Department of Physics and Astronomy

University of Heidelberg

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Tobias Podszus

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On the High-Energy Behaviour

of Stong-Field QED

in an Intense Plane Wave

This Master thesis has been carried out by Tobias Podszus

at the

Max-Planck Institut für Kernphysik

under the supervision of

Herrn Priv.-Doz. Antonino Di Piazza

Zum Hochenergieverhalten der stark-Feld QED in einer intensiven ebenen Welle:

Wir behandeln das Hoch-Energie Verhalten der QED in starken elektromagnetischen Hintergrundfeldern in Form von ebenen Wellen, generiert durch einen Laserpuls. Frühere Arbeiten in diesem Bereich deuteten darauf hin, dass unter diesen Bedingungen die Kopplungskonstante der QED für hohe Energien mit der 2/3-Potenz der Energie skaliert und nicht, wie typisch für normale Vakuum-QED, logarithmisch mit der Energie. Diese Berechnungen gelten jedoch nur im Grenzwert für geringe Laserfrequenzen und konstante Feldstärken. Wir zeigen hier, dass dieser Grenzwert jedoch nicht mit dem Grenzwert für hohe Energien kommutiert und demnach die Skalierung mit der 2/3-Potenz nur im Grenzwert eines konstanten und gekreuzten elektromagnetischen Feldes Gültigkeit besitzt. Des weiteren berechnen wir den asymptotischen Ausdruck für den Polarisations- und Massen-Operator in einem starken Laserpuls im Grenzwert für hochenergetische Photonen, bzw. Elektronen und erhalten, dass diese mit dem Logarithmus zum Quadrat mit der Energie skalieren. Damit zeigen wir anschließend auch, dass die Wahrscheinlichkeiten für nicht-lineare Breit-Wheeler Paarproduktion und für nicht-lineare Compton-Streuung, ähnlich zur Vakuum-QED, logarithmisch mit der Energie skalieren.

On the high-energy behaviour of strong-field QED in an intense plane wave:

We study the high-energy behaviour of QED in a strong plane wave electromagnetic background field generated by a laser pulse. Earlier calculations in this field hinted that under this circumstances the coupling constant of QED may increase with the 2/3-power of the energy scale for high energies and not logarithmic like in normal vacuum QED. Nevertheless, this calculations were performed in the limit of low laser frequencies or constant-crossed-fields. We show in this work that this limit does not commute with the high-energy limit and thus the power-law scaling just pertains to the constant-crossed field limit. Further we calculate the asymptotic expression of the polarization and mass operator in a strong laser pulse in the limit of high energetic photons and electrons, respectively, and obtain that they scale double logarithmic with the energy scale. Using this we show that also the probability for non-linear Breit-Wheeler pair production and for non-linear Compton scattering scales logarithmic with the energy like in vacuum QED.

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Contents

1 Introduction

Nowadays the best tested theory with the most precise calculations in physics is quantum electrodynamics (QED) [Hanneke et al., 2008, Sturm et al., 2011]. This is due to the smallness of the QED coupling constant $\alpha = e^2/\hbar c$ at ordinary energies. Here $e < 0$ is the electron charge, $\hbar$ the Planck-constant, c the light velocity, and we are working in units where $4\pi\epsilon_0 = 1$ with ϵ_0 being the electric permittivity of vacuum. For energies of the order of the electron rest energy $mc^2 = 0.511\text{MeV}$, with m being the electron mass, the fine structure constant is about $\alpha \approx 1/137$. Since every higher quantum loop correction scales with the small parameter α, we are allowed to use perturbation theory and thus reduce the calculations for a given precision to a finite number of Feynman diagrams. Nevertheless, these Feynman diagrams contain divergences and have to be renormalised to give finite results, which leads to the effective coupling constant increasing with higher energies and featuring a pole at $\Lambda_{\text{QED}} \approx mc^2 \exp(3\pi/2\alpha) \approx 10^{277}\text{GeV}$, called Landau pole [Berestetskii et al., 1982, Schwartz, 2014, Weigand, 2018]. Thus, strictly speaking, QED is only an effective description of the electromagnetic sector for energies below the Landau pole rather than a fundamental theory and it can not correctly describe the microscopic degrees of freedom beyond this scale. But due to the large value of the Landau pole, this plays practically no role. Further we already know that QED as group U(1) is confined with the weak force into the electroweak theory in a group SU(2)xU(1) for energies in the order of 10^2GeV and thus much below the Landau pole [Schwartz, 2014]. The large value of the Landau pole is due to the fact that radiative corrections increase logarithmic in vacuum QED [Berestetskii et al., 1982, Schwartz, 2014, Weigand, 2018].

Since the results of QED calculations were in agreement with experiment with high accuracy, one tries to test QED under other extreme parameter conditions to give it the chance to fail. One way is e.g. by testing it in the presence of intense electromagnetic background fields. Due to quantum fluctuations the QED vacuum consists of the fluctuation of virtual electron-positron-pairs. Thus strong fields can influence the vacuum by polarisation of the virtual pairs, which leads to effects like birefringence or other non-linear effects in physical processes [Di Piazza et al., 2012]. If the field becomes so strong that an electron gains an energy compared to its rest energy in a distance shorter than its Compton-wavelength, the vacuum becomes unstable and the virtual electron-positron-pairs can become real particles. This field strength is called critical field of QED and is given by $E_{\text{crit}} = m^2c^3/\hbar|e| = 1.3 \times 10^{16}\text{V/cm}$ [Ritus, 1985, Sauter, 1931, Heisenberg and Euler, 1936, Schwinger, 1951, Di Piazza et al., 2012].

Since this field strength is extremely large, it was not possible to reach it until today in an experimental setup. One prospective tool to reach it in future could be laser pulses. The critical field strength corresponds to a laser intensity of $I_{\rm crit} = E_{\rm crit}^2/4\pi = 4.6 \times 10^{29}{\rm W/cm}^2$, where we use here and in the following units in which $c = 1 = \hbar$. Recent lasers can reach peak intensities in the order of $10^{22}{\rm W/cm}^2$ [Yanovsky et al., 2008] and new laser facilities try to reach peak intensities in the order of $10^{23} - 10^{24}{\rm W/cm}^2$ [Papadopoulos et al., 2016, Center for Relativistic Laser Science , CoReLS, Extreme Light Infrastructure , ELI] in near future. Although this is far away from the critical intensity, and thus from an unstable vacuum, for moving particles it can be still enough since here the effective field strength in the rest frame of the particle is important [Mitter, 1975, Ritus, 1985]. This is indicated by the Lorentz and gauge invariant non-linearity parameter χ_0 for electrons/positrons and κ_0 for photons, respectively, which can be understood for the electrons/positrons as being the ratio of the effective field strength in the Lorentz boosted rest frame of the electron/positron over the critical field strength. In that manner with an electron/positron or photon with an energy of 500MeV counter-propagating to the laser pulse one could reach effectively the critical field strength, i.e. $\chi_0 \gtrsim 1$ or $\kappa_0 \gtrsim 1$, already with intensities around $10^{23}{\rm W/cm}^2$ [Mitter, 1975, Ritus, 1985, Podszus and Di Piazza, 2019]. Since electron beams with higher energies were already produced one could imagine to principally enter a regime were $\chi_0 \gg 1$ [Bula et al., 1996, Burke et al., 1997, Blackburn et al., 2018, Baumann et al., 2018, Yakimenko et al., 2019].

So far, calculations to radiative corrections in strong field QED only exist for leading order corrections or special background field configurations. This is due to the fact that for certain field strengths interactions with the background field become rather large and one can not directly use perturbation theory like in normal vacuum QED. Thus, for physical processes one has to take infinitely many Feynman diagrams with background field interactions into account. This problem can be circumvented by working in the so-called Furry picture where all interactions with the background field are already taken into account exactly by including them in the quantisation of the electron-positron-field [Furry, 1951]. Nevertheless, the obtained equations in this picture become quite complicated already for simple Feynman diagrams and exact solutions can be obtained only in the case of certain limits.

One of these limits is the low-frequency or constant-crossed-field (CCF) limit. A CCF is a constant and uniform electromagnetic field $(\boldsymbol{E}_0, \boldsymbol{B}_0)$ in which the two Lorentz invariants $\boldsymbol{E}_0^2 - \boldsymbol{B}_0^2$ and $\boldsymbol{E}_0 \cdot \boldsymbol{B}_0$ vanish. The interesting point is that calculations hinted that in the high energy regime, i.e. for $\chi_0 \gg 1$ or $\kappa_0 \gg 1$, of the CCF limit the effective coupling of QED scales as $\alpha\chi_0^{2/3}$ or $\alpha\kappa_0^{2/3}$ [Ritus, 1970, Narozhny, 1979, 1980, Morozov et al., 1981]. This behaviour is called 'Ritus-Narozhny conjecture'. Since in those limits the energy of the system enters the expression only over χ_0 or κ_0, this implies that in the CCF radiative corrections for high-energies scale with the power of $\chi_0^{2/3}$ or $\kappa_0^{2/3}$ and thus behave quiet different to normal vacuum QED where radiative corrections increase logarithmic with the energy scale.

Although this calculations are strictly only valid for the special case of CCF, they become more relevant due to the so-called local-constant-field limit often used in numerical simulations. It states that for low-frequency plane waves the background field can assumed to be locally constant. The basic idea behind this is that in this case the formation length of physical processes is much smaller than the typical laser wavelength. In that way one can use for physical processes the probabilities obtained in a CCF and average over the phase-dependent plane-wave profile [Ritus, 1985]. The question is, if this power-law scaling for high energies is just a special feature of the CCF limit or if radiative corrections in general scale in a power-law for strong field QED. This would imply that strong field QED behaves qualitatively different than vacuum QED and the effective coupling constant would be in the order of unity for $\kappa_0 \approx 10^3$. Thus strong field QED would behave like a strongly coupled theory for energies much below the Landau pole of vacuum QED, which seems to be strange since it is derived from vacuum QED. Thus we ask in a first point, if the high energy limit and the CCF limit commute.

In this work we will show that the low-frequency/CCF and the high-energy limit do not commute and identify the parameter discriminating between both. Thus, strictly speaking, the power-law scaling for high-energies is only valid in the CCF case, and we show instead that in the high-energy limit of a general plane wave radiative corrections increase logarithmic as in vacuum. This can be understood as the formation length of radiative corrections become larger than the typical laser wavelength in the high-energy limit. Further we will calculate the high-energy asymptotic expressions of the polarisation and mass operator in a strong pulsed background field, starting from their general expressions in a plane wave background field first calculated in Ref. [Baier et al., 1976b, Becker and Mitter, 1975, Baier et al., 1976a].

The thesis is structured as the following. In the first section we show the considered set-up, explain in more detail QED in background fields and the Furry picture, and introduce the notation. In Section (3) we will investigate the polarisation operator, demonstrating on it the non-commutativity of the CCF and high-energy limit, and calculate its high-energy asymptotic. In Section (4) we pass to the mass operator and calculate its high-energy asymptotic. In both, Section (3) and Section (4), the calculations for the high-energy asymptotic were performed for simplicity in a special pulse shape for the background field. In Section (5) we will generalise these results to arbitrary pulse shapes. The asymptotic expressions are compared with numerical calculations to test their applicability in Section (6). The conclusions of this thesis are in Section (7).

2 QED in a strong plane-wave background field

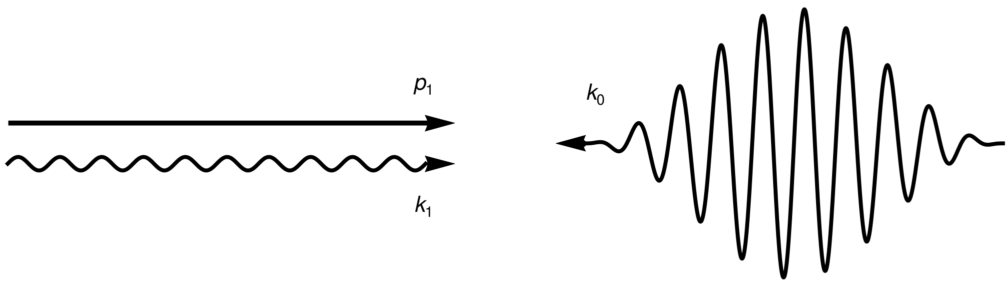

Figure 2.1: The figure shows the set up of the considered system. Either a single electron of momentum p_1 or a single photon of momentum k_1 collides head on with a strong laser pulse with momentum k_0. After the collision the electron has momentum p_2 or the photon has momentum k_2, respectively.

We are dealing with the following situation showed in figure (2.1). In an idealized case we have either a single incoming electron with four-momentum p_1^μ or a single incoming photon with four-momentum k_1^μ which collides head on with a strong linearly polarized electromagnetic plane wave laser pulse of four-momentum k_0^μ. 'Strong' means in this case that the laser pulse contains so many photons that we have to take them into account exactly in our calculations. The electrons or photons can interact during the collision with some of the laser background photons due to quantum fluctuations, such that the outgoing electron has four-momentum p_2^μ or the outgoing photon has four-momentum k_2^μ. The leading order quantum corrections of the electron or photon are given by the mass and polarisation operator, respectively. Their interaction with background photons is shown by the Feynman diagrams in figure (2.2) and (2.3). It can be imagined as one background photon interacts with the electron propagator or states at different positions inside the mass or polarisation operator. Of course we can also have higher interaction orders with more than one background photon and with different combinations of their positions. For the polarisation operator the interaction with only one background photon is not possible due to the Furry theorem (C parity conservation) and thus the first correction starts there directly with the interaction of two background photons [Furry, 1937]. Since we deal with a strong background field, the number of background photons interacting with the electron or photon should be much smaller than the total number of background photons in a laser pulse, and we can assume that the laser pulse stays unaffected by the interaction. The sum of all the interacting Feynman diagrams describes the leading order quantum correction in a background field and can be obtained by working in the Furry picture. There one uses, instead of the normal electron propagators and states, the so called Volkov-electron propagators and states, represented in figure (2.2) and (2.3) by double lines. To simplify the calculations and for a better comparison with results calculated for the CCF limit, we assume in our calculations that the four-momentum p_1^μ (k_1^μ) of the in-coming electron (photon) is identical with the four-momentum p_2^μ (k_2^μ) of the out-going electron (photon), i.e. $p_1^\mu = p_2^\mu = p^\mu$ ($k_1^\mu = k_2^\mu = k^\mu$), and that both are on-shell, i.e. $p_1^2 = p_2^2 = p^2 = m^2$ ($k_1^2 = k_2^2 = k^2 = 0$).

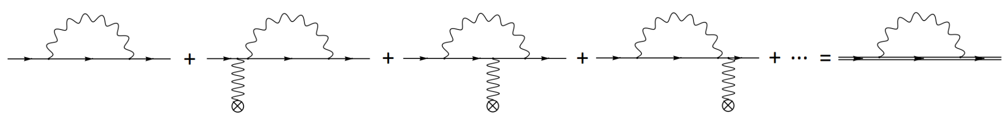

Figure 2.2: The sum of the Feynman diagrams of the one-loop vacuum mass operator and all its possible interactions with the background photons (indicated by the crossed circle) is given by the one-loop mass operator in the Furry picture. The double line indicates that the Volkov-states and -propagators are used, containing all possible interactions of the fermion line with the background field.

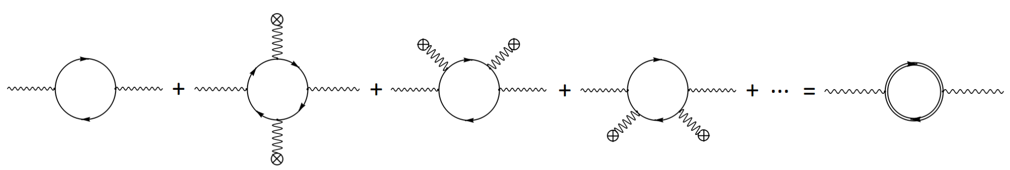

Figure 2.3: The sum of the Feynman diagrams of the one-loop vacuum polarisation operator and all its possible interactions with the background photons (indicated by the crossed circle) is given by the one-loop polarisation operator in the Furry picture. The double line indicates that the Volkov-propagator is used, containing all possible interactions of the fermion line with the background field.

2.1 QED and the Furry picture

2.1.1 Lagrangian and Dirac equation

The Lagrangian of QED is given by

$$\mathcal{L}_{QED} = -\frac{1}{16\pi}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{\lambda}{8\pi}(\partial_\mu \mathcal{A}^\mu)^2 + \bar{\Psi}[i\gamma_\mu\partial^\mu - m]\Psi - e\bar{\Psi}\gamma_\mu\mathcal{A}^\mu\Psi, \tag{2.1}$$

with Ψ and $\mathcal{A}$ being the Dirac fermion field and the photon field, respectively [Weigand, 2018, Berestetskii et al., 1982]. The gauge fixing parameter λ ensures that the Lorentz-gauge constraint $\partial_\mu \mathcal{A}^\mu = 0$ is taken into account when the quantisation of the photon field is performed (which leads to only two physical degrees of freedom). The electromagnetic field tensor is given by $\mathcal{F}^{\mu\nu} = \partial^\mu\mathcal{A}^\nu - \partial^\nu\mathcal{A}^\mu$ and γ_μ is the Dirac gamma matrix. From this Lagrangian we obtain for the fermion field the Dirac equation

$$(i\gamma_\mu\partial^\mu - e\gamma_\mu\mathcal{A}^\mu - m)\Psi = 0 \tag{2.2}$$

and for the photon field (we are working in Feynman Gauge with $\lambda = 1$ at the end) [Weigand, 2018, Berestetskii et al., 1982]

$$\partial^2\mathcal{A}^\mu - (1-\lambda)\partial^\mu(\partial\mathcal{A}) = 4\pi e\bar{\Psi}\gamma^\mu\Psi. \tag{2.3}$$

2.1.2 Photon-field splitting

In the presence of a strong laser field the term $-e\bar{\Psi}\gamma_\mu\mathcal{A}^\mu\Psi$ in the Lagrangian becomes large and can not be treated perturbatively in the quantisation procedure. Nevertheless we can quantise the spinor field in the presence of the strong background field by working in the so-called Furry picture [Mackenroth, 2014, Meuren, 2015, Berestetskii et al., 1982, Furry, 1951]. For this we split the electromagnetic potential into two components, i.e.

$$\mathcal{A}^\mu = A_0^\mu + A_{\text{rad}}^\mu. \tag{2.4}$$

Here A_0^μ is the electromagnetic potential of the external field. Since it is effectively not effected by the interaction, it resembles the vacuum expectation value of $\mathcal{A}^\mu$. We can treat it classically and do not have to quantise it. A_{rad}^μ resembles the quantum fluctuations around A_0^μ (e.g. like incoming or outgoing photons or virtual photons in loops) and has to be quantised [Mackenroth, 2014, Meuren, 2015, Berestetskii et al., 1982]. In that way the fermion field can be quantised in the presence of the background field such that the fermion states and propagators in the Furry picture take all the interactions of the fermion with the background field exactly into account. This is done by solving the Dirac equation for the corresponding background field A_0^μ for the fermion states and propagators, which are called then Volkov-states and Volkov-propagators, respectively. The interaction term in the Furry picture thus depends only on the weak field A_{rad}^μ scaling with the small parameter α and can be taken perturbatively. Thus Feynman diagrams can be calculated like in vacuum QED just by taking the corresponding expressions of fermion Volkov-states and Volkov-propagators in the Furry picture [Mackenroth, 2014, Meuren, 2015, Berestetskii et al., 1982].

2.2 Plane wave background field

The photons of the plane wave background field are on-shell ($k_0^2 = 0$) and we assume without loss of generality that they propagate along the direction $\boldsymbol{n}$, with $\boldsymbol{n}$ being a unit vector, i.e. $\boldsymbol{n}^2 = 1$. Thus we can write the laser four-momentum as $k_0^\mu = (\omega_0, \boldsymbol{k_0}) = \omega_0(1, \boldsymbol{n})$, where ω_0 is the central laser angular frequency. Since we deal with a plane wave, the electromagnetic potential $A_0^\mu(\varphi)$ of the laser pulse only depends on the phase $\varphi = (k_0 x)$ with x being the space-time four position vector. Due to gauge invariance, we can always bring the electromagnetic potential to the form $A_0^\mu(\varphi) = (0, \boldsymbol{A_0}(\varphi))$. The Lorentz gauge condition $\partial_\mu A_0^\mu(\varphi) = 0$ reduces for plane waves to $0 = k_{0\mu} A_0^\mu(\varphi) = -\omega_0 \boldsymbol{n}\boldsymbol{A}_0(\varphi)$, such that the potential only depends on two physical parameters in form of transverse polarisation directions [Berestetskii et al., 1982, Weigand, 2018]. Now we can introduce two new unit vectors $\boldsymbol{e}$ and $\boldsymbol{b}$ which are perpendicular to each other and to $\boldsymbol{n}$, i.e. $\boldsymbol{e}^2 = 1 = \boldsymbol{b}^2$, $\boldsymbol{e} \cdot \boldsymbol{b} = 0$, and $\boldsymbol{n} = \boldsymbol{e} \times \boldsymbol{b}$. With these unit vectors we define two new four-vectors by $e_e^\mu = (0, \boldsymbol{e})$ and $e_b^\mu = (0, \boldsymbol{b})$ such that we can write the electromagnetic potential in the form $A_0^\mu(\varphi) = A_{0,e}\psi_e(\varphi)e_e^\mu + A_{0,b}\psi_b(\varphi)e_b^\mu$. Here $A_{0,j}$ with $j \in \{e, b\}$ is the amplitude of the electromagnetic potential and $\psi_j(\varphi)$ being the pulse shape. Since we deal in this work only with linear polarized background fields, we can bring always, by a special choice of $\boldsymbol{e}$ and $\boldsymbol{b}$, the potential into the form $A_0^\mu(\varphi) = A_0\psi(\varphi)e_e^\mu$. The pulse shape $\psi(\varphi)$ can be arbitrary as long as it is well-behaved and goes sufficiently fast to zero for $\varphi \to \pm\infty$. Thus notice that we will consider here not the case of an infinite long monochromatic plane wave field. The field tensor $F_0^{\mu\nu}(\varphi)$ is given by $F_0^{\mu\nu}(\varphi) = \partial^\mu A_0^\nu(\varphi) - \partial^\nu A_0^\mu(\varphi) = F_0^{\mu\nu}\psi'(\varphi)$, with $F_0^{\mu\nu} = A_0(k_0^\mu e_e^\nu - k_0^\nu e_e^\mu)$, where here and in the following the prime denotes the derivative of the function with respect to its argument. Thus the amplitude $A_0 > 0$ is proportional to the amplitude $F_0^{\mu\nu}$ of the plane wave field. Further we see that the i-th component of the electric field is $E_0^i = F_0^{0i} = A_0\omega_0 e_e^i$ with $i \in \{1, 2, 3\}$, thus $\boldsymbol{e}$ corresponds to the direction of the electric field, and the i-th component of the magnetic field is $B_0^i = -\frac{1}{2}\epsilon^{ijl}F_{0,il} = -\frac{1}{2}\epsilon^{ijl}A_0\omega_0 n_j e_{e,l} = -\frac{1}{2}A_0\omega_0(\boldsymbol{n} \times \boldsymbol{e})^i = -\frac{1}{2}A_0\omega_0\boldsymbol{b}^i$, with $i, j, l \in \{1, 2, 3\}$ and the Levi-civita symbol ϵ^{ijl}, thus $\boldsymbol{b}$ corresponds to the direction of the magnetic field.

2.3 Light cone coordinates

Since the pulse shape of the plane wave only depends on the phase $\varphi = (k_0 x) = \omega_0(t - \boldsymbol{n} \cdot \boldsymbol{x})$ with $x^\mu = (t, \boldsymbol{x})$, it makes sense to change the coordinate system in a way that it just depends on one coordinate vector instead of two like in canonical coordinates [Meuren, 2015]. For this we use the vectors $\boldsymbol{n}$, $\boldsymbol{e}$ and $\boldsymbol{b}$ which were already introduced in the previous section. As a basis for our light-cone coordinates we choose the four-vectors

$$k_0^\mu = \omega_0(1, \boldsymbol{n}), \quad \bar{k}_0^\mu = \frac{1}{2\omega_0}(1, -\boldsymbol{n}), \quad e_e^\mu = (0, \boldsymbol{e}), \quad e_b^\mu = (0, \boldsymbol{b}). \tag{2.5}$$

They fulfil the conditions

$$k_0^2 = 0 = \bar{k}_0^2, \quad k_0\bar{k}_0 = 1, \quad k_0 e_j = 0 = \bar{k}_0 e_j, \quad e_i e_j = -\delta_{ij}, \tag{2.6}$$

with $i, j \in \{e, b\}$, and we can construct the metric in light-cone coordinates by $g^{\mu\nu} = k_0^\mu \bar{k}_0^\nu + \bar{k}_0^\mu k_0^\nu - e_e^\mu e_e^\nu - e_b^\mu e_b^\nu$. The contraction between two four-vectors x^μ and y_μ can be written as $x_\mu y^\mu = x^\nu g_{\nu\mu} y^\mu = x_+ y^+ + x_- y^- + x_{\mathrm{I}} y^{\mathrm{I}} + x_{\mathrm{II}} y^{\mathrm{II}}$, where we label the components of the light-cone coordinates by $+$, $-$, I, and II. Let J be the transformation matrix such that $y^{\mu'} = J^{\mu'}{}_\nu y^\nu$ with $\mu' \in \{+, -, \mathrm{I}, \mathrm{II}\}$ and J^{-1} its inverse. From the metric and the contraction we obtain their components are given by

$$\begin{gathered} J^+{}_\mu = \bar{k}_{0\mu}, \quad J^-{}_\mu = k_{0\mu}, \quad J^{\mathrm{I}}{}_\mu = e_{e\mu}, \quad J^{\mathrm{II}}{}_\mu = e_{b\mu}, \\ J^{-1\mu}{}_+ = k_0^\mu, \quad J^{-1\mu}{}_- = \bar{k}_0^\mu, \quad J^{-1\mu}{}_{\mathrm{I}} = -e_e^\mu, \quad J^{-1\mu}{}_{\mathrm{II}} = -e_b^\mu. \end{gathered} \tag{2.7}$$

Thus we see that we can transform an arbitrary four-vector, e.g. x^μ, to components of light-cone coordinates by contraction with the light-cone basis

$$x^- = xk_0 = x_+, \quad x^+ = x\bar{k}_0 = x_-, \quad x^{\mathrm{I}} = xe_e = -x_{\mathrm{I}}, \quad x^{\mathrm{II}} = xe_b = -x_{\mathrm{II}}. \tag{2.8}$$

Notice that in this way $k_0^- = 0$, $k_0^+ = 1$, $k_0^{\mathrm{I}} = 0$, and $k_0^{\mathrm{II}} = 0$ and that the phase $\varphi = xk_0 = x^-$, such that in light-cone coordinates the plane wave indeed depends only on one coordinate vector component. We can write the contraction between two arbitrary four-vectors x_μ and y^μ in light-cone coordinates now also as

$$x_\mu y^\mu = x^+ y^- + x^- y^+ - x^\perp y^\perp, \tag{2.9}$$

where we use the short notation $x^\perp y^\perp = x^{\mathrm{I}} y^{\mathrm{I}} + x^{\mathrm{II}} y^{\mathrm{II}}$. Since the determinant of J is unity, the four-dimensional integral in light-cone coordinates becomes

$$\int d^4x = \int dx^+ dx^- dx^{\mathrm{I}} dx^{\mathrm{II}}. \tag{2.10}$$

Further we use for δ-functions the notation

$$\delta^{(-,\perp)}(x) = \delta(x^-)\delta^2(\boldsymbol{x}^\perp) = \delta(x^-)\delta(x^{\mathrm{I}})\delta(x^{\mathrm{II}}). \tag{2.11}$$

2.4 Light-cone related coordinate vectors

Due to the Ward-Takahashi identity the polarisation operator vanishes by contraction with the four-momentum of the incoming or outgoing photon [Meuren, 2015], i.e.

$$k_{1\mu}T^{\mu\nu}(k_1,k_2) = 0 = T^{\mu\nu}(k_1,k_2)k_{2\nu}. \tag{2.12}$$

Thus it makes sense to express the polarisation operator in a basis related to the four-momentum of the incoming/outgoing photon. At this point we will introduce the two complete sets of four-vectors k_1^μ, Q_1^μ, $\Lambda_{1,e}^\mu$, $\Lambda_{1,b}^\mu$ and k_2^ν, Q_2^ν, $\Lambda_{2,e}^\nu$, $\Lambda_{2,b}^\nu$ with

$$Q_1^\mu = \frac{k_0^\mu k_1^2 - k_1^\mu(k_0k_1)}{(k_0k_1)}, \qquad Q_2^\nu = \frac{k_0^\nu k_2^2 - k_2^\nu(k_0k_2)}{(k_0k_2)}. \tag{2.13}$$

and

$$\Lambda_{j,e}^\mu = \frac{(k_0^\mu e_e^\nu - k_0^\nu e_e^\mu)k_{j\nu}}{(k_0k_j)}, \qquad \Lambda_{j,b}^\mu = \frac{(k_0^\mu e_b^\nu - k_0^\nu e_b^\mu)k_{j\nu}}{(k_0k_j)}, \tag{2.14}$$

with $j \in \{1,2\}$. The four-vectors of each set are orthogonal to each other such that

$$k_jQ_j = 0, \quad k_j\Lambda_{j,e} = 0 = k_j\Lambda_{j,b}, \quad Q_j\Lambda_{j,e} = 0 = Q_j\Lambda_{j,b}, \quad \Lambda_{j,e}\Lambda_{j,b} = 0. \tag{2.15}$$

Therefore to fulfill the Ward-Takahashi identity given in Eq. (2.12), the polarisation operator can be constructed along the four-vectors Q_1^μ, $\Lambda_{1,e}^\mu$, $\Lambda_{1,b}^\mu$ and Q_2^ν, $\Lambda_{2,e}^\nu$, $\Lambda_{2,b}^\nu$. If $k_1^- = k_2^- = k^-$ and $\boldsymbol{k}_1^\perp = \boldsymbol{k}_2^\perp = \boldsymbol{k}^\perp$ one can use the short notation $\Lambda_{1,e}^\mu = \Lambda_{2,e}^\mu = \Lambda_e^\mu$ and $\Lambda_{1,b}^\mu = \Lambda_{2,b}^\mu = \Lambda_b^\mu$.

2.5 Lorentz and gauge invariant parameters

There are three Lorentz and gauge invariant parameters describing the above presented set-up. The first parameter is the energy scale, $\eta_0 = (k_0p)/m^2$ for electrons and $\theta_0 = (k_0k)/m^2 = (k_0^2+k^2)/2m^2$ for photons, thus it is for on-shell photons equal to twice the total energy square in the centre-of-momentum system of the photon and the laser photon in units of m^2. For photons it is also related to the Mandelstam variable s by $s = 2m^2\theta_0$.

The second parameter is the intensity parameter $\xi_0 = |e|A_0/m = |e|E_0/m\omega_0$, which is proportional to the amplitude of the electromagnetic potential E_0 of the laser pulse and thus also related to its intensity. Furthermore, ξ_0 corresponds to the energy an electron at rest gains due to the Lorenz force in one laser cycle and in units of m^2. Thus it becomes relativistic after one laser cycle if ξ_0 is in the order of unity [Di Piazza et al., 2012]. Also the probability for the absorption of n background photons scales as $\propto \xi_0^{2n}$ for $\xi_0 \lesssim 1$ [Mackenroth, 2014]. Thus the formalism of vacuum perturbation theory becomes inapplicable for ξ_0 close to unity, and one has to work in the already mentioned Furry picture if one wants to perform analytical calculations in a perturbative way.

The third parameter is the quantum non-linearity parameter, which is given by the product of energy scale and intensity parameter and is named $\chi_0 = \eta_0\xi_0 = \sqrt{|(F_{0,\mu\nu}p^\nu)^2|}/(mE_{\text{crit}})$ for electrons and $\kappa_0 = \theta_0\xi_0 = \sqrt{|(F_{0,\mu\nu}k^\nu)^2|}/(mE_{\text{crit}})$ for photons. This parameter is related to the effective field strength in units of the critical field strength of QED in a physical process initiated by the electron or photon. For an electron it reduces to be the ratio of the experienced electric field in its rest frame over the critical field strength. Thus it is a sign for the non-linear behaviour of a system [Mitter, 1975, Ritus, 1985].

2.6 Volkov-state and Volkov-propagator

For intense strong background fields with ξ_0 close to unity electrons become strongly coupled to the background photons such that normal perturbation theory becomes inapplicable. Nevertheless, by quantising the electron in the Furry picture, one can reobtain the perturbative formalism [Berestetskii et al., 1982, Mackenroth, 2014, Meuren, 2015, Meuren et al., 2013]. To obtain the electron state in the Furry picture we have to find an exact solution of the Dirac equation in the plane wave background field, i.e. we have to find a solution for $\Psi_{\text{Furry},p}(x)$ in

$$(i\gamma_\mu\partial^\mu - e\gamma_\mu A_0^\mu(\varphi) - m)\Psi_{\text{Furry},p}(x) = 0. \tag{2.16}$$

We choose as a boundary condition that the background field is first switched-on for $t \to -\infty$ such that $A_0 \to 0$ and $\Psi_{\text{Furry},p}(x) \to \Psi_p$ for $\varphi \to -\infty$, where Ψ_p is the solution of the free Dirac equation

$$\Psi_p(x) = e^{-ipx}\frac{u_p}{\sqrt{2\varepsilon_p}}. \tag{2.17}$$

Here $p^\mu = (\varepsilon_p, \boldsymbol{p})$, and the constant bispinor u_p satisfies the condition $(\gamma_\mu p^\mu - m)u_p = 0$ and is normalised as $\bar{u}_p u_p = 2m$. With those boundary conditions we can solve the Dirac equation in a plane wave. The solution, which is the Volkov state, is given by

$$\Psi_{\text{Furry},p}(x) = E_p(x)\frac{u_p}{\sqrt{2\varepsilon_p}} \tag{2.18}$$

with the so-called Ritus matrix

$$E_p(x) = \left[1 + \frac{e\gamma_\mu k_0^\mu \gamma_\nu A_0^\nu(\varphi)}{2k_0 p}\right] e^{iS_p(x)} \tag{2.19}$$

and the phase

$$S_p(x) = -px - \int_{-\infty}^{\varphi} d\phi \left[\frac{epA_0(\phi)}{kp} - \frac{e^2A^2(\phi)}{2kp}\right]. \tag{2.20}$$

Note that the phase is equivalent to the classical action of an electron in a plane wave field, although the Volkov states are an exact solution of the Dirac equation.

The conjugate of the Volkov state is then given by

$$\bar{\Psi}_{\text{Furry},p}(x) = \frac{\bar{u}_p}{\sqrt{2\varepsilon_p}} \bar{E}_p(x) \tag{2.21}$$

with

$$\bar{E}_p(x) = \left[1 + \frac{e\gamma_\nu A_0^\nu(\varphi)\gamma_\mu k_0^\mu}{2k_0 p}\right] e^{-iS_p(x)} \tag{2.22}$$

and $\bar{u}_p(\gamma_\mu p^\mu - m) = 0$. The Ritus matrices $E_p(x)$ and $\bar{E}_p(x)$ are orthogonal and form a complete set, i.e.

$$\int \frac{d^4p}{(2\pi)^4} \bar{E}_p(x) E_p(y) = \delta^4(x-y), \tag{2.23}$$

$$\int d^4x \bar{E}_p(x) E_q(x) = (2\pi)^4 \delta^4(p-q). \tag{2.24}$$

The Volkov-propagator is the Green's function of the Dirac equation in the plane wave background field and thus defined as

$$(i\gamma_\mu \partial^\mu - e\gamma_\mu A_0^\mu(\varphi) - m) G_{\text{Furry}}(x,y) = \delta^4(x-y). \tag{2.25}$$

A solution of this equation, which is called then Volkov-propagator, is given by

$$iG_{\text{Furry}}(x,y) = \lim_{\epsilon\to 0} i \int \frac{d^4p}{(2\pi)^4} E_p(x) \frac{\gamma_\mu p^\mu + m}{p^2 - m^2 + i\epsilon} \bar{E}_p(y). \tag{2.26}$$

Note that in general the expressions of the fermion field are spin dependent. Since we deal with electrons of identical spin in the in- and out-state, we do not consider spin related changes. To simplify the notation we therefore drop the spin indices and only introduce the average spin $\zeta/2$ for the mass operator. Nevertheless the spin is taken into account in the expression of the mass operator taken from Baier et al. [1976a].

2.7 Feynman rules in the Furry picture

The Volkov-state (Eq. (2.18)) and Volkov-propagator (Eq. (2.26)) for an electron in the Furry picture are already given in the previous section. The state and the propagator of the photon are not effected by the background field and thus are, in the Furry picture, similar to vacuum QED [Berestetskii et al., 1982, Weigand, 2018, Meuren et al., 2013, Meuren, 2015]. With the polarisation four-vector ε^μ ($\varepsilon^\mu \varepsilon_\mu^* = -1$), four-momentum $k^\mu = (\omega, \boldsymbol{k})$, and in Feynman gauge ($\lambda = 1$) they are given by

$$A^\mu_{\text{rad},k} = \sqrt{4\pi} \frac{\varepsilon^\mu}{\sqrt{2\omega}} e^{-ikx}, \tag{2.27}$$

$$-iD_{\mu\nu}(x-y) = -i \lim_{\epsilon\to 0} \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{4\pi g_{\mu\nu}}{k^2 + i\epsilon}. \tag{2.28}$$

Since interactions with the background field are already implemented in the Volkov-states and Volkov-propagators, also the vertex in the Furry picture is equal to vacuum QED, because the only left over interactions are interactions of fermions with photons of the A^{μ}_{rad} part of the electromagnetic potential. Thus we have to add for every vertex an $-ie\gamma^{\mu}$ and have to integrate over the corresponding position four-vector. Further we have to insert a $-\text{Tr}[...]$ for every closed fermion loop.

In that way we can construct the polarisation and mass operator in figures (2.2) and (2.3). We start with the polarisation operator $P(k_1, k_2)$ which is given by $iP(k_1, k_2) = T(k_1, k_2) = \varepsilon_{\mu}\varepsilon^{*}_{\nu}T^{\mu\nu}(k_1, k_2)$ with

$$T^{\mu\nu}(k_1, k_2) = 4\pi \int d^4x \int d^4y(-1)\text{Tr}\left[e^{-ik_1x}(-ie\gamma^{\mu})iG_{\text{Furry}}(x, y) \right. \\ \left. \times(-ie\gamma^{\nu})iG_{\text{Furry}}(y, x)e^{ik_2y}\right]. \tag{2.29}$$

$T^{\mu\nu}(k_1, k_2)$ is divergent and has to be renormalised. For this we write it in the way

$$T^{\mu\nu}(k_1, k_2) = \left[T^{\mu\nu}(k_1, k_2) - T^{\mu\nu}_{A_0=0}(k_1, k_2)\right] + T^{\mu\nu}_{A_0=0}(k_1, k_2), \tag{2.30}$$

with $T^{\mu\nu}_{A_0=0}(k_1, k_2)$ being the polarisation operator with $A_0 = 0$, thus for normal vacuum QED without a background field. In that way the first term in brackets $T^{\mu\nu}(k_1, k_2) - T^{\mu\nu}_{A_0=0}(k_1, k_2) \equiv iP^{\mu\nu}_{f}(k_1, k_2)$ is already finite and contains all corrections due to the background field. The second term has to be renormalised, but the renormalisation is already known, since it is the polarisation operator of normal vacuum QED, and it vanishes in our case, since we are dealing with identical and on-shell photons [Baier et al., 1976b, Berestetskii et al., 1982].

The mass operator, presented in figure (2.2), for on-shell incoming and outgoing electrons, having the same averaged spin $\zeta/2$, is given by

$$-iM_{\zeta}(p_1, p_2) = \int d^4x \int d^4y \bar{\Psi}_{\text{Furry},p_2}(x)(-ie\gamma^{\mu})iG_{\text{Furry}}(x, y) \\ \times (-i)D_{\mu\nu}(x - y)(-ie\gamma^{\nu})\Psi_{\text{Furry},p_1}(y). \tag{2.31}$$

Again $M_{\zeta}(p_1, p_2)$ is divergent and has to be renormalised. We write it in the way

$$M_{\zeta}(p_1, p_2) = [M_{\zeta}(p_1, p_2) - M_{\zeta,A_0=0}(p_1, p_2)] + M_{\zeta,A_0=0}(p_1, p_2) \tag{2.32}$$

with $M_{\zeta,A_0=0}(p_1, p_2)$ being the mass operator for $A_0 = 0$, so for normal vacuum QED without a background field. In that way the first term in brackets $M_{\zeta}(p_1, p_2) - M_{\zeta,A_0=0}(p_1, p_2) \equiv M_{f,\zeta}(p_1, p_2)$ is already finite and contains all corrections due to the background field. The second term has to be renormalised, but the renormalisation is already known from normal vacuum QED and it vanishes in our case, since we are dealing with identical and on-shell electrons and consider the mass operator being on the mass shell [Baier et al., 1976a, Berestetskii et al., 1982].

3 CCF Limit and High-Energy Asymptotic of the One-Loop Polarization Operator in a Plane Wave

3.1 The One-Loop Polarization Operator in a Plane Wave

We are starting from the general expression of the one-loop polarization operator in a plane wave, which was first calculated in [Baier et al., 1976b, Becker and Mitter, 1975]. For technical reasons we use here a symmetrical form calculated later in [Meuren et al., 2013]:

$$T^{\mu\nu}(k_1,k_2) - T^{\mu\nu}_{A_0=0}(k_1,k_2) = -i4\pi^2\alpha\delta^{(-,\perp)}(k_1-k_2)\int_{-1}^{+1}dv\int_0^\infty \frac{d\tilde{s}}{\tilde{s}}\int_{-\infty}^{+\infty}dz^- \\ [c_1\Lambda_e^\mu\Lambda_b^\nu + c_2\Lambda_b^\mu\Lambda_e^\nu + c_3\Lambda_e^\mu\Lambda_e^\nu + c_4\Lambda_b^\mu\Lambda_b^\nu + c_5 Q_e^\mu Q_b^\nu]e^{i\Phi} \tag{3.1}$$

with

$$c_1 = 2m^2\xi_e\xi_b\left(\frac{\tilde{s}}{4\mu}X_{eb} - \frac{\tilde{s}v^2}{4\mu}X_{be}\right)e^{i\tilde{s}\beta}, \tag{3.2}$$

$$c_2 = 2m^2\xi_e\xi_b\left(\frac{\tilde{s}}{4\mu}X_{be} - \frac{\tilde{s}v^2}{4\mu}X_{eb}\right)e^{i\tilde{s}\beta}, \tag{3.3}$$

$$c_3 = 2m^2\left[\frac{\tilde{s}}{4\mu}(\xi_e^2 Z_e + \xi_b^2 Z_b) + \xi_e^2 X_{ee}\right]e^{i\tilde{s}\beta} - \left(\frac{i}{\tilde{s}} + \frac{(k_1k_2)}{2}\right)\left(e^{i\tilde{s}\beta}-1\right), \tag{3.4}$$

$$c_4 = 2m^2\left[\frac{\tilde{s}}{4\mu}(\xi_e^2 Z_e + \xi_b^2 Z_b) + \xi_b^2 X_{bb}\right]e^{i\tilde{s}\beta} - \left(\frac{i}{\tilde{s}} + \frac{(k_1k_2)}{2}\right)\left(e^{i\tilde{s}\beta}-1\right), \tag{3.5}$$

$$c_5 = -\frac{2\mu}{\tilde{s}}\left(e^{i\tilde{s}\beta}-1\right), \tag{3.6}$$

$$e^{i\Phi} = \exp\{i[(k_2^+ - k_1^+)z^- + \mu(k_1k_2) - \tilde{s}m^2]\}, \tag{3.7}$$

and

$$\mu = \frac{1}{4}\tilde{s}(1-v^2), \tag{3.8}$$

$$e^{i\tilde{s}\beta} = \exp\Big\{ - i\tilde{s}m^2 \sum_{i=e,b} \xi_i^2 (J_i - I_i^2) \Big\}, \tag{3.9}$$

$$I_i = \frac{1}{2}\int_{-1}^{+1} d\lambda \; \psi_i((k_0z) - \lambda\mu(k_0k)), \tag{3.10}$$

$$J_i = \frac{1}{2}\int_{-1}^{+1} d\lambda \; \psi_i^2((k_0z) - \lambda\mu(k_0k)), \tag{3.11}$$

$$X_{ij} = [I_i - \psi_i((k_0z) + \mu(k_0k))][I_j - \psi_j((k_0z) - \mu(k_0k))], \tag{3.12}$$

$$Z_i = \frac{1}{2}[\psi_i((k_0z) - \mu(k_0k)) - \psi_i((k_0z) + \mu(k_0k))]^2. \tag{3.13}$$

with $i, j \in \{e, b\}$. For a linearly-polarized plane wave we can set without restriction $\xi_e = \xi_0$ and $\xi_b = 0$. Therefore c_1 and c_2 vanish and we can set $\psi = \psi_e$ and $\psi_b = 0$. Further the plane wave just depends on the phase $k_0z = k_0^+z^- = \varphi$, such that we change the variable to the phase φ, and we are dealing with on-shell incoming and outgoing photons where the four-momentum k_1^μ ($k_1^2 = 0$) of the incoming photon coincides with the four-momentum k_2^μ of the outgoing photon, i.e. $k_1^\mu = k_2^\mu = k^\mu$. Thus terms proportional to $(k_1k_2) = k^2 = 0$ vanish. This allows us also to ignore in the following the term c_5, because the term $Q_1^\mu Q_2^\nu$ turns out to be proportional to $k_1^\mu k_2^\nu = k^\mu k^\nu$ for on-shell external photons. Thus it would not contribute to any physical measurable, since they depend on the transition probability $T = \varepsilon_\mu \varepsilon_\nu^* T^{\mu\nu}$ and do to gauge invariance $\varepsilon_\mu k^\mu = 0 = \varepsilon_\nu^* k^\nu$ [Berestetskii et al., 1982]. Therefore we end with

$$T^{\mu\nu}(k_1,k_2) - T^{\mu\nu}_{A_0=0}(k_1,k_2) = -i4\pi^2\alpha\frac{1}{k_0^+}\delta^{(-,\perp)}(k_1-k_2) \int_{-1}^{+1} dv \int_0^\infty \frac{d\tilde{s}}{\tilde{s}} \int_{-\infty}^{+\infty} d\varphi \; [c_3\Lambda_e^\mu\Lambda_e^\nu + c_4\Lambda_b^\mu\Lambda_b^\nu]e^{-i\tilde{s}m^2} \tag{3.14}$$

with

$$c_3 = 2m^2\left[\frac{1}{(1-v^2)}\xi_0^2 Z + \xi_0^2 X\right]e^{-i\tilde{s}m^2\xi_0^2Q^2} - \frac{i}{\tilde{s}}\left(e^{-i\tilde{s}m^2\xi_0^2Q^2} - 1\right), \tag{3.15}$$

$$c_4 = 2m^2\left[\frac{1}{(1-v^2)}\xi_0^2 Z\right]e^{-i\tilde{s}m^2\xi_0^2Q^2} - \frac{i}{\tilde{s}}\left(e^{-i\tilde{s}m^2\xi_0^2Q^2} - 1\right), \tag{3.16}$$

where

$$Q^2 = \frac{1}{2}\int_{-1}^{+1} d\lambda\ \psi^2(\varphi - \lambda\mu(k_0k)) - \frac{1}{4}\left[\int_{-1}^{+1} d\lambda\ \psi(\varphi - \lambda\mu(k_0k))\right]^2, \tag{3.17}$$

$$\begin{aligned} X = &\left[\frac{1}{2}\int_{-1}^{+1} d\lambda\ \psi(\varphi - \lambda\mu(k_0k)) - \psi(\varphi + \mu(k_0k))\right] \\ &\times \left[\frac{1}{2}\int_{-1}^{+1} d\lambda\ \psi(\varphi - \lambda\mu(k_0k)) - \psi(\varphi - \mu(k_0k))\right], \end{aligned} \tag{3.18}$$

$$Z = \frac{1}{2}\left[\psi(\varphi - \mu(k_0k)) - \psi(\varphi + \mu(k_0k))\right]^2. \tag{3.19}$$

Since $k_0^- = 0 = k_0^\perp$ and thus $(k_0k) = k_0^+k^-$, we can absorb $1/k_0^+$ into the δ^- function by $\frac{1}{k_0^+}\delta(k_1^- - k_2^-) = \delta((k_0k_1) - (k_0k_2))$. We perform the substitution $\tau = \mu(k_0k) = \frac{1}{4}\tilde{s}(1-v^2)(k_0k)$, thus $d\tau = d\tilde{s}\frac{1}{4}(1-v^2)(k_0k)$, and obtain

$$\begin{aligned} T^{\mu\nu}(k_1,k_2) - T^{\mu\nu}_{A_0=0}(k_1,k_2) = &-i4\pi^2\alpha\delta^2(\boldsymbol{k}_{1,\perp} - \boldsymbol{k}_{2,\perp})\delta((k_0k_1) - (k_0k_2)) \\ &\int_{-1}^{+1} dv \int_0^\infty \frac{d\tau}{\tau}\int_{-\infty}^{+\infty} d\varphi\ [c_3\Lambda_e^\mu\Lambda_e^\nu + c_4\Lambda_b^\mu\Lambda_b^\nu]e^{-i\frac{4m^2}{(1-v^2)}\frac{\tau}{(k_0k)}} \end{aligned} \tag{3.20}$$

with

$$\begin{aligned} c_3 = &2m^2\left[\frac{1}{(1-v^2)}\xi_0^2 Z(\varphi,\tau) + \xi_0^2 X(\varphi,\tau)\right] e^{-i\frac{4m^2}{(1-v^2)}\frac{\tau}{(k_0k)}\xi_0^2Q^2(\varphi,\tau)} \\ &- \frac{i(1-v^2)(k_0k)}{4\tau}\left(e^{-i\frac{4m^2}{(1-v^2)}\frac{\tau}{(k_0k)}\xi_0^2Q^2(\varphi,\tau)} - 1\right), \end{aligned} \tag{3.21}$$

$$\begin{aligned} c_4 = &2m^2\left[\frac{1}{(1-v^2)}\xi_0^2 Z(\varphi,\tau)\right] e^{-i\frac{4m^2}{(1-v^2)}\frac{\tau}{(k_0k)}\xi_0^2Q^2(\varphi,\tau)} \\ &- \frac{i(1-v^2)(k_0k)}{4\tau}\left(e^{-i\frac{4m^2}{(1-v^2)}\frac{\tau}{(k_0k)}\xi_0^2Q^2(\varphi,\tau)} - 1\right), \end{aligned} \tag{3.22}$$

and

$$Q^2(\varphi,\tau) = \frac{1}{2\tau}\int_{-\tau}^{+\tau} d\tau'\ \psi^2(\varphi - \tau') - \frac{1}{4\tau^2}\left[\int_{-\tau}^{+\tau} d\tau'\ \psi(\varphi - \tau')\right]^2, \tag{3.23}$$

$$\begin{aligned} X(\varphi,\tau) = &\left[\frac{1}{2\tau}\int_{-\tau}^{+\tau} d\tau'\ \psi(\varphi - \tau') - \psi(\varphi + \tau)\right] \\ &\times \left[\frac{1}{2\tau}\int_{-\tau}^{+\tau} d\tau'\ \psi(\varphi - \tau') - \psi(\varphi - \tau)\right], \end{aligned} \tag{3.24}$$

$$Z(\varphi,\tau) = \frac{1}{2}\left[\psi(\varphi - \tau) - \psi(\varphi + \tau)\right]^2. \tag{3.25}$$

Here we additionally used for $Q^2(\varphi,\tau)$ and $X(\varphi,\tau)$ the substitution $\tau' = \lambda\tau$. At this point one should also notice, that the function $Q^2(\varphi,\tau)$ has the structure of a variance of $\psi(\varphi)$ since one could interpret the integrals as $\mathrm{E}[\psi^2] - \mathrm{E}[\psi]^2 = \mathrm{Var}[\psi]$, where E[...] denotes the expectation value. Thus the function $Q^2(\varphi,\tau)$ is always non-negative, i.e. $Q^2(\varphi,\tau) \geq 0$. Further the functions $Q^2(\varphi,\tau)$, $X(\varphi,\tau)$, and $Z(\varphi,\tau)$ are independent of v and all other terms only depend on $(1-v^2)$ or v^2. Therefore we can rewrite the integral of v by splitting the integral into the two regions $v \in [0,1]$ and $v \in [-1,0]$, substituting in the lower part from -1 to 0, $v = -v'$ and setting after this $v' = v$. In this way we obtain that the integral in $v \in [-1,1]$ is equal to twice the integral in $v \in [0,1]$. Next we can substitute $u^2 = 1 - v^2$, $dv = -\frac{u}{v}du = -\frac{u}{\sqrt{1-u^2}}du$, $u(v=0) = 1$, and $u(v=1) = 0$ and obtain

$$\begin{aligned} T^{\mu\nu}(k_1,k_2) - T^{\mu\nu}_{A_0=0}(k_1,k_2) = & -i8\pi^2\alpha\delta^2(\boldsymbol{k}_{1,\perp} - \boldsymbol{k}_{2,\perp})\delta((k_0k_1) - (k_0k_2)) \\ & \int_{-\infty}^{+\infty} d\varphi \int_0^{\infty} \frac{d\tau}{\tau} \int_0^1 du \, \frac{u}{\sqrt{1-u^2}} [c_3\Lambda_e^\mu\Lambda_e^\nu + c_4\Lambda_b^\mu\Lambda_b^\nu] e^{-i\frac{4\tau}{\theta_0 u^2}} \end{aligned} \tag{3.26}$$

with

$$\begin{aligned} c_3 = & 2m^2\left[\frac{1}{u^2}\xi_0^2 Z(\varphi,\tau) + \xi_0^2 X(\varphi,\tau)\right] e^{-i\frac{4\tau}{\theta_0 u^2}\xi_0^2 Q^2(\varphi,\tau)} \\ & - \frac{iu^2\theta_0 m^2}{4\tau}\left(e^{-i\frac{4\tau}{\theta_0 u^2}\xi_0^2 Q^2(\varphi,\tau)} - 1\right), \end{aligned} \tag{3.27}$$

$$\begin{aligned} c_4 = & 2m^2\left[\frac{1}{u^2}\xi_0^2 Z(\varphi,\tau)\right] e^{-i\frac{4\tau}{\theta_0 u^2}\xi_0^2 Q^2(\varphi,\tau)} \\ & - \frac{iu^2\theta_0 m^2}{4\tau}\left(e^{-i\frac{4\tau}{\theta_0 u^2}\xi_0^2 Q^2(\varphi,\tau)} - 1\right). \end{aligned} \tag{3.28}$$

Here we use the energy scale parameter θ_0, which is given by $\theta_0 = (k_0k)/m^2$. We perform now the substitution $\rho = \frac{1}{u^2}$, $du = -\frac{1}{2}\rho^{-3/2}d\rho$, $\rho(u=1) = 1$, and $\rho(u = 0) = \infty$ and get

$$\begin{aligned} T^{\mu\nu}(k_1,k_2) - T^{\mu\nu}_{A_0=0}(k_1,k_2) = & -i4\pi^2\alpha\delta^2(\boldsymbol{k}_{1,\perp} - \boldsymbol{k}_{2,\perp})\delta((k_0k_1) - (k_0k_2)) \\ & \int_{-\infty}^{+\infty} d\varphi \int_0^{\infty} \frac{d\tau}{\tau} \int_1^{\infty} \frac{d\rho}{\rho^{3/2}} \, \frac{1}{\sqrt{\rho-1}} [c_3\Lambda_e^\mu\Lambda_e^\nu + c_4\Lambda_b^\mu\Lambda_b^\nu] e^{-i\frac{4\rho\tau}{\theta_0}} \end{aligned} \tag{3.29}$$

with

$$\begin{aligned} c_3 = & 2m^2\left[\rho\xi_0^2 Z(\varphi,\tau) + \xi_0^2 X(\varphi,\tau)\right] e^{-i\frac{4\rho\tau}{\theta_0}\xi_0^2 Q^2(\varphi,\tau)} \\ & - \frac{i\theta_0 m^2}{4\rho\tau}\left(e^{-i\frac{4\rho\tau}{\theta_0}\xi_0^2 Q^2(\varphi,\tau)} - 1\right), \end{aligned} \tag{3.30}$$

$$\begin{aligned} c_4 = & 2m^2\left[\rho\xi_0^2 Z(\varphi,\tau)\right] e^{-i\frac{4\rho\tau}{\theta_0}\xi_0^2 Q^2(\varphi,\tau)} \\ & - \frac{i\theta_0 m^2}{4\rho\tau}\left(e^{-i\frac{4\rho\tau}{\theta_0}\xi_0^2 Q^2(\varphi,\tau)} - 1\right). \end{aligned} \tag{3.31}$$

Using the relation $P^{\mu\nu} = -iT^{\mu\nu}$ we rewrite the field dependent part of the polarisation operator as $P_f^{\mu\nu}(k_1,k_2) = -i\left[T^{\mu\nu}(k_1,k_2) - T^{\mu\nu}_{A_0=0}(k_1,k_2)\right]$ such that

$$P_f^{\mu\nu}(k_1,k_2) = (2\pi)^3\delta^2(\boldsymbol{k}_{1,\perp} - \boldsymbol{k}_{2,\perp})\delta((k_0k_1) - (k_0k_2)) \sum_{l=e,b} P_l(k)\Lambda_l^\mu(k)\Lambda_l^\nu(k), \tag{3.32}$$

with

$$\begin{aligned} P_e(k) = &-\frac{\alpha}{2\pi}m^2 \int_{-\infty}^{\infty} d\varphi \int_0^{\infty} \frac{d\tau}{\tau} \int_1^{\infty} \frac{d\rho}{\rho^{3/2}} \frac{1}{\sqrt{\rho - 1}} \\ &\times \Big\langle 2\xi_0^2 [X(\varphi,\tau) + \rho Z(\varphi,\tau)] e^{-i\frac{4\tau\rho}{\theta_0}[1+\xi_0^2 Q^2(\varphi,\tau)]} \\ &- i\frac{\theta_0}{4\tau\rho} \Big\{ e^{-i\frac{4\tau\rho}{\theta_0}[1+\xi_0^2 Q^2(\varphi,\tau)]} - e^{-i\frac{4\tau\rho}{\theta_0}} \Big\} \Big\rangle, \end{aligned} \tag{3.33}$$

$$\begin{aligned} P_b(k) = &-\frac{\alpha}{2\pi}m^2 \int_{-\infty}^{\infty} d\varphi \int_0^{\infty} \frac{d\tau}{\tau} \int_1^{\infty} \frac{d\rho}{\rho^{3/2}} \frac{1}{\sqrt{\rho - 1}} \Big\langle 2\xi_0^2 \rho Z(\varphi,\tau) e^{-i\frac{4\tau\rho}{\theta_0}[1+\xi_0^2 Q^2(\varphi,\tau)]} \\ &- i\frac{\theta_0}{4\tau\rho} \Big\{ e^{-i\frac{4\tau\rho}{\theta_0}[1+\xi_0^2 Q^2(\varphi,\tau)]} - e^{-i\frac{4\tau\rho}{\theta_0}} \Big\} \Big\rangle. \end{aligned} \tag{3.34}$$

We see that the expressions for the polarisation operator of both polarisation directions are identical, except that $P_e(k)$ additionally has a term containing the function $X(\varphi, \tau)$ in the integrand. Thus it is enough to consider $P_e(k)$ in the following calculations. The result for $P_b(k)$ can be obtained by subtracting the contribution of the $X(\varphi, \tau)$ term from the result of $P_e(k)$.

3.2 Low-Frequency or Constant-Crossed-Field Limit vs High-Energy Limit

We want to proof now if the low-frequency or CCF limit and the high energy limit commute. For this we first perform the low-frequency/CCF limit of the polarisation operator. To realize the limit of a low background laser frequency, so $\omega_0 \to 0$ and $k_0 \to 0$, with a constant field strength in a Lorentz invariant way we have to send $\xi_0 = |e|/m\ (E_0/\omega_0) \to \infty$ and $\theta_0 = (k_0 k)/m^2 \to 0$, such that the product of both stays constant, i.e. $\kappa_0 = \xi_0\theta_0 = \sqrt{(F_{0,\mu\nu}k^\nu)^2}/(mE_{\text{crit}}) = const.$ In this limit the phase in the exponential functions in P_e and P_b in Eqs. (3.33) and (3.34) becomes large. Since the exponent is proportional to $1/\theta_0 \propto m^2$ and m^2 is assumed to have a small negative imaginary part, i.e. $m^2 = m^2 - i0$, the whole integrand is damped down exponentially for large exponents. Since the exponent is proportional to τ the integral in τ get its main contribution for small values of τ. Thus we can expand the functions $X(\varphi, \tau)$, $Z(\varphi, \tau)$, and $Q^2(\varphi, \tau)$ for $\tau \ll 1$. After the expansion we can perform the integral in τ and obtain for the expressions of the polarisation operator in the CCF limit [Meuren et al., 2013, Ritus, 1972, Narozhny, 1969]

$$P_{e,\text{CCF}}(k) = -\frac{\alpha}{3\pi}m^2 \int_{-\infty}^{\infty} d\varphi \int_1^{\infty} \frac{d\rho}{\rho^{3/2}} \frac{4\rho - 1}{\sqrt{\rho - 1}} g\left(\frac{4\rho}{\kappa(\varphi)}\right), \tag{3.35}$$

$$P_{b,\text{CCF}}(k) = -\frac{\alpha}{3\pi}m^2 \int_{-\infty}^{\infty} d\varphi \int_1^{\infty} \frac{d\rho}{\rho^{3/2}} \frac{4\rho + 2}{\sqrt{\rho - 1}} g\left(\frac{4\rho}{\kappa(\varphi)}\right), \tag{3.36}$$

where $g(z) = z^{-2/3} df(z)/dz$, and

$$f(z) = i \int_0^\infty dt\, e^{-i(tz+t^3/3)} = \pi[\mathrm{Gi}(z) + i\mathrm{Ai}(z)]. \tag{3.37}$$

$\mathrm{Ai}(z)$ and $\mathrm{Gi}(z)$ are the Airy and Scorer functions [Wolfram MathWorld] and $\kappa(\varphi) = \kappa_0 \, |\psi'(\varphi)|$. Here and in the following a primed function indicates the derivative with respect to its argument. If we go then to the limit of high energies by sending $\kappa_0 \to \infty$ we obtain that both $P_{e,\mathrm{CCF}}(k)$ and $P_{b,\mathrm{CCF}}(k)$ scale as $\alpha \int d\varphi \kappa^{2/3}(\varphi)$ in the real and imaginary part. Thus the high energy regime of the CCF limit shows a power-law scaling with the energy which is also known as 'Ritus-Narozhny conjecture' [Meuren et al., 2013]. This behaviour is quite different to vacuum QED where radiative corrections increase logarithmic. Since all the calculations are based on QED this different behaviour seems to be strange and thus the question comes up if this power-law scaling is a general behaviour of the high energy regime in strong field QED or if it just pertains to the CCF limit. To see in a first instance if the Ritus-Narozhny conjecture also holds for general strong field QED we start with the question if the high-energy limit, which is done in the second step, commutes with the low-frequency limit in the first step. To see if they did, we perform now the high-energy limit in the sense that the energy of the incoming photon becomes large, thus sending $k \to \infty$. We can reach this with the three lorentz and gauge invariant parameters by sending now $\theta_0 = (k_0 k)/m^2 \to \infty$ and $\kappa_0 = \sqrt{(F_{0,\mu\nu} k^\nu)^2}/(m E_{\mathrm{crit}}) \to \infty$ such that $\xi_0 = \kappa_0/\theta_0 = |e|/m \; (E_0/\omega_0)$ stays constant. In this situation the phase in the exponential function in P_e and P_b in Eqs. (3.33) and (3.34) becomes small since now $\xi_0^2/\theta_0 \ll 1$. Thus the integral in τ gets also contributions for large values of τ and we can not expand the functions $X(\varphi, \tau)$, $Z(\varphi, \tau)$, and $Q^2(\varphi, \tau)$ any more for small τ, like we did it in the CCF limit. Thus the high energy and the low-frequency/CCF limit do not commute. The parameter discriminating between both limits is exactly $r_0 = \xi_0^2/\theta_0$, which is much larger than unity in the low-frequency/CCF limit ($r_0 \gg 1$) and much smaller unity in the high energy limit ($r_0 \ll 1$). This shows that the power-law scaling only pertains to the high energy regime of the low-frequency/CCF limit and thus radiative corrections in general strong field QED can show a different scaling. In the following we want to investigate how the high energy regime of QED scales in an intense plane wave laser pulse by calculating the high energy asymptotic of the one-loop polarisation operator.

3.3 Calculation of the High-Energy Asymptotic

Now we calculate the high-energy asymptotic expression of the one-loop polarization operator in a plane wave laser pulse. Thus we are working in the limit of $\theta_0 \to \infty$. We start here again with the Eqs. (3.33) and (3.34). As mentioned above it is enough to consider $P_e(k)$, since the terms in $P_b(k)$ are included in $P_e(k)$. All integrals in ρ have the form

$$I_n = \int_1^\infty \frac{d\rho}{\rho^{3/2-n}} \frac{e^{-ia\rho}}{\sqrt{\rho - 1}}, \tag{3.38}$$

with $n = -1, 0, +1$ and $\mathrm{Im}[a] < 0$ (due to $\theta_0^{-1} \propto m^2 \to m^2 - i0$). Here is either $a = a_0(\tau) = 4\tau/\theta_0$ or $a = a_0(\tau) + a_f(\varphi,\tau)$, with $a_f(\varphi,\tau) = \frac{4\tau}{\theta_0}\xi_0^2 Q^2(\varphi,\tau)$. Using the substitution $\rho' = \rho - 1$ we can write this integral as

$$I_n = \int_0^\infty d\rho'\ (1+\rho')^{n-3/2}(\rho')^{-1/2}e^{-ia(\rho'+1)}. \tag{3.39}$$

Those integrals are analytically solved by the confluent Hypergeometric function of the second kind [Wolfram MathWorld] defined as

$$U(b,c,z) = \frac{1}{\Gamma(b)}\int_0^\infty dt\ (1+t)^{c-b-1}t^{b-1}e^{-zt}, \tag{3.40}$$

where $\Gamma(b)$ is the gamma function, which is defined as $\Gamma(b) = \int_0^\infty dt\ t^{b-1}e^{-t}$ [Wolfram MathWorld]. Thus the integrals are given by

$$I_{-1}(a) = e^{-ia}\sqrt{\pi}\ U\left(\frac{1}{2}, -1, ia\right), \tag{3.41}$$

$$I_0(a) = e^{-ia}\sqrt{\pi}\ U\left(\frac{1}{2}, 0, ia\right), \tag{3.42}$$

$$I_1(a) = e^{-ia}\sqrt{\pi}\ U\left(\frac{1}{2}, 1, ia\right), \tag{3.43}$$

which is in agreement with the results obtained in Podszus and Di Piazza [2019] Eqs. (11)-(13). Thus we can write $P_e(k)$ in Eq. (3.33) as

$$\begin{aligned} P_e(k) = &-\frac{\alpha}{2\pi}m^2\int_{-\infty}^{\infty}d\varphi\int_0^\infty\frac{d\tau}{\tau}\Big\{2\xi_0^2\Big[X(\varphi,\tau)I_0(a_0(\tau)+a_f(\varphi,\tau)) \\ &+ Z(\varphi,\tau)I_1(a_0(\tau)+a_f(\varphi,\tau))\Big] \\ &- i\frac{\theta_0}{4\tau}\Big[I_{-1}(a_0(\tau)+a_f(\varphi,\tau)) - I_{-1}(a_0(\tau))\Big]\Big\}. \end{aligned} \tag{3.44}$$

To perform the integral over φ we have to define the pulse-shape function $\psi(\varphi)$, which is needed for the functions $X(\varphi,\tau)$, $Z(\varphi,\tau)$, and $Q^2(\varphi,\tau)$. We define it as $\psi(\varphi) = -\sinh(\varphi)/\cosh^2(\varphi)$ [Mackenroth and Di Piazza, 2011], shown also in figure (3.1). In Section (5) we will generalize the results also to arbitrary pulse-shape functions. The function above describes a one-cycle, finite pulsed field and is chosen as prototype, since it performs the oscillation and the damping at $\varphi \to \pm\infty$ of the field within a single function. Since $\int_{-\infty}^{+\infty} d\tau'\ \psi^2(\varphi-\tau')$ is finite and the pulse function goes sufficiently fast to zero such that $\lim_{\tau\to\pm\infty}\int_{-\infty}^{\infty} d\varphi\ \psi(\varphi)\psi(\varphi+\tau) = 0$, one obtains that also $\tau Q^2(\varphi,\tau) = \frac{1}{2}\int_{-\tau}^{+\tau} d\tau'\ \psi^2(\varphi-\tau') - \frac{1}{4\tau}\left[\int_{-\tau}^{+\tau} d\tau'\ \psi(\varphi-\tau')\right]^2$ is finite for all values of $\tau \in [0,\infty]$. Because we are looking at the high-energy limit, i.e. $\theta_0 \gg 1$ and $\xi_0 = const$ such that $\xi_0^2/\theta_0 \ll 1$, we can expand the I_n-functions for $a_f(\varphi,\tau) \ll 1$. In leading order we can approximate $I_1(a_0(\tau) + a_f(\varphi,\tau)) \approx$

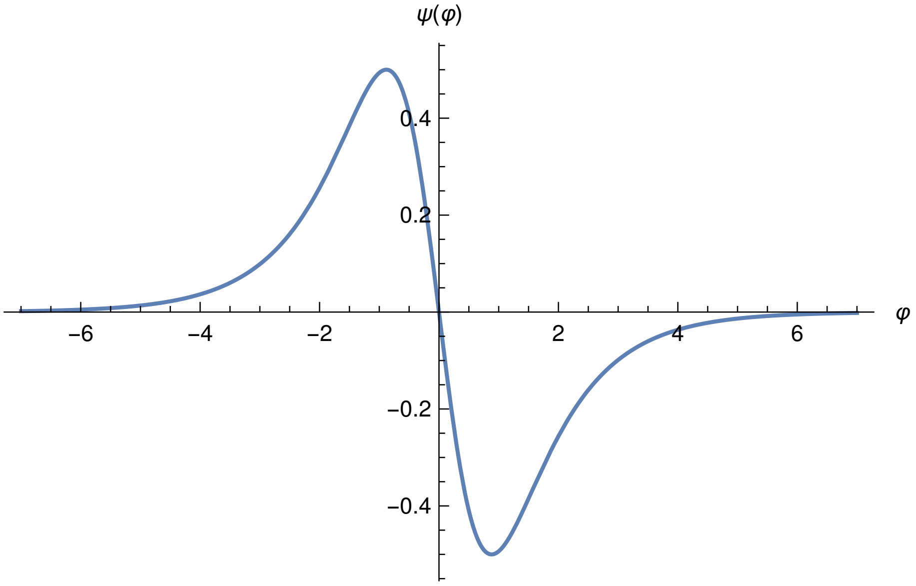

Figure 3.1: Plot of the one-cycle, finite pulse-shape function $\psi(\varphi) = -\sinh(\varphi)/\cosh^2(\varphi)$.

$I_1(a_0(\tau))$, $I_0(a_0(\tau) + a_f(\varphi,\tau)) \approx I_0(a_0(\tau))$ and $I_{-1}(a_0(\tau) + a_f(\varphi,\tau)) - I_{-1}(a_0(\tau)) \approx I'_{-1}(a_0(\tau))a_f(\varphi,\tau) = -iI_0(a_0(\tau))a_f(\varphi,\tau)$. On a first point this should be a good approximation, since we get for every higher order a factor $a_f(\varphi,\tau) \propto \xi_0^2/\theta_0$ in front such that these terms should be suppressed by $1/\theta_0$. Nevertheless we have to proof at the end if this is indeed the case. By only taking the first non-vanishing order terms the polarisation operator is given by

$$P_e(k) = -\frac{\alpha m^2 \xi_0^2}{\pi} \int_{-\infty}^{\infty} d\varphi \int_0^{\infty} \frac{d\tau}{\tau} \Big\{ I_0(a_0(\tau))X(\varphi,\tau) + I_1(a_0(\tau))Z(\varphi,\tau) - \frac{1}{2} I_0(a_0(\tau))Q^2(\varphi,\tau) \Big\}. \tag{3.45}$$

Notice that, since ξ_0 only arises in $a_f(\varphi,\tau) \propto \xi_0^2$, the expression is proportional to ξ_0^2 and coincides with the leading order term of the expansion in the limit $\xi_0 \to 0$. Since the only dependence on φ occurs now in the functions $X(\varphi,\tau)$, $Z(\varphi,\tau)$, and $Q^2(\varphi,\tau)$, we can perform the integrals over φ, which are given by

$$\mathcal{I}_X(\tau) = \int_{-\infty}^{\infty} d\varphi \; X(\varphi,\tau) = \frac{1}{\tau^2} - 2\tau \frac{3+\cosh(4\tau)}{\sinh^3(2\tau)}, \tag{3.46}$$

$$\mathcal{I}_Z(\tau) = \int_{-\infty}^{\infty} d\varphi \; Z(\varphi,\tau) = \frac{2}{3} + \frac{\tau\coth(\tau) - 1}{\sinh^2(\tau)} + \frac{\tau\tanh(\tau) - 1}{\cosh^2(\tau)}, \tag{3.47}$$

$$\mathcal{I}_{Q^2}(\tau) = \int_{-\infty}^{\infty} d\varphi \; Q^2(\varphi,\tau) = \frac{2}{3} - \frac{1}{\tau^2} + \frac{2}{\tau}\frac{1}{\sinh(2\tau)}, \tag{3.48}$$

and we obtain for the leading order contribution of the polarisation operator

$$
\begin{aligned}
P_e(k) = &- \frac{\alpha m^2 \xi_0^2}{\pi} \int_0^\infty \frac{d\tau}{\tau} e^{-2i\tau/\theta_0} \Bigg\{ \mathrm{K}_0\left(\frac{2i\tau}{\theta_0}\right) \mathcal{I}_Z(\tau) \\
&+ \sqrt{\pi}\, \mathrm{W}_{-1/2,1/2}\left(\frac{4i\tau}{\theta_0}\right) \left[\mathcal{I}_X(\tau) - \frac{1}{2}\mathcal{I}_{Q^2}(\tau) \right] \Bigg\}.
\end{aligned}
\tag{3.49}
$$

Here we use that $\mathrm{W}_{-1/2,1/2}\left(\frac{4i\tau}{\theta_0}\right) = e^{-2i\tau/\theta_0} U\left(\frac{1}{2}, 0, \frac{4i\tau}{\theta_0}\right)$ and that $\mathrm{K}_0\left(\frac{2i\tau}{\theta_0}\right) = \sqrt{\pi} e^{-2i\tau/\theta_0} U\left(\frac{1}{2}, 1, \frac{4i\tau}{\theta_0}\right)$ with $\mathrm{W}_{b,c}(ia)$ being the Whittaker-W-function and $\mathrm{K}_b(a)$ being the modified Bessel function of second kind [Wolfram MathWorld]. Now we have to perform the integration in τ. For a clearer computation we divide the integral into the three parts

$$
\mathscr{I}_X = \sqrt{\pi} \int_0^\infty \frac{d\tau}{\tau} e^{-2i\tau/\theta_0} \mathrm{W}_{-1/2,1/2}\left(\frac{4i\tau}{\theta_0}\right) \mathcal{I}_X(\tau), \tag{3.50}
$$

$$
\mathscr{I}_Z = \int_0^\infty \frac{d\tau}{\tau} e^{-2i\tau/\theta_0} \mathrm{K}_0\left(\frac{2i\tau}{\theta_0}\right) \mathcal{I}_Z(\tau), \tag{3.51}
$$

$$
\mathscr{I}_{Q^2} = \int_0^\infty \frac{d\tau}{\tau} e^{-2i\tau/\theta_0} \mathrm{W}_{-1/2,1/2}\left(\frac{4i\tau}{\theta_0}\right) \mathcal{I}_{Q^2}(\tau). \tag{3.52}
$$

To calculate the asymptotic values of these integrals we divide the integral over $\tau \in [0, \infty]$ into two regions by the parameter τ_0, such that we have two integrals with $\tau \in [0, \tau_0]$ and $\tau \in [\tau_0, \infty]$. Further we assume for τ_0 that it fulfills the relation $1 \ll \tau_0 \ll \theta_0$ [Bender and Orszag, 1999]. Since we are working in the limit $\theta_0 \to \infty$ it is always possible to find such a τ_0. This assumption allows us now to expand in both integral regions, $\tau \in [0, \tau_0]$ and $\tau \in [\tau_0, \infty]$, for different parameters. In the region $\tau \in [0, \tau_0]$ it is always $\tau \leq \tau_0$. If we divide the assumption by θ_0 we obtain that $\tau_0/\theta_0 \ll 1$ and thus also $\tau/\theta_0 \ll 1$. In this way we can expand in the region $\tau \in [0, \tau_0]$ for small values of τ/θ_0 such that [Inc.]

$$
e^{-2i\tau/\theta_0} \mathrm{W}_{-1/2,1/2}\left(\frac{4i\tau}{\theta_0}\right) \overset{\tau/\theta_0 \ll 1}{\approx} \frac{2}{\sqrt{\pi}} \tag{3.53}
$$

and

$$
\mathrm{K}_0\left(\frac{2i\tau}{\theta_0}\right) \overset{\tau/\theta_0 \ll 1}{\approx} -\gamma - \ln\left(i\frac{\tau}{\theta_0}\right). \tag{3.54}
$$

Here $\gamma = 0.577\ldots$ is the Euler-Mascheroni constant [Wolfram MathWorld]. In the region $\tau \in [\tau_0, \infty]$ it is always $\tau \geq \tau_0$. Since the assumption already states that $\tau_0 \gg 1$, also $\tau \gg 1$, and we can expand in this region $\mathcal{I}_X(\tau)$, $\mathcal{I}_Z(\tau)$, and $\mathcal{I}_{Q^2}(\tau)$ for large τ, given by

$$
\mathcal{I}_X(\tau \gg 1) \approx 0 \tag{3.55}
$$

$$
\mathcal{I}_Z(\tau \gg 1) \approx \frac{2}{3} \tag{3.56}
$$

$$
\mathcal{I}_{Q^2}(\tau \gg 1) \approx \frac{2}{3}. \tag{3.57}
$$

At this point the integrals of both regions can be calculated and we will do this computation now for all three parts in Eqs. (3.50) - (3.52). The simplest integral to compute is $\mathscr{I}_X$ since $\mathcal{I}_X(\tau)$ vanishes for large τ and thus we obtain

$$\mathscr{I}_X \approx 2\int_0^{\tau_0} \frac{d\tau}{\tau}\mathcal{I}_X(\tau) \overset{\tau_0 \gg 1}{\approx} -2/3. \tag{3.58}$$

Here we neglect at the end all terms suppressed by $1/\tau_0$ since $\tau_0 \gg 1$. For $\mathscr{I}_{Q^2}$ in Eq. (3.52) we divide the integral by τ_0 and use for both integrals the different approximations explained above

$$\mathscr{I}_{Q^2} = \frac{2}{\sqrt{\pi}}\int_0^{\tau_0} \frac{d\tau}{\tau}\mathcal{I}_{Q^2}(\tau) + \frac{2}{3}\int_{\tau_0}^{\infty} \frac{d\tau}{\tau} e^{-2i\tau/\theta_0}\mathrm{W}_{-1/2,1/2}\left(\frac{4i\tau}{\theta_0}\right). \tag{3.59}$$

Both integrals can be solved. By performing a partial integration we obtain for the first integral

$$\int_0^{\tau_0} \frac{d\tau}{\tau}\mathcal{I}_{Q^2}(\tau) = \frac{2}{3}\ln(\tau_0) + C_{Q^2,1}, \tag{3.60}$$

where the constant $C_{Q^2,1}$ is defined as

$$C_{Q^2,1} = \int_0^{\infty} d\tau \ \ln(\tau)\mathcal{I'}_{Q^2}(\tau) \approx 0.218\ldots. \tag{3.61}$$

For the second integral we obtain, only keeping terms not suppressed by $1/\theta_0$,

$$\int_{\tau_0}^{\infty} \frac{d\tau}{\tau} e^{-2i\tau/\theta_0}\mathrm{W}_{-1/2,1/2}\left(\frac{4i\tau}{\theta_0}\right) \approx -\frac{2}{\sqrt{\pi}}\left[2 + \gamma + i\frac{\pi}{2} + \ln(\tau_0) - \ln(\theta_0)\right]. \tag{3.62}$$

Putting both results together we get

$$\mathscr{I}_{Q^2} = \frac{4}{3\sqrt{\pi}}\left[\ln(\theta_0) - 2 - \gamma - i\frac{\pi}{2}\right] - \frac{2}{\sqrt{\pi}}C_{Q^2,1}. \tag{3.63}$$

We see that both $\ln(\tau_0)$, appearing in the individual integrals, cancel each other such that the result at the end does not depend on the point where we split the integral into two. This should be of course the case, since we introduce the parameter τ_0 artificially and thus it should not appear in the final results. The last and most complicated term in the polarisation operator is $\mathscr{I}_Z$. After dividing the integral in $\mathscr{I}_Z$ in Eq. (3.51) by τ_0, we first expand in the region $\tau \in [0, \tau_0]$ only the Bessel function for small τ/θ_0 and remain the exponential term unchanged,

$$\mathscr{I}_Z = \int_0^{\tau_0} \frac{d\tau}{\tau} e^{-2i\tau/\theta_0}\left[-\gamma - \ln\left(i\frac{\tau}{\theta_0}\right)\right]\mathcal{I}_Z(\tau) + \frac{2}{3}\int_{\tau_0}^{\infty} \frac{d\tau}{\tau} e^{-2i\tau/\theta_0}\mathrm{K}_0\left(\frac{2i\tau}{\theta_0}\right). \tag{3.64}$$

Now we can rewrite the first integral by $\int_0^{\tau_0} \ldots = \int_0^{\infty} \ldots - \int_{\tau_0}^{\infty} \ldots$ and the second by $\int_{\tau_0}^{\infty} \ldots = \int_0^{\infty} \ldots - \int_0^{\tau_0} \ldots$ and use again the expansions for $\mathcal{I}_Z$ and K_0, respectively, in the integrals depending on τ_0. We can combine then both and end with the three integrals

$$\mathscr{I}_Z = \int_0^\infty \frac{d\tau}{\tau} e^{-2i\tau/\theta_0} \left[-\gamma - \ln\left(i\frac{\tau}{\theta_0} \right) \right] \mathcal{I}_Z(\tau) - \frac{2}{3} \int_0^\infty \frac{d\tau}{\tau} e^{-2i\tau/\theta_0} \left[-\gamma - \ln\left(i\frac{\tau}{\theta_0} \right) \right] + \frac{2}{3} \int_0^\infty \frac{d\tau}{\tau} e^{-2i\tau/\theta_0} \mathrm{K}_0\left(\frac{2i\tau}{\theta_0} \right). \tag{3.65}$$

The second and the third integral can be solved numerically. For this we first have to perform the substitution $\tilde{\tau} = 2i\tau/\theta_0$ such that they do not depend on the parameter θ_0 any more. They are given by the constant term

$$C_K = \int_0^\infty \frac{d\tilde{\tau}}{\tilde{\tau}} e^{-\tilde{\tau}} \left[\mathrm{K}_0(\tilde{\tau}) + \gamma + \ln\left(\frac{\tilde{\tau}}{2} \right) \right] \approx 0.240\ldots. \tag{3.66}$$

Notice that this constant is independent of the pulse shape and thus its numerical value stays unchanged for other pulse shape functions. In that way we can write $\mathscr{I}_Z$ as

$$\mathscr{I}_Z = \frac{2}{3} C_K + \int_0^\infty \frac{d\tau}{\tau} e^{-2i\tau/\theta_0} \left[-\gamma - \ln\left(i\frac{\tau}{\theta_0} \right) \right] \mathcal{I}_Z(\tau). \tag{3.67}$$

The remaining integral can be solved by dividing again the integral by τ_0, but now with expanding the exponential term for small τ/θ_0 by $e^{-2i\tau/\theta_0} \approx 1$ in the region $\tau \in [0, \tau_0]$. We have

$$\mathscr{I}_Z = \int_0^{\tau_0} \frac{d\tau}{\tau} \left[-\gamma - \ln\left(i\frac{\tau}{\theta_0} \right) \right] \mathcal{I}_Z(\tau) + \frac{2}{3} \int_{\tau_0}^\infty \frac{d\tau}{\tau} e^{-2i\tau/\theta_0} \left[-\gamma - \ln\left(i\frac{\tau}{\theta_0} \right) \right] + \frac{2}{3} C_K. \tag{3.68}$$

The first integral can be solved by using that $\int_0^{\tau_0} \ldots = \int_0^\infty \ldots - \int_{\tau_0}^\infty \ldots$, a partial integration and the approximations for $\mathcal{I}_Z(\tau)$. It is given by

$$\int_0^{\tau_0} \frac{d\tau}{\tau} \left[-\gamma - \ln\left(i\frac{\tau}{\theta_0} \right) \right] \mathcal{I}_Z(\tau) \approx -\frac{1}{3} \ln^2(\tau_0) + \frac{1}{2} C_{Z,2} + \left[\frac{2}{3} \ln(\tau_0) - C_{Z,1} \right] \left[-\gamma - i\frac{\pi}{2} + \ln(\theta_0) \right], \tag{3.69}$$

where we define the two constant terms by

$$C_{Z,1} = \int_0^\infty d\tau \ \ln(\tau) \mathcal{I}'_Z(\tau) \approx -0.781\ldots, \tag{3.70}$$

$$C_{Z,2} = \int_0^\infty d\tau \ \ln^2(\tau) \mathcal{I}'_Z(\tau) \approx 0.579\ldots. \tag{3.71}$$

Note that both constants depend on the pulse shape, such that their numerical value is different for other pulse shape functions. For the second integral we use that $1/\theta_0 \propto m^2 = m^2 - i0$ has a small negative imaginary part and only take terms not suppressed by $1/\theta_0$

$$\frac{2}{3}\int_{\tau_0}^{\infty}\frac{d\tau}{\tau}e^{-2i\tau/\theta_0}\left[-\gamma - \ln\left(i\frac{\tau}{\theta_0}\right)\right] = -\frac{5}{36}\pi^2 + \frac{1}{3}\Big[\gamma^2 + i\gamma\pi - \ln^2(2) + (\ln(\tau_0) - \ln(\theta_0))(2\gamma + i\pi + \ln(\tau_0) - \ln(\theta_0))\Big]. \tag{3.72}$$

Putting both results together we obtain for the leading order in θ_0

$$\begin{aligned}\mathscr{I}_Z =& \frac{1}{3}\ln^2(\theta_0) - \left(\frac{2}{3}\gamma + i\frac{\pi}{3} + C_{Z,1}\right)\ln(\theta_0) + \frac{1}{3}\left(\gamma^2 + i\pi\gamma - \frac{5}{12}\pi^2 - \ln^2(2)\right)\\ &+ \left(\gamma + i\frac{\pi}{2}\right)C_{Z,1} + \frac{1}{2}C_{Z,2} + \frac{2}{3}C_K.\end{aligned} \tag{3.73}$$

At this point we have calculated all terms of the polarisation operator in the first non-vanishing order. As mentioned before we have to proof now, if higher order contributions of the functions I_n, which we neglect after Eq. (3.44), are indeed suppressed by $1/\theta_0$. These higher order contributions are given by

$$\tilde{\mathscr{I}}_X = \int_0^{\infty}\frac{d\tau}{\tau}\sum_{n=1}^{\infty}\frac{\xi_0^{2n}}{n!}\left(\frac{4\tau}{\theta_0}\right)^n \frac{d^n I_0(a)}{da^n}\bigg|_{a=4\tau/\theta_0}\mathcal{I}_{X,n}(\tau), \tag{3.74}$$

$$\tilde{\mathscr{I}}_Z = \int_0^{\infty}\frac{d\tau}{\tau}\sum_{n=1}^{\infty}\frac{\xi_0^{2n}}{n!}\left(\frac{4\tau}{\theta_0}\right)^n \frac{d^n I_1(a)}{da^n}\bigg|_{a=4\tau/\theta_0}\mathcal{I}_{Z,n}(\tau), \tag{3.75}$$

$$\tilde{\mathscr{I}}_{Q^2} = \frac{i}{\sqrt{\pi}}\int_0^{\infty}\frac{d\tau}{\tau}\sum_{n=2}^{\infty}\frac{\xi_0^{2n-2}}{n!}\left(\frac{4\tau}{\theta_0}\right)^{n-1} \frac{d^n I_{-1}(a)}{da^n}\bigg|_{a=4\tau/\theta_0}\mathcal{I}_{Q^2,n}(\tau). \tag{3.76}$$

Here we extract already the integral over φ into the functions $\mathcal{I}_{X,n}(\tau)$, $\mathcal{I}_{Z,n}(\tau)$ and $\mathcal{I}_{Q^2,n}(\tau)$ given by

$$\mathcal{I}_{X,n}(\tau) = \int_{-\infty}^{\infty} d\varphi X(\varphi,\tau)Q^{2n}(\varphi,\tau), \tag{3.77}$$

$$\mathcal{I}_{Z,n}(\tau) = \int_{-\infty}^{\infty} d\varphi Z(\varphi,\tau)Q^{2n}(\varphi,\tau), \tag{3.78}$$

$$\mathcal{I}_{Q^2,n}(\tau) = \int_{-\infty}^{\infty} d\varphi Q^{2n}(\varphi,\tau). \tag{3.79}$$

The functions $\mathcal{I}_{X,n}(\tau)$, $\mathcal{I}_{Z,n}(\tau)$ for $n \geq 1$ and $\mathcal{I}_{Q^2,n}(\tau)$ for $n \geq 2$ are only different from zero for $\tau \approx 1$ and tend to zero both for $\tau \to 0$ and for $\tau \to \infty$, which can be easily ascertained numerically. Therefore, in the limit $\theta_0 \to \infty$ we can expand the remaining terms in Eqs. (3.74) - (3.76) for $\tau/\theta_0 \ll 1$ or $a \ll 1$. For the exponential terms e^{-ia} in the definition of the functions $I_{-1}(a)$, $I_0(a)$, and $I_1(a)$ it is therefore enough to directly approximate them by their leading order, i.e. $e^{-ia} \approx 1$, since higher order terms or derivatives on them just lead to terms sub-leading in θ_0. Thus the derivatives $\frac{d^n}{da^n} I_c(a)$ are given for small a approximately by the n-th derivative of the Hypergeometric function [Inc.], i.e.

$$\begin{aligned}\frac{d^n}{da^n} I_c(a) &\approx \sqrt{\pi} \frac{d^n}{da^n} U\left(\frac{1}{2}, c, ia\right) + \mathcal{O}(a) \\ &= (-i)^n \Gamma\left(\frac{1}{2}+n\right) U\left(\frac{1}{2}+n, c+n, ia\right) + \mathcal{O}(a),\end{aligned} \tag{3.80}$$

where $c \in \{-1, 0, 1\}$. Since at the end $a = a_0(\tau)$, we can expand the hypergeometric function U for small $a \ll 1$. It is given in leading order by [Inc.]

$$\begin{aligned}U\left(\frac{1}{2}+n, c+n, ia\right) \overset{a\to 0}{\approx} (ia)^{-c-n} &\left(\frac{\Gamma(c+n-1)}{\Gamma(1/2+n)} ia + \mathcal{O}(a^2)\right) \\ &+ \left(\frac{\Gamma(1-c-n)}{\Gamma(3/2-c)} + \mathcal{O}(a)\right).\end{aligned} \tag{3.81}$$

Now we can put the equations (3.81), (3.80) and (3.74) - (3.76) together and set $a = 4\tau/\theta_0$. At this point we see that the leading order of $\tilde{\mathscr{I}}_X$ and $\tilde{\mathscr{I}}_{Q^2}$ scales as $1/\theta_0$

$$\begin{aligned}\tilde{\mathscr{I}}_X \approx \int_0^\infty \frac{d\tau}{\tau} \sum_{n=1}^\infty \frac{(-\xi_0^2)^n}{n!} &\left[i\frac{4\tau}{\theta_0} \Gamma(n-1) \right. \\ &\left. + \left(i\frac{4\tau}{\theta_0}\right)^n \Gamma(1-n) \frac{\Gamma(1/2+n)}{\Gamma(3/2)} + \mathcal{O}(\theta_0^{-2}) \right] \mathcal{I}_{X,n}(\tau),\end{aligned} \tag{3.82}$$

$$\begin{aligned}\tilde{\mathscr{I}}_{Q^2} \approx \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{d\tau}{\tau} \sum_{n=2}^\infty \frac{(-\xi_0^2)^{n-1}}{n!} &\left[i\frac{4\tau}{\theta_0} \Gamma(n-2) \right. \\ &\left. + \left(i\frac{4\tau}{\theta_0}\right)^{n-1} \Gamma(2-n) \frac{\Gamma(1/2+n)}{\Gamma(5/2)} + \mathcal{O}(\theta_0^{-2}) \right] \mathcal{I}_{Q^2,n}(\tau).\end{aligned} \tag{3.83}$$

Note that $\Gamma(n-1) + \Gamma(1-n) \overset{n\to 1}{\to} -2\gamma$ and $\Gamma(n-2) + \Gamma(2-n) \overset{n\to 2}{\to} -2\gamma$ [Inc.]. Thus both terms are sub-leading in θ_0 and do not contribute to the leading order result for the polarization operator. Instead the term $\tilde{\mathscr{I}}_Z$ is indeed not suppressed by $1/\theta_0$ and contributes to the leading order of the polarization operator. By putting (3.81) into (3.80) and setting $c = 1$ we see that in leading order $\frac{d^n}{da^n} I_1(a) \approx (-1)^n (n-1)!/a^n$ such that $\tilde{\mathscr{I}}_Z$ can be written as [Inc.]

$$\begin{aligned}\tilde{\mathscr{I}}_Z = \tilde{\mathscr{I}}_Z(\xi_0) &\approx \int_{-\infty}^\infty d\varphi \int_0^\infty \frac{d\tau}{\tau} Z(\varphi,\tau) \sum_{n=1}^\infty \frac{1}{n} \left[-\xi_0^2 Q^2(\varphi,\tau)\right]^n \\ &= -\int_{-\infty}^\infty d\varphi \int_0^\infty \frac{d\tau}{\tau} Z(\varphi,\tau) \ln\left[1 + \xi_0^2 Q^2(\varphi,\tau)\right].\end{aligned} \tag{3.84}$$

For this expression we can not solve both integrals analytically. Note that $Q^2(\varphi,\tau) \geq 0$ as mentioned below Eq. (3.23). Thus with the results in Eqs. (3.58), (3.63), (3.73), and (3.84) the high-energy asymptotic of the polarization operator in leading order of θ_0 is given by

$$P_e(k) = -\frac{\alpha m^2 \xi_0^2}{3\pi}\Bigg[\ln^2(\theta_0) - (2\gamma + i\pi + 2 + 3C_{Z,1})\ln(\theta_0) + \gamma^2 + i\pi\gamma - \frac{5}{12}\pi^2 - \ln^2(2) + 2 + 3\tilde{\mathscr{I}}_Z(\xi_0) + \left(\gamma + i\frac{\pi}{2}\right)(C_{Z,1}+2) + \frac{3}{2}C_{Z,2} + 2C_K + 3C_{Q^2,1}\Bigg], \tag{3.85}$$

$$P_b(k) = P_e(k) - \frac{2\alpha m^2 \xi_0^2}{3\pi}. \tag{3.86}$$

We see that the polarisation operator scales double logarithmic with the energy scale and thus shows a similar behaviour like in vacuum QED. Notice that the double logarithm only appears in the real part of the polarisation operator, whereas the imaginary part scales logarithmic. Further we see that the main contributions to the polarisation operator come from linear interactions with the background field, since the expression is proportional to ξ_0^2, except of the function $\tilde{\mathscr{I}}_Z(\xi_0)$. But in the definition of this function in Eq. (3.84) we see that higher non-linear interaction terms just contribute logarithmic to the asymptotic expression. At this point we can also calculate the probability for non-linear Breit-Wheeler pair production. Due to the optical theorem it is related to the imaginary part of the one-loop polarization operator [Meuren, 2015, Berestetskii et al., 1982, Meuren et al., 2013, Reiss, 1962, Nikishov and Ritus, 1964] and given by

$$P_{BW} = \frac{1}{m^2\theta_0}\mathrm{Im}\left[\frac{P_e(k)+P_b(k)}{2}\right] = \frac{\alpha}{3}\frac{\xi_0^2}{\theta_0}\left[\ln(\theta_0) - \gamma - 1 - \frac{3}{2}C_{Z,1}\right]. \tag{3.87}$$

This probability scales logarithmic with the energy scale, but is suppressed by $1/\theta_0$ such that it goes to zero for high energies. Therefore radiative corrections in a plane wave laser pulse scale logarithmic like in vacuum QED and the power-law scaling pertains to the high energy regime in the CCF limit.

4 High-Energy Asymptotic of the One-Loop Mass Operator in a Plane Wave

In the previous section we saw on the example of the polarisation operator that the CCF limit and the high energy limit do not commute. The argumentation works similar for the mass operator, such that we do not present the non-commutativity in this section. Rather we directly start with the computation of the high energy asymptotic of the one-loop mass operator in an intense plane wave laser pulse. The calculations for the mass operator follow the same structure as for the polarization operator. We start with the general expression given in [Baier et al., 1976a]. Like in the case of the polarisation operator we assume also here that the incoming and outgoing electrons are identical $p_1^\mu = p_2^\mu = p^\mu$ with an average spin $\boldsymbol{\zeta}_1/2 = \boldsymbol{\zeta}_2/2 = \boldsymbol{\zeta}/2$ and on-shell ($p^2 = m^2$). Also here the vacuum part of the mass operator vanishes after renormalisation for on-shell particles [Berestetskii et al., 1982] and we can express the field dependent part of the mass operator in a linear polarized plane wave ($\xi_2 = 0$ and $\xi_1 = \xi_0$) by $M_{f,\zeta}(p_1, p_2) = (2\pi)^3\delta^2(\boldsymbol{p}_{1,\perp} - \boldsymbol{p}_{2,\perp})\delta((k_0p_1) - (k_0p_2))M_\zeta(p)$, with $M_\zeta(p) = \sum_{j=1}^5 M_{j,\zeta}(p)$. The functions $M_{j,\zeta}(p)$ are given by

$$M_{1,\zeta}(p) = \frac{\alpha}{2\pi}m\int_{-\infty}^{\infty}d\varphi\int_0^\infty\frac{d\tau}{\tau}\int_0^1 du(1+u)\Big\{e^{-i\frac{\tau u m^2}{2(k_0p)(1-u)}\left[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)\right]} - e^{-i\frac{\tau u m^2}{2(k_0p)(1-u)}}\Big\}, \tag{4.1}$$

$$M_{2,\zeta}(p) = \frac{\alpha}{4\pi}m\xi_0^2\int_{-\infty}^{\infty}d\varphi\int_0^\infty\frac{d\tau}{\tau}\int_0^1 du\Delta^2(\varphi,\tau)e^{-i\frac{\tau u m^2}{2(k_0p)(1-u)}\left[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)\right]}, \tag{4.2}$$

$$M_{3,\zeta}(p) = \frac{\alpha}{4\pi}m\xi_0^2\int_{-\infty}^{\infty}d\varphi\int_0^\infty\frac{d\tau}{\tau}\int_0^1 du\frac{u^2}{1-u}R(\varphi,\tau)e^{-i\frac{\tau u m^2}{2(k_0p)(1-u)}\left[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)\right]}, \tag{4.3}$$

$$M_{4,\zeta}(p) = \frac{\alpha}{4\pi}m\xi_0^2\int_{-\infty}^{\infty}d\varphi\int_0^\infty\frac{d\tau}{\tau}\int_0^1 du\frac{u}{1-u}S(\varphi,\tau)e^{-i\frac{\tau u m^2}{2(k_0p)(1-u)}\left[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)\right]}, \tag{4.4}$$

$$M_{5,\zeta}(p) = i\frac{\alpha}{4\pi}\frac{(s_\mu f_0^{*\mu\nu}p_\nu)}{\eta_0}\int_{-\infty}^{\infty}d\varphi\int_0^\infty\frac{d\tau}{\tau}\int_0^1 du u\Delta(\varphi,\tau)e^{-i\frac{\tau u m^2}{2(k_0p)(1-u)}\left[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)\right]}. \tag{4.5}$$

Here we already performed the substitution $\tau = 2(k_0p)u(1-u)s$ and neglect terms proportional to $\xi_2 \propto a_2 = 0$ to obtain this expression from the expression (3.31) in

[Baier et al., 1976a]. Further we use the functions

$$
\begin{aligned}
\Delta(\varphi,\tau) &= \psi(\varphi-\tau)-\psi(\varphi), && (4.6)\\
\tilde{Q}^2(\varphi,\tau) &= \frac{1}{\tau}\int_0^\tau d\tau'\Delta^2(\varphi,\tau') - \frac{1}{\tau^2}\left[\int_0^\tau d\tau'\Delta(\varphi,\tau')\right]^2, && (4.7)\\
R(\varphi,\tau) &= \left[\Delta(\varphi,\tau)-\frac{2}{\tau}\int_0^\tau d\tau'\Delta(\varphi,\tau')\right]\frac{1}{\tau}\int_0^\tau d\tau'\Delta(\varphi,\tau'), && (4.8)\\
S(\varphi,\tau) &= \frac{1}{\tau}\int_0^\tau d\tau'\Delta^2(\varphi,\tau'), && (4.9)
\end{aligned}
$$

and define the spin four-vector by [Berestetskii et al., 1982]

$$
s^\mu = \left(\frac{\boldsymbol{p}\cdot\boldsymbol{\zeta}}{m}, \boldsymbol{\zeta} + \frac{(\boldsymbol{p}\cdot\boldsymbol{\zeta})\boldsymbol{p}}{m(\varepsilon_p+m)}\right), \qquad (4.10)
$$

and the field pseudo-tensor amplitude in units of the critical field by $f_0^{*\mu\nu} = (1/2)\,\epsilon^{\mu\nu\lambda\rho}F_{0,\lambda\rho}/E_{\text{crit}}$. The energy scale parameter η_0 is defined analogously to θ_0 in the case of the polarization operator as $\eta_0 = (k_0 p)/m^2$. Since we want to calculate the high energy asymptotic of the mass operator, we will work in the limit $\eta_0 \to \infty$. Further we notice that, as for the polarisation operator, the function $\tilde{Q}^2(\varphi,\tau)$ is related to the variance and is thus always non-negative, i.e. $\tilde{Q}^2(\varphi,\tau) \geq 0$. We perform now on the $M_{j,\zeta}(p)$ the substitution $v = 1-u$, with $dv = -du$, and obtain

$$
M_{1,\zeta}(p) = \frac{\alpha}{2\pi}m\int_{-\infty}^{\infty}d\varphi\int_0^\infty\frac{d\tau}{\tau}\int_0^1 dv(2-v)\left\{e^{-i\frac{\tau}{2\eta_0}\frac{1-v}{v}\left[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)\right]} - e^{-i\frac{\tau}{2\eta_0}\frac{1-v}{v}}\right\}, \qquad (4.11)
$$

$$
M_{2,\zeta}(p) = \frac{\alpha}{4\pi}m\xi_0^2\int_{-\infty}^{\infty}d\varphi\int_0^\infty\frac{d\tau}{\tau}\int_0^1 dv\Delta^2(\varphi,\tau)e^{-i\frac{\tau}{2\eta_0}\frac{1-v}{v}\left[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)\right]}, \qquad (4.12)
$$

$$
M_{3,\zeta}(p) = \frac{\alpha}{4\pi}m\xi_0^2\int_{-\infty}^{\infty}d\varphi\int_0^\infty\frac{d\tau}{\tau}\int_0^1 dv\frac{(1-v)^2}{v}R(\varphi,\tau)e^{-i\frac{\tau}{2\eta_0}\frac{1-v}{v}\left[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)\right]}, \qquad (4.13)
$$

$$
M_{4,\zeta}(p) = \frac{\alpha}{4\pi}m\xi_0^2\int_{-\infty}^{\infty}d\varphi\int_0^\infty\frac{d\tau}{\tau}\int_0^1 dv\frac{1-v}{v}S(\varphi,\tau)e^{-i\frac{\tau}{2\eta_0}\frac{1-v}{v}\left[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)\right]}, \qquad (4.14)
$$

$$
M_{5,\zeta}(p) = i\frac{\alpha}{4\pi}\frac{(s_\mu f_0^{*\mu\nu}p_\nu)}{\eta_0}\int_{-\infty}^{\infty}d\varphi\int_0^\infty\frac{d\tau}{\tau}\int_0^1 dv(1-v)\Delta(\varphi,\tau)e^{-i\frac{\tau}{2\eta_0}\frac{1-v}{v}\left[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)\right]}. \qquad (4.15)
$$

Now we perform the substitution $y = \frac{1}{v}$, with $dv = -\frac{1}{y^2}dy$, $y(v=0) = \infty$, and $y(v=1) = 1$, and get

$$\begin{aligned}M_{1,\zeta}(p) =& \frac{\alpha}{2\pi} m \int_{-\infty}^{\infty} d\varphi \int_0^{\infty} \frac{d\tau}{\tau} \int_1^{\infty} \frac{dy}{y^2}\left(2 - \frac{1}{y}\right)\left\{e^{-i\frac{\tau}{2\eta_0}(y-1)\left[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)\right]}\right. \\ &\left. - e^{-i\frac{\tau}{2\eta_0}(y-1)}\right\},\end{aligned} \tag{4.16}$$

$$M_{2,\zeta}(p) = \frac{\alpha}{4\pi} m\xi_0^2 \int_{-\infty}^{\infty} d\varphi \int_0^{\infty} \frac{d\tau}{\tau} \int_1^{\infty} \frac{dy}{y^2} \Delta^2(\varphi,\tau) e^{-i\frac{\tau}{2\eta_0}(y-1)\left[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)\right]}, \tag{4.17}$$

$$\begin{aligned}M_{3,\zeta}(p) =& \frac{\alpha}{4\pi} m\xi_0^2 \int_{-\infty}^{\infty} d\varphi \int_0^{\infty} \frac{d\tau}{\tau} \int_1^{\infty} \frac{dy}{y^2}\left(y - 2 + \frac{1}{y}\right) \\ & \times R(\varphi,\tau) e^{-i\frac{\tau}{2\eta_0}(y-1)\left[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)\right]},\end{aligned} \tag{4.18}$$

$$M_{4,\zeta}(p) = \frac{\alpha}{4\pi} m\xi_0^2 \int_{-\infty}^{\infty} d\varphi \int_0^{\infty} \frac{d\tau}{\tau} \int_1^{\infty} \frac{dy}{y^2}(y-1) S(\varphi,\tau) e^{-i\frac{\tau}{2\eta_0}(y-1)\left[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)\right]}, \tag{4.19}$$

$$\begin{aligned}M_{5,\zeta}(p) =& i\frac{\alpha}{4\pi} \frac{(s_\mu f_0^{*\mu\nu} p_\nu)}{\eta_0} \int_{-\infty}^{\infty} d\varphi \int_0^{\infty} \frac{d\tau}{\tau} \int_1^{\infty} \frac{dy}{y^2}\left(1 - \frac{1}{y}\right) \\ & \times \Delta(\varphi,\tau) e^{-i\frac{\tau}{2\eta_0}(y-1)\left[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)\right]}.\end{aligned} \tag{4.20}$$

The integrals over y have the structure $\int_1^{\infty} dy \frac{1}{y^n} e^{-iya}$ with $n \in 1,2,3$ and $\mathrm{Im}[a] < 0$ (since $1/\eta_0 \propto m^2 = m^2 - i0$), which can be solved by the Γ-function defined as [Wolfram MathWorld]

$$\Gamma(0, ia) = \int_1^{\infty} dy \frac{1}{y} e^{-iya}. \tag{4.21}$$

Integrals with $n > 1$ can be solved by partial integration and the ones of interest are given by

$$\int_1^{\infty} dy \frac{1}{y^2} e^{-iya} = e^{-ia} - ia\Gamma(0, ia), \tag{4.22}$$

$$\int_1^{\infty} dy \frac{1}{y^3} e^{-iya} = \frac{1}{2}(1 - ia)e^{-ia} - \frac{1}{2}a^2\Gamma(0, ia). \tag{4.23}$$

Nevertheless we perform at last the substitution $x = y - 1$ on the functions $M_{j,\zeta}(p)$, to simplify our notation, and we introduce, like for the polarization operator, the variables $\tilde{a}_0(\tau)=\tau/2\eta_0$ and $\tilde{a}_f(\varphi,\tau)=\tau\xi_0^2\tilde{Q}^2(\varphi,\tau)/2\eta_0$. We end with

$$\begin{aligned}M_{1,\zeta}(p)=&\frac{\alpha}{2\pi}m\int_{-\infty}^{\infty}d\varphi\int_0^{\infty}\frac{d\tau}{\tau}\left\{I_{0,1}\left(\tilde{a}_0(\tau)+\tilde{a}_f(\varphi,\tau)\right)-I_{0,1}\left(\tilde{a}_0(\tau)\right)\right.\\&\left.+2I_{1,1}\left(\tilde{a}_0(\tau)+\tilde{a}_f(\varphi,\tau)\right)-2I_{1,1}\left(\tilde{a}_0(\tau)\right)\right\},\end{aligned}\tag{4.24}$$

$$M_{2,\zeta}(p)=\frac{\alpha}{4\pi}m\xi_0^2\int_{-\infty}^{\infty}d\varphi\int_0^{\infty}\frac{d\tau}{\tau}I_{0,0}\left(\tilde{a}_0(\tau)+\tilde{a}_f(\varphi,\tau)\right)\Delta^2(\varphi,\tau),\tag{4.25}$$

$$M_{3,\zeta}(p)=\frac{\alpha}{4\pi}m\xi_0^2\int_{-\infty}^{\infty}d\varphi\int_0^{\infty}\frac{d\tau}{\tau}I_{2,1}\left(\tilde{a}_0(\tau)+\tilde{a}_f(\varphi,\tau)\right)R(\varphi,\tau),\tag{4.26}$$

$$M_{4,\zeta}(p)=\frac{\alpha}{4\pi}m\xi_0^2\int_{-\infty}^{\infty}d\varphi\int_0^{\infty}\frac{d\tau}{\tau}I_{1,0}\left(\tilde{a}_0(\tau)+\tilde{a}_f(\varphi,\tau)\right)S(\varphi,\tau),\tag{4.27}$$

$$M_{5,\zeta}(p)=i\frac{\alpha}{4\pi}\frac{(s_\mu f_0^{*\mu\nu}p_\nu)}{\eta_0}\int_{-\infty}^{\infty}d\varphi\int_0^{\infty}\frac{d\tau}{\tau}I_{1,1}\left(\tilde{a}_0(\tau)+\tilde{a}_f(\varphi,\tau)\right)\Delta(\varphi,\tau).\tag{4.28}$$

Here we additionally introduce the function

$$I_{n,d}(a)=\int_0^{\infty}\frac{dx}{(1+x)^2}\frac{x^n}{(1+x)^d}e^{-iax},\tag{4.29}$$

with n and d being non-negative integers. By a re-substitution and the equations (4.21)-(4.23) one can express these functions in terms of gamma functions,

$$I_{0,0}(a)=1-iae^{ia}\Gamma(0,ia),\tag{4.30}$$

$$I_{0,1}(a)=\frac{1}{2}[1-ia-a^2e^{ia}\Gamma(0,ia)],\tag{4.31}$$

$$I_{1,0}(a)=-1+(1+ia)e^{ia}\Gamma(0,ia),\tag{4.32}$$

$$I_{1,1}(a)=\frac{1}{2}[1+ia+a(a-2i)e^{ia}\Gamma(0,ia)],\tag{4.33}$$

$$I_{2,1}(a)=\frac{1}{2}[-3-ia+(2+4ia-a^2)e^{ia}\Gamma(0,ia)].\tag{4.34}$$

To perform the integral over φ we define now, as in the case of the polarisation operator, the pulse-shape function $\psi(\varphi)$ as $\psi(\varphi)=-\sinh(\varphi)/\cosh^2(\varphi)$ [Mackenroth and Di Piazza, 2011], shown in figure (3.1). In Section (5) we will generalize also this results to arbitrary pulse-shape functions. Again, since $\int_{-\infty}^{+\infty}d\tau'\,\psi^2(\varphi-\tau')$ is finite and the pulse function goes sufficiently fast to zero such that $\lim_{\tau\to\pm\infty}\int_{-\infty}^{\infty}d\varphi\,\psi(\varphi)\psi(\varphi+\tau)=0$, one obtains that also $\tau\tilde{Q}^2(\varphi,\tau)=\int_0^\tau d\tau'\,\psi^2(\varphi-\tau')-\frac{1}{\tau}\left[\int_0^\tau d\tau'\,\psi(\varphi-\tau')\right]^2$ is finite for all values of $\tau\in[0,\infty]$. Because we are looking at the high-energy limit, i.e. $\eta_0\gg1$ and $\xi_0=const$ such that $\xi_0^2/\theta_0\ll1$, we can expand the $I_{n,d}$-functions for $\tilde{a}_f(\varphi,\tau)\ll1$. Thus in leading order we can approximate $I_{0,0}(\tilde{a}_0(\tau)+\tilde{a}_f(\varphi,\tau))\approx I_{0,0}(\tilde{a}_0(\tau))$, $I_{1,0}(\tilde{a}_0(\tau)+\tilde{a}_f(\varphi,\tau))\approx I_{1,0}(\tilde{a}_0(\tau))$, $I_{1,1}(\tilde{a}_0(\tau)+\tilde{a}_f(\varphi,\tau))\approx I_{1,1}(\tilde{a}_0(\tau))$, $I_{2,1}(\tilde{a}_0(\tau)+\tilde{a}_f(\varphi,\tau))\approx I_{2,1}(\tilde{a}_0(\tau))$, $I_{0,1}(\tilde{a}_0(\tau)+\tilde{a}_f(\varphi,\tau))-I_{0,1}(\tilde{a}_0(\tau))\approx I'_{0,1}(\tilde{a}_0(\tau))\tilde{a}_f(\varphi,\tau)$, and $I_{1,1}(\tilde{a}_0(\tau)+\tilde{a}_f(\varphi,\tau))-I_{1,1}(\tilde{a}_0(\tau))\approx I'_{1,1}(\tilde{a}_0(\tau))\tilde{a}_f(\varphi,\tau)$. Then the leading order terms of the mass operator are given by

$$M_{1,\zeta}(p) = \frac{\alpha}{2\pi} m\xi_0^2 \int_0^\infty d\tau \frac{d}{d\tau} \left[I_{0,1}\left(\tilde{a}_0(\tau)\right) + 2I_{1,1}\left(\tilde{a}_0(\tau)\right)\right] \mathcal{I}_{\tilde{Q}^2}(\tau), \tag{4.35}$$

$$M_{2,\zeta}(p) = \frac{\alpha}{4\pi} m\xi_0^2 \int_0^\infty \frac{d\tau}{\tau} I_{0,0}\left(\tilde{a}_0(\tau)\right) \mathcal{I}_{\Delta^2}(\tau), \tag{4.36}$$

$$M_{3,\zeta}(p) = \frac{\alpha}{4\pi} m\xi_0^2 \int_0^\infty \frac{d\tau}{\tau} I_{2,1}\left(\tilde{a}_0(\tau)\right) \mathcal{I}_{R}(\tau), \tag{4.37}$$

$$M_{4,\zeta}(p) = \frac{\alpha}{4\pi} m\xi_0^2 \int_0^\infty \frac{d\tau}{\tau} I_{1,0}\left(\tilde{a}_0(\tau)\right) \mathcal{I}_{S}(\tau), \tag{4.38}$$

$$M_{5,\zeta}(p) = i\frac{\alpha}{4\pi} \frac{(s_\mu f_0^{*\mu\nu} p_\nu)}{\eta_0} \int_0^\infty \frac{d\tau}{\tau} I_{1,1}\left(\tilde{a}_0(\tau)\right) \mathcal{I}_{\Delta}(\tau), \tag{4.39}$$

where we already perform the integral over φ using the functions

$$\mathcal{I}_{\tilde{Q}^2}(\tau) = \int_{-\infty}^{\infty} d\varphi\, \tilde{Q}^2(\varphi,\tau) = \frac{2}{3} - \frac{4}{\tau^2} + \frac{4}{\tau}\frac{1}{\sinh(\tau)}, \tag{4.40}$$

$$\mathcal{I}_{\Delta^2}(\tau) = \int_{-\infty}^{\infty} d\varphi\, \Delta^2(\varphi,\tau) = \frac{4}{3} + \frac{8}{\sinh(\tau)}\left[\frac{\tau}{\sinh^2(\tau)} - \coth(\tau) + \frac{\tau}{2}\right], \tag{4.41}$$

$$\mathcal{I}_{R}(\tau) = \int_{-\infty}^{\infty} d\varphi\, R(\varphi,\tau) = -\frac{2}{3} - \frac{8}{\tau^2} + \frac{4}{\sinh(\tau)}\left[\frac{\tau}{\sinh^2(\tau)} + \coth(\tau) + \frac{\tau}{2}\right], \tag{4.42}$$

$$\mathcal{I}_{S}(\tau) = \int_{-\infty}^{\infty} d\varphi\, S(\varphi,\tau) = \frac{4}{3} + \frac{4}{\tau}\frac{1-\tau\coth(\tau)}{\sinh(\tau)}, \tag{4.43}$$

$$\mathcal{I}_{\Delta}(\tau) = \int_{-\infty}^{\infty} d\varphi\, \Delta(\varphi,\tau) = 0. \tag{4.44}$$

We see here that $M_{5,\zeta}(p)$ vanishes, and thus we will not consider it in the next calculation steps of the leading order terms. We have to solve now the integral over τ for the terms $M_{1,\zeta}(p)$ to $M_{4,\zeta}(p)$. We do this by introducing the variable τ_0 and splitting the integral region into two, $\tau \in [0,\tau_0]$ and $\tau \in [\tau_0,\infty]$, like we do it in the calculations for the polarization operator, assuming again that $1 \ll \tau_0 \ll \eta_0$ [Bender and Orszag, 1999]. In this way we can expand functions in the region $\tau \in [0,\tau_0]$ for small values of τ/η_0 and in the region $\tau \in [\tau_0,\infty]$ for large values of τ. For the expansions of small τ/η_0 we use that $\Gamma[0, i\tau/(2\eta_0)] \overset{\tau/\eta_0 \ll 1}{\approx} -\gamma - \ln(i\tau/(2\eta_0))$ [Inc.].

Starting with $M_{1,\zeta}(p)$ this means $\frac{d}{d\tau}\left[I_{0,1}\left(\tau/2\eta_0\right) + 2I_{1,1}\left(\tau/2\eta_0\right)\right] \overset{\tau/\eta_0 \ll 1}{\approx} 0$, since the leading term is proportional to τ/η_0, and $\mathcal{I}_{\tilde{Q}^2}(\tau \gg 1) \approx 2/3$. Thus

$$M_{1,\zeta}(p) \approx \frac{\alpha}{3\pi} m\xi_0^2 \int_{\tau_0}^\infty d\tau \frac{d}{d\tau}\left[I_{0,1}\left(\frac{\tau}{2\eta_0}\right) + 2I_{1,1}\left(\frac{\tau}{2\eta_0}\right)\right] \approx -\frac{\alpha}{2\pi} m\xi_0^2, \tag{4.45}$$

where we neglect terms proportional to $\tau_0/\eta_0 \ll 1$ in the result. For $M_{2,\zeta}(p)$ we can divide the integral into two and approximate $I_{0,0}\left(\tau/2\eta_0\right) \overset{\tau/\eta_0 \ll 1}{\approx} 1$ and $\mathcal{I}_{\Delta^2}(\tau \gg 1) \approx$

4/3, and obtain

$$M_{2,\zeta}(p) \approx \frac{\alpha}{4\pi} m\xi_0^2 \left\{ \int_0^{\tau_0} \frac{d\tau}{\tau} \mathcal{I}_{\Delta^2}(\tau) + \frac{4}{3} \int_{\tau_0}^{\infty} \frac{d\tau}{\tau} I_{0,0}\left(\frac{\tau}{2\eta_0}\right) \right\}. \tag{4.46}$$

To solve the first integral we have to rewrite it by $\int_0^{\tau_0} \ldots = \int_0^{\infty} \ldots - \int_{\tau_0}^{\infty} \ldots$ and perform a partial integration on the integral from 0 to ∞. We obtain

$$\int_0^{\tau_0} \frac{d\tau}{\tau} \mathcal{I}_{\Delta^2}(\tau) = \frac{4}{3} \ln(\tau_0) - C_{\Delta^2,1}, \tag{4.47}$$

where the constant term is given by

$$C_{\Delta^2} = \int_0^{\infty} d\tau \ln(\tau) \mathcal{I}'_{\Delta^2}(\tau) \approx -0.637\ldots. \tag{4.48}$$

For the second integral we directly find that the integral is given by the function

$$\frac{4}{3} \int_{\tau_0}^{\infty} \frac{d\tau}{\tau} I_{0,0}\left(\frac{\tau}{2\eta_0}\right) = -\frac{4}{3} \left[e^{\frac{i\tau}{2\eta_0}} \Gamma\left(0, \frac{i\tau}{2\eta_0}\right) \right]_{\tau_0}^{\infty} \approx -\frac{4}{3} \left(\gamma + \ln\left(\frac{i\tau_0}{2\eta_0}\right) \right). \tag{4.49}$$

Combining both results we obtain

$$M_{2,\zeta}(p) \approx \frac{\alpha}{3\pi} m\xi_0^2 \left[\ln(2\eta_0) - \gamma - i\frac{\pi}{2} - \frac{3}{4} C_{\Delta^2} \right]. \tag{4.50}$$

For $M_{3,\zeta}(p)$ we can approximate $I_{2,1}\left(\frac{\tau}{2\eta_0}\right) \overset{\tau/\eta_0 \ll 1}{\approx} -\frac{3}{2} - \gamma - \ln\left(\frac{i\tau}{2\eta_0}\right)$ and $\mathcal{I}_R(\tau \gg 1) \approx -2/3$ and obtain the two integrals

$$M_{3,\zeta}(p) \approx \frac{\alpha}{4\pi} m\xi_0^2 \left\{ -\int_0^{\tau_0} \frac{d\tau}{\tau} \left[\frac{3}{2} + \gamma + \ln\left(\frac{i\tau}{2\eta_0}\right) \right] \mathcal{I}_R(\tau) - \frac{2}{3} \int_{\tau_0}^{\infty} \frac{d\tau}{\tau} I_{2,1}\left(\frac{\tau}{2\eta_0}\right) \right\}. \tag{4.51}$$

To solve the first integral we again rewrite it by $\int_0^{\tau_0} \ldots = \int_0^{\infty} \ldots - \int_{\tau_0}^{\infty} \ldots$, perform a partial integration on the integral from 0 to ∞, and obtain

$$\begin{aligned} -\int_0^{\tau_0} \frac{d\tau}{\tau} \left[\frac{3}{2} + \gamma + \ln\left(\frac{i\tau}{2\eta_0}\right) \right] \mathcal{I}_R(\tau) = \left(\frac{3}{2} + \gamma + \ln\left(\frac{i}{2\eta_0}\right) \right) \left[C_{R,1} + \frac{2}{3} \ln(\tau_0) \right] \\ + \frac{1}{2} C_{R,2} + \frac{1}{3} \ln^2(\tau_0). \end{aligned} \tag{4.52}$$

Here the constant terms are given by

$$C_{R,1} = \int_0^{\infty} d\tau \ln(\tau) \mathcal{I}'_R(\tau) \approx -0.347\ldots, \tag{4.53}$$

$$C_{R,2} = \int_0^{\infty} d\tau \ln^2(\tau) \mathcal{I}'_R(\tau) \approx 0.154\ldots. \tag{4.54}$$

For the second integral we find that

$$\int_{\tau_0}^{\infty}\frac{d\tau}{\tau}\left[I_{2,1}\left(\frac{\tau}{2\eta_0}\right)-e^{\frac{i\tau}{2\eta_0}}\Gamma\left(0,\frac{i\tau}{2\eta_0}\right)\right]=\left[\left(\frac{3}{2}+\frac{i\tau}{4\eta_0}\right)e^{\frac{i\tau}{2\eta_0}}\Gamma\left(0,\frac{i\tau}{2\eta_0}\right)\right]_{\tau_0}^{\infty} \approx\frac{1}{2}+\frac{3}{2}\left(\gamma+\ln\left(\frac{i\tau_0}{2\eta_0}\right)\right) \tag{4.55}$$

and that

$$\int_{\tau_0}^{\infty}\frac{d\tau}{\tau}e^{\frac{i\tau}{2\eta_0}}\Gamma\left(0,\frac{i\tau}{2\eta_0}\right)\approx\frac{1}{2}\Big\{\gamma^2+\frac{\pi^2}{4}+\ln^2(\tau_0)+2\ln(\tau_0)\left(\gamma+\ln\left(\frac{i}{2\eta_0}\right)\right) +i\pi\gamma-(2\gamma+i\pi)\ln(2\eta_0)+\ln^2(2\eta_0)\Big\}, \tag{4.56}$$

where we expand in both cases for $\tau_0/\eta_0 \ll 1$ and only keep terms not suppressed by $1/\eta_0$ to obtain these results. Putting all together we get

$$M_{3,\zeta}(p)=-\frac{\alpha}{12\pi}m\xi_0^2\Big[\ln^2(2\eta_0)-(3+2\gamma+i\pi-3C_{R,1})\ln(2\eta_0)+1 +3\gamma+\gamma^2+\frac{3}{2}i\pi+i\pi\gamma+\frac{\pi^2}{4}-\frac{3}{2}C_{R,1}(3+2\gamma+i\pi)-\frac{3}{2}C_{R,2}\Big]. \tag{4.57}$$

The calculation of $M_{4,\zeta}(p)$ is similar to the previous one of $M_{3,\zeta}(p)$. Here we approximate $I_{1,0}\left(\frac{\tau}{2\eta_0}\right)\overset{\tau/\eta_0\ll 1}{\approx}-1-\gamma-\ln\left(\frac{i\tau}{2\eta_0}\right)$ and $\mathcal{I}_S(\tau\gg 1)\approx 4/3$ and obtain the two integrals

$$M_{4,\zeta}(p)\approx\frac{\alpha}{4\pi}m\xi_0^2\left\{-\int_0^{\tau_0}\frac{d\tau}{\tau}\left[1+\gamma+\ln\left(\frac{i\tau}{2\eta_0}\right)\right]\mathcal{I}_S(\tau)+\frac{4}{3}\int_{\tau_0}^{\infty}\frac{d\tau}{\tau}I_{1,0}\left(\frac{\tau}{2\eta_0}\right)\right\}. \tag{4.58}$$

We rewrite again the first integral by $\int_0^{\tau_0}\ldots=\int_0^{\infty}\ldots-\int_{\tau_0}^{\infty}\ldots$ and perform a partial integration on the integral from 0 to ∞, and get

$$-\int_0^{\tau_0}\frac{d\tau}{\tau}\left[1+\gamma+\ln\left(\frac{i\tau}{2\eta_0}\right)\right]\mathcal{I}_S(\tau)=\left(1+\gamma+\ln\left(\frac{i}{2\eta_0}\right)\right)\left[C_{S,1}-\frac{4}{3}\ln(\tau_0)\right] +\frac{1}{2}C_{S,2}-\frac{2}{3}\ln^2(\tau_0), \tag{4.59}$$

where the constant terms are given by

$$C_{S,1}=\int_0^{\infty}d\tau\ln(\tau)\mathcal{I}_S'(\tau)\approx 0.695\ldots, \tag{4.60}$$

$$C_{S,2}=\int_0^{\infty}d\tau\ln^2(\tau)\mathcal{I}_S'(\tau)\approx 1.02\ldots. \tag{4.61}$$

For the second integral we find that

$$\int_{\tau_0}^{\infty}\frac{d\tau}{\tau}\left[I_{1,0}\left(\frac{\tau}{2\eta_0}\right)-e^{\frac{i\tau}{2\eta_0}}\Gamma\left(0,\frac{i\tau}{2\eta_0}\right)\right]=\left[e^{\frac{i\tau}{2\eta_0}}\Gamma\left(0,\frac{i\tau}{2\eta_0}\right)\right]_{\tau_0}^{\infty} \approx\gamma+\ln\left(\frac{i\tau_0}{2\eta_0}\right), \tag{4.62}$$

where we expand for $\tau_0/\eta_0 \ll 1$ and only keep terms not suppressed by $1/\eta_0$. The integral over the Γ-function was already calculated in Eq. (4.56). Putting all together we obtain

$$\begin{aligned} M_{4,\zeta}(p) =&\frac{\alpha}{6\pi}m\xi_0^2\Bigg[\ln^2(2\eta_0)-\left(2+2\gamma+i\pi+\frac{3}{2}C_{S,1}\right)\ln(2\eta_0) \\ &+2\gamma+\gamma^2+i\pi+i\pi\gamma+\frac{\pi^2}{4}+\frac{3}{2}C_{S,1}\left(1+\gamma+i\frac{\pi}{2}\right)+\frac{3}{4}C_{S,2}\Bigg]. \end{aligned} \tag{4.63}$$

At this point we have calculated all terms of the mass operator in the first non-vanishing order. As for the polarisation operator we have to take now the higher order contributions into account and proof if they give terms which are not suppressed by $1/\eta_0$. The higher order contributions come from terms of the Taylor expansion for small $\tilde{a}_f(\varphi,\tau)$

$$I_{n,d}(\tilde{a}_0(\tau_0)+\tilde{a}_f(\varphi,\tau))=\sum_{j=0}^{\infty}\frac{1}{j!}\,\frac{d^j}{da^j}I_{n,d}(a)\bigg|_{a=\tilde{a}_0(\tau)}\tilde{a}_f^j(\varphi,\tau). \tag{4.64}$$

Thus the higher order contributions are given by

$$\delta M_{1,\zeta}(p)=\frac{\alpha}{2\pi}m\int_0^{\infty}\frac{d\tau}{\tau}\sum_{j=2}^{\infty}\frac{\xi_0^{2j}}{j!}\left(\frac{\tau}{2\eta_0}\right)^j\frac{d^j}{da^j}\left[I_{0,1}(a)+2I_{1,1}(a)\right]\bigg|_{a=\tau/2\eta_0}\mathcal{I}_{\tilde{Q}^2,j}(\tau), \tag{4.65}$$

$$\delta M_{2,\zeta}(p)=\frac{\alpha}{4\pi}m\xi_0^2\int_0^{\infty}\frac{d\tau}{\tau}\sum_{j=1}^{\infty}\frac{\xi_0^{2j}}{j!}\left(\frac{\tau}{2\eta_0}\right)^j\frac{d^j}{da^j}I_{0,0}(a)\bigg|_{a=\tau/2\eta_0}\mathcal{I}_{\Delta^2,j}(\tau), \tag{4.66}$$

$$\delta M_{3,\zeta}(p)=\frac{\alpha}{4\pi}m\xi_0^2\int_0^{\infty}\frac{d\tau}{\tau}\sum_{j=1}^{\infty}\frac{\xi_0^{2j}}{j!}\left(\frac{\tau}{2\eta_0}\right)^j\frac{d^j}{da^j}I_{2,1}(a)\bigg|_{a=\tau/2\eta_0}\mathcal{I}_{R,j}(\tau), \tag{4.67}$$

$$\delta M_{4,\zeta}(p)=\frac{\alpha}{4\pi}m\xi_0^2\int_0^{\infty}\frac{d\tau}{\tau}\sum_{j=1}^{\infty}\frac{\xi_0^{2j}}{j!}\left(\frac{\tau}{2\eta_0}\right)^j\frac{d^j}{da^j}I_{1,0}(a)\bigg|_{a=\tau/2\eta_0}\mathcal{I}_{S,j}(\tau), \tag{4.68}$$

$$\delta M_{5,\zeta}(p)=i\frac{\alpha}{4\pi}\frac{(s_\mu f_0^{*\mu\nu}p_\nu)}{\eta_0}\int_0^{\infty}\frac{d\tau}{\tau}\sum_{j=1}^{\infty}\frac{\xi_0^{2j}}{j!}\left(\frac{\tau}{2\eta_0}\right)^j\frac{d^j}{da^j}I_{1,1}(a)\bigg|_{a=\tau/2\eta_0}\mathcal{I}_{\Delta,j}(\tau), \tag{4.69}$$

where we extract already the integral over φ into the functions

$$\mathcal{I}_{\tilde{Q}^2,j}(\tau) = \int_{-\infty}^{\infty} d\varphi\, \tilde{Q}^{2j}(\varphi,\tau), \tag{4.70}$$

$$\mathcal{I}_{\Delta^2,j}(\tau) = \int_{-\infty}^{\infty} d\varphi\, \Delta^2(\varphi,\tau)\tilde{Q}^{2j}(\varphi,\tau), \tag{4.71}$$

$$\mathcal{I}_{R,j}(\tau) = \int_{-\infty}^{\infty} d\varphi\, R(\varphi,\tau)\tilde{Q}^{2j}(\varphi,\tau), \tag{4.72}$$

$$\mathcal{I}_{S,j}(\tau) = \int_{-\infty}^{\infty} d\varphi\, S(\varphi,\tau)\tilde{Q}^{2j}(\varphi,\tau), \tag{4.73}$$

$$\mathcal{I}_{\Delta,j}(\tau) = \int_{-\infty}^{\infty} d\varphi\, \Delta(\varphi,\tau)\tilde{Q}^{2j}(\varphi,\tau). \tag{4.74}$$

To see if the higher order terms are sub-leading in η_0 it is enough to proof, if higher derivatives scale as $\frac{d^j}{da^j} I_{n,d}(a) \propto \frac{1}{a^j}$, since every higher order term of the Taylor expansion obtains already a term $\tilde{a}_f(\varphi,\tau) \propto \frac{\tau}{\eta_0}\tilde{Q}^2(\varphi,\tau) \ll 1$ for every order. If we have a look on the functions $I_{n,d}$ given by the equations (4.30) to (4.34) we see that they contain five different kinds of terms: constant ones, terms proportional to a, terms $\propto e^{ia}\Gamma(0,ia)$, $\propto ae^{ia}\Gamma(0,ia)$, and $\propto a^2 e^{ia}\Gamma(0,ia)$. Derivatives on the constant or $\propto a$ terms can never scale as $\propto 1/a^j$, as well as derivatives on a^2 or e^{ia}. Thus we have to consider only terms where the derivative acts on $\Gamma(0,ia)$, since the derivative of it is given by $\frac{d}{da}\Gamma(0,ia) = -\frac{1}{a}e^{-ia}$ [Inc.]. Hence only derivatives on the term $\propto e^{ia}\Gamma(0,ia)$ can scales as $\frac{d^j}{da^j}e^{ia}\Gamma(0,ia) \approx \left(-\frac{1}{a}\right)^j (j-1)! + \mathcal{O}(a^{2-j})$. These terms occur only in the functions $I_{1,0}(a)$ and $I_{2,1}(a)$, which means that the higher order contributions $\delta M_{3,\zeta}(p)$ and $\delta M_{4,\zeta}(p)$ can contribute to the leading order, whereas the higher order contributions $\delta M_{1,\zeta}(p)$, $\delta M_{2,\zeta}(p)$, and $\delta M_{5,\zeta}(p)$ are sub-leading in η_0. In the Taylor series of $I_{1,0}(\tilde{a}_0(\tau_0) + \tilde{a}_f(\varphi,\tau))$ and $I_{2,1}(\tilde{a}_0(\tau_0) + \tilde{a}_f(\varphi,\tau))$ we therefore have to consider the higher order contributions given in leading order of η_0 by [Inc.]

$$\begin{aligned}
\sum_{j=1}^{\infty} \frac{1}{j!} e^{ia} \frac{d^j}{da^j}\Gamma(0,ia)\bigg|_{a=\tilde{a}_0(\tau)} \tilde{a}_f^j(\varphi,\tau) &= \sum_{j=1}^{\infty} \frac{1}{j!}\left(-\frac{1}{\tilde{a}_0(\tau)}\right)^j (j-1)!\,\tilde{a}_f^j(\varphi,\tau) \\
&= \sum_{j=1}^{\infty} \frac{1}{j}\left(-\xi_0^2\tilde{Q}^2(\varphi,\tau)\right)^j \\
&= -\ln\left[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)\right].
\end{aligned} \tag{4.75}$$

We rewrite the leading order corrections of $M_{3,\zeta}(p)$ and $M_{4,\zeta}(p)$ as $\delta M_{3,\zeta}(p) = \frac{\alpha}{4\pi} m\xi_0^2 \tilde{\mathscr{I}}_R(\xi_0)$ and $\delta M_{4,\zeta}(p) = \frac{\alpha}{4\pi} m\xi_0^2 \tilde{\mathscr{I}}_S(\xi_0)$, where

$$\tilde{\mathscr{I}}_R(\xi_0) = -\int_{-\infty}^{\infty} d\varphi \int_0^{\infty} \frac{d\tau}{\tau} R(\varphi,\tau)\ln[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)], \tag{4.76}$$

$$\tilde{\mathscr{I}}_S(\xi_0) = -\int_{-\infty}^{\infty} d\varphi \int_0^{\infty} \frac{d\tau}{\tau} S(\varphi,\tau)\ln[1+\xi_0^2\tilde{Q}^2(\varphi,\tau)], \tag{4.77}$$

which depend just on the parameter ξ_0 and on the pulse shape. Thus the asymptotic expression of the mass operator $M_\zeta(p) = \sum_{j=1}^{4} M_{j,\zeta}(p) + \delta M_{3,\zeta}(p) + \delta M_{4,\zeta}(p)$ is, using the results of Eqs. (4.45), (4.50), (4.57), and (4.63), given by

$$\begin{aligned} M_\zeta(p) =& \frac{\alpha}{12\pi} m\xi_0^2 \Big\{ \ln^2(2\eta_0) + [3 - 2\gamma - i\pi - 3(C_{R,1} + C_{S,1})] \ln(2\eta_0) \\ &+ \frac{3}{2}[(2\gamma + i\pi)(C_{R,1} + C_{S,1}) + 3C_{R,1} + 2C_{S,1} - 2C_{\Delta^2} + C_{R,2} + C_{S,2}] \\ &+ 3[\tilde{\mathscr{I}}_R(\xi_0) + \tilde{\mathscr{I}}_S(\xi_0)] - 7 - 3\gamma + \gamma^2 - \frac{3}{2} i\pi + i\pi\gamma + \frac{\pi^2}{4} \Big\} . \end{aligned} \tag{4.78}$$

We see that the mass operator scales, like the polarisation operator in the previous section, double logarithmic with the energy scale and thus behaves similar to vacuum QED. Again the double logarithm only appears in the real part, whereas the imaginary part of the mass operator scales logarithmic. Further also here the main contributions to the mass operator come from linear interactions with the background field, since the expression is proportional to ξ_0^2, except of the functions $\tilde{\mathscr{I}}_R(\xi_0)$ and $\tilde{\mathscr{I}}_S(\xi_0)$, which indicate a logarithmic contribution of higher non-linear interaction terms to the asymptotic expression. At this point we can also compute the probability for non-linear Compton-scattering, which is by the optical theorem related to the imaginary part of the one-loop mass operator [Mackenroth and Di Piazza, 2011, Ivanov et al., 2004, Boca and Florescu, 2009, Harvey et al., 2009, Meuren, 2015]. Its high energy asymptotic is thus given by

$$P_C = -\frac{2}{m\eta_0} \mathrm{Im} M_0(p) = \frac{\alpha}{6} \frac{\xi_0^2}{\eta_0} \left[\ln(2\eta_0) + \frac{3}{2} - \gamma - \frac{3}{2}(C_{R,1} + C_{S,1}) \right] . \tag{4.79}$$

This probability scales, like the probability for non-linear Breit-Wheeler pair production in the previous section, logarithmic with the energy, but is suppressed by $1/\eta_0$, such that it goes to zero for high energies. Again this logarithmic scaling is in agreement with the scaling of radiative corrections in vacuum QED and thus the power law scaling pertains to the high energy regime of the CCF limit. The parameter discriminating between both limits is $s_0 = \xi_0^2/\eta_0$, which is much larger than unity for the low-frequency/CCF limit ($s_0 \gg 1$) and much smaller than unity for the high-energy limit ($s_0 \ll 1$).

5 Generalization of the results to arbitrary, finite pulse shapes

In this section we will generalize the asymptotic expressions, found in the previous sections, to arbitrary pulse shapes $\psi(\varphi)$. But to be able to perform the calculations in this generalized manner, the pulse has to be finite. More precise this means for us, that the integrals

$$W_\psi = \int_{-\infty}^{\infty} d\varphi\, \psi^2(\varphi), \tag{5.1}$$

$$W_{\psi'} = \int_{-\infty}^{\infty} d\varphi\, \psi'^2(\varphi), \tag{5.2}$$

$$W_{\psi''} = \int_{-\infty}^{\infty} d\varphi\, \psi''^2(\varphi) \tag{5.3}$$

have to be finite and that the pulse goes sufficiently fast to zero such that

$$\lim_{\tau\to\pm\infty} \int_{-\infty}^{\infty} d\varphi\, \psi(\varphi)\psi(\varphi+\tau) = 0. \tag{5.4}$$

The first and the last assumption were already stressed in the series expansion of the polarization and mass operator below Eq. (3.44) and Eq. (4.34), respectively.

5.1 Polarization operator

We will generalize here the results for the polarization operator. For the polarization operator we introduce in Section (3) the pulse shape below Eq. (3.44), where we will start at this point, staying with the same notation. Since W_ψ is finite and the pulse goes fast enough to zero such that Eq. (5.4) holds, also here the quantity $\tau Q^2(\varphi,\tau)$ is finite. We can expand in the limit $\theta_0 \to \infty$ the Eq. (3.44) for $a_f(\varphi,\tau) \ll 1$ and obtain in leading order Eq. (3.49), but now with the general expressions of $\mathcal{I}_X(\tau)$, $\mathcal{I}_Z(\tau)$, and $\mathcal{I}_{Q^2}(\tau)$ given in Eqs. (3.46) - (3.48). At this point we can show that the functions $\mathcal{I}_X(\tau)$, $\mathcal{I}_Z(\tau)$, and $\mathcal{I}_{Q^2}(\tau)$ tend to zero quadratically for small τ. We do this by starting, e.g., with inserting the expression of $X(\varphi,\tau)$ in Eq. (3.24) in the general expression of $\mathcal{I}_X(\tau)$ given by

$$\begin{aligned}\mathcal{I}_X(\tau) = \int_{-\infty}^{\infty} d\varphi &\left[\frac{1}{2}\int_{-1}^{1} dy\ \psi(\varphi-\tau y) - \psi(\varphi-\tau)\right] \\ &\times \left[\frac{1}{2}\int_{-1}^{1} dy\ \psi(\varphi-\tau y) - \psi(\varphi+\tau)\right].\end{aligned} \tag{5.5}$$

For $\tau \ll 1$ we can Taylor expand the pulse shape function up to quadratic terms as $\psi(\varphi + x) \overset{x \ll 1}{\approx} \psi(\varphi) + x\psi'(\varphi) + \frac{x^2}{2}\psi''(\varphi)$, with x being $-\tau y$ or $\pm\tau$. After performing the integrals in dy, multiplying out the brackets, and only keeping terms up to the order of τ^2, one ends up with

$$\mathcal{I}_X(\tau) \overset{\tau \ll 1}{\approx} -W_{\psi'}\tau^2. \tag{5.6}$$

Proceeding analogously for $\mathcal{I}_Z(\tau)$ and $\mathcal{I}_{Q^2}(\tau)$, one obtains

$$\mathcal{I}_Z(\tau) \overset{\tau \ll 1}{\approx} 2W_{\psi'}\tau^2, \tag{5.7}$$

$$\mathcal{I}_{Q^2}(\tau) \overset{\tau \ll 1}{\approx} \frac{1}{3}W_{\psi'}\tau^2. \tag{5.8}$$

All three functions tend to zero quadratically. This is important since we used in the calculations of some integrals in Section (3) that $\mathcal{I}(\tau)\ln^2(\tau) \to 0$ for $\tau \to 0$. Similarly we can perform the limit $\tau \to \infty$ for the functions $\mathcal{I}_X(\tau)$, $\mathcal{I}_Z(\tau)$, and $\mathcal{I}_{Q^2}(\tau)$. Here we have to use substitutions of the kind $\varphi' = \varphi - \tau y$ and Eq. (5.4). By using this we obtain that

$$\lim_{\tau\to\infty} \mathcal{I}_X(\tau) = 0, \tag{5.9}$$

$$\lim_{\tau\to\infty} \mathcal{I}_Z(\tau) = W_\psi, \tag{5.10}$$

$$\lim_{\tau\to\infty} \mathcal{I}_{Q^2}(\tau) = W_\psi. \tag{5.11}$$

With the limits of $\tau \ll 1$ and $\tau \to \infty$ it is already possible to perform the integration over τ in the expressions of the functions $\mathscr{I}_X$, $\mathscr{I}_Z$, and $\mathscr{I}_{Q^2}$ in Eqs. (3.50) - (3.52) by following the same line like in Section (3), but of course taking now the general expressions for $\mathcal{I}_X(\tau)$, $\mathcal{I}_Z(\tau)$, and $\mathcal{I}_{Q^2}(\tau)$. Starting with $\mathscr{I}_X$ we obtain

$$\mathscr{I}_X \approx 2\int_0^\infty \frac{d\tau}{\tau}\mathcal{I}_X(\tau), \tag{5.12}$$

since again $\mathcal{I}_X(\tau)$ vanishes for large τ such that we can directly expand everything for small τ/θ_0. For $\mathscr{I}_{Q^2}$ and $\mathscr{I}_Z$ we have to divide again the integration region into two parts, $\tau \in [0, \tau_0]$ and $\tau \in [\tau_0, \infty]$ with $1 \ll \tau_0 \ll \theta_0$, and use different expansions for the both regions. Proceeding along the same steps as in Section (3), we obtain for them

$$\mathscr{I}_{Q^2} = \frac{2W_\psi}{\sqrt{\pi}}\left[\ln(\theta_0) - 2 - \gamma - i\frac{\pi}{2}\right] - \frac{2}{\sqrt{\pi}}C_{Q^2,1}, \tag{5.13}$$

$$\begin{aligned}\mathscr{I}_Z =& \frac{W_\psi}{2}\left[\ln^2(\theta_0) - \left(2\gamma + i\pi + \frac{2}{W_\psi}C_{Z,1}\right)\ln(\theta_0) + \gamma^2 + i\pi\gamma - \frac{5}{12}\pi^2 - \ln^2(2)\right] \\ &+ \left(\gamma + i\frac{\pi}{2}\right)C_{Z,1} + \frac{1}{2}C_{Z,2} + W_\psi C_K,\end{aligned} \tag{5.14}$$

where C_K is given by Eq. (3.66) and $C_{Q^2,1}$, $C_{Z,1}$, and $C_{Z,2}$ are given by the general definitions in the Eqs. (3.61), (3.70), and (3.71), respectively. Note that $C_{Q^2,1}$, $C_{Z,1}$, and $C_{Z,2}$ depend on the pulse shape such that their numerical value will be different for different pulse shape functions. Instead C_K is independent of the pulse shape such that its numerical value is fixed. Now we have to proof if the higher order terms in Eqs. (3.74) - (3.76) give contributions to the leading order. At this point we just have to convince our self that also for an arbitrary, finite pulse the functions $\mathcal{I}_{X,n}(\tau)$, $\mathcal{I}_{Z,n}(\tau)$ for $n \geq 1$ and $\mathcal{I}_{Q^2,n}(\tau)$ for $n \geq 2$, defined in Eqs.(3.77) to (3.79), tend to zero both for $\tau \to 0$ and for $\tau \to \infty$. Obviously they tend to zero for $\tau \to 0$, since the functions $X(\varphi, \tau)$, $Z(\varphi, \tau)$, and $Q^2(\varphi, \tau)$ tend to zero for $\tau \to 0$. For large τ we can use the fact that the pulse is finite and tends to zero for $\pm\infty$. Nevertheless the pulse has a finite width ϕ, which could be much larger than unity, i.e. $\phi \gg 1$. Thus the integral get its mayor contribution for $\tau \lesssim \phi$ and the approximations in Section (3) are just valid for $\theta_0 \gg \phi$, which we assume in the following. In this case we can expand the remaining functions in Eqs. (3.74) - (3.76) for $\tau/\theta_0 \ll 1$. The further calculation is exactly the same as in Section (3), since it does not depend on the pulse shape. Thus the only higher order contribution is given by $\tilde{\mathscr{I}}_Z(\xi_0)$ given in Eq. (3.84). Summing up all terms we obtain for the high-energy asymptotics of the one-loop polarization operator in an arbitrary, finite plane wave pulse

$$\begin{aligned} P_e(k) = &- \frac{\alpha m^2 \xi_0^2 W_\psi}{2\pi} \left\{ \ln^2(\theta_0) - \left(2\gamma + i\pi + 2 + \frac{2}{W_\psi} C_{Z,1} \right) \ln(\theta_0) \right. \\ &+ \frac{2}{W_\psi} \tilde{\mathscr{I}}_Z(\xi_0) + \gamma^2 + i\pi\gamma - \frac{5}{12}\pi^2 - \ln^2(2) + 4 + 2\left(\gamma + i\frac{\pi}{2} + C_K \right) \\ &\left. + \frac{2}{W_\psi} \left[\left(\gamma + i\frac{\pi}{2} \right) C_{Z,1} + \frac{1}{2} C_{Z,2} + C_{Q^2,1} + \mathscr{I}_X \right] \right\}, \end{aligned} \tag{5.15}$$

$$P_b(k) = P_e(k) + \frac{\alpha m^2 \xi_0^2}{\pi} \mathscr{I}_X. \tag{5.16}$$

Again, we can compute the probability for non-linear Breit-Wheeler pair production by using the optical theorem. For an arbitrary, finite pulse it is given by

$$P_{BW} = \frac{\alpha}{2} \frac{\xi_0^2 W_\psi}{\theta_0} \left[\ln(\theta_0) - \gamma - 1 - \frac{1}{W_\psi} C_{Z,1} \right]. \tag{5.17}$$

The results of $P_e(k)$, $P_b(k)$, and P_{BW} in an arbitrary pulse shape are, except of the numerical value of some constants, exactly the same as for the pulse shape used in Section (3). Thus all conclusions also hold here. The only additional assumption is that θ_0 has to be much larger as the width of the Laser pulse.

5.2 Mass operator

We will generalize now the results for the mass operator. Therefore we start in Section (4) below Eq. (4.34), where the pulse shape is introduced the first time in the calculations. We use the same notation as in Section (4). Again, do to Eqs. (5.1) and (5.4) the quantity $\tau\tilde{Q}^2(\varphi,\tau)$ is finite. Thus in the limit $\eta_0 \to \infty$ we can expand the Eqs. (4.24) - (4.28) for $\tilde{a}_f(\varphi,\tau) \ll 1$, and obtain in leading order the Eqs. (4.35) - (4.39), but now with the general expressions for $\mathcal{I}_{\tilde{Q}^2}(\tau)$, $\mathcal{I}_{\Delta^2}(\tau)$, $\mathcal{I}_R(\tau)$, $\mathcal{I}_S(\tau)$, and $\mathcal{I}_\Delta(\tau)$ given in Eqs. (4.40) - (4.44). Again $\mathcal{I}_\Delta(\tau) = 0$, since with the substitution $\varphi' = \varphi - \tau$ one obtains that $\int_{-\infty}^{\infty} d\varphi\psi(\varphi-\tau) = \int_{-\infty}^{\infty} d\varphi'\psi(\varphi')$. For the other terms we can show, like in the previous subsection, that they tend to zero at least quadratically for small τ. We do this by a Taylor expansion of the pulse shape function up to third order in τ, such that for $\tau \ll 1$ we approximate $\psi(\varphi-\tau) \approx \psi(\varphi) - \tau\psi'(\varphi) + \frac{\tau^2}{2}\psi''(\varphi) - \frac{\tau^3}{6}\psi'''(\varphi)$. After performing the integrals, multiplying all terms out, using partial integrations in the integration over φ to express all pulse shape depending terms by W_ψ, $W_{\psi'}$ or $W_{\psi''}$, using the fact that the pulse shape and its derivatives go to zero for $\varphi \to \infty$, and keeping only the leading terms in τ, we obtain

$$\mathcal{I}_{\tilde{Q}^2}(\tau) \overset{\tau\ll 1}{\approx} \frac{1}{12} W_{\psi'}\tau^2, \tag{5.18}$$

$$\mathcal{I}_{\Delta^2}(\tau) \overset{\tau\ll 1}{\approx} W_{\psi'}\tau^2, \tag{5.19}$$

$$\mathcal{I}_R(\tau) \overset{\tau\ll 1}{\approx} -\frac{1}{72} W_{\psi''}\tau^4, \tag{5.20}$$

$$\mathcal{I}_S(\tau) \overset{\tau\ll 1}{\approx} \frac{1}{3} W_{\psi'}\tau^2. \tag{5.21}$$

Similarly we can perform the limit $\tau \to \infty$. Using substitutions of the kind $\varphi' = \varphi - x$ with x being τ or τ' and using Eq. (5.4) we get

$$\lim_{\tau\to\infty} \mathcal{I}_{\tilde{Q}^2}(\tau) = W_\psi, \tag{5.22}$$

$$\lim_{\tau\to\infty} \mathcal{I}_{\Delta^2}(\tau) = 2W_\psi, \tag{5.23}$$

$$\lim_{\tau\to\infty} \mathcal{I}_R(\tau) = -W_\psi, \tag{5.24}$$

$$\lim_{\tau\to\infty} \mathcal{I}_S(\tau) = 2W_\psi. \tag{5.25}$$

Since all functions tend to zero at least quadratically for $\tau \to 0$ and are constant for $\tau \to \infty$, we can perform the integral over τ in $M_{1,\varsigma}(p)$ to $M_{4,\varsigma}(p)$ given by Eqs. (4.35) - (4.38). We do this by introducing again the parameter τ_0, with $1 \ll \tau_0 \ll \eta_0$, and splitting the integral region into two, $\tau \in [0,\tau_0]$ and $\tau \in [\tau_0,\infty]$, and using the approximation $\tau/\eta_0 \ll 1$ and $\tau \gg 1$, respectively. Following the same line as in Section (4), but now of course with the general expression and limits for the functions $\mathcal{I}_{\tilde{Q}^2}(\tau)$, $\mathcal{I}_{\Delta^2}(\tau)$, $\mathcal{I}_R(\tau)$, and $\mathcal{I}_S(\tau)$ we obtain

$$M_{1,\zeta}(p) =-\frac{3\alpha}{4\pi}m\xi_0^2 W_\psi, \tag{5.26}$$

$$M_{2,\zeta}(p) =\frac{\alpha}{2\pi}m\xi_0^2\left\{W_\psi\left[\ln(2\eta_0)-\gamma-i\frac{\pi}{2}\right]-\frac{1}{2}C_{\Delta^2}\right\}, \tag{5.27}$$

$$\begin{aligned}M_{3,\zeta}(p) =&-\frac{\alpha}{8\pi}m\xi_0^2 W_\psi\Bigg[\ln^2(2\eta_0)-\left(3+2\gamma+i\pi-\frac{2}{W_\psi}C_{R,1}\right)\ln(2\eta_0)+1+3\gamma\\&+\gamma^2+\frac{3}{2}i\pi+i\pi\gamma+\frac{\pi^2}{4}-\frac{1}{W_\psi}C_{R,1}(3+2\gamma+i\pi)-\frac{1}{W_\psi}C_{R,2}\Bigg],\end{aligned} \tag{5.28}$$

$$\begin{aligned}M_{4,\zeta}(p) =&\frac{\alpha}{4\pi}m\xi_0^2 W_\psi\Bigg[\ln^2(2\eta_0)-\left(2+2\gamma+i\pi+\frac{1}{W_\psi}C_{S,1}\right)\ln(2\eta_0)+2\gamma\\&+\gamma^2+i\pi+i\pi\gamma+\frac{\pi^2}{4}+\frac{1}{W_\psi}C_{S,1}\left(1+\gamma+i\frac{\pi}{2}\right)+\frac{1}{2W_\psi}C_{S,2}\Bigg].\end{aligned} \tag{5.29}$$

Here C_{Δ^2}, $C_{R,1}$, $C_{R,2}$, $C_{S,1}$, and $C_{S,2}$ are given by the general expressions in Eqs. (4.48), (4.53), (4.54), (4.60), and (4.61), respectively. Since they depend on the pulse shape, their numerical value will be different for different pulse shape functions.

At this point we have to proof if higher order terms of the expansion are suppressed by $1/\eta_0$, like we do it in Section (4). Since there the argumentation does not depend on the pulse shape, the results are also valid for general pulse shapes and thus the only relevant higher order contributions are $\delta M_{3,\zeta}(p)=\frac{\alpha}{4\pi}m\xi_0^2\tilde{\mathscr{I}}_R(\xi_0)$ and $\delta M_{4,\zeta}(p)=\frac{\alpha}{4\pi}m\xi_0^2\tilde{\mathscr{I}}_S(\xi_0)$ with $\tilde{\mathscr{I}}_R(\xi_0)$ and $\tilde{\mathscr{I}}_S(\xi_0)$ given in Eqs. (4.76) and (4.77), respectively.

Summing all terms up we obtain for the high-energy asymptotics of the one-loop mass operator $M_\zeta(p)=\sum_{j=1}^4 M_{j,\zeta}(p)+\delta M_{3,\zeta}(p)+\delta M_{4,\zeta}(p)$ in an arbitrary, finite plane wave pulse

$$\begin{aligned}M_\zeta(p) =&\frac{\alpha}{8\pi}m\xi_0^2 W_\psi\Bigg\{\ln^2(2\eta_0)+\left[3-2\gamma-i\pi-\frac{2}{W_\psi}(C_{R,1}+C_{S,1})\right]\ln(2\eta_0)\\&+\frac{2}{W_\psi}[\tilde{\mathscr{I}}_R(\xi_0)+\tilde{\mathscr{I}}_S(\xi_0)]-7-3\gamma+\gamma^2-\frac{3}{2}i\pi+i\pi\gamma+\frac{\pi^2}{4}\\&+\frac{1}{W_\psi}[(2\gamma+i\pi)(C_{R,1}+C_{S,1})+3C_{R,1}+2C_{S,1}-2C_{\Delta^2}+C_{R,2}+C_{S,2}]\Bigg\}.\end{aligned} \tag{5.30}$$

At this point we can again compute the probability for non-linear Compton-scattering, which is, due to the optical theorem, related to the imaginary part of the mass operator. For an arbitrary, finite plane wave pulse it is given by

$$P_C=\frac{\alpha}{4}\frac{\xi_0^2 W_\psi}{\eta_0}\left[\ln(2\eta_0)+\frac{3}{2}-\gamma-\frac{1}{W_\psi}(C_{R,1}+C_{S,1})\right]. \tag{5.31}$$

Also here all conclusions of Section (4) hold for these results, too, since they are identical except of the numerical value of some constants.

6 Numerical proof of the results

We want to test now our asymptotic expressions by numerical calculations. We use for the numerical calculations the exact expressions for the polarization or mass operator, but exand them to leading order in the limit $\xi_0 \to 0$. In that way we can neglect higher non-linear interaction terms and the expressions in the limit $\xi_0 \to 0$ goes coincide with the leading order expressions in the high energy limit. We will first show the results for the pulse shape $\psi(\varphi) = -\sinh(\varphi)/\cosh^2(\varphi)$ used in the first two sections and afterwards we take the pulse shape $\psi(\varphi) = \sin^2\left(\frac{\varphi}{2N}\right)\sin(\varphi)$ for $\varphi \in [0, 2N\pi]$ and zero otherwise. Here N denotes the number of cycles and we will use for the calculations $N = 5$ and $N = 10$ to see the influence of the cycle number.

For the pulse shape function $\psi(\varphi) = -\sinh(\varphi)/\cosh^2(\varphi)$ in the case of the polarization operator the analytical asymptotic expression is given by Eqs. (3.85) and (3.86) just without the term $\tilde{\mathscr{I}}_Z(\xi_0)$ and with the numerical values of the constant terms given in Eqs. (3.61), (3.66), (3.70), and (3.71). The numerical calculations are performed using the exact expression of the polarization operator in Eqs. (3.33) and (3.34), but in the limit $\xi_0 \to 0$ such that the expression coincides with the leading term of the high energy expansion, which is e.g. for $P_e(k)$ given in Eq. (3.45). The results for the real and imaginary part of $P_e(k)$ and $P_b(k)$ are shown in figure 6.1.

We see in figure 6.1 that the asymptotic results nicely approach to the exact ones for large values of θ_0 and this in all cases, for the real and imaginary part and for $P_e(k)$ and $P_b(k)$. Further we see that, as expected from the asymptotic results, the imaginary part of $P_e(k)$ and $P_b(k)$ go co-inside and the real parts differ for large θ_0 by a constant term of 2/3.

We do now the same proof with the one-loop mass operator and the pulse shape $\psi(\varphi) = -\sinh(\varphi)/\cosh^2(\varphi)$. The analytical asymptotic expression is given by Eq. (4.78) just without the terms $\tilde{\mathscr{I}}_R(\xi_0)$ and $\tilde{\mathscr{I}}_S(\xi_0)$ and with the numerical values of the constant terms given in Eqs. (4.48), (4.53), (4.54), (4.60), and (4.61). The numerical calculations are performed using the exact expression of the mass operator in Eqs. (4.24) - (4.27), but in the limit $\xi_0 \to 0$ such that the expression coincides with the leading term of the high energy expansion given in Eqs. (4.35) - (4.38). The results are shown in figure 6.2.

We see in figure 6.2 that the asymptotic results nicely approach to the exact ones for large values of η_0 in both cases, for the real and the imaginary part of $M_\zeta(p)$.

Now we take a different pulse shape function, more precise the pulse shape $\psi(\varphi) = \sin^2\left(\frac{\varphi}{2N}\right)\sin(\varphi)$ for $\varphi \in [0, 2N\pi]$ and zero otherwise. Here N denotes the number of

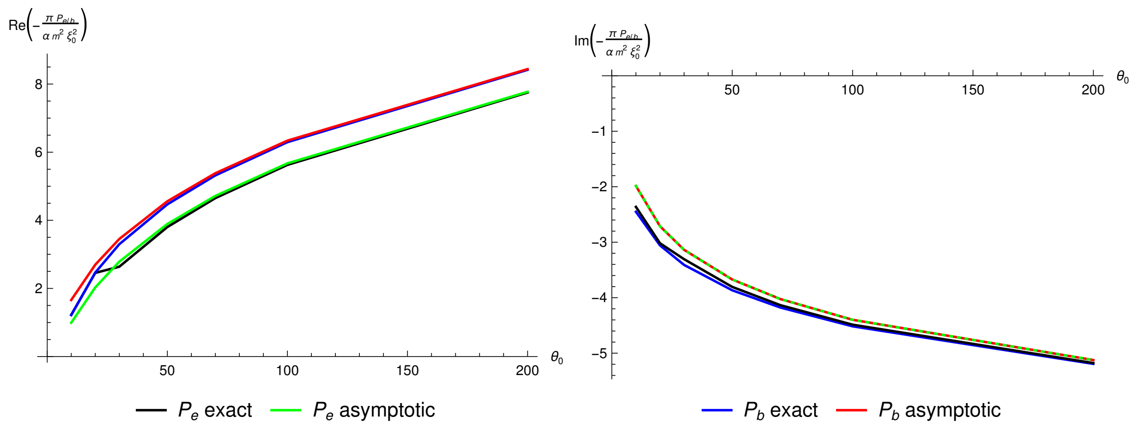

Figure 6.1: On the left plot we see the real part and on the right plot the imaginary part of $P_e(k)$ and $P_b(k)$ in units of $-\alpha m^2 \xi_0^2/\pi$. We use as a pulse shape $\psi(\varphi) = -\sinh(\varphi)/\cosh^2(\varphi)$ like in Section (3). In black and blue the exact numeric results of Eqs. (3.33) and (3.34) are shown and in green and red the asymptotic results of Eqs. (3.85) and (3.86). Since the asymptotic expressions for the imaginary part of $P_e(k)$ and $P_b(k)$ are the same, they are represented as a green red line in the right plot. For both, the exact and the asymptotic results, we assume that the parameter ξ_0 is sufficiently small such that they can be approximated to be proportional to ξ_0^2.

cycles. We will use for the calculations $N = 5$ and $N = 10$ cycles to see the influence of the cycle number on the high energy behaviour of the polarization operator. Since the analytic result just depends over W_ψ and some constant terms on the pulse shape the polarization and mass operator are influenced similarly by different cycle numbers and we therefore only consider here the polarization operator. The analytical result is given by Eqs. (5.15) and (5.16). Here we have to calculate now for $N = 5$ and $N = 10$ cycles the constant terms $C_{Q^2,1}$, $C_{Z,1}$, $C_{Z,2}$, W_ψ, and $\mathscr{I}_X$ from the general expressions in Eqs. (3.61), (3.70), (3.71), (5.1), and (5.12). They are given for $N = 5$ by [Inc.]

$$C_{Q^2,1} \approx 1.392\ldots, \tag{6.1}$$

$$C_{Z,1} \approx -7.442\ldots, \tag{6.2}$$

$$C_{Z,2} \approx 4.641\ldots, \tag{6.3}$$

$$W_\psi = \frac{15}{8}\pi, \tag{6.4}$$

$$\mathscr{I}_X \approx -5.890\ldots, \tag{6.5}$$

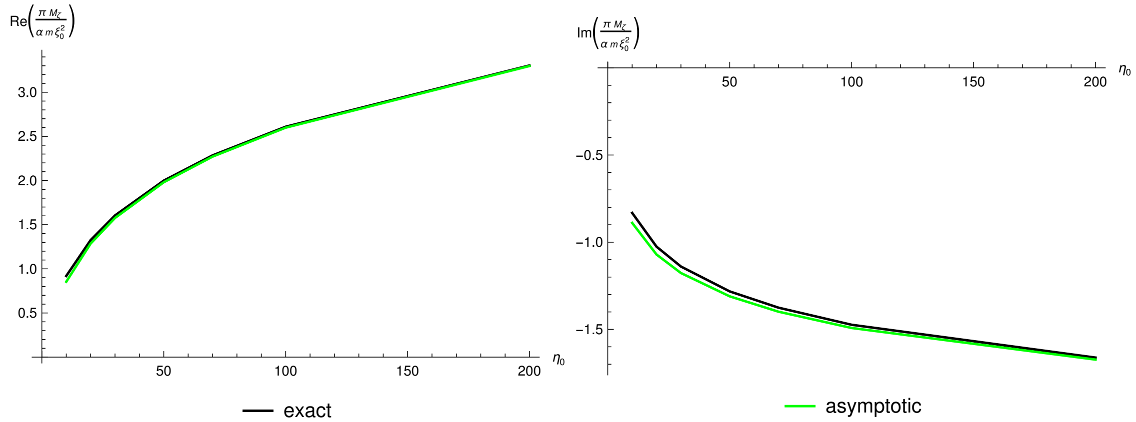

Figure 6.2: On the left plot we see the real part and on the right plot the imaginary part of $M_\zeta(p)$ in units of $\alpha m\xi_0^2/\pi$. We use as a pulse shape $\psi(\varphi) = -\sinh(\varphi)/\cosh^2(\varphi)$ like in Section (4). In black the exact numeric result of Eqs. (4.1) - (4.4) is shown and in green the asymptotic result of Eq. (4.78). For both, the exact and the asymptotic results, we assume that the parameter ξ_0 is sufficiently small such that they can be approximated to be proportional to ξ_0^2.

and for $N = 10$ by

$$C_{Q^2,1} \approx 2.725\ldots, \tag{6.6}$$
$$C_{Z,1} \approx -14.94\ldots, \tag{6.7}$$
$$C_{Z,2} \approx 9.312\ldots, \tag{6.8}$$
$$W_\psi = \frac{15}{4}\pi, \tag{6.9}$$
$$\mathscr{I}_X \approx -11.78\ldots. \tag{6.10}$$

C_K does not depend on the pulse shape and its numerical value is given in Eq. (3.66). For the numerical calculations we use the exact expression of the polarization operator in Eqs. (3.33) and (3.34), and expand them to leading order in the limit $\xi_0 \to 0$ such that their expression coincides with the leading term of the high energy expansion, which is e.g. for $P_e(k)$ given in Eq. (3.45). Here the functions $\mathcal{I}_X(\tau)$, $\mathcal{I}_Z(\tau)$, and $\mathcal{I}_{Q^2}(\tau)$ have to be calculated analytically for the special pulse shape of course. We obtain that they are given by [Inc.]

$$\begin{aligned}\mathcal{I}_X(\tau) = \frac{5\pi}{4608\tau^2}\Bigg\{&901 + 9(-25+32\tau^2)\cos\left(\frac{8\tau}{5}\right)\\ &+576(-1+2\tau^2)\cos(2\tau) + 4(-25+72\tau^2)\cos\left(\frac{12\tau}{5}\right)\\ &-24\tau\left[15\sin\left(\frac{8\tau}{5}\right) + 48\sin(2\tau) + 10\sin\left(\frac{12\tau}{5}\right)\right]\Bigg\},\end{aligned} \tag{6.11}$$

$$\mathcal{I}_Z(\tau) = -\frac{5}{16}\pi\left[-6 + \cos\left(\frac{8\tau}{5}\right) + 4\cos(2\tau) + \cos\left(\frac{12\tau}{5}\right)\right], \tag{6.12}$$

$$\begin{aligned}\mathcal{I}_{Q^2}(\tau) = \frac{5\pi}{4608\tau^2}\Bigg[&-901 + 1728\tau^2 + 225\cos\left(\frac{8\tau}{5}\right)\\ &+576\cos(2\tau) + 100\cos\left(\frac{12\tau}{5}\right)\Bigg]\end{aligned} \tag{6.13}$$

for $N = 5$ and by

$$\begin{aligned}\mathcal{I}_X(\tau) = \frac{5\pi}{78408\tau^2}\Bigg\{&29702 + 121(-50+81\tau^2)\cos\left(\frac{9\tau}{5}\right)\\ &+19602(-1+2\tau^2)\cos(2\tau) + 81(-50+121\tau^2)\cos\left(\frac{11\tau}{5}\right)\\ &-198\tau\left[55\sin\left(\frac{9\tau}{5}\right) + 198\sin(2\tau) + 45\sin\left(\frac{11\tau}{5}\right)\right]\Bigg\},\end{aligned} \tag{6.14}$$

$$\mathcal{I}_Z(\tau) = -\frac{5}{8}\pi\left[-6 + \cos\left(\frac{9\tau}{5}\right) + 4\cos(2\tau) + \cos\left(\frac{11\tau}{5}\right)\right], \tag{6.15}$$

$$\begin{aligned}\mathcal{I}_{Q^2}(\tau) = \frac{5\pi}{39204\tau^2}\Bigg[&-14851 + 29403\tau^2 + 3025\cos\left(\frac{9\tau}{5}\right)\\ &+9801\cos(2\tau) + 2025\cos\left(\frac{11\tau}{5}\right)\Bigg]\end{aligned} \tag{6.16}$$

for $N = 10$. Using this we can compute the remaining integral in τ numerically. The results are shown for the real part of $P_e(k)$ and $P_b(k)$ in figure 6.3 and for the imaginary part in figure 6.4.

We see in figure 6.3 that the asymptotic results for the real part of $P_e(k)$ in the left plot and $P_b(k)$ in the right plot nicely approach to the exact ones for large values of θ_0 in both cases, for $N = 5$ and $N = 10$ cycles. Further we see that in both plots the results for $N = 10$ are approximately twice as large as the results for $N = 5$, which is due to the fact that W_ψ scales with the number of cycles. Again the real part of $P_e(k)$ and $P_b(k)$ differ for large θ_0 by a constant term depending on the value of $\mathscr{I}_X$.

In figure 6.4 we see the imaginary part of $P_e(k)$ in the left plot and $P_b(k)$ in the right plot. The asymptotic results for both, $P_e(k)$ and $P_b(k)$, nicely approach to

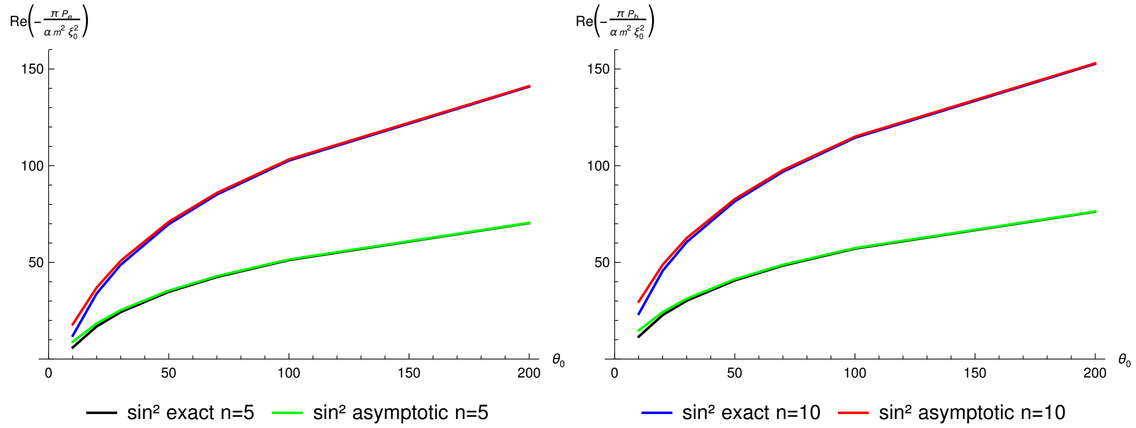

Figure 6.3: Shown are the real parts, on the left plot for $P_e(k)$ and on the right plot for $P_b(k)$, both in units of $-\alpha m^2\xi_0^2/\pi$. We use as a pulse shape $\psi(\varphi) = \sin^2(\varphi/2N)\sin(\varphi)$ for $\varphi \in [0, 2N\pi]$ and zero otherwise. In black and blue the exact numeric result and in green and red the asymptotic result is shown for $N = 5$ and $N = 10$, respectively. The exact results were calculated from Eqs. (3.33) and (3.34) and the asymptotic results are from the Eqs. (5.15) and (5.16) for the left and the right plot, respectively. For both, the exact and the asymptotic results, we assume that the parameter ξ_0 is sufficiently small such that they can be approximated to be proportional to ξ_0^2.

the exact ones for large values of θ_0 for both cycle numbers, $N = 5$ and $N = 10$. Further we see that in both plots the results for $N = 10$ are twice as large as the results for $N = 5$, which is due to the fact that W_ψ scales with the number of cycles. Again the imaginary parts of $P_e(k)$ and $P_b(k)$ are identical for the corresponding cycle numbers.

Thus we see that, considering only linear interaction terms with the background field ($\xi_0 \ll 1$), our analytical expressions for the high energy asymptotic indeed represent the true behaviour for large energy scales θ_0 or η_0.

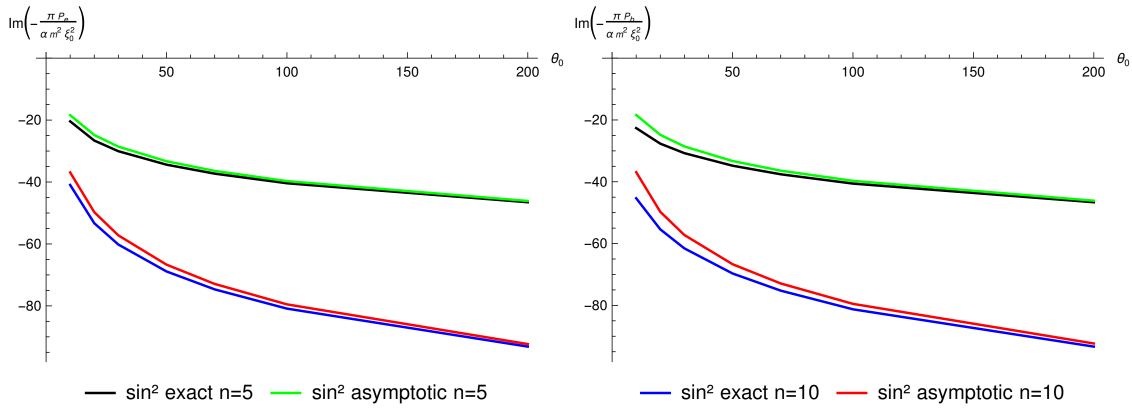

Figure 6.4: Shown are the imaginary parts, on the left plot for $P_e(k)$ and on the right plot for $P_b(k)$, both in units of $-\alpha m^2 \xi_0^2/\pi$. We use as a pulse shape $\psi(\varphi) = \sin^2(\varphi/2N)\sin(\varphi)$ for $\varphi \in [0, 2N\pi]$ and zero otherwise. In black and blue the exact numeric result and in green and red the asymptotic result is shown for $N = 5$ and $N = 10$, respectively. The exact results were calculated from Eqs. (3.33) and (3.34) and the asymptotic results are from the Eqs. (5.15) and (5.16) for the left and the right plot, respectively. For both, the exact and the asymptotic results, we assume that the parameter ξ_0 is sufficiently small such that they can be approximated to be proportional to ξ_0^2.

7 Conclusion

We considered the polarization and mass operator in a plane wave laser pulse with identical and on-shell incoming and outgoing photons and electrons, respectively. We showed that the low frequency or CCF limit, with $\xi_0 \to \infty$ and either $\theta_0 \to 0$ and κ_0 finite or $\eta_0 \to 0$ and χ_0 finite, and the high-energy limit, with ξ_0 finite and either $\theta_0 \to \infty$ and $\kappa_0 \to \infty$ or $\eta_0 \to \infty$ and $\chi_0 \to \infty$, of these operators does not commute. This is due to the parameter $r_0 = \xi_0^2/\theta_0$ or $s_0 = \xi_0^2/\eta_0$, respectively, which are much greater than unity for the low-frequency or CCF limit and much smaller than unity for the high-energy limit. Thus the high-energy behaviour in the low frequency or CCF limit, which scales with the (2/3)-power of the energy, pertains only to this limit and represents not the high energy behaviour of general strong field QED. We calculated then also the high-energy asymptotics of the polarization and mass operator. Both show a double logarithmic increase with the energy scale. The main contributions come from linear interaction terms, since the expression is proportional to ξ_0^2. Terms from higher non-linear interactions with the background field arise in the $\widetilde{\mathscr{I}}$-functions and lead just to logarithmic contributions to the leading order terms in θ_0 or η_0. Further we calculated, by using the optical theorem, from the asymptotic expressions of the polarisation and mass operator the probabilities for non-linear Breit-Wheeler pairproduction and non-linear Compton scattering. Both scale logarithmic with the energy scale, but are suppressed by $1/\theta_0$ or $1/\eta_0$, respectively. This behaviour is quite similar to vacuum QED, where radiative corrections increase logarithmic with the energy scale. We want to point out that the interesting power-law behaviour of the high energy regime in a CCF can nevertheless be tested experimentally. For this one just has to make sure, that the parameter $r_0 = \xi_0^2/\theta_0$ or $s_0 = \xi_0^2/\eta_0$ is much greater than unity. This can for example be reached by using low frequencies and high field strength amplitudes for the background field laser.

Appendix

A Bibliography

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 26.06.2019 ....