68/7, 1/2, -3/5, -46/3$ |
| 2 | $-114/7, 4/5$ |
| 501971 | 36 |
| 3132760 | 122/3 |
| 37950512 | 238/5 |
| 42987520 | 48 |

Table 3. $m$ and central charge

For each $c$ given in Table 3 together with $c = 0, -22/5$, we can obtain indicial equations and indicial roots as in Table 4 (the first column provides candidate conformal

weights). Since the sets of indicial roots for $c = 36, 122/3, 238/5$ and $48$ contain non-rational numbers, theses cases are excluded by the condition (A). For data in 5th–8th

| $c$ | Indices | Indices+$c/24$ |
|---|---|---|
| $0$ | $0, 2, (-1 \pm \sqrt{-7 - 4\alpha_1})/2$ | $0, 2, (-1 \pm \sqrt{-7 - 4\alpha_1})/2$ |
| $-22/5$ | $-1/60, 11/60, (25 \pm \sqrt{1114 - 3600\alpha_1})/60$ | $-1/5, 0, (14 \pm \sqrt{1114 - 3600\alpha_1})/60$ |
| $-68/7$ | $-1/42, 5/42, 17/42, 1/2$ | $-3/7, -2/7, 0, 2/21$ |
| $1/2$ | $-1/48, 1/24, 23/48, 1/2$ | $0, 1/16, 1/2, 25/48$ |
| $-46/3$ | $-1/36, 1/12, 11/36, 23/36$ | $-2/3, -5/9, -1/3, 0$ |
| $-3/5$ | $-1/40, 1/40, 9/40, 31/40$ | $-1/20, 0, 1/5, 3/4$ |
| $-114/7$ | $-1/28, 3/28, 1/4, 19/28$ | $-5/7, -4/7, -3/7, 0$ |
| $4/5$ | $-1/30, 1/30, 11/30, 19/30$ | $0, 1/15, 2/5, 2/3$ |

Table 4.   Central charges and indices

rows of Table 4, we can find corresponding VOAs.

PROPOSITION 14.   *Let $V$ be a vertex operator algebra with central charge $c$, which satisfies conditions* (A)–(C). *If $c$ is either $-46/3$ or $-3/5$, then $V$ is isomorphic to $L(c, 0)$. If $c$ is either $-114/7$ or $4/5$, then $V$ is isomorphic to $\hat{L}(c, 3)$.*

PROOF. If the central charge $c$ is either $c = -46/3$ or $c = -3/5$, Table 4 shows that the corresponding monic MLDE coincides with $L_{2,9}(\vartheta)f = 0$ and $L_{3,5}(\vartheta)f = 0$ in (5), respectively. Since any conformal weight of simple $L(c, 0)$-module is not a positive integer (see also Table 1), it follows from Theorem 8 that $V$ is isomorphic to $L(c, 0)$.

If the central charge $c$ is either $-114/7$ or $4/5$, then it follows from Table 2 and Table 4 that the corresponding monic MLDE has order 2 or 3 (see eq.(16)), respectively. Since any conformal weight of simple $\hat{L}(c, 3)$-module is not a positive integer by Table 2, it follows from Theorem 13 that a VOA $V$ is isomorphic to $\hat{L}(c, 3)$.            □

### 6.   Main results

In this section we show that a VOA which satisfies (A) the central charge and conformal weights are rational numbers, (B) there is a monic modular linear differential equation of order 4 whose space of solutions coincides with the space spanned by characters of all simple $V$-modules and (C) $\dim V_0 = 1$, $\dim V_1 = 0$ and $\dim V_2 = 1$, is isomorphic to one of $L(-46/3, 0)$, $L(-3/5, 0)$, $\hat{L}(-114/7, 3)$ and $\hat{L}(4/5, 3)$. Since we have already proved in Proposition 14 that central charges $-46/3, 1/2, -114/7$ and $4/5$ have corresponding VOAs, it suffices to exclude the cases $c = 0, -22/5, -68/7, 1/2$.

Observe that each central charge in the first 4 rows in Table 4 is equal to $c_{p,q}$ with $(p-1)(q-1)/2 < 4$. We prove that there are no VOAs with central charge $c_{p,q}$ satisfying the conditions (A)–(C) for such pairs $(p, q)$.

The indicial equation of the monic MLDE (17) is $\Psi(\rho) = \rho^4 - \rho^3 + \alpha_1\rho^2 + \alpha_2\rho + \alpha_3$ (coefficients $\alpha_1$, $\alpha_2$ and $\alpha_3$ are uniquely determined by indical roots in Table 4). Let $\lambda$ be an indicial root and *suppose that other indicial roots do not sit in $\lambda + \mathbb{Z}_{>0}$*. Then a solution $q^\lambda(1 + \sum_{n=1}^\infty a_n q^n)$ with index $\lambda$ is uniquely determined by the recursive relations

$$(24) \quad a_n = \Psi(\lambda + n)^{-1} \sum_{m=1}^{n} \Big\{ e_{2,m}(n - m + \lambda)^3 - \big(3m e_{2,m} + \alpha_1 e_{4,m}\big)(n - m + \lambda)^2$$

$$+ \Big(m^2 e_{2,m} + \frac{1}{2}\alpha_1 m e_{4,m} - \alpha_2 e_{6,m}\Big)(n - m + \lambda) - \alpha_3 e_{8,m} \Big\} a_{n-m}$$

for each $n \geq 1$.

REMARK.   If the indicial equation has multiple roots, then there is a logarithmic solution (see [11, Chapter XVI]). Since characters of simple $V$-modules are power series in $q$, the condition (B) implies that any solution of the monic MLDE (24) does not involve logarithmic terms in $q$. Thus it is enough to consider the case that indicial equations of monic MLDEs of the form (17) do not have multiple roots.

**(a) Central charge** $c_{2,3} = 0$**.**   The monic MLDE whose set of indicial roots is $\{0, 2, (-1 \pm \sqrt{-7 - 4\alpha_1})/2\}$ is given by

$$(25) \quad D^4(f) - E_2 D^3(f) + (3E_2' + \alpha_1 E_4)D^2(f) - \Big(E_2'' + \frac{\alpha_1}{2}E_4' + (2\alpha_1 + 4)E_6\Big)D(f) = 0.$$

The constant function 1 is a solution with index 0. It follows from the last remark that the indicial equation does not have multiple roots. We prove that VOAs with central charge 0, which satisfy the conditions (A)–(C) do not exist.

Let $V$ be a VOA with central charge 0, which satisfies the conditions (A)–(C). Then the character $\mathrm{ch}_V = \sum_{n=0}^{\infty}(\dim V_n)q^n = 1 + q^2 + mq^3 + \cdots$ is a solution of (25). Then (21) implies $\alpha_1 = -2(7m + 664)/(m + 912)$. Because all indicial roots are rational numbers, it follows that the indicial roots are

$$(26) \qquad \frac{-1 \pm \sqrt{-7 - 4\alpha_1}}{2} = -\frac{1}{2} \pm \frac{\sqrt{(49m - 1072)(m + 912)}}{m + 912}$$

are rational numbers. Since $m \geq 0$ and the right-hand side of (26) is a rational number, there exists an integer $r$ such that $(49m - 1072)(m + 912) = r^2$, which is equivalent to $(49m - 7r + 21808)(49m + 7r + 21808) = 52349440$. By solving this equation, we have the list of $m$

$$(27) \qquad \begin{aligned} &\{24, 78, 88, 232, 375, 708, 1194, 1285, 1636, 2608, 2733, 4236, 7908, 12400, \\ &\quad 23838, 66328, 102282, 533733\}. \end{aligned}$$

We can verify by using (26) and (27) on a case-by-case basis that the set of indicial roots does not contain integers greater than 2. Therefore, the solution $f_2$ with index 2 and the first coefficient 1 is uniquely determined by recursive relations (24) as

$$(28) \quad f_2 = q^2 + mq^3 + \frac{9(853m^2 + 23392m - 371328)}{8(m + 2342)}q^4$$

$$+ \frac{3083022m^2 + 254932344m^2 - 1529365568m - 46472052736}{(m + 2342)(9m + 13928)}q^5 + O(q^6).$$

Since 1 is a solution of (25) and $\mathrm{ch}_V = 1 + q^2 + mq^3 + O(q^4)$ by the condition (C), we see that $\mathrm{ch}_V - 1$ is a solution of the monic MLDE with index 2 and the first co-

efficient 1, which shows that $\mathrm{ch}_V = 1 + f_2$. Therefore, all Fourier coefficients of $f_2$ must be non-negative integers. Substituting each number of (27) into Fourier coefficients of the solution $f_2$, we can show that the Fourier coefficient of $q^4$ is an integer if and only if $m \in \{24, 232, 1636, 2608\}$ and that the Fourier coefficient of $q^5$ is not an integer[1] for $m = 1636$ and $2608$, and then $m$ is 24 or 232.

For $m = 24$, the set of indicial roots is $\{0, 2, -2/3, -1/3\}$. Since there are no indicial roots in the set $-2/3 + \mathbb{Z}_{>0}$ the solution $f_{-2/3}$ with index $-2/3$ is uniquely determined by recursive relations (24) as

$$f_{-2/3} = q^{-2/3}(1 - 1448q - 51748q^2 + O(q^3)).$$

Since characters of simple $V$-modules are contained in the space of solutions of (25) (the condition (B)), a scalar multiple of the solution $f_{-2/3}$ must coincide with a character of a simple $V$-module. However, this is impossible since the Fourier coefficients of $f_{-2/3}$ have alternate signature.

For $m = 232$, the corresponding monic MLDE is given by

$$(29) \qquad \vartheta_0^4(f) - \frac{155}{36}E_4\vartheta_0^2(f) + \frac{715}{216}E_6\vartheta_0(f) = 0$$

and the set of indicial roots of the monic MLDE is $\{-2, 0, 1, 2\}$.

Let $F_1 = -(E'_{10} + 264\eta^{24})/\eta^{22}$ and $F_2 = (E_4^3 - 720\eta^{24})'/\eta^{26}$. Then $\eta^{22}F_1$ and $\eta^{26}F_2$ are extremal quasimodular forms of depth 1 and weight 12 and 14, respectively ([13, Theorem 2.1]). We show that the product $F_1F_2$ is a solution of the monic MLDE (29).

DEFINITION. A **quasimodular form** of depth $r$ and weight $k$ is a function $f_0 + f_1E_2 + f_2E_2^2 + \cdots + f_rE_2^r$, where $f_i$ is a modular form of weight $k - 2i$ and $f_r \neq 0$. A quasimodular form of depth $r$ and weight $k$ is called **extremal** if its $q$-expansion $\sum_{n=0}^{\infty} a_n q^n$ satisfies

$$a_0 = a_1 = \cdots = a_{m-2} = 0, \ a_{m-1} \neq 0, \quad m = \sum_{i=0}^{r} \dim \mathcal{M}_{k-2i},$$

where $\mathcal{M}_i$ is the space of modular forms of weight $i$ on $SL_2(\mathbb{Z})$.

It is not difficult to verify that

$$(30) \qquad \vartheta_0^2(F_1) - \frac{143}{144}E_4F_1 = 0 \text{ and } E_4\vartheta_0^2(F_2) + \frac{1}{3}E_6\vartheta_0(F_2) - \frac{143}{144}E_2^2F_2 = 0.$$

Since the space of solutions of a (not necessarily monic) MLDE is invariant under the slash action of $SL_2(\mathbb{Z})$ (the proof given in [17, pp.385] works), the functions

$$F_1(-1/\tau) = \frac{1}{2\pi\sqrt{-1}}(2\pi\sqrt{-1}\tau F_1 - E_{10}),$$

$$F_2(-1/\tau) = \frac{1}{2\pi\sqrt{-1}}(2\pi\sqrt{-1}\tau F_2 + E_4^3 - 720\eta^{24}),$$

---

[1]They are 3607786048/29 and 1523763136/5, respectively

are solutions of (30), respectively. It then follows from (30) that

$$\vartheta_0(F_1 F_2) = \vartheta_0(F_1) F_2 + F_1 \vartheta_0(F_2),$$

$$\vartheta_0^2(F_1 F_2) = 4\vartheta_0(F_1)\vartheta_0(F_2) - \frac{E_6}{3E_4} F_1 \vartheta_0(F_2),$$

$$\vartheta_0^4(F_1 F_2) = -\frac{715}{216} E_6 \vartheta_0(F_1) F_2 - \frac{1025}{216} E_6 F_1 \vartheta_0(F_2) + \frac{155}{9} E_4 \vartheta_0(F_1)\vartheta_0(F_2),$$

which shows that $F_1 F_2$ is a solution of the monic MLDE (29). Because the space of solutions of (29) is invariant under the action of $SL_2(\mathbb{Z})$, the function

$$F_1(-1/\tau) F_2(-1/\tau) = \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 (2\pi\sqrt{-1}\tau F_1 - E_{10})(2\pi\sqrt{-1}\tau F_2 + E_4^3 - 720\eta^{24})$$

is also a solution of (29) by the modular invariance property. Since the condition (B) implies that the space of solutions of (29) is spanned by power series in $q$, this is a contradiction.

Summarizing these arguments, we have the proposition.

PROPOSITION 15.    *Vertex operator algebras with central charge $c = 0$ satisfying* (A)–(C) *do not exist.*

(b) **Central charge $c_{2,5} = -22/5$.** The associated monic MLDE (17) is given by

(31)
$$D^4(f) - E_2 D^3(f) + (3E_2' + \alpha_1 E_4) D^2(f)$$
$$- \left\{ E_2'' + \frac{\alpha_1}{2} E_4' - \left(\frac{17}{675} - \frac{\alpha_1}{6}\right) E_6 \right\} D(f) + \left(\frac{1793}{4320000} - \frac{11\alpha_1}{3600}\right) E_8 f = 0.$$

The set of indicial roots is obtained as $\{-1/60,\ 11/60,\ (25 \pm \sqrt{1114 - 3600\alpha_1})/60\}$. Further, it follows by (18) and (19) that (31) is rewritten as

(32)
$$\vartheta_6 \circ \vartheta_4 \circ L_{2,5}(\vartheta) f + \left(\alpha_1 - \frac{121}{400}\right) E_4 L_{2,5}(\vartheta) f = 0.$$

Proposition 6 and Table 1 together with the monic MLDE $L_{2,5}(\vartheta) f = 0$ in (5) imply that the space of solutions of (32) contains characters $\chi_{1,1}$ and $\chi_{1,2}$ of simple $L(-22/5, 0)$-modules. Therefore, we see that $\chi_{1,1}$ is a solution with index $11/60$ of the monic MLDE (31).

PROPOSITION 16.    *Vertex operator algebras with central charge $c = -22/5$ satisfying the conditions* (A)–(C) *do not exist.*

PROOF. Suppose that there is a VOA $V$ with central charge $-22/5$, which satisfies the conditions (A)–(C). Then the condition (B) implies that $\mathcal{X}(V)$ coincides with the space of solutions of the monic MLDE (31) whose set of indicial roots is $\{-1/60,\ 11/60,\ (25 \pm \sqrt{1114 - 3600\alpha_1})/60\}$. If there are no indicial roots in $11/60 + \mathbb{Z}_{>0}$, then it follows from the condition (C) that the character of $V$ is $\chi_{1,1}$ and that $V \cong L(-22/5, 0)$ by Corollary 2 of Theorem 5, which contradicts to the condition (B) since the number of

simple $L(-22/5, 0)$-modules is 2.

Suppose that there is an indicial root in $11/60 + \mathbb{Z}_{>0}$. Then there exists a positive integer $\ell$ such that $(25 + \sqrt{1114 - 3600\alpha_1})/60 = 11/60 + \ell$. Then the set of indicial roots is

$$\left\{ \frac{13}{20} - \ell, \ -\frac{1}{60}, \ \frac{11}{60}, \ \frac{11}{60} + \ell \right\}.$$

Since there are no indicial roots in $13/20 - \ell + \mathbb{Z}_{>0}$, the solution $f_{13/20-\ell}$ with index $13/20 - \ell$ is written as

$$f_{13/20-\ell} = q^{13/20-\ell} \left( 1 - \frac{6(13500\ell^4 - 20475\ell^3 + 9525\ell^2 - 968\ell - 182)}{(3\ell - 5)(15\ell - 22)(15\ell - 11)} q + O(q^2) \right)$$

by the recursive relations (24), and a scalar multiple of $f_{13/20-\ell}$ is a character of a simple $V$-module. However, it follows that $(13500\ell^4 - 20475\ell^3 + 9525\ell^2 - 968\ell - 182)/(3\ell - 5)(15\ell - 22)(15\ell - 11)$ is positive for all $\ell \in \mathbb{Z}_{>0}$, which shows that any scalar multiple of $f_{13/20-\ell}$ cannot be a character of a simple $V$-module since the second Fourier coefficient is negative. Thus we have proved the proposition. $\square$

**(c) Central charge $c_{2,7} = -68/7$.** The associated monic MLDE (17) is written as

$$
\begin{aligned}
(33) \qquad & D^4(f) - E_2 D^3(f) + \left( 3E_2' + \frac{2}{7} E_4 \right) D^2(f) \\
& \qquad - \left( E_2'' + \frac{1}{7} E_4' + \frac{619}{37044} E_6 \right) D(f) - \frac{85}{148176} E_8 f = 0,
\end{aligned}
$$

and the set of indicial roots is $\{-1/42, \ 5/42, \ 17/42, \ 1/2\}$. It follows from (5) and (19) that the left-hand side of this monic MLDE is rewritten as

$$(34) \qquad \vartheta_6 \left( \vartheta_0^3(f) - \frac{5}{252} E_4 \vartheta_0(f) + \frac{85}{74088} E_6 f \right) = \vartheta_6 \circ L_{2,7}(\vartheta) f.$$

Therefore, the character $\chi_{1,1}$ of $L(-68/7, 0)$ is a solution of (33) by Table 1 and (5).

PROPOSITION 17.    *Vertex operator algebras with central charge $c = -68/7$ satisfying the conditions* (A)–(C) *do not exist.*

PROOF. Suppose that there is a VOA with central charge $-68/7$, which satisfies the conditions (A)–(C). Since there are no indicial roots which sits in $17/42 + \mathbb{Z}_{>0}$, the character of $V$ must be $\chi_{1,1}$. Thus it follows from Corollary 2 of Theorem 5 that $V$ is isomorphic to $L(-68/7, 0)$ by Lemma 2. However, the space spanned by characters of all simple $L(-68/7, 0)$-modules is 3-dimensional, which contradicts to the condition (B). $\square$

**(d) Central charge $c_{3,4} = 1/2$.** The associated monic MLDE is given by

$$
\begin{aligned}
(35) \qquad & D^4(f) - E_2 D^3(f) + \left( 3E_2' + \frac{199}{768} E_4 \right) D^2(f) \\
& \qquad - \left( E_2'' + \frac{199}{1536} E_4' + \frac{229}{55296} E_6 \right) D(f) - \frac{23}{110592} E_8 f = 0.
\end{aligned}
$$

The set of incidial roots is $\{-1/48, 23/48, 1/24, 1/2\}$. It follows from (5) and (19) that the left-hand side of this monic MLDE is rewritten as

$$\vartheta_6 \circ \left(\vartheta_0^3(f) - \frac{107}{2304}E_4\vartheta_0(f) + \frac{23}{55296}E_6 f\right) = \vartheta_6 \circ L_{3,4}(\vartheta)f.$$

Therefore, the character $\chi_{1,1}$ of $L(1/2, 0)$ is a solution of the monic MLDE (35) by Table 1 and (5).

PROPOSITION 18.    *Vertex operator algebras with central charge $1/2$ satisfying the conditions (A)–(C) do not exist.*

PROOF. Suppose that there is a VOA $V$ with central charge $1/2$ satisfying the conditions (A)–(C). Since there are no indicial roots which sit in $1/48 + \mathbb{Z}_{>0}$, the character of $V$ must be $\chi_{1,1}$ by Lemma 2. Thus it follows from Corollary 2 of Theorem 5 that $V$ is isomorphic to $L(1/2, 0)$. However, the space spanned by characters of all simple $L(1/2, 0)$-modules is 3-dimensional, which contradicts to the condition (B).    □

THEOREM 19.    *Let $V$ be a vertex operator algebra. Suppose that* (A) *the central charge and conformal weights are rational numbers,* (B) *there is a monic modular linear differential equation of order 4 whose space of solutions coincides with the space spanned by characters of all simple $V$-modules and* (C) $\dim V_0 = 1$, $\dim V_1 = 0$ *and* $\dim V_2 = 1$. *Then $V$ is isomorphic to one of the minimal models $L(-46/3, 0)$, $L(-3/5, 0)$, and the $\mathbb{Z}_2$-graded simple current extensions $\hat{L}(-114/7, 3)$, $\hat{L}(4/5, 3)$ of $L(-114/7, 0)$, $L(4/5, 0)$ by their simple modules with conformal weight 3, respectively.*

PROOF. The conditions (A), (B) and (C) imply the rational numbers which are the candidates for central charges and indicial roots of monic MLDEs of order 4 as in Table 4. Then Propositions 14–18 prove the theorem.    □

As a corollary of the theorem, we obtain

COROLLARY.    *Let $V$ be a rational and $C_2$-cofinite vertex operator algebra such that $V_0 = \mathbb{C}\mathbf{1}$, $\dim V_1 = 0$ and $\dim V_2 = 1$. Suppose that there is a monic modular linear differential equation of order 4 whose space of solutions coincides with the space spanned by characters of all simple $V$-modules. Then $V$ is isomorphic to one of the minimal models with central charge $-46/3$, $-3/5$, and the $\mathbb{Z}_2$-graded simple current extensions of minimal models of central charge $-114/7$, $4/5$.*

PROOF. Since $V$ is rational and $C_2$-cofinite, it is known in [4, Theorem 11.3] that the central charge and conformal weights are rational numbers. Moreover, any VOA with $V_0 = \mathbb{C}\mathbf{1}$ is of CFT type (see Proposition 3). Then the assertion follows from Theorem 19.    □

## 7.  Final remark

In this paper we classified vertex operator algebras $V$ **of CFT type** whose spaces of characters of simple modules coincide with the space of solutions of monic MLDEs of order 4 under the condition on the dimensions of lower homogeneous subspaces of

VOAs, namely, $\dim V_1 = 0$ and $\dim V_2 = 1$. This condition provides that such VOAs are isomorphic to one of minimal models $L(c_{2,9}, 0)$, $L(c_{3,5}, 0)$ and extensions of minimal models $L(c_{5,6}, 0)$, $L(c_{3,14}, 0)$ by simple modules with conformal weight 3. Therefore, this result characterizes minimal models with central charges $c_{2,9}$ and $c_{3,5}$ in terms of monic MLDEs and dimensions of $V_1$ and $V_2$. Particularly, monic MLDEs associated with data which correspond central charges and conformal weights played crucial role.

Our results (Theorem 8 and 13) can be basically used to determine a VOA whose space spanned by characters of all simple modules coincides with the space of solutions of monic MLDEs of order $n$ under the condition that the first $n-1$ coefficients of character of $V$ coincide with those of the character of $V(c_{p,q}, 0)$ *when we suppose that the central charge is $c_{p,q}$*. However, if we omit the condition that the central charge is $c_{p,q}$, then $\mathbb{Z}_2$-orbifold models of lattice VOAs and for monic MLDEs of orders 3 and 5 appear ([24]). The problem characterizing all VOAs associated to monic MLDEs of higher dimensional minimal models is far from at the moment. We need to develop another methods.

The first obstruction appears for 6th order case since the space of modular form of weight 12 is 2-dimensional (The lower weight spaces have dimension 1). However, our method works if we change the condition (B) to the character of $V$ coincides with $q^{-c/24}\bigl(1 + q^2 + q^3 + 2q^4 + 2q^5 + mq^6 + O(q^7)\bigr)$ ([22]), which is the first 6th part of $\mathrm{ch}_{V(c,0)}$. Moreover, combining the condition on simple modules, we state:

**Conjectures.** Let $V$ be a simple vertex operator algebra. Suppose that (I) the character $\mathrm{ch}_V$ of $V$ coincides with $q^{-c/24} \prod_{n>1}(1-q^n)^{-1}$ up to the first $d$ terms with $d = (p-1)(q-1)-1$, (II) any simple $V$-module except $V$ has a character of the form $q^{h-c/24} \prod_{1+q+O(q^2)}$ and (III) the space spanned by characters of all simple $V$-modules is $n = (p-1)(q-1)/2$-dimensional and coincides with the space of a monic modular linear differential equation. Then $V$ is isomorphic to the minimal model $L(c_{p,q}, 0)$.

Unfortunately our prescription does not work on solving this conjecture since the dimension of $M_k(\Gamma)$ is given by $d_k = \sum_{i=0}^{k} \dim M_{2i}(\Gamma)$ (see Table 5). Therefore, the general monic MLDE of order $n$ has $d_{2n}$ linear parameters. However, while we have only $(d_{2n}+2)$-simultaneous equations. Therefore, we cannot use our method or generalized methods to the higher order cases. For instance, by the Table 5 we can apply our method at most 9th order case.

| Weight | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | $\cdots$ | 34 | 36 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\dim M_k(\Gamma)$ | 0 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 2 | 2 | $\cdots$ | 3 | 4 |
| $d_k$ | 0 | 1 | 2 | 3 | 4 | 6 | 7 | 9 | 11 | 13 | $\cdots$ | 32 | 36 |

Table 5.   Dimensions of the spaces of modular forms

Final observation is about Theorem 8 in §3. This theorem can be generalized even in the cases that the set of conformal weights of the simple $L(c_{p,q}, 0)$-modules has a positive integer. Therefore it seems to be very possible to classify VOAs with central charge $c_{p,q}$.

## A.   Appendix by Don Zagier

We will show that the Diophantine equation

$$1050\,c^5 + (5m + 31020)\,c^4 + (-703m + 275600)\,c^3 + (32992m + 673104)\,c^2$$
$$+ (-517172m + 504352)\,c + (3984m - 210432) \,=\, 0$$

has precisely sixteen solutions with $(c, m) \in \mathbb{Q} \times \mathbb{Z}$, seven with $m \in \{1, 2\}$, four with $m > 500\,000$, and five with $m < -100\,000$, as listed below in increasing order of $c$:

| $c$ | $-114/7$ | $-46/3$ | $-68/7$ | $-22/5$ | $-3/5$ | $1/2$ | $4/5$ |
|---|---|---|---|---|---|---|---|
| $m$ | 2 | 1 | 1 | 1 | 1 | 1 | 2 |

| $c$ | 36 | 122/3 | 238/5 | 48 |
|---|---|---|---|---|
| $m$ | 501 971 | 3 132 760 | 37 950 512 | 42 987 520 |

| $c$ | 50 | 53 | 58 | 60 | 102 |
|---|---|---|---|---|---|
| $m$ | $-1\,453\,535\,852$ | $-12\,059\,329$ | $-2\,661\,904$ | $-1\,837\,492$ | $-177\,972$ |

For the proof, we first notice that setting $c = x/210$ transforms the equation into

$$x^5 + (m + 6204)\,x^4 + 42\,(-703m + 275600)\,x^3 + 141120\,(2062m + 42069)\,x^2$$
$$+ 1852200\,(504352 - 517172m)\,x + 18670176000\,(83m - 4384) = 0\,.$$

For $m \in \mathbb{Z}$ this is a monic polynomial in $x$ with integer coefficients, so that any solution with $x$ rational in fact has $x$ integral. Next, we make the substitution $m = n - x - 35730$. In terms of the new variables, the equation to be solved takes on the form $A(x)\,n + B(x) = 0$, where

$$A(x) \,=\, x^4 - 29526x^3 + 290989440x^2 - 957905978400x + 1549624608000\,,$$
$$B(x) \,=\, 420\,(1846547x^3 - 22460023656x^2 + 81488964629520x - 132023660227200)$$

with $\deg(B) < \deg(A)$. (The coefficients in the substitution $m \mapsto n$ were chosen precisely to achieve this.) Since neither $A(x)$ nor $B(x)$ has an integral root (in fact, both are irreducible), the rational function $F(x) = B(x)/A(x)$ takes on non-zero rational values for $x \in \mathbb{Z}$, and our problem is to find all integral values in this set. Since $F$ vanishes at infinity, and all non-zero integers are at least 1 in absolute value, it is already clear that there are only finitely of these and hence only finitely many solutions $(c, m) \in \mathbb{Q} \times \mathbb{Z}$ of the original equation. A brute force calculation would be time-consuming because the set of integers $x$ for which $|F(x)| \geq 1$ is $[-775\,532\,377,\ 775\,567\,102]$, with cardinality nearly two billion. But we do not have to check the integrality of $F(x)$ for all integers $x$ in this range. Instead, we check it only for the much smaller range $|x| < 50000$. This takes only 0.8 seconds with the software package PARI and yields precisely 16 values of $x$ with $F(x) \in \mathbb{Z}$, which after transforming back via $(c, m) = (x/420, -F(x) - x - 35730)$ correspond to the 16 pairs $(c, m)$ tabulated above. We then observe that for $|x| > 50000$ the value of $F(x)$ belongs to the interval $[-11407,\ 23179]$, and since $F(x) - n$ is irreducible for all $n$ in this interval (as PARI checks in 2.3 seconds), there are no further solutions.

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