CALCULATING THE MORDELL-WEIL RANK OF ELLIPTIC THREEFOLDS AND THE COHOMOLOGY OF SINGULAR HYPERSURFACES

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Abstract. In this paper we give a method for calculating the rank of a general elliptic curve over the field of rational functions in two variables. We reduce this problem to calculating the cohomology of a singular hypersurface in a weighted projective 4-space. We then give a method for calculating the cohomology of a certain class of singular hypersurfaces, extending work of Dimca for the isolated singularity case.

1. Introduction

Throughout this paper we work over the field of complex numbers $\mathbf{C}$. We study families $\pi : X \to S$ of elliptic curves over rational surfaces, i.e., X is a smooth threefold, S a smooth rational surface and π is a flat morphism admitting a section $\sigma_0 : S \to X$. Throughout this paper we will assume that X is not birational to a product $E \times S'$ with E an elliptic curve and S' a rational surface.

The two main invariants of π are its configuration of singular fibers and the Mordell-Weil group $\mathrm{MW}(\pi)$ consisting of *rational* sections of π. Unlike the configuration of singular fibers the Mordell-Weil group is a birational invariant (in the sense of Section 2).

The configuration of singular fibers is well-understood. The general fiber of π is an elliptic curve over $\mathbf{C}(S)$, in particular we have an equation of the form

$$y^2 = x^3 + Ax + B, \text{ where } A, B \in \mathbf{C}(S). \tag{1}$$

The singular fibers lie over the curve Δ given by the zero and pole divisor of $4A^3 + 27B^2$. The fiber-type over a general point p of some irreducible component of Δ can be easily calculated using Tate's algorithm. The fiber-type over a special point can be calculated using the work of Miranda [19].

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In this paper we concentrate on the Mordell-Weil group $\mathrm{MW}(\pi)$. Using the Shioda-Tate-Wazir formula [26, Theorem 4.2] one can relate the rank of $\mathrm{MW}(\pi)$ to the Picard numbers $\rho(S)$ and $\rho(X)$ and the type of singular fibers of π over a general point of each component of Δ. In general it turns out to be rather hard to calculate $\rho(X)$ directly. Even in the case of elliptic surfaces it is a difficult problem to calculate $\rho(X)$ for a given example, this can only be done in very specific cases, see e.g. [15].

The main idea is the following: every elliptic threefold over a rational surface (with a section) has a model as a hypersurface Y of degree $6n$ in the weighted projective space $\mathbf{P} := \mathbf{P}(2n, 3n, 1, 1, 1)$, for some n. The existence of such a model (with minimal n) is a direct consequence of the existence of a (global minimal) Weierstrass equation for an elliptic curve over the function field $\mathbf{C}(S)$ of S. Whenever we refer to a minimal model in this paper we mean the model given by a minimal Weierstrass equation, not to a minimal model in the sense of Mori theory. In general, this threefold Y is singular. In the first part of this paper we show

Theorem 1.1. *Let $\pi : X \to S$ be an elliptic threefold X over a rational surface S and let Y be a minimal model of X/S in $\mathbf{P}(2n, 3n, 1, 1, 1)$. Assume that $H^4(Y, \mathbf{Q})$ has a pure weight 4 Hodge structure. Then*

$$\operatorname{rank} \mathrm{MW}(\pi) = \operatorname{rank}(H^{2,2}(H^4(Y, \mathbf{C})) \cap H^4(Y, \mathbf{Z})) - 1.$$

In particular, this theorem shows that a multiple of a Hodge class is algebraic.

The advantage of this theorem is that we can relate the computation of $\mathrm{MW}(\pi)$ to a computation for a hypersurface in weighted projective space. The latter problem is indeed doable as we will show in the second part of the paper.

The assumption that $H^4(Y, \mathbf{Q})$ has a pure weight 4 Hodge structure is very weak. We do not know of examples such that $H^4(Y, \mathbf{Q})$ does not have a pure weight 4 Hodge structure. Later on we will describe a large class of elliptic threefolds for which we have a method to calculate $H^4(Y, \mathbf{Q})$. Each member Y of this class has a pure weight 4 Hodge structure on $H^4(Y, \mathbf{Q})$.

For a complete proof we refer to Section 4. Here we only give a sketch of the proof: from [19] we get a factorization of the birational map $Y \dashrightarrow X$. This factorization is sufficiently explicit to relate the difference $\rho(X) - \rho(S)$ to $H^{2,2}(H^4(Y, \mathbf{C})) \cap H^4(Y, \mathbf{Z})$. The configuration of singular fibers of π is relatively easy to compute. Applying the Shioda-Tate-Wazir formula then yields the proof.

If X is chosen sufficiently general then Y is quasismooth and hence a V-manifold. Using this one can show that $h^4(Y) = 1$. Theorem 1.1 then implies $\operatorname{rank} \mathrm{MW}(\pi) = 0$. For this reason we shall focus in this paper on non-quasismooth hypersurfaces.

A more explicit form of the above remark is the following (see Corollary 4.4):

Corollary 1.2. *Let $\pi : X \to S$ be an elliptic threefold associated with a hypersurface defined by*

$$y^2 = x^3 + Px + Q$$

with $P \in \mathbf{C}[z_0, z_1, z_2]_{4n}$ and $Q \in \mathbf{C}[z_0, z_1, z_2]_{6n}$, such that

(1) *the curve $\Delta : 4P^3 + 27Q^2 = 0$ is reduced with only double points as singularities and Q vanishes at each of these double points or*
(2) *P is identically zero and $Q = 0$ defines a smooth curve of degree $6n$ in $\mathbf{P}^2$.*

Then $\operatorname{rank} \mathrm{MW}(\pi) = 0$.

Theorem 1.1 implies the following two results: if we call $\delta = h^4(Y) - 1$ the defect of Y then $\operatorname{rank} \mathrm{MW}(\pi) \leq \delta$. (The notion of defect for singular hypersurfaces is due to Clemens [3].) Moreover, one can show that $\mathrm{MW}(\pi) \otimes \mathbf{Q}$ is isomorphic to the group of Weil Divisors on Y modulo the Cartier Divisors tensored with $\mathbf{Q}$.

In the case of elliptic surfaces $\psi : E \to \mathbf{P}^1$ one has a theorem similar to Theorem 1.1. However, we are not aware of any statement concerning elliptic surfaces similar to Corollary 1.2. The reason for this is the following: let T be a surface in weighted projective space corresponding to ψ. The degree of T is divisible by 6. Set $n = \deg(T)/6$. One can show that $\operatorname{rank} \mathrm{MW}(\psi) = \operatorname{rank}(H^{1,1}(H^2(T, \mathbf{C})) \cap H^2(T, \mathbf{Z})) - 1$ and $h^{2,0}(H^2(T, \mathbf{C})) = n - 1$. In this case, using Noether-Lefschetz theory, one can obtain a series of statements on the Mordell-Weil rank of a very general elliptic surface: e.g., one obtains statements on the Mordell-Weil rank for a very general degree $6n$ elliptic surface, and results on the dimension of the locus of elliptic surfaces with fixed Mordell-Weil-rank [4, 16]. However, if $n > 1$ then $h^{2,0}(E) > 0$ and hence it seems hard to calculate $\operatorname{rank}(H^{1,1}(H^2(E, \mathbf{C})) \cap H^2(E, \mathbf{Z})) - 1$ in concrete examples. This is the key obstruction for proving results similar to Corollary 1.2.

To calculate the rank of $\mathrm{MW}(\pi)$ we need to calculate the group $H^4(Y, \mathbf{C})$ together with its Hodge structure. If Y has only isolated singularities and all singularities are semi-weighted homogeneous hypersurface singularities then this can be done by applying a method of Dimca [8]. However, Y might have non-isolated singularities. It turns out in our situation that at a general point of a one-dimensional component of Y_{sing} we have a transversal ADE surface singularity. We extend Dimca's method to a class of hypersurfaces with non-isolated singularities:

For the calculation of $H^4(Y, \mathbf{C})$ there is no reason to assume that the hypersurface comes from an elliptic fibration, i.e., at this stage we work in the following context: let $\mathbf{P} = \mathbf{P}(w_0, w_1, w_2, w_3, w_4)$ be a 4-dimensional weighted projective space and set $w = w_0 + w_1 + w_2 + w_3 + w_4$. We call a degree d hypersurface $Y \subset \mathbf{P}$ *admissible* if Y is defined by a weighted homogeneous polynomial $f \in \mathbf{C}[x_0, x_1, x_2, x_3, x_4]$, such that

(1) Y intersects $\mathbf{P}_{\text{sing}}$ transversally, i.e., if Σ is the locus where all the partials of f vanish, then $\Sigma \cap \mathbf{P}_{\text{sing}} = \emptyset$. ($Y$ will still have singularities along $\mathbf{P}_{\text{sing}}$, these arise from the construction of the weighted projective space and are finite quotient singularities.)
(2) Y is smooth in codimension 1.
(3) In codimension 2 the threefold Y has only transversal ADE surface singularities.
(4) In codimension 3 all singularities are contact equivalent to a weighted homogeneous hypersurface singularity (cf. Remark 7.2).

To formulate our theorem concerning the calculation of the cohomology groups we have to introduce some notation: we define $\mathcal{P}$ as the set of all points $p \in \Sigma$, such that (Y, p) is not a transversal ADE surface singularity. Now let $f_p \in \mathbf{C}[y_0, y_1, y_2, y_3]$ be such that $(f_p, 0)$ is contact equivalent to (Y, p), where f_p is weighted homogeneous of degree d_p and w_p is the sum of the weights. In particular, $f_p = 0$ defines a surface in some weighted projective 3-space.

Let $R(f_p)$ be the Jacobian ring of f_p. If (Y, p) is an isolated singularity we set $\tilde{R}(f_p) = R(f_p)$. If (Y, p) is not an isolated singularity, then $\tilde{R}(f_p)$ is defined as follows: the equation $f_p = 0$ determines a surface $S \subset \mathbf{P}(v_0, v_1, v_2, v_3)$, which has finitely many singularities $(S, q_1), \ldots, (S, q_t)$. Let M_j be the Milnor-algebra of (S, q_j) and set $\mu := \sum_j \dim M_j$ to be the total Milnor number. Let $h_1, \ldots, h_\mu$ be polynomials of degree $2d_p - w_p$, such that their image under the natural (surjective) map $R(f_p)_{2d-w} \to \oplus_j M_j$ spans $\oplus_j M_j$ and set $\tilde{R}(f_p) = R(f_p)/(h_1, \ldots, h_\mu)$.

Using that $f_p = 0$ is contact equivalent to $(Y, 0)$ one obtains a natural map $R(f)_{kd-w} \to R(f_p)_{kd_p - w_p}$ for $k = 1, 2$.

The following theorem is a combination of Proposition 7.7 and several results from Section 8.

Theorem 1.3. *Let Y be an admissible hypersurface. Then*

$$H^1(Y, \mathbf{Q}) = H^5(Y, \mathbf{Q}) = 0 \text{ and}$$
$$H^0(Y, \mathbf{Q}) = \mathbf{Q}, H^2(Y, \mathbf{Q}) = \mathbf{Q}(-1), H^6(Y, \mathbf{Q}) = \mathbf{Q}(-3).$$

The group $H^4(Y, \mathbf{Q})$ has a pure weight 4 Hodge structure, with vanishing $h^{4,0}$ and $h^{0,4}$ and

$$\begin{aligned} h^{3,1}(H^4(Y, \mathbf{C})) &= \dim \operatorname{coker}(R(f)_{d-w} \to \oplus_{p \in \mathcal{P}} \tilde{R}(f_p)_{d_p - w_p}) \\ h^{2,2}(H^4(Y, \mathbf{C})) &= \dim \operatorname{coker}(R(f)_{2d-w} \to \oplus_{p \in \mathcal{P}} \tilde{R}(f_p)_{2d_p - w_p}). \end{aligned}$$

Combining Theorems 1.1 and 1.3 we obtain the following (see also Section 9)

Theorem 1.4. *Let $\pi : X \to S$ be an elliptic threefold, such that S is a rational surface, and the associated threefold $Y \subset \mathbf{P}$ is admissible. Assume that the map*

$$R(f)_{d-w} \to \oplus_{p \in \mathcal{P}} \tilde{R}(f_p)_{d_p - w_p}$$

is surjective. Then

$$\operatorname{rank} \mathrm{MW}(\pi) = \dim \operatorname{coker}(R(f)_{2d-w} \to \oplus_{p\in\mathcal{P}} \tilde{R}(f_p)_{2d_p-w_p}).$$

Remark 1.5. The only non-zero Betti number that we have not determined so far is $h^3(Y)$. Usually, one is able to calculate $e(Y)$ and one can use this to determine $h^3(Y)$.

Remark 1.6. If Y is not admissible then our method fails. In this case the first step would be to calculate the local cohomology $H^i_p(Y, \mathbf{Q})$ of such a singularity. To our knowledge there is no method which works for a large class of such singularities.

This theorem can be used to classify elliptic threefolds with small numerical invariants. In [17] we classify the possibilities for $\mathrm{MW}(\pi)$ if $n = 1$ and the j-invariant of the fibers of π is constant.

Our method is similar to Dimca's, but differs from recent methods such as work by Cynk [5], Rams [22], Grooten-Steenbrink [13], and the classical work of Clemens [3], Werner [28], Schoen [23] and van Geemen-Werner [11].

The differences between the methods of the papers quoted above and ours are the following: in all cases the method is applied to a smaller class of singularities, namely in the isolated singularity case Rams deals with isolated A_k, D_m, E_n singularities. In the non-isolated case, Grooten-Steenbrink deal with transversal A_1 singularities and singularities of the type $w^2 = xyz$ and $zw = x^2y$. The other papers deal with a subset of these singularities.

The restriction on the type of singularity (by Rams and by Grooten-Steenbrink) implies that $(R_{f_p})_{d-w} = 0$ for all singularities. In particular, $H^4(Y, \mathbf{Q})$ is a pure $(2, 2)$ Hodge structure. In most of these cases a map $\psi : H^4(\mathbf{P} \setminus Y, \mathbf{C}) \to V$ is constructed, where V is a certain vector space, such that $\operatorname{coker}(\psi) \cong H^4(Y, \mathbf{C})$. We use $V = \oplus_{p\in\mathcal{P}}(R_{f_p})_{2d-w}$, whereas in the above mentioned articles vector spaces of higher dimension are used. In the isolated singularity case we can explain this as follows: Rams takes $V = \oplus_{p\in\mathcal{P}} H^4(F_p, \mathbf{C})$, where F_p is the Milnor fiber of (Y, p). In this language our choice of V corresponds to $\oplus_{p\in\mathcal{P}} H^4(F_p, \mathbf{C})_0$, the subspace fixed by the monodromy. For an isolated A_k-singularity one has that $h^4(F_p, \mathbf{C}) = k$, whereas $h^4(F_p, \mathbf{C})_0 = 0$ or 1. Choosing a smaller dimensional space is of computational advantage.

The organization of this paper is as follows. In Section 2 we recall some standard facts on elliptic fibrations over rational varieties. In Section 3 we discuss some results of Miranda from [19] that allow us to describe the rational map $X \dashrightarrow Y$. In Section 4 we give proofs of Theorem 1.1 and Corollary 1.2. In Section 5 we recall some standard results on the cohomology of hypersurfaces Y in weighted projective space. In the case of non-quasismooth hypersurfaces we use the Poincaré residue map to calculate the cohomology of the smooth part of Y. In Sections 6, 7 and 8 we relate the cohomology of the smooth part of Y and some local cohomology with the cohomology of Y. This enables us to prove Theorem 1.3. In Section 9 we summarize our method to calculate the Mordell-Weil group. The remaining sections are devoted to applications of our method. In Section 10 we discuss how one can obtain the results of Grooten and Steenbrink [13] by our approach. In Section 11 we calculate the Mordell-Weil rank of three elliptic threefolds. In Section 12 we calculate the Mordell-Weil rank of a class of elliptic Calabi-Yau threefolds which were constructed by Hirzebruch. This calculation allows us to compute all the Hodge numbers of these threefolds.

Part 1. Relation between the Mordell-Weil group and cohomology of singular hypersurfaces

2. Set-up

Definition 2.1. An *elliptic threefold* is a quadruple (X, S, π, σ_0), with X a smooth projective threefold, S a smooth projective surface, $\pi : X \to S$ a flat morphism, such that the generic fiber is a genus 1 curve and σ_0 is a section of π.

The *Mordell-Weil group* of π, denoted by $\mathrm{MW}(\pi)$, is the group of rational sections $\sigma : S \dashrightarrow X$ with identity element σ_0.

Recall that a morphism $\pi : X \to S$ (with X a smooth projective threefold and S a smooth projective surface) is flat if and only if all fibers have dimension one. Clearly $\mathrm{MW}(\pi)$ is a birational invariant, in the sense that if $\pi_i : X_i \to S_i$, $i = 1, 2$ are elliptic threefolds such that there exist an birational isomorphism $\psi : X_1 \stackrel{\sim}{\dashrightarrow} X_2$ mapping the general fiber of π_1 to the general fiber of π_2 then $\psi^* : \mathrm{MW}(\pi_2) \to \mathrm{MW}(\pi_1)$ is well-defined and is an isomorphism.

The following technical definition will be needed

Definition 2.2. Let $\pi : X \to S$ be an elliptic threefold. An effective divisor $D \subset X$ is called *fibral* if $\pi(D) \subset S$ is a curve.

We shall frequently make use of the following fundamental result:

Theorem 2.3 (Shioda-Tate-Wazir, [26, Theorem 4.2])**.** *Let $\pi : X \to S$ be an elliptic threefold then*

$$\rho(X) = \rho(S) + f + \operatorname{rank} \mathrm{MW}(\pi) + 1$$

where f is the number of irreducible surfaces F in X such that $\pi(F)$ is a curve, and $F \cap \sigma_0(S) = \emptyset$.

Using Lefschetz' (1,1) theorem and Poincaré duality we can rephrase the Shioda-Tate-Wazir formula as

$$\operatorname{rank} \mathrm{MW}(\pi) = \operatorname{rank} H^{2,2}(X, \mathbf{C}) \cap H^4(X, \mathbf{Z}) - f - \rho(S) - 1.$$

In general this is hard to compute. Theorem 1.1 says that the analogous formula also holds if we replace X by a minimal (singular) Weierstrass model. In this case one has tools to compute the right hand side.

We shall now describe in some detail how to associate to an elliptic threefold $\pi : X \to S$ a hypersurface in weighted projective 4-space. Here we restrict ourselves to the case where S is a rational surface. In this case we can find a hypersurface Y of degree $6n$ in $\mathbf{P}(2n, 3n, 1, 1, 1)$ which is birational to X as follows: the morphism π establishes $\mathbf{C}(X)$ as a field extension of $\mathbf{C}(S) = \mathbf{C}(z_1, z_2)$. The field $\mathbf{C}(X)$ is the function field of an elliptic curve over $\mathbf{C}(z_1, z_2)$, i.e., $\mathbf{C}(X) = \mathbf{C}(x, y, z_1, z_2)$ where

$$y^2 = x^3 + f_1(z_1, z_2)x + f_2(z_1, z_2) \tag{2}$$

with $f_1, f_2 \in \mathbf{C}(z_1, z_2)$. Without loss of generality we may assume that (2) is a global minimal Weierstrass equation, i.e., f_1, f_2 are polynomials and there is no polynomial $g \in \mathbf{C}[z_1, z_2]$ such that g^4 divides f_1 and g^6 divides f_2.

To obtain a hypersurface in $\mathbf{P}(2n, 3n, 1, 1, 1)$ we need to find a weighted homogeneous polynomial. Let $n = \lceil \max\{\deg(f_1)/4, \deg(f_2)/6\} \rceil$ and define P and Q as the polynomials

$$P = z_0^{4n} f_1(z_1/z_0, z_2/z_0), \quad Q = z_0^{6n} f_2(z_1/z_0, z_2/z_0).$$

Then

$$y^2 = x^3 + P(z_0, z_1, z_2)x + Q(z_0, z_1, z_2)$$

defines a hypersurface Y of degree $6n$ in $\mathbf{P} := \mathbf{P}(2n, 3n, 1, 1, 1)$. Let Σ be the locus where all the partial derivatives of the defining equation vanish. Consider the projection $\tilde{\psi} : \mathbf{P}(2n, 3n, 1, 1, 1) \dashrightarrow \mathbf{P}^2$ with center $L = \{z_0 = z_1 = z_2 = 0\}$ and its restriction $\psi = \tilde{\psi}|_Y$ to Y. Then there exists a diagram

$$\begin{array}{ccc} X & \dashrightarrow & Y \\ \downarrow \pi & & \downarrow \psi \\ S & \dashrightarrow & \mathbf{P}^2. \end{array}$$

Note that $Y \cap L = \{(1 : 1 : 0 : 0 : 0)\}$. If $n = 1$ then $\mathbf{P}_{\text{sing}}$ consists of two points, none of which lie on Y. If $n > 1$ then an easy calculation in local coordinates shows that $\mathbf{P}_{\text{sing}}$ is precisely L, that Σ and L are disjoint and that Y has an isolated singularity at $(1 : 1 : 0 : 0 : 0)$. For any n we have that ψ is not defined at $(1 : 1 : 0 : 0 : 0)$. Let $\tilde{\mathbf{P}}$ be the blow-up of $\mathbf{P}$ along L. Let X_0 be the strict transform of Y in $\tilde{\mathbf{P}}$. An easy calculation in local coordinates shows that $X_0 \to Y$ resolves the singularity of Y at $(1 : 1 : 0 : 0 : 0)$ and that the induced map $\pi_0 : X_0 \to S_0$ with $S_0 = \mathbf{P}^2$ is a morphism. Moreover, all fibers of π_0 are irreducible curves.

3. Miranda's construction

The threefolds X_0 and X are birational and one might therefore ask for a precise sequence of birational morphisms relating X_0 and X. This question might be too hard. A slightly weaker problem is solved by Miranda: starting with $\pi_0 : X_0 \to S_0$ Miranda [19] produces a smooth elliptic threefold $\pi' : X' \to S'$ birational to π. Actually, Miranda produces a series $\{\pi_i : X_i \to S_i\}$

where $\{\pi_{i+1} : X_{i+1} \to S_{i+1}\}$ can be obtained from $\{\pi_i : X_i \to S_i\}$ by applying one of the following three types of birational transformations:

(1) S_{i+1} is the blow-up of S_i in a point p of the discriminant curve of π, i.e., with $\pi_i^{-1}(p)$ a singular curve. Then we define X_{i+1} as the fiber product of X_i with S_{i+1} over S_i:

$$\begin{array}{ccc} X_{i+1} := X_i \times_{S_i} S_{i+1} & \longrightarrow & X_i \\ \downarrow & & \downarrow \\ S_{i+1} := \operatorname{Bl}_p S_i & \longrightarrow & S_i. \end{array}$$

This procedure is applied in the following two cases

(a) To simplify the geometry: let $\Delta_i \subset S_i$ be the (reduced) discriminant curve of π_i. After applying this procedure sufficiently many times, we may assume that each irreducible component of Δ_i is smooth, and that Δ_i has only ordinary double points as singularities.

(b) Suppose X_i has an isolated singularity in the fiber of $p \in S_i$. Blowing up this singularity would yield a non-flat morphism. Instead, if we apply this base change procedure we get a curve of singular points in X_{i+1}.

(2) Even when we start with a minimal local equation, we might obtain a non-minimal equation, i.e., it might happen that X_i has, in one of its charts, a local equation of the form by $y^2 = x^3 + u^4 f_1 x + u^6 f_2$, where $f_1, f_2 \in \mathbf{C}[z_0, z_1]$ and $u \in \mathbf{C}[z_0, z_1] \setminus \mathbf{C}$ is irreducible. In this chart the elliptic fibration is given by $(x, y, z_0, z_1) \mapsto (z_0, z_1)$, which can be interpreted as projection onto the plane $x = y = 0$. Note that after applying the first operation sufficiently many times, we can assume that $x = y = u = 0$ is a smooth irreducible curve. We need to get rid of the factor u^4 and u^6 in the equation, which can be done as follows:

(a) Blow up $C_i : x = y = u = 0$, yielding a threefold X_{i+1} with local equation $y^2 = ux^3 + u^3 f_1 x + u^4 f_2$ in one of the charts. An easy calculation shows that in the other two "new" charts we have that X_{i+1} is smooth.

(b) Blow up $C_{i+1} : x = y = u = 0$, yielding a (non-normal) threefold X_{i+2} with local equation $y^2 = u^2 x^3 + u^2 f_1 x + u^2 f_2$ in one of the charts.

(c) Blow up the surface $R_{i+2} : u = y = 0$, yielding a threefold X_{i+3} with local equation $y^2 = x^3 + f_1 x + f_2$ in one of the charts.

(d) If we patch all the local charts together, we see that the fiber over a point in $\{u = 0\}$ is a reducible curve, consisting of two rational curves and one elliptic curve. Actually $\pi_{i+3}^{-1}(\{u = 0\})$ consists of three irreducible components, two of them are ruled surfaces over $C : \{u = 0\}$, the third is an elliptic surface. We can contract the two ruled surfaces, obtaining X_{i+5}.

An easy calculation in local coordinates shows that both $X_{i+3} \to X_{i+4}$ and $X_{i+4} \to X_{i+5}$ are blow-ups with center a smooth curve contained in the smooth locus.

The base surface remains unchanged, i.e., $S_i = S_{i+1} = \cdots = S_{i+5}$. The geometric construction is summarized in the following table:

Threefold	Singular locus	Important divisor
X_i	C_i (curve)	$F_i = \pi_i^{-1}(\{u = 0\})$
$X_{i+1} = \mathrm{Bl}_{C_i}(X_i)$	C_{i+1} (curve)	$E_{i+1}/C = \mathbf{P}^1$ – bdle.
$X_{i+2} = \mathrm{Bl}_{C_{i+1}}(X_{i+1})$	$R_{i+2} = E_{i+2}$ (surface)	$E_{i+2}/C = \mathbf{P}^1$ – bdle.
$X_{i+3} = \mathrm{Bl}_{R_{i+2}}(X_{i+2})$	$\emptyset$	E_{i+3} = elliptic surface double cover of E_{i+2}
$X_{i+4} = \mathrm{Con}_{E_{i+1}}(X_{i+3})$		
$X_{i+5} = \mathrm{Con}_{F_i}(X_{i+4})$		

When we contract E_{i+1}, F_i we mean that we contract the strict transform of E_{i+1}, F_i.

(3) To resolve singularities: X_{i+1} is obtained by blowing up a curve C inside the singular locus of X_i such that C_{red} is smooth. Set $S_{i+1} = S_i$ and π_{i+1} to be the composition $X_{i+1} \to X_i \xrightarrow{\pi_i} S_i$.

Note that by using the defining equation one can show that at a general point of C_{red} one has a transversal ADE surface singularity.

These three steps should be applied in the following order:

(1) Apply step 1, to obtain a fibration with nice properties: i.e., repeat step 1 until $\Delta_{i,\text{red}} \subset S_i$ has at most nodes as singularities and the j-function $j : S_i \dashrightarrow \mathbf{P}^1$ is a morphism.

At this stage we obtain a Weierstrass fibration i.e., there exists a line bundle $\mathcal{L}_i$ on S_i and sections $A \in H^0(S_i, \mathcal{L}_i^{\otimes 4}), B \in H^0(S_i, \mathcal{L}_i^{\otimes 6})$ such that $X_i = \{Y^2Z = X^3 + AXZ^2 + BZ^3\} \subset \mathbf{P}(\mathcal{O} \oplus \mathcal{L}_i^{-2} \oplus \mathcal{L}_i^{-3})$. We can consider $A = 0$ and $B = 0$ as curves inside S_i. Repeat step 1 until the reduced curves underlying $A = 0$ and $B = 0$ have at most ordinary double points as singularities.

(2) Apply step 2, until there is no curve $C \subset S_i$ such that A vanishes along C with order at least 4, and B vanishes along C with order at least 6.

(3) Apply step 3, until X_i has only isolated singularities or is smooth. If X_i is smooth then stop.

(4) Apply step 1 for each of the isolated singularities of X_i. The outcome of this is a threefold whose singular locus consist of finitely many smooth irreducible curves which are all disjoint.

(5) If necessary apply step 2.

(6) Go to point (3).

From this description it is not at all clear why this procedure should terminate. For this fact we refer to [19].

Remark 3.1. Miranda uses a slightly different order and he uses a fourth type of modification, namely the contraction of $\mathbf{P}^1 \times \mathbf{P}^1$ to a $\mathbf{P}^1$. We indicate now why this does not influence the termination of this procedure.

The extra modification is applied if X_i has an isolated A_1 singularity at $p \in X_i$. We can then first blow up X_i in p. The exceptional divisor E is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. The morphism $\pi_{i+1} : X_{i+1} \to S_{i+1} = S_i$ has a fiber with a two-dimensional component, contradicting flatness. This can be resolved by contracting E to $\mathbf{P}^1$, a so-called "small resolution". The problem is that the space X_{i+2} obtained in this way is a priori only an algebraic space, rather than an algebraic variety. To determine whether X_{i+2} is actually an algebraic variety one needs to consider the global geometry of X_{i+2}.

To avoid this problem we choose a different procedure: namely we blow up S_i in $\pi_i(p)$ and then base change. The threefold X_{i+1} now has a curve C of singularities. Then we blow up C and obtain a threefold X_{i+2}. A direct calculation in local coordinates shows that X_{i+2} is smooth in a neighborhood of the exceptional divisor of $X_{i+2} \to X_{i+1}$. We give a sketch of this calculation: in local coordinates (X_i, p) is given by $t_1^2+t_2^2+t_3^2+t_4^2 = 0$. If we use the base change procedure, we obtain a curve $C \subset X_{i+1}$ of singularities. A straightforward calculation shows that at a general point of C we have a local equation of the form $s_1^2 + s_2^2 + s_3^2 = 0$, i.e., we have a transversal A_1 surface singularity, except for two points on C where we have a local equation of the form $s_1^2 + s_2^2 + s_4 s_3^2 = 0$ (a so-called pinch point). Here C is given by the equation $s_1 = s_2 = s_3 = 0$.

Following the above algorithm, we now need to blow up C. A calculation in local coordinates shows that the threefold X_{i+2} obtained in this way is smooth in a neighborhood of the exceptional divisor.

In order to show that our procedure terminates, note that one could follow Miranda's algorithm until one has only isolated A_1-singularities left. It is clear that the above procedure then resolves all the remaining singularities.

4. Comparing Mordell-Weil ranks

Starting with an elliptic threefold $\pi : X \to S$ we found a hypersurface $Y \subset \mathbf{P}(2n, 3n, 1, 1, 1)$. Applying Miranda's construction to Y gives us an elliptic threefold $\pi' : X' \to S'$. We now want to express $\operatorname{rank} \mathrm{MW}(\pi) = \operatorname{rank} \mathrm{MW}(\pi')$ in terms of invariants of Y. For this we use the following result:

Theorem 4.1. *Let V and $\tilde{V}$ be complex varieties. Let $\varphi : \tilde{V} \to V$ be a proper birational morphism. Let $\mathcal{Z} \subset V$ be a closed subvariety such that φ restricted to $\tilde{V} \setminus \pi^{-1}(\mathcal{Z})$ is injective. Set $E := \pi^{-1}(\mathcal{Z})$. Then there is an exact sequence of Mixed Hodge structures*

$$\cdots \to H^{i-1}(E, \mathbf{Q}) \to H^i(V, \mathbf{Q}) \to H^i(\tilde{V}, \mathbf{Q}) \oplus H^i(\mathcal{Z}, \mathbf{Q}) \to H^i(E, \mathbf{Q}) \to \dots.$$

Proof. See [21, Corollary 5.37]. □

Lemma 4.2. *Let V be a threefold, $C \subset V$ be a smooth curve contained in the smooth locus of V. Let $\tilde{V}$ be the blow-up of V along C, let E be the exceptional divisor and $\iota : E \to \tilde{V}$ be the inclusion. Then*

$$\iota^* : H^3(\tilde{V}, \mathbf{Q}) \to H^3(E, \mathbf{Q})$$

is surjective.

Proof. Let $\psi : V_1 \to V$ be a resolution of singularities of V and let E_1 be the exceptional divisor of ψ. Since C is contained in the smooth locus we have that $\psi^{-1}(C)$ is isomorphic to C. Let $\psi_1 : \tilde{V_1} \to V_1$ be the blow-up of V_1 along $\psi^{-1}(C)$. Equivalently, $\tilde{V_1} = \tilde{V} \times_V V_1$.

The exceptional divisor of ψ_1 is isomorphic to E and the exceptional divisor of $\tilde{V_1} \to V$ is isomorphic to the disjoint union of E and E_1. Denote $\Sigma = V_{\text{sing}}$.

From Theorem 4.1 we get the following exact sequence

$$\cdots \to H^3(\tilde{V_1}, \mathbf{Q}) \to H^3(E, \mathbf{Q}) \to H^4(V_1, \mathbf{Q}) \to \dots.$$

Since V_1 and E are smooth we have that $H^3(E, \mathbf{Q})$ has a pure weight 3 Hodge structure and $H^4(V_1, \mathbf{Q})$ has a pure weight 4 Hodge structure. Hence the map $H^3(E, \mathbf{Q}) \to H^4(V_1)$ is the zero map and $H^3(\tilde{V_1}, \mathbf{Q}) \to H^3(E, \mathbf{Q})$ is surjective. Consider now the exact sequence of Theorem 4.1 for $\psi_1 \circ \psi$:

$$\cdots \to H^3(\tilde{V_1}, \mathbf{Q}) \oplus H^3(\Sigma, \mathbf{Q}) \to H^3(E_1, \mathbf{Q}) \oplus H^3(E, \mathbf{Q}) \to H^4(V, \mathbf{Q}) \to \dots$$

Since $H^3(\tilde{V_1}, \mathbf{Q}) \to H^3(E, \mathbf{Q})$ is surjective we obtain that $H^3(E, \mathbf{Q}) \to H^4(V, \mathbf{Q})$ is the zero map.

Consider now the exact sequence of Theorem 4.1 for $\tilde{V} \to V$:

$$\cdots \to H^3(\tilde{V}, \mathbf{Q}) \to H^3(E, \mathbf{Q}) \to H^4(V, \mathbf{Q}) \to \dots$$

Since $H^3(E, \mathbf{Q}) \to H^4(V, \mathbf{Q})$ is the zero map we obtain that $H^3(\tilde{V}, \mathbf{Q}) \to H^3(E, \mathbf{Q})$ is surjective. □

Theorem 4.3. *Let $Y \subset \mathbf{P}$ be a minimal Weierstrass fibration and let $\pi : X \to S$ be an elliptic threefold, birational to Y. Assume that $H^4(Y, \mathbf{Q})$ has a pure weight 4 Hodge structure. Then*

$$\operatorname{rank} \operatorname{MW}(\pi) = \operatorname{rank} \left(H^{2,2}(H^4(Y, \mathbf{C})) \cap H^4(Y, \mathbf{Z}) \right) - 1$$

and $H^5(Y, \mathbf{Q}) \cong H^5(X, \mathbf{Q})$.

Proof. Since both $\operatorname{rank} \operatorname{MW}(\pi)$ and $H^5(X, \mathbf{Q})$ are birational invariants of smooth fibred threefolds, it suffices to prove this statement for the elliptic threefold $\pi' : X' \to S'$ obtained from Miranda's procedure. Then by the Shioda-Tate-Wazir formula and Lefschetz (1,1) one has

$$\begin{aligned}
\operatorname{rank} \operatorname{MW}(\pi) &= \rho(X') - \rho(S') - f - 1 \\
&= \operatorname{rank} H^2(X', \mathbf{Z}) \cap H^{1,1}(X', \mathbf{C}) - \rho(S') - f - 1 \\
&= \operatorname{rank} H^4(X', \mathbf{Z}) \cap H^{2,2}(X', \mathbf{C}) - \rho(S') - f - 1
\end{aligned}$$

where f is the number of independent fibral divisors, not intersecting the image of the zero section.

Let $\pi_i : X_i \to S_i$ be the associated sequence of modifications. Let f_i denote the number of independent fibral divisors of π_i, not intersecting the zero-section. We will show by induction that for each i we have that $H^4(X_i, \mathbf{Q})$ has a pure weight 4 Hodge structure and that

$$\text{rank}\left(H^{2,2}(H^4(X_i,\mathbf{C})) \cap H^4(X_i,\mathbf{Z})\right) - \rho(S_i) - f_i - 1 \tag{3}$$

is independent of i.

This suffices for the first statement: for the elliptic threefold in the final step of Miranda's construction we have that (3) equals $\operatorname{rank} \mathrm{MW}(\pi)$ by the Shioda-Tate-Wazir formula.

Now consider (3) for $i = 0$. From $S_0 = \mathbf{P}^2$ we get $\rho_0(S_0) = 1$. Since all fibers of π_0 are irreducible, we get $f_0 = 0$. Finally, Theorem 4.1 applied to $X_0 \to Y$ yields an exact sequence of $\mathbf{Q}$-MHS

$$H^3(E,\mathbf{Q}) \to H^4(Y,\mathbf{Q}) \to H^4(X,\mathbf{Q}) \to H^4(E,\mathbf{Q}) \to H^5(X,\mathbf{Q}).$$

Since $E \cong \mathbf{P}^2$ we get $H^3(E,\mathbf{Q}) = 0$ and $H^4(E,\mathbf{Q}) = \mathbf{Q}(-2)$. Also the map $H^4(X,\mathbf{Q}) \to H^4(E,\mathbf{Q})$ is non-zero, hence we get

$$0 \to H^4(Y,\mathbf{Q}) \to H^4(X_0,\mathbf{Q}) \to \mathbf{Q}(-2) \to 0.$$

In particular, $H^4(X_0, \mathbf{Q})$ has a pure weight 4 Hodge structure and

$$\begin{aligned} & \text{rank}\left(H^{2,2}(H^4(X_0,\mathbf{C})) \cap H^4(X_0,\mathbf{Z})\right) - \rho(S_0) - f_0 - 1 \\ = \; & \text{rank}\left(H^{2,2}(H^4(X_0,\mathbf{C})) \cap H^4(X_0,\mathbf{Z})\right) - 2 \\ = \; & \text{rank}\left(H^{2,2}(H^4(Y,\mathbf{C})) \cap H^4(Y,\mathbf{Z})\right) - 1. \end{aligned}$$

To prove that (3) is actually independent of i, we consider each of the three types of modifications mentioned in Miranda's construction separately. In each case we apply Theorem 4.1 several times without mentioning it explicitly:

(1) Consider the first type of modification, i.e. we blow up a point $p \in \Delta \subset S_i$ and then base change. For the proper modification $X_{i+1} \to X_i$ we have that $\mathcal{Z} = C \subset X_i$ is a curve of arithmetic genus 1, i.e., C is either a union of k rational curves, a cuspidal rational curve or a nodal rational curve. In the last two cases we set $k = 1$. Using the universal property of the fiber product we obtain that the exceptional divisor $E \subset X_{i+1}$ is isomorphic to a product $C \times \mathbf{P}^1$. Using our induction hypothesis on $H^4(X_i, \mathbf{Q})$ (i.e., that it is of pure weight 4) and that $H^3(E, \mathbf{Q})$ has no classes of weight ≥ 4 [21, Theorem 5.39], the exact sequence of Theorem 4.1 yields the following exact sequence

$$0 \to H^4(X_i,\mathbf{Q}) \to H^4(X_{i+1},\mathbf{Q}) \to H^4(E,\mathbf{Q}) = \mathbf{Q}(-2)^k.$$

Each of the k irreducible components of $C \times \mathbf{P}^1$ yields a class ξ_j in $H^4(X_{i+1}, \mathbf{Q})$. I.e., we have

$$\mathrm{span}\{\xi_1, \dots, \xi_k\} \subset H^4(X_{i+1}, \mathbf{Q}) \to H^4(E, \mathbf{Q})$$

Clearly $\dim H^4(E, \mathbf{Q}) = k$ and the ξ_j map to a basis of $H^4(E, \mathbf{Q})$. In particular, the ξ_j are independent in $H^4(X_{i+1}, \mathbf{Q})$ and the map $H^4(X_{i+1}, \mathbf{Q}) \to H^4(E, \mathbf{Q})$ is surjective. The conclusion is that

$$\mathrm{rank}\left(H^{2,2}(H^4(X_{i+1}, \mathbf{C})) \cap H^4(X_{i+1}, \mathbf{Z})\right) =$$
$$= k + \mathrm{rank}\left(H^{2,2}(H^4(X_i, \mathbf{C})) \cap H^4(X_i, \mathbf{Z})\right),$$

$f_{i+1} = f_i + k - 1$ and $\rho(S_{i+1}) = \rho(S_i) + 1$, and hence the quantity (3) is unchanged.

(2) The second modification consists of two blow-ups of a curve, the blow-up of a rational surface and two blow-down morphisms. We consider first the blow-up of a curve in X_i, and the blow-up of the curve in X_{i+1}. A reasoning very similar to the previous case yields that $H^4(X_{i+1}, \mathbf{Q})$ and $H^4(X_{i+2}, \mathbf{Q})$ have a pure weight 4 Hodge structure, that classes of type $(2,2)$ are added to $H^4(X_{i+1}, \mathbf{Z})$ and $H^4(X_{i+2}, \mathbf{Z})$ and that $f_{i+2} = f_{i+1} + 1 = f_i + 2$. I.e., the quantity (3) is unchanged.

Consider now the third step, the blow-up of a rational surface. In this case both $\mathcal{Z}$ and E are irreducible surfaces and we have an isomorphism $H^4(\mathcal{Z}, \mathbf{Q}) \to H^4(E, \mathbf{Q})$. Since $H^3(E, \mathbf{Q})$ has Hodge weights at most 3 [21, Theorem 5.39] and $H^4(X_{i+2}, \mathbf{Q})$ has a pure weight 4 Hodge structure, Theorem 4.1 implies that we have an isomorphisms $H^4(X_{i+2}, \mathbf{Q}) \to H^4(X_{i+3}, \mathbf{Q})$. Hence $H^4(X_{i+1}, \mathbf{Q})$ is of pure weight 4 and all entries in (3) remain unchanged.

The final two steps are the contraction of the two ruled surfaces. I.e., $X_{i+3} \to X_{i+4}$ and $X_{i+4} \to X_{i+5}$ are blow-ups of curves. In the previous section it is argued that these curves are smooth and lie in the smooth locus of X_{i+4} and X_{i+5}.

Combining Lemma 4.2 with the exact sequence of Theorem 4.1 yields exact sequences

$$0 \to H^4(X_{i+4}, \mathbf{Q}) \to H^4(X_{i+3}, \mathbf{Q}) \to H^4(E_{i+1}, \mathbf{Q}) \to \dots$$

and

$$0 \to H^4(X_{i+5}, \mathbf{Q}) \to H^4(X_{i+4}, \mathbf{Q}) \to H^4(F_i, \mathbf{Q}) \to \dots$$

(notation as in the previous section.)

In particular, $H^4(X_{i+4}, \mathbf{Q})$ and $H^4(X_{i+5}, \mathbf{Q})$ have pure weight 4 Hodge structures. As above, one can show that the class of E_{i+1} (resp. F_i) in $H^4(X_{i+3}, \mathbf{Q})$ (resp. $H^4(X_{i+4}, \mathbf{Q})$) is mapped to a nonzero element in $H^4(E_{i+1}, \mathbf{Q})$ (resp. $H^4(F_i, \mathbf{Q})$). Hence these maps are surjective, i.e., $H^4(X_{i+5}, \mathbf{Z})$ has rank 1 smaller than $H^4(X_{i+4}, \mathbf{Z})$, and the difference is a class of type $(2,2)$. Similarly,

$H^4(X_{i+4},\mathbf{Z})$ has rank 1 smaller than $H^4(X_{i+3},\mathbf{Z})$, and the difference is a class of type $(2,2)$. Moreover, $f_{i+3}=f_{i+4}+1=f_{i+5}+2$, hence the quantity (3) is unchanged.

(3) The third modification is to blow up a curve C inside $X_{i,\text{sing}}$ such that C_{red} is smooth. The exceptional divisor of such a blow up is not necessarily irreducible, say it has k irreducible components, hence $H^4(E,\mathbf{Q})=\mathbf{Q}(-2)^k$. Each component of E yields a class ξ_j in $H^4(X_{i+1},\mathbf{Q})$ and the same argument as above shows that $H^4(X_{i+1},\mathbf{Q})$ has pure weight 4 and that the classes ξ_j are independent. Hence $f_{i+1}=f_i+k$ and rank $\left(H^{2,2}(H^4(X_i,\mathbf{C}))\cap H^4(X_i,\mathbf{Z})\right)$ increases by k. Since $S_{i+1}=S_i$ we have proved that (3) remains unchanged.

To prove that $H^5(Y,\mathbf{Q})\cong H^5(X,\mathbf{Q})$, note that in all three cases the map $H^4(X_i,\mathbf{Q})\to H^4(E,\mathbf{Q})$ is surjective. Since $h^5(\mathcal{Z},\mathbf{Q})=h^5(E,\mathbf{Q})=0$ it follows from Theorem 4.1 that $H^5(X_i,\mathbf{Q})\cong H^5(X_{i+1},\mathbf{Q})$ for all i. □

Corollary 4.4. *Let $\pi: X\to S$ be an elliptic threefold associated with a hypersurface*

$$y^2=x^3+Px+Q \textit{ or } y^2=x^3+R$$

with $P\in\mathbf{C}[z_0,z_1,z_2]_{4n}$ and $Q\in\mathbf{C}[z_0,z_1,z_2]_{6n}$, such that

(1) *the curve $\Delta: 4P^3+27Q^2=0$ is reduced, Δ has only double points as singularities, and Q vanishes at each of these double points or*
(2) *P is identical zero and $Q=0$ defines a smooth curve of degree $6n$ in $\mathbf{P}^2$.*

Then $\operatorname{rank}\operatorname{MW}(\pi)=0$.

Proof. Using Lefschetz hyperplane Theorem [9, Theorem B22] we obtain that $h^2(Y)=1$. An easy calculation shows that our assumptions on P and Q are equivalent to Y being quasismooth. Then [9, Corollary B19] states that $H^i(Y,\mathbf{Q})$ satisfies Poincaré duality, hence

$$h^4(Y)=h^2(Y)=1$$

and $\operatorname{rank}\operatorname{MW}(\pi)=0$. □

Part 2. Cohomology of hypersurfaces in $\mathbf{P}$

5. Cohomology of hypersurfaces in $\mathbf{P}$: general results

In this section let Y be an irreducible and reduced hypersurface of degree d in some weighted projective space $\mathbf{P}$ of dimension $n+1$ defined by the polynomial g. Let $\Sigma\subset\mathbf{P}$ denote the locus where all the partials of g vanish. We assume that Σ does not intersect $\mathbf{P}_{\text{sing}}$, i.e., Y intersects the singular locus of $\mathbf{P}$ transversally. As usual we set $\dim\emptyset=-1$.

For an arbitrary hypersurface Y the following form of Lefschetz' hyperplane theorem holds:

Proposition 5.1 ([9, Corollary B22]). *We have the following isomorphisms for the cohomology of Y:*

(1) $H^i(Y,\mathbf{Q}) \cong H^i(\mathbf{P},\mathbf{Q})$ *for* $i \leq n-1$.
(2) $H^i(Y,\mathbf{Q}) \cong H^i(\mathbf{P},\mathbf{Q})$ *for* $n+2+\dim\Sigma \leq i \leq 2n$.

In all our applications we have $\dim\Sigma \leq 1$. Our main interest lies in the case where $\Sigma \neq \emptyset$, but we start by discussing what happens in the case $\Sigma = \emptyset$, i.e., we assume for the moment that Y is quasismooth.

We can calculate the cohomology for such Y as follows: from Proposition 5.1 it follows that $H^i(Y,\mathbf{Q}) \cong H^i(\mathbf{P},\mathbf{Q})$ for $i \neq n, 2n+2$. Since $\dim Y = n$ we have that $H^{2n+2}(Y,\mathbf{Q}) = 0$. Hence it remains to calculate $H^n(Y,\mathbf{Q})$. The Poincaré residue map

$$H^{n+1}(\mathbf{P}\setminus Y,\mathbf{C})(1) \to H^n(Y,\mathbf{C})_{\text{prim}}$$

is an isomorphism (see e.g. [27, Section 6.1.1].) The left hand side can be calculated using ideas of Griffiths [12], extended to weighted projective spaces by Steenbrink [24]:

Let $U := \mathbf{P}\setminus Y$. Since U is affine we have that

$$H^k(U,\mathbf{C}) = H^0(U,\Omega^k_U)/dH^0(U,\Omega^{k-1}_U).$$

Note that

$$H^0(U,\Omega^k_U) \cong \cup_{i\geq 0} H^0(\mathbf{P},\Omega^k_{\mathbf{P}}(iY)).$$

For $\omega \in H^0(U,\Omega^k_U)$ define $\operatorname{ord}_Y(\omega) := \min\{i\colon \omega \in H^0(\mathbf{P},\Omega^k_{\mathbf{P}}(iY)\}$. Let $P^\bullet$ be the filtration defined by

$$P^sH^0(U,\Omega^k_U) = \{\omega \in H^0(U,\Omega^k_U)\colon \operatorname{ord}_Y(\omega) \leq k-s+1\}.$$

Since $d(P^sH^0(U,\Omega^{k-1}_U)) \subset P^sH^0(U,\Omega^k_U)$ this induces a filtration $P^\bullet$ on $H^k(U,\mathbf{C})$, called the polar filtration.

Theorem 5.2 (Griffiths-Steenbrink [24, Section 4]). *The Hodge Filtration $F^\bullet$ on $H^{n+1}(U,\mathbf{C})$ coincides with the filtration $P^\bullet$.*

If we drop the assumption that Y is quasismooth then we get the following weaker

Theorem 5.3 (Deligne-Dimca [6]). *For any hypersurface $Y \subset \mathbf{P}$ we have*

$$P^sH^k(U,\mathbf{C}) \supset F^sH^k(U,\mathbf{C}).$$

There exist examples for which both filtrations differ, see [9, Remark 6.1.33], [10].

Remark 5.4. Since $H^{n+1}(U,\mathbf{C}) = F^1H^{n+1}(U,\mathbf{C})$ it follows from the above theorem that $H^{n+1}(U,\mathbf{C}) = P^1H^{n+1}(U,\mathbf{C})$. This implies that every class of $H^{n+1}(U,\mathbf{C})$ has pole order at most $n+1$.

Our main interest lies in the case where $k = n+1 = \dim U$. In this case we can make this more explicit. The de Rham complex with filtration $P^\bullet$ yields a spectral sequence $E_r^{p,q}$. Essentially, Griffiths and Steenbrink show that this spectral sequence degenerates at E_1 in the case that Y is quasismooth. This yields an isomorphism

$$\operatorname{Gr}^p_F H^{n+1}(U,\mathbf{C}) = \operatorname{Gr}^p_P H^{n+1}(U,\mathbf{C}) = \frac{H^0(\mathbf{P},\Omega^{n+1}_{\mathbf{P}}((n+2-p)Y)}{H^0(\mathbf{P},\Omega^{n+1}_{\mathbf{P}}((n+1-p)Y)) + dH^0(\mathbf{P},\Omega^{n}_{\mathbf{P}}((n+1-p)Y))}.$$

Recall that g is a defining polynomial for Y. Let x_i denote the coordinates on $\mathbf{P}$ of weight w_i and let $w=\sum w_i$. Set

$$\Omega := \left(\prod_j x_j\right)\sum(-1)^i w_i \frac{dx_0}{x_0}\wedge\frac{dx_1}{x_1}\wedge\dots\wedge\widehat{\frac{dx_i}{x_i}}\wedge\dots\wedge\frac{dx_{n+1}}{x_{n+1}}.$$

Then $H^0(\mathbf{P},\Omega^{n+1}(kY))$ is generated (as $\mathbf{C}$-vector space) by

$$\omega_f := \frac{f}{g^k}\Omega$$

where $\deg(f)=kd-w$. An easy calculation shows that $\omega_f\in H^0(\mathbf{P},\Omega^{n+1}(k-1)Y)+dH^0(\mathbf{P},\Omega^n(kY))$ if and only if f is in the ideal $(g_{x_0},\dots,g_{x_n})$ generated by the partial derivatives of g. Let $R(g)$ be the Jacobian ring

$$\mathbf{C}[x_0,\dots,x_{n+1}]/(g_{x_0},\dots,g_{x_{n+1}}).$$

Combining this observation with Theorem 5.2 yields:

Proposition 5.5. *We have natural identifications between $R(g)_{id-w}$ and*

$$\operatorname{Gr}^{n+2-i}_P H^{n+1}(U,\mathbf{C})\cong \operatorname{Gr}^{n+2-i}_F H^{n+1}(U,\mathbf{C})\cong \operatorname{Gr}^{n+1-i}_F H^{n}(Y,\mathbf{C})_{\text{prim}}.$$

We will now extend these results in certain cases to the non-quasismooth situation: suppose that $Y\subset\mathbf{P}$ is a hypersurface in a weighted projective space $\mathbf{P}$ of dimension $n+1$ defined by a weighted homogeneous equation $g=0$. Write $Y^*=Y\setminus\Sigma$ and let $\mathbf{P}^*=\mathbf{P}\setminus\Sigma$ where, as before, Σ is defined by the vanishing of the partials of g. Note that, since we have assumed that Y intersects $\mathbf{P}_{\text{sing}}$ transversally, we have $\Sigma\cap\mathbf{P}_{\text{sing}}=\emptyset$. In particular, $U=\mathbf{P}^*\setminus Y^*=\mathbf{P}\setminus Y$.

In generalizing the approach described above we encounter the following problems:

(1) The Poincaré residue map is not an isomorphism.
(2) We can still define the filtered de Rham complex and construct the spectral sequence $E_r^{p,q}$. This sequence, however, does not degenerate at E_1 but at a higher step.
(3) The polar filtration and the Hodge filtration differ.

The following approach is similar to [8], where Dimca studied hypersurfaces with isolated singularities. The exact sequence of the pair (Y,Y^*) reads as

$$\dots\to H^k_\Sigma(Y,\mathbf{Q})\to H^k(Y,\mathbf{Q})\to H^k(Y^*,\mathbf{Q})\to H^{k+1}_\Sigma(Y,\mathbf{Q})\to\dots \tag{4}$$

This is a sequence of Mixed Hodge structures by [21, Proposition 5.47].

From now we on assume that $n = 2$ and $\dim \Sigma \leq 0$ or $n = 3$ and $\dim \Sigma \leq 1$. This will be the case in all our applications. By Proposition 5.1 the only interesting cohomology groups are $H^i(Y, \mathbf{Q})$ for $i = n, n+1, n+2$. We will study these groups by using (4). In this section we focus on the calculation of $H^i(Y^*, \mathbf{Q})$. The calculation of $H^i_\Sigma(Y, \mathbf{Q})$ will then be done in the following sections.

We start by relating the cohomology of Y^* to the cohomology of U and Σ. For this we need the notion of primitive cohomology. If $V \subset \mathbf{P}$ is a quasi-projective subvariety of codimension c, we define $H^i(V, \mathbf{Q})_{\text{prim}}$ to be the kernel of the natural map $H^i(V, \mathbf{Q}) \to H^{i+2c}(\mathbf{P}, \mathbf{Q})(c)$, given by repeated cupping with the hyperplane class.

In the quasismooth case we can relate $H^i(Y^*, \mathbf{C})_{\text{prim}}$ to $H^{i+1}(U, \mathbf{C})$ by using the Poincaré residue map. In the non-quasismooth case this is more subtle.

Proposition 5.6. *We have the following:*

(1) *Suppose* $n = 2$ *and* $\dim \Sigma = 0$*, then*
$$H^2(Y^*, \mathbf{Q})_{\text{prim}} \cong H^3(U, \mathbf{Q})(1);\ H^3(Y^*, \mathbf{Q}) \cong \mathbf{Q}(-2)^{\#\Sigma - 1}$$
$$\text{and } H^4(Y^*, \mathbf{Q}) = 0.$$

(2) *Suppose* $n = 3$ *and* $\dim \Sigma = 0$*, then*
$$H^3(Y^*, \mathbf{Q}) \cong H^4(U, \mathbf{Q})(1);\ H^4(Y^*, \mathbf{Q}) \cong \mathbf{Q}(-2)$$
$$\text{and } H^5(Y^*, \mathbf{Q}) \cong \mathbf{Q}(-3)^{\#\Sigma - 1}.$$

(3) *Suppose* $n = 3$ *and* $\dim \Sigma = 1$*, then*
$$0 \to H^4(U, \mathbf{Q})(1) \to H^3(Y^*, \mathbf{Q}) \to H^2(\Sigma, \mathbf{Q})^*_{\text{prim}}(-3) \to 0$$
is exact. Moreover
$$H^4(Y^*, \mathbf{Q}) \cong H^1(\Sigma, \mathbf{Q})^*(-3) \text{ and } H^5(Y^*, \mathbf{Q}) \cong H^0(\Sigma, \mathbf{Q})^*_{\text{prim}}(-3).$$

Before proving Proposition 5.6 we shall prove some auxiliary results.

Proposition 5.7. *We have a Thom-type isomorphism*
$$T : H^k(Y^*, \mathbf{Q}) \to H^{k+2}(\mathbf{P}^*, U, \mathbf{Q})(1). \tag{5}$$

Proof. The map T is induced by the Thom isomorphism on the (punctured) affine cones over Y^*, $\mathbf{P}^*$ and U. For the precise construction we refer to [8, Section 2]. □

Consider now the long exact sequence of MHS of the pair $(\mathbf{P}^*, U)$:
(6)
$$\ldots \to H^k(\mathbf{P}^*, U, \mathbf{Q}) \xrightarrow{j^*} H^k(\mathbf{P}^*, \mathbf{Q}) \xrightarrow{i^*} H^k(U, \mathbf{Q}) \to H^{k+1}(\mathbf{P}^*, U, \mathbf{Q}) \to \ldots$$

Lemma 5.8. *We have that*
$$H^k(\mathbf{P}^*, U, \mathbf{Q}) \cong H^k(\mathbf{P}^*, \mathbf{Q})$$

for $k > n+2$ *and that*

$$H^k(Y^*, \mathbf{Q}) \cong H^{k+2}(\mathbf{P}^*, \mathbf{Q})(1)$$

for $k > n$.

Proof. Since U is affine we have $H^i(U, \mathbf{Q}) = 0$ for $i \geq n+2$, hence the first isomorphism follows from sequence (6). The second isomorphism follows from the Thom isomorphism combined with the first isomorphism. □

Using that $\mathbf{P}^*$ is a V-manifold we can relate $H^k(\mathbf{P}^*)$ to the cohomology of Σ:

Lemma 5.9. *If* $\dim \Sigma = 0$ *then*

$$H^i(\mathbf{P}^*, \mathbf{Q}) \cong \begin{cases} 0 & \text{for } i = 2n+2 \\ H^0(\Sigma, \mathbf{Q})^*_{\text{prim}}(-n-1) & \text{for } i = 2n+1 \\ H^i(\mathbf{P}, \mathbf{Q}) & \text{for } i < 2n+1 \end{cases}$$

as MHS and if $\dim \Sigma = 1$ *then*

$$H^i(\mathbf{P}^*, \mathbf{Q}) = \begin{cases} 0 & \text{for } i = 2n+2 \\ H^0(\Sigma, \mathbf{Q})^*_{\text{prim}}(-n-1) & \text{for } i = 2n+1 \\ H^1(\Sigma, \mathbf{Q})^*(-n-1) & \text{for } i = 2n \\ H^2(\Sigma, \mathbf{Q})^*_{\text{prim}}(-n-1) & \text{for } i = 2n-1 \\ H^i(\mathbf{P}, \mathbf{Q}) & \text{for } i < 2n-1 \end{cases}$$

as MHS.

Proof. We have the Gysin exact sequence

$$0 \to H^0_c(\mathbf{P}^*, \mathbf{Q}) \to H^0_c(\mathbf{P}, \mathbf{Q}) \to H^0_c(\Sigma, \mathbf{Q}) \to H^1_c(\mathbf{P}^*, \mathbf{Q}) \to \dots$$

Note that $\mathbf{P}$ and Σ are compact. If $\dim \Sigma = 0$ then it follows immediately from the Gysin sequence that

$$H^i_c(\mathbf{P}^*, \mathbf{Q}) = \begin{cases} 0 & i = 0 \\ H^0(\Sigma, \mathbf{Q})_{\text{prim}} & i = 1 \\ H^i(\mathbf{P}, \mathbf{Q}) & i > 1. \end{cases}$$

If $\dim \Sigma = 1$ it follows that

$$H^i_c(\mathbf{P}^*, \mathbf{Q}) = \begin{cases} 0 & i = 0 \\ H^0(\Sigma, \mathbf{Q})_{\text{prim}} & i = 1 \\ H^1(\Sigma, \mathbf{Q}) & i = 2 \\ H^2(\Sigma, \mathbf{Q})_{\text{prim}} & i = 3 \\ H^i(\mathbf{P}, \mathbf{Q}) & i > 3. \end{cases}$$

Since $\mathbf{P}$ is a V-manifold, the same holds for $\mathbf{P}^*$ and we can apply Poincaré duality to obtain the lemma. □

We are now in a position to prove Proposition 5.6.

Proof of Proposition 5.6. Suppose that $n = 2$ and $\dim \Sigma = 0$. Then we have

$$\begin{aligned} H^3(Y^*, \mathbf{Q}) &\cong H^5(\mathbf{P}^*, U, \mathbf{Q})(1) \cong H^5(\mathbf{P}^*, \mathbf{Q})(1) \\ &\cong H^0(\Sigma, \mathbf{Q})_{\text{prim}}(-2)^* \cong \mathbf{Q}(-2)^{\#\Sigma - 1}. \end{aligned}$$

The first isomorphism is the Thom-isomorphism (Proposition 5.7), the second isomorphism comes from Lemma 5.8, the third isomorphism comes from Lemma 5.9 and the fourth isomorphism is immediate. Similarly, one has $H^4(Y^*, \mathbf{Q}) \cong H^6(\mathbf{P}^*, U, \mathbf{Q})(1) = 0$. To calculate $H^2(Y^*, \mathbf{Q})$ consider the long exact sequence (6) of the pair $(\mathbf{P}^*, U)$:

$$\ldots \to H^3(\mathbf{P}^*, \mathbf{Q}) \to H^3(U, \mathbf{Q}) \to H^4(\mathbf{P}^*, U, \mathbf{Q}) \to H^4(\mathbf{P}^*, \mathbf{Q}) \to \ldots.$$

It follows from Lemma 5.9 that $H^3(\mathbf{P}^*, \mathbf{Q}) \cong H^3(\mathbf{P}, \mathbf{Q}) = 0$. From the same lemma it follows that $H^4(\mathbf{P}^*, \mathbf{Q}) \cong H^4(\mathbf{P}, \mathbf{Q})$. Since U is affine and of dimension 3, we have that $H^4(U, \mathbf{Q}) = 0$. Finally, the Thom-isomorphism yields $H^4(\mathbf{P}^*, U, \mathbf{Q}) \cong H^2(Y^*, \mathbf{Q})(-1)$. Combining everything gives

$$0 \to H^3(U, \mathbf{Q}) \to H^2(Y^*, \mathbf{Q})(-1) \to H^4(\mathbf{P}, \mathbf{Q}) \to 0$$

whence $H^3(U, \mathbf{Q})(1) \cong H^2(Y^*, \mathbf{Q})_{\text{prim}}$.

In the case $n = 3$ we can proceed similarly: combining the Thom isomorphism with Lemmas 5.8 and 5.9 yields the following isomorphisms:

$$H^5(Y^*, \mathbf{Q}) \cong H^7(\mathbf{P}^*, \mathbf{Q})(1) \cong H^0(\Sigma, \mathbf{Q})^*_{\text{prim}}(-3).$$

If $\dim \Sigma = 0$ then

$$H^4(Y^*, \mathbf{Q}) \cong H^6(\mathbf{P}^*, \mathbf{Q})(1) \cong H^6(\mathbf{P}, \mathbf{Q})(1) = \mathbf{Q}(-2)$$

and if $\dim \Sigma = 1$ then

$$H^4(Y^*, \mathbf{Q}) \cong H^6(\mathbf{P}^*, \mathbf{Q})(1) \cong H^1(\Sigma, \mathbf{Q})^*(-3).$$

The calculation of $H^3(Y^*, \mathbf{Q})$ is slightly more complicated. We have an exact sequence

$$H^4(\mathbf{P}^*, \mathbf{Q}) \to H^4(U, \mathbf{Q}) \to H^5(\mathbf{P}^*, U, \mathbf{Q}) \to H^5(\mathbf{P}^*, \mathbf{Q}) \to H^5(U, \mathbf{Q}) = 0.$$

From Lemma 5.9 it follows that $H^5(\mathbf{P}^*, \mathbf{Q}) \cong H^2(\Sigma, \mathbf{Q})^*_{\text{prim}}(-3)$. From the same lemma it follows that $H^4(\mathbf{P}^*, \mathbf{Q}) \cong H^4(\mathbf{P}, \mathbf{Q})$. Since $H^4(\mathbf{P}, \mathbf{Q}) \to H^4(U, \mathbf{Q})$ is the zero-map, we obtain, after applying the Thom-isomorphism, the following short exact sequence

$$0 \to H^4(U, \mathbf{Q})(1) \to H^3(Y^*, \mathbf{Q}) \to H^2(\Sigma, \mathbf{Q})^*_{\text{prim}}(-3) \to 0.$$

To finish the proof, note that if $\dim \Sigma = 0$ then $H^0(\Sigma, \mathbf{Q})_{\text{prim}} = \mathbf{Q}^{\#\Sigma - 1}$ and $H^2(\Sigma, \mathbf{Q})_{\text{prim}} = 0$. In particular, $H^4(U, \mathbf{Q})(1) \cong H^3(Y^*, \mathbf{Q})$ in this case. □

Remark 5.10. Later on we will show that the contribution of $H^\bullet(\Sigma, \mathbf{Q})$ to $H^\bullet(Y^*, \mathbf{Q})$ is irrelevant for the calculation of $H^4(Y, \mathbf{Q})$.

Remark 5.11. To finish our analysis of $H^n(Y^*,\mathbf{Q})$ we give a set of generators for $H^{n+1}(U,\mathbf{C})$. Recall that we have the pole order filtration on $\Omega_U^\bullet$, inducing a filtration on $H^i(U,\mathbf{C})$.

As explained above, the pole filtration on the de Rham complex yields a spectral sequence. Remark 5.4 implies that $P^1H^{n+1}(U,\mathbf{C}) = H^{n+1}(U,\mathbf{C})$. From this it follows easily that

$$\oplus_{p=0}^{n+1} E_1^{n+1-p,p} \to H^{n+1}(U,\mathbf{C})$$

is surjective. An easy calculation (the same as in the quasismooth case) shows that

$$\oplus_{p=0}^{n+1} E_1^{n+1-p,p} = \oplus_{k=1}^{n+1} R(g)_{dk-w}.$$

The right hand side is finite dimensional and generates $H^{n+1}(U,\mathbf{C})$. Moreover, the direct sum decomposition is the same as the direct sum decomposition with respect to the graded pieces of the polar filtration.

A summary of our results is the following:

Proposition 5.12. *Suppose $n=3$. Let C be the cokernel of $H^4(U,\mathbf{Q}) \to H^4_\Sigma(Y,\mathbf{Q})$. Suppose C is a pure weight 4 Hodge structure, with trivial $(4,0)$ and $(0,4)$-part. Then the cokernel of*

$$\psi_1 : R_{d-w}(g) \to H^4_\Sigma(Y,\mathbf{C})$$

contains $F^3C_\mathbf{C}$. The cokernel of

$$\psi_2 : R_{2d-w}(g) \oplus R_{d-w}(g) \to F^2H^4_\Sigma(Y,\mathbf{C})$$

contains $F^2C_\mathbf{C}$. Moreover, if ψ_1 is surjective, then C has a pure $(2,2)$-Hodge structure with

$$\dim C = \dim \operatorname{coker}(R_{2d-w}(g) \to H^4_\Sigma(Y^*,\mathbf{C})).$$

Proof. Since $P^4H^4(U,\mathbf{C})$ consists of forms of pole order 0, we have that $P^4H^4(U,\mathbf{C})$ and $H^0(\mathbf{P},\Omega^4_\mathbf{P})$ are isomorphic. Since this group vanishes we have that $P^4H^4(U,\mathbf{C}) = 0$. Since $F^3H^4(U,\mathbf{C}) \subset P^3H^4(U,\mathbf{C})$ (by Theorem 5.3) it follows that

$$P^3H^4(U,\mathbf{C}) = \operatorname{Gr}^3_P H^4(U,\mathbf{C}) \to \operatorname{Gr}^3_F H^4(U,\mathbf{C})$$

is surjective. Since $R_{d-w}(g)$ surjects onto $P^3H^4(U,\mathbf{C})$ we obtain that $h^{3,1}(C)$ equals the dimension of the cokernel of

$$R_{d-w}(g) \to \operatorname{Gr}^3_F H^4_\Sigma(Y,\mathbf{C}).$$

Similarly one obtains that $h^{3,1}(C) + h^{2,2}(C)$ equals the dimension of the cokernel

$$R_{d-w}(g) \oplus R_{2d-w}(g) \to F^2H^4_\Sigma(Y,\mathbf{C}).$$

Finally, if ψ_1 is surjective then $0 = h^{3,1}(C) = h^{1,3}(C)$. Hence C is of pure type $(2,2)$ and

$$\begin{aligned}\dim C_\mathbf{C} = \dim \operatorname{Gr}^2_F C_\mathbf{C} &= \dim \operatorname{coker}(R_{2d-w}(g) \to \operatorname{Gr}^2_F H^4_\Sigma(Y,\mathbf{C}))\\ &= \dim \operatorname{coker}(R_{2d-w}(g) \to H^4_\Sigma(Y,\mathbf{C})).\end{aligned}$$

□

Remark 5.13. The above proof could be slightly simplified if $P^\bullet = F^\bullet$. However, there exist degree 5 surfaces in $\mathbf{P}^4$ with one singularity, namely an ordinary double point, such that $F^\bullet \neq P^\bullet$. See [10].

6. Cohomology of a surface with isolated ADE-singularities

Let $S \subset \mathbf{P}$ be a surface in a 3-dimensional weighted projective space given by an equation $g = 0$, such that the set Σ, the locus where all partials of g vanish, is finite and all singularities of S at points of Σ are of type A_k, D_m or E_n. As usual we set $S^* = S \setminus \Sigma$. We want to calculate $H^2(S, \mathbf{Q})_{\text{prim}}$ and for this reason compare it to a quasismooth surface $\tilde{S}$ of the same degree as S.

Lemma 6.1. *Let μ be the total Milnor number of S. We have that $H^i(S, \mathbf{Q})$ has a pure Hodge structure of weight i and*

$$h^{p,q}(S) = \begin{cases} h^{p,q}(\tilde{S}) & \text{if } (p,q) \neq (1,1) \\ h^{1,1}(\tilde{S}) - \mu & \text{if } (p,q) = (1,1). \end{cases}$$

Proof. We first remark that the statement follows from the Lefschetz Hyperplane Theorem 5.1 for all $p + q \neq 2, 3$.

Consider the long exact sequence of the pair (S, S^*)

$$\begin{aligned} \dots \quad \to \quad & H^3_\Sigma(S, \mathbf{Q}) \to H^3(S, \mathbf{Q}) \to H^3(S^*, \mathbf{Q}) \\ \to \quad & H^4_\Sigma(S, \mathbf{Q}) \to H^4(S, \mathbf{Q}) \to H^4(S^*, \mathbf{Q}) \to \dots \end{aligned}$$

from e.g. [8, Example 1.9] it follows that $H^3_\Sigma(S, \mathbf{Q}) = 0$. For each $p \in \Sigma$ we have that (S, p) is given locally by a weighted homogeneous equation. In particular, we can find a small neighborhood X of p such that X is a cone over a projective curve, and $X^* = X \setminus \{p\}$ is a $\mathbf{C}^*$-bundle over this curve. It follows directly from the Leray-spectral sequence that $H^3(X^*, \mathbf{Q}) = H^1(\mathbf{C}^*, \mathbf{Q}) \otimes H^2(X, \mathbf{Q}) = H^2(X^*, \mathbf{Q})(-1)$. From the long exact sequence of the pair (X, X^*) and the fact that X is contractible it follows that $H^4_p(S, \mathbf{Q}) = H^4_p(X, \mathbf{Q}) = H^3(X^*, \mathbf{Q}) = \mathbf{Q}(-2)$.

Using Proposition 5.6 the above exact sequence simplifies to

$$0 \to H^3(S, \mathbf{Q}) \to \mathbf{Q}(-2)^{\#\Sigma - 1} \to \mathbf{Q}(-2)^{\#\Sigma} \to \mathbf{Q}(-2) \to 0.$$

In particular, $H^3(S, \mathbf{Q}) = 0$. The same argument with $\Sigma = \emptyset$ also shows $H^3(\tilde{S}, \mathbf{Q}) = 0$. It remains to show that $H^2(S, \mathbf{Q})$ has a pure Hodge structure and to determine the Hodge numbers of $H^2(S, \mathbf{Q})$.

Let S' be a minimal resolution of the singularities of S that are contained in Σ. The exceptional locus E consist of a union of smooth rational curves. Each connected component has an intersection matrix of type ADE. We want to apply Theorem 4.1 with $\mathcal{Z} = \Sigma$ and exceptional locus E. Since the singularities are rational we have $h^1(E, \mathbf{Q}) = 0$. In particular, $H^2(S, \mathbf{Q}) \hookrightarrow H^2(S', \mathbf{Q})$. Since $H^2(S', \mathbf{Q})$ has pure weight 2 Hodge structure the same holds for $H^2(S, \mathbf{Q})$.

Again using that S has rational singularities it follows that $h^{2,0}(S) = h^{2,0}(\tilde{S})$ and $h^{0,2}(S) = h^{0,2}(\tilde{S})$ (see e.g., [25, Introduction]). Since $e(S) = e(\tilde{S}) - \mu$ (e.g., by [9, Corollary 5.4.4]), the lemma follows. □

As argued in Section 5, we can express the Hodge numbers of $\tilde{S}$ in terms of the Jacobian ideal of $\tilde{g}$, where $\tilde{g}$ is an equation for $\tilde{S}$. Let $d = \deg(\tilde{g})$ and $w = \sum w_i$. Let $R(\tilde{g})$ be the Jacobian ring of $\tilde{g}$. Then $h^{2,0}(\tilde{S}) = h^{0,2}(\tilde{S}) = \dim R(\tilde{g})_{d-w} = \dim R(\tilde{g})_{3d-w}$ and $h^{1,1}(\tilde{S}) = \dim R(\tilde{g})_{2d-w}$.

We want to calculate $H^2(S, \mathbf{C})$ together with the Hodge filtration. From Proposition 5.6 it follows that $H^3(U, \mathbf{C})(1) \cong H^2(S, \mathbf{C})_{\text{prim}}$. In [25] it is proven that the Hodge and polar filtration coincide in this case.

Let g be an equation for S and let $R(g)$ be Jacobian Ring of S. Then we have surjections

$$R(g)_{d-w} \to H^{2,0}(S, \mathbf{C}),\ R(g)_{3d-w} \to H^{0,2}(S, \mathbf{C})$$

and

$$R(g)_{2d-w} \to H^{1,1}(S, \mathbf{C})_{\text{prim}}$$

(cf. the results in Section 5, in particular, Remark 5.11).

In [25] this statement is made more precise. For each singularity (S, p) let g_p be a local equation and let $R(g_p)$ be the Jacobian ring of g_p. Note that $R(g_p)$ is naturally isomorphic to the Milnor algebra of (S, p). Let $\pi_p : R(g) \to R(g_p)$ be the natural projection. Then

Theorem 6.2 (Steenbrink [25])**.** *The Poincaré residue map induces the following isomorphisms*

$$H^{2,0}(S, \mathbf{C}) \cong R_{d-w}(g)$$

and

$$H^{1,1}(S, \mathbf{C})_{\text{prim}} \cong \{f \in R_{2d-w}(g) \colon f \in \ker(\pi_p)\ \forall p \in \Sigma\}.$$

Proof. This is a reformulation of the main result of [25]. We show how this statement can be obtained from the result in [25]. In the introduction of [25] it is argued that $H^{2,0}(S) \cong R_{d-w}(g)$. In Section 5 of [25] it is moreover shown that $\dim R_{2d-w}(g) = \dim R_{2d-w}(\tilde{g})(= h^{1,1}(\tilde{S})_{\text{prim}})$. As argued in Section 5 the map

$$R_{2d-w}(g) \to H^{1,1}(S)_{\text{prim}}$$

is surjective. Using these two facts and $h^{1,1}(S) = h^{1,1}(\tilde{S}) - \mu$ we get that the kernel of

$$R_{2d-w}(g) \to H^{1,1}(S, \mathbf{C})_{\text{prim}}$$

has dimension μ.

We will now construct a section to this map. Let $j : S \setminus \Sigma \to S$ be the inclusion. Let $\tilde{\Omega}^p_S = j_* \Omega^p_{S \setminus \Sigma}$ and let $\mathcal{T}$ be the cokernel of $d : \Omega^1(S) \to \Omega^2(2S)$. Then $\mathcal{T}$ is a skyscraper sheaf supported at Σ. At each $p \in \Sigma$ we have that the stalk $\mathcal{T}_p$ is isomorphic to the Tjurina algebra of (S, p), which is by definition isomorphic to $R(g_p)$. Since S has only ADE singularities we have for each

$p \in \Sigma$ that the Milnor algebra and the Tjurina algebra of (S,p) coincide, in particular, $h^0(S,\mathcal{T}_p) = \mu$.

Consider the exact sequence (from [25, Corollary 17])

$$0 \to H^1(S,\tilde{\Omega}^1_S)_{\text{prim}} \to R_{2d-w}(g) \to H^0(S,\mathcal{T}) \to H^2(S,\tilde{\Omega}^1_S) \to 0.$$

As argued in [25] we have that $H^2(S,\tilde{\Omega}^1_S) \subset H^3(S,\mathbf{C}) = 0$.

Hence this exact sequence reduces to

$$0 \to H^1(S,\tilde{\Omega}^1_S)_{\text{prim}} \to R_{2d-w}(g) \to \oplus_{p\in\Sigma} R(g_p) \to 0.$$

In [25] it is then argued that $H^1(S,\tilde{\Omega}^1_S) = H^{1,1}(S,\mathbf{C})$. Hence the above map provides the desired section. (The fact that $H^{1,1}(S) \to R_{2d-w}(g) \to H^{1,1}(S)$ is actually the identity follows from the construction of the first map in [25].) □

Remark 6.3. Steenbrink's point of view is different from the approach taken by Dimca. In the previous section we constructed a surjection from $R_{2d-w}(g)$ onto $H^{1,1}(S,\mathbf{C})$, whereas Steenbrink constructs an injection from $H^{1,1}(S,\mathbf{C})$ to $R_{2d-w}(g)$, which is a section of the former map.

To unite the two approaches we can do the following. Let μ be the total Milnor number of S. Fix μ polynomials $h_1,\dots,h_\mu$ of degree $2d-w$ such that their image spans $\oplus_{p\in\Sigma} R(g_p)$. Set $\tilde{R}(g) := R(g)/(h_1,\dots,h_\mu)$. Then $H^{2,0}(Y,\mathbf{C}) \cong \tilde{R}_{d-w}(g)$ and $H^{1,1}(Y,\mathbf{C}) \cong \tilde{R}_{2d-w}(g)$.

Remark 6.4. Suppose $p \in \Sigma$ has a non-trivial stabilizer group, i.e., $\tilde{p} := (x_0,x_1,x_2,x_3)$ is a lift of p to $\mathbf{C}^4$ and the stabilizer subgroup $G_p \subset \mathbf{C}^*$ of $\tilde{p}$ is non-trivial.

Without loss of generality we can assume that $\tilde{p} = (1,\alpha,0,0)$. Suppose $f(x_0,x_1,x_2,x_3)$ is a defining polynomial for S. Let $g(x_1,x_2,x_3) = f(1,x_1+\alpha,x_2,x_3)$. If G_p consists of one element then the Milnor algebra of (S,p) equals $\mathbf{C}\{x_1,x_2,x_3\}/(g_{x_1},g_{x_2},g_{x_3})$. However, if $\#G_p > 1$ then the Milnor algebra of (S,p) equals

$$\left(\mathbf{C}\{x_1,x_2,x_3\}/(g_{x_1},g_{x_2},g_{x_3})\right)^{G_p}.$$

7. Calculation of $H^4_\Sigma(Y,\mathbf{C})$, local information

In this and the following section we assume that Y is an admissible hypersurface in a weighted projective space $\mathbf{P}(w_0,\dots,w_4)$ (cf. the Introduction) given by $f=0$. Let $\Sigma \subset \mathbf{P}(w_0,\dots,w_4)$ be the locus where all partials of f vanish.

Since Y is admissible we can find for every $p \in \Sigma$ a weighted homogeneous polynomial g_p (with weights $w_{1,p}$, $w_{2,p}$, $w_{3,p}$, $w_{4,p}$ and degree d_p) such that

1. (Y,p) is contact equivalent to $(\{g_p=0\},0) \subset (\mathbf{C}^4,0)$;
2. the surface $S := \{g_p = 0\} \subset \mathbf{P}(w_{1,p},w_{2,p},w_{3,p},w_{4,p})$ has finitely many ADE-singularities.

Remark 7.1. The conditions on the singularities of Y are very mild. For example in the case of elliptic threefolds we considered hypersurfaces of the form $y^2 = x^3 + Px + Q$, with $(P, Q) \in \mathbf{C}[z_0, z_1, z_2]_{4n} \times \mathbf{C}[z_0, z_1, z_2]_{6n}$. For fixed n the locus where the conditions on the singularities are not satisfied has a large codimension. E.g., in the isolated singularity case the most frequently occuring singularities such as ADE threefold singularities are all weighted homogeneous singularities.

Remark 7.2. Recall that two singularities $(\{f_1 = 0\}, 0)$ and $(\{f_2 = 0\}, 0)$ are contact equivalent if and only if

$$\mathbf{C}\{x_1, \ldots, x_n\}/(f_1) \cong \mathbf{C}\{x_1, \ldots, x_n\}/(f_2).$$

If f_1 (and f_2) are isolated singularities then f_1 and f_2 are contact equivalent if and only if their Milnor algebras are isomorphic. If we assume that f_1 is weighted homogeneous then, by the Euler formula, we get $f_1 + J(f_1) = J(f_1)$, hence the Tjurina algebra and the Milnor algebra of f_1 are isomorphic.

It turns out that if f_2 is isolated and contact equivalent to a weighted homogeneous singularity f_1 then it is also right equivalent to f_1, and hence the Tjurina algebra of f_2 is isomorphic to the Tjurina algebra of f_1. This implies that in the isolated case we could reword our condition on (Y, p) by saying that the Milnor number and the Tjurina number of (Y, p) coincide. (Details of this reasoning can be found in [7, Theorem 7.42] and [14, Section 9.1].)

For non-isolated singularities we are not aware of such a simple reformulation.

Remark 7.3. Note that the surface S satisfies the hypothesis of the previous section. We define $S^* = S \setminus \Sigma_p$ where Σ_p is the locus where all the partials of g_p vanish. Let $X \subset \mathbf{C}^4$ be the zero set of g_p, i.e. the affine cone over the surface S.

Lemma 7.4.

$$H^i_p(Y, \mathbf{Q}) \cong H^i_0(X, \mathbf{Q}).$$

Proof. This follows directly from the definition of contact equivalence. □

Let Σ' be the singular locus of X and set $X^* = X \setminus \{0\}$. In this section we relate $H^\bullet_0(X, \mathbf{Q})$ to $H^\bullet(S, \mathbf{Q})$.

Lemma 7.5. *For $i > 1$ we have isomorphisms*

$$H^i_0(X, \mathbf{Q}) \cong H^{i-1}(X^*, \mathbf{Q}).$$

Moreover,

$$H^i_0(X, \mathbf{Q}) = 0$$

for $i = 0, 1$.

Proof. Since X is the affine cone over $S \subset \mathbf{P}(w_{1,p}, w_{2,p}, w_{3,p}, w_{4,p})$ it is contractible and hence $H^i(X, \mathbf{Q}) = 0$ for $i > 0$. The long exact sequence of the pair (X, X^*) therefore yields an isomorphism

$$H^i_0(X, \mathbf{Q}) \cong H^{i-1}(X^*, \mathbf{Q})$$

for $i > 1$. Clearly, the natural map

$$H^0(X, \mathbf{Q}) \to H^0(X^*, \mathbf{Q})$$

is an isomorphism. Since $H^1(X, \mathbf{Q}) = 0$ the same sequence gives that both $H^0_0(X, \mathbf{Q})$ and $H^1_0(X, \mathbf{Q})$ vanish. □

The cone X^* is a $\mathbf{C}^*$-fibration over S. Recall from Section 6 that $H^i(S, \mathbf{Q})$ vanishes unless $i = 0, 2, 4$ and that $H^0(S, \mathbf{Q}) = \mathbf{Q}$, $H^4(S) = \mathbf{Q}(-2)$. The Hodge structure on $H^2(S, \mathbf{Q})$ can be calculated by Theorem 6.2. This enables us to calculate the Hodge structure of $H^\bullet_0(X, \mathbf{Q})$.

Proposition 7.6. *We have that*

$$H^i_0(X, \mathbf{Q}) = \begin{cases} H^2(S, \mathbf{Q})_{\text{prim}} & \textit{for } i = 3 \\ H^2(S, \mathbf{Q})_{\text{prim}}(-1) & \textit{for } i = 4 \\ \mathbf{Q}(-3) & \textit{for } i = 6 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Consider the E_2 part of the Leray spectral sequence for $X^* \to S$:

$H^1(\mathbf{C}^*, \mathbf{Q})$	$\mathbf{Q}(-1)$	0	$H^2(S, \mathbf{Q})(-1)$	0	$\mathbf{Q}(-3)$
$H^0(\mathbf{C}^*, \mathbf{Q})$	$\mathbf{Q}$	0	$H^2(S, \mathbf{Q})$	0	$\mathbf{Q}(-2)$
	$H^0(S, \mathbf{Q})$	$H^1(S, \mathbf{Q})$	$H^2(S, \mathbf{Q})$	$H^3(S, \mathbf{Q})$	$H^4(S, \mathbf{Q})$

The only possible non-zero differentials are the maps $\mathbf{Q}(-1) \to H^2(S, \mathbf{Q})$ and $H^2(S, \mathbf{Q})(-1) \to \mathbf{Q}(-2)$. We will show below that these maps are actually injective, respectively surjective. Assuming this for the moment it follows that the E_3-terms equals

$H^1(\mathbf{C}^*)$	0	0	$H^2(S, \mathbf{Q})_{\text{prim}}(-1)$	0	$\mathbf{Q}(-3)$
$H^0(\mathbf{C}^*)$	$\mathbf{Q}$	0	$H^2(S, \mathbf{Q})_{\text{prim}}$	0	0
	$H^0(S)$	$H^1(S)$	$H^2(S)$	$H^3(S)$	$H^4(S)$

and the spectral sequence degenerates at E_3. Hence $H^i(X^*, \mathbf{Q}) \cong \oplus_j E_3^{i-j,j}$ and thus

$$H^i(X^*, \mathbf{Q}) = \begin{cases} \mathbf{Q} & \text{for } i = 0 \\ 0 & \text{for } i = 1 \\ H^2(S, \mathbf{Q})_{\text{prim}} & \text{for } i = 2 \\ H^2(S, \mathbf{Q})_{\text{prim}}(-1) & \text{for } i = 3 \\ 0 & \text{for } i = 4 \\ \mathbf{Q}(-3) & \text{for } i = 5. \end{cases}$$

By Lemma 7.5 we have $H^i_0(X, \mathbf{Q}) = H^{i-1}(X^*, \mathbf{Q})$ for $i > 1$ and thus we obtain the proposition.

It remains to show that the differential $\mathbf{Q}(-1) \to H^2(S, \mathbf{Q})$ is injective and that the differential $H^2(S, \mathbf{Q})(-1) \to \mathbf{Q}(-2)$ is surjective.

Let $\tilde{X}$ be the blow-up of X at 0. Then $\tilde{X}$ is a $\mathbf{C}$-fibration over S. Note that S admits Poincaré duality (a consequence of Lemma 6.1). Using that $H^i_c(\mathbf{C}^*,\mathbf{Z})=0$ for $i\neq 1$ it follows that the Leray-Spectral sequence (for cohomology with compact support) associated with $\tilde{X}\to S$ degenerates at E_2 and we get that $H^{6-i}_c(\tilde{X},\mathbf{Q})\cong H^i(S,\mathbf{Q})(-1)$. Similarly, we get that $H^i(\tilde{X},\mathbf{Q})=H^i(S,\mathbf{Q})$.

Let $E\subset\tilde{X}$ be the exceptional divisor. Then $E\cong S$ and $\tilde{X}\setminus E=X^*$. Consider the following part of the Gysin exact sequence:

$$\begin{aligned} H^1(E,\mathbf{Q})=0 \quad &\to \quad H^2_c(X^*,\mathbf{Q})\to H^2_c(\tilde{X},\mathbf{Q})\to H^2(E,\mathbf{Q}) \\ &\to \quad H^3_c(X^*,\mathbf{Q})\to H^3_c(\tilde{X},\mathbf{Q})=0. \end{aligned}$$

The map $H^2_c(\tilde{X},\mathbf{Q})\to H^2(E,\mathbf{Q})$ is induced by a map from integral cohomology. Let $h\in H^2(E,\mathbf{Z})$ be the hyperplane class. From the Leray spectral sequence it follows that $H^2_c(\tilde{X},\mathbf{Z})=H^0(E,\mathbf{Z})\otimes H^2_c(\mathbf{C},\mathbf{Z})$. Let $h_1\in H^2_c(\tilde{X},\mathbf{Z})$ be $[E]$ times a generator of $H^2_c(\mathbf{C},\mathbf{Z})$. Let $\iota: E\to\tilde{X}$ be the inclusion. Then it is easy to see that $\iota^*(h_1)=-h$. Hence the map ι^* is not constant and since $h^2_c(\tilde{X},\mathbf{Q})=h^4(S,\mathbf{Q})=1$ it follows that ι^* is injective. From the Gysin exact sequence it follows that $H^2_c(X^*,\mathbf{Q})=0$ and that $h^3_c(X^*)=h^2(E)-1$. Assume for the moment that X^* is smooth, i.e., E is quasismooth. Using Poincaré duality we get that $h^3(X^*)=h^2(E)-1$. Since $H^3(X^*,\mathbf{Q})$ equals

$$\ker(H^2(E,\mathbf{Q})(-1)\to H^4(\tilde{X},\mathbf{Q}))=\ker(H^2(S,\mathbf{Q})(-1)\to\mathbf{Q}(-2))$$

it follows that the differential $H^2(S,\mathbf{Q})(-1)\to\mathbf{Q}(-2)$ is surjective.

For the other differential we can proceed similarly:

$$\begin{aligned} H^3(E,\mathbf{Q})=0 \quad &\to \quad H^4_c(X^*,\mathbf{Q})\to H^4_c(\tilde{X},\mathbf{Q})\to H^4(E,\mathbf{Q}) \\ &\to \quad H^5_c(X^*,\mathbf{Q})\to H^5_c(\tilde{X},\mathbf{Q})=0. \end{aligned}$$

The map $H^4_c(\tilde{X},\mathbf{Q})\to H^4(E,\mathbf{Q})$ is again induced by a map on integral cohomology, and the class of h times a generator of $H^2_c(\mathbf{C},\mathbf{Z})$ is mapped to a nonzero multiple of a generator of $H^4(E,\mathbf{Z})$. This implies that $h^4_c(X^*)=h^4_c(\tilde{X})-h^4(E)=h^2(E)-1$. Using Poincaré duality we get that the differential $\mathbf{Q}(-1)\to H^2(S,\mathbf{Q})$ is injective, provided that S is quasismooth.

If S is not quasismooth then we can find a family of quasismooth surfaces S_λ degenerating to S for $\lambda=0$. Now for $\lambda\neq 0$, we have that the differential

$$\mathbf{Q}(-1)\to H^2(S_\lambda,\mathbf{Q})$$

is induced by a non-zero map $H^2(\tilde{X}_\lambda,\mathbf{Z})\to H^2(E_\lambda,\mathbf{Z})$. Let h_λ be a family of generators of $H^2(E_\lambda,\mathbf{Z})$ and let h'_λ be a family of generators of $H^2(\tilde{X}_\lambda,\mathbf{Z})$. Then h'_λ is mapped to $-h_\lambda$. By taking the limit $\lambda\to 0$, we see that h'_0 is mapped to $-h_0$, hence $H^2(\tilde{X}_0,\mathbf{Q})\to H^2(E,\mathbf{Q})$ is injective, and from this it follows that $\mathbf{Q}(-1)\to H^2(S,\mathbf{Q})$ is injective. A similar argument shows that also $H^2(S,\mathbf{Q})\to\mathbf{Q}(-2)$ is surjective. This finishes the proof. □

The following proposition will be useful for our purposes

Proposition 7.7. *Let Y, p, d_p be as above. Let $w_p = w_{1,p}+w_{2,p}+w_{3,p}+w_{4,p}$. Then $H^4_p(Y,\mathbf{Q})$ has a pure weight 4 Hodge structure without $(0,4)$ and $(4,0)$-component. We have*

$$F^3H^4_p(Y,\mathbf{C}) \cong \tilde{R}_{d_p-w_p}(g_p)$$

and

$$F^2H^4_p(Y,\mathbf{C})/F^3H^4_p(Y,\mathbf{C}) \cong \tilde{R}_{2d_p-w_p}(g_p)$$

where $\tilde{R}$ is obtained from R as explained in Remark 6.3.

Proof. This is a combination of Lemma 7.4, Proposition 7.6 and Theorem 6.2. □

Proposition 7.8. *Let (Y, p) be a transversal ADE surface singularity. Then $H^6_p(Y,\mathbf{Q}) = \mathbf{Q}(-3)$ and $H^i_p(Y) = 0$ for $i \neq 6$.*

Proof. For simplicity we assume that (Y, p) is an A_k-singularity. Using Lemma 7.4 it suffices to prove the statement for (Y, p) given by

$$x_1^2 + x_2^2 + x_3^{k+1} = 0.$$

This equation defines a surface $S \subset \mathbf{P}(k+1, k+1, 2, 1)$ of degree $2k+2$ with an isolated A_k singularity in $(0 : 0 : 0 : 1)$.

From Lemma 7.4 and Proposition 7.6 it follows that it suffices to prove that $H^2(S,\mathbf{Q})_{\text{prim}} = 0$. We start by calculating $h^2(\tilde{S})$ for a quasismooth surface $\tilde{S}$ of the same degree, e.g., $\tilde{g} := x_1^2 + x_2^2 + x_3^{k+1} + x_4^{2k+2} = 0$. This can be done by calculating the dimension of several graded pieces of the Jacobian ring of $\tilde{Y}$. The sum of the weights equals $2k + 5$, hence we are interested in $h^{2,0}(\tilde{S}) = \dim R(\tilde{g})_{-3} = 0$, $h^{0,2}(\tilde{S}) = R(\tilde{g})_{4k+1} = 0$ and

$$h^{1,1}(\tilde{S}) = \dim R(\tilde{g})_{2k-1} = \dim \operatorname{span}\{[x_3^i x_4^j] \colon 2i + j = 2k - 1\} = k.$$

Hence $h^2(\tilde{S})_{\text{prim}} = k$. Since $\mu(Y, p) = k$, we get $h^2(S)_{\text{prim}} = h^2(\tilde{S})_{\text{prim}} - \mu(Y, p) = 0$. This finishes the A_k case.

For D_m, E_n singularities one can proceed similarly. □

8. Glueing local information

Let $\mathbf{P}$ be a four dimensional weighted projective space and let $Y \subset \mathbf{P}$ be a hypersurface, given by $f = 0$. Let Σ be the locus where all the partials of f vanish. We assume the usual conditions, i.e., $\Sigma \cap \mathbf{P}_{\text{sing}} = \emptyset$, $\dim \Sigma \leq 1$ and that at a general point of any one dimensional component of Σ we have a transversal ADE surface singularity. Finally, let $\mathcal{P} \subset \Sigma$ be the set of points $p \in \Sigma$ such that (Y, p) is not a transversal ADE surface singularity.

We want to use the previous section to relate $H^4(Y,\mathbf{Q})_{\text{prim}}$ to the cokernel of $H^4(U,\mathbf{Q})(1) \to \oplus_{p\in\mathcal{P}} H^4_p(Y,\mathbf{Q})$. In this section all considerations are topological. For this reason we work with $\mathbf{Q}$ coefficients and use $H^i(\cdot)$ as shorthand for $H^i(\cdot,\mathbf{Q})$.

For each point $p \in \mathcal{P}$, fix a small contractible neighborhood $U_p \subset \Sigma$. Let $\Sigma_1 := \Sigma \setminus \cup_{p\in\mathcal{P}} U_p$ be the complement of the U_p. Note that Σ_1 is a closed Riemann surface with boundary embedded in $\mathbf{P}$.

Lemma 8.1. *We have that*

$$H^4_{\Sigma_1}(Y) \cong H^2(\Sigma_1)^*(-3),\ H^5_{\Sigma_1}(Y) \cong H^1(\Sigma_1)^*(-3)$$

and

$$H^6_{\Sigma_1}(Y) \cong H^0(\Sigma_1)^*(-3).$$

Proof. Take a finite open covering $\mathcal{U} := \{V_i\}$ of Σ_1 such that each V_i is homeomorphic to a disc with boundary S^1, in particular each V_i is contractible. Let $D_i = \overline{V_i}$ be the closure in the complex topology. It is easy to show that we can find such a covering with the property that each intersection $D_{i_1} \cap D_{i_1} \cap \cdots \cap D_{i_k}$ is empty or contractible.

We now proceed by induction. If $\#\mathcal{U} = 1$, then Σ_1 is contractible. Hence $H^0(\Sigma_1) = \mathbf{Q}$ and all other cohomology groups of Σ_1 vanish. In this case we have a deformation retract $(Y, Y \setminus \Sigma_1)$ to $(Y', Y' \setminus \{p\})$ where (Y', p) is a transversal ADE surface singularity. From this it follows that $H^i_{\Sigma_1}(Y) \cong H^i_p(Y')$. From Proposition 7.8 it follows that $H^6_p(Y') = \mathbf{Q}(-3)$ and all other local cohomology groups vanish. Hence the statement is true in this case.

Assume now $\#\mathcal{U} = k$, let $\Sigma_0 = \cup_{1\leq i\leq k-1} D_i$. We have two Mayer-Vietoris sequences (one is dual to the usual Mayer-Vietoris sequence, the other is Mayer-Vietoris for cohomology with support), namely

$$\begin{array}{ccccccc}
H^i(D_k\cap\Sigma_0)^* & \longrightarrow & H^i(D_k)^*\oplus H^i(\Sigma_0)^* & \longrightarrow & H^i(\Sigma_1)^* & \longrightarrow & H^{i-1}(D_k\cap\Sigma_0)^* \\
\downarrow\wr & & \downarrow\wr & & \downarrow & & \downarrow\wr \\
H^{6-i}_{D_k\cap\Sigma_0}(Y)(3) & \longrightarrow & H^{6-i}_{D_k}(Y)(3)\oplus H^{6-i}_{\Sigma_0}(Y)(3) & \longrightarrow & H^{6-i}_{\Sigma_1}(Y)(3) & \longrightarrow & H^{6-(i-1)}_{D_k\cap\Sigma_0}(Y)(3)
\end{array}$$

The first two vertical maps are isomorphisms by the induction hypothesis. From the five-lemma it follows that $\dim H^i(\Sigma) = \dim H^{6-i}_\Sigma(Y)$, which yields the lemma. ☐

Lemma 8.2. *We have that*

$$H^6_\Sigma(Y) \cong H^0(\Sigma)^*(-3) \text{ and } H^5_\Sigma(Y) \cong H^1(\Sigma)^*(-3).$$

Proof. Let $D_p = \overline{U_p}$. Using that D_p is contractible we have that $H^i_{D_p}(Y) \cong H^i_p(Y)$. From Proposition 7.6 it follows that $H^6_p(Y) = \mathbf{Q}(-3)$ and also that $H^5_p(Y) = 0$.

Let $\Sigma_2 = \cup\overline{U_p}$. Since $\overline{U_p}$ is contractible we have that $H^1(\Sigma_2) = 0$ and $H^5_{\Sigma_2}(Y) = \oplus H^5_p(Y) = 0$. In a similar way we get $H^6_{\Sigma_2}(Y) = \mathbf{Q}(-3)^{\#\mathcal{P}} = H^0(\Sigma_2)^*(-3)$.

Along $D := \Sigma_1 \cap \Sigma_2$, which is union of circles, we have transversal ADE surface singularities. A reasoning as in Lemma 8.1 shows that $H^5_D(Y) \cong H^1(Y)^*$ and $H^6_D(Y) \cong H^0(D)^*$.

As in the previous lemma we can consider the two Mayer-Vietories sequences (the vertical arrows are isomorphisms by either the above discussion or using Lemma 8.1)

$$\begin{array}{ccccccc}
H^1(D)^* & \longrightarrow & H^1(\Sigma_1)^* \oplus H^1(\Sigma_2)^* & \longrightarrow & H^1(\Sigma)^* & \longrightarrow & \cdots \\
\downarrow \wr & & \downarrow \wr & & \downarrow & & \\
H^5_D(Y)(3) & \longrightarrow & H^5_{\Sigma_1}(Y)(3) \oplus H^5_{\Sigma_2}(Y)(3) & \longrightarrow & H^5_\Sigma(Y)(3) & \longrightarrow & \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^0(D)^* & \longrightarrow & H^0(\Sigma_1)^* \oplus H^0(\Sigma_2)^* & \longrightarrow & H^0(\Sigma)^* & \longrightarrow & 0 \\
& & \downarrow \wr & & \downarrow \wr & & \downarrow & & \\
\cdots & \longrightarrow & H^6_D(Y)(3) & \longrightarrow & H^6_{\Sigma_1}(Y)(3) \oplus H^6_{\Sigma_2}(Y)(3) & \longrightarrow & H^6_\Sigma(Y)(3) & \longrightarrow & 0
\end{array}$$

An application of the five-lemma yields the proof. □

Lemma 8.3. *Suppose* $\dim \Sigma = 1$. *Then* $H^5(Y) = 0$ *and* $H^4(Y^*) \to H^5_\Sigma(Y)$ *is an isomorphism.*

Proof. Consider the exact sequence of the pair (Y, Y^*)

$$\begin{aligned} H^4(Y) &\to H^4(Y^*) \to H^5_\Sigma(Y) \to H^5(Y) \\ &\to H^5(Y^*) \to H^6_\Sigma(Y) \to H^6(Y) \to H^6(Y^*) = 0. \end{aligned}$$

Note that it follows from Proposition 5.6 that $H^5(Y^*) = H^0(\Sigma)^*_{\text{prim}}(-3)$. Using Lemma 8.2 it follows that $h^5(Y^*) = h^6_\Sigma(Y) - h^6(Y)$, hence the map $H^5(Y^*) \to H^6_\Sigma(Y)$ is injective.

From Proposition 5.6 it follows that $H^4(Y^*)$ is isomorphic to $H^1(\Sigma)^*(-3)$. From Lemma 8.2 it follows that $H^5_\Sigma(Y)$ is isomorphic to $H^1(\Sigma)^*(-3)$. Hence $H^4(Y^*)$ and $H^5_\Sigma(Y)$ have the same dimension.

Note that the possible Hodge weights of $H^4(Y^*) \cong H^1(\Sigma)^*(-3)$ are 5 and 6, where $H^4(Y)$ has Hodge weights at most 4 [21, Theorem 5.39]. Hence $H^4(Y) \to H^4(Y^*)$ is the zero-map, $H^4(Y^*) \cong H^5_\Sigma(Y)$ and $H^5(Y) = 0$. □

Theorem 8.4. *We have that*

$$H^4(Y)_{\text{prim}} = \text{coker}(H^4(U)(1) \to \oplus_{p\in\mathcal{P}} H^4_p(Y)).$$

Proof. Suppose first that $\dim \Sigma = 0$. Then $\mathcal{P} = \Sigma$.

Consider the exact sequence

$$H^3(Y^*) \to H^4_\Sigma(Y) \to H^4(Y) \to H^4(Y^*).$$

From Proposition 5.6 it follows $H^4(Y^*)_{\text{prim}} = 0$ and $H^3(Y^*) \cong H^4(U)(1)$, hence we have an exact sequence

$$H^4(U)(1) \to H^4_\Sigma(Y) \to H^4(Y)_{\text{prim}} \to 0.$$

This proves the case $\dim \Sigma = 0$.

Suppose that $\dim \Sigma = 1$. Consider the diagram (where both the horizontal and the vertical sequence are exact)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^4(U)(1) & \longrightarrow & H^3(Y^*) & \longrightarrow & H^2(\Sigma)^*(-3)_{\text{prim}} & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & H^4_\Sigma(Y) & & & & \\ & & & & \downarrow & & & & \\ & & & & H^4(Y) & & & & \\ & & & & \downarrow & & & & \\ & & & & 0. & & & & \end{array}$$

The horizontal sequence comes from Proposition 5.6, the vertical sequence is part of the long exact sequence of the pair (Y, Y^*). From Lemma 8.3 it follows that $H^4_\Sigma(Y) \to H^4(Y)$ is surjective.

We start by constructing a map $H^4_\Sigma(Y) \to H^2(\Sigma)^*(-3)$: let $\tilde{Y}$ be a resolution of all singularities contained in Σ of Y. Let E be the exceptional divisor. Then there is a natural map $H^2(\Sigma) \to H^2(E)$. Since $\tilde{Y}$ is smooth we have that $H^i_E(\tilde{Y}) = H^{6-i}(E)^*(-3)$. The resolution $(\tilde{Y}, E) \to (Y, \Sigma)$ induces a natural map $H^i_\Sigma(Y) \to H^i_E(\tilde{Y})$. Composing the maps as follows

$$H^4_\Sigma(Y) \to H^4_E(\tilde{Y}) \cong H^2(E)^*(-3) \to H^2(\Sigma)^*(-3)$$

yields a map $H^4_\Sigma(Y) \to H^2(\Sigma)^*(-3)$. It is easy to check that the composition $H^3(Y^*) \to H^4_\Sigma(Y) \to H^2(\Sigma)^*(-3)_{\text{prim}}$ is the same map as the map $H^3(Y^*) \to H^2(\Sigma)^*(-3)$ in the above diagram.

Let K be the kernel of the map $H^4_\Sigma(Y)_{\text{prim}} \to H^2(\Sigma)^*_{\text{prim}}(-3)$. The above diagram shows that

$$H^4(Y)_{\text{prim}} = \operatorname{coker} H^3(Y^*) \to H^4_\Sigma(Y)_{\text{prim}} = \operatorname{coker} H^4(U)(1) \to K.$$

The final equality is a consequence of the snake lemma.

Hence it remains to show that

$$K \cong \oplus_{p \in \mathcal{P}} H^4_p(Y).$$

Let $\Sigma_2 := \cup \overline{U_p}$ and $D = \Sigma_1 \cap \Sigma_2$. Note that D is a union of circles. Consider the Mayer-Vietoris sequence

$$\begin{array}{rcl} H^4_D(Y) & \to & H^4_{\Sigma_1}(Y) \oplus H^4_{\Sigma_2}(Y) \to \\ & \to & H^4_\Sigma(Y) \to H^5_D(Y) \to H^5_{\Sigma_1}(Y) \oplus H^5_{\Sigma_2}(Y) \to H^5_\Sigma(Y) \end{array}$$

Note that $H^5_\Sigma(Y) = H^1(\Sigma)^*(-3)$ by Lemma 8.2. Note also that that $H^5_{\Sigma_2}(Y) = H^1(\Sigma_2)^*(-3)$ by a reasoning similar to the one in the proof of Lemma 8.1. Since $H^1(\Sigma_2) = 0$ it follows that $H^5_{\Sigma_2} = 0$.

Since we have transversal ADE singularities along D and Σ_1 this sequence becomes (after tensoring with $\mathbf{Q}(3)$)

$$\begin{aligned} 0 = H^2(D)^* &\to H^2(\Sigma_1)^* \oplus H^4_{\Sigma_2}(Y)(3) \to \\ &\to H^4_{\Sigma}(Y)(3) \to H^1(D)^* \to H^1(\Sigma_1) \to H^1(\Sigma)^* \to \dots \end{aligned}$$

Since Σ_1 is a deformation retract of $\Sigma \setminus \mathcal{P}$ we obtain the following exact sequence: (dualized sequence of the pair $(\Sigma, \Sigma \setminus \mathcal{P})$)

$$0 \to H^2(\Sigma_1)^* \to H^2(\Sigma)^* \to \oplus_{p\in\mathcal{P}} H^2_p(\Sigma)^* \to H^1(\Sigma_1)^* \to H^1(\Sigma)^*$$

This yields a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2(\Sigma_1)^* \oplus H^4_{\Sigma_2}(Y)(3) & \xrightarrow{\varphi_1} & H^4_{\Sigma}(Y)(3) & \longrightarrow & H^1(D)^* & \longrightarrow & H^1(\Sigma_1)^* \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow = \\ 0 & \longrightarrow & H^2(\Sigma_1)^* & \xrightarrow{\varphi_2} & H^2(\Sigma)^* & \longrightarrow & \oplus_{p\in\mathcal{P}} H^2_p(\Sigma)^* & \longrightarrow & H^1(\Sigma_1)^* \end{array}$$

Here, the map $H^4_{\Sigma_2}(Y)(3) \to H^2(\Sigma)^*$ is the unique map, making this diagram commutative.

Using that $g_p = 0$ is weighted homogeneous we get that (Σ, p) is locally a set of m lines through p. In particular, $U_p \setminus \{p\}$ can be retracted to $\overline{U_p} \cap \Sigma_1$. Taking direct sums over all $p \in \mathcal{P}$ this shows that $H^i(\Sigma_2 \setminus \mathcal{P}) \cong H^i(D)$. Since for each $p \in \mathcal{P}$ we have that U_p is contractible we get a natural isomorphism

$$H^{i+1}_{\mathcal{P}}(\Sigma) \cong H^i(\Sigma_2 \setminus \mathcal{P}) \cong H^i(D).$$

Hence the above diagram simplifies to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2(\Sigma_1)^* \oplus H^4_{\Sigma_2}(Y) & \xrightarrow{\varphi_1} & H^4_{\Sigma}(Y) & \longrightarrow & \operatorname{coker}\varphi_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \sim & & \\ 0 & \longrightarrow & H^2(\Sigma_1)^* & \xrightarrow{\varphi_2} & H^2(\Sigma) & \longrightarrow & \operatorname{coker}\varphi_2 & \longrightarrow & 0 \end{array}$$

(The main point here is that $\operatorname{coker}\varphi_1 \cong \operatorname{coker}\varphi_2$.) From this diagram it follows that $H^4_{\Sigma_2}(Y) = \ker(H^4_{\Sigma}(Y) \to H^2(\Sigma)^*) = \ker(H^4_{\Sigma}(Y)_{\text{prim}} \to H^2(\Sigma)^*_{\text{prim}})$.

Since the $D_p := \overline{U_p}$ are contractible, there exists a deformation retract from $Y \setminus \Sigma_2$ to $Y \setminus \mathcal{P}$, hence $H^4_{\Sigma_2}(Y) \cong H^4_{\mathcal{P}}(Y)$, which yields the proof. $\square$

9. Method for calculating $MW(\pi)$

In this section we present a method to calculate the Mordell-Weil rank of a general elliptic threefold.

We start by identifying the set Σ and a finite subset $\mathcal{P}'$ containing the set $\mathcal{P}$ (cf. the previous section.)

Proposition 9.1. *Suppose we have a threefold $Y \subset \mathbf{P}(2n, 3n, 1, 1, 1)$ defined by the vanishing of $g := -y^2 + x^3 + Px + Q$, where P and Q are homogeneous polynomials in z_0, z_1, z_2 of degree $4n$ and $6n$. Suppose Y is minimal.*

Let Δ be the curve defined by $4P^3+27Q^2=0$ and Δ_1 be the underlying reduced curve. Let $\psi : \mathbf{P}(2n,3n,1,1,1) \to \mathbf{P}^2$ be the projection onto the plane $x=y=0$. Take $\mathcal{P}$ to be the set defined in Section 8. Then $\psi(\mathcal{P})$ is contained in $\Delta_{1,\text{sing}} \cup \mathcal{Q}_1 \cup \mathcal{Q}_2$ where

$$\mathcal{Q}_1 := \{q \in \Delta_{1,\text{smooth}} \colon q \text{ is an isolated zero of } P|_{\Delta_1}\}.$$

and

$$\mathcal{Q}_2 := \left\{ q \in \Delta_{1,\text{smooth}} \colon \begin{array}{l} P \text{ and } \Delta_1 \text{ have a common component } C \\ \text{containing } q, \operatorname{ord}_C(P)=2 \text{ and } \operatorname{ord}_q(P)\geq 3. \end{array} \right\}.$$

Proof. If all the partials of g vanish at p then, in particular, $\partial g/\partial x$ and $\partial g/\partial y$ vanish, hence p is a singular point of $\overline{\psi|_Y^{-1}\psi(p)}$ and $\overline{\psi(\Sigma)} \subset \Delta_1$. Moreover, if $p \in \Sigma$, then p is the unique singular point of $\overline{\psi|_Y^{-1}(\psi(p))}$.

For a general point q on a component C of Δ one can find the transversal type of the singularity along the corresponding component of Σ by Tate's algorithm. For more details we refer to [19]. We will use Tate's algorithm to identify the set of points where we do not have a transversal surface singularity.

I_ν-fiber. Suppose C is a component of Δ of multiplicity ν and $P\mid_C \not\equiv 0$. We show now that if $p \in \mathcal{P}$ then $q := \psi(p)$ is either in $\Delta_{1,\text{sing}}$ or $P(q)=0$ (i.e., $q \in \mathcal{Q}_1$).

For each $q \in C$ we have that $\overline{\psi^{-1}(q)}$ has precisely one singular point. Let Σ' be the union of all these points. Let $t=0$ be an equation for C and let s be a second local coordinate.

An easy calculation show that at a general point of C the x-coordinate of p equals $-3Q(s,t)/2P(s,t)$. As long as $P(s,t)\neq 0$ we can move the point $x=-3Q(s,t)/2P(s,t), y=0$ to $(0,0)$. This yields a new local equation of Y, namely

$$8P^3y^2 = 8P^3x^3 - 36PQ^2x^2 + 2P\Delta x - Q\Delta.$$

Since $\Delta(s,t)=t^\nu h(s,t)$, we have that (Y,p) is equivalent to the singularity

$$y^2 = x^2 + t^\nu x + t^\nu$$

unless $h(t,s)P(t,s)Q(t,s)=0$. For degree reasons we can disregard $t^\nu x$, hence we have a transversal $A_{\nu-1}$ singularity unless $h(t,s)P(t,s)Q(t,s)=0$. Since $\Delta = 4P^3+27Q^2$ we have that then $h(t,s)P(t,s)=0$.

I_ν^*-fiber, $\nu>0$. Suppose C is a component of Δ with multiplicity $6+\nu$ and that $\operatorname{ord}_C(P)=2, \operatorname{ord}_C(Q)=3$. Let $t=0$ be an equation for C and let s be a second local coordinate. I.e., we can write $P(s,t)=t^2P_1(s,t)$ and $Q(s,t)=t^3Q_1(s,t)$. As above, we move the point $(-3tQ_1(s,t)/P(s,t),0)$ to $(0,0)$. Then we get a local equation of the form

$$8P_1(t)^3y^2 = 8P_1(t)^3x^3 - 36tP_1(t)Q_1(t)^2x^2 + 2t^2P_1(t)\Delta_2(t)x - t^3Q_1(t)\Delta_2(t).$$

Where $\Delta_2(t,s)=\Delta(t,s)/t^6$. Then $\Delta_2 = 4P_1(t,s)^3 + 27Q_1(t,s)^2 = t^\nu h(t,s)$ for some h. This local equation is equivalent to a transversal $D_{4+\nu}$-singularity, unless $P_1(t,s)Q_1(t,s)h(t,s)=0$. A reason similar to the I_ν case shows that either $p \in \Delta_{1,\text{sing}}$ or P_1 and Q_1 vanish at q, which implies that $P = t^2 P_1$ vanish at least up to order 3 at q, i.e., $q \in \mathcal{Q}_2$.

Exceptional cases $II, III, IV, I_0^*, IV^*, III^*, II^*$.

Of these we do only the most difficult cases II^*, III^*, the other cases being very similar.

Case II^*: from Tate's algorithm it follows that we have a local equation of the form

$$y^2 = x^3 + t^4 P_1(s,t)x + t^5 Q_1(s,t)$$

such that $Q_1(s,t)$ does not vanish at a general point of C. Hence $\Delta(s,t) = t^{10}(4t^2 P_1(s,t)^3 + 27 Q_1(s,t)^5)$. This is a transversal E_8 singularity unless $Q_1(t,s)$ vanishes, but then q is a singular point of Δ_1.

Case III^*: from Tate's algorithm it follows that we have a local equation of the form

$$y^2 = x^3 + t^3 P_1(s,t)x + t^5 Q_1(s,t)$$

such that $P_1(s,t)$ does not vanish at a general point of C. Hence $\Delta(s,t) = t^9(4P_1(s,t)^3 + 27tQ_1(s,t)^2)$. This is a transversal E_7 singularity unless $P_1(s,t)$ vanishes, but then q is a singular point of Δ_1. □

Lemma 9.2. *Suppose $q \in \mathbf{P}^2$ is such that $P(q) = 0$ and q is an isolated double point of Δ. Then $\mathcal{P} \cap \psi^{-1}(q) = \emptyset$.*

Proof. Using that $\Delta = 4P^3 + 27Q^2$ and our assumptions on Δ and P we obtain that $Q = 0$ is a smooth reduced curve in a neighborhood of q and that $Q = 0$ does not have a common component with $P = 0$ or $\Delta = 0$ in a neighborhood of p. I.e., we have a local equation of the form

$$y^2 = x^3 + Px + s.$$

If Σ and $\psi^{-1}(q)$ intersect, then the fiber needs to be singular at that point, i.e., $(x,y,t,s) = (0,0,0,0)$, However, it is easy to see that Y is smooth at this point, hence $\psi^{-1}(q) \cap \Sigma = \emptyset$. □

For a Weierstrass equation $g := -y^2 + x^3 + Px + Q$ let $\mathcal{Q} := (\Delta_{1,\text{sing}} \cup \mathcal{Q}_1 \cup \mathcal{Q}_2) \setminus \mathcal{Q}_3$, where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are defined as in Proposition 9.1 and

$$\mathcal{Q}_3 = \{q \in \Delta_{1,\text{sing}} \colon P(q) = 0 \text{ and } q \text{ is an isolated double point of } \Delta\}.$$

Let

$$\mathcal{P}' := \bigcup_{q \in \mathcal{Q}} \overline{\psi|_Y^{-1}(q)_{\text{sing}}} \subset Y.$$

Note that $\mathcal{P}'$ is a finite set and contains the set $\mathcal{P}$ of the previous section.

Procedure 9.3. *Given an equation $y^2 = x^3 + Px + Q$ with homogeneous polynomials $P \in \mathbf{C}[z_0,z_1,z_2]_{4n}, Q \in \mathbf{C}[z_0,z_1,z_2]_{6n}$ such that there is no $u \in \mathbf{C}[z_0,z_1,z_2] \setminus \mathbf{C}$ with $u^4 | P$ and $u^6 | Q$.*

1. *Set $Y = \{(x,y,z_0,z_1,z_2) \in \mathbf{P}(2n,3n,1,1,1) \colon y^2 = x^3 + Px + Q\}$.*
2. *Determine the set $\mathcal{P}' \subset Y$ defined above.*

(3) *For each* $p \in \mathcal{P}'$ *check whether* (Y, p) *is contact equivalent to a weighted homogeneous hypersurface singularity* (Y', p').

If not, then stop, otherwise fix weights $w_{1,p}, w_{2,p}, w_{3,p}, w_{4,p}$ *and a weighted homogeneous polynomial* $g_p \in \mathbf{C}[y_1, y_2, y_3, y_4]$ *such that* (Y, p) *is contact equivalent to* $(\{g_p = 0\}, 0)$. *Fix also a map* $(Y, p) \to (\{g_p = 0\}, 0)$. *Let* $d_p := \deg g_p$, $w_p := \sum w_{i,p}$.

(4) *For each* $p \in \mathcal{P}'$ *let* $R(g_p)$ *be the Jacobian ring of* g_p. *If* (Y, p) *is an isolated singularity then set* $\tilde{R}(g_p) = R(g_p)$. *If* (Y, p) *is not an isolated singularity then* $\tilde{R}$ *is defined as in Remark 6.3.*

(5) *Calculate the dimension* r_1 *of the cokernel of the natural map*

$$\mathbf{C}[x, y, z_0, z_1, z_2]_{7n-3} \to \oplus_{p \in \mathcal{P}'} \tilde{R}(g_p)_{2d_p - w_p}.$$

(6) *Calculate the dimension* r_0 *of the cokernel of the natural map*

$$\mathbf{C}[x, y, z_0, z_1, z_2]_{n-3} \to \oplus_{p \in \mathcal{P}'} \tilde{R}(g_p)_{d_p - w_p}.$$

(7) *If* $r_0 = 0$ *then* $\operatorname{rank} \mathrm{MW}(\pi) = r_1$.

(8) *If* $r_0 > 0$ *then* $\operatorname{rank} \mathrm{MW}(\pi) \leq r_1$.

Proof. As is shown above $\mathcal{P}'$ is finite and contains $\mathcal{P}$. For each $p \in \mathcal{P}' \setminus \mathcal{P}$ we have that (Y, p) is smooth or a transversal ADE surface singularity. By 7.8 it follows that $H^4_p(Y, \mathbf{Q}) = 0$. Hence to calculate the cokernel of $H^4(U, \mathbf{Q})(1) \to \oplus_{q \in \mathcal{P}} H^4_q(Y, \mathbf{Q})$, we can replace $\mathcal{P}$ by $\mathcal{P}'$.

We proceed by calculating $h^{3,1}(H^4(Y, \mathbf{C}))$ and $h^{2,2}(H^4(Y, \mathbf{C}))$. Combining Proposition 7.7 with Theorem 8.4 yields that

(1) $h^{3,1}(H^4(Y, \mathbf{C})) \leq r_0$ and $h^{2,2}(H^4(Y, \mathbf{C}))_{\mathrm{prim}} \leq r_1$.

(2) If $r_0 = 0$ then $h^{3,1}(H^4(Y, \mathbf{C})) = h^{4,0}(H^4(Y, \mathbf{C})) = 0$. Since $H^4(Y, \mathbf{Q})$ has a pure weight 4 Hodge structure it follows that $h^{1,3}(H^4(Y, \mathbf{C})) = h^{0,4}(H^4(Y, \mathbf{C})) = 0$, hence $H^4(Y, \mathbf{C})$ is of pure type $(2, 2)$ and

$$\operatorname{rank} H^4(Y, \mathbf{C})_{\mathrm{prim}} \cap H^{2,2}(H^4(Y))_{\mathrm{prim}} = r_1.$$

Applying Theorem 4.3 finishes the proof. □

Remark 9.4. An elliptic curve E over $\mathbf{C}(t_1)$ is for trivial reasons also an elliptic curve over $\mathbf{C}(t_1, t_2)$. We discuss what the outcome of our method is, if we apply it to such Y. Note that Y is defined as the zero-set of

$$-y^2 + x^3 + P(z_0, z_1)x + Q(z_0, z_1)$$

i.e., Y is a cone over an elliptic surface. Here we assume that n is such that $\deg(P) = 4n$ and $\deg(Q) = 6n$. The discriminant curve is a union of lines through $(0 : 0 : 1)$. From this it follows that $\mathcal{P}' = \{(0 : 0 : 0 : 0 : 1)\}$. For simplicity assume that the $(0 : 0 : 0 : 0 : 1)$ is an isolated singularity.

For $p = (0 : 0 : 0 : 0 : 1)$ we have a *local* equation

$$-v^2 + u^3 + P(s, t)u + Q(s, t) = 0 \tag{7}$$

i.e., we have $d_p = 6n$ and $w_p = 5n+2$. Our algorithm tells us that we should calculate the dimension r_1 of the cokernel of

$$\mathbf{C}[x, y, z_0, z_1, z_2]_{7n-3} \to \tilde{R}(g_p)_{7n-2}$$

and calculate the dimension r_0 of the cokernel of

$$\mathbf{C}[x, y, z_0, z_1, z_2]_{n-3} \to \oplus_{p \in \mathcal{P}'} \tilde{R}(g_p)_{n-2}.$$

It is easy to see that both maps are the zero map. In particular, our method tells us that

$$\operatorname{rank} \mathrm{MW}(\pi) \leq r_1 = \dim R(g_p)_{7n-2} = h^{1,1}(S)_{\text{prim}}$$

where S is the elliptic surface defined by (7). Of course, we could obtain this inequality directly, i.e., by applying the Shioda-Tate formula to S.

Part 3. **Examples**

10. Example of Grooten-Steenbrink

Grooten and Steenbrink studied the family of threefolds

$$g := -W^2 + A_2 X_0^2 + 2A_1 X_0 (X_1 X_3 - X_2^2) + A_0 (X_1 X_3 - X_2^2)^2 = 0$$

where $A_i \in \mathbf{C}[X_1, X_2, X_3]_i$. This defines a degree 4 threefold Y in $\mathbf{P} = \mathbf{P}(1, 1, 1, 1, 2)$.

For general A_0, A_1, A_2 Grooten and Steenbrink proved that $h^4(Y) = 2$. Here we will give a proof of this by our methods. We shall use the notation of the previous sections.

Lemma 10.1. *Let $A_i \in \mathbf{C}[X_1, X_2, X_3]_i$ for $i = 0, 1, 2$. Assume that $A_2 A_0 - A_1^2$ defines a smooth conic intersecting the conic $X_1 X_3 - X_0^2$ in four distinct points. Then*

1. *The locus Σ is given by $W = X_0 = X_1 X_3 - X_2^2 = 0$.*
2. *If $p \in \Sigma$ is such that $A_2(p)A_0(p) - A_1^2(p) \neq 0$ then (Y, p) is a transversal A_1-singularity.*
3. *If $p \in \Sigma$ is such that $A_2(p)A_0(p) - A_1^2(p) = 0$ then (Y, p) is a pinch point. There are precisely four such points.*

Proof. An easy calculation shows that the partials of F vanish if and only if $W = 0, X_0 = 0, X_1 X_3 - X_2^2 = 0$, yielding the first claim.

Note that the conic $X_1 X_3 - X_2^2 = 0$ is smooth. Hence we can parameterize this conic by a local coordinate t. Let s be a second local coordinate in the plane $W = X_0 = 0$, such that $s = 0$ is a local equation for the conic $X_1 X_3 - X_2^2 = 0$. Then we have a local equation for Y of the form

$$w^2 = A_2 x_0^2 + 2A_1 x_0 s + A_0 s^2.$$

If p is such that $A_2(p)A_0 - A_1^2(p) \neq 0$ then this defines a transversal A_1-singularity. This gives the second claim.

If p is such that $A_2(p)A_0 - A_1(p)^2 = 0$ then we change the coordinate t such that $t = 0$ is an equation for $A_2A_0 - A_1^2$. Then Y has a local equation of the form $w^2 = x_0^2 + ts^2$, hence we have a pinch point.

The pinch points are precisely the point in the intersection of $A_2(p)A_0 - A_1(p)^2 = 0$ with Σ, i.e., the intersection of two plane conics. Our assumptions on the A_i yield that there are exactly four distinct intersection points, i.e., there are four pinch pints. This finishes the proof. □

Proposition 10.2. *Let $A_i \in \mathbf{C}[X_1, X_2, X_3]_i$ for $i = 0, 1, 2$. Assume that $A_2A_0 - A_1^2$ defines a smooth conic intersecting the conic $X_1X_3 - X_0^2$ in four distinct points. Then $h^4(Y) = 2$.*

Proof. Lemma 10.1 shows that the set $\mathcal{P}$ of Theorem 8.4 consists precisely of the four pinch points p_1, p_2, p_3, p_4. A local equation for a pinch point p is $g_p := x_1x_2 - x_3^2x_4 = 0$. If we set $\mathrm{wt}(x_1) = 2$ and let all other variables have weight 1, then we get a weighted homogeneous equation of degree 3. The surface S defined by g_p has an A_1-singularity at $q : x_1 = x_2 = x_3 = 0$. This implies that we can apply Theorem 6.2 to calculate the cohomology of S:

$$H^{2,0}(S, \mathbf{C}) = R(g_p)_{-2} = 0,\ H^{1,1}(S, \mathbf{C})_{\mathrm{prim}} = \{f \in R(g_p)_1 : f(q) = 0\}.$$

To determine the latter group, note that x_1 and x_2 are in the Jacobian ideal of g_p, hence $R(g_p)_1 = \langle \overline{x_3}, \overline{x_4}\rangle$ and

$$\tilde{R}(g_p)_1 = H^{1,1}(S, \mathbf{C})_{\mathrm{prim}} = \mathbf{C}\overline{x_3}.$$

We would like to calculate $H^4(Y, \mathbf{Q})$. By Proposition 7.7 and Theorem 8.4 we have that $h^{3,1}(Y, \mathbf{C})$ is at most the dimension of the cokernel

$$R(g)_{-2} \to \oplus_{i=1}^4 R(g_{p_i})_{-2}$$

hence $h^{3,1}(Y, \mathbf{C}) = 0$.

From the same results it follows that $h^{2,2}(Y, \mathbf{C})$ equals the dimension of the cokernel

$$\psi_2 : R(g)_2 \to \oplus_{p_i \in \mathcal{P}} \tilde{R}(g_{p_i})_1.$$

An easy calculation shows that for $f \in R(g)_2$ we have

$$\psi_2(f) = (f_{X_0}(p_1), f_{X_0}(p_2), f_{X_0}(p_3), f_{X_0}(p_4)).$$

Since f is a a degree 2 polynomial it follows that $f_{X_0} = a_0X_0 + a_1X_1 + a_2X_2 + a_3X_3$ for some $a_i \in \mathbf{C}$. All points of $\mathcal{P}$ lie in the subspace given by $W = X_0 = 0$, i.e., are of the form $(0 : \alpha_i : \beta_i : \gamma_3 : 0)$. This means that the image of ψ_2 is of the form

$$\mathrm{span}(a_1\alpha_i + a_2\beta_i + a_3\gamma_i : i = 1, 2, 3).$$

Hence the image of ψ_2 has dimension at most 3. If the dimension of the image were less than 3 then the above description shows that the four points in $\mathcal{P}$ are collinear, contradicting the fact that they lie on an irreducible conic, hence the image has dimension 3, and the cokernel is one dimensional. This implies that $H^4(Y, \mathbf{Q}) = \mathbf{Q}(-2)^2$, and the defect of Y equals 1. □

11. Three easy examples

We give three examples of rational elliptic threefolds. The first and third threefold have $\operatorname{rank} \mathrm{MW}(\pi) = 2$ and the second threefold satisfies $\operatorname{rank} \mathrm{MW}(\pi) = 0$.

Example 11.1. Consider $y^2 = x^3 + z_0^6 + z_1^2 z_2^4$. The set Σ consists of $p = (0:0:0:0:1)$ and $q = (0:0:0:1:0)$. Local equations for p and q are $v^2 = u^3 + t_1^6 + s_1^2$ and $v^2 = u^3 + t_2^6 + s_2^4$, both singularities are weighted homogeneous with weights $(3,1,2,3)$ and $(3,2,4,6)$ and of degree 6 and 12 respectively. In particular $w_p = 9, w_q = 15$. Hence $R(g_p)_{d_p - w_p} = R(g_q)_{d_q - w_q} = 0$, and $H^4(Y, \mathbf{Q})$ has a pure $(2,2)$ Hodge structure by Proposition 7.7 with Theorem 8.4.

Note that

$$R(g_p)_{2d_p - w_p} = R(g_p)_3 = \mathbf{C}\overline{t_1^3} \oplus \mathbf{C}\overline{t_1 u}$$

and

$$R(g_q)_{2d_q - w_q} = R(g_q)_9 = \mathbf{C}\overline{u t_2 s_2} \oplus \mathbf{C}\overline{t_2^3 s_2}.$$

This implies that $\mathbf{C}[x,y,z_0,z_1,z_2]_4 \to H^4_p(Y, \mathbf{C})$ is

$$f \mapsto (\delta_{z_0}^3 f(p), \delta_x \delta_{z_0} f(p))$$

and $\mathbf{C}[x,y,z_0,z_1,z_2]_4 \to H^4_q(Y, \mathbf{C})$ is the map

$$f \mapsto (\delta_{z_0}^3 \delta_{z_2} f(q), \delta_x \delta_{z_0} \delta_{z_2} f(q)).$$

Write $f \in \mathbf{C}[x,y,z_0,z_1,z_2]_4$ as $\sum_I a_I w^I$. Then it follows that

$$\psi_2(f) = ((6a_{00301}, a_{10101}), (6a_{00301}, a_{10101}))\,.$$

Hence the image of ψ has dimension 2. From this it follows that the cokernel is also 2-dimensional and $\operatorname{rank} \mathrm{MW}(\pi) = 2$.

Let $\omega = e^{2\pi i/3}$. Note that the sections $(x,y) = (z_0^2, z_1 z_2^2)$ and $(x,y) = (\omega z_0^2, z_1 z_2^2)$ are independent in $\mathrm{MW}(\pi)$, hence generate a finite index subgroup of $\mathrm{MW}(\pi)$.

To determine the torsionpart of $\mathrm{MW}(\pi)$, fix a general line $\ell \subset \mathbf{P}^2$. Then $\pi_\ell : \pi^{-1}(\ell) \to \ell$ is a rational elliptic surface with 6 I_2-fibers. From e.g. [20] it follows that $MW(\pi_\ell) \cong \mathbf{Z}^8$, in particular it has no torsion. Since $\mathrm{MW}(\pi) \to MW(\pi_\ell)$ is injective it follows that $\mathrm{MW}(\pi)$ has also no torsion.

Example 11.2. Consider $y^2 = x^3 + x^2 + f$, where f is the product of six distinct lines, and no three lines pass through one point. The set $\mathcal{P}$ is precisely the set of points $(0:0:\alpha:\beta:\gamma)$, where $(\alpha:\beta:\gamma)$ is an intersection point of two of these lines.

All singularities are of type A_1, with local equation $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$. From this it follows that $R(f_p)_{2d_p - w_p} = \mathbf{C}\overline{1}$.

Fix some coordinates $(\alpha_i : \beta_i : \gamma_i)$ for $p_i \in \mathcal{P}$. Then $\operatorname{rank} \mathrm{MW}(\pi)$ equals

$$\dim \operatorname{coker} \mathbf{C}[x,y,z_0,z_1,z_2]_4 \to \mathbf{C}^{15}$$

where we map f to $\oplus f(0,0,\alpha_i,\beta_i,\gamma_i)$.

Since $\dim \mathbf{C}[x,y,z_0,z_1,z_2]_4 = 15$, this cokernel is non-zero precisely when there exists a degree 4 plane curve C containing all the p_i. Such a curve does not exist since for each of the six lines L_j we have $\#C \cap L_j \geq 5$, hence $C \supset L_j$. Since there are six distinct lines, this implies $\deg(C) \geq 6$, contradicting $\deg(C) = 4$.

Hence the cokernel is trivial and $\operatorname{rank} \mathrm{MW}(\pi) = 0$. Using a reasoning similar as in the previous example it follows that the torsion part of $\mathrm{MW}(\pi)$ is also trivial, hence $\mathrm{MW}(\pi) = 0$.

Example 11.3. Consider the elliptic threefold Y

$$y^2 + x^3 + z_0^2 z_2^2 (z_0 z_2 - z_1^2).$$

The locus Σ of Y is given by $y = w = z_0 z_2 = 0$, i.e., is 1-dimensional.

The discriminant curve is $z_0^2 z_2^2 (z_0 z_2 - z_1^2)$. The set $\mathcal{P}'$ consists of three points $p_1 = (0:0:1:0:0)$, $p_2 = (0:0:0:1:0)$, $p_3 = (0:0:0:0:1)$. Note that Σ is one dimensional in this case.

At p_1 and p_3 we have a local equation of the form

$$v^2 = u^3 + t^2 s^2 + s^3$$

Set weights for s,t,u,v as $2,1,2,3$. Then this equation is weighted homogeneous of degree 6, and

$$R(g_p)_{d_p - w_p} = 0, \; R(g_p)_{2d_p - w_p} = \operatorname{span}\{\overline{t^4}, \overline{s^2}, \overline{rt^2}, \overline{rs}\}.$$

Along $v = u = s = 0$ we have a transversal A_2-singularity. The Milnor algebra of an isolated A_2-singularity $v^2 + u^3 + t^2$ is generated by 1 and u. If we homogenize these two monomials we get t^4 and ut^2. Hence

$$\tilde{R}(g_p)_{2d_p - w_p} = R(g_p)_{2d_p - w_p} / (\overline{t^4}, \overline{ut^2}) = \operatorname{span}\{\overline{s^2}, \overline{us}\}.$$

For $p = p_1$ we have that, after homogenizing, s^2 corresponds to $z_0^2 z_2^2$ and xs corresponds to xz_0z_2. For $p = p_3$ we get similarly that $\tilde{R}_{g_p}$ is generated by $\overline{z_0^2 z_2^2}$ and $\overline{xz_0z_2}$.

At $p = p_2$ we have a local equation of the form

$$v^2 = u^3 + t^2 s^2$$

If we set weights for s,t,u,v as $2,2,1,3$ we get a weighted homogeneous equation of degree 12. Again $R(g_p)_{d_p - w_p} = 0$. We get that $R(g_p)_{2d_p - w_p}$ is four dimensional, and that

$$\tilde{R}(g_p)_{2d_p - w_p} = 0.$$

This implies that $r_0 = 0$ and r_1 is the cokernel of

$$\mathbf{C}[x,y,z_0,z_1,z_2]_4 \to \tilde{R}(g_{p_1})_4 \oplus \tilde{R}(g_{p_3})_4.$$

Since both summands have the same generators it turns out that the cokernel has dimension 2. In particular, $\operatorname{rank} \mathrm{MW}(\pi)$ is 2. The sections $(x = \omega^i z_0 z_2, y = z_0 z_1 z_2)$ for $i = 0,1$ generate a finite-index subgroup of $\mathrm{MW}(\pi)$.

In order to determine the torsion subgroup of $\mathrm{MW}(\pi)$: fix a general line ℓ in $\mathbf{P}^2$ and consider $\pi_\ell : \pi^{-1}(\ell) \to \ell$. Then $\pi^{-1}(\ell)$ is a rational elliptic surface with $2IV$ fibers and $2II$ fibers. Such an elliptic surface has tivial torsion subgroup [20], hence $\mathrm{MW}(\pi)$ has no torsion.

12. An application

The following construction of Calabi-Yau threefolds is due to F. Hirzebruch and was communicated to us by N. Yui. Some of the details of the construction were worked out in the Diplomarbeit [2] of N. Behrens.

Construction 12.1. Let S be a del Pezzo surface, i.e., the blow-up of $\mathbf{P}^2$ in m points $p_1, \dots p_m$ in general position (meaning no three points on a line, and no six points on a conic), $0 \leq m \leq 8$. By E_i we denote the exceptional divisors of the blow-down morphism $\varphi : S \to \mathbf{P}^2$. Let L be the pullback to S of a general line in $\mathbf{P}^2$.

We consider the anti-canonical line bundle $\mathcal{L} = \omega_S^{-1} = \mathcal{O}(3L - \sum E_i)$ and define the rank 3 bundle $\mathcal{E} = \mathcal{O} \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3}$. Then $\mathbf{P}(\mathcal{E})$ is a $\mathbf{P}^2$-bundle over S. We use Grothendieck's definition of projective space, in particular $p_*\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) = \mathcal{E}$ where p is the bundle projection. Fix sections

$$
\begin{aligned}
X := (0,1,0) &\in H^0(\mathcal{L}^2 \oplus \mathcal{O} \oplus \mathcal{L}^{-1}) = H^0(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \otimes \mathcal{L}^2), \\
Y := (0,0,1) &\in H^0(\mathcal{L}^3 \oplus \mathcal{L} \oplus \mathcal{O}) = H^0(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \otimes \mathcal{L}^3), \\
Z := (1,0,0) &\in H^0(\mathcal{O} \oplus \mathcal{L}^{-2} \oplus \mathcal{L}^{-3}) = H^0(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)).
\end{aligned}
$$

For general sections g_2, g_3 in $H^0(\mathcal{L}^4)$ and $H^0(\mathcal{L}^6)$ respectively, the equation

$$
Y^2 Z = 4X^3 + g_2 X Z^2 + g_3 Z^3 \tag{8}
$$

defines a *smooth* hypersurface W in $\mathbf{P}(\mathcal{E})$. Note that W is in the linear system defined by the anti-canonical line bundle $\omega_{\mathbf{P}(\mathcal{E})}^{-1} = (p^*\mathcal{L}^6) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E})}(3)$. The projection onto S defines an elliptic fibration $\pi : W \to S$ with a section.

Lemma 12.2. *The threefold W has trivial canonical bundle.*

Proof. Since

$$
\omega_{\mathbf{P}(\mathcal{E})} = p^*(\omega_S \otimes \det \mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E}))}(-3) = p^*\mathcal{L}^{-6} \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E}))}(-3)
$$

and $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(W_7) = p^*\mathcal{L}^6 \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E}))}(3)$ it follows from the adjunction formula that

$$
\omega_{W_7} = \omega_{\mathbf{P}(\mathcal{E})}(W_7)|_{W_7} = \mathcal{O}_{W_7}.
$$

□

In [2] a detailed proof of the following result is given:

Theorem 12.3 ([2, Theorem 2.35])**.** *Let $r = \operatorname{rank} \mathrm{MW}(\pi)$. Then W has the following Hodge numbers:*

1. $h^{1,0}(W) = h^{0,1}(W) = h^{2,0}(W) = h^{0,2}(W) = 0$,
2. $h^{1,3}(W) = h^{3,1}(W) = 0$,
3. $h^{0,3}(W) = h^{3,0}(W) = 1$,
4. $h^{1,1}(W) = m + 2 + r$,

(5) $h^{1,2}(W) = h^{2,1}(W) = 272 - 29m + r$.

The topological Euler characteristic $e(W) = -540 + 60m$.

Remark 12.4. The fact that $h^{1,0}(W) = h^{2,0}(W) = 0$ and that $\omega_{W_7} = \mathcal{O}_{W_7}$ implies that W_7 is a Calabi-Yau threefold. For Calabi-Yau threefolds finding their mirror partner is of particular interest. The line bundle $(p^*\mathcal{L}^6) \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E})}(3)$ is not an ample line bundle. (This follows e.g., since $\pi_*(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \otimes \mathcal{L}^2) = \mathcal{E} \otimes \mathcal{L}^2 = \mathcal{L}^2 \oplus \mathcal{O}_S \oplus \mathcal{L}^{-1}$.) Hence we are not in a position where Batyrev's mirror construction [1] can be applied directly. In order to find a mirror family it is first of all necessary to compute the Hodge numbers of W. This was the motivation behind [2].

To actually find the Hodge numbers we need to determine the rank of $\mathrm{MW}(\pi)$. In [2] it is conjectured that $r = 0$ for all such W. We apply our methods to prove this conjecture. We first calculate the Mordell-Weil rank by computing $h^4(Y)$. In the second half of this section we illustrate our methods by determining all Hodge numbers by going through the various constructions, thus avoiding a direct reference to Theorem 12.3.

We know that W is birational to a hypersurface Y of degree $6n$ in some weighted projective space $\mathbf{P}(2n, 3n, 1, 1, 1)$. For $n = 1, 2$ such a threefold is a deformation of a rational variety. Since W is a Calabi-Yau hypersurface we have $n \geq 3$.

Lemma 12.5. *There exists a degree 18 hypersurface* Y *in* $\mathbf{P}(6, 9, 1, 1, 1)$*, birational to* W *and such that* Y_{sing} *consists of* $(1 : 1 : 0 : 0 : 0)$ *and* m *isolated semi-weighted homogeneous hypersurface singularities with Milnor number 50. For each of these singularities we have that* $H^4_p(Y, \mathbf{Q}) \cong \mathbf{Q}(-2)^8$.

Proof. We need to consider g_2, g_3 as functions on $\mathbf{P}^2$, rather than elements in $H^0(S, \mathcal{L}^i)$. Since $\varphi_*\mathcal{L} = \mathcal{O}(3) \otimes \mathcal{I}_{p_1,\dots,p_m}$, it follows that $g_2 \in H^0(\mathcal{O}(12) \otimes \mathcal{I}^4_{p_1,\dots,p_m})$ and $g_3 \in H^0(\mathcal{O}(18) \otimes \mathcal{I}^6_{p_1,\dots,p_m})$. Let P and Q be the associated weighted homogeneous polynomials of degree 12 and 18 respectively. Then

$$y^2 = x^3 + Px + Q \tag{9}$$

defines a degree 18 hypersurface Y in $\mathbf{P}(6, 9, 1, 1, 1)$ birational to W.

Let $\tilde{\psi} : \mathbf{P} \to \mathbf{P}^2$ be the projection from $\{z_0 = z_1 = z_2 = 0\}$ to the plane $\{x = y = 0\}$. Then $\psi = \tilde{\psi}|_Y$ corresponds to the elliptic fibration on W. Note that p is defined on $Y \setminus \{(1 : 1 : 0 : 0 : 0)\}$. Since W is smooth all singularities (besides $(1 : 1 : 0 : 0 : 0)$) lie in $\psi^{-1}(p_i)$ for $i = 1, \dots m$.

Equation (9) shows that $\overline{\psi^{-1}(p_i)}$ has equation $Y^2Z = X^3 + P(p_i)XZ^2 + Q(p_i)Z^3$. In particular, $\overline{\psi^{-1}(p_i)}$ is an irreducible and reduced cubic plane curve and it has at most one singularity. Since Y is singular at $q_i = (0 : 0 : p_i)$, the same holds for $\overline{\psi^{-1}(p_i)}$, and there are no other singular points on $Y \setminus \{(1 : 1 : 0 : 0 : 0)\}$.

We proceed by calculating the Milnor number of (Y, q_i). A local equation for Y around q_i is

$$v^2 = 4u^3 + h_4(t, s)u + h_6(t, s) + h.o.t.$$

An easy calculation, using that W is smooth, shows that the lowest degree part

$$v^2 = 4u^3 + h_4(t,s)u + h_6(t,s)$$

defines a quasismooth surface in $\mathbf{P}(2,3,1,1)$. In particular, (Y, q_i) is a semi-weighted homogeneous hypersurface singularity, i.e., we may ignore the higher order terms.

To calculate the Milnor number of (Y, q_i) we need to consider the Jacobian ring R of the defining equation of the singularity. Using Lemma 12.6 (proven below) it follows that

$$\sum \dim R_d t^d = 1 + 2t + 4t^2 + 6t^3 + 8t^4 + 8t^5 + 8t^6 + 6t^7 + 4t^8 + 2t^9 + t^{10}.$$

Hence $\mu = \dim R = 50$.

To calculate the local cohomology it suffices to determine $\dim R_{d-w} = R_{-1}$ and $\dim R_{2d-w} = \dim R_5$. The former space is 0, the latter space is 8-dimensional. Now apply Proposition 7.6 and Theorem 6.2. □

Lemma 12.6. *Let $f \in \mathbf{C}[x_0, \dots, x_{n+1}]$ be a weighted homogeneous polynomial of degree d with weights $w_0, \dots, w_{n+1}$. Assume that each w_i divides d and that $f = 0$ has at most an isolated singularity at the origin. Let R be the Jacobian ring of f. Then*

$$\sum_k \dim R_k t^k = \prod \frac{t^{d-w_i} - 1}{t^{w_i} - 1}.$$

Proof. Since $f = 0$ has at most a singularity at the origin it follows that the partials of f form a regular sequence in $\mathbf{C}[x_0, \dots, x_{n+1}]$. This implies that R is resolved by its Koszul complex. An easy calculation yields the proof. □

For the rest of this section, let Y be the degree $6n$ hypersurface in $\mathbf{P}(6,9,1,1,1)$ constructed in the proof above. In particular, $Y \cap \{z_1 = z_2 = z_3 = 0\} = \{(1:1:0:0:0)\}$. Let $q_i = (0:0:p_i)$.

The form of the singularity (Y, q_i) allows us to use Dimca's results. For this we first prove the following two lemmas.

Lemma 12.7. *Let $T \subset \mathbf{P}(6,9,1,1,1)$ be a quasismooth hypersurface of degree 18. Then $h^3(T) = 546$ and the topological Euler characteristic $e(T) = -542$.*

Proof. Since the topology of quasismooth hypersurfaces is invariant under deformation, it suffices to prove this statement for T given by

$$f := y^2 + x^3 + z_0^{18} + z_1^{18} + z_2^{18}.$$

Let R be the Jacobian ring of f. Using Griffiths-Steenbrink (see Section 5) we know that

$$h^3(T) = \dim R_0 + \dim R_{18} + \dim R_{36} + \dim R_{54}.$$

An easy calculation shows that $\dim R_0 = \dim R_{54} = 1$ and $\dim R_{18} = \dim R_{36} = 272$. Hence $h^3(T) = 546$. From Lefschetz' hyperplane theorem (Proposition 5.1) it follows that $h^i(T) = 1$ for $i = 0, 2, 4, 6$ and all other Betti numbers vanish. From this the equality $e(T) = 4 - 546 = -542$ follows. $\square$

Lemma 12.8. *The topological Euler characteristic $e(Y)$ of Y equals $-542 + 50m$.*

Proof. Let T be a quasismooth hypersurface of the same degree of Y. From e.g. [9, Corollary 5.4.4] it follows that

$$e(Y) = e(T) + \mu$$

where μ is the total Milnor number of Y, i.e., the sum of the Milnor numbers of the singularities of Y besides $(1 : 1 : 0 : 0 : 0)$. From Lemma 12.5 and Lemma 12.7 it follows that $e(Y) = -542 + 50m$. $\square$

Using the Lefschetz hyperplane theorem (Proposition 5.1) we obtain that

$$h^0(Y) = h^2(Y) = h^6(Y) = 1 \text{ and } h^1(Y) = h^5(Y) = 0.$$

Hence $h^3(Y) = 546 - 50m + h^4(Y) - 1$.

To calculate $h^4(Y)$ we use Dimca's method. For this we need some results on linear systems on $\mathbf{P}^2$.

Definition 12.9. Let $L_d(k^m)$ be the linear system of degree d curves having a point of order k at $p_1, \dots, p_m$. The defect of $L_d(k^m)$ equals $m\frac{k(k+1)}{2} - \operatorname{codim}_{\mathbf{C}[z_0,z_1,z_2]_d} L_d(k^m)$, i.e., the difference between the expected codimension and the actual codimension.

We are interested in $L_{18}(6^m)$ and $L_{12}(4^m)$, in the case that the m points are the p_i.

Proposition 12.10. *For $k > 0$ we have that the linear system $L_{3k}(k^m)$ has no defect.*

Proof. Note that $L_{3k}(k^m)$ is isomorphic to $H^0(S, \mathcal{O}_S(3kH - k\sum E_m))$. Set $D = 3H - \sum E_i$ and let C be an irreducible smooth curve in $|D|$. (Such a curve exists since the p_i are in general position and $m \leq 8$.) Since C is the strict transform of a degree 3 curve in $\mathbf{P}^2$ we have that $g(C) = 1$.

Let $\mathcal{L} = \mathcal{O}(D)|_C$. Then $\deg(\mathcal{L}) = D^2 = 9 - m > 0$. Using $g(C) = 1$ we find for $t > 0$ that $h^0(\mathcal{L}^t) = t(9 - m)$ and $h^1(\mathcal{L}^{\otimes t}) = 0$.

Consider now the long exact sequence in cohomology associated to

$$0 \to \mathcal{O}_S((t-1)D) \to \mathcal{O}_S(tD) \to \mathcal{L}^{\otimes t} \to 0.$$

Since for $t \geq 1$ we have that $h^1(\mathcal{L}^{\otimes t}) = 0$, we find that $h^1(\mathcal{O}_S(tD)) \leq h^1(\mathcal{O}_S((t-1)D))$. Note that for $t = 1$ we have that $h^1(\mathcal{O}_S((t-1)D)) = h^{0,1}(S) = 0$. Combining this yields that $h^1(\mathcal{O}_S(tD)) = 0$ for $t \geq 0$. This implies that

$$h^0(\mathcal{O}_S(tD)) = h^0(\mathcal{O}_S((t-1)D)) + h^0(\mathcal{L}^{\otimes t}) = h^0(\mathcal{O}_S((t-1)D)) + t(9 - m)$$

whence

$$h^0(\mathcal{O}_S(tD)) = \frac{t(t+1)(9-m)}{2} + h^0(\mathcal{O}_S) = \frac{t(t+1)(9-m)}{2} + 1.$$

The expected dimension of $L_{3k}(k^m)$ equals

$$\frac{(3k+1)(3k+2)}{2} - m\frac{k(k+1)}{2} = \frac{k(k+1)(9-m)}{2} + 1.$$

This implies that $L_{3k}(k^m)$ has the expected dimension and thus $L_{3k}(k^m)$ has no defect. □

Proposition 12.11. *We have that* $h^4(Y) = 1$, *hence* $h^3(Y) = 546 - 50m$.

Proof. From Dimca's work, (the dimension zero case of Sections 7 and 8), it follows that the primitive cohomology $H^4(Y, \mathbf{Q})_{\text{prim}}$ is isomorphic to the cokernel of

$$H^4(\mathbf{P} \setminus Y, \mathbf{Q}) \to \oplus_{q_i} H^4_{q_i}(Y, \mathbf{Q}).$$

From Lemma 12.5 we know that $H^4_{q_i}(Y, \mathbf{Q}) = \mathbf{Q}(-2)^8$.

A local equation of (Y, q_i) (see the proof of Lemma 12.5) is

$$f_{q_i} := -v^2 + 4u^3 + h_{4,i}(t,s)u + h_{6,i}(t,s).$$

This equation is weighted homogeneous. Moreover, we know that this is an equation of a quasismooth surface. Let $R(f_{q_i})$ denote the Jacobian ring of f_{q_i}.

From Proposition 7.7 and Theorem 8.4 it follows that the cokernel of $H^4(\mathbf{P} \setminus Y, \mathbf{C}) \to \oplus H^4_{q_i}(Y, \mathbf{C})$ equals the cokernel of $\operatorname{Gr}^2_P H^4(\mathbf{P} \setminus Y, \mathbf{C}) \to \oplus H^4_{q_i}(Y, \mathbf{C})$. Using the natural maps

$$\mathbf{C}[z_0, z_1, z_2]_{12} x \oplus \mathbf{C}[z_0, z_1, z_2]_{18} \twoheadrightarrow R(f)_{18} \twoheadrightarrow \operatorname{Gr}^2_P H^4(\mathbf{P} \setminus Y, \mathbf{C})$$

it follows that it suffices to prove that

$$\mathbf{C}[z_0, z_1, z_2]_{12} x \oplus \mathbf{C}[z_0, z_1, z_2]_{18} \to \oplus H^4_{q_i}(Y, \mathbf{C}) = \oplus_i R(f_{q_i})_5 \tag{10}$$

is surjective.

Define $T_{q,m,d} : \mathbf{C}[z_0, z_1, z_2]_d \to \mathbf{C}^{m(m+1)/2}$ to be the $(m-1)$st part of the Taylor expansion around $(\alpha_1, \alpha_2, \alpha_3)$ for some fixed lift of $q \in \mathbf{P}^2$ to $\mathbf{C}^3$. Then the map form (10) can be factored as

$$\mathbf{C}[z_0, z_1, z_2]_{12} x \oplus \mathbf{C}[z_0, z_1, z_2]_{18} \stackrel{\oplus(T_{q_i,4,12} \oplus T_{q_i,6,18})}{\longrightarrow} \oplus_i \left(\mathbf{C}^{10} \oplus \mathbf{C}^{21}\right) \to \oplus R(f_{q_i})_5.$$

The first map is surjective by Proposition 12.10 and the second map is surjective since it is a projection. From this the lemma follows. □

Applying Theorem 4.3 yields:

Corollary 12.12. *We have* $\operatorname{rank} \mathrm{MW}(\pi) = 0$.

Remark 12.13. Actually, $\mathrm{MW}(\pi) = 0$: let $\ell \subset \mathbf{P}^2$ be a general line. Then $\pi_\ell : \pi^{-1}(\ell) \to \ell$ is an elliptic surface with 36 I_1 fibers. (This follows from the fact that the discriminant curve is reduced.) Suppose $\mathrm{MW}(\pi_\ell)$ has a torsion section of order k, then one can factor the j-map over $X_1(k) \to X(1)$ since this map is ramified at ∞ with ramifaction index k it turns out that π_ℓ has a fiber of type I_{km} of I^*_{km} for some $m \geq 1$. Since all fibers of π_ℓ are of type

I_1 it follows that $\mathrm{MW}(\pi_\ell)$ has trivial torsion part, hence $\mathrm{MW}(\pi)$ has trivial torsion.

Remark 12.14. We can now determine all Hodge number of W by applying 12.3. In particular, $h^{1,1}(W) = m + 2$ and $h^{2,1}(W) = 272 - 29m$.

We would like to illustrate the techniques used in the proof of Theorem 4.3 by explicitly factorizing the rational map $W \dashrightarrow Y$. This explicit factorization yields also the Hodge numbers of W without using the results of [2].

To study the behavior of the cohomology groups we factor $W \dashrightarrow Y$ as a series of proper modifications and inverse proper modifications. For each modification $\mathcal{Z}$ is the center of the modification and E is the exceptional divisor.

(1) We start by blowing up $(1 : 1 : 0 : 0 : 0)$. Denote the obtained threefold by W_1. In this case $\mathcal{Z} = \{pt\}$, and $E \cong \mathbf{P}^2$ and W_1 is smooth near E.
(2) Base change with S. We denote the resulting threefold by W_2. In this case $\mathcal{Z}$ is the union of m cuspidal cubic curves C and E is the disjoint union of m copies $C \times \mathbf{P}^1$. Applying Theorem 4.1 we obtain that $h^4(W_2) = h^4(W_1) + m$ and $h^2(W_2) = h^2(W_1) + m$ and all other Betti number remain unchanged.
(3) The singular locus of W_2 consists of m disjoint curves C_i (topologically a $\mathbf{P}^1$), namely m copies of $\{c\} \times \mathbf{P}^1 \subset C \times \mathbf{P}^1$, where C is the cuspidal curve from the previous point and $c \in C$ is the cusp.

    Let $s = 0$ be a local equation for one of the exceptional curves in S. Then a local equation of W_2 is of the form
$$y^2 = x^3 + s^4 f_1(s,t)x + s^6 f_2(s,t).$$
    Blowing up $\{x = y = s = 0\}$ yields a $\mathbf{P}^1$-bundle over $\{s = 0\}$. Let W_3 be the blow-up of all m curves in the singular locus. Then $\mathcal{Z}$ is the disjoint union of m copies of $\mathbf{P}^1$ and E is the disjoint union of m ruled surfaces. The ruling on each of the irreducible components of E gives a class in $H^4(W_3, \mathbf{Q})$, in total yielding m independent classes in $H^4(W_3, \mathbf{Q})$. Each irreducible component of E yields a class in $H^2(W_3, \mathbf{Q})$, hence in total we get m classes in $H^2(W_3, \mathbf{Q})$. Applying Theorem 4.1 we obtain that $h^4(W_3) = h^4(W_2) + m$ and $h^2(W_3) = h^2(W_2) + m$, all other Betti number are invariant.
(4) The singular locus of W_3 consists of m disjoint curves (topologically a $\mathbf{P}^1$), each of which lies in one of the exceptional divisors of the previous step. Let $s = 0$ denote the image of such a curve in some open set in S. Then a local equation of W_3 is of the form
$$y^2 = sx^3 + s^3 f_1(s,t)x + s^4 f_2(s,t).$$
    Blowing up $\{x = y = s = 0\}$ yields a $\mathbf{P}^1$-bundle over $\{s = 0\}$. Let W_4 be the blow-up of all m curves in the singular locus. Then $\mathcal{Z}$ is

the disjoint union of m copies of $\mathbf{P}^1$ and E is the disjoint union of m ruled surfaces. As above the ruling on each of the irreducible components of E gives a class in $H^4(W_4, \mathbf{Q})$, in total yielding m independent classes in $H^4(W_4, \mathbf{Q})$. Each irreducible component of E yields a class $H^2(W_4, \mathbf{Q})$, hence in total we get m classes in $H^2(W_4, \mathbf{Q})$. Applying Theorem 4.1 we obtain that $h^4(W_4) = h^4(W_3) + m$ and $h^2(W_4) = h^2(W_3) + m$, all other Betti number are invariant.

(5) The singular locus of W_4 consists of m disjoint surfaces (topologically a ruled surface F with base a rational curve in S). This locus is precisely the exceptional divisor of $W_4 \to W_3$.

Let $s = 0$ denote the image of such a surface in some open set in S. Then a local equation of W_4 is of the form

$$y^2 = s^2x^3 + s^2 f_1(s,t)x + s^2 f_2(s,t).$$

Blowing up $\{y = s = 0\}$ yields a smooth threefold W_5. In this case $\mathcal{Z}$ consists of m copies of F, and E consists of m elliptic surfaces $\mathcal{R}_i$. A calculation in local coordinates shows that each $\mathcal{R}_i$ is a relatively minimal rational elliptic surfaces, hence the Betti numbers of $\mathcal{R}$ equal $1, 0, 10, 0, 1$ by [18].

The Betti numbers of F are $1, 0, 2, 0, 1$. From these Betti numbers and Theorem 4.1 it follows that $h^4(W_4) = h^4(W_5)$. Since $h^2(W_4) = h^4(W_4)$ we obtain by Poincaré duality on the smooth threefold W_5 that $h^2(W_4) = h^2(W_5)$. The exact sequence from Theorem 4.1 reads

$$0 \to H^2(F, \mathbf{Q})^m \to H^2(S, \mathbf{Q})^m \to H^3(W_4, \mathbf{Q}) \to H^3(W_5, \mathbf{Q}) \to 0,$$

whence $h^3(W_4) - h^3(W_5) = 8m$. The kernel of the map $H^3(W_4, \mathbf{Q}) \to H^3(W_5, \mathbf{Q})$ is an $8m$-dimensional weight 2 sub-Hodge structure of $H^3(W_4, \mathbf{Q})$.

(6) The threefold W_5 is smooth, but the elliptic fibration $\pi_5 : W_5 \to S$ is not minimal. Let E_i be one of the (fixed) exceptional curves in S. Then over each point $p \in E_i$ we have a reducible fiber, consisting of three components, namely two rational curves and one elliptic curve. Actually $\pi_5^{-1}(E_i)$ is the union of three surfaces, two of which are $\mathbf{P}^1$-bundles and one is a rational elliptic surface.

To obtain a minimal elliptic threefold we need to contract the ruled surfaces over the E_i. Recall that over E_i we have two ruled surfaces, namely the strict transform of $\pi_1^{-1}(E_i)$ and the strict transform of the exceptional divisor of $W_2 \to W_1$. In order to obtain a smooth threefold we first need to contract the strict transform of the first exceptional divisor first, denote the threefold obtained in this way by W_6. Denote by W_7 the threefold obtained by contracting the second ruled surface.

Both contractions are the inverse of a proper modification, where $\mathcal{Z}$ consists of m copies of $\mathbf{P}^1$ and E consists of m ruled surfaces.

Applying Theorem 4.3 yields that $h^2(W_i) = h^2(W_{i-1}) - m$ and $h^4(W_i) = h^4(W_{i-1}) - m$, for $i = 6, 7$.

A summary of the changes in cohomology is the following:

	Y	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_7
h^0	1								1
h^1	0								0
h^2	1	$+1$	$+m$	$+m$	$+m$		$-m$	$-m$	$m+2$
h^3	$546-50m$					$-8m$			$546-58m$
h^4	1	$+1$	$+m$	$+m$	$+m$		$-m$	$-m$	$m+2$
h^5	0								0
h^6	1								1

We started with $W \subset \mathbf{P}(\mathcal{E})$ and we ended up with another smooth threefold W_7. It is by no means clear that it is again isomorphic to W. We show now that W_7 can indeed be embedded in $\mathbf{P}(\mathcal{E})$ and that W and W_7 are actually isomorphic.

For this we have to consider the notion of a Weierstrass fibration. Given an elliptic fibration $\pi : X \to S$ with a zero section $\sigma_0 : S \to X$, we can define a hypersurface $Y \subset \mathbf{P}(\mathcal{O} \oplus \mathcal{N}^{-2} \oplus \mathcal{N}^{-3})$, with $\mathcal{N} = \pi_* N_{S_0/X}^{-1}$ and $S_0 = \sigma_0(S)$, e.g., see [19, Section 1].

The main difference between Y and X is that all components of fibers not intersecting the zero section are contracted. A threefold Y obtained by this procedure is called a Weierstrass fibration. The line bundle $\mathcal{N}$ is called the associated line bundle.

Lemma 12.15. *The following holds for Weierstrass fibrations:*

1. *Let $\psi_1 : Y_1 \to S_1$ be a Weierstrass fibration with line bundle $\mathcal{N}_1$, let $\varphi : S_2 \to S_1$ be a morphism and let $Y_2 = Y_1 \times_{S_1} S_2 \to S_2$ be the base changed Weierstrass-fibration with associated line bundle $\mathcal{N}_2$. Assume that* Pic(S_2) *does not contain an element of order* 2. *Then $\mathcal{N}_2 \cong \varphi^*\mathcal{N}_1$.*
2. *Let $\psi_3 : Y_3 \to S_3$ be a non-minimal Weierstrass fibration with line bundle $\mathcal{N}_3$. Assume that it has an equation of the form*
$$y^2 = x^3 + u^4 f_1 x + u^6 f_2$$
where $u = 0$ defines a reduced divisor D. Assume that Pic(S_3) *does not contain an element of order* 2. *Then the Weierstrass fibration given by*
$$y^2 = x^3 + f_1 x + f_2$$
has associated line bundle $\mathcal{N}_3(-D)$.

Proof. (1) A Weierstrass fibration is a fibration of the form
$$y^2 z = x^3 + Axz^2 + Bz^3$$
with $A \in H^0(\mathcal{N}^4)$, $B \in H^0(\mathcal{N}^6)$. After base change we get an equation
$$y^2 z = x^3 + A_2 xz^2 + B_2 z^3$$

where $A_2 = f^*A$ and $B_2 = f^*B$, hence the associated line bundle is $\mathcal{N}_1^{\otimes 4} = f^*\mathcal{N}_1^{\otimes 4}$ and $\mathcal{N}_2^{\otimes 6} = f^*\mathcal{N}_1^{\otimes 6}$, whence $\mathcal{N}_2^{\otimes 2} = f^*\mathcal{N}_1^{\otimes 2}$. Since $\mathrm{Pic}(S_2)$ has no elements of order 2, this implies that $\mathcal{N}_2 = f^*\mathcal{N}_1$.

(2) Similarly, if we minimize

$$y^2 z = x^3 + A_3 x z^2 + B_3 z^3$$

then we have for the new threefold that

$$y^2 z = x^3 + A_4 x z^2 + B_4 z^3$$

satisfies $A_4 = A_3/u^4$ and $B_4 = B_3/u^6$. Hence $(\mathcal{N}_4)^{\otimes i} = (\mathcal{N}_3(-D))^{\otimes i}$, for $i = 4, 6$, which yields the statement. □

Proposition 12.16. *The threefold W_7 can be embedded in $\mathbf{P}(\mathcal{E})$ and it is isomorphic to W.*

Proof. Of the threefolds W_i we considered above, three are actually Weierstrass fibrations, namely W_1, W_2 and W_7.

Since $P \in H^0(\mathcal{O}_{\mathbf{P}^2}(12))$ and $Q \in H^0(\mathcal{O}_{\mathbf{P}^2}(18))$, we have that W_1 has associated line bundle $\mathcal{N}_1 = \mathcal{O}_{\mathbf{P}^2}(3)$.

Using Lemma 12.15 we get that $\mathcal{N}_2 = \mathcal{O}_S(3H)$. In the minimalization process $W_2 \to W_7$ we minimalize over $D = E_1 + E_2 + \cdots + E_m$. Applying the above lemma yields that $\mathcal{N}_7 = \mathcal{O}_S(3H - \sum E_i) = \omega_S^{-1} = \mathcal{L}$. This implies that $W_7 \subset \mathbf{P}(\mathcal{E})$. Yielding the first claim.

To show that the procedures $W \mapsto Y$ and $Y \mapsto W_7$ are each-other's inverse, we start by describing the former one. We start by taking g_2, g_3 in $H^0(\mathcal{L}^4)$ and $H^0(\mathcal{L}^6)$. We consider these two sections as functions on S. In order to consider them as function on $\mathbf{P}^2$ we multiply them with u^4 and u^6, where $u = 0$ is a defining equation for D, the union of all the exceptional divisors. The functions $u^4 g_2$ and $u^4 g_3$ are pullbacks of functions from $\mathbf{P}^2$, say $u^4 g_2 = \varphi^* P$ and $u^6 g_3 = \varphi^* Q$. Then Y is given by $y^2 = 4x^3 + Px + Q$.

The first part of the proof shows that $Y \mapsto W_7$ is exactly the inverse. □

From Lemma 12.2 it follows that W_7 has trivial canonical bundle. Hence $h^{3,0}(W_7) = h^0(W_7, \Omega^3) = 1$. By Serre duality we get $H^2(W_7, \mathcal{O}_{W_7}) \cong H^1(W_7, \mathcal{O}_{W_7})^*$. We already observed that $h^1(W_7) = 0$, hence

$$H^2(W_7, \mathcal{O}_{W_7}) \cong H^1(W_7, \mathcal{O}_{W_7})^* = 0.$$

This finishes the determination of all Hodge numbers. To conclude we give the complete Hodge diamond of W (and W_7):

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 272-29m & & 272-29m & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & . \end{array}$$

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