The entanglement and relative entropy of a chiral fermion on the torus

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(Dated: June 7, 2019)

We derive the entanglement entropy of chiral fermions on the circle at arbitrary temperature. The spin-sector contribution depends only on the total length of the entangling region, regardless of the configuration of the intervals. Thus mutual information is independent of the spin boundary conditions, a property that does not hold for higher Rényi mutual information. Together with the modular Hamiltonian, this provides a systematic way of obtaining relative entropy on the torus.

arXiv:1906.02207v1 [hep-th] 5 Jun 2019

I. INTRODUCTION

Entanglement is perhaps the most characteristic feature of quantum theory. Although there exists many measures for it, the most important one is the *entanglement entropy*. Given a global state defined via a density matrix ρ and a subsystem A of it, the reduced density matrix on A is defined as $\rho_A := \mathrm{tr}_B(\rho)$, where B is the complement of A. Then, the entanglement entropy between A and B is simply the von Neumann entropy of the subsystem,

$$S(A) = -\mathrm{tr}_A\left(\rho_A \log \rho_A\right) . \tag{1}$$

In practice however, computing the logarithm in (1) is generically very challenging. Therefore, it is customary in QFT to resort to the *replica trick*. This consists of first computing a closely related quantity - the Rényi entropy for integer index n:

$$S_n(A) = \frac{1}{1-n} \log \mathrm{tr}\left(\rho_A^n\right) . \tag{2}$$

This requires to compute the partition function of the theory on the $n-$th cover of the manifold of interest, where the n copies are glued cyclically along the entangling region A. In order for this strategy to work, it is crucial that we are able to find the analytic continuation of the Rényi entropy (2) away from integer n and take the limit $n \to 1$. To the best of our knowledge such analytic continuation for the entanglement entropy of free fermions on the torus (finite temperature on the circle) is not known for arbitrary torus modulus.

In this paper, we derive the entanglement entropy of the chiral fermion on the torus via a different route - by computing the entropy directly from the resolvent. This avoids the need of finding the analytic continuation of the Rényi entropies. The Rényi entropies for integer n where first computed in [1] for a single interval, and in [2] for multiple intervals. The analytic continuation to the entanglement entropy however was obtained for the high and low temperature expansions.

The resolvent for the chiral fermion on the torus was recently obtained in [3] and also in [4]. As direct applications of the resolvent and modular Hamiltonian, we study entanglement entropy, mutual information and relative entropy. Interestingly, while Rényi mutual information for $n \geq 2$ is known to be spin-dependent [2], we show that mutual information ($n = 1$) is not. This hints towards some underlying factorisation structure of the reduced density matrix. On the other hand, relative entropy (or Kullback-Leiber divergence) does depend on the choice of boundary conditions, due to the expectation value of the modular Hamiltonian involved.

The paper is organised as follows. In section II we review the resolvent method, the main tool that we will apply to later computations. In section III we go straight to the calculation of the entanglement entropy, and discuss the mutual information. In section IV we consider relative entropy and illustrate some properties taking a small interval expansion. We summarise and discuss our results in section VI.

II. REVIEW OF THE RESOLVENT

Let us quickly summarise the definition of the *resolvent* and the mains steps involved in its derivation for the fermions on the torus [3]. A global state ρ for the system on a generic Riemann surface determines the equal-time correlation function $G(x, y) := \langle \psi(x)\psi^\dagger(y)\rangle$. The *resolvent* of the correlator is defined via:

$$R(\xi) := (G + \xi - 1/2)^{-1} \tag{3}$$

where we always leave the space-time dependence of R implicit. In (3), the inverse must be understood in the sense of distributions, i.e. as integrated agains regular test functions. Eq. (3) thus involves solving an integral equation for R.

One of the central results of [3] is the resolvent of a single chiral fermion on the torus for an arbitrary number p of disjoint intervals $A = \cup_{j=1}^{p}(a_j, b_j)$, for the $\nu = 2$ (R-NS) and $\nu = 3$ (NS-NS) boundary conditions. We shall not focus on its derivation, but rather quickly review its structure and explain its properties. Explicitly it is given

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by

$$R(\xi) = \frac{\delta(x-y)}{\xi - 1/2} - \frac{F(x,y;\xi)}{(\xi-1/2)^2} \tag{4}$$

where $x, y \in A$ are any two points on the entangling region, and

$$F(x,y;\xi) = \frac{\xi - 1/2}{\xi + 1/2} \left(\frac{\Omega(x|\tau)}{\Omega(y|\tau)}\right)^{ih} G_\nu(x-y|\tau, Lh) \tag{5}$$

where each term is defined as

$$h = \frac{1}{2\pi} \log \frac{\xi + 1/2}{\xi - 1/2} \tag{6}$$

$$G_\nu(z|\tau,\mu) = \frac{\eta^3(\tau)}{i\vartheta_1(z|\tau)} \frac{\vartheta_\nu(z - i\mu|\tau)}{\vartheta_\nu(-i\mu|\tau)} \tag{7}$$

$$\Omega(x|\tau) = -\prod_{j=1}^{p} \frac{\vartheta_1(x - a_j|\tau)}{\vartheta_1(x - b_j|\tau)}\,. \tag{8}$$

Here, $L = \sum_j |b_j - a_j|$ is the total length of all intervals, $q = \exp i\tau$ the nome, and our conventions for the Dedekind eta and the Jacobi elliptic functions are described in the Appendix. As usual, we fix by convenience the periods of the torus to be $1, \tau$. Since physics on the torus depends only on τ, we can always recover the result for a different spatial circle by rescaling τ. We shall restrict to purely imaginary modulus $\tau = i\beta$ where β is the inverse temperature, keeping in mind that the general case can be recovered by analytic continuation.

The knowledge of the resolvent was further used to derive the modular Hamiltonian for an arbitrary number of intervals on the torus. The modular Hamiltonian K of a density matrix is defined via

$$\rho = \frac{e^{-K}}{\mathrm{tr}\,(e^{-K})} \tag{9}$$

and has found numerous applications in many body quantum systems [5–7], quantum information [8–10], quantum field theory [11–16], modular theory [17, 18] and the AdS/CFT correspondence [19–24].

The modular Hamiltonian for the chiral fermion on the torus exhibits a surprisingly interesting structure. Even for a single interval, the modular flow couples any given point to an infinite but discrete set of other points. These accumulate near the boundaries of the interval, where their contribution becomes increasingly damped or 'red-shifted' as they approach the endpoints. In the limit of zero temperature, these points 'condense' regularly in the interval, giving rise to continuous non-locality [3].

III. ENTANGLEMENT ENTROPY

We now determine the entanglement entropy on the torus of an arbitrary set of disjoint intervals. As mentioned above, once the resolvent is obtained, the entanglement entropy follows by a trace formula [11]:

$$S = -\mathrm{tr} \int_{1/2}^{\infty} d\xi \left[(\xi - 1/2)\,(R(\xi) - R(-\xi)) - \frac{2\xi}{\xi + 1/2} \right]$$

Replacing the resolvent (4) one finds

$$S = -\int_A dx \lim_{y\to x} \int_{1/2}^{\infty} d\xi \times (\xi - 1/2) \left[-\frac{F(\xi)}{(\xi-1/2)^2} + \frac{F(-\xi)}{(\xi+1/2)^2} \right]. \tag{10}$$

Next, we substitute F from (5), and the term inside the brackets becomes

$$G_\nu(x-y|\tau, Lh) \left(\frac{\Omega(x)}{\Omega(y)}\right)^{ih} - G_\nu(x-y|\tau, -Lh) \left(\frac{\Omega(x)}{\Omega(y)}\right)^{-ih}$$

In the limit $y \to x$, $G(x-y|\tau, Lh)$ diverges as the UV propagator, but these contributions cancel, leaving a well defined expression. To extract the finite piece we use the Laurent series (A3)-(A4) provided in the appendix. Then, the limit $y \to x$ of this last equations yields

$$-\frac{h}{\pi} \partial_x \log \Omega(x|\tau) + \Sigma_\nu(\lambda|\tau) \tag{11}$$

Here, the first term is position-dependent but identical for each spin sector. On the other hand, the second term is space independent but varies according to the spin, and is given by a Laurent expansion

$$\Sigma_3(\lambda|\tau) = 2 \sum_{\substack{k\geq 1 \\ \text{odd}}} p_k(\lambda|\tau) \tag{12}$$

$$\Sigma_2(\lambda|\tau) = \frac{\lambda^2 - 1}{\lambda^2 + 1} + 2 \sum_{\substack{k\geq 2 \\ \text{even}}} p_k(\lambda|\tau) \tag{13}$$

where we used the convenient variable

$$\lambda = e^{\pi L h}$$

and the coefficients are rational functions of the nome $q = e^{i\pi\tau}$

$$p_k(\lambda|\tau) = \frac{q^k}{\lambda^{-2} + q^k} - \frac{q^k}{\lambda^2 + q^k}$$

Thus, we learn from (11) that the entanglement entropy decomposes into a spin-independent part $S^{(0)}$ and a spin-dependent one $S^{(\nu)}$

$$S = S^{(0)} + S^{(\nu)} \tag{14}$$

In the next subsections we consider them separately.

A. Spin-independent entropy

We start by rederiving the results known in [1, 2]. Let us start by focusing on the first contribution in (11), which does not depend on the boundary conditions. Introducing it back into (10), the two integrals decouple:

$$S^{(0)} = \frac{1}{2\pi^2}\int_{1/2}^{\infty} d\xi \frac{\log\left(\frac{\xi+1/2}{\xi-1/2}\right)}{\xi+1/2} \times \int_A dx\, \partial_x \log\Omega(x|\tau)$$

We see that only boundary terms from the spacial integral contribute, one per each endpoint of the intervals. This gives the final result for the spin-independent entropy, which takes a form very reminiscent of the one described in [11],

$$S^{(0)} = \frac{1}{6}\Big(\sum_{i,j}\log|\vartheta_1(b_i-a_j|\tau)| - \sum_{i<j}\log|\vartheta_1(b_i-b_j|\tau)\vartheta_1(a_i-a_j|\tau)| - p\log|\vartheta_1(\epsilon|\tau)|\Big) \tag{15}$$

but with respect to the Jacobi ϑ_1 function. Here we introduced a UV regulator $\epsilon > 0$, such that we integrate only within $(a_i+\epsilon, b_i-\epsilon)$, avoiding the endpoints of each interval.

B. Spin-dependent entropy

In this section we derive our first new result: the contribution to entanglement entropy that does depend on the choice of boundary conditions. As mentioned above, we restrict to the sectors $\nu = 2, 3$.

Let us start with $\nu = 3$ or (NS-NS) which has no zero mode. Plugging the second term of (11) back into (10), we have

$$S^{(3)} = L\int_{1/2}^{\infty}\frac{d\xi}{\xi+1/2}\Sigma_3(\lambda|\tau) \tag{16}$$

Notice here that the integral over the entangling region was trivial and gives a global prefactor of L, the total length of the regions. The remaining integral is however much more challenging and finding a closed form goes beyond the scope of this paper. However, in order to bring it into a more explicit form, let us move to the more convenient variable

$$\Lambda = \lambda^2 = \left(\frac{\xi+1/2}{\xi-1/2}\right)^L \tag{17}$$

in terms of which the entropy reads

$$S^{(3)} = 2\sum_{\substack{k\geq 1\\ \text{odd}}}\int_1^{\infty}\frac{d\Lambda}{\Lambda(\Lambda^{1/L}-1)}\left(\frac{q^k}{\Lambda+q^k} - \frac{q^k}{\Lambda^{-1}+q^k}\right) \tag{18}$$

Notice the integral is regular since $L < 1$, as we fixed the length of the spatial circle to unity. We have found no analytic expression for (18), but it can easily be dealt with numerically.

The case $\nu = 2$ or (R-NS) is special, for it has a zero mode. This manifests in the presence of the additional term in (13), and gives an entropy of

$$S^{(2)} = \int_1^{\infty}\frac{d\Lambda}{\Lambda(\Lambda^{1/L}-1)} \times \left(\frac{\Lambda-1}{\Lambda+1} + 2\sum_{\substack{k\geq 2\\ \text{even}}}\frac{q^k}{\Lambda+q^k} - \frac{q^k}{\Lambda^{-1}+q^k}\right) \tag{19}$$

which can again be computed numerically.

The main result of this section is that the spin-dependent contribution to the entropy depends only on the total length of the entangling region. It does not depend on any other details of the subregion. This remarkable simplification is not true for the Rényi entropies [2].

C. Mutual information

The mutual information between two disjoint intervals A and B is another important information theory quantity. It it a measure of the correlation between two distributions, here the two reduced density matrices ρ_A and ρ_B. It is defined via

$$I(A,B) := S(A) + S(B) - S(AB) \tag{20}$$

Remarkably, as pointed out in the previous section, the spin-dependent part to the entropy depends on the entangling region only via its total length L. Therefore, its contribution to the mutual information vanishes

$$I(A,B) = S^{(0)}(A) + S^{(0)}(B) - S^{(0)}(AB) \tag{21}$$

or equivalently

$$S^{(\nu)}(A) + S^{(\nu)}(B) - S^{(\nu)}(AB) = 0\,. \tag{22}$$

This is one the main results of this paper: mutual information is insensible to the fermion boundary conditions. As was explicitly shown in [2] this does not hold for the Rényi mutual information $I_n(A,B)$ for $n \geq 2 \in \mathbb{Z}$.

This property, together with the spin-independent entropy (15) implies that even on the torus - as in the plane [25] - the mutual information is additive:

$$I(A,BC) = I(A,B) + I(A,C) \tag{23}$$

And therefore, as in the plane, this implies that the three-partite information [26]

$$I(A,B,C) := S(A) + S(B) + S(C) - S(AB) - S(AC) - S(BC) + S(ABC) \tag{24}$$

vanishes identically,

$$I(A,B,C) = 0\,. \tag{25}$$

The mutual information between the two non-intersecting intervals (a_1, b_1) and (a_2, b_2) is explicitly given by

$$I(A,B) = \frac{1}{6}\log\left|\frac{\vartheta_1(a_2-a_1|\tau)\vartheta_1(b_2-b_1|\tau)}{\vartheta_1(b_2-a_1|\tau)\vartheta_1(b_1-a_2|\tau)}\right| \tag{26}$$

In Fig. 1 we plot the mutual information (26) for different temperatures, as we continuously vary the separation between the two intervals.

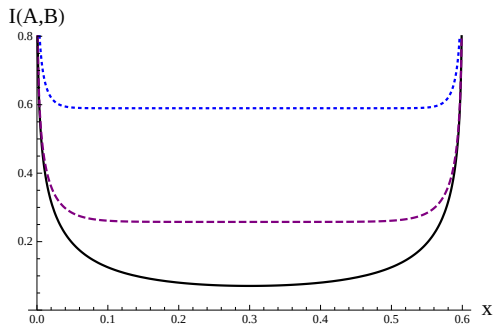

FIG. 1: Mutual information between the intervals $A = (0, 0.2)$ and $B = (x, x + 0.2)$ as we vary their separation x, for $\beta = 3$ (solid black), $\beta = 0.16$ (dashed purple), $\beta = 0.07$ (dotted blue).

IV. RELATIVE ENTROPY

Another essential concept in information theory is *relative entropy*, also known as the Kullback-Leibler divergence. It is a measure of the distinguishability between two probability distributions ρ and σ. Although it is not symmetric, it has a number of properties that make it fundamental due to its connection to many other information functions. It is defined by

$$D(\rho|\sigma) = \mathrm{tr}\,(\rho\log\rho) - \mathrm{tr}\,(\rho\log\sigma)\ . \tag{27}$$

Relative entropy is always positive and vanishes only if $\rho = \sigma$. As is well known, this can be rewritten as

$$D(\rho|\sigma) = \Delta\langle K\rangle - \Delta S \tag{28}$$

in terms of the variations of the modular Hamiltonian $K_\sigma = -\log\sigma$ and the entanglement entropy

$$\Delta\langle K\rangle = \langle K_\sigma\rangle_\rho - \langle K_\sigma\rangle_\rho$$
$$\Delta S = S(\rho) - S(\sigma)\,.$$

Here, σ is the reference state since we used the modular Hamiltonian associated to σ instead of ρ. The convenience of (28) is that presented in this form it closely resembles the first law of thermodynamics. Given that we have both the modular Hamiltonian from [3] and the entanglement entropy from section III, we are in position to compute the relative entropy directly using (28).

Consider the following class of density distributions. Given a global thermal state at some modulus τ, the reduced density matrix on the subregion A provides a reference state $\sigma = f(\tau)$. If we now start from a global state at some different modulus τ' and reduce to the same region A, this will produce another state $\rho = f(\tau')$. We will consider the relative entropy between two such states.

We start by reviewing the structure of the modular Hamiltonian and then computing its expectation value. As referred to in the introduction, an explicit expression of the modular Hamiltonian for chiral fermions on the torus was found in [3]. It contains a local and a bi-local term:

$$K = K^{\mathrm{loc}} + K^{\text{bi-loc}} \tag{29}$$

As an operator, the local part is spin-independent and takes the standard geometric form,

$$K^{\mathrm{loc}} = \int_A dx\,\beta(x)T(x) \tag{30}$$

where the stress-tensor of the fermion is

$$T(x) = \frac{i}{2}\left(\psi^\dagger\partial_x\psi - \psi\partial\psi^\dagger\right)(x) \tag{31}$$

and $\beta(x)$ is known as the *entanglement temperature*, and is given by

$$\beta(x) = \frac{2\pi\beta}{2\pi + \beta\partial_x\log\Omega(x|\tau)}\,. \tag{32}$$

The entanglement temperature is the natural generalisation of the more familiar Unruh-temperature measured by an accelerated observer in the vacuum. Close to the endpoints of each interval, $\beta(x) \sim x$ and the modular Hamiltonian resembles that of Rindler space.

The bi-local term was the novel feature found for fermions on the torus. It couples a given point x to an infinite but discrete set of other points $x_k(x)$ on the entangling region,

$$K_\pm^{\text{bi-loc}} = \int_A dx \sum_{k\in\mathbb{Z}} (\pm 1)^k\,\tilde\beta(x, x_k(x))\psi^\dagger(x)\psi(x_k(x)) \tag{33}$$

Here, the points x_k are solutions to the transcendental equation

$$x - x_k + \beta g(x, x_k) - k = 0 \tag{34}$$

where

$$g(x,y) = \frac{1}{2\pi L}\log\frac{\Omega(x|\tau)}{\Omega(y|\tau)} \tag{35}$$

and $\Omega(z|\tau)$ as defined in (8). The bi-local entanglement temperature is

$$\tilde{\beta}(x,y) = \frac{i\pi}{L\,(1+\beta\partial_x g(x,y))\sinh \pi g(x,y)}\,. \tag{36}$$

Now we consider the expectation value of each term in the modular Hamiltonian, which we will eventually replace back into the relative entropy (28). We start with the local term (30). Notice that the expectation value acts only on the operator T in (31), while the entanglement temperature $\beta(x)$ is fixed by the reference state σ (at modulus τ) and remains the same for the perturbed state ρ. As expected, $\langle T\rangle$ is divergent due to operators evaluated at identical space-time positions, but relative entropy is well defined since these UV divergences cancel in (28). Using the shorthand notation

$$\Delta G_\nu(z) = G_\nu(z|\tau') - G_\nu(z|\tau) \tag{37}$$

the variation of the local term reads

$$\Delta\langle K^{\rm loc}\rangle = -i\partial_z \Delta G_\nu(z)\Big|_{z=0}\int_A dx\,\beta(x)\,. \tag{38}$$

Although we have not found a closed expression for the integral, it is easy to compute it numerically. Notice that while $K^{\rm loc}$ as an operator is independent of the spin sector, its expectation value is not. Indeed, the prefactor depends on the derivative of the propagator G_ν. As we comment in the discussion, this implies that relative entropy is not spin-independent.

On the other hand, the variation of the bi-local contribution is

$$\Delta\langle K^{\text{bi-loc}}\rangle = \int_A dx \sum_{k\in\mathbb{Z}} (-1)^{\pm}\tilde{\beta}(x,x_k(x))\Delta G_\nu(x-x_k(x)) \tag{39}$$

This integral is technically difficult to deal with: one must first determine the solutions $x_k(x)$ to (34) - a transcendental equation involving elliptic functions - and then perform the integration. The evaluation of such expressions goes beyond the scope of this paper, and will be investigated in another work.

Finally, the variation of the entanglement entropy has also two contributions,

$$\Delta S = S(\tau') - S(\tau)\,. \tag{40}$$

Putting everything together, the relative entropy is given by:

$$D(\rho|\sigma) = \Delta\langle K^{\rm loc}\rangle + \Delta\langle K^{\text{bi-loc}}\rangle - \Delta S^{(0)} - \Delta S^{(\nu)} \tag{41}$$

where each term is given in (38), (39), (15), (18), (19). As emphasised above, in practice the evaluation of (41) is very challenging and we leave it as future work.

A. An example: the thermal cylinder

In order to illustrate how to compute in principle this relative entropy, let us simplify things and perform the explicit calculation for a single interval of length L, in the limit when the size of the spatial circle is much larger than both the entangling region and the temporal cycle. We view this as an exercise combining known tools rather than a new result. Then, our expressions for the modular Hamiltonian and entanglement entropy on the torus reduce to the well known universal result on the thermal cylinder [13]:

$$K(\beta) = \beta\int_0^L dx \frac{\sinh\frac{\pi(L-x)}{\beta}\sinh\frac{\pi x}{\beta}}{\sinh\frac{\pi L}{\beta}} T \tag{42}$$

$$S(\beta) = \frac{1}{6}\log\left(\frac{\beta}{\pi\epsilon}\sinh\frac{\pi L}{\beta}\right)\,, \tag{43}$$

where T is again the fermionic stress tensor. Notice that in this case the modular flow is purely local. As described above, by varying the inverse temperature β of the parent state, we get a one-parameter family of states defined on A. We can compute the relative entropies between such states.

For this simplified case it is easy to obtain (38) from the fermionic propagator, and find an explicit expression for the relative entropy:

$$D(\beta|\beta') = \frac{\beta}{24}\left(\frac{1}{\beta'^2}-\frac{1}{\beta^2}\right)\left(-\beta+\pi L\coth\frac{\pi L}{\beta}\right) + \frac{1}{6}\log\frac{\beta\sinh\frac{\pi L}{\beta}}{\beta'\sinh\frac{\pi L}{\beta'}} \tag{44}$$

Finally, it is illustrative to plot this function for a fixed reference $\beta = 2$ and interval length $L = 1$, while we vary β', see Fig. 2.

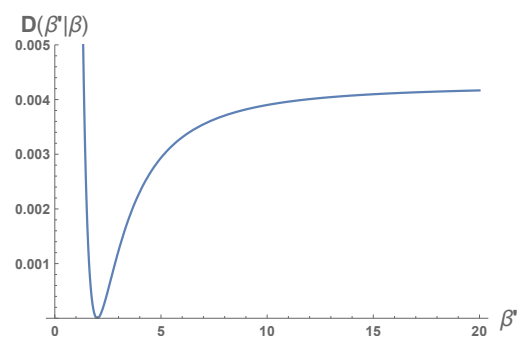

FIG. 2: Relative entropy between the states obtained by reducing a thermal state on the line at temperature β' and β respectively, to a single interval of length $L = 1$. The reference temperature associated to σ is $\beta = 2$.

As can be seen from the figure, the distribution obtained from a parent state at low β' (high temperature)

is very distinguishable from the finite temperature state. On the other hand, the relative entropy with respect to the vacuum $\beta' \to \infty$ asymptotes to a constant. Relative entropy only vanishes at $\beta' = \beta$, as it should, and the fact that the curve has vanishing slope at that point is the *first law of entanglement*. Again, we take this as a pedagogical first step in the explicit computation of relative entropy, and leave for future work the more involved calculations on the torus.

V. THE RESOLVENT WITH A CHEMICAL POTENTIAL

In order to repeat the calculations for the two remaining cases $\nu = 1, 4$ (R-R and NS-R), let us derive the resolvent in these sectors. We will do this explicitly for $\nu = 1$, the other case is entirely analogous. Consider first the Green's function

$$G(z|\tau,\mu) = \sum_{k\in\mathbb{Z}} \frac{\mathrm{e}^{-2\pi \mathrm{i} kz}}{1-\mathrm{e}^{2\pi(\mathrm{i}k\tau-\mu)}}, \tag{45}$$

where we introduced a *chemical potential* μ to regularize the divergent term at $k = 0$. The above series does not converge pointwise, however, it does converge in the sense of distributions: Starting from the geometric series

$$\frac{1}{2\mathrm{i}}\cot \pi z = \frac{1}{2}\frac{\mathrm{e}^{\mathrm{i}\pi z}+\mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z}-\mathrm{e}^{-\mathrm{i}\pi z}} = \frac{1}{2} + \sum_{k\geq 1} \mathrm{e}^{-2\pi \mathrm{i} kz}$$

for $\Im[z] < 0$, we find

$$\begin{aligned} G(z|\tau,\mu) &= \frac{1}{2\mathrm{i}}\cot \pi z - \frac{1}{2} + \frac{1}{1-\mathrm{e}^{-2\pi\mu}} \\ &+ \sum_{k\geq 1}\Big[\frac{\mathrm{e}^{2\pi \mathrm{i} kz}}{1-\mathrm{e}^{-2\pi(\mathrm{i}k\tau+\mu)}} + \frac{\mathrm{e}^{-2\pi \mathrm{i} kz}}{1-\mathrm{e}^{2\pi(\mathrm{i}k\tau-\mu)}} - \mathrm{e}^{-2\mathrm{i}\pi kz}\Big], \end{aligned}$$

which can be rewritten as

$$\begin{aligned} G(z|\tau,\mu) &= \frac{1}{2\mathrm{i}}\cot \pi z + \frac{1}{2}\frac{1+\mathrm{e}^{-2\pi\mu}}{1-\mathrm{e}^{-2\pi\mu}} \\ &- \sum_{k\geq 1}\Big[\frac{\mathrm{e}^{2\pi \mathrm{i} kz}\mathrm{e}^{2\pi \mathrm{i} k\tau}}{\mathrm{e}^{-2\pi\mu}-\mathrm{e}^{2\pi \mathrm{i} k\tau}} - \frac{\mathrm{e}^{-2\pi \mathrm{i} kz}\mathrm{e}^{2\pi \mathrm{i} k\tau}}{\mathrm{e}^{2\pi\mu}-\mathrm{e}^{2\pi \mathrm{i} k\tau}}\Big]. \end{aligned}$$

The right hand side is now absolutely convergent on the strip $-\Im[\tau] < \Im[z] < \Im[\tau]$ and yields

$$G_\nu(z|\tau,\mu) = \frac{\eta^3(\tau)}{\mathrm{i}\vartheta_1(z|\tau)}\frac{\vartheta_\nu(z-\mathrm{i}\mu|\tau)}{\vartheta_\nu(-\mathrm{i}\mu|\tau)} \tag{46}$$

for general spin structure $\nu = 1, 2, 3, 4$.

As can be readily seen from the quasiperiodicities of the theta functions, eq. (46) satisfies the generalized KMS condition

$$G(z+\tau|\tau,\mu) = \pm \mathrm{e}^{-2\pi\mu} G(z|\tau,\mu). \tag{47}$$

We can thus use the methodology of [3] to derive the resolvent as (4) with

$$F(x,y;\xi) = \frac{\xi-1/2}{\xi+1/2}\left[\frac{\Omega(x)}{\Omega(y)}\right]^{\mathrm{i}h} G(x-y|\tau,\mu+Lh). \tag{48}$$

Note that, unlike the Green's function (46), the resolvent is well defined in the limit $\mu \to 0$.

Having found the resolvent, we can now in principle reapeat our calculations for the cases $\nu = 1, 4$. However, there is one caveat: Because of the bose statistics in (45), G has an unbounded spectrum and thus we have to find new valid formulae for the modular Hamiltonian, entanglement, and relative entropy in terms of the resolvent. While this is easy for the modular Hamiltonian, we have yet to find good expressions for the entropies.

VI. DISCUSSION

In this short note we have described some applications of the modular Hamiltonian found in [3] for the chiral fermion on the circle at finite temperature. First, we found an exact expression for the entanglement entropy for multiple intervals, valid for any torus modulus τ, i.e. generic spatial and thermal circles. This is a generalisation of the results in [1, 2], where the authors provided the entanglement entropy in the high and low temperature expansions. Whereas the spin-independent contribution to the entanglement entropy was known from these works, the novel result of this paper was to derive the analytic continuation of the spin-dependent term for an arbitrary torus.

Remarkably, this piece depends on the entangling intervals only via the total length of the region $L = \sum_j |b_j - a_j|$, and not the details of their configuration. This implies that mutual information does not depend on the spin sector, a property that is not shared by its Rényi generalisations. Since mutual information is equal to the relative entropy between the union state and the product state,

$$I(A,B) = D(\rho_{AB}|\rho_A \otimes \rho_B) \tag{49}$$

and this only vanishes $\rho_{AB} = \rho_A \otimes \rho_B$, our result suggests an interesting factorisation property of the spin-dependent component of the density matrix.

The third result of this paper is an explicit formula for the relative entropy $D(\rho|\sigma)$ of the free fermion on the torus. As the reference state, we chose the reduced density matrix σ on multiple intervals coming from a global state on the torus of modulus $\tau = i\beta$. The target state is obtained in the same way but starting from a different temperature β'. This provides a one-parameter family of states for computing relative entropy. Although in practice the integrals involved in the computation are very challenging, mainly due to the non-local terms involved, the final expression (41) is explicit and can be investigated numerically.

Moreover, we also observed that while both the modular Hamiltonian as an operator and the entanglement entropy separate into a universal and spin-dependent parts, this does not hold for the relative entropy. This is simply because the computation of relative entropy involves expectation values which yields Green's functions in the modular Hamiltonians. Since relative entropy is a measure of the distinguishability of two distributions, this implies that fermions with some boundary conditions are more distinguishable than others.

Finally, we showed that the propagator of the fermion on the torus with a chemical potential allows to compute the resolvent for the spin sectors $\nu = 1, 4$ which were not considered in [3]. Moreover, the propagator-like term that appears in the resolvent for the $\nu = 2, 3$ cases corresponds precisely to the propagator with a chemical potential. This is due to the identical boundary conditions imposed in both problems.

VII. ACKNOWLEDGMENTS

PF is financially supported by the DFG project DFG HI 744/9-1. The work of IR is funded by the Gravity, Quantum Fields and Information group at AEI, which is generously supported by the Alexander von Humboldt Foundation and the Federal Ministry for Education and Research through the Sofja Kovalevskaja Award. IR also acknowledges the hospitality of Perimeter Institute, where part of this work was done.

Appendix A

The work with the Dedekind eta function given by

$$\eta(\tau) := e^{\pi i\tau/12} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n\tau}\right) \tag{A1}$$

and the Jacobi theta functions

$$\vartheta_3(z|\tau) := \sum_{k\in\mathbb{Z}} e^{i\pi\tau n^2} e^{2\pi i z} \tag{A2}$$

The propagators appearing in (7) are given by the following Laurent series,

$$G_3(z|\tau, Lh) = \frac{1}{2i\sin(\pi z)} + \sum_{\substack{k\geq 1\\ \text{odd}}} p_k\left(z, \lambda|q\right) \tag{A3}$$

$$G_2(z|\tau, Lh) = \frac{1}{2i}\text{cotan}(\pi z) + \frac{1}{2}\frac{\lambda^2 - 1}{\lambda^2 + 1} + \sum_{\substack{k\geq 2\\ \text{even}}} p_k\left(z, \lambda|q\right) \tag{A4}$$

where again $\lambda = e^{\pi Lh}$, and we defined

$$p_k(z, \lambda|q) = \frac{w^k q^k}{\lambda^{-2} + q^k} - \frac{w^{-k} q^k}{\lambda^2 + q^k}\,. \tag{A5}$$

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