FINITENESS OF IRREDUCIBLE HOLOMORPHIC ETA QUOTIENTS OF A GIVEN LEVEL

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Abstract. We show that for any positive integer N, there are only finitely many holomorphic eta quotients of level N, none of which is a product of two holomorphic eta quotients other than 1 and itself. This result is an analog of Zagier's conjecture/ Mersmann's theorem which states that: Of any given weight, there are only finitely many irreducible holomorphic eta quotients, none of which is an integral rescaling of another eta quotient. We construct such eta quotients for all cubefree levels. In particular, our construction demonstrates the existence of irreducible holomorphic eta quotients of arbitrarily large weights.

1. Introduction

The Dedekind eta function is defined by the infinite product:

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) \text{ for all } z \in \mathfrak{H}, \tag{1.1}$$

where $q^r = q^r(z) := e^{2\pi i r z}$ for all r and $\mathfrak{H} := \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$. Eta is a holomorphic function on $\mathfrak{H}$ with no zeros. This function has its significance in Number Theory. For example, $1/\eta$ is the generating function for the ordinary partition function $p : \mathbb{N} \to \mathbb{N}$ (see [1]) and the constant term in the Laurent expansion at 1 of the Epstein zeta function ζ_Q attached to a positive definite quadratic form Q is related via the Kronecker limit formula to the value of η at the root of the associated quadratic polynomial in $\mathfrak{H}$ (see [9]). The value of η at such a quadratic irrationality of discriminant $-D$ is also related via the Lerch/Chowla-Selberg formula to the values of the Gamma function with arguments in $D^{-1}\mathbb{N}$ (see [23]). In fact, the eta function comes up naturally in many other areas of Mathematics (see the Introduction in [4] for a brief overview of them).

The function η is a modular form of weight $1/2$ with a multiplier system on $\mathrm{SL}_2(\mathbb{Z})$ (see [12]). An eta quotient f is a finite product of the form

$$\prod \eta_d^{X_d}, \tag{1.2}$$

where $d \in \mathbb{N}$, η_d is the rescaling of η by d, defined by

$$\eta_d(z) := \eta(dz) \quad \text{for all } z \in \mathfrak{H} \tag{1.3}$$

and $X_d \in \mathbb{Z}$. Eta quotients naturally inherit modularity from η: The eta quotient f in (1.2) transforms like a modular form of weight $\frac{1}{2}\sum_d X_d$ with a

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multiplier system on suitable congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$: The largest among these subgroups is

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \pmod{N} \right\}, \tag{1.4}$$

where

$$N := \operatorname{lcm}\{d \in \mathbb{N} \mid X_d \neq 0\}. \tag{1.5}$$

We call N the *level* of f. Since η is non-zero on $\mathfrak{H}$, the eta quotient f is holomorphic if and only if f does not have any pole at the cusps of $\Gamma_0(N)$.

An *eta quotient on* $\Gamma_0(M)$ is an eta quotient whose level divides M. Let f, g and h be nonconstant holomorphic eta quotients on $\Gamma_0(M)$ such that $f = g \times h$. Then we say that f is *factorizable on* $\Gamma_0(M)$. We call a holomorphic eta quotient f of level N *quasi-irreducible* (resp. *irreducible*), if it is not factorizable on $\Gamma_0(N)$ (resp. on $\Gamma_0(M)$ for all multiples M of N).

Irreducible holomorphic eta quotients were first considered by Zagier, who conjectured (see [24]) that: *There are only finitely many primitive and irreducible holomorphic eta quotients of a given weight.* An eta quotient f is called *primitive* if no eta quotient h and no integer $\nu > 1$ satisfy the equation $f = h_\nu$, where $h_\nu(z) := h(\nu z)$ for all $z \in \mathfrak{H}$. Zagier's conjecture was established by his student Mersmann in an excellent *Diplomarbeit* [15]. I gave a simplified proof of this theorem in [8]. Unfortunately, none of the existing proofs of Mersmann's finiteness theorem yield an explicit upper bound for the levels of primitive and irreducible holomorphic eta quotients of a given weight. However, another approach would be to look at the problem from the dual perspective, where instead of considering holomorphic eta quotients of a given weight, we consider holomorphic eta quotients of a given level. If one could obtain a nontrivial estimate for the least possible weight for a primitive and irreducible holomorphic eta quotient of level N, that would immediately imply an effective proof of Mersmann's finiteness theorem. For example, if p is a prime and if $N = p$ or p^2, then the least possible weight of a primitive and irreducible holomorphic eta quotient of level N is $(p-1)/2$ (see Section 6.3 in [7]). Though the notions of irreducibility and quasi-irreducibility of holomorphic eta quotients are conjecturally equivalent (see [4]), in practice irreducibility of a holomorphic eta quotient is much harder to determine than its quasi-irreducibility. However, since every irreducible holomorphic eta quotient is quasi-irreducible, the least possible weight of a primitive and irreducible holomorphic eta quotient of level N is bounded below by the least possible weight of a primitive and quasi-irreducible holomorphic eta quotient of level N. We denote the later by $k_{\min}(N)/2$. With a huge amount of numerical evidence similar to what we see below in Table 1, we speculate that

Conjecture 1. *For a positive integer N, if there exists a primitive and irreducible holomorphic eta quotient of weight $k/2$ and level N, then*

$$4k^2 \geq \max_{\substack{p^n \| N \\ p \text{ prime}}} np. \tag{1.6}$$

Mersmann's finiteness theorem is equivalent to say that for each $k \in \mathbb{N}$, there exists an $M_k \in \mathbb{N}$ such that if there exists a primitive and irreducible

Table 1. N vs. $k_{\min}(N)$

N	$k_{\min}$	N	$k_{\min}$	N	$k_{\min}$	N	$k_{\min}$
$2\cdot 3$	1	$2^2\cdot 13$	4	$3\cdot 5\cdot 7$	3	$2^2\cdot 7^2$	3
$2\cdot 5$	2	$2\cdot 3^3$	2	$2^2\cdot 3^3$	2	$2^3\cdot 5^2$	3
$2^2\cdot 3$	1	$2^3\cdot 7$	2	$3\cdot 37$	11	$11\cdot 19$	11
$2\cdot 7$	3	$3\cdot 19$	6	$2^4\cdot 7$	2	$2^3\cdot 3^3$	2
$3\cdot 5$	2	$2^2\cdot 3\cdot 5$	2	$2^3\cdot 3\cdot 5$	2	$2^4\cdot 3\cdot 5$	2
2^4	2	$3^2\cdot 7$	2	$2\cdot 3^2\cdot 7$	2	3^5	3
$2\cdot 3^2$	2	2^6	2	2^7	3	$2^2\cdot 3^2\cdot 7$	2
$2^2\cdot 5$	2	$2\cdot 3\cdot 11$	3	$2\cdot 5\cdot 13$	3	2^8	3
$3\cdot 7$	3	$2^2\cdot 17$	4	$7\cdot 19$	8	$2^5\cdot 3^2$	2
$2\cdot 11$	4	$2\cdot 5\cdot 7$	2	$3^3\cdot 5$	3	$2^2\cdot 3^4$	2
$2^3\cdot 3$	2	$2^3\cdot 3^2$	2	$2^3\cdot 17$	3	$2^4\cdot 3^3$	2
$2\cdot 13$	5	$2\cdot 37$	13	$2\cdot 3\cdot 23$	3	2^9	3
$2^2\cdot 7$	3	$3\cdot 5^2$	3	$2^2\cdot 5\cdot 7$	2	$2^6\cdot 3^2$	2
$2\cdot 3\cdot 5$	2	$2\cdot 3\cdot 13$	3	$2^4\cdot 3^2$	2	5^4	5
2^5	2	$2^4\cdot 5$	2	$2^2\cdot 37$	8	3^6	3
$2\cdot 17$	6	3^4	3	$2\cdot 3\cdot 5^2$	2	$2^8\cdot 3$	2
$2^2\cdot 3^2$	2	$2^2\cdot 3\cdot 7$	2	$2\cdot 7\cdot 11$	3	2^{10}	3
$2\cdot 19$	7	$5\cdot 17$	6	$2\cdot 3^4$	3	$17\cdot 97$	21
$3\cdot 13$	5	$2^3\cdot 11$	2	$2^3\cdot 3\cdot 7$	2	2^{11}	3
$2^3\cdot 5$	2	$2\cdot 3^2\cdot 5$	2	$2\cdot 5\cdot 17$	4	3^7	5
$2\cdot 3\cdot 7$	2	$2\cdot 47$	16	$2^2\cdot 43$	9	7^4	7
$2^2\cdot 11$	3	$2^5\cdot 3$	2	$2^4\cdot 11$	2	5^5	5
$3^2\cdot 5$	2	$2\cdot 7^2$	3	$2^2\cdot 3^2\cdot 5$	2	2^{12}	3
$2\cdot 23$	8	$3^2\cdot 11$	4	$2\cdot 7\cdot 13$	3	3^8	5
$2^4\cdot 3$	2	$2^2\cdot 5^2$	2	$2^3\cdot 23$	3	2^{13}	3
$2\cdot 5^2$	2	$2\cdot 3\cdot 17$	3	$3^3\cdot 7$	3	2^{14}	3
$3\cdot 17$	6	$2^3\cdot 13$	3	$2^6\cdot 3$	2	2^{15}	4

holomorphic eta quotient of weight less than or equal to $k/2$ and level N, then N divides M_k. From [24], we know that $M_1 = 12$. Also, from Corollary A.1 in the appendix of [7], we know*

$$2^8\cdot 3^3\cdot 5^2\cdot 7\cdot 11 \,|\, M_2. \tag{1.7}$$

In particular, the truth of the above conjecture would imply

$$M_2 \,|\, 2^8\cdot 3^5\cdot 5^3\cdot 7^2\cdot 11\cdot 13. \tag{1.8}$$

Since there are only finitely many holomorphic eta quotients of a given weight and level (see 3.12), knowing M_k is equivalent to having a complete list of primitive and irreducible holomorphic eta quotients of weight $k/2$, up to a bounded amount of computation. About thirty years ago, Zagier gave such a list for eta quotients of weight $1/2$ (see [24]), the exhaustiveness

*Unlike the general case, irreducibility of a holomorphic eta quotient of weight 1 is rather easy to determine, because a holomorphic eta quotient of weight 1 and level N is irreducible if and only if it is not factorizable on $\Gamma_0(\mathrm{lcm}(N, 12))$ (see Lemma 1 in [4]). In particular, the irreducibility of the holomorphic eta quotients listed in Appendix A in [7] could be easily verified.

of which was also established by Mersmann (see [15, 5]). Though Mersmann's finiteness theorem predicts the existence of similar lists of eta quotients for any given weight, till now we do not even have such an exhaustive list for primitive and irreducible holomorphic eta quotients of weight 1 (for an incomplete list, see Appendix A in [7]). On the other hand, it is much easier to list *all* the irreducible holomorphic eta quotients of a given level! For example, let us look at the following cases: Since the only holomorphic eta quotients of level 1 are the powers of eta, η is the only irreducible holomorphic eta quotient of level 1. In general, since the weight of any holomorphic eta quotient is at least $1/2$, each eta quotient of weight $1/2$ is irreducible. In particular, for any $p \in \mathbb{N}$, the eta quotient η_p is irreducible. Again, from Corollary 2 in [4], we know that for any prime p, the holomorphic eta quotients η^p/η_p and η_p^p/η are irreducible (the irreducibility of the later also follows from Lemma 7 below). It is easy to show that any other holomorphic eta quotient of level p except these three is factorizable on $\Gamma_0(p)$. So, the above three are the only irreducible holomorphic eta quotients of a prime level p. Here, we shall show that the finiteness of irreducible holomorphic eta quotients of a given level also holds in general. This in particular, implies that the maximum of the weights of the irreducible holomorphic eta quotients of level N is bounded above with respect to N. Conversely, since the valence formula implies that there are only finitely many holomorphic eta quotients of a given level and weight (see 3.12), the finiteness of irreducible holomorphic eta quotients of a given level is also implied by such an upper bound. In particular, the finiteness of quasi-irreducible holomorphic eta quotients of a given level (see Theorem 1), has an application in [4], in showing that the levels of the factors of a holomorphic eta quotient f are bounded above in terms of the level of f.

Before ending this section, let us compare the situation with that of the modular forms with the trivial multiplier system. Note that the notions of irreducibility and factorizability also makes sense if we replace "holomorphic eta quotients" with "modular forms" above. For example, the modular form $\Delta := \eta^{24}$ of level 1 is not factorizable into a product of modular forms with the trivial multiplier system on $\mathrm{SL}_2(\mathbb{Z})$, since Δ is a cusp form of the least possible weight on the full modular group. However, Δ is factorizable on $\Gamma_0(2)$:

$$\Delta = \eta^8\eta_2^8 \times \frac{\eta^{16}}{\eta_2^8}. \tag{1.9}$$

From (3.6), it follows readily that η^p/η_p is holomorphic for each prime p. In particular, so is the rightmost eta quotient in (1.9). Also, from Newman's criteria (see [16, 17] or [19]), it follows that the multiplier systems of both of the eta quotients on the right hand side of (1.9) are trivial.

For $k \in 2\mathbb{N}$, we define the normalized Eisenstein series E_k by

$$E_k(z) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n, \tag{1.10}$$

where the function $\sigma_{k-1}:\mathbb{N}\to\mathbb{N}$ is given by

$$\sigma_{k-1}(n):=\sum_{d|n} d^{k-1} \tag{1.11}$$

and the k-th Bernoulli number B_k is defined by

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}\frac{B_k}{k!}\cdot t^k. \tag{1.12}$$

For each even integer $k>2$, E_k is a modular form of weight k on $\mathrm{SL}_2(\mathbb{Z})$ (see [24]). Since there are no nonzero modular forms of odd weight or weight 2 with the trivial multiplier system on $\mathrm{SL}_2(\mathbb{Z})$, neither E_4 nor E_6 is factorizable on $\mathrm{SL}_2(\mathbb{Z})$. However, since $E_6(i)=0$ and $E_4(e^{2\pi i/3})=0$ and since the valence formula (3.9) for $\Gamma_0(2)$ (resp. for $\Gamma_0(4)$) implies that the modular form

$$f_1:=\eta_2(i)^{24}\frac{\eta^{16}}{\eta_2^8}-\eta(i)^{24}\frac{\eta_2^{16}}{\eta^8},\quad\text{resp.}\quad f_2:=\eta_4(e^{\frac{2\pi i}{3}})^8\frac{\eta^8}{\eta_2^4}-\eta(e^{\frac{2\pi i}{3}})^8\frac{\eta_4^8}{\eta_2^4} \tag{1.13}$$

of weight 4 on $\Gamma_0(2)$ (resp. of weight 2 on $\Gamma_0(4)$) only has a simple zero at i in $\Gamma_0(2)\backslash\mathfrak{H}$ (resp. at $e^{2\pi i/3}$ in $\Gamma_0(4)\backslash\mathfrak{H}$), it follows that f_1 is a nontrivial factor of E_6 on $\Gamma_0(2)$ (resp. f_2 is a nontrivial factor of E_4 on $\Gamma_0(4)$). It is easy to check that for all integers $N>1$, the stabilizers of i and $e^{2\pi i/3}$ in $\Gamma_0(N)$ are trivial. The holomorphy of the eta quotients in the above linear combinations follows trivially from (3.15), once one notes the outermost columns of the matrix in (3.17). The triviality of the multiplier systems of these eta quotients follows again from Newman's criteria (see [16, 17] or [19]).

Also, it follows from the valence formula (3.9) for $\mathrm{SL}_2(\mathbb{Z})$ that every modular form with the trivial multiplier system on $\mathrm{SL}_2(\mathbb{Z})$ has a unique factorization of the form:

$$C_0\,E_4^aE_6^b\prod_{t\in\mathbb{C}^*}(E_4^3-tE_6^2)^{c_t}, \tag{1.14}$$

for some $C_0\in\mathbb{C}$ and some nonnegative integers a,b,c_t, where c_t is zero for all but finitely many t. In particular, any modular form with the trivial multiplier system and of weight greater than 12 on $\mathrm{SL}_2(\mathbb{Z})$ is factorizable on $\mathrm{SL}_2(\mathbb{Z})$. We have $E_4^3-E_6^2=1728\Delta$ (see [24]), which is factorizable on $\Gamma_0(2)$ (see 1.9). Clearly, $E_4^3-tE_6^2$ is nonzero at ∞ for all $t\neq1$. The valence formula for $\mathrm{SL}_2(\mathbb{Z})$ implies that for each $t\in\mathbb{C}^*\smallsetminus\{1\}$, $E_4^3-tE_6^2$ vanishes only at one point z_t in a fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$. Since t is nonzero and since E_4 and E_6 have no common zeros, neither $E_4(z_t)$ nor $E_6(z_t)$ is zero. In particular, the stabilizer of z_t in $\mathrm{SL}_2(\mathbb{Z})$ (hence, also in $\Gamma_0(2)$) is trivial. It follows that $E_4^3-tE_6^2$ only has a simple zero at z_t. In particular, for each $t\in\mathbb{C}^*\smallsetminus\{1\}$, the modular form above is not factorizable on $\mathrm{SL}_2(\mathbb{Z})$. Now, the valence formula (3.9) for $\Gamma_0(2)$ implies that for each such t, $E_4^3-tE_6^2$ has three distinct zeros on $\Gamma_0(2)\backslash\mathfrak{H}$, one of which is equivalent to z_t under the action of $\Gamma_0(2)$ on $\mathfrak{H}$, whereas the modular form

$$\eta_2(z_t)^{24}\frac{\eta^{16}}{\eta_2^8}-\eta(z_t)^{24}\frac{\eta_2^{16}}{\eta^8} \tag{1.15}$$

of weight 4 on $\Gamma_0(2)$ only has a simple zero at z_t in $\Gamma_0(2)\backslash\mathfrak{H}$. It follows that for $t \in \mathbb{C}^* \smallsetminus \{1\}$, the modular form above is a factor of $E_4^3 - tE_6^2$ on $\Gamma_0(2)$. Thus, every modular form on $\mathrm{SL}_2(\mathbb{Z})$ is factorizable on $\Gamma_0(2) \cap \Gamma_0(4) = \Gamma_0(4)$. However, since the smallest weight of which nonzero modular forms with the trivial multiplier system exist is 2, the modular form of weight 2 on $\Gamma_0(N)$ defined by

$$NE_2(Nz) - E_2(z) \tag{1.16}$$

(see [10]) is irreducible for all $N > 1$. It is not known whether for any level N, there exist any "irreducible modular form" of weight greater than 2. On the contrary, it follows from Theorem 3 below (or from Corollary 2 in [4]) that there exist irreducible holomorphic eta quotients (i. e. which are not products of other holomorphic eta quotients) of arbitrarily large weights. We shall also see some irreducibility criteria for holomorphic eta quotients in [6].

2. The results

In order to state our main results, first we introduce some notations. Denote the set of positive integers by $\mathbb{N}$. For $N \in \mathbb{N}$, by $\wp_N$ we denote the set of prime divisors of N. For a divisor d of N, we say that d *exactly divides* N and write $d \| N$ if $\gcd(d, N/d) = 1$. Below in Corollary 1, we provide an upper bound for the weights of the irreducible holomorphic eta quotients of a given level in terms of the function $\kappa : \mathbb{N} \to \mathbb{N}$ defined by

$$\kappa(N) = \varphi\left(\mathrm{rad}(N)\right) \prod_{\substack{p \in \wp_N \\ p^n \| N}} \big((n-1)(p-1)+2\big), \tag{2.1}$$

where φ denotes Euler's totient function and $\mathrm{rad}(N)$ denotes the product of the distinct prime divisors of N. Let $\mathrm{d} : \mathbb{N} \to \mathbb{N}$ denote the divisor function. Clearly, we have $\kappa(N) \leq \varphi(\mathrm{rad}(N))^2 \cdot \mathrm{d}(N)$. Below in Theorem 1 (resp. Corollary 1), we provide an upper bound $\varOmega(N)$ (resp. $\varOmega_0(N)$) for the number of the holomorphic eta quotients which are not factorizable on $\Gamma_0(N)$ (resp. irreducible holomorphic eta quotients of level N). We define the functions $\varOmega, \varOmega_0 : \mathbb{N} \to \mathbb{N}$ by $\varOmega(1) = \varOmega_0(1) = 1$ and for $N > 1$,

$$\begin{aligned} \varOmega(N) = \prod_{\substack{p \in \wp_N \\ p^n \| N}} p^{2\,\mathrm{d}(N)} \Big(\frac{p^2-1}{p^4}\Big)^{\mathrm{d}(N/p^n)} - \frac{1}{d(N)!} \prod_{\substack{p \in \wp_N \\ p^n \| N}} \frac{(p^2-1)^{\mathrm{d}(N)}}{(p+1)^{2\,\mathrm{d}(N/p^n)}} \\ + 2\Big(\mathrm{d}(N) - \sum_{\substack{p \in \wp_N \\ p^n \| N}} \mathrm{d}(N/p^n)\Big) - 2^{\omega(N)}(\omega(N)-1) \end{aligned} \tag{2.2}$$

and

$$\varOmega_0(N) = \varOmega(N) - 2\,\mathrm{d}(N) + 2^{\omega(N)} + 1, \tag{2.3}$$

where $\omega(N)$ denotes the number of distinct prime divisors of N. It is rather easy to show that $\varOmega(N)$ (and hence, also $\varOmega_0(N)$) is bounded above by $\mathrm{rad}(N)^{2\,\mathrm{d}(N)}$ for all N (see Lemma 5 below). We say that a holomorphic eta quotient f is *divisible* by a holomorphic eta quotient g if f/g is holomorphic. We shall show that

Theorem 1. *For all $N \in \mathbb{N}$, the following assertions hold:*

(*a*) *The weight of any holomorphic eta quotient on $\Gamma_0(N)$ which is not factorizable on $\Gamma_0(N)$ is less than $\kappa(N)/2$.*
(*b*) *The number of nonconstant holomorphic eta quotients on $\Gamma_0(N)$ which are not factorizable on $\Gamma_0(N)$ is bounded above by $\Omega(N)$.*
(*c*) *There are at most $\Omega_0(N)$ quasi-irreducible holomorphic eta quotients of level N.*

In particular, since any irreducible holomorphic eta quotient is quasi-irreducible, from the above theorem we conclude:

Corollary 1. *For all $N \in \mathbb{N}$, the following assertions hold:*

(*a*) *The weight of any irreducible holomorphic eta quotient of level N is less than $\kappa(N)/2$.*
(*b*) *The number of irreducible holomorphic eta quotients of level N is bounded above by $\Omega_0(N)$.* □

In fact, $\kappa(N)/2$ is the smallest possible weight for an eta quotient f such that f/g is holomorphic for all holomorphic eta quotients g which are not factorizable on $\Gamma_0(N)$:

Theorem 2. *For all $N \in \mathbb{N}$, there exists a holomorphic eta quotient F_N of weight $\kappa(N)/2$ on $\Gamma_0(N)$ such that a holomorphic eta quotient h on $\Gamma_0(N)$ is divisible by F_N if and only if h is divisible by all the holomorphic eta quotients on $\Gamma_0(N)$ which are not factorizable on $\Gamma_0(N)$.*

In the above theorem, the uniqueness of the eta quotient F_N readily follows from the claim. We shall see F_N explicitly in (4.2). We recall that the Reducibility Conjecture (see Conjecture 1′ in [4]) states: *Every quasi-irreducible holomorphic eta quotient is irreducible.* Since holomorphic eta quotients on $\Gamma_0(N)$ which are not factorizable on $\Gamma_0(N)$ are in particular quasi-irreducible, it follows from the above theorem that

Corollary 2. *If the Reducibility Conjecture* (*Conjecture* 1′ *in* [4]) *holds, then for all $N \in \mathbb{N}$, there exists a holomorphic eta quotient F_N of weight $\kappa(N)/2$ on $\Gamma_0(N)$ such that a holomorphic eta quotient h on $\Gamma_0(N)$ is divisible by F_N if and only if h is divisible by all the irreducible holomorphic eta quotients on $\Gamma_0(N)$.* □

We shall also show that

Theorem 3. *For $N \in \mathbb{N}$ and for any divisor t of $N/\mathrm{rad}(N)$, there is an irreducible holomorphic eta quotient of weight*

$$\frac{1}{2}\varphi(\mathrm{rad}(N))\,\varphi(\mathrm{rad}(\gcd(t, N/t)))$$

on $\Gamma_0(N)$. In particular, for $t = N/\mathrm{rad}(N)$, there exists an irreducible holomorphic eta quotient of level N and of the weight as above.

3. Notations and the basic facts

For $N \in \mathbb{N}$, by $\mathcal{D}_N$ we denote the set of divisors of N. For $X \in \mathbb{Z}^{\mathcal{D}_N}$, we define the eta quotient η^X by

$$\eta^X := \prod_{d \in \mathcal{D}_N} \eta_d^{X_d}, \tag{3.1}$$

where X_d is the value of X at $d \in \mathcal{D}_N$ whereas η_d denotes the rescaling of η by d. Clearly, the level of η^X divides N. In other words, η^X transforms like a modular form on $\Gamma_0(N)$. We define the summatory function $\sigma : \mathbb{Z}^{\mathcal{D}_N} \to \mathbb{Z}$ by

$$\sigma(X) := \sum_{d \in \mathcal{D}_N} X_d. \tag{3.2}$$

Since η is of weight $1/2$, the weight of η^X is $\sigma(X)/2$ for all $X \in \mathbb{Z}^{\mathcal{D}_N}$.

Recall that an eta quotient f on $\Gamma_0(N)$ is holomorphic if it does not have any poles at the cusps of $\Gamma_0(N)$. Under the action of $\Gamma_0(N)$ on $\mathbb{P}^1(\mathbb{Q})$ by Möbius transformation, for $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$, we have

$$[a : b] \sim_{\Gamma_0(N)} [a' : \gcd(N, b)] \tag{3.3}$$

for some $a' \in \mathbb{Z}$ which is coprime to $\gcd(N, b)$ (see [10]). We identify $\mathbb{P}^1(\mathbb{Q})$ with $\mathbb{Q} \cup \{\infty\}$ via the canonical bijection that maps $[\alpha : \lambda]$ to α/λ if $\lambda \neq 0$ and to ∞ if $\lambda = 0$. For $s \in \mathbb{Q} \cup \{\infty\}$ and a weakly holomorphic modular form f on $\Gamma_0(N)$, the order of f at the cusp s of $\Gamma_0(N)$ is the exponent of q^{1/w_s} occurring with the first nonzero coefficient in the q-expansion of f at the cusp s, where w_s is the width of the cusp s (see [10, 18]). The following is the set of the equivalence classes of the cusps of $\Gamma_0(N)$ (see [10, 14]):

$$\mathcal{S}_N := \left\{ \frac{a}{t} \in \mathbb{Q} \;\middle|\; t \in \mathcal{D}_N,\ a \in \mathbb{Z},\ \gcd(a, t) = 1 \right\} / \sim, \tag{3.4}$$

where $\dfrac{a}{t} \sim \dfrac{b}{t}$ if and only if $a \equiv b \pmod{\gcd(t, N/t)}$. For $d \in \mathcal{D}_N$ and for $s = \dfrac{a}{t} \in \mathcal{S}_N$ with $\gcd(a, t) = 1$, we have

$$\operatorname{ord}_s(\eta_d; \Gamma_0(N)) = \frac{N \cdot \gcd(d, t)^2}{24 \cdot d \cdot \gcd(t^2, N)} \in \frac{1}{24}\mathbb{N} \tag{3.5}$$

(see [14]). It is easy to check the above inclusion when N is a prime power. The general case follows by multiplicativity (see 3.13 and 3.16). It follows that for all $X \in \mathbb{Z}^{\mathcal{D}_N}$, we have

$$\operatorname{ord}_s(\eta^X; \Gamma_0(N)) = \frac{1}{24} \sum_{d \in \mathcal{D}_N} \frac{N \cdot \gcd(d, t)^2}{d \cdot \gcd(t^2, N)} X_d. \tag{3.6}$$

In particular, that implies

$$\operatorname{ord}_{a/t}(\eta^X; \Gamma_0(N)) = \operatorname{ord}_{1/t}(\eta^X; \Gamma_0(N)) \tag{3.7}$$

for all $t \in \mathcal{D}_N$ and for all the $\varphi(\gcd(t, N/t))$ inequivalent cusps of $\Gamma_0(N)$ represented by rational numbers of the form $\dfrac{a}{t} \in \mathcal{S}_N$ with $\gcd(a, t) = 1$. Let

$\psi(N)$ denote the index of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$. Then $\psi:\mathbb{N}\to\mathbb{N}$ is given by

$$\psi(N):=N\cdot\prod_{\substack{p|N\\ p\text{ prime}}}\left(1+\frac{1}{p}\right) \tag{3.8}$$

(see [10]). The *valence formula* for $\Gamma_0(N)$ (see [3, 18]) states:

$$\sum_{P\in\Gamma_0(N)\backslash\mathfrak{H}}\frac{1}{n_P}\cdot\mathrm{ord}_P(f)\,+\,\sum_{s\in\mathcal{S}_N}\mathrm{ord}_s(f\,;\Gamma_0(N))=\frac{k\cdot\psi(N)}{24}\,, \tag{3.9}$$

where $k\in\mathbb{Z}$, $f:\mathfrak{H}\to\mathbb{C}$ is a meromorphic function that transforms like a modular forms of weight $k/2$ on $\Gamma_0(N)$ which is also meromorphic at the cusps of $\Gamma_0(N)$ and n_P is the number of elements in the stabilizer of P in the group $\Gamma_0(N)/\{\pm I\}$, where $I\in\mathrm{SL}_2(\mathbb{Z})$ denotes the identity matrix. In particular, if f is an eta quotient, then from (3.9) we obtain

$$\sum_{s\in\mathcal{S}_N}\mathrm{ord}_s(f\,;\Gamma_0(N))=\frac{k\cdot\psi(N)}{24}\,, \tag{3.10}$$

because eta quotients do not have poles or zeros on $\mathfrak{H}$. it follows from (3.10) and from (3.7) that for an eta quotient f of weight $k/2$ on $\Gamma_0(N)$, the valence formula further reduces to

$$\sum_{t\,|N}\varphi(\gcd(t,N/t))\cdot\mathrm{ord}_{1/t}(f\,;\Gamma_0(N))=\frac{k\cdot\psi(N)}{24}\,. \tag{3.11}$$

Since $\mathrm{ord}_{1/t}(f\,;\Gamma_0(N))\in\frac{1}{24}\mathbb{Z}$ (see 3.5), from (3.11) it follows that of any particular weight, there are only finitely many holomorphic eta quotients on $\Gamma_0(N)$. More precisely, the number of holomorphic eta quotients of weight $k/2$ on $\Gamma_0(N)$ is at most the number of solutions of the following equation

$$\sum_{t\in\mathcal{D}_N}\varphi(\gcd(t,N/t))\cdot x_t=k\cdot\psi(N) \tag{3.12}$$

in nonnegative integers x_t.

We define the *order map* $\mathcal{O}_N:\mathbb{Z}^{\mathcal{D}_N}\to\frac{1}{24}\mathbb{Z}^{\mathcal{D}_N}$ of level N as the map which sends $X\in\mathbb{Z}^{\mathcal{D}_N}$ to the ordered set of orders of the eta quotient η^X at the cusps $\{1/t\}_{t\in\mathcal{D}_N}$ of $\Gamma_0(N)$. Also, we define the *order matrix* $A_N\in\mathbb{Z}^{\mathcal{D}_N\times\mathcal{D}_N}$ of level N by

$$A_N(t,d):=24\cdot\mathrm{ord}_{1/t}(\eta_d\,;\Gamma_0(N)) \tag{3.13}$$

for all $t,d\in\mathcal{D}_N$. For example, for a prime power p^n, we have

$$A_{p^n}=\begin{pmatrix} p^n & p^{n-1} & p^{n-2} & \cdots & p & 1\\ p^{n-2} & p^{n-1} & p^{n-2} & \cdots & p & 1\\ p^{n-4} & p^{n-3} & p^{n-2} & \cdots & p & 1\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 1 & p & p^2 & \cdots & p^{n-1} & p^{n-2}\\ 1 & p & p^2 & \cdots & p^{n-1} & p^n \end{pmatrix}. \tag{3.14}$$

By linearity of the order map, we have

$$\mathcal{O}_N(X) = \frac{1}{24} \cdot A_N X\,. \tag{3.15}$$

For $r \in \mathbb{N}$, if $Y, Y' \in \mathbb{Z}^{\mathcal{D}_N^r}$ is such that $Y - Y'$ is nonnegative at each element of $\mathcal{D}_N^r$, then we write $Y \geq Y'$. In particular, for $X \in \mathbb{Z}^{\mathcal{D}_N}$, the eta quotient η^X is holomorphic if and only if $A_N X \geq 0$.

From (3.13) and (3.5), we note that $A_N(t,d)$ is multiplicative in N, t and d. Hence, it follows that

$$A_N = \bigotimes_{\substack{p^n \| N \\ p \text{ prime}}} A_{p^n}\,, \tag{3.16}$$

where by $\otimes$, we denote the Kronecker product of matrices.*

It is easy to verify that for a prime power p^n, the matrix A_{p^n} is invertible with the tridiagonal inverse:

$$A_{p^n}^{-1} = \frac{1}{p^n \cdot (p - \frac{1}{p})} \begin{pmatrix} p & -p & & & & \\ -1 & p^2+1 & -p^2 & & \text{\huge 0} & \\ & -p & p\cdot(p^2+1) & -p^3 & & \\ & & \ddots & \ddots & \ddots & \\ & \text{\huge 0} & & -p^2 & p^2+1 & -1 \\ & & & & -p & p \end{pmatrix}, \tag{3.17}$$

where for each positive integer $j < n$, the nonzero entries of the column $A_{p^n}^{-1}(\_\,, p^j)$ are the same as those of the column $A_{p^n}^{-1}(\_\,, p)$ shifted down by $j-1$ entries and multiplied with $p^{\min\{j-1, n-j-1\}}$. More precisely,

$$p^n \cdot (p - \frac{1}{p}) \cdot A_{p^n}^{-1}(p^i, p^j) =$$

$$\begin{cases} p & \text{if } i = j = 0 \text{ or } i = j = n \\ -p^{\min\{j, n-j\}} & \text{if } |i-j| = 1 \\ p^{\min\{j-1, n-j-1\}} \cdot (p^2+1) & \text{if } 0 < i = j < n \\ 0 & \text{otherwise.} \end{cases} \tag{3.18}$$

For general N, the invertibility of the matrix A_N now follows by (3.16). Hence, any eta quotient on $\Gamma_0(N)$ is uniquely determined by its orders at the set of the cusps $\{1/t\}_{t \in \mathcal{D}_N}$ of $\Gamma_0(N)$. In particular, for distinct $X, X' \in \mathbb{Z}^{\mathcal{D}_N}$, we have $\eta^X \neq \eta^{X'}$. The last statement is also implied by the uniqueness of q-series expansion: Let $\eta^{\widehat{X}}$ and $\eta^{\widehat{X}'}$ be the *eta products* (i. e. $\widehat{X}, \widehat{X}' \geq 0$) obtained by multiplying η^X and $\eta^{X'}$ with a common denominator. The claim follows by induction on the weight of $\eta^{\widehat{X}}$ (or equivalently, the weight of $\eta^{\widehat{X}'}$) when we compare the corresponding first two exponents of q occurring in the q-series expansions of $\eta^{\widehat{X}}$ and $\eta^{\widehat{X}'}$.

*Kronecker product of matrices is not commutative. However, since any given ordering of the primes dividing N induces a lexicographic ordering on $\mathcal{D}_N$ with which the entries of A_N are indexed, Equation (3.16) makes sense for all possible orderings of the primes dividing N.

4. The finiteness

In this section, we prove the finiteness of irreducible holomorphic eta quotients of a given level (see the corollary of Theorem 1).

Let A_N be the order matrix of level N (see 3.13). From the invertibility of A_N, it follows trivially that for each $t \in \mathcal{D}_N$, there is an eta quotient which vanishes nowhere except at the cusps a/t of $\Gamma_0(N)$ for all integers a which are coprime to t (see Corollary 1.42 in the Preliminaries of [7]) *. Let $B_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ be the matrix whose columns are made of the exponents of these eta quotients. A little more precise description of B_N is as follows: Since all the entries of A_N^{-1} are rational (see 3.16, 3.17), for each $t \in \mathcal{D}_N$, there exists a smallest positive integer $m_{t,N}$ such that $m_{t,N} \cdot A_N^{-1}(\_\,, t)$ has integer entries, where $A_N^{-1}(\_\,, t)$ denotes the column of A_N indexed by $t \in \mathcal{D}_N$. We define $B_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ by

$$B_N(\_\,, t) := m_{t,N} \cdot A_N^{-1}(\_\,, t) \text{ for all } t \in \mathcal{D}_N. \tag{4.1}$$

Clearly, B_N is invertible over $\mathbb{Q}$. Recall that for $X \in \mathbb{Z}^{\mathcal{D}_N}$, η^X is holomorphic if and only if $A_N X \geq 0$ (see 3.15). We define the eta quotient F_N by

$$F_N := \prod_{t \in \mathcal{D}_N} \eta^{B_N(\_\,, t)}. \tag{4.2}$$

The lemma below follows immediately:

Lemma 1. *For $N \in \mathbb{N}$, let F_N be as defined above. Then for $X \in \mathbb{Z}^{\mathcal{D}_N}$, both of the eta quotients $f := \eta^X$ and F_N/f are holomorphic if and only if $X \in B_N \cdot [0,1]^{\mathcal{D}_N}$.* □

Let $X \in \mathbb{Z}^{\mathcal{D}_N} \smallsetminus \{0\}$ be such that η^X is a holomorphic eta quotient which is not factorizable on $\Gamma_0(N)$. Define $Y \in \mathbb{Z}^{\mathcal{D}_N}$ by $Y := B_N^{-1} X$. Suppose, for some $t \in \mathcal{D}_N$, we have $Y_t \geq 1$. Then η^X is divisible by the nonconstant holomorphic eta quotient $\eta^{B_N(\_\,, t)}$. Since η^X is not factorizable on $\Gamma_0(N)$, we conclude that $X = B_N(\_\,, t)$. Thus, we have proved that

Lemma 2. *For $N \in \mathbb{N}$, let $B_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ be as defined in (4.1). For $X \in \mathbb{Z}^{\mathcal{D}_N}$, if η^X is a holomorphic eta quotient which is not factorizable on $\Gamma_0(N)$, then either $X \in B_N \cdot [0,1)^{\mathcal{D}_N}$ or $X = B_N(\_\,, t)$ for some $t \in \mathcal{D}_N$.* □

Since for $N \in \mathbb{N}$, there are only finitely many lattice points in the bounded polytope $B_N \cdot [0,1)^{\mathcal{D}_N}$, from Lemma 2 it follows that there are only finitely many holomorphic eta quotients on $\Gamma_0(N)$ which are not factorizable on $\Gamma_0(N)$ (e. g. the irreducible holomorphic eta quotients whose levels divide N).

Proof of Theorem 1.(a). Let f be a holomorphic eta quotient on $\Gamma_0(N)$ which is not factorizable on $\Gamma_0(N)$. From the above lemma, we see that the weight of f is at most equal to the maximum value of $\sigma(X)/2$, where X varies over $B_N \cdot [0,1]^{\mathcal{D}_N}$ and σ is as defined in (3.2). Since for all $t \in \mathcal{D}_N$,

*The invertibility of the order matrix (and hence, the existence of such eta quotients) has been known classically. For example, see Satz 8 in [15], Proposition 3.2 in [12], the proof of Theorem 3 in [13] or the proof of Theorem 2 in [19].

the sum of all the entries in the column $B_N(\_\,,t)$ of B_N is positive (see 4.8), it follows that

$$\max_{X\in B_N\cdot[0,1]^{\mathcal{D}_N}} \sigma(X) = \sum_{t\in\mathcal{D}_N} \sigma(B_N(\_\,,t)).$$

Hence, it suffices to show that

$$\kappa(N) = \sum_{d\in\mathcal{D}_N} \sigma(B_N(\_\,,t)). \tag{4.3}$$

Since for $N\in\mathbb{N}$ and $t\in\mathcal{D}_N$, all the entries of the columns $A_N^{-1}(\_\,,t)$ are multiplicative in N and t (see 3.16), so is the smallest positive integer $m_{t,N}$ such that $m_{t,N}\cdot A_N^{-1}(\_\,,t)\in\mathbb{Z}^{\mathcal{D}_N}$ (see Lemma 4 in [4]). Hence, from the multiplicativity of $A_N^{-1}(d,t)$ in N, d and t (see 3.16), it follows that $B_N(d,t)$ (see 4.1) is also multiplicative in N, d and t. That implies:

$$B_N = \bigotimes_{\substack{p\in\wp_N\\ p^n\| N}} B_{p^n}, \tag{4.4}$$

where $\wp_N$ denotes the set of prime divisors of N. For a prime p, from (4.1) and (3.17), we have

$$B_{p^n} = \begin{pmatrix} p & -p & & & & \\ -1 & p^2+1 & -p & & \text{\huge 0} & \\ & -p & p^2+1 & -p & & \\ & & \ddots & \ddots & \ddots & \\ & \text{\huge 0} & & -p & p^2+1 & -1 \\ & & & & -p & p \end{pmatrix}. \tag{4.5}$$

Summing up the entries of each column of B_{p^n}, we get:

$$\sigma(B_{p^n}(\_\,,p^j)) = \begin{cases} p-1 & \text{if } j=0 \text{ or } j=n\\ (p-1)^2 & \text{otherwise.}\end{cases} \tag{4.6}$$

Since (4.4) implies that

$$B_N(\_\,,t) = \bigotimes_{\substack{p\in\wp_N\\ p^j\| t}} B_{p^n}(\_\,,p^j) \text{ for all } d\in\mathcal{D}_N, \tag{4.7}$$

from (4.6) we get:

$$\begin{aligned}\sigma(B_N(\_\,,t)) &= \prod_{\substack{p\in\wp_N\\ p^j\| t}} \sigma(B_{p^n}(\_\,,p^j)) \\ &= \Big(\prod_{\substack{p\in\wp_N\\ p\nmid\gcd(t,N/t)}} (p-1)\Big)\cdot\prod_{\substack{p\in\wp_N\\ p\mid\gcd(t,N/t)}} (p-1)^2 \\ &= \varphi(\mathrm{rad}(N))\cdot\varphi(\mathrm{rad}(\gcd(t,N/t))).\end{aligned} \tag{4.8}$$

Since $\varphi(\operatorname{rad}(\gcd(t,N/t)))$ is multiplicative in $t\in\mathcal{D}_N$, the summatory function $N\mapsto\sum_{t\in\mathcal{D}_N}\varphi(\operatorname{rad}(\gcd(t,N/t)))$ is multiplicative in N. So,

$$\begin{aligned}(4.9)\qquad \sum_{t\in\mathcal{D}_N}\varphi(\operatorname{rad}(\gcd(t,N/t))) &= \prod_{\substack{p\in\wp_N\\ p^n\|N}}\sum_{j=0}^{n}\varphi(\operatorname{rad}(p^{\min\{j,n-j\}}))\\ &= \prod_{\substack{p\in\wp_N\\ p^n\|N}}((n-1)(p-1)+2).\end{aligned}$$

Now, (4.3) follows from (4.8) and (4.9).

The only $X\in B_N\cdot[0,1]^{\mathcal{D}_N}$ with $\sigma(X)=\kappa(N)$ is $X=\sum_{t\in\mathcal{D}_N}B_N(\_\,,t)$. Since $N>1$, it follows trivially from Lemma 2, that for such an X, the holomorphic eta quotient η^X is factorizable on $\Gamma_0(N)$. □

Proof of Theorem 1.(b). In Lemma 2, we saw that each holomorphic eta quotient which is not factorizable on $\Gamma_0(N)$ correspond either to a column of B_N or to a lattice point in the fundamental parallelepiped $B_N\cdot[0,1)^{\mathcal{D}_N}$. Clearly, the number of the columns of B_N is $\mathrm{d}(N)$. In Lemma 3 below, we show that the number of lattice points in a fundamental parallelepiped of B_N is

$$(4.10)\qquad \varOmega'(N):=\prod_{\substack{p\in\wp_N\\ p^n\|N}}p^{2\,\mathrm{d}(N)}\Big(\frac{p^2-1}{p^4}\Big)^{\mathrm{d}(N/p^n)}.$$

However, there also exist lattice points in the fundamental parallelepiped $B_N\cdot[0,1)^{\mathcal{D}_N}$ which correspond to some holomorphic eta quotients which are factorizable on $\Gamma_0(N)$. For example, if X is a lattice point outside the unit sphere in $\mathbb{R}^{\mathrm{d}(N)}$ such that all its entries are nonnegative, then η^X is clearly factorizable on $\Gamma_0(N)$. In Lemma 4 below, we show that the number of such lattice points in $B_N\cdot[0,1)^{\mathcal{D}_N}$ is at least

$$\begin{aligned}(4.11)\qquad \varOmega''(N):=&\frac{1}{\mathrm{d}(N)!}\prod_{\substack{p^n\|N\\ p\text{ prime}}}\frac{(p^2-1)^{\mathrm{d}(N)}}{(p+1)^{2\,\mathrm{d}(N/p^n)}}+2\sum_{\substack{p\in\wp_N\\ p^n\|N}}\mathrm{d}(N/p^n)\\ &-2^{\omega(N)}(\omega(N)-1)-\mathrm{d}(N),\end{aligned}$$

where $\omega(N)$ denotes the number of distinct prime divisors of N. Hence, we conclude that the number of holomorphic eta quotients which are not factorizable on $\Gamma_0(N)$ is bounded above by

$$(4.12)\qquad \varOmega(N)=\mathrm{d}(N)+\varOmega'(N)-\varOmega''(N).$$

□

Now, we prove the lemmas which were necessary in the above proof:

Lemma 3. *There are exactly $\varOmega'(N)$ lattice points in a fundamental parallelepiped of B_N, where $\varOmega':\mathbb{N}\to\mathbb{N}$ is as defined in (4.10).*

Proof. Since the number of integer points in a fundamental parallelepiped of a nonsingular integer matrix is equal to the volume of the parallelepiped (see Theorem 2 in [2]), it suffices to show that the determinant of B_N is $\Omega'(N)$. Indeed, for a prime number p and a positive integer n, we have $\det(B_{p^n}) = \Omega'(p^n)$ which follows trivially after transforming B_{p^n} (see 4.5) to the following matrix through elementary column operations

$$\begin{pmatrix} p & & & & & & & & & \\ -1 & p^2 & & & & & & & & \\ & -p & p^2 & & & & & & \mathbf{0} & \\ & & \ddots & \ddots & & & & & & \\ & & & -p & p^2 & & & & & \\ & & & & -p & p^2-1 & -p & & & \\ & & & & & & p^2 & -p & & \\ & \mathbf{0} & & & & & & \ddots & \ddots & \\ & & & & & & & p^2 & -p & \\ & & & & & & & & p^2 & -1 \\ & & & & & & & & & p \end{pmatrix}$$

and from the fact that for square matrices A and D, we have

$$\det \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det(A)\det(D).$$

Since for any two matrices $A_{m\times m}$ and $B_{n\times n}$,

$$\det(A \otimes B) = \det(A)^n \det(B)^m \tag{4.13}$$

(see [11]), the general case now follows by induction on the number of prime divisors of N (see 4.4), $\square$

Lemma 4. *Let $\Omega'' : \mathbb{N} \to \mathbb{N}$ be as defined in (4.11). In the fundamental parallelepiped $B_N \cdot [0,1)^{\mathcal{D}_N}$, there are at least $\Omega''(N)$ lattice points with nonnegative coordinates, none of which lies on the unit sphere in $\mathbb{R}^{\mathcal{D}_N}$.*

Proof. From (4.1), it follows that B_N is invertible for all $N \in \mathbb{N}$. For $n \in \mathbb{N}$ and a prime p, the matrix B_{p^n} is as in (4.5). It is easy to verify that

$$B_{p^n}^{-1} = \frac{1}{p^n \cdot (p - \frac{1}{p})} \begin{pmatrix} p^n & p^{n-1} & p^{n-2} & \cdots & p & 1 \\ p^{n-2} & p^{n-1} & p^{n-2} & \cdots & p & 1 \\ p^{n-3} & p^{n-2} & p^{n-1} & \cdots & p^2 & p \\ p^{n-4} & p^{n-3} & p^{n-2} & \cdots & p^3 & p^2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ p & p^2 & p^3 & \cdots & p^{n-2} & p^{n-3} \\ 1 & p & p^2 & \cdots & p^{n-1} & p^{n-2} \\ 1 & p & p^2 & \cdots & p^{n-1} & p^n \end{pmatrix}. \tag{4.14}$$

Clearly, the axes-intercepts of the fundamental parallelepiped $B_N \cdot [0,1]^{\mathcal{D}_N}$ is given by the reciprocals of the diagonal entries of B_N^{-1}. Hence, from (4.4) and

(4.14), it follows that the coordinates of the furthest points in $B_N \cdot [0,1]^{\mathcal{D}_N}$ on the axes of $\mathbb{R}^{\mathcal{D}_N}$ are given by the columns of the matrix

$$\bigotimes_{\substack{p \in \wp_N \\ p^n \| N}} \begin{pmatrix} p - \frac{1}{p} & & & & & \\ & p^2 - 1 & & & \text{\huge 0} & \\ & & p^2 - 1 & & & \\ & & & \ddots & & \\ & \text{\huge 0} & & & p^2 - 1 & \\ & & & & & p - \frac{1}{p} \end{pmatrix}. \tag{4.15}$$

In particular, $B_N \cdot [0,1]^{\mathcal{D}_N}$ contains the simplex S_N which is the convex hull of the origin and the points in $\mathbb{R}^{\mathcal{D}_N}$ whose coordinates are given by the columns of the following matrix

$$C_N = \bigotimes_{\substack{p \in \wp_N \\ p^n \| N}} \begin{pmatrix} p - 1 & & & & & \\ & p^2 - 1 & & & \text{\huge 0} & \\ & & p^2 - 1 & & & \\ & & & \ddots & & \\ & \text{\huge 0} & & & p^2 - 1 & \\ & & & & & p - 1 \end{pmatrix}. \tag{4.16}$$

The number of lattice points in the $\mathrm{d}(N)$-dimensional rectangular parallelepiped $P_N := C_N \cdot [0,1)^{\mathcal{D}_N}$ is clearly the same as its volume. i. e. $\det(C_N)$. Since the ratio of the volumes of S_N and P_N is $1/d(N)!$ (see [20]), the simplex S_N contains at least

$$\frac{(\det C_N)}{d(N)!} \tag{4.17}$$

lattice points excluding all the vertices of S_N except the origin. From (4.15) and (4.16), it follows that for $t \in \mathcal{D}_N$, $\gcd(t, N/t)$ is divisible by $\mathrm{rad}(N)$ if and only if

$$C_N(\_\,, t) \in B_N \cdot [0,1]^{\mathcal{D}_N} \smallsetminus B_N \cdot [0,1)^{\mathcal{D}_N}. \tag{4.18}$$

In other words, the number of nonzero vertices of S_N which are contained in $B_N \cdot [0,1)^{\mathcal{D}_N}$ is the same as the number of $t \in \mathcal{D}_N$ such that $\mathrm{rad}(N)$ does not divide $\gcd(t, N/t)$. It is easy to show that the number of such divisors of N is

$$2 \sum_{\substack{p \in \wp_N \\ p^n \| N}} \mathrm{d}(N/p^n) - 2^{\omega(N)}(\omega(N) - 1), \tag{4.19}$$

where $\omega(N)$ denotes the number of distinct prime divisors of N. Again, from (4.16) and (4.13), it follows that

$$\det(C_N) = \prod_{\substack{p^n \| N \\ p \text{ prime}}} \frac{(p^2 - 1)^{\mathrm{d}(N)}}{(p+1)^{2\,\mathrm{d}(N/p^n)}} \tag{4.20}$$

From (4.11), (4.17), (4.19) and (4.20), we obtain that there are at least $\Omega''(N)+\mathrm{d}(N)$ lattice points in $B_N\cdot[0,1)^{\mathcal{D}_N}$ with nonnegative coordinates. However, exactly $\mathrm{d}(N)$ among these points lie on intersections of the unit sphere with the axes of $\mathbb{R}^{\mathcal{D}_N}$. $\square$

Proof of Theorem 1.(c). From Lemma 2, we recall that for $X\in\mathbb{Z}^{\mathcal{D}_N}$, if η^X is a holomorphic eta quotient which is not factorizable on $\Gamma_0(N)$, then either $X\in B_N\cdot[0,1)^{\mathcal{D}_N}$ or $X=B_N(\_\,,t)$ for some $t\in\mathcal{D}_N$. The parallelepiped $B_N\cdot[0,1)^{\mathcal{D}_N}$ contains $\mathrm{d}(N)$ points which lie at the intersections of the unit sphere with the axes of $\mathbb{R}^{\mathcal{D}_N}$. These points corresponds to the rescalings of η by the divisors of N. In particular, these are eta quotients of weight $1/2$. So, each of these $\mathrm{d}(N)$ rescalings of η are irreducible, whereas only one of them, viz. η_N is of level N.

Next, we count the number of eta quotients of the form $\eta^{B_N(\_\,,t)}$ which are of level N. For a prime p, from (4.5) we see that the eta quotient $\eta^{B_{p^n}(\_\,,p^j)}$ is of level p^n if and only if $j\geq n-1$. Hence, from (4.7) it follows that for $N\in\mathbb{N}$, the eta quotient $\eta^{B_N(\_\,,t)}$ is of level N if and only if for each prime divisor p of N, we have $p^{n-1}|t$, where $n\in\mathbb{N}$ is such that $p^n\|N$. It is trivial to note that the number of such divisors t of N is $2^{\omega(N)}$, where $\omega(N)$ denotes the number of prime divisors of N. Thus, among the $\mathrm{d}(N)$ columns of B_N, only $2^{\omega(N)}$ correspond to eta quotients of level N. $\square$

In the following, we provide a rather uncomplicated function which dominates $\Omega(N)$ for all N:

Lemma 5. *Let $\Omega:\mathbb{N}\to\mathbb{N}$ be as defined in (2.2). For all $N\in\mathbb{N}$, we have*
$$\Omega(N)\leq\mathrm{rad}(N)^{2\,\mathrm{d}(N)}.$$

Proof. From the proof of Theorem 1.(b), we see that $\Omega(N)<\Omega'(N)+\mathrm{d}(N)$ for all $N>1$. By induction on the number of prime divisors of N, it follows easily that $\Omega'(N)+N^2\leq\mathrm{rad}(N)^{2\,\mathrm{d}(N)}$ for all $N>1$. $\square$

5. The common multiple with the least weight

In the previous section, we saw that if a holomorphic eta quotient on $\Gamma_0(N)$ is not factorizable on $\Gamma_0(N)$, then its weight is at most equal to $\kappa(N)/2$. In this section, we show that $\kappa(N)/2$ is the smallest possible weight for an eta quotient f such that for each holomorphic eta quotient g which is not factorizable on $\Gamma_0(N)$, f/g is holomorphic (see Theorem 2).

Lemma 6. *For $N\in\mathbb{N}$ and $t\in\mathcal{D}_N$, the holomorphic eta quotient $\eta^{B_N(\_\,,t)}$ is not factorizable on $\Gamma_0(N)$, where $B_N\in\mathbb{Z}^{\mathcal{D}_N\times\mathcal{D}_N}$ is as defined in (4.1).*

Proof. For $t\in\mathcal{D}_N$ and for $Y=A_N\cdot B_N(\_\,,t)\in\mathbb{Z}^{\mathcal{D}_N}$, from (4.1) we get
$$Y_d=\begin{cases} m_{t,N} & \text{if } d=t\\ 0 & \text{otherwise}\end{cases}$$
for all $d\in\mathcal{D}_N$. Recall that for $X\in\mathbb{Z}^{\mathcal{D}_N}$, η^X is holomorphic if and only if $A_NX\geq0$ (see 3.15). Suppose, $\eta^{B_N(\_\,,t)}$ is factorizable on $\Gamma_0(N)$. Then there are $X',X''\in\mathbb{Z}^{\mathcal{D}_N}\smallsetminus\{0\}$ with $B_N(\_\,,t)=X'+X''$ such that $A_NX'\geq0$

and $A_N X'' \geq 0$. Hence, there exist $m', m'' > 0$ with $m_{t,N} = m' + m''$ such that for $d \in \mathcal{D}_N$, we have

$$(A_N X')_d = \begin{cases} m' & \text{if } d = t, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (A_N X'')_d = \begin{cases} m'' & \text{if } d = t, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, we have $X' = m' \cdot A_N^{-1}(\_\,, d)$ and $X'' = m'' \cdot A_N^{-1}(\_\,, d)$. Since $m', m'' < m_{t,N}$ and since $m_{t,N}$ is the smallest positive integer such that $m_{t,N} \cdot A_N^{-1}(\_\,, t) \in \mathbb{Z}^{\mathcal{D}_N}$, we conclude that $X' \notin \mathbb{Z}^{\mathcal{D}_N}$ and $X'' \notin \mathbb{Z}^{\mathcal{D}_N}$. Thus, we get a contradiction! Hence, $\eta^{B_N(\_\,,t)}$ is not factorizable on $\Gamma_0(N)$. □

Proof of Theorem 2. Let F_N be the same as in (4.2). Then Lemma 1 and Lemma 2 together imply that if a holomorphic eta quotient h on $\Gamma_0(N)$ is divisible by F_N, then it is divisible by all the holomorphic eta quotients on $\Gamma_0(N)$ which are not factorizable on $\Gamma_0(N)$.

Conversely, let a holomorphic eta quotient h on $\Gamma_0(N)$ be divisible by each holomorphic eta quotient g on $\Gamma_0(N)$ which is not factorizable on $\Gamma_0(N)$. Let $B_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ be as defined in (4.1). Then Lemma 6 implies that h is divisible by $\eta^{B_N(\_\,,t)}$ for all $t \in \mathcal{D}_N$. So in particular, we have

$$\text{(5.1)} \qquad \operatorname{ord}_{1/t}(h; \Gamma_0(N)) \geq \operatorname{ord}_{1/t}(\eta^{B_N(\_\,,t)}; \Gamma_0(N)) = \operatorname{ord}_{1/t}(F_N; \Gamma_0(N)),$$

where the last equality holds since F_N is the product of all the eta quotients $\{\eta^{B_N(\_\,,t)}\}_{t \in \mathcal{D}_N}$, and since (4.1) and (3.15) together imply that $\eta^{B_N(\_\,,t)}$ has nonzero order only at the cusp $1/t$ of $\Gamma_0(N)$. Since any eta quotient on $\Gamma_0(N)$ is uniquely determined by its orders at the set of the cusps $\{1/t\}_{t \in \mathcal{D}_N}$ of $\Gamma_0(N)$, from (5.1) it follows that h is divisible by F_N. □

6. Examples of irreducible holomorphic eta quotients

In this section, we shall show that there exist holomorphic eta quotients of arbitrarily large weights (see Theorem 3).

Lemma 7. *For $N \in \mathbb{N}$ and $t \in \mathcal{D}_{N/\mathrm{rad}(N)}$, the holomorphic eta quotient $\eta^{B_N(\_\,,t)}$ is irreducible, where $B_N \in \mathbb{Z}^{\mathcal{D}_N \times \mathcal{D}_N}$ is as defined in (4.1).*

Proof. From Theorem 3 in [4], we know that a holomorphic eta quotient on $\Gamma_0(N)$ is reducible only if it is factorizable on some $\Gamma_0(M)$ for some multiple M of N with $\mathrm{rad}(M) = \mathrm{rad}(N)$. Suppose, for some $t \in \mathcal{D}_{N/\mathrm{rad}(N)}$ the holomorphic eta quotient $\eta^{B_N(\_\,,t)}$ is reducible. Then there exists a multiple M of N with $\mathrm{rad}(M) = \mathrm{rad}(N)$ such that $\eta^{B_N(\_\,,t)}$ is factorizable on $\Gamma_0(M)$. Since $t \in \mathcal{D}_{N/\mathrm{rad}(N)} \subseteq \mathcal{D}_{M/\mathrm{rad}(M)}$, it follows from (4.7) and (4.5) that $B_M(d,t) = B_N(d,t)$ for all $d|N$ and $B_M(d,t) = 0$ if $d \nmid N$. In other words, we have $\eta^{B_N(\_\,,t)} = \eta^{B_M(\_\,,t)}$ which is not factorizable on $\Gamma_0(M)$ by Lemma 6. Thus, we get a contradiction! Hence, for all $t \in \mathcal{D}_{N/\mathrm{rad}(N)}$, $\eta^{B_N(\_\,,t)}$ is irreducible. □

Proof of Theorem 3. Since for all $X \in \mathbb{Z}^{\mathcal{D}_N}$, the weight of the eta quotient η^X is $\sigma(X)/2$, the theorem follows immediately from Lemma 7, (4.8) and from the fact that for $t = N/\mathrm{rad}(N)$, the eta quotient $\eta^{B_N(\_\,,t)}$ is of level N (see 4.7 and 4.5). □

Appendix: Comparison of the weights

By $k_{\max}(N)/2$, we denote the maximum of the weights of holomorphic eta quotients of level N which are not factorizable on $\Gamma_0(N)$. Let p be a prime. From the discussion about holomorphic eta quotients on $\Gamma_0(p)$ in Section 1, it follows that $k_{\max}(p) = p-1$. Also, from Theorem 6.4 in [7], we know $k_{\max}(p^2) = (p-1)^2$. With the support of a huge amount of experimental data, we make the following conjecture:

Conjecture 2. (*a*) *For each prime number p, all the irreducible holomorphic eta quotients of level p^3 are rescalings of eta quotients of smaller levels. In particular, we have $k_{\max}(p^3) = (p-1)^2$.*
(*b*) *For each odd prime p and for all integers $n > 3$, we have*

$$k_{\max}(p^n) = (n-1)(p-1)^2 - 2^{r_n}\left(\left\lfloor\frac{n}{2}\right\rfloor(p-1)-1\right), \tag{6.1}$$

where $r_n \in \{0,1\}$ is the residue of n modulo 2.

For all odd primes p and for all integers $n > 3$, in [6] we see examples of irreducible holomorphic eta quotients of level p^n and of the same weight as in (6.1) (see Corollary 1 and (2.1) in [6]). However, the catch of the above problem is to show that any holomorphic eta quotient of level p^n whose weight is greater than the quantity given in (6.1), must be reducible (see Conjecture 1 in [6] and Theorem 2 in [4]).

In the table below, we compare $k_{\max}(N)$ with $\kappa(N)$ for several $N \in \mathbb{N}$, where $\kappa(N)/2$ is the weight of the eta quotient F_N which we defined in Theorem 2 (see also 4.2). Since we have already discussed above the cases of odd prime powers as well as those of 2^n for $n \leq 3$, we omit such levels from the following table.

Table 2. $k_{\max}(N)$ vs. $\kappa(N)$

N	$k_{\max}$	κ	N	$k_{\max}$	κ	N	$k_{\max}$	κ
$2\cdot 3$	2	8	$3\cdot 13$	24	96	$2\cdot 37$	36	144
$2\cdot 5$	4	16	$2^3\cdot 5$	8	32	$2\cdot 3\cdot 13$	45	192
$2^2\cdot 3$	3	12	$2\cdot 3\cdot 7$	23	96	$2^4\cdot 5$	11	40
$2\cdot 7$	6	24	$2^2\cdot 11$	13	60	$5\cdot 17$	64	256
$3\cdot 5$	8	32	$3^2\cdot 5$	18	64	$2^3\cdot 11$	20	80
2^4	2	5	$2\cdot 23$	22	88	$2\cdot 47$	46	184
$2\cdot 3^2$	5	16	$2^4\cdot 3$	6	20	$2^5\cdot 3$	8	24
$2^2\cdot 5$	5	24	$2\cdot 5^2$	17	48	$2\cdot 7^2$	37	96
$3\cdot 7$	12	48	$3\cdot 17$	32	128	$3^2\cdot 11$	45	160
$2^3\cdot 3$	5	16	$2^2\cdot 13$	16	72	$2^2\cdot 5^2$	25	72
$2\cdot 13$	12	48	$2\cdot 3^3$	7	24	$2\cdot 3\cdot 17$	60	256
$2^2\cdot 7$	8	36	$2^3\cdot 7$	12	48	$2^3\cdot 13$	24	96
$2\cdot 3\cdot 5$	15	64	$3\cdot 19$	36	144	$3\cdot 5\cdot 7$	56	384
2^5	2	6	2^6	3	7	$3\cdot 37$	72	288
$2\cdot 17$	16	64	$2\cdot 3\cdot 11$	38	160	$2^4\cdot 7$	18	60
$2^2\cdot 3^2$	6	24	$2^2\cdot 17$	20	96	2^7	3	8
$2\cdot 19$	18	72	$2\cdot 5\cdot 7$	33	192	$7\cdot 19$	108	432

N	$k_{\max}$	κ	N	$k_{\max}$	κ	N	$k_{\max}$	κ
$3^3 \cdot 5$	32	96	$2^4 \cdot 11$	30	100	$17 \cdot 97$	1536	6144
$2^3 \cdot 17$	34	128	$11 \cdot 19$	180	720	2^{11}	6	12
$2^2 \cdot 37$	48	216	2^8	4	9	2^{12}	7	13
$2 \cdot 3^4$	13	32	2^9	5	10	2^{13}	7	14
$2 \cdot 5 \cdot 17$	85	512	$2 \cdot 503$	502	2008	2^{14}	9	15
$2^2 \cdot 43$	56	252	2^{10}	6	11	2^{15}	9	16

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