

Linear independence of powers of singular moduli of degree 3

Florian Luca, Antonin Riffaut

August 9, 2018

Abstract

We show that two distinct singular moduli $j(\tau), j(\tau')$, such that for some positive integers m, n the numbers $1, j(\tau)^m$ and $j(\tau')^n$ are linearly dependent over \mathbb{Q} generate the same number field of degree at most 2. This completes a result of Riffaut, who proved the above theorem except for two explicit pair of exceptions consisting of numbers of degree 3. The purpose of this article is to treat these two remaining cases.

1 Introduction

Let j be the classical j -function on the Poincaré plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$. A *singular modulus* is a number of the form $j(\tau)$, where $\tau \in \mathbb{H}$ is a complex algebraic number of degree 2. It is known that $j(\tau)$ is an algebraic integer and Class Field Theory tells that

$$[\mathbb{Q}(j(\tau)) : \mathbb{Q}] = [\mathbb{Q}(\tau, j(\tau)) : \mathbb{Q}(\tau)] = h_{\Delta}$$

is the class number of the order $\mathcal{O}_{\Delta} = \mathbb{Z}[(\Delta + \sqrt{\Delta})/2]$, where Δ is the discriminant of the minimal polynomial of τ over \mathbb{Z} . Moreover, $\mathbb{Q}(\tau, j(\tau))/\mathbb{Q}(\tau)$ is an abelian Galois extension with Galois group (canonically) isomorphic to the class group of the order \mathcal{O}_{Δ} . One can also interpret \mathcal{O}_{Δ} as the automorphism ring of the lattice $\langle 1, \tau \rangle$, or of the corresponding elliptic curve. For all details, see, for instance, [7, §7 and §11].

Starting from the ground-breaking article of André [2], equations involving singular moduli were studied by many authors, see [1, 5, 10] for a historical account and further references. In particular, Kühne [8] proved that the equation $x + y = 1$ has no solutions in singular moduli x, y , and Bilu et al. [4] proved the same conclusion holds for the equation $xy = 1$. These results were generalized in [1] and [5]. In [1], solutions of all linear equations $Ax + By = C$, with $A, B, C \in \mathbb{Q}$, were determined. Here is the main result of [1].

Theorem 1.1 (Allombert et al. [1]). *Let x, y be two singular moduli, and A, B, C rational numbers with $AB \neq 0$. Assume that $Ax + By = C$. Then we have one of the following options:*

(trivial case) $A + B = C = 0$ and $x = y$;

(rational case) $x, y \in \mathbb{Q}$;

(quadratic case) $x \neq y$ and x, y generate the same number field over \mathbb{Q} of degree 2.

This result is best possible, since in both the rational case and the quadratic case of Theorem 1.1, one easily finds $A, B, C \in \mathbb{Q}$ such that $AB \neq 0$ and $Ax + By = C$. Moreover, the lists of singular moduli of degrees 1 and 2 over \mathbb{Q} are widely available or can be easily generated using a suitable computer package, like PARI [11]. In particular, there are 13 rational singular moduli, and 29 pairs of \mathbb{Q} -conjugate singular moduli of degree 2; see [5, Section 1] for more details. This means that Theorem 1.1 gives a completely explicit characterization of all solutions.

In [10], Riffaut generalized Theorem 1.1 by introducing exponents; that is, instead of equation $Ax + By = C$, he considered the more general equation $Ax^m + By^n = C$, where the positive integer exponents m, n are unknown as well. He proved that, if $x \neq y$, then x, y generate the same number field of degree $h \leq 3$, and $h = 3$ is possible only if either $\{\Delta, \Delta'\} = \{-4 \cdot 23, -23\}$, or $\{\Delta, \Delta'\} = \{-4 \cdot 31, -31\}$, where Δ, Δ' denote the respective discriminants of x and y . In this article, we eliminate these two remaining cases. Here is the statement of our result.

Theorem 1.2. *Let $x = j(\tau), y = j(\tau')$ be two singular moduli of respective discriminants Δ and Δ' , and m, n two positive integers. If $\{\Delta, \Delta'\} = \{-4 \cdot 23, -23\}$ or $\{\Delta, \Delta'\} = \{-4 \cdot 31, -31\}$, then the numbers $1, x^m, y^n$ are linearly independent over \mathbb{Q} .*

Consequently, Theorem 1.2 together with [10, Theorem 1.5] completely solve the above equation for distinct singular moduli, and we deduce the following Theorem.

Theorem 1.3. *Let $x = j(\tau), y = j(\tau')$ be two distinct singular moduli of respective discriminants Δ and Δ' , and m, n two positive integers. Assume that $Ax^m + By^n = C$, for some $A, B, C \in \mathbb{Q}^\times$. Then x and y generate the same number field over \mathbb{Q} of degree at most 2.*

As previously, this result is now best possible for distinct singular moduli, since if $h \leq 2$, then for all exponents m, n , one easily finds $A, B, C \in \mathbb{Q}^\times$ such that $Ax^m + By^n = C$. However, our current methods are still not able to handle the case $x = y$, which is equivalent to the following question: can a singular modulus of degree 3 or higher be a root of a trinomial with rational coefficients? Much about trinomials is known, but this knowledge is still insufficient to rule out such a possibility. Otherwise, the assumption $C \neq 0$ is seemingly restrictive, but in fact, the case $C = 0$ is contained in [10, Theorem 1.6].

Our calculations were performed using the PARI/GP package [11]. The sources are available from the second author.

2 Preliminaries

Below we briefly recall some basic facts about the conjugates of a singular modulus and the height of an algebraic number.

Fields generated by a power of a singular modulus

Let $j(\tau)$ be a singular modulus of discriminant Δ . It is well-known that the conjugates of $j(\tau)$ over \mathbb{Q} can be described explicitly; see, for instance, [10, Subsection 2.2]. In particular, $j(\tau)$ admits one real conjugate which has the property that it is much larger in absolute value than all its other conjugates, called the *dominant j -value* of discriminant Δ . As a useful consequence, a singular modulus and any of its powers generate the same field over \mathbb{Q} ; see [10, Lemma 2.6], a statement which we reproduce below.

Lemma 2.1. *Let x be a singular modulus of discriminant Δ , with $|\Delta| \geq 11$, and n a non-zero integer. Then $\mathbb{Q}(x) = \mathbb{Q}(x^n)$.*

The height of a non-zero algebraic number

Let α be a non-zero algebraic number of degree d over \mathbb{Q} , and $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$ all its conjugates in $\overline{\mathbb{Q}}$. The logarithmic height of α , denoted by $h(\alpha)$, is defined to be

$$h(\alpha) = \frac{1}{d} \left(\log |a| + \sum_{k=1}^d \log \max\{1, |\alpha_k|\} \right),$$

where a is the leading coefficient of the minimal polynomial of α in \mathbb{Z} . In particular, $\log |a| = 0$ when α is an algebraic integer.

Here are some useful properties of the logarithmic height.

- For any non-zero algebraic number α and $\lambda \in \mathbb{Q}^*$, we have $h(\alpha^\lambda) = |\lambda| h(\alpha)$. In particular, $h(1/\alpha) = h(\alpha)$. See [6, Lemma 1.5.18].
- For any two non-zero algebraic numbers α and β , we have $h(\alpha\beta) \leq h(\alpha) + h(\beta)$.

3 Linear forms in two logarithms

Let α be an algebraic number with $|\alpha| = 1$ but not a root of unity and m a positive integer. We are interested in estimating the quantity $\lambda = 1 - \alpha^m$, which is closely related to a linear form in two logarithms.

Laurent, Mignotte and Nesterenko describe in [9] a lower bound on the absolute value of a general linear form in two logarithms, see [9, Théorème 3]. In our particular case, Mignotte give in [3] a slight sharpening of this bound. The following Theorem is a corollary of [3, Theorems A.1.2 and A.1.3].

Theorem 3.1. *Let α be a complex algebraic number with $|\alpha| = 1$, but not a root of unity, and m a positive integer. There exists an effective computable constant $c_1(\alpha) > 0$, depending only on the degree d of α over \mathbb{Q} and its logarithmic height $h(\alpha)$, such that*

$$|1 - \alpha^m| > 0.99e^{-c_1(\alpha)(\log m)^2}.$$

Proof. We briefly detail the proof, especially to explain how to compute $c_1(\alpha)$ in terms of d and $h(\alpha)$.

We apply [3, Theorems A.1.2 and A.1.3] to the linear form

$$\Lambda = 2i\pi - m \log \alpha,$$

where we choose the principal complex logarithm (defined on $\mathbb{C} \setminus \mathbb{R}^-$) for $\log \alpha$. We have

$$\log |\Lambda| > -(9.03\mathcal{H}^2 + 0.23)(Dh(\alpha) + 25.84) - 2\mathcal{H} - 2 \log \mathcal{H} - 0.7D + 2.07,$$

where $D = d/2$ and $\mathcal{H} = D(\log m - 0.96) + 4.49 \leq c'_1(d) \log m$ for $m \geq 13$, with

$$c'_1(d) = D + \max \left\{ 0, \frac{4.49 - 0.96D}{\log 13} \right\} > 0.$$

Hence,

$$\begin{aligned} \log |\Lambda| &> -(\log m)^2 \left(9.03c'_1(d)^2(Dh(\alpha) + 25.84) + \frac{2c'_1(d)}{\log m} + \frac{2 \log \log m}{(\log m)^2} \right. \\ &\quad \left. + \frac{0.23(Dh(\alpha) + 25.84) + 2 \log c'_1(d) + 0.7D - 2.07}{(\log m)^2} \right) > -c_1(\alpha)(\log m)^2, \end{aligned}$$

with

$$\begin{aligned} c_1(\alpha) &= 9.03c'_1(d)^2(Dh(\alpha) + 25.84) + \frac{2c'_1(d)}{\log 13} + \frac{2 \log \log 13}{(\log 13)^2} \\ &\quad + \frac{0.23(Dh(\alpha) + 25.84) + 2 \log c'_1(d) + 0.7D - 2.07}{(\log 13)^2}. \end{aligned}$$

It follows that

$$|1 - \alpha^m| > \frac{e^{-c_1(\alpha)(\log m)^2}}{1 + e^{-c_1(\alpha)(\log m)^2}} > 0.99e^{-c_1(\alpha)(\log m)^2},$$

resulting from the mean value theorem. ■

In practice, if α is explicitly known (as an algebraic number in number field L), it is then possible to compute effectively $c_1(\alpha)$ for $m \geq 13$. For $m < 13$, one just has to estimate directly $|1 - \alpha^m|$.

Another way of estimating $1 - \alpha^m$ is to reduce it modulo a prime ideal \mathfrak{p} of \mathcal{O}_L . More precisely, we want to evaluate its valuation $v_{\mathfrak{p}}(1 - \alpha^m)$ at \mathfrak{p} ; for an element $z \in L$, we write $v_{\mathfrak{p}}(z)$ instead of $v_{\mathfrak{p}}(z\mathcal{O}_L)$ for more simplicity. This can be obtained as follows.

Proposition 3.2. *Let α be an algebraic integer that is not a root of unity in a number field L of degree d , and m a positive integer. Let \mathfrak{p} be a prime ideal of \mathcal{O}_L over a prime number p . Assume that $\mathfrak{p} \nmid \alpha$. Denote by m_0 the order of α in $\mathcal{O}_L/\mathfrak{p}$, that is the least positive integer such that $1 - \alpha^{m_0} = 0 \pmod{\mathfrak{p}}$, and $v_0 = v_{\mathfrak{p}}(1 - \alpha^{m_0})$. Then, assuming $p > d + 1$, we have*

$$v_{\mathfrak{p}}(1 - \alpha^m) = \begin{cases} 0 & \text{if } m_0 \nmid m \\ sv_{\mathfrak{p}}(p) + v_0 & \text{if } m = m_0p^s r, \gcd(p, r) = 1. \end{cases}$$

Proof. If $m_0 \nmid m$, it is clear that $1 - \alpha^m \not\equiv 0 \pmod{\mathfrak{p}}$; hence, $v_{\mathfrak{p}}(1 - \alpha^m) = 0$. Otherwise, write $m = m_0p^s r$ with $\gcd(p, r) = 1$. We proceed by induction on $s \geq 0$. For $s = 0$, factoring $1 - \alpha^m$ gives

$$1 - \alpha^m = (1 - \alpha^{m_0}) \left(\sum_{l=0}^{r-1} \alpha^{m_0 l} \right).$$

Since $\alpha^{m_0 l} \equiv 1 \pmod{\mathfrak{p}}$, for all $l \in \{0, \dots, r-1\}$, we deduce

$$v_{\mathfrak{p}}(1 - \alpha^m) = v_{\mathfrak{p}}(1 - \alpha^{m_0 r}) + v_{\mathfrak{p}}(r) = v_0.$$

We now let $\beta = \alpha^{r m_0}$ and treat the case $s = 1$. Writing $\beta = 1 + \lambda$, where $\lambda \in \mathfrak{p}$, we have that

$$\frac{\beta^p - 1}{\beta - 1} = \frac{(1 + \lambda)^p - 1}{\lambda} = \sum_{k=1}^{p-1} \binom{p}{k} \lambda^{k-1} + \lambda^{p-1}.$$

In the right-hand side, we have that $v_{\mathfrak{p}}(\lambda) \geq 1$, and $v_{\mathfrak{p}}(\lambda^{p-1}) \geq (p-1) > d \geq v_{\mathfrak{p}}(p)$, so

$$v_{\mathfrak{p}}\left(\sum_{k=1}^{p-1} \binom{p}{k} \lambda^{k-1} + \lambda^{p-1}\right) = v_{\mathfrak{p}}(p).$$

Hence, for $s = 1$, we have

$$v_{\mathfrak{p}}(1 - \alpha^m) = v_{\mathfrak{p}}(1 - \alpha^{m_0 r}) + v_{\mathfrak{p}}\left(\frac{\beta^p - 1}{\beta - 1}\right) = v_0 + v_{\mathfrak{p}}(p).$$

The statement now follows by induction on s , where the induction step from s to $s+1$ is done as above (by replacing α by α^{p^s}). \blacksquare

4 Proof of Theorem 1.2

Let $x = j(\tau), y = j(\tau')$ be two singular moduli of respective discriminants Δ and Δ' , with $\{\Delta, \Delta'\} = \{-4 \cdot 23, -23\}$ or $\{\Delta, \Delta'\} = \{-4 \cdot 31, -31\}$, such that

$$Ax^m + By^n = C \tag{4.1}$$

for some $A, B, C \in \mathbb{Q}^\times$ and m, n positive integers.

Both x and y are of degree 3 over \mathbb{Q} , and admit one real conjugate corresponding to the dominant j -value, and two complex conjugates. If x is real, then y is also real. Indeed, if not, then, together with (4.1), we have

$$Ax^m + B\bar{y}^n = C.$$

We obtain that $y^n = \bar{y}^n$, which contradicts Lemma 2.1.

The equation (4.1) implies that $\mathbb{Q}(x^m) = \mathbb{Q}(y^n)$; hence, $\mathbb{Q}(x) = \mathbb{Q}(y)$ by Lemma 2.1. In particular, the Galois orbit of (x, y) over \mathbb{Q} has exactly 3 elements, and each conjugate of x occurs exactly once as the first coordinate of a point in the orbit, just as each conjugate of y occurs exactly once as the second coordinate.

We denote by $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ the conjugates of (x, y) , with x_1, y_1 real, and x_2, x_3 , respectively y_2, y_3 , are complex conjugates. By (4.1) again, the points (x_i^m, y_i^n) , $i \in \{1, 2, 3\}$, are collinear. We can write the relation of collinearity of these points in one of the following two ways:

$$\begin{vmatrix} 1 & x_1^m & y_1^n \\ 1 & x_2^m & y_2^n \\ 1 & x_3^m & y_3^n \end{vmatrix} = 0; \tag{4.2}$$

$$\left(\frac{x_1}{x_2}\right)^{-m} \left(\frac{y_1}{y_2}\right)^n = \frac{1 - \left(\frac{y_3}{y_2}\right)^n - \left(\frac{x_3}{x_1}\right)^m}{1 - \left(\frac{y_3}{y_1}\right)^n - \left(\frac{x_3}{x_2}\right)^m}. \tag{4.3}$$

We focus first on the case $\{\Delta, \Delta'\} = \{-4 \cdot 23, -23\}$, and we detail afterwards the slight differences in the treatment of the case $\{\Delta, \Delta'\} = \{-4 \cdot 31, -31\}$. We denote by L the Galois closure of $\mathbb{Q}(x) = \mathbb{Q}(y)$, which by definition contains all x_i 's and y_i 's.

As announced above, we consider the case $\Delta = 4\Delta' = -4 \cdot 23$.

Using PARI, one can find a prime ideal \mathfrak{p} of \mathcal{O}_L over $p = 23$ such that $\mathfrak{p} | x_2 \mathcal{O}_L$, $\mathfrak{p} | x_3 \mathcal{O}_L$, but $\mathfrak{p} \nmid x_1 y_2 y_3 \mathcal{O}_L$. Hence, modulo \mathfrak{p}^m , the equation (4.2) becomes

$$1 - \alpha^n = 0 \pmod{\mathfrak{p}^m},$$

with $\alpha = y_3/y_2$. On the one hand, we deduce that $m \leq v_{\mathfrak{p}}(1 - \alpha^n)$. On the other hand, we apply Proposition 3.2, checking first that $1 - \alpha = 0 \pmod{\mathfrak{p}}$, $v_{\mathfrak{p}}(1 - \alpha) = 1$, $v_{\mathfrak{p}}(p) = 2 < 6 < 22 = p - 1$; writing $m = p^s r$ with $\gcd(p, r) = 1$, we get

$$v_{\mathfrak{p}}(1 - \alpha^m) = s v_{\mathfrak{p}}(p) + 1 = 2s + 1.$$

Consequently,

$$m \leq 2 \frac{\log n}{\log 23} + 1. \quad (4.4)$$

Next, we want to estimate the expression on the right-hand side of (4.3) in terms of m and n (in fact, only in terms of n thanks to (4.4)), in order to obtain a bound on n . The principal difficulty is to find a lower bound of the absolute value of its denominator. Since y_3/y_1 is pretty close to 0, it depends essentially on the quantity $1 - \beta^m$ with $\beta = x_3/x_2$. Noticing that $|\beta| = 1$ and β is not a root of unity, then according to Theorem 3.1, there exists a constant $c_1(\beta) > 0$ such that

$$|1 - \beta^m| > 0.99e^{-c_1(\beta)(\log m)^2}.$$

Explicitly, for $m \geq 13$, we can choose $c_1(\beta) = 4973.14$. It follows that

$$\begin{aligned} \left| 1 - \left(\frac{y_3}{y_1}\right)^n - \left(\frac{x_3}{x_2}\right)^m \right| &> 0.99e^{-4973.15(\log m)^2} - \left|\frac{y_3}{y_1}\right|^n \\ &> 0.99e^{-4973.14(\log(2\frac{\log n}{\log 23} + 1))^2} - \left|\frac{y_3}{y_1}\right|^n \end{aligned}$$

(recall the inequality (4.4)). By a quick calculation, we observe that the last term of the previous inequality is positive provided that $n > 2074$. More specifically, if $n > 2075$, then

$$\left| 1 - \left(\frac{y_3}{y_1}\right)^n - \left(\frac{x_3}{x_2}\right)^m \right| > 0.98e^{-4973.14(\log(2\frac{\log n}{\log 23} + 1))^2}.$$

Finally, for $m \geq 13$ and $n > 2075$, we have

$$\left|\frac{x_1}{x_2}\right|^{-m} \left|\frac{y_1}{y_2}\right|^n \leq 2.05e^{4973.14(\log(2\frac{\log n}{\log 23} + 1))^2},$$

and

$$\begin{aligned} -\left(2\frac{\log n}{\log 23} + 1\right) \log \left|\frac{x_1}{x_2}\right| + n \log \left|\frac{y_1}{y_2}\right| &\leq \log 2.05 \\ &\quad + 4973.14 \left(\log \left(2\frac{\log n}{\log 23} + 1\right)\right)^2. \end{aligned}$$

This last inequality yields $n \leq 2092$, and then (4.4) gives $m \leq 5$. This is in contradiction with the previous assumptions $m \geq 13$ and $n > 2075$. Therefore, either $m < 13$ or $n \leq 2075$. In both cases, $m < 13$, and for each possible m , we can explicitly compute a constant $c_2(m)$ such that

$$\left| \frac{x_1}{x_2} \right|^{-m} \left| \frac{y_1}{y_2} \right|^n \leq c_2(m).$$

This allows to bound n . The table below summarizes all constants $c_2(m)$ and all bounds we obtain.

Table 4.1: Constants $c_2(m)$ and bounds on n for each $m < 13$, in the case $\Delta = 4\Delta' = -4 \cdot 23$

m	$c_2(m)$	Upper bound of n
1	1.15	2
2	1.21	5
3	11.97	8
4	1.10	10
5	1.28	13
6	6.00	16
7	1.07	18
8	1.38	21
9	4.02	24
10	1.04	26
11	1.50	29
12	3.04	32

Again, inequality (4.4) eliminates all entries of Table 4.1 with $m \geq 3$. Consequently, either $m = 1$ and $n \leq 2$, or $m = 2$ and $n \leq 5$. For each of these remaining couples (m, n) , a direct calculation shows that the determinant in equation (4.2) does not vanish.

To finish, we repeat this process for the case $\Delta = 4\Delta' = -4 \cdot 31$. In this case, one can find a prime ideal \mathfrak{p} of \mathcal{O}_L over $p = 11$ such that $\mathfrak{p} | x_2 \mathcal{O}_L$, $\mathfrak{p} | x_3 \mathcal{O}_L$, but $\mathfrak{p} \nmid x_1 y_2 y_3 \mathcal{O}_L$ as before, and we get

$$m \leq \frac{\log n}{\log 11} + 2. \quad (4.5)$$

We obtain as well, for $m \geq 13$ and $n > 1440$,

$$\left| \frac{x_1}{x_2} \right|^{-m} \left| \frac{y_1}{y_2} \right|^n \leq 2.05e^{4820.16(\log(\frac{\log n}{\log 11} + 2))^2},$$

then

$$-\left(\frac{\log n}{\log 11} + 2\right) \log \left| \frac{x_1}{x_2} \right| + n \log \left| \frac{y_1}{y_2} \right| \leq \log 2.05 + 4820.16 \left(\log \left(\frac{\log n}{\log 11} + 2 \right) \right)^2,$$

which yields $n \leq 1720$ and $m \leq 5$; again a contradiction. For each possible $m < 13$, we compute a constant $c_2(m)$ as defined above, and we deduce a bound on n . Here is the table:

Table 4.2: Constants $c_2(m)$ and bounds on n for each $m < 13$, in the case $\Delta = 4\Delta' = -4 \cdot 31$

m	$c_2(m)$	Upper bound of n
1	1.13	3
2	1.25	6
3	6.17	10
4	1.06	13
5	1.44	16
6	3.13	19
7	1.02	22
8	1.76	26
9	2.13	29
10	1.01	32
11	2.33	36
12	1.65	39

Inequality (4.5) eliminates all entries of Table 4.2 with $m \geq 3$. Consequently, either $m = 1$ and $n \leq 3$, or $m = 2$ and $n \leq 6$. Each of these remaining possibilities can be excluded by a direct calculation showing that the respective determinant does not vanish.

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